Ultimate Boundedness of a Neural Network Identifier for Nonlinear Dynamics via Robust Adaptive Update

Donghwa Hong

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1 Problem Formulation

1.1 System and Identifier Dynamics

The nonlinear system is described by:

$$\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{x}(t) + g(\boldsymbol{x}, u) + h(\boldsymbol{x}, u) \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a known Hurwitz matrix, $g(\cdot)$ is an known nonlinear function and $h(\cdot)$ is an unknown nonlinear function. The NN identifier is:

$$\dot{\hat{\boldsymbol{x}}}(t) = \mathbf{A}\hat{\boldsymbol{x}}(t) + g(\boldsymbol{x}, u) + \hat{h}(\hat{\bar{\boldsymbol{x}}}) \tag{2}$$

where the NN output is $\hat{h}(\hat{x}, u) = \hat{\mathbf{W}}^T\sigma(\hat{\mathbf{V}}^T\hat{\bar{\boldsymbol{x}}})$. Here, $\hat{\mathbf{W}} \in \mathbb{R}^{h\times n}$ and $\hat{\mathbf{V}} \in \mathbb{R}^{d\times h}$ are the estimated weight matrices. The NN input $\hat{\bar{\boldsymbol{x}}}$ is constructed from the estimated states $\hat{\boldsymbol{x}}$ and other available signals.

1.2 Error Dynamics

Defining the errors $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}$, $\tilde{\mathbf{W}} = \mathbf{W} - \hat{\mathbf{W}}$, and $\tilde{\mathbf{V}} = \mathbf{V} - \hat{\mathbf{V}}$, the error dynamics are given by:

$$\dot{\tilde{\boldsymbol{x}}} = \mathbf{A}\tilde{\boldsymbol{x}} + \tilde{\mathbf{W}}^T\sigma(\hat{\mathbf{V}}^T\hat{\bar{\boldsymbol{x}}}) + w(t) \tag{3}$$

where $w(t) = \mathbf{W}^T(\sigma(\mathbf{V}^T\bar{\boldsymbol{x}}) - \sigma(\hat{\mathbf{V}}^T\hat{\bar{\boldsymbol{x}}})) + \epsilon(x)$ is the lumped disturbance term.

Assumption 1. *The ideal weights $\mathbf{W}, \mathbf{V}$ are bounded. The disturbance $w(t)$ is bounded by $\|w(t)\| \leq w_M$. The activation function σ and its derivatives are bounded.*

Assumption 2 (Open-Loop Stability)**.** *The open-loop system* (1) *is stable, which implies that the state vector $\boldsymbol{x}(t)$ is bounded in L_∞.*

2 Update Law and Stability Analysis

2.1 Update Laws

Theorem 1. *Consider the plant model* (1) *and the identifier model* (2)*. Given Assumption 2, if the weights of the NLPNN are updated according to*

$$\dot{\hat{\mathbf{W}}} = -\eta_1\left(\frac{\partial J}{\partial \hat{\mathbf{W}}}\right) - \rho_1\|\tilde{\boldsymbol{x}}\|\hat{\mathbf{W}}, \tag{4}$$

$$\dot{\hat{\mathbf{V}}} = -\eta_2\left(\frac{\partial J}{\partial \hat{\mathbf{V}}}\right) - \rho_2\|\tilde{\boldsymbol{x}}\|\hat{\mathbf{V}}, \tag{5}$$

where $\eta > 0$ is the learning rate, $J = \frac{1}{2}\int_0^t e^{-\lambda(t-\tau)}\|\tilde{\boldsymbol{x}}(\tau)\|^2\,\mathrm{d}\tau$ is the objective function and ρ is a small positive number, then $\tilde{\boldsymbol{x}}$, $\tilde{\mathbf{W}}$, and $\tilde{\mathbf{V}} \in L_\infty$.

Proof. Since the cost functional is of an integral form, we first introduce the filtered error signal $\boldsymbol{z}(t)$ to construct the update laws.

$$\boldsymbol{z}(t) = \int_0^t e^{-\lambda(t-\tau)}\tilde{\boldsymbol{x}}(\tau)\,\mathrm{d}\tau \tag{6}$$

$$\dot{\boldsymbol{z}} = -\lambda\boldsymbol{z} + \tilde{\boldsymbol{x}} \tag{7}$$

Let us define

$$\mathrm{net}_{\hat{\mathbf{V}}} = \hat{\mathbf{V}}\hat{\boldsymbol{x}} \tag{8}$$

$$\mathrm{net}_{\hat{\mathbf{W}}} = \hat{\mathbf{W}}\sigma(\hat{\mathbf{V}}\hat{\boldsymbol{x}}). \tag{9}$$

Therefore, by using the chain rule $\frac{\partial J}{\partial \hat{\mathbf{W}}}$ and $\frac{\partial J}{\partial \hat{\mathbf{V}}}$ can be computed according to

$$\frac{\partial J}{\partial \hat{\mathbf{W}}} = \frac{\partial J}{\partial \mathrm{net}_{\hat{\mathbf{W}}}} \cdot \frac{\partial \mathrm{net}_{\hat{\mathbf{W}}}}{\partial \hat{\mathbf{W}}}$$

$$\frac{\partial J}{\partial \hat{\mathbf{V}}} = \frac{\partial J}{\partial \mathrm{net}_{\hat{\mathbf{V}}}} \cdot \frac{\partial \mathrm{net}_{\hat{\mathbf{V}}}}{\partial \hat{\mathbf{V}}},$$

where

$$\frac{\partial J}{\partial \mathrm{net}_{\hat{\mathbf{W}}}} = \frac{\partial J}{\partial \tilde{\boldsymbol{x}}}\frac{\partial \tilde{\boldsymbol{z}}}{\partial \hat{\boldsymbol{x}}}\frac{\partial \hat{\boldsymbol{x}}}{\partial \mathrm{net}_{\hat{\mathbf{W}}}} = -\boldsymbol{z}^T\frac{\partial \hat{\boldsymbol{x}}}{\partial \mathrm{net}_{\hat{\mathbf{W}}}},$$

$$\frac{\partial J}{\partial \mathrm{net}_{\hat{\mathbf{V}}}} = \frac{\partial J}{\partial \tilde{\boldsymbol{x}}}\frac{\partial \tilde{\boldsymbol{x}}}{\partial \hat{\boldsymbol{x}}}\frac{\partial \hat{\boldsymbol{x}}}{\partial \mathrm{net}_{\hat{\mathbf{V}}}} = -\boldsymbol{z}^T\frac{\partial \hat{\boldsymbol{x}}}{\partial \mathrm{net}_{\hat{\mathbf{V}}}} \tag{10}$$

and

$$\frac{\partial \mathrm{net}_{\hat{\mathbf{W}}}}{\partial \hat{\mathbf{W}}} = \sigma(\hat{\mathbf{V}}\hat{\boldsymbol{x}})$$

$$\frac{\partial \mathrm{net}_{\hat{\mathbf{V}}}}{\partial \hat{\mathbf{V}}} = \hat{\boldsymbol{x}}. \tag{11}$$

We modify the original BP algorithm such that the static approximations of $\frac{\partial \hat{\boldsymbol{x}}}{\partial \mathrm{net}_{\hat{\mathbf{W}}}}$ and $\frac{\partial \hat{\boldsymbol{x}}}{\partial \mathrm{net}_{\hat{\mathbf{V}}}}$ ($\dot{\hat{\boldsymbol{x}}} = 0$) can be used.

$$\frac{\partial \hat{\boldsymbol{x}}}{\partial \mathrm{net}_{\hat{\mathbf{W}}}} \approx -\mathbf{A}^{-1}$$

$$\frac{\partial \hat{\boldsymbol{x}}}{\partial \mathrm{net}_{\hat{\mathbf{V}}}} \approx -\mathbf{A}^{-1}\hat{\mathbf{W}}(\mathbf{I} - \boldsymbol{\Lambda}(\hat{\mathbf{V}}\hat{\boldsymbol{x}})), \tag{12}$$

where

$$\boldsymbol{\Lambda}(\hat{\mathbf{V}}\hat{\boldsymbol{x}}) = \mathrm{diag}\{\sigma_i^2(\hat{\mathbf{V}}_i\hat{\boldsymbol{x}})\}, \quad i = 1, 2, \ldots, m. \tag{13}$$

$$\dot{\hat{\mathbf{W}}} = -\eta_1\left(z^T\boldsymbol{A}^{-1}\right)^T\sigma(\hat{\mathbf{V}}^T\hat{\bar{\boldsymbol{x}}})^T - \rho_1\|\tilde{\boldsymbol{x}}\|\hat{\mathbf{W}} \tag{14}$$

$$\tag{15}$$

$$\dot{\hat{\mathbf{V}}} = -\eta_2\hat{\bar{\boldsymbol{x}}}\left(\boldsymbol{z}^T\boldsymbol{A}^{-1}\hat{\mathbf{W}}(\mathbf{I} - \boldsymbol{\Lambda}(\hat{\mathbf{V}}\hat{\boldsymbol{x}}))\right)^T - \rho_2\|\tilde{\boldsymbol{x}}\|\hat{\mathbf{V}} \tag{16}$$

$$\tag{17}$$

where $\eta_W, \eta_V, \rho_1, \rho_2 > 0$ are design parameters. By using (14) and (15) in terms of $\tilde{\mathbf{W}}$ may be written as

$$\dot{\tilde{\mathbf{W}}} = \eta_1(\boldsymbol{z}^T\mathbf{A}^{-1})^T(\sigma(\hat{\boldsymbol{x}}))^T + \rho_1\|\tilde{\boldsymbol{x}}\|\hat{\mathbf{W}}. \tag{18}$$

$$\dot{\tilde{\mathbf{V}}} = \eta_2\hat{\bar{\boldsymbol{x}}}\left(\boldsymbol{z}^T\boldsymbol{A}^{-1}\hat{\mathbf{W}}\,\mathrm{diag}(\sigma'(\hat{\mathbf{V}}^T\hat{\bar{\boldsymbol{x}}}))\right)^T - \rho_2\|\tilde{\boldsymbol{x}}\|\hat{\mathbf{V}} \tag{19}$$

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2.2 Lyapunov Stability

Theorem 2. *For the system given by* (3) *with the update laws* (15)-(17)*, all signals in the system* ($\tilde{\boldsymbol{x}}, \tilde{\mathbf{W}}, \tilde{\mathbf{V}}$) *are Uniformly Ultimately Bounded.*

Proof. The stability proof is conducted in two steps using a cascaded system approach.

We first prove the boundedness of the state error $\tilde{\boldsymbol{x}}$ and the output layer weight error $\tilde{\mathbf{W}}$. This is possible because the term $\sigma_v = \sigma(\hat{\mathbf{V}}^T\hat{\tilde{\boldsymbol{x}}})$ in the error dynamics (3) is always bounded, regardless of the value of $\hat{\mathbf{V}}$, due to the bounded nature of the activation function σ.

Consider the Lyapunov function candidate for the first subsystem:

$$L = \frac{1}{2}\tilde{\boldsymbol{x}}^T\mathbf{P}_1\tilde{\boldsymbol{x}} + \frac{1}{2}\,\mathrm{tr}(\tilde{\mathbf{W}}^T\rho^{-1}\tilde{\mathbf{W}}) + \frac{1}{2}\int_0^t e^{-\lambda(t-\tau)}\tilde{\boldsymbol{x}}(\tau)^T\mathbf{P}_2\tilde{\boldsymbol{x}}(\tau)\,\mathrm{d}\tau \tag{20}$$

Its time derivative, after substituting the error dynamics, is:

$$\dot{L} = -\frac{1}{2}\tilde{\boldsymbol{x}}^T(\mathbf{Q}_1 - \mathbf{P}_2)\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{x}}^T\mathbf{P}_1(\tilde{\mathbf{W}}^T\sigma_v + w) + \mathrm{tr}(\dot{\tilde{\mathbf{W}}}^T\rho^{-1}\tilde{\mathbf{W}}) - \lambda L_{\mathrm{int}} \tag{21}$$

where $\sigma_v = \sigma(\hat{\mathbf{V}}^T\hat{\tilde{\boldsymbol{x}}})$, $\mathbf{Q} = \mathbf{Q}_1 - \mathbf{P}_2 > 0$, and $L_{\mathrm{int}} = \frac{1}{2}\int_0^t e^{-\lambda(t-\tau)}\tilde{\boldsymbol{x}}(\tau)^T\mathbf{P}_2\tilde{\boldsymbol{x}}(\tau)\,\mathrm{d}\tau$. We substitute the update law (15) using $\dot{\tilde{\mathbf{W}}} = -\dot{\hat{\mathbf{W}}}$.

$$\begin{aligned}
\mathrm{tr}(\dot{\tilde{\mathbf{W}}}^T\rho^{-1}\tilde{\mathbf{W}}) &= \mathrm{tr}\left(\left(\eta_W\mathbf{A}^{-T}\boldsymbol{z}\sigma_v^T + \rho\|\tilde{\boldsymbol{x}}\|\hat{\mathbf{W}}\right)^T\rho^{-1}\tilde{\mathbf{W}}\right)\\
&= \eta_W\,\mathrm{tr}(\sigma_v\boldsymbol{z}^T\mathbf{A}^{-1}\rho^{-1}\tilde{\mathbf{W}}) + \|\tilde{\boldsymbol{x}}\|\,\mathrm{tr}(\hat{\mathbf{W}}^T\tilde{\mathbf{W}})
\end{aligned}$$

We expand the leakage term by substituting $\hat{\mathbf{W}} = \mathbf{W} - \tilde{\mathbf{W}}$:

$$\begin{aligned}
\|\tilde{\boldsymbol{x}}\|\,\mathrm{tr}(\hat{\mathbf{W}}^T\tilde{\mathbf{W}}) &= \|\tilde{\boldsymbol{x}}\|\,\mathrm{tr}((\mathbf{W} - \tilde{\mathbf{W}})^{\mathbf{T}}\tilde{\mathbf{W}})\\
&= -\|\tilde{\boldsymbol{x}}\|\|\tilde{\mathbf{W}}\|^2 + \|\tilde{\boldsymbol{x}}\|\|\mathbf{W}\|\|\tilde{\mathbf{W}}\|
\end{aligned}$$

Substituting this back into the $\dot{L}$ expression:

$$\begin{aligned}
\dot{L} \leq &-\frac{1}{2}\tilde{\boldsymbol{x}}^T\mathbf{Q}\tilde{\boldsymbol{x}} - \|\tilde{\boldsymbol{x}}\|\|\tilde{\mathbf{W}}\|^2 - \lambda L_{\mathrm{int}}\\
&+ \tilde{\boldsymbol{x}}^T\mathbf{P}_1(\tilde{\mathbf{W}}^T\sigma_v + w) + \eta_W\,\mathrm{tr}(\sigma_v\boldsymbol{z}^T\mathbf{A}^{-1}\rho^{-1}\tilde{\mathbf{W}}) + \|\tilde{\boldsymbol{x}}\|\|\mathbf{W}\|\|\tilde{\mathbf{W}}\|
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
|\tilde{\boldsymbol{x}}^T\mathbf{P}_1\tilde{\mathbf{W}}^T\sigma_v| &\leq \|\tilde{\boldsymbol{x}}\|\|\mathbf{P}_1\|(\|\tilde{\mathbf{W}}\|\sigma_M + \bar{w})\\
\|\tilde{\boldsymbol{x}}\|\|\mathbf{W}\|\|\hat{\mathbf{W}}\| &\leq \|\tilde{\boldsymbol{x}}\|\|\tilde{\mathbf{W}}\|W_M\\
|\eta_W\,\mathrm{tr}(\sigma_v\boldsymbol{z}^T\mathbf{A}^{-1}\rho^{-1}\tilde{\mathbf{W}})| &\leq \eta_W\|\sigma_v\|\|\boldsymbol{z}\|\|\mathbf{A}^{-1}\rho^{-1}\|\|\tilde{\mathbf{W}}\|\\
&\leq \eta_W\sigma_M\frac{\sqrt{n}}{\lambda}\|\tilde{\boldsymbol{x}}\|\|A^{-1}\rho^{-1}\|\|\tilde{\mathbf{W}}\|.
\end{aligned}$$

where $\|W\| \leq W_M$, $\|\sigma(\hat{\tilde{\boldsymbol{x}}})\| \leq \sigma_M$, and because $\boldsymbol{z}(t)$ is the state of the first-order filter (6) driven by $\tilde{\boldsymbol{x}}(t)$, its 2-norm satisfies

$$\begin{aligned}\|\boldsymbol{z}(t)\| &= \left\|\int_0^t e^{-\lambda(t-\tau)}\tilde{\boldsymbol{x}}(\tau)\,\mathrm{d}\tau\right\| \\ &\leq \int_0^t e^{-\lambda(t-\tau)}\|\tilde{\boldsymbol{x}}(\tau)\|\,\mathrm{d}\tau \\ &\leq \|\tilde{\boldsymbol{x}}\|_\infty \int_0^t e^{-\lambda(t-\tau)}\,\mathrm{d}\tau \\ &= \frac{1-e^{-\lambda t}}{\lambda}\,\|\tilde{\boldsymbol{x}}\|_\infty \\ &\leq \frac{\sqrt{n}}{\lambda}\,\|\tilde{\boldsymbol{x}}\|_\infty \\ &\leq \frac{\sqrt{n}}{\lambda}\,\|\tilde{\boldsymbol{x}}\| \quad (\text{since } \|\tilde{\boldsymbol{x}}\|_\infty \leq \|\tilde{\boldsymbol{x}}\|).\end{aligned} \tag{22}$$

with n denoting the state dimension.

Then, the inequality becomes:

$$\begin{aligned}\dot{L} \leq &-\frac{1}{2}\lambda_{\min}(\mathbf{Q})\|\tilde{\boldsymbol{x}}\|^2 - \|\tilde{\boldsymbol{x}}\|\|\tilde{\mathbf{W}}\|^2 - \lambda L_{\text{int}} \\ &+ \|\tilde{\boldsymbol{x}}\|\|\mathbf{P}_1\|(\|\tilde{\mathbf{W}}\|\sigma_M + \bar{w}) + \|\tilde{\boldsymbol{x}}\|W_M\|\tilde{\mathbf{W}}\| + \eta_W\sigma_M\frac{\sqrt{n}}{\lambda}\|\tilde{\boldsymbol{x}}\|_\infty\|\mathbf{A}^{-1}\|\|\rho^{-1}\|\|\tilde{\mathbf{W}}\|\end{aligned} \tag{23}$$

By completing the squares for the terms involving $\|\hat{W}\|$, we look for conditions on $\|x\|$ which are independent of the neural network weights error and also make the time derivative of the Lyapunov candidate negative.

$$\begin{aligned}\dot{L} \leq &-\|\tilde{\boldsymbol{x}}\|\|\tilde{\mathbf{W}}\|^2 + \left((\|\mathbf{P}_1\|\sigma_M + W_M + \eta_W\sigma_M\frac{\sqrt{n}}{\lambda}\|\mathbf{A}^{-1}\rho_1^{-1}\|)\|\tilde{\boldsymbol{x}}\|\right)\|\tilde{\mathbf{W}}\| \\ &-\frac{1}{2}\lambda_{\min}(\mathbf{Q})\|\tilde{\boldsymbol{x}}\|^2 + \|\tilde{\boldsymbol{x}}\|\|\mathbf{P}_1\|w_M - \lambda L_{\text{int}}\end{aligned}$$

Let $k_b = \|\mathbf{P}_1\|\sigma_M + W_M + \eta_W\sigma_M\frac{\sqrt{n}}{\lambda}\|\mathbf{A}^{-1}\rho_1^{-1}\|$. The terms involving $\tilde{\mathbf{W}}$ are of the form $-(\|\tilde{\boldsymbol{x}}\|)\|\tilde{\mathbf{W}}\|^2 + (k_b\|\tilde{\boldsymbol{x}}\|)\|\tilde{\mathbf{W}}\|$. By completing the square, this is bounded above by $\frac{(k_b\|\tilde{\boldsymbol{x}}\|)^2}{4|\tilde{\boldsymbol{x}}\|} = \frac{k_b^2}{4}\|\tilde{\boldsymbol{x}}\|$. The final inequality for $\dot{L}$ is:

$$\dot{L} \leq -\frac{1}{2}\lambda_{min}(\mathbf{Q})\|\tilde{\boldsymbol{x}}\|^2 - \lambda L_{\text{int}} + \left(\|\mathbf{P}_1\|\bar{w} + \frac{k_b^2}{4}\right)\|\tilde{\boldsymbol{x}}\| \tag{24}$$

To find a sufficient condition that guarantees $\dot{L} \leq 0$ and subsequently derive the ultimate bound, we can analyze a simpler upper bound. Since the term $-\lambda L_{int}$ is always non-positive, it can be omitted from the right-hand side while the inequality still holds. The analysis thus proceeds with the remaining terms :

$$\|\tilde{\boldsymbol{x}}\| \geq \frac{2\left(\|\mathbf{P}_1\|\bar{w} + (\|\mathbf{P}_1\|\sigma_M + W_M + \eta_W\sigma_M\frac{\sqrt{n}}{\lambda}\|\mathbf{A}^{-1}\rho_1^{-1}\|)^2\right)}{\lambda_{min}(\mathbf{Q})} = b \tag{25}$$

Furthermore, the above condition on $\|\tilde{\boldsymbol{x}}\|$ guarantees the negative semi-definiteness of $\dot{L}$ and therefore, ultimate boundedness of $\tilde{\boldsymbol{x}}$. In fact, $\dot{L}$ is negative definite outside the ball with radius b.

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Remark 1.