

Ultimate Boundedness of a Neural Network Identifier for Nonlinear Dynamics via Robust Adaptive Update

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1 Problem Formulation

1.1 System and Identifier Dynamics

The nonlinear system is described by:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + g(\mathbf{x}, u) + h(\mathbf{x}, u) \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state, $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a known Hurwitz matrix, $g(\cdot)$ is an known nonlinear function and $h(\cdot)$ is an unknown nonlinear function. The NN identifier is:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + g(\mathbf{x}, u) + \hat{h}(\hat{\mathbf{x}}) \quad (2)$$

where the NN output is $\hat{h}(\hat{\mathbf{x}}, u) = \hat{\mathbf{W}}^T \sigma(\hat{\mathbf{V}}^T \hat{\mathbf{x}})$. Here, $\hat{\mathbf{W}} \in \mathbb{R}^{h \times n}$ and $\hat{\mathbf{V}} \in \mathbb{R}^{d \times h}$ are the estimated weight matrices. The NN input $\hat{\mathbf{x}}$ is constructed from the estimated states $\hat{\mathbf{x}}$ and other available signals.

1.2 Error Dynamics

Defining the errors $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$, $\tilde{\mathbf{W}} = \mathbf{W} - \hat{\mathbf{W}}$, and $\tilde{\mathbf{V}} = \mathbf{V} - \hat{\mathbf{V}}$, the error dynamics are given by:

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \tilde{\mathbf{W}}^T \sigma(\hat{\mathbf{V}}^T \hat{\mathbf{x}}) + w(t) \quad (3)$$

where $w(t) = \mathbf{W}^T (\sigma(\mathbf{V}^T \tilde{\mathbf{x}}) - \sigma(\hat{\mathbf{V}}^T \hat{\mathbf{x}})) + \epsilon(x)$ is the lumped disturbance term.

Assumption 1. *The ideal weights \mathbf{W}, \mathbf{V} are bounded. The disturbance $w(t)$ is bounded by $\|w(t)\| \leq w_M$. The activation function σ and its derivatives are bounded.*

Assumption 2 (Open-Loop Stability). *The open-loop system (1) is stable, which implies that the state vector $\mathbf{x}(t)$ is bounded in L_∞ .*

2 Update Law and Stability Analysis

2.1 Update Laws

Theorem 1. *Consider the plant model (1) and the identifier model (2). Given Assumption 2, if the weights of the NLPNN are updated according to*

$$\dot{\hat{\mathbf{W}}} = -\eta_1 \left(\frac{\partial J}{\partial \hat{\mathbf{W}}} \right) - \rho_1 \|\tilde{\mathbf{x}}\| \hat{\mathbf{W}}, \quad (4)$$

$$\dot{\hat{\mathbf{V}}} = -\eta_2 \left(\frac{\partial J}{\partial \hat{\mathbf{V}}} \right) - \rho_2 \|\tilde{\mathbf{x}}\| \hat{\mathbf{V}}, \quad (5)$$

where $\eta > 0$ is the learning rate, $J = \frac{1}{2} \int_0^t e^{-\lambda(t-\tau)} \|\tilde{\mathbf{x}}(\tau)\|^2 d\tau$ is the objective function and ρ is a small positive number, then $\tilde{\mathbf{x}}, \tilde{\mathbf{W}},$ and $\tilde{\mathbf{V}} \in L_\infty$.

Proof. Since the cost functional is of an integral form, we first introduce the filtered error signal $\mathbf{z}(t)$ to construct the update laws.

$$\mathbf{z}(t) = \int_0^t e^{-\lambda(t-\tau)} \tilde{\mathbf{x}}(\tau) d\tau \quad (6)$$

$$\dot{\mathbf{z}} = -\lambda \mathbf{z} + \tilde{\mathbf{x}} \quad (7)$$

Let us define

$$\text{net}_{\hat{\mathbf{V}}} = \hat{\mathbf{V}} \hat{\mathbf{x}} \quad (8)$$

$$\text{net}_{\hat{\mathbf{W}}} = \hat{\mathbf{W}} \sigma(\hat{\mathbf{V}} \hat{\mathbf{x}}). \quad (9)$$

Therefore, by using the chain rule $\frac{\partial J}{\partial \hat{\mathbf{W}}}$ and $\frac{\partial J}{\partial \hat{\mathbf{V}}}$ can be computed according to

$$\begin{aligned} \frac{\partial J}{\partial \hat{\mathbf{W}}} &= \frac{\partial J}{\partial \text{net}_{\hat{\mathbf{W}}}} \cdot \frac{\partial \text{net}_{\hat{\mathbf{W}}}}{\partial \hat{\mathbf{W}}} \\ \frac{\partial J}{\partial \hat{\mathbf{V}}} &= \frac{\partial J}{\partial \text{net}_{\hat{\mathbf{V}}}} \cdot \frac{\partial \text{net}_{\hat{\mathbf{V}}}}{\partial \hat{\mathbf{V}}}, \end{aligned}$$

where

$$\begin{aligned} \frac{\partial J}{\partial \text{net}_{\hat{\mathbf{W}}}} &= \frac{\partial J}{\partial \tilde{\mathbf{x}}} \frac{\partial \tilde{\mathbf{z}}}{\partial \tilde{\mathbf{x}}} \frac{\partial \hat{\mathbf{x}}}{\partial \text{net}_{\hat{\mathbf{W}}}} = -\mathbf{z}^T \frac{\partial \hat{\mathbf{x}}}{\partial \text{net}_{\hat{\mathbf{W}}}}, \\ \frac{\partial J}{\partial \text{net}_{\hat{\mathbf{V}}}} &= \frac{\partial J}{\partial \tilde{\mathbf{x}}} \frac{\partial \tilde{\mathbf{z}}}{\partial \tilde{\mathbf{x}}} \frac{\partial \hat{\mathbf{x}}}{\partial \text{net}_{\hat{\mathbf{V}}}} = -\mathbf{z}^T \frac{\partial \hat{\mathbf{x}}}{\partial \text{net}_{\hat{\mathbf{V}}}} \end{aligned} \quad (10)$$

and

$$\begin{aligned} \frac{\partial \text{net}_{\hat{\mathbf{W}}}}{\partial \hat{\mathbf{W}}} &= \sigma(\hat{\mathbf{V}} \hat{\mathbf{x}}) \\ \frac{\partial \text{net}_{\hat{\mathbf{V}}}}{\partial \hat{\mathbf{V}}} &= \hat{\mathbf{x}}. \end{aligned} \quad (11)$$

We modify the original BP algorithm such that the static approximations of $\frac{\partial \hat{\mathbf{x}}}{\partial \text{net}_{\hat{\mathbf{W}}}}$ and $\frac{\partial \hat{\mathbf{x}}}{\partial \text{net}_{\hat{\mathbf{V}}}}$ ($\dot{\hat{\mathbf{x}}} = 0$) can be used.

$$\begin{aligned} \frac{\partial \hat{\mathbf{x}}}{\partial \text{net}_{\hat{\mathbf{W}}}} &\approx -\mathbf{A}^{-1} \\ \frac{\partial \hat{\mathbf{x}}}{\partial \text{net}_{\hat{\mathbf{V}}}} &\approx -\mathbf{A}^{-1} \hat{\mathbf{W}} (\mathbf{I} - \mathbf{\Lambda}(\hat{\mathbf{V}} \hat{\mathbf{x}})), \end{aligned} \quad (12)$$

where

$$\mathbf{\Lambda}(\hat{\mathbf{V}} \hat{\mathbf{x}}) = \text{diag}\{\sigma_i^2(\hat{\mathbf{V}}_i \hat{\mathbf{x}})\}, \quad i = 1, 2, \dots, m. \quad (13)$$

$$\dot{\hat{\mathbf{W}}} = -\eta_1 (z^T \mathbf{A}^{-1})^T \sigma(\hat{\mathbf{V}}^T \hat{\mathbf{x}})^T - \rho_1 \|\tilde{\mathbf{x}}\| \hat{\mathbf{W}} \quad (14)$$

$$(15)$$

$$\dot{\hat{\mathbf{V}}} = -\eta_2 \hat{\mathbf{x}} \left(z^T \mathbf{A}^{-1} \hat{\mathbf{W}} (\mathbf{I} - \mathbf{\Lambda}(\hat{\mathbf{V}} \hat{\mathbf{x}})) \right)^T - \rho_2 \|\tilde{\mathbf{x}}\| \hat{\mathbf{V}} \quad (16)$$

$$(17)$$

where $\eta_W, \eta_V, \rho_1, \rho_2 > 0$ are design parameters. By using (14) and (15) in terms of $\tilde{\mathbf{W}}$ may be written as

$$\dot{\tilde{\mathbf{W}}} = \eta_1 (z^T \mathbf{A}^{-1})^T (\sigma(\hat{\mathbf{x}}))^T + \rho_1 \|\tilde{\mathbf{x}}\| \tilde{\mathbf{W}}. \quad (18)$$

$$\dot{\tilde{\mathbf{V}}} = \eta_2 \hat{\mathbf{x}} \left(z^T \mathbf{A}^{-1} \tilde{\mathbf{W}} \text{diag}(\sigma'(\hat{\mathbf{V}}^T \hat{\mathbf{x}})) \right)^T - \rho_2 \|\tilde{\mathbf{x}}\| \tilde{\mathbf{V}} \quad (19)$$

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2.2 Lyapunov Stability

Theorem 2. *For the system given by (3) with the update laws (15)-(17), all signals in the system $(\tilde{\mathbf{x}}, \tilde{\mathbf{W}}, \tilde{\mathbf{V}})$ are Uniformly Ultimately Bounded.*

Proof. The stability proof is conducted in two steps using a cascaded system approach.

We first prove the boundedness of the state error $\tilde{\mathbf{x}}$ and the output layer weight error $\tilde{\mathbf{W}}$. This is possible because the term $\sigma_v = \sigma(\hat{\mathbf{V}}^T \hat{\mathbf{x}})$ in the error dynamics (3) is always bounded, regardless of the value of $\hat{\mathbf{V}}$, due to the bounded nature of the activation function σ .

Consider the Lyapunov function candidate for the first subsystem:

$$L = \frac{1}{2} \tilde{\mathbf{x}}^T \mathbf{P}_1 \tilde{\mathbf{x}} + \frac{1}{2} \text{tr}(\tilde{\mathbf{W}}^T \rho^{-1} \tilde{\mathbf{W}}) + \frac{1}{2} \int_0^t e^{-\lambda(t-\tau)} \tilde{\mathbf{x}}(\tau)^T \mathbf{P}_2 \tilde{\mathbf{x}}(\tau) d\tau \quad (20)$$

Its time derivative, after substituting the error dynamics, is:

$$\dot{L} = -\frac{1}{2} \tilde{\mathbf{x}}^T (\mathbf{Q}_1 - \mathbf{P}_2) \tilde{\mathbf{x}} + \tilde{\mathbf{x}}^T \mathbf{P}_1 (\tilde{\mathbf{W}}^T \sigma_v + w) + \text{tr}(\dot{\tilde{\mathbf{W}}}^T \rho^{-1} \tilde{\mathbf{W}}) - \lambda L_{\text{int}} \quad (21)$$

where $\sigma_v = \sigma(\hat{\mathbf{V}}^T \hat{\mathbf{x}})$, $\mathbf{Q} = \mathbf{Q}_1 - \mathbf{P}_2 > 0$, and $L_{\text{int}} = \frac{1}{2} \int_0^t e^{-\lambda(t-\tau)} \tilde{\mathbf{x}}(\tau)^T \mathbf{P}_2 \tilde{\mathbf{x}}(\tau) d\tau$. We substitute the update law (15) using $\dot{\tilde{\mathbf{W}}} = -\dot{\hat{\mathbf{W}}}$.

$$\begin{aligned} \text{tr}(\dot{\tilde{\mathbf{W}}}^T \rho^{-1} \tilde{\mathbf{W}}) &= \text{tr} \left(\left(\eta_W \mathbf{A}^{-T} \mathbf{z} \sigma_v^T + \rho \|\tilde{\mathbf{x}}\| \hat{\mathbf{W}} \right)^T \rho^{-1} \tilde{\mathbf{W}} \right) \\ &= \eta_W \text{tr}(\sigma_v \mathbf{z}^T \mathbf{A}^{-1} \rho^{-1} \tilde{\mathbf{W}}) + \|\tilde{\mathbf{x}}\| \text{tr}(\hat{\mathbf{W}}^T \tilde{\mathbf{W}}) \end{aligned}$$

We expand the leakage term by substituting $\hat{\mathbf{W}} = \mathbf{W} - \tilde{\mathbf{W}}$:

$$\begin{aligned} \|\tilde{\mathbf{x}}\| \text{tr}(\hat{\mathbf{W}}^T \tilde{\mathbf{W}}) &= \|\tilde{\mathbf{x}}\| \text{tr}((\mathbf{W} - \tilde{\mathbf{W}})^T \tilde{\mathbf{W}}) \\ &= -\|\tilde{\mathbf{x}}\| \|\tilde{\mathbf{W}}\|^2 + \|\tilde{\mathbf{x}}\| \|\mathbf{W}\| \|\tilde{\mathbf{W}}\| \end{aligned}$$

Substituting this back into the \dot{L} expression:

$$\begin{aligned} \dot{L} &\leq -\frac{1}{2} \tilde{\mathbf{x}}^T \mathbf{Q} \tilde{\mathbf{x}} - \|\tilde{\mathbf{x}}\| \|\tilde{\mathbf{W}}\|^2 - \lambda L_{\text{int}} \\ &\quad + \tilde{\mathbf{x}}^T \mathbf{P}_1 (\tilde{\mathbf{W}}^T \sigma_v + w) + \eta_W \text{tr}(\sigma_v \mathbf{z}^T \mathbf{A}^{-1} \rho^{-1} \tilde{\mathbf{W}}) + \|\tilde{\mathbf{x}}\| \|\mathbf{W}\| \|\tilde{\mathbf{W}}\| \end{aligned}$$

Moreover, we have

$$\begin{aligned} |\tilde{\mathbf{x}}^T \mathbf{P}_1 \tilde{\mathbf{W}}^T \sigma_v| &\leq \|\tilde{\mathbf{x}}\| \|\mathbf{P}_1\| (\|\tilde{\mathbf{W}}\| \sigma_M + \bar{w}) \\ \|\tilde{\mathbf{x}}\| \|\mathbf{W}\| \|\hat{\mathbf{W}}\| &\leq \|\tilde{\mathbf{x}}\| \|\tilde{\mathbf{W}}\| W_M \\ |\eta_W \text{tr}(\sigma_v \mathbf{z}^T \mathbf{A}^{-1} \rho^{-1} \tilde{\mathbf{W}})| &\leq \eta_W \|\sigma_v\| \|\mathbf{z}\| \|\mathbf{A}^{-1} \rho^{-1}\| \|\tilde{\mathbf{W}}\| \\ &\leq \eta_W \sigma_M \frac{\sqrt{n}}{\lambda} \|\tilde{\mathbf{x}}\| \|\mathbf{A}^{-1} \rho^{-1}\| \|\tilde{\mathbf{W}}\|. \end{aligned}$$

where $\|\mathbf{W}\| \leq W_M$, $\|\sigma(\hat{\mathbf{x}})\| \leq \sigma_M$, and because $\mathbf{z}(t)$ is the state of the first-order filter (6) driven

by $\tilde{\mathbf{x}}(t)$, its 2-norm satisfies

$$\begin{aligned}
\|\mathbf{z}(t)\| &= \left\| \int_0^t e^{-\lambda(t-\tau)} \tilde{\mathbf{x}}(\tau) d\tau \right\| \\
&\leq \int_0^t e^{-\lambda(t-\tau)} \|\tilde{\mathbf{x}}(\tau)\| d\tau \\
&\leq \|\tilde{\mathbf{x}}\|_\infty \int_0^t e^{-\lambda(t-\tau)} d\tau \\
&= \frac{1 - e^{-\lambda t}}{\lambda} \|\tilde{\mathbf{x}}\|_\infty \\
&\leq \frac{\sqrt{n}}{\lambda} \|\tilde{\mathbf{x}}\|_\infty \\
&\leq \frac{\sqrt{n}}{\lambda} \|\tilde{\mathbf{x}}\| \quad (\text{since } \|\tilde{\mathbf{x}}\|_\infty \leq \|\tilde{\mathbf{x}}\|).
\end{aligned} \tag{22}$$

with n denoting the state dimension.

Then, the inequality becomes:

$$\begin{aligned}
\dot{L} \leq & -\frac{1}{2} \lambda_{\min}(\mathbf{Q}) \|\tilde{\mathbf{x}}\|^2 - \|\tilde{\mathbf{x}}\| \|\tilde{\mathbf{W}}\|^2 - \lambda L_{\text{int}} \\
& + \|\tilde{\mathbf{x}}\| \|\mathbf{P}_1\| (\|\tilde{\mathbf{W}}\| \sigma_M + \bar{w}) + \|\tilde{\mathbf{x}}\| W_M \|\tilde{\mathbf{W}}\| + \eta_W \sigma_M \frac{\sqrt{n}}{\lambda} \|\tilde{\mathbf{x}}\|_\infty \|\mathbf{A}^{-1}\| \|\rho^{-1}\| \|\tilde{\mathbf{W}}\|
\end{aligned} \tag{23}$$

By completing the squares for the terms involving $\|\tilde{\mathbf{W}}\|$, we look for conditions on $\|\tilde{\mathbf{x}}\|$ which are independent of the neural network weights error and also make the time derivative of the Lyapunov candidate negative.

$$\begin{aligned}
\dot{L} \leq & -\|\tilde{\mathbf{x}}\| \|\tilde{\mathbf{W}}\|^2 + \left((\|\mathbf{P}_1\| \sigma_M + W_M + \eta_W \sigma_M \frac{\sqrt{n}}{\lambda} \|\mathbf{A}^{-1} \rho_1^{-1}\|) \|\tilde{\mathbf{x}}\| \right) \|\tilde{\mathbf{W}}\| \\
& - \frac{1}{2} \lambda_{\min}(\mathbf{Q}) \|\tilde{\mathbf{x}}\|^2 + \|\tilde{\mathbf{x}}\| \|\mathbf{P}_1\| w_M - \lambda L_{\text{int}}
\end{aligned}$$

Let $k_b = \|\mathbf{P}_1\| \sigma_M + W_M + \eta_W \sigma_M \frac{\sqrt{n}}{\lambda} \|\mathbf{A}^{-1} \rho_1^{-1}\|$. The terms involving $\tilde{\mathbf{W}}$ are of the form $-(\|\tilde{\mathbf{x}}\|) \|\tilde{\mathbf{W}}\|^2 + (k_b \|\tilde{\mathbf{x}}\|) \|\tilde{\mathbf{W}}\|$. By completing the square, this is bounded above by $\frac{(k_b \|\tilde{\mathbf{x}}\|)^2}{4 \|\tilde{\mathbf{x}}\|} = \frac{k_b^2}{4} \|\tilde{\mathbf{x}}\|$. The final inequality for \dot{L} is:

$$\dot{L} \leq -\frac{1}{2} \lambda_{\min}(\mathbf{Q}) \|\tilde{\mathbf{x}}\|^2 - \lambda L_{\text{int}} + \left(\|\mathbf{P}_1\| \bar{w} + \frac{k_b^2}{4} \right) \|\tilde{\mathbf{x}}\| \tag{24}$$

To find a sufficient condition that guarantees $\dot{L} \leq 0$ and subsequently derive the ultimate bound, we can analyze a simpler upper bound. Since the term $-\lambda L_{\text{int}}$ is always non-positive, it can be omitted from the right-hand side while the inequality still holds. The analysis thus proceeds with the remaining terms :

$$\|\tilde{\mathbf{x}}\| \geq \frac{2 \left(\|\mathbf{P}_1\| \bar{w} + (\|\mathbf{P}_1\| \sigma_M + W_M + \eta_W \sigma_M \frac{\sqrt{n}}{\lambda} \|\mathbf{A}^{-1} \rho_1^{-1}\|)^2 \right)}{\lambda_{\min}(\mathbf{Q})} = b \tag{25}$$

Furthermore, the above condition on $\|\tilde{\mathbf{x}}\|$ guarantees the negative semi-definiteness of \dot{L} and therefore, ultimate boundedness of $\tilde{\mathbf{x}}$. In fact, \dot{L} is negative definite outside the ball with radius b .

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Remark 1.