Theoretical Foundation of KSIF Package

운용3팀 유승현

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1 Strategic Vector Space

We can define Strategy as a linear map as follows.

Definition 1. Strategy Strategy is a function $\S : \mathbb{T} \bigotimes \mathbb{F} \bigotimes \mathbb{D} \to \mathbb{T} \bigotimes$

Problem 1. Show that the $C^0[k_{min}, k_{max}]$ with $\|\cdot\|_\infty$ is Normed Vector Space.

Proof. Without loss of generality, we can set $k_{min} = 0$, and $k_{max} = 1$. Let's define $(+)$ and scalar multiplication. All functions below are the elements of $C^0[0,1]$.

(addition) $(f+g)(x) = f(x) + g(x), ^\forall x \in [0,1]$.
(scalar multiplication) $(af)(x) = af(x), ^\forall x \in [0,1]$.

Then the following holds, because $(+)$ is commutative, and associative.

(1) $f + g = g + f$;
(2) $(f + g) + h = f + (g + h)$;
(3) there exists an zero function $0 \in C^0[0,1]$ such that $0 + f = f$;
(4) for each f in $C^0[0,1]$ there exists an function $-f \in C^0[0,1]$ such that $f + (-f) = 0$;
(5) $c(f + g) = cf + cg, ^\forall c \in \mathbb{R}$;
(6) $(c + d)f = cf + df, ^\forall c, d \in \mathbb{R}$;
(7) $c(df) = (cd)f, ^\forall c, d \in \mathbb{R}$;
(8) $1f = f$;

So, the $C^0[0,1]$ is vector space with addition and scalar multiplication. Also, $C^0[0,1]$ is normed vector space with sup norm $\|\cdot\|_\infty$, because following holds.

(1) $\|f\|_\infty \geq 0$ and $\|f\|_\infty = 0$ iff $f = 0$;
$(\because)$ $\|f\|_\infty \geq |f(x)| \geq 0$ for all $x \in [0,1]$, and if $\|f\|_\infty = 0$, then for all $x \in [0,1], f(x) = 0$, i.e. $f(x) = 0$. Opposite direction is trivial.
(2) $\|cf\|_\infty = |c|\|f\|_\infty$ for all $c \in \mathbb{R}$;
$(\because)$ $\|cf\|_\infty = \sup_x |cf(x)| = \sup_x |c||f(x)| = |c| \sup_x |f(x)| = |c|\|f\|_\infty$.

(3) $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$ (triangle inequality);
$(\because)$ $\|f+g\|_\infty = \sup_x |f(x)+g(x)| \le \sup_x(|f(x)|+|g(x)|) \le \sup_x |f(x)| + \sup_x |g(x)| = \|f\|_\infty + \|g\|_\infty$.

So, the $C^0[0,1]$ is normed vector space. □

Problem 2. (Proposition 2) Show any compact set K in $\mathbb{R}^n$ is complete.

Proof. According to *Heine-Borel theorem*, A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded. So, We have to show all *Cauchy Sequence* in K converges to a point in K. Because of K is compact, all *Cauchy Sequence* $\{x_n\}$ in K has a convergent subsequence $\{x_{n_k}\}$ which converges to x in K. Then, we can show that $\{x_n\}$ converges to x.

$(\because)$ We can find N such that,

$$|x_n - x| \le |x_n - x_{n_k}| + |x_{n_k} - x| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $n \ge N$ and $k \ge N_k \ge N$. In here, we use $n_k \ge k$ to show $|x_n - x_{n_k}| \le \frac{\varepsilon}{2}$
So, K is complete. □

Problem 3. (Proof of Theorem 1)

Theorem 1. *Let $C^0[k_{min}, k_{max}]$ be the set of all continuous functions $f : [k_{min}, k_{max}] \to \mathbb{R}$ with the sup norm, $\|f\| = \sup_{k \in [k_{min}, k_{max}]} |f(k)|$. Then $C^0[k_{min}, k_{max}]$ is a Banach space (complete normed vector space).*

Proof. We can use,

Proposition 1. *if $f_n \in C^0[k_{min}, k_{max}]$ and $f_n \to f$ uniformly as $n \to \infty$, then $f \in C^0[k_{min}, k_{max}]$.*

Proposition 2. *any compact set K in $\mathbb{R}^n$ is complete.*

By **Problem 1**, it is suffices to show that $C^0[k_{min}, k_{max}]$ is complete, i.e. Cauchy Sequence $\{f_n\}$ converges to $f \in C^0[k_{min}, k_{max}]$. For all $x \in [k_{min}, k_{max}]$, there exist N such that,

$$|f_n(x) - f_m(x)| \le \sup_{y \in [k_{min}, k_{max}]} |f_n(y) - f_m(y)| = \|f_n - f_m\| \le \varepsilon$$

for all $n, m \ge N$.
So, $\{f_n(x)\}$ is Cauchy Sequence.
And we know that $\mathbb{R}$ is complete, that means we can find converging points $f(x)$ of the Cauchy sequence $\{f_n(x)\}$ for each $x \in [k_{min}, k_{max}]$.

Then we can show $\{f_n\} \to f$ uniformly as $n \to \infty$, because we can find sufficiently large N such that,

$$\begin{aligned}|f_n(x) - f(x)| &\le |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\le \|f_n - f_m\| + |f_m(x) - f(x)| \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

for $n \geq N$, and for all $x \in [k_{min}, k_{max}]$. In here, we are free to choose m (which must be depend on x) arbitrary large, that is $m \geq N_x \geq N$. By **Proposition 1**, $f \in C^0[k_{min}, k_{max}]$. So, $C^0[k_{min}, k_{max}]$ is a Banach space.

□

Problem 4. Show Bellman Operator T satisfies *monotonicity* and *discounting.*

Definition 2. Bellman Operator $\mathbf{T}$ is,

$$(\mathbf{T}V)(k_t) = \max_{k_{t+1} \in X} u(A_t(k_t)^\alpha + (1-\delta)k_t - k_{t+1}) + \beta V(k_{t+1})$$

with concave u and V.

a (monotonicity)

Proof. Let any f and $g \in C^0[k_{min}, k_{max}]$ satisfying $f(k) \leq g(k)$ for all $k \in [k_{min}, k_{max}]$. Then,

$$\begin{aligned}
(\mathbf{T}f)(k) &= \max_{k' \in X} u(A_t(k)^\alpha + (1-\delta)k - k') + \beta f(k') \\
&\leq \max_{k' \in X} u(A_t(k)^\alpha + (1-\delta)k - k') + \beta g(k') \\
&= (\mathbf{T}g)(k)
\end{aligned}$$

Here, the inequality holds because $\beta > 0$. □

b (discounting)

Proof. For all $f \in C^0[k_{min}, k_{max}]$, $a \geq 0$, and $k \in [k_{min}, k_{max}]$,

$$\begin{aligned}
[\mathbf{T}(f+a)](k) &= \max_{k' \in X}\{u(A_t(k)^\alpha + (1-\delta)k - k') + \beta(f(k') + a)\} \\
&= \max_{k' \in X}\{u(A_t(k)^\alpha + (1-\delta)k - k') + \beta f(k') + \beta a\} \\
&= \max_{k' \in X}\{u(A_t(k)^\alpha + (1-\delta)k - k') + \beta f(k')\} + \beta a \\
&= (\mathbf{T}f)(k) + \beta a
\end{aligned}$$

So if we set β as modulus, then $[\mathbf{T}(f+a)](k) \leq (\mathbf{T}f)(k) + \beta a$ holds.

□