Some setting

Bayes ' theorem

$$P(s \mid m) = \frac{P(m \mid s)\, P(s)}{P(m)}$$

Posterior

$$\frac{1}{\frac{\text{Gamma}[r]}{r!}} \text{Exp}[-\lambda] \frac{\lambda^r}{r!} \frac{1}{\lambda}$$

$$\text{post} = \frac{e^{-\lambda} \lambda^{-1+r}}{\text{Gamma}[r]}$$

$$\frac{e^{-\lambda} \lambda^{-1+r}}{\text{Gamma}[r]}$$

$$\int_0^\infty \text{Exp}[-\lambda] \frac{\lambda^r}{r!} \frac{1}{\lambda}\, d\lambda$$

$$\text{ConditionalExpression}\left[\frac{\text{Gamma}[r]}{r!}, \text{Re}[r] > 0\right]$$

$$\text{Solve}\left[D\left[\frac{e^{-\lambda} \lambda^{-1+r}}{\text{Gamma}[r]}, \lambda\right] == 0, \lambda\right]$$

$$\{\{\lambda \to -1 + r\}\}$$

$$\frac{e^{-\lambda} \lambda^{-1+r}}{\text{Gamma}[r]} \text{ /. } \{\lambda \to -1 + r\} \text{ // Simplify}$$

Laplace approximation

Prefactor, constant

$$\frac{e^{1-r} (-1 + r)^{-1+r}}{(r - 1)!} \text{ // Simplify}$$

Log Posterior

```
L = Log[ e^(-λ) λ^(-1+r) / Gamma[r] ]
```

$$\text{Log}\left[\frac{e^{-\lambda}\,\lambda^{-1+r}}{\text{Gamma}[r]}\right]$$

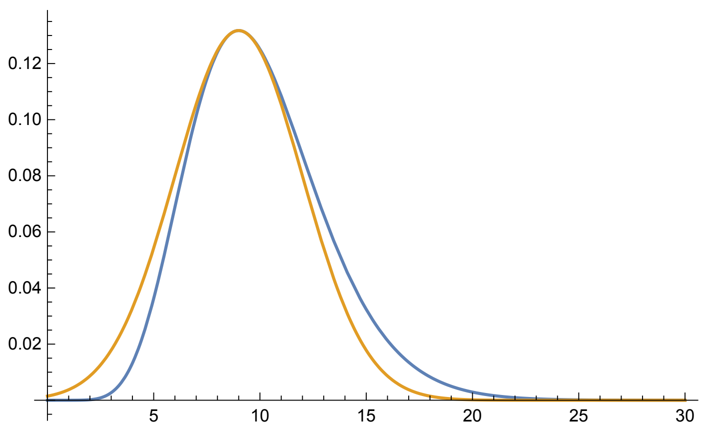
Exponential factor

```
D[D[L, λ], λ] // Simplify
```

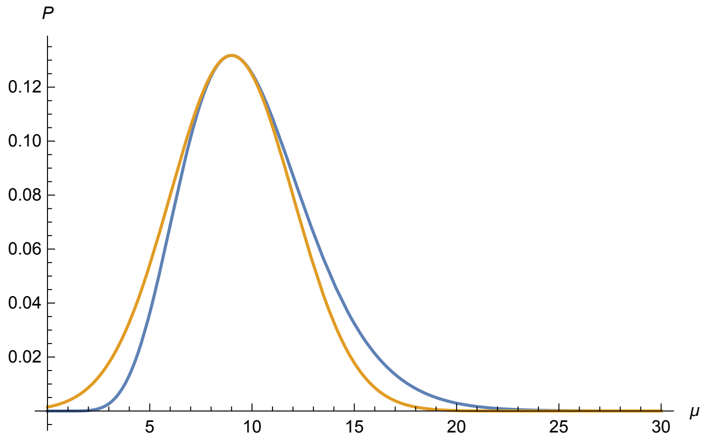
$$\frac{1-r}{\lambda^2}$$

Plot of posterior and its Laplace approximation.

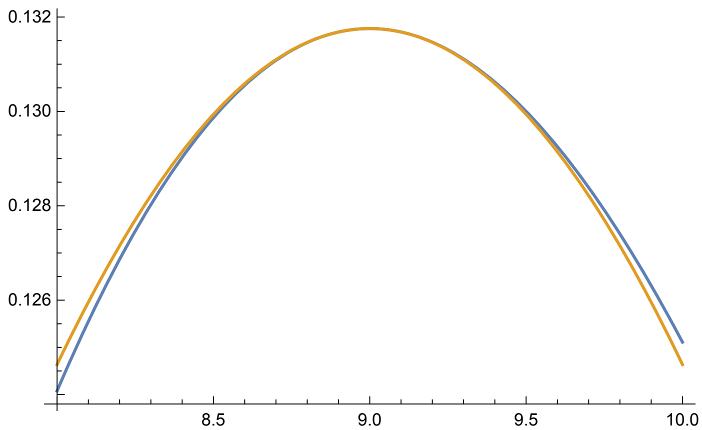
```
c = 1/(r - 1); r = 10;
Plot[{Exp[-λ] λ^r/r! 1/λ r, e^(1-r) (-1 + r)^(-1+r)/(r - 1)! Exp[-c/2 (λ - (r - 1))^2]}, {λ, 0, 30}]
```



```
Show[%2, AxesLabel → {HoldForm[μ], HoldForm[P]},
 PlotLabel → None, LabelStyle → {GrayLevel[0]}]
```

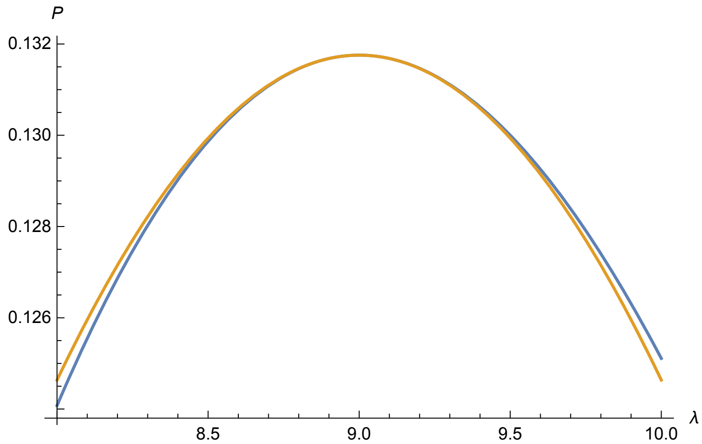


r = 10; Plot$\left[\left\{\text{Exp}[-\lambda]\ \frac{\lambda^r}{r!}\ \frac{1}{\lambda}\ r,\ \frac{e^{1-r}\ (-1+r)^{-1+r}}{(r-1)!}\ \text{Exp}\left[-\frac{c}{2}\ (\lambda-(r-1))^2\right]\right\},\ \{\lambda,\ 8,\ 10\}\right]$



Look into the tip

```
Show[%39, AxesLabel → {HoldForm[λ], HoldForm[P]},
 PlotLabel → None, LabelStyle → {GrayLevel[0]}]
```




```
Clear[r]
```

$$\left\{\text{Exp}[-\lambda]\,\frac{\lambda^r}{r!}\,r\right\}\ /.\ \lambda \to \text{Exp}[x]$$

$$\left\{\frac{e^{-e^x}\,(e^x)^r\,r}{r!}\right\}$$

$$r = 10;\ \text{Plot}\left[\frac{e^{-e^x}\,(e^x)^r\,r}{r!},\ \{x, 0, 4\}\right]$$