Calculation of signatures

CRISM workshop: Statistics for Differential Equations Driven by Rough Paths

Jeremy Reizenstein

Centre for Complexity Science, University of Warwick

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Signatures

The signature of a path is a set of iterated integrals.
Consider a path in $\mathbb{R}^3$ parameterised by the variable t ranging from 0 to 1, given by

$$t \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$$

Then, for example, the element 2,3 of the signature is

$$\int_0^1 \left[\int_0^t \gamma_2'(s)\, ds \right] \gamma_3'(t)\, dt = \int_0^1 \int_0^t d\gamma_2(s)\, d\gamma_3(t)$$

and element 2,1,2 of the signature is

$$\int_0^1 \int_0^t \int_0^s d\gamma_2(r)\, d\gamma_1(s)\, d\gamma_2(t).$$

Signatures

The mth level of the signature of a path in $\mathbb{R}^d$ given as a function from $[a, b]$ is the d^m values of the elements with m integrated integrals. It is denoted $X_{a,b}^m$ and takes values in $(\mathbb{R}^d)^{\otimes m}$. Given a piecewise linear path, we can compute the first m levels of its signature.

- For a straight path with displacement x, the signature is

$$\left(1, x, \frac{x \otimes x}{2!}, \frac{x \otimes x \otimes x}{3!}, \dots\right) \qquad \cancel{\Omega\left(\binom{d+m-1}{d}\right)}\Omega(d^m)$$

- Chen's identity for the signature of the concatenation of paths, $a \leq b \leq c$

$$X_{a,c}^m = \sum_{k=0}^{m} X_{a,b}^k \otimes X_{b,c}^{m-k} \qquad \Omega(md^m)$$

Log-Signature demonstration

There is redundancy in the signature. For example, in $\mathbb{R}^2$, the first four levels of the signature look like this

$$1 + (\cdot\ \cdot) + \begin{pmatrix} (\cdot\ \cdot) \\ (\cdot\ \cdot) \end{pmatrix} + \left(\begin{pmatrix} (\cdot\ \cdot) \\ (\cdot\ \cdot) \end{pmatrix} \begin{pmatrix} (\cdot\ \cdot) \\ (\cdot\ \cdot) \end{pmatrix} \right) + \begin{pmatrix} \left(\begin{pmatrix} (\cdot\ \cdot) \\ (\cdot\ \cdot) \end{pmatrix} \begin{pmatrix} (\cdot\ \cdot) \\ (\cdot\ \cdot) \end{pmatrix} \right) \\ \left(\begin{pmatrix} (\cdot\ \cdot) \\ (\cdot\ \cdot) \end{pmatrix} \begin{pmatrix} (\cdot\ \cdot) \\ (\cdot\ \cdot) \end{pmatrix} \right) \end{pmatrix}$$

- that is $2 + 4 + 8 + 16 = 30$ numbers while the log signature is only $2 + 1 + 2 + 3 = 8$ numbers.

Log-Signature domain

The log signature lives in the free Lie algebra, a subspace of the tensor space. We can pick any basis to express it, e.g. in the Lyndon basis.

$$a\mathbf{1} \quad + \quad b\mathbf{2} \quad + \quad c\mathbf{12} \quad + \quad d\mathbf{112} \quad + \quad e\mathbf{122}$$

$$a\begin{pmatrix}1\\0\end{pmatrix}+b\begin{pmatrix}0\\1\end{pmatrix}+c\begin{pmatrix}0&1\\-1&0\end{pmatrix}+d\begin{pmatrix}\begin{pmatrix}0&1\\-2&0\end{pmatrix}\\\begin{pmatrix}1&0\\0&0\end{pmatrix}\end{pmatrix}+e\begin{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}\\\begin{pmatrix}0&-2\\1&0\end{pmatrix}\end{pmatrix}$$

is the log of

$$1+\begin{pmatrix}a\\b\end{pmatrix}+\frac{1}{2}\begin{pmatrix}a^2&ab+2c\\ab-2c&b^2\end{pmatrix}+\frac{1}{6}\begin{pmatrix}\begin{pmatrix}a^3&a^2b+6d+3ac\\a^2b-12d&ab^2+6e+3bc\end{pmatrix}\\\begin{pmatrix}a^2b+6d-3ac&ab^2-12e\\ab^2+6e-3bc&b^3\end{pmatrix}\end{pmatrix}+\cdots$$

Detour on Lyndon words

The Lyndon basis gets an element from each Lyndon-word using a procedure which splits a Lyndon word into its longest Lyndon suffix and the rest, e.g.

$$
\begin{aligned}
&\underline{1122} \to [1, \underline{122}] \to [1, [\underline{12}, 2]] \to [[1, [1, 2]], 2] \\
&= [[1, 12 - 21], 2] \\
&= [1(12 - 21) - (12 - 21)1, 2] \\
&= [(112 - 121) - (121 - 211), 2] \\
&= ((112 - 121) - (121 - 211))2 \\
&\qquad - 2((112 - 121) - (121 - 211)) \\
&= 22121 - 21212 + 1122 - 2211
\end{aligned}
$$

Log-Signature domain

The log signature of a path with displacement x is just x. BCH formula expresses $A \bullet B := \log(\exp A \otimes \exp B)$ as

$$A + B + [A, B]/2 + \big([A, [A, B]] + [B, [B, A]]\big)/12 + \cdots$$

$$\begin{aligned}&(a\mathbf{1} + b\mathbf{2} + c\mathbf{12} + d\mathbf{112} + e\mathbf{122} + \cdots) \bullet (a'\mathbf{1} + b'\mathbf{2})\\ &= (a\mathbf{1} + b\mathbf{2} + c\mathbf{12} + \cdots) + (a'\mathbf{1} + b'\mathbf{2})\\ &\qquad + (ab'\mathbf{12} - a'b\mathbf{12} - a'c\mathbf{112} + cb'\mathbf{122} + \cdots)/2 + \cdots\\ &= (a + a')\mathbf{1} + (b + b')\mathbf{2} + (c + \tfrac{ab' - a'b}{2})\mathbf{12} + \cdots\end{aligned}$$

Anagrams

		123	132	213	231	312	321
Lyndon	123	1	-1	0	-1	0	1
	132	0	1	-1	1	-1	0

Code example

```
void joinSegmentToSignatureInPlace(
  FLogSignature<2,3>& a, const FSegment<2>& b){
    a[4] +=0.5*a[2]*b[1]
            +0.0833333333333*a[1]*a[1]*b[0]
            -0.0833333333333*a[0]*a[1]*b[1]
            -0.0833333333333*a[1]*b[0]*b[1]
            +0.0833333333333*a[0]*b[1]*b[1];
    a[3] +=-0.0833333333333*a[0]*b[0]*b[1]
            +0.0833333333333*a[1]*b[0]*b[0]
            +0.0833333333333*a[0]*a[0]*b[1]
            -0.5*a[2]*b[0]
            -0.0833333333333*a[0]*a[1]*b[0];
    a[2] +=0.5*a[0]*b[1]
            -0.5*a[1]*b[0];
    a[1] += b[1];
    a[0] += b[0];
}
```

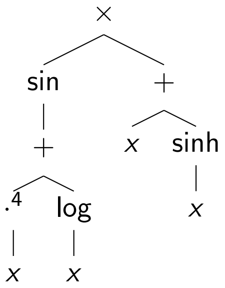
Some comparative timings

(D,M)	(2,6)	(2,10)	(3,10)	(5,5)	(10,4)
Time (s) for 100 (log) signatures of length-100 paths:					
Signature	0.021	0.201	8.005	0.212	0.404
Compiled	0.005	0.491	276.174	0.325	0.578
Projection	0.028	0.389	16.491	0.389	0.829
Time (s) for 100 (log) signatures of length-**1000** paths:					
Signature	0.017	2.427	100.572	2.400	4.935
Compiled	0.042	4.931	2857.072	3.341	5.845
Projection	0.211	3.285	123.143	3.297	7.701
Time (s) for a single call of the preparation function:					
Compiled	0.107	0.805	439.38	0.372	1.466
Projection	0.001	0.037	4.386	0.103	0.156

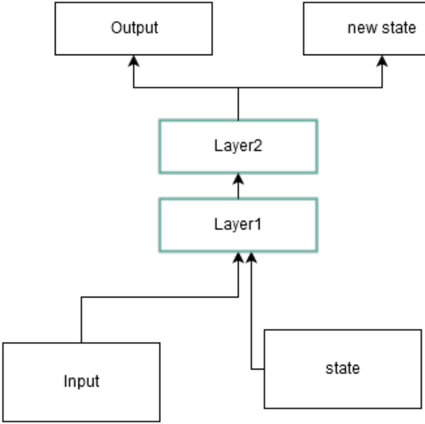
Autodifferentiation

Consider evaluating, for some fixed value of x,

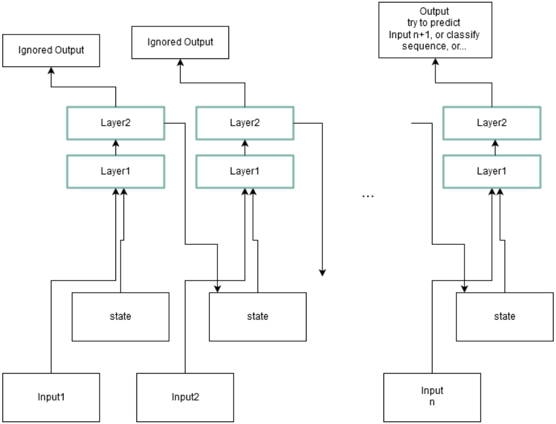
$$\frac{d}{dx}\left[\left(\sin(x^4 + \log x)\right)(x + \sinh x)\right]$$



Recurrent Neural Networks



Recurrent Neural Networks



Thanks!

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Dr Ben Graham