SIMATS

School of Engineering

Engineering Mathematics I

Science & Humanities

Saveetha Institute of Medical And Technical Sciences,Chennai.

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DIFFERENTIAL CALCULUS

BASIC FORMULAE :

1. $d/dx\,(c) = 0$
2. $d/dx\,(e^x) = e^x$
3. $d/dx\,(\sin x) = \cos x$
4. $d/dx\,x^n = nx^{n-1}$
5. $d/dx\,a^x = a^x \log a$
6. $d/dx\,\cos x = -\sin x$
7. $d/dx\,\tan x = \sec^2 x$
8. $d/dx\,\cot x = -\operatorname{cosec}^2 x$
9. $d/dx\,\log_a x = \frac{\log_a e}{x}$
10. $d/dx\,\log x = 1/x$
11. $d/dx\,\sin^{-1} x = 1/\sqrt{1-x^2}$
12. $d/dx\,\cos^{-1} x = -1/\sqrt{1-x^2}$

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

Product Rule :

$$d/dx\,[uv] = uv' + vu'$$

where

$u = f(x)$ $\quad v = g(x)$

$u' = f'(x)$ $\quad v' = g'(x)$

Problem : 1

$$d/dx\,(x^3+2x)e^x$$

Solution :

$$= (x^3+2x)\,d/dx\,e^x + e^x\,d/dx\,(x^3+2x)$$

$$= (x^3+2x)e^x + e^x(3x^2+2)$$

$$= e^x[x^3+2x+3x^2+2]$$

$$= e^x[x^3+3x^2+2x+2]$$

Problem : 2

$$d/dx\,(x^2 \sin x)$$

Solution :

$$= x^2\,d/dx\,(\sin x) + \sin x\,d/dx\,(x^2)$$

$$= x^2 \cos x + 2x \sin x$$

θ	0°	30°	45°	60°	90°
$\operatorname{cosec}\theta$	∞	2	$\sqrt{2}$	$2/\sqrt{3}$	1
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞
$\cot\theta$	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0

QUOTIENT RULE :

$$d/dx\,(u/v) = \frac{uv' - vu'}{v^2}$$

where

$u = f(x)$ $\quad v = g(x)$

$u' = f'(x)$ $\quad v' = g'(x)$

Problem : 1

Find $d/dx\left(\frac{e^{2x}+1}{\sin x}\right)$

Solution :

$$= \frac{\sin x\,d/dx\,(e^{2x}+1) - (e^{2x}+1)\,d/dx\,(\sin x)}{\sin^2 x}$$

$$= \frac{2e^{2x}\sin x - \cos x\,e^{2x} - \cos x}{\sin^2 x}$$

$$\therefore d/dx\left(\frac{e^{2x}+1}{\sin x}\right) = \frac{e^{2x}[2\sin x + \cos^2 x]}{\sin^2 x}$$

Problem : 2

Find $d/dx\left(\frac{x^2-1}{x^2+1}\right)$

Solution :

$$y = \frac{x^2-1}{x^2+1}\ ;\ \frac{dy}{dx} = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{2x(x^2+1-x^2+1)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

CHAIN RULE :

$$d/dx\,[f(g(x))] = f'[g(x)]\cdot[g'(x)]$$

Problem :

$$d/dx\,(\cos^3(\log x^2))$$

Solution :

$$= [3\cos^2(\log x^2)]\,[-\sin(\log x^2)]\,[1/x^2 \cdot 2x]$$

$$= -[3\cos^2(\log x^2)] \times [\sin(\log x^2)] \times 2/x$$

$$= -6/x\,[\cos^2(\log x^2)]\,[\sin(\log x^2)]$$

Allied Angle Table :

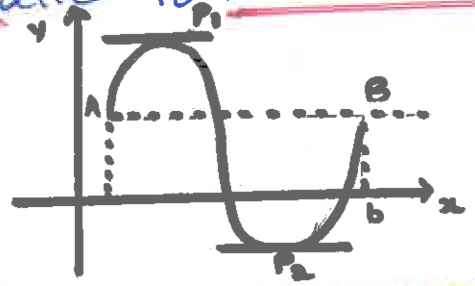
α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$
$-\theta$	$-\sin\theta$	$+\cos\theta$	$-\tan\theta$
$90^\circ-\theta$	$+\cos\theta$	$+\sin\theta$	$+\cot\theta$
$90^\circ+\theta$	$+\cos\theta$	$-\sin\theta$	$-\cot\theta$
$180^\circ-\theta$	$+\sin\theta$	$-\cos\theta$	$-\tan\theta$
$180^\circ+\theta$	$-\sin\theta$	$-\cos\theta$	$+\tan\theta$
$270^\circ-\theta$	$-\cos\theta$	$-\sin\theta$	$+\cot\theta$
$270^\circ+\theta$	$-\cos\theta$	$+\sin\theta$	$-\cot\theta$
$360^\circ-\theta$	$-\sin\theta$	$+\cos\theta$	$-\tan\theta$
$360^\circ+\theta$	$+\sin\theta$	$+\cos\theta$	$+\tan\theta$

MEAN VALUE THEOREMS

ROLLE'S THEOREM

Let f be a real function on $[a,b]$. $f(x)$ is a continuous on $[a,b]$. $f(x)$ is derivable in (a,b). $f(a)=f(b)$ $\exists c \in (a,b)$, $f'(c)=0$

If $y=f(x)$, be a continuous curve with end points A and B, having tangent at every point b/w A and B and the ordinates A and B are Equal there exist atleast one point p on the curve b/w A and B ∍ the tangent at P is parallel to the X-axis



Example 1:

$f(x)=3x^4-4x^2+5$

$f(x)$ continuous in $[-1,1]$

$f(x)$ derivative in $[-1,1]$

$f(-1)=f(1)=4$

$f'(c)=0 \Rightarrow c=\pm\sqrt{2/3}$

Conditions of rolle's theorem is satisfied

Example 2:

$f(x)=\cos 1/x$ on $[-1,1]$

$f(x)$ continuous on $[-1,1]$

$f(x)$ derivative on $[-1,1]$

$f(-1)=f(1)=\cos 1$

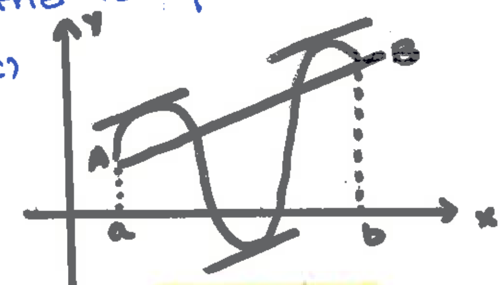
But f is not continuous at $x=0$

Rolle's theorem cannot be applied

LAGRANGE'S MEAN VALUE THEOREM

Let f be a real function on $[a,b]$. $f(x)$ is continuous in $[a,b]$. $f(x)$ is derivable in (a,b). $f(a)\neq f(b)$ $\exists c \in (a,b)$ ∍ $f'(c)=\frac{f(b)-f(a)}{b-a}$. If $y=f(x)$ is a continuous curve with A and B as end points and at each point b/t A and B, the curve has a tangent then there is atleast one point P on curve b/w A and B at which the tangent is parallel to the chord AB. $A=(a,f(a))$ $B=(b,f(b))$. So, the slope of chord $AB=\frac{f(b)-f(a)}{b-a}$

The slope of the tangent at the point $P(c,f(c))$ is $f'(c)$



Example 1:

$f(x)=x^3+3$

$f(x)$ continuous in $[1,2]$

$f(x)$ derivative in $[1,2]$

$f(1)=2$, $f(2)=10$

$f(1)\neq f(2)$

$f'(c)=\frac{f(2)-f(1)}{2-1}=8$

Lagrange's Mean value theorem is satisfied

Example 2:

$f(x)=|x|$

$f(x)$ continuous in $[-2,3]$

$f(x)$ is not differentiable at $x=0$

$f(x)$ is not differentiable in $[-2,3]$

Langrange's Mean value theorem cannot be applied.

CAUCHY'S MEAN VALUE THEOREM

Let $f(x)$ & $g(x)$ are continuous in $[a,b]$, $f(x)$ and $g(x)$ are differentiable in (a,b)

$g'(x)\neq 0$ $\forall x \in (a,b)$ $\exists$ a, $c \in (a,b)$

$\Rightarrow \frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$

Example 1:

$f(x)=e^x$, $g(x)=e^{-x}$ in (a,b)

$f(x)$ & $g(x)$ continuous in $[a,b]$

$f(x)$ & $g(x)$ differentiable in (a,b)

$g'(x)\neq 0$ $\forall x \in (a,b)$

$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$

$e^{a+b}=e^{2c}$

$c=\frac{a+b}{2}$

Example 2:

$f(x)=x^3$, $g(x)=x^2$ in $[1,2]$

$f(x)$ & $g(x)$ continuous in $[1,2]$

$f(x)$ & $g(x)$ differentiable in $(1,2)$

$g'(x)\neq 0$, $\forall x \in (1,2)$

$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}=\frac{3c}{2}$

$c=\frac{14}{9} \in (1,2)$

TAYLOR'S SERIES

COMPUTE : $f'(x), f''(x), f'''(x), \dots$

COMPUTE : $f(a), f'(a), f''(a), \dots$

FORMULA : $f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$

PROBLEM : 1

$f(x) = \log_e x$ in powers of $(x-1)$

Solution:

$f(x) = \log_e x$, $a = 1$

$f(x) = \log_e x \qquad f(1) = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \qquad f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$

$$\log_e x = (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots$$

Expand $f(x) = e^x$ in powers of $x-4$.

Solution:

$$e^x = e^4 \left\{ 1 + (x-4) + \frac{1}{2}(x-4)^2 + \frac{1}{6}(x-4)^3 + \dots \right\}$$

PROBLEM : 2

$f(x) = \cos x$ in powers of $x - \frac{\pi}{2}$

Solution:

$f(x) = \cos x \qquad a = \frac{\pi}{2}$

$f(x) = \cos x \qquad f(\pi/2) = 0$

$f'(x) = -\sin x \qquad f'(\pi/2) = -1$

$f''(x) = -\cos x \qquad f''(\pi/2) = 0$

$f'''(x) = \sin x \qquad f'''(\pi/2) = 1$

$f^{IV}(x) = \cos x \qquad f^{IV}(\frac{\pi}{2}) = 0$.

$$\cos x = (x - \tfrac{\pi}{2}) + \frac{1}{3!}(x - \tfrac{\pi}{2})^3 + \dots$$

Expand $f(x) = \sin x$ in powers of $x - \frac{\pi}{4}$

Solution:

$$\sin x = \frac{1}{\sqrt{2}} \left\{ 1 + x - \frac{\pi}{4} + \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 (-1) - \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3 + \dots \right\}$$

MACLAURIN'S SERIES

COMPUTE : $f(x), f'(x), f''(x), f'''(x), \dots$

COMPUTE : $f(0), f'(0), f''(0), f'''(0), \dots$

FORMULA : $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots$

PROBLEM 1 :

Expand $\log(1+x)$ in power of x

Solution

$f(x) = \log(1+x)$

$f(x) = \log(1+x) \qquad f(0) = 0$

$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \qquad f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \qquad f'''(0) = 2$

$f^{IV}(x) = \frac{-6}{(1+x)^4} \qquad f^{IV}(0) = -6$

$$\log(1+x) = 0 + \frac{x}{1!} - \frac{x^2}{2!} + \dots$$

Ex:3 Expand $x e^{\cos x}$ in Powers of x.

PROBLEM 2 :

Expand $e^{\sin x}$ in Powers of x

Solution

$f(x) = e^{\sin x}$

$f(x) = e^{\sin x} \qquad f(0) = 1$

$f'(x) = e^{\sin x} \cos x \qquad f'(0) = 1$

$f''(x) = f'(x)\cos x - f(x)\sin x \qquad f''(0) = 1$

$f'''(x) = f''(x)\cos x - 2f'(x)\sin x - f(x)\cos x \qquad f'''(0) = 0$

$$e^{\sin x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} - \dots$$

Ex:4 Expand $e^{\cos x}$ in Powers of x.

REMARK : Taylor's Series (or) Maclaurin's Series can be applied only for functions that are continuously differentiable.

INDETERMINATE FORMS AND L' HOSPITAL'S RULE

Limit of a function

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

Indeterminate Forms

$$\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ 0\cdot\infty,\ 0^0,\ \infty^0,\ 1^\infty$$

L' Hospital's Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Again indeterminate Form

$$\lim_{x \to \infty} \frac{f''(x)}{g''(x)}$$

Problem 1: Evaluate

$$\lim_{x \to 0} \frac{\sin x}{x}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{\sin 0}{0} = \frac{0}{0}$$

Apply L' Hospital's Rule

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{\cos 0}{1}$$

$$= 1$$

Problem 2:

Evaluate $\lim_{x \to 0} \frac{\sin x - \sin^{-1} x}{x^2}$

$$\lim_{x \to 0} \frac{\sin x - \sin^{-1} x}{x^2}$$

$$= \frac{\sin 0 - \sin^{-1} 0}{0} = \frac{0}{0}$$

indeterminate Form

Apply L' Hospital's Rule

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\cos x - \frac{1}{\sqrt{1-x^2}}}{2x}$$

$$= \frac{0}{0}$$

Again $\lim_{x \to 0} \frac{f''(x)}{g''(x)}$

$$= \lim_{x \to 0} \frac{-\sin x - \frac{0 - \frac{1}{2\sqrt{1-x^2}}(-2x)}{1-x^2}}{2}$$

$$= \frac{0}{2} = 0$$

Ex: Evaluate $\lim_{x \to} \frac{x^2}{e^x}$

Problem 3:

Evaluate $\lim_{x \to \infty} \frac{x^n}{e^x}$

$$\lim_{x \to \infty} \frac{x^n}{e^x} = \frac{\infty^n}{e^\infty} = \frac{\infty}{\infty}$$

Apply L'Hospital's Rule

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{n x^{n-1}}{e^x}$$

$$= \frac{\infty}{\infty}$$

Again $\lim_{x \to \infty} \frac{f''(x)}{g''(x)}$

$$\lim_{x \to \infty} \frac{n(n-1)\, x^{n-2}}{e^x} = \frac{\infty}{\infty}$$

Continuing this manner

$$\lim_{x \to \infty} \frac{n(n-1)\ldots 1 x^0}{e^x}$$

$$\Rightarrow \lim_{x \to \infty} \frac{n!}{e^x} = \frac{n!}{\infty} = 0$$

Ex: Evaluate

$$\lim_{x \to 0} \frac{\cos x - \cos^{-1} x}{x^2}$$

Problem 4:

Evaluate $\lim_{x \to 0} \frac{e^{ax} - e^{bx}}{x}$

$$\lim_{x \to 0} \frac{e^{ax} - e^{bx}}{x} = \frac{1-1}{0} = \frac{0}{0}$$

Apply L'Hospital's Rule

$$\lim_{x \to 0} \frac{ae^{ax} - be^{bx}}{1} = \frac{a-b}{1} = a-b$$

Problem 5:

Evaluate $\lim_{x \to -3} \frac{x^2-9}{2x^2+7x+3}$

$$\lim_{x \to -3} \frac{x^2-9}{2x^2+7x+3} = \frac{(-3+3)(-3-3)}{18-21+3}$$

$$= \frac{0}{0}$$

Apply L'Hospital's Rule

$$\lim_{x \to -3} \frac{2x}{4x+7} = \frac{2(-3)}{4(-3)+7}$$

$$= \frac{-6}{-12+7} = \frac{-6}{-5}$$

$$= \frac{6}{5}$$

MAXIMA AND MINIMA IN 1D

PROCEDURE:

* put $f'(x)=0$
* Find Stationary points.
* Compute $f''(x)$
* $f''(a)<0$
  'a' is Maxima
* $f''(a)>0$
  'a' is minima
* $f''(a)=0$
  Test Fails

Example 1:
What is the value of the function $(x-1)(x-2)^2$ at its maxima?

Solution:

$f(x)=(x-1)(x-2)^2$

$f(x)=(x-1)(x^2+4-4x)$

$f(x)=(x^3-5x^2+8x-4)$

$f'(x)=3x^2-10x+8$

$f'(x)=0$

$3x^2-10x+8=0$

$(3x-4)(x-2)=0$

$x=\frac{4}{3},2$

Now $f''(x)=6x-10$

$f''\left(\frac{4}{3}\right)=6\times\frac{4}{3}-10<0$

$f''(2)=12-10>0$

Hence at $x=\frac{4}{3}$ the function will occupy Maximum

Maximum $=f(4/3)$

Maximum Value $=\frac{4}{27}$

Example 2:
check whether the function $x^2\log x$ in the interval $(1,e)$ has a point of Maximum or Minimum

Solution:

$f(x)=x^2\log x$

$f'(x)=2x\log x+x$

$f''(x)=2(1+\log x)+1$

$f''(1)=3+2\log_e 1$

$f''(e)=3+2\log_e e$

$\Rightarrow f(x)$ has local Minimum at $\frac{1}{\sqrt{e}}$

$\rightarrow$ But x lies only in interval $(1,e)$ so that $y_2=\sqrt{x}$ has not extremum in $(1,e)$.

Hence, neither a point of Maximum or Minimum

Example 3:
Check Whether the function is Maximum or Minimum
$f(x)=x^3-3x^2-9x+12$

Solution:

$f(x)=x^3-3x^2-9x+12$

$f'(x)=3x^2-6x-9$

$f'(x)=0$

$3x^2-6x-9=0$

$(x+1)(x-3)=0$

$x=-1,3$

$f''(x)=6x-6$

$f''(-1)=-6-6=-12<0$

$f(x)$ is Maximum

$f''(3)=18-6=12>0$

$f(x)$ is Minimum.

INTEGRAL CALCULUS

Basic formulae

1. $\int k\,dx = kx + c$
2. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c$
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
4. $\int \frac{f'(x)}{f(x)} dx = \log_e f(x) + c$
5. $\int e^{ax} dx = \frac{e^{ax}}{a} + c \; ; a \neq 0$
6. $\int \frac{1}{x} dx = \log x + c$
7. $\int \cos(ax)\,dx = \frac{\sin ax}{a} + c$, $a \neq 0$
8. $\int \sin(ax)\,dx = \frac{-\cos ax}{a} + c$, $a \neq 0$
9. $\int \sec^2 x\,dx = \tan x + c$
10. $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
11. $\int e^{ax} \sin bx\,dx =$ $\frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$
12. $\int e^{ax} \cos bx\,dx =$ $\frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$

Problems:

1) $\int 2022\,dx = 2022x + c$
2) $\int (3x+4)^2 dx = \frac{(3x+4)^3}{9} + c$
3) $\int x^{2022} dx = \frac{x^{2023}}{2023} + c$
4) $\int \frac{2x}{x^2+2} dx = \log(x^2+2) + c$
5) $\int e^{-3x} dx = \frac{e^{-3x}}{-3} + c$
6) $\int \cos(2x)\,dx = \frac{\sin 2x}{2} + c$
7) $\int \sin(3x)\,dx = \frac{-\cos 3x}{3} + c$
8) $\int e^{2x} \cos 3x\,dx =$ $\frac{e^{2x}}{13}[2\cos 3x + 3 \sin 3x]$
9) $\int e^{4x} \sin 2x\,dx =$ $\frac{e^{4x}}{20}[4 \sin 2x - 2\cos 2x]$
10) $\int \sec^2 3x\,dx$ $= \tan 3x + c$

Integration By Parts

$\int u\,dv = uv - \int v\,du$

Problem:

1) $\int x^2 \tan^{-1} x\,dx$

Solution:

$u = \tan^{-1} x \qquad dv = x^2 dx$

$du = \frac{1}{1+x^2} dx \qquad v = \frac{x^3}{3}$

$\int x^2 \tan^{-1} x\,dx = \tan^{-1} x \left(\frac{x^3}{3}\right) - \int \frac{x^3}{3} \cdot \frac{dx}{1+x^2}$

$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3}\left[x^3 \log(1+x^2) - \int \log(1+x^2) . 3x^2\right]$

$\Rightarrow \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \log_e (1+x^2) + c$

Ex: Evaluate $\int x^3 e^x dx$

Evaluate $\int x^2 \log x\,dx$

BERNOULLI'S FORMULAE

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

Problem:

1) $\int x^3 e^{-2x} dx$

Solution:

$u = x^3 \qquad v\,dx = e^{-2x} dx$

$u' = 3x^2 \qquad v_1 = \frac{e^{-2x}}{-2}$

$u'' = 6x \qquad v_2 = \frac{e^{-2x}}{4}$

$u''' = 6 \qquad v_3 = \frac{e^{-2x}}{-8}$

$u^{iv} = 0 \qquad v_4 = \frac{e^{-2x}}{16}$

$\int x^3 e^{-2x} dx = \left(x^3 \times \frac{e^{-2x}}{-2}\right) - \left(3x^2 \times \frac{e^{-2x}}{4}\right) + 6x\left(\frac{e^{-2x}}{-8}\right) - 6\left(\frac{e^{-2x}}{16}\right)$

$\Rightarrow -\frac{e^{-2x}}{8}[4x^3 + 6x^2 + 6x + 3] + c$

DEFINITE INTEGRAL PROPERTIES

PROPERTY : 1

$$\int_a^b f(x)dx = \int_a^b f(t)\,dt$$

PROBLEM :

Prove That :

$$\int_0^{\pi/2} \cos x\,dx = \int_0^{\pi/2} \cos t\,dt$$

LHS:

$$\int_0^{\pi/2} \cos x\,dx = [\sin x]_{x=0}^{x=\pi/2}$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin(0)$$

$$= 1 - 0$$

$$= 1$$

RHS:

$$\int_0^{\pi/2} \cos t\,dt = [\sin t]_{t=0}^{t=\pi/2}$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin(0)$$

$$= 1 - 0$$

$$= 1$$

$\therefore$ LHS = RHS

$$\therefore \int_0^{\pi/2} \cos x\,dx = \int_0^{\pi/2} \cos t\,dt$$

PROPERTY : 2

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

PROBLEM

Prove that

$$\int_2^3 x^3 dx = -\int_3^2 x^3 dx$$

LHS

$$\int_2^3 x^3 dx = \left[\frac{x^4}{4}\right]_{x=2}^{x=3}$$

$$= \frac{81}{4} - \frac{16}{4}$$

$$= \frac{65}{4}$$

RHS

$$\int_3^2 x^3 dx = \left[\frac{x^4}{4}\right]_{x=3}^{x=2}$$

$$= \frac{16}{4} - \frac{81}{4}$$

$$= -\frac{65}{4}$$

$\therefore$ LHS = RHS

$$\therefore \int_2^3 x^3 dx = -\int_3^2 x^3 dx$$

PROPERTY : 3

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

PROBLEM

Prove that

$$\int_0^2 e^x dx = \int_0^2 e^{(a-x)} dx$$

LHS

$$\int_0^2 e^x dx = [e^x]_{x=0}^{x=2}$$

$$= e^2 - e^0$$

$$= e^2 - 1$$

RHS

$$\int_0^2 e^{(a-x)} dx = \left[\frac{e^{(a-x)}}{-1}\right]_{x=0}^{x=2}$$

$$= -e^{(2-2)} + e^{2-0}$$

$$= -e^0 + e^2$$

$$= e^2 - 1$$

$\therefore$ LHS = RHS

$$\therefore \int_0^2 e^x dx = \int_0^2 e^{(a-x)} dx$$

PROPERTY : 4

$$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$$

iff $f(2a-x) = f(x)$

PROBLEM

Prove that

$$\int_0^{2\pi} \cos x\,dx = 2\int_0^{\pi} \cos x\,dx$$

$f(x) = \cos x$, $a = \pi$

$f(2\pi - x) = \cos(2\pi - x)$

$= \cos x$

$\therefore f(x) = f(2\pi - x)$

LHS

$$\int_0^{2\pi} \cos x\,dx = [\sin x]_0^{2\pi}$$

$$= \sin(2\pi) - \sin(0)$$

$$= 0$$

RHS

$$2\int_0^{\pi} \cos x\,dx = 2[\sin x]_0^{\pi}$$

$$= 2[\sin(\pi) - \sin(0)]$$

$$= 2[0]$$

$$= 0$$

$\therefore$ LHS = RHS

$$\int_0^{2\pi} \cos x\,dx = 2\int_0^{\pi} \cos x\,dx$$

PROPERTY : 5

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx \qquad a < c < b$$

PROBLEM

Prove That

$$\int_2^3 x^{-2}dx + \int_3^4 x^{-2}dx = \int_2^4 x^{-2}dx$$

LHS

$$= \left[-\frac{1}{x}\right]_2^3 + \left[-\frac{1}{x}\right]_3^4$$

$$= \left(-\frac{1}{3} + \frac{1}{2}\right) + \left(-\frac{1}{4} + \frac{1}{3}\right)$$

$$= \frac{1}{4}$$

RHS

$$\int_2^4 x^{-2}dx = \left[-\frac{1}{x}\right]_2^4$$

$$= -\frac{1}{4} + \frac{1}{2}$$

$$= \frac{1}{4}$$

LHS = RHS

PROPERTY : 6

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \quad \text{(Even)}$$

$$\int_{-a}^{a} f(x)dx = 0 \quad \text{(odd}$$

DOUBLE INTEGRALS

CONSTANT LIMIT

Problem 1

$$\int_0^3\int_0^2 e^{x+y}\,dy\,dx$$

Solution:

$$\text{Let } I = \int_0^3\int_0^2 e^x e^y\,dy\,dx$$
$$= \int_0^3 e^x\,dx \int_0^2 e^y\,dy$$
$$= \left[e^x\right]_0^3 \left[e^y\right]_0^2$$
$$= \left[e^3 - e^0\right]\left[e^2 - e^0\right]$$
$$= \left[e^3 - 1\right]\left[e^2 - 1\right]$$

Problem: 2

$$\int_0^1\int_1^2 x(x+y)\,dy\,dx$$

Solution:

$$\int_0^1\int_1^2 x(x+y)\,dy\,dx = \int_0^1\int_1^2 \left[x^2+xy\right]dy\,dx$$
$$= \int_0^1 \left[x^2y + \frac{xy^2}{2}\right]_{y=1}^{y=2} dx \quad \text{Diff w.r.to 'x'}$$
$$= \int_0^1 \left[(2x^2+2x)-\left(x^2+\frac{x}{2}\right)\right]dx$$
$$= \left[\frac{x^3}{3} + \frac{3x^2}{4}\right]_0^1 = \frac{13}{12}$$

VARIABLE LIMIT

Problem 1

$$\int_0^1\int_0^{1-x} y\,dy\,dx$$

Solution:

$$I = \int_0^1\int_0^{1-x} y\,dy\,dx$$
$$= \int_0^1 \left[y^2/2\right]_0^{1-x} dx = \int_0^1 \frac{(1-x^2)^2}{2}dx$$
$$= \int_0^1 \frac{(1-x)^2}{2}dx = \frac{1}{2}\left[\frac{(1-x)^3}{-3}\right]_0^1$$
$$= \frac{1}{2}\left[0+\frac{1}{3}\right] = \frac{1}{6}$$

Problem: 2

Evaluate $\int_0^5\int_0^{x^2} x(x^2+y^2)\,dx\,dy$

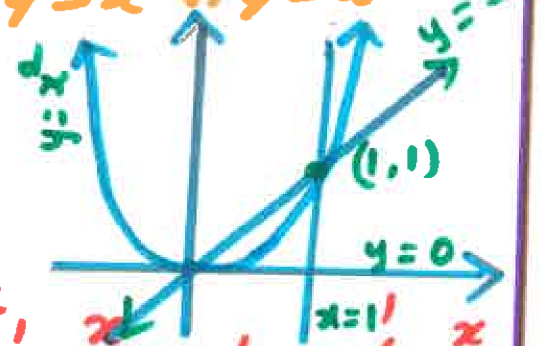
Problem's solution:

$$I = \int_0^5\int_0^{x^2} x(x^2+y^2)\,dy\,dx$$
$$= \int_0^5\int_0^{x^2} (x^3+xy^2)\,dy\,dx = \int_0^5\left[x^3y + \frac{xy^3}{3}\right]_0^{x^2}dx$$
$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right)dx = \left[\frac{x^6}{6} + \frac{x^8}{24}\right]_0^5$$
$$= \left[\frac{5^6}{6} + \frac{5^8}{24}\right] = 5^6\left[\frac{1}{6} + \frac{5^2}{24}\right]$$
$$= 5^6\left[\frac{4+25}{24}\right] = 5^6\left[29/24\right]$$

AREA $\quad A = \iint_R dx\,dy$

Problem 1

Find the area which bounded by $y=x$ & $y=x^2$



Solution

$x : 0$ to 2

$y : x^2$ to x

Required Area,

$$= \iint dx\,dy = \int_0^1\int_{x^2}^{x} dy\,dx = \int_0^1 (y)_{x^2}^{x} dx$$
$$= \int_0^1 (x-x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

Problem: 3

Evaluate $\iint_R xy\,dx\,dy$, where R is the domain bounded by x axis, ordinate $x=2a$ & the curve $x^2=4ay$.

Solution:

$x=2a$, $x^2=4ay \rightarrow$ ① & ②

sub ① & ②, we get $(2a)^2=4ay$

$\Rightarrow y=a$, x varies from $x=2\sqrt{ay}$ to $x=2a$ & y varies from $y=0$ to $y=a$

$\therefore$ Required area $= \int_0^a\int_{2\sqrt{ay}}^{2a} xy\,dx\,dy$

$$= \int_0^a \left[\frac{yx^2}{2}\right]_{x=2\sqrt{ay}}^{x=2a} dy = \left(a^4 - \frac{2a^4}{3}\right) = \frac{a^4}{3}$$

Triple Integrals

Constant Limit:

Problem: 1

$$\int_0^1\int_0^2\int_1^2 x^2 y z\,dx\,dy\,dz$$

$$=\int_0^1 z\,dz\int_0^2 y\,dy\int_1^2 x^2\,dx$$

$$=\left[\frac{z^2}{2}\right]_0^1\left[\frac{y^2}{2}\right]_0^2\left[\frac{x^3}{3}\right]_1^2$$

$$=\frac{1}{2}\cdot\frac{4}{2}\left[\frac{8}{3}-\frac{1}{3}\right]=\frac{7}{3}$$

Problem: 2

$$\int_0^1\int_0^2\int_0^e dy\,dx\,dz$$

$$=\int_0^1\int_0^2\left[\int_0^e dy\right]dx\,dz$$

$$=\int_0^1\int_0^2 e\,dx\,dz$$

$$=e\int_0^1[x]_0^2\,dz$$

$$=2e\,[z]_0^1$$

$$=2e$$

Variable Limit:

Problem: 1

$$\int_0^{\log 2}\int_0^x\int_0^{x+y} e^{(x+y+z)}\,dx\,dy\,dz$$

$$=\int_0^{\log 2}\int_0^x e^{x+y}\,(e^z)_0^{x+y}\,dy\,dx$$

$$=\int_0^{\log 2}\int_0^x\left(e^{2x+2y}-e^{x+y}\right)dy\,dx$$

$$=\int_0^{\log 2}\left[e^{2x}\left(\frac{e^{2y}}{2}\right)_0^x-e^x\left(e^y\right)_0^x\right]dx$$

$$=\frac{1}{2}\int_0^{\log 2}\left(e^{4x}-3e^{2x}+2e^x\right)dx$$

$$=\frac{1}{2}\left[\frac{e^{4\log_e 2}}{4}-\frac{3}{2}e^{2\log_e 2}+2e^{\log_e 2}-\frac{3}{4}\right]$$

$$=\frac{1}{2}\left[\frac{e^{\log_e 16}}{4}-\frac{3}{2}e^{\log_e 4}+2e^{\log_e 2}-\frac{3}{4}\right]$$

$$=\frac{1}{2}\left[\frac{16}{4}-\frac{12}{2}+4-\frac{3}{4}\right]$$

$$=1-\frac{3}{8}$$

$$=5/8$$

Problem: 2

$$\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}\frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$$

$$=\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\sin^{-1}\frac{z}{\sqrt{a^2-x^2-y^2}}\right]_0^{\sqrt{a^2-x^2-y^2}}dy\,dx$$

$$=\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\sin^{-1}1-\sin^{-1}0\right]dy\,dx$$

$$=\int_0^a\int_0^{\sqrt{a^2-x^2}}\frac{\pi}{2}\,dy\,dx$$

$$=\frac{\pi}{2}\int_0^a[y]_0^{\sqrt{a^2-x^2}}\,dx$$

$$=\frac{\pi}{2}\int_0^a\sqrt{a^2-x^2}\,dx$$

$$=\frac{\pi}{2}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\frac{\pi}{2}\left[0+\frac{a^2}{2}\sin^{-1}1-0\right]$$

$$=\frac{\pi}{2}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2}$$

$$=\frac{\pi^2a^2}{8}$$

Volume:

$$V=\iiint dv=\iiint dx\,dy\,dz$$

Problem:

$$x^2+y^2+z^2=a^2$$

$z=0$ to $z=\sqrt{a^2-x^2-y^2}$

$y=0$ to $y=\sqrt{a^2-x^2}$

$x=0$ to $x=a$

$$V=8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}dz\,dy\,dx$$

$$=8\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$=8\int_0^a\left[\frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}+\frac{y}{2}\sqrt{a^2-x^2-y^2}\right]_0^{\sqrt{a^2-x^2}}dx$$

$$=8\int_0^a\left[\frac{a^2-x^2}{2}\cdot\frac{\pi}{2}\right]dx$$

$$=2\pi\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$=2\pi\left[\frac{2}{3}a^3\right]$$

$$=\frac{4}{3}\pi a^3\ \text{cubic units}$$

EULER'S THEOREM $\left\{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu\right\}$

Partial Derivatives:

A function $f(x,y)$ which depend on two variables 'x' and 'y' where x and y are independent on each other.

(i.e) $\boxed{\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y}}$

Problem : 1 $u=e^x \sin y$

$u_x=\frac{\partial u}{\partial x}=e^x\sin y$

$u_y=\frac{\partial u}{\partial y}=e^x\cos y$

Problem : 2 If $x^3+y^3=3axy$ then find dy/dx.

Soln: $f(x,y)=x^3+y^3-3axy$

$\frac{\partial f}{\partial x}=3x^2-3ay$; $\frac{\partial f}{\partial y}=3y^2-3ax$

$\frac{dy}{dx}=\frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}=\frac{-3[x^2-ay]}{3[y^2-ax]}$

$\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$

Problem : 3 $w=x^2\log y$

Soln: $w_x=2x\log y$

$w_y=x^2/y$

Problem : 1 Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u$, where $\log u=\frac{x^3+y^3}{3x+4y}$.

Soln: $z=\log u=\frac{x^3+y^3}{3x+4y}$

$\Rightarrow \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[3+4\left(\frac{y}{x}\right)\right]}=x^2\left[\frac{1+\left(\frac{y}{x}\right)^3}{3+4\left(\frac{y}{x}\right)}\right]$

$\therefore z$ is a homogeneous function of degree 2 in x and y. By Euler's theorem we get $x\frac{dz}{dx}+\frac{dz}{dy}=2z\rightarrow (1)$

$$\boxed{\begin{gathered}x\cdot\frac{1}{u}\frac{\partial u}{\partial x}+y\cdot\frac{1}{u}\frac{\partial u}{\partial y}=2\log u\\ \text{(or) } x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u\end{gathered}}$$

Problem : 2 If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ Prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$.

Soln: Gn, $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$;

$\tan u=\frac{x^3+y^3}{x-y}$. By Euler's theorem

$x\frac{\partial}{\partial x}\tan u+y\frac{\partial}{\partial y}\tan u=2\tan u$

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\tan u\frac{1}{\sec^2 u}$

$$\boxed{\therefore x\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=\sin 2u}$$

Problem : 3 If $u=\cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ Prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$.

Soln: Gn, $u=\cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

$\cos u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$ [By Euler's theorem]

$x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}=nf$

$x\cdot\frac{\partial}{\partial x}(\cos u)+y\cdot\frac{\partial}{\partial y}(\cos u)=\frac{1}{2}\cos u$

$\sin u\left[x\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}\right]=\frac{1}{2}\cos u$

$$\boxed{x\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u.}$$

Problem : 4 If $u=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ Prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$.

Soln: $u=\sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]\Rightarrow \sin u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$

let $f(x,y)=\frac{x+y}{\sqrt{x}+\sqrt{y}}$ $\therefore f(x,y)=\sin u$

$f(tx,ty)=\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}=t^{1/2}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

f is a homogeneous function of degree ½. By Euler's theorem,

$x\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}=\frac{1}{2}f$.

$$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u.}$$

Problem : 5 $u=(x^2+y^2)^{-1/2}$

Soln: $\frac{\partial u}{\partial x}=-\frac{1}{2}(x^2+y^2)^{-3/2}\cdot 2x$

$x\cdot\frac{\partial u}{\partial x}=\frac{-x^2}{(x^2+y^2)^{-3/2}}$

$\frac{\partial u}{\partial y}=-\frac{1}{2}(x^2+y^2)^{-3/2}\cdot 2y=\frac{-y}{(x^2+y^2)^{-3/2}}$

$y\cdot\frac{\partial u}{\partial y}=\frac{-y^2}{(x^2+y^2)^{-3/2}}$

$x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=\frac{-(x^2+y^2)}{(x^2+y^2)^{-1/2}}=-u$

$\boxed{\therefore \text{Euler's theorem verified.}}$

Problem : 6 let $u=\log\frac{x^4+y^4}{x+y}$. By using Euler's theorem, $x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=3$

Soln: $u=\log\left(\frac{x^4+y^4}{x+y}\right)$

$e^u=\frac{x^4+y^4}{x+y}=f(x,y)\rightarrow (1)$

$f(tx,ty)=\frac{t^4x^4+t^4y^4}{tx+ty}=t^3\left(\frac{x^4+y^4}{x+y}\right)$

$\therefore f$ is a homogeneous function of degree 3. (ie) $t^3f(x,y)$. By using Euler's theorem we get,

$x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=3f$; $f(x,y)=e^u$

$\therefore e^u x\cdot\frac{\partial u}{\partial x}+e^u y\cdot\frac{\partial u}{\partial y}=3e^u$.

$$\boxed{x\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}=3.}$$

JACOBIAN

DEFINITION:

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

PROPERTIES:

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

$u=u(r,s)$, $v=v(r,s)$, $r=(x,y)$, $s=(x,y)$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

u & v depends on each other

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = 0.$$

PROBLEMS

1. If $u = x^2+1$, $v = y^2-2$ find $\frac{\partial(u,v)}{\partial(x,y)}$

SOLUTION

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 2y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy$$

2. If $u = \frac{y^2}{x}$, $v = \frac{x^2}{y}$ Find $\frac{\partial(u,v)}{\partial(x,y)}$

SOLUTION:

$\frac{\partial u}{\partial x} = \frac{-y^2}{x^2}$, $\frac{\partial v}{\partial x} = \frac{2x}{y}$, $\frac{\partial u}{\partial y} = \frac{2y}{x}$, $\frac{\partial v}{\partial y} = \frac{-x^2}{y^2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{-y^2}{x^2} & \frac{2y}{x} \\ \frac{2x}{y} & \frac{-x^2}{y^2} \end{vmatrix} = -3$$

3. If $u = x-y$, $v = y-z$, $w = z-x$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

SOLUTION:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

4. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

SOLUTION:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix} = 4$$

5. If $x = r\sin\theta\cos\phi$, $z = r\cos\theta$, $y = r\sin\theta\sin\phi$ Find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

SOLUTION:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta.$$

6. If $u = u(x,y)$, $v = v(x,y)$ then P.T. $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

SOLUTION

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = J_1 \times J_2$$

$$J_1 \times J_2 = \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y} \end{vmatrix}$$

$$J_1 \times J_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

7. If $x = e^u\cos v$, $y = e^u\sin v$ P.T $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$

SOLUTION:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} e^u\cos v & e^u\sin v \\ -e^u\sin v & e^u\cos v \end{vmatrix}$$

$$J_1 = \frac{\partial(x,y)}{\partial(u,v)} = e^{2u}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$J_2 = \frac{1}{x^2+y^2} = \frac{1}{e^{2u}}$$

$$\therefore J_1 \times J_2 = e^{2u} \times \frac{1}{e^{2u}} = 1$$

On simplification,

$$x^2+y^2 = e^{2u} \Rightarrow u = \frac{1}{2}\log(x^2+y^2)$$

$$\frac{y}{x} = \tan v \Rightarrow v = \tan^{-1}\left(\frac{y}{x}\right)$$

8. If $x = u$, $y = u\tan v$, $z = w$ then P.T. $\frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$

Solution:

$$J_1 = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & \tan v & 0 \\ 0 & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$J_1 = u\sec^2 v$$

$u = x$, $v = \tan^{-1}\left(\frac{y}{x}\right)$, $w = z$

$$J_2 = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$J_2 = \frac{1}{u\sec^2 v} \qquad \therefore J_1 \times J_2 = 1.$$

9. check whether $f_1 = x^2+y^2+z^2$, $f_2 = xy+yz+zx$, $f_3 = (x+y+z)^2$ are functionally dependant.

Solution

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} 2x & y+z & 2(x+y+z) \\ 2y & z+x & 2(x+y+z) \\ 2z & x+y & 2(x+y+z) \end{vmatrix}$$

$$= 4(x+y+z)\begin{vmatrix} x & y+z & 1 \\ y & z+x & 1 \\ z & x+y & 1 \end{vmatrix}$$

$$= 4(x+y+z)\begin{vmatrix} x & 1 & 1 \\ y & 1 & 1 \\ z & 1 & 1 \end{vmatrix} = 0.$$

$\Rightarrow f_1, f_2, f_3$ depends on each other.

They are related by

$$f_3 = f_1 + 2f_2.$$

2-D Extrema
Maxima & Minima

Unconstrained:

$f(x,y)=0$; $f_x(a,b)=0$
$f_y(a,b)=0$; $f_{xx}(a,b)=A$
$f_{yy}(a,b)=C$

$AC-B^2>0$, $A<0 \Rightarrow$ Maxima
$AC-B^2>0$, $A>0 \Rightarrow$ Minima
$AC-B^2<0 \Rightarrow$ Saddle point (neither max nor min)

Problem 1 Find the minimum value of $f(x,y)=x^2-xy+y^2-2x+y$

Sol Given $f(x,y)=x^2-xy+y^2-2x+y$

$f_x = 2x-y-2$ $\quad A=2$
$f_y = -x+2y+1$ $\quad B=-1$
$\quad C=2$

By Solving, $x=1$, $y=0$
$(1,0)$ - Extrema point
$(AC-B^2)_{(1,0)}=3>0$

$\therefore$ Min Value is $f(x,y)_{(1,0)}=-1$.

Constrained:

Method of Lagrange's multipli -ers:

Problem 1 A rectangular box open at the top, is to have a volume of 32 cc. Find the dimensions of the box, that requires the least material for its construction.

Sol Given: Volume of rectangle = 32cc

To find: Dimensions of box.
Surface Area $= xy+2yz+2zx$
Volume $= xyz = 32$

Auxilary function:

$F(x,y,z) = xy+2yz+2zx+\lambda(xyz-32)$
$F_x = \frac{\partial F}{\partial x} = y+2z+\lambda yz$
$F_y = \frac{\partial F}{\partial y} = x+2z+\lambda xz$
$F_z = \frac{\partial F}{\partial z} = 2x+2y+\lambda xy$

When F is Extremum.
$F_x=0 \Rightarrow \frac{1}{z}+\frac{2}{y}=-\lambda$
$F_y=0 \Rightarrow \frac{1}{z}+\frac{2}{x}=-\lambda$
$F_z=0 \Rightarrow \frac{2}{y}+\frac{2}{x}=-\lambda$

By solving above equations:

$x=4$
$y=4$
$z=2$
Dimensions of rectangle box.

Problem Find minimum value of $x^2+y^2+z^2$ subject to $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.

Sol: Let $f=x^2+y^2+z^2$
$\phi=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$

Auxilary function:

$F(x,y,z)=f+\lambda\phi$
$F(x,y,z)=x^2+y^2+z^2+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$
$F_x=\frac{\partial F}{\partial x}=0 \Rightarrow x=\left(\frac{\lambda}{2}\right)^{1/3}$
$F_y=\frac{\partial F}{\partial y}=0 \Rightarrow y=\left(\frac{\lambda}{2}\right)^{1/3}$
$F_z=\frac{\partial F}{\partial z}=0 \Rightarrow z=\left(\frac{\lambda}{2}\right)^{1/3}$

Put $x=y=z$ in Given Equation

$x=3$, $y=3$, $z=3$

$\therefore$ Minimum Value is

$x^2+y^2+z^2 = 3^2+3^2+3^2$
$=27$

Rank Of The Matrices

The number of non-zero elements in a row (or) column of a matrix is called rank of a matrix. $\rho(A)$

To find $\rho(A)$ for 2x2 matrix

$|A| \neq 0$

$\rho(A)=2$

Problem

$$A=\begin{bmatrix}3 & 4\\ 5 & 6\end{bmatrix}$$

$|A| = 18-20=-2 \neq 0$

$\therefore \rho(A)=2$

$|A| = 0$

$\rho(A)=1$

Problem

$$A=\begin{bmatrix}8 & 4\\ 4 & 2\end{bmatrix}$$

$= 16-16=0$

$\rho(A)=1$

To find $\rho(A)$ for 3x3 matrix

$|A| \neq 0$ — $\rho(A)=3$

$|A| \neq 0$ — $\rho(A) \neq 3$

(2x2) $|A|=0$ — $\rho(A)=1$

(2x2) $|A| \neq 0$ — $\rho(A)=2$

Problem

$$A=\begin{bmatrix}2 & -1 & -1\\ 1 & 2 & 1\\ 4 & -7 & -5\end{bmatrix}$$

$|A|=0$: $\rho(A) \neq 3$

It may be rank 1 or 2.

So find $|A|_{2\times 2} = 5 \neq 0$

$\therefore \rho(A)=2$

System of Linear Equations

$$a_{11}x+a_{12}y+a_{13}z=b_1$$
$$a_{21}x+a_{22}y+a_{23}z=b_2$$
$$a_{31}x+a_{32}y+a_{33}z=b_3$$

$AX=B$

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix} \rightarrow [A,B]=\left[\begin{array}{ccc|c}a_{11} & a_{12} & a_{13} & b_1\\ a_{21} & a_{22} & a_{23} & b_2\\ a_{31} & a_{32} & a_{33} & b_3\end{array}\right] \rightarrow \rho(A,B)$$

$\rightarrow \rho(A)$

Consistent

Unique Solution

$\rho(A)=\rho(A,B)=n$ (no. of unknowns)

Infinite number of solutions

$\rho(A)=\rho(A,B)<n$ (no. of unknowns)

Inconsistent

No solution

$\rho(A) \neq \rho(A,B)$

Problem

$$x+y+z=6;\quad x+2y-2z=-3;\quad 2x+3y+z=11;$$

$$A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -2\\ 2 & 3 & 1\end{bmatrix} \qquad [A,B]=\begin{bmatrix}1 & 1 & 1 & 6\\ 1 & 2 & -2 & -3\\ 2 & 3 & 1 & 11\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -3\\ 0 & 1 & -1\end{bmatrix} \quad R_2\to R_2-R_1,\ R_3\to R_3-2R_1 \quad \sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 1 & -3 & -9\\ 0 & 1 & -1 & -1\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & -2\\ 0 & 1 & -1\end{bmatrix} \quad R_2\to R_2-R_3 \quad \sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 0 & -2 & -8\\ 0 & 1 & -1 & -1\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -1\\ 0 & 0 & -2\end{bmatrix} \quad R_2 \sim R_3 \quad \sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 1 & -1 & -1\\ 0 & 0 & -2 & -8\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix} \quad R_3\to \frac{R_3}{-2} \quad \sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 1 & -1 & -1\\ 0 & 0 & 1 & 4\end{bmatrix}$$

$\rho(A)=\rho(A,B)=3$

The system of equations are consistent and have unique solution.

The reduced system is

$z=4$; $y-z=-1$; $x+y+z=6$.

$x=-1$; $y=3$; $z=4$;

Problem

$$x+2y+z=2;\quad 2x-y-z=2;\quad 4x-7y-5z=2$$

$$[A,B]=\begin{bmatrix}1 & 2 & 1 & 2\\ 2 & -1 & -1 & 2\\ 4 & -7 & -5 & 2\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & -5 & -3 & -2\\ 0 & -15 & -9 & -6\end{bmatrix} \quad R_2\to R_2-2R_1,\ R_3\to R_3-4R_1$$

$$\sim\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & -5 & -3 & -2\\ 0 & 0 & 0 & 0\end{bmatrix} \quad R_3\to R_3-3R_2$$

$$A\sim\begin{bmatrix}1 & 2 & 1\\ 0 & -5 & -3\\ 0 & 0 & 0\end{bmatrix}$$

$\rho(A)=2$; $\rho(A,B)=2$

$\rho(A)=\rho(A,B)=2<3\ (n)$

The reduced system is

$x+2y+z=2$

$-5y-3z=2 \Rightarrow 5y+3z=2$

$z=k$

$x=\frac{1}{5}(k+6)$; $y=\frac{1}{5}(2-3k)$

Problem

$$2x+y+5z=4;\quad 3x-2y+2z=2;\quad 5x-8y-4z=1$$

$$[A,B]=\begin{bmatrix}2 & 1 & 5 & 4\\ 3 & -2 & 2 & 2\\ 5 & -8 & -4 & 1\end{bmatrix}$$

$R_2\to 2R_2-3R_1$

$R_3\to 2R_3-5R_1$

$$\sim\begin{bmatrix}2 & 1 & 5 & 4\\ 0 & -7 & -11 & -8\\ 0 & -21 & -33 & -18\end{bmatrix}$$

$R_3\to R_3-3R_2$

$$\sim\begin{bmatrix}2 & 1 & 5 & 4\\ 0 & -7 & -11 & -8\\ 0 & 0 & 0 & 6\end{bmatrix}$$

$\rho(A)=2$; $\rho(A,B)=3$

$\rho(A) \neq \rho(A,B)$

No solution

GAUSS-ELIMINATION METHOD

DEFINITION:

Co efficient matrix converted to upper triangular matrix.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$AX = B$

AUGUMENTED MATRIX

$$[A/B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} a & b & c & p \\ 0 & d & e & q \\ 0 & 0 & f & r \end{array}\right]$$

$$ax + by + cz = p$$
$$dy + cz = q$$
$$fz = r$$

(Back substitution method).

$$x + 2y + z = 3$$
$$2x + 3y + 3z = 10$$
$$3x - y + 2z = 13$$

$AX = B$

$$(A,B) = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right] \begin{array}{l} R_2 + (-2)R_1 \\ R_3 + (-3)R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right]$$

$$(A,B) \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

$$x + 2y + z = 3$$
$$-y + z = 4$$
$$-8z = -24$$

GAUSS-JORDAN METHOD

DEFINITION:-

co-efficient matrix converted to unit matrix.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$AX = B$

AUGUMENTED MATRIX

$$[A/B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} a & 0 & 0 & d \\ 0 & b & 0 & e \\ 0 & 0 & c & f \end{array}\right]$$

$$ax + 0y + 0z = d$$
$$0x + by + 0z = e$$
$$0x + 0y + cz = f$$

Problem

$$x + 2y + z = 3$$
$$2x + 3y + 3z = 10$$
$$3x - y + 2z = 13$$

$AX = B$

$$(A,B) = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] R_3\left(\frac{1}{8}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right] \begin{array}{l} R_{13}(3) \\ R_{23}(1) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

$$x = 2$$
$$y = -1$$
$$z = 3$$

LU DECOMPOSITION

FOR 2X2 MATRIX

LINEAR SYSTEM OF EQUATION

$a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

$\boxed{Ax = B}$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\boxed{A = LU}$

$L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$

$Ax = B$

$LUx = B \longrightarrow ①$

$Ux = y \longrightarrow ②$

$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

From ① and ② $\boxed{Ly = B}$

Solving we get y_1, y_2

Substitute y in $\boxed{Ux = y}$

Solving we get

$\boxed{x_1 \text{ and } x_2}$

PROBLEM

$2x_1 + 3x_2 = 13$

$3x_1 + 4x_2 = 18$

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$$

$\boxed{Ax = B}$

Let $\boxed{A = LU}$

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{bmatrix}$$

$u_{11} = 2, u_{12} = 3, l_{21}u_{11} = 3$
$\Rightarrow l_{21} = \frac{3}{2}$

$l_{21}u_{21} + u_{22} = 4 \Rightarrow u_{22} = -\frac{1}{2}$

$L = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix}$

$\boxed{LUx = B}$, Let $Ux = y$.

$Ly = B$

$$\begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 18 \end{bmatrix}$$

Solving, $y_1 = 13, y_2 = -\frac{3}{2}$

Now $\boxed{Ux = y}$

$$\begin{bmatrix} 2 & 3 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 13 \\ -\frac{3}{2} \end{bmatrix}$$

Solving $x_1 = 2$ and $x_2 = 3$

The solutions are $(2, 3)$

FOR 3X3 MATRIX

Linear system of equation

$a_{11} + a_{12} + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

$\boxed{Ax = B}$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\quad \boxed{A = LU}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Let $LUx = B \longrightarrow ①$
$Ux = y \longrightarrow ②$

$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

FROM ① and ② $\boxed{Ly = B}$

Solving we get y_1, y_2, y_3

Substitute y in $\boxed{Ux = y}$

Solving we get, $\boxed{x_1, x_2, x_3}$.

PROBLEM

$x_1 + x_2 + x_3 = 1$

$3x_1 + x_2 - 3x_3 = 5$

$x_1 - 2x_2 - 5x_3 = 10$

$\boxed{Ax = B}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

$\boxed{A = LU}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Solving, $u_{22} = -2, u_{23} = -6, u_{33} = 3$
$l_{21} = 3, l_{31} = 1, l_{32} = 3/2$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 3/2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

$LUx = B, \quad Ux = y \Rightarrow Ly = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 3/2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

solving we get $y_1 = 1, y_2 = 2, y_3 = 6$

$Ux = y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

Solving $x_1 = 6, x_2 = -7, x_3 = 2$

The solutions are $(6, -7, 2)$

Types of Matrices

Matrix

Arrangement of m rows and n columns order: m×n

$A = [a_{ij}]_{m\times n}$

i (Elements in rows)
j (Elements in Columns)

Square Matrix

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ 8 & 7 & 6 \end{bmatrix}_{3\times 3}$

No. of rows = No. of columns

Rectangular Matrix

$\begin{bmatrix} 5 & 4 \\ -6 & 1 \\ 3 & 7 \end{bmatrix}_{3\times 2}$

Triangular Matrix

Upper Triangular Matrix

$a_{ij} = 0$ if $i > j$

Example:

$\begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Lower Triangular Matrix

$a_{ij} = 0$ if $i < j$

Example:

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

Row Matrix

Matrix with single row.

$[5 \ \ -2 \ \ 3]$

Column Matrix

Matrix with single column

$\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

Unit Matrix (or) Identity Matrix

All diagonal elements are equal to one

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

Orthogonal Matrix

$AA^T = A^TA = I$. If $|A| = 1$, then matrix A is proper.

Zero Matrix (or) Null matrix

All rows (or) columns are equal to zero.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

Diagonal Matrix

All the elements except main diagonal elements are zero

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$

Skew Symmetric Matrix

A matrix A is skew Symmetric if $A = -A^T$

$A = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & 5 \\ -2 & -5 & 0 \end{bmatrix}$; $-A^T = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & 5 \\ -2 & -5 & 0 \end{bmatrix}$

Symmetric Matrix

A matrix A is symmetric, if $A = A^T$

$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ $A^T = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Scalar matrix

Scalar matrix: Diagonal elements are equal. other elements are zero. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3}$

Operations on Matrices

Addition of Matrices

$A + B = [c_{ij}]$

Example:

$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 5 \end{bmatrix}_{2\times 3}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix}_{2\times 3}$

$A+B = \begin{bmatrix} -1+1 & 2+2 & 3+3 \\ 0+1 & 1+0 & 5-2 \end{bmatrix}$

$A+B = \begin{bmatrix} 0 & 4 & 6 \\ 1 & 1 & 3 \end{bmatrix}$

Subtraction of Matrices

$A + (-B) = [c_{ij}]$

$A - B = [c_{ij}]$

$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

$A - B = \begin{bmatrix} -1-1 & 2-2 \\ 0-1 & 1-0 \end{bmatrix}$

$A - B = \begin{bmatrix} -2 & 0 \\ -1 & 1 \end{bmatrix}$

Matrix Multiplications

Product of every row matrix with every column matrix

$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 6 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ -2 & 6 \end{bmatrix}$

$= \begin{bmatrix} 2-4-6 & 2-3+18 \\ 0+24-8 & 0+18+24 \end{bmatrix}$

$= \begin{bmatrix} -8 & 19 \\ 16 & 42 \end{bmatrix}$

Scalar Product

Each entry in the matrix is multiplied by the scalar.

$2\begin{bmatrix} 1 & 0 & -3 \\ 4 & 2 & 7 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 & -6 \\ 8 & 4 & 14 \end{bmatrix}$

Properties Of Eigen Values

Property – 1

$\lambda_1 + \lambda_2 + \dots + \lambda_n = tr(A)$

Example:

$$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

$\lambda_1 = 7, \; \lambda_2 = 3$

$$\boxed{\begin{aligned} \lambda_1 + \lambda_2 &= tr(A) \\ 7+3 &= 5+5 \\ 10 &= 10 \end{aligned}}$$

Property – 2

$det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

Example:

$$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

$\lambda_1 = 7, \; \lambda_2 = 3$

$$\boxed{\begin{aligned} \lambda_1 \cdot \lambda_2 &= det(A) \\ 7 \cdot 3 &= 25 - 4 \\ 21 &= 21 \end{aligned}}$$

Property – 3

$A^m = \lambda_1^n, \lambda_2^n, \dots \lambda_n^m$

Example

Find the Eigen Value of A^3

$$\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

$\lambda_1 = 2, \; \lambda_2 = 3, \; \lambda_3 = 2$

$A^3 = \lambda_1^3, \lambda_2^3, \lambda_3^3$

$A^3 = (2^3), (3^3), (2^3)$

$\boxed{A^3 = 8, 27, 8}$

Property – 4

EIGEN VALUES of A^{-1} are

$A^{-1} = \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots \frac{1}{\lambda_r}$

Example

If 2 & 3 are Eigen Values of A^{-1}

$$\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

$\lambda_1 = 2, \; \lambda_2 = 3, \; \lambda_3 = \gamma$

Property – 5

EIGEN VALUES of A and A^T are same

$A = A^T$

$|A - \lambda I| = |A^T - \lambda I|$

Example

$$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

$\boxed{A = A^T}$

$\lambda_1 = 7 \qquad \lambda_1 = 7$

$\lambda_2 = 3 \qquad \lambda_2 = 3$

Eigen Value = A

$$\begin{bmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix} = |A - \lambda I|$$

$= 25 - 10\lambda + \lambda^2 - 4$

$= \lambda^2 - 10\lambda + 21$

$\boxed{\lambda = 7, 3}$

$$\boxed{\begin{aligned} |A - \lambda I| &= |A^T - \lambda I| \\ \lambda = 7, 3 &= \lambda = 7, 3 \end{aligned}}$$

Property – 6

$\alpha_0 \lambda^2 + \alpha_1 \lambda + \alpha_2 =$

$\alpha_0 A^2 + \alpha_1 A + \alpha_2 I$

Example

$$\boxed{\begin{aligned} AX &= \lambda X \\ AX - KX &= \lambda X - KX \\ (A - KI)X &= (\lambda - K)X \end{aligned}}$$

$\lambda - K = A - KI$

$A^2 x = \lambda^2 x$

$\alpha_0 (A^2 x) = \alpha_0 (\lambda^2 x)$

$\alpha_1 (Ax) = \alpha_1 (\lambda x)$

$\alpha_0 (A^2 x) + \alpha_1 (Ax) =$

$\alpha_0 (\lambda^2 x) + \alpha_1 (\lambda x)$

$$\boxed{\begin{aligned} &\alpha_0 \lambda^2 + \alpha_1 \lambda + \alpha_2 = \\ &\alpha_0 A^2 + \alpha_1 A + \alpha_2 I \end{aligned}}$$

Property – 7

EIGEN VALUES of triangular matrices are precisely its primary diagonal elements

Example:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Soln:

Here diagonal elements are 1, 2, 0

So, EIGEN VALUES

$\boxed{\lambda = 1, 2, 0}$

Example

$$\begin{bmatrix} 7 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix}$$

Soln:

Here diagonal elements are 7, 1, 9

EIGEN VALUES $\lambda = 7, 1, 9$

SYMMETRIC MATRICES $(A=A^T)$ EIGEN VALUES AND EIGEN VECTORS

FOR 2X2 MATRIX

$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$

$\boxed{S_1 = a_{11}+a_{22}}$

$\boxed{S_2 = |A|}$

characteristic Equation

$\lambda^2 - S_1\lambda + S_2 = 0$

EIGEN VALUES C.E = 0

Find λ_1 and λ_2

Eigen vector

$(A-\lambda I)X_1 = 0$

$(A-\lambda I)X_2 = 0$

PROBLEM:

$A=\begin{bmatrix} 1 & 2 \\ 2 & 1\end{bmatrix}$

$S_1 = 2,\ S_2 = -3$

C.E $\lambda^2 - 2\lambda - 3 = 0$

EIGEN VALUES $\lambda_1 = -1, \lambda_2 = 3$

EIGEN VECTORS

case(i) $\lambda_1 = -1$

$\boxed{X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

case (ii) $\lambda_2 = 3$

$\boxed{X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

FOR 3X3 MATRIX

DISTINCT EIGEN VALUES

$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix}$

$S_1 = a_{11}+a_{22}+a_{33}$

S_2 = Sum of minor of Main diagonal

$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33}\end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}$

$\boxed{S_3 = |A|}$

EIGEN VALUES

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

C.E = 0 Find $\lambda_1, \lambda_2, \lambda_3$

EIGEN VECTORS

$[A-\lambda I]X_1 = 0$

$[A-\lambda I]X_2 = 0$

$[A-\lambda I]X_3 = 0$

PROBLEM

$A=\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5\end{bmatrix}$

$S_1 = 18,\ S_2 = 99,\ S_3 = 162$

characteristic equation

$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

EIGEN VALUES

$\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = 9$

Eigen vector $[A-\lambda I]X_1 = 0$

case (i) $\lambda = 3$

$4x_1 - 2x_2 + 0x_3 = 0$

$-2x_1 + 3x_2 - 2x_3 = 0$

$0x_1 - 2x_2 + 2x_3 = 0$

$\boxed{X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$

Case (ii) $\lambda = 6$

$x_1 - 2x_2 + 0x_3 = 0$

$-2x_1 + 0x_2 - 2x_3 = 0$

$0x_1 - 2x_2 - x_3 = 0$

$\boxed{X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}}$

case (iii) $\lambda = 9$

$-2x_1 - 2x_2 + 0x_3 = 0$

$-2x_1 - 3x_2 - 2x_3 = 0$

$0x_1 - 2x_2 - 4x_3 = 0$

$\boxed{X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}$

RESULT: EIGEN VALUES

$\lambda = 3, 6, 9$

EIGEN VECTORS

$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

FOR 3X3 MATRIX REPEATED EIGEN VALUES

PROBLEM:

$A=\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3\end{bmatrix}$

EIGEN VALUES $\lambda = 1, 4, 4$

$S_1 = 9, S_2 = 24, S_3 = 16$

characteristic equation

$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

Eigen values are Repeated

case (i) $\lambda = 1$

EIGEN VECTOR

$[A-\lambda I]X_1 = 0$

$2x_1 + x_2 + x_3 = 0 \rightarrow ①$

$x_1 + 2x_2 - x_3 = 0 \rightarrow ②$

$x_1 - x_2 + 2x_3 = 0 \rightarrow ③$

From ① & ②

$\boxed{x_1 = -1,\ x_2 = 1,\ x_3 = 1}$

$\boxed{X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$

case (ii) $\boxed{\lambda = 4}$

EIGEN VECTOR

$[A-\lambda I]X_2 = 0$

$-x_1 + x_2 + x_3 = 0$

$\Rightarrow x_1 - x_2,\ x_3 = 0 \rightarrow ④$

only one Equation

Put $x_3 = 0, x_2 = 1$

$\Rightarrow x_1 = 1$

$\boxed{X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$

$X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \perp$ to X_2

$a + b = 0 \Rightarrow b = -a$

$① \Rightarrow a - b - c = 0$

choosing $a = 1, b = -1, c = 2$

$\boxed{X_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}$

RESULT:

For Repeated Eigen values 1, 4, 4

Eigen vectors

$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$X_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Diagonalization of Matrix

A square matrix A is said to be **diagonalisable** if there exists a non-singular matrix P, such that $P^{-1}AP = D$, where D is a diagonal matrix. The matrix P is called a modal matrix of A.

Problem:- Repeated Eigen Values

Diagonalize $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 3 \end{pmatrix}$

Characteristic equation is

$S_1 = 9;$ $\quad \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_2 = 24;$ $\quad \Rightarrow \lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

$S_3 = 16;$ $\quad \lambda = 1, 4, 4.$

Eigen Vectors: $(A - \lambda I)X = 0$

$X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad X_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Normalized Vectors:

$$\begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}; \quad P^{-1} = \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$A.P = \begin{bmatrix} 1/\sqrt{3} & 4/\sqrt{2} & 4/\sqrt{6} \\ 1/\sqrt{3} & 4/\sqrt{2} & -\frac{4}{\sqrt{6}} \\ 1/\sqrt{3} & 0 & 8/\sqrt{6} \end{bmatrix}$$

$$D = P^{-1}AP$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Problem:- Non Repeated Eigen Values

Diagonalize $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 18;$ $\quad \Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$S_2 = 45;$ $\quad \lambda = 0, 3, 15.$

$S_3 = 0;$

Eigen Vectors: $(A - \lambda I)X = 0$

$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 2 \\ +1 \\ -2 \end{pmatrix}; \quad X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Normalized Vectors:

$$\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}; \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}; \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}; \quad P^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$A.P = \begin{bmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Problem:-

The eigen vectors of a 3x3 real symmetric matrix A corresponding to the eigen values 2, 3, 6 are $[1, 0, -1]^T$, $[1, 1, 1]^T$, $[-1, 2, -1]^T$ respectively. Find the matrix A.

Given A is symmetric and the eigen values are different.

So, the eigen vectors are orthogonal pairwise.

The normalized eigen vectors are

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}$$

Normalized modal matrix is

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

By Orthogonal reduction theorem,

$$P^T A P = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix},$$

since 2, 3, 6 are the eigen values.

$$P^T A P = D \Rightarrow P D P^T = A.$$

$$A = PDP^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} & \sqrt{3} & -\sqrt{6} \\ 0 & \sqrt{3} & 2\sqrt{6} \\ -\sqrt{2} & \sqrt{3} & -\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

which is the required matrix.

Remark: If $X_1 = (a_1, b_1, c_1)$ and $X_2 = (a_2, b_2, c_2)$ be two 3-dimensional vectors, they are orthogonal if their dot product is 0.

$\Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$; ie $X_1^T X_2 = 0 = X_2^T X_1$

CAYLEY HAMILTON THEOREM

Every Square matrix satisfies its own characteristic equation.

Verification of CHT

Properties: To find the inverse of a non-singular matrix A. To find higher integral power of A.

2x2 matrix

The characteristic equation is

$$\boxed{\lambda^2 - s_1\lambda + s_2 = 0} \quad -(1)$$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Find s_1, s_2

$s_1 = a_{11} + a_{22}$ Trace of A

$s_2 = |A|$

Replace λ by A in (1)

By CHT, $A^2 - s_1 A + s_2 I = 0$

Find $A^2 = A \cdot A$

Substitute, A^2, s_1, s_2

$A^2 - s_1 A + s_2 = 0$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Problem Find Eigen values and Eigen Vector of $\begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix}$.

Characteristic Equation is

$\lambda^2 - 6\lambda + 5 = 0$.

$A^2 = A \cdot A = \begin{pmatrix} 1 & 0 \\ 24 & 25 \end{pmatrix}$

$A^2 - 6\lambda + 5I = \begin{pmatrix} 1 & 0 \\ 24 & 25 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Hence Verified.

3x3 matrix

Characteristic equation

$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Find $s_1 = a_{11} + a_{22} + a_{33}$

$s_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$s_3 = |A|$

By CHT.

$A^3 - s_1 A^2 + s_2 A - s_3 I = 0$

Find: $A^2 = A.A$

$A^3 = A^2 . A$

Sub s_1, s_2, s_3, A^2, A^3 to

$A^3 - s_1 A^2 + s_2 A - s_3 I = 0$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence proved.

Problem: $A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

The characteristic equation is.

$\lambda^3 - 11A^2 + 38A - 40I = 0$

$A^2 = \begin{pmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{pmatrix}$

$A^3 = \begin{pmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{pmatrix}$

By CHT,

$A^3 - s_1 A^2 + s_2 A - s_3 I = 0$

$\begin{pmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{pmatrix} - 11\begin{pmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{pmatrix}$

$+ 38\begin{pmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

$- \begin{pmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Hence Verified

Finding A^{-1}

$A^3 - 11A^2 + 38A - 40I = 0$

$\div A \quad A^2 - 11A + 38I - 40A^{-1} = 0$

$40A^{-1} = A^2 - 11A + 38I$

$A^2 = \begin{pmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 7 & -7 & 11 \end{pmatrix}$

Substite A^2, A

$40A^{-1} = A^2 - 11A + 38I$

$40A^{-1} = \begin{pmatrix} 9 & 7 & 5 \\ -9 & 25 & -9 \\ 10 & 7 & 11 \end{pmatrix}$

$-11\begin{pmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} + \begin{pmatrix} 38 & 0 & 0 \\ 0 & 38 & 0 \\ 0 & 0 & 38 \end{pmatrix}$

$A^{-1} = \frac{1}{40}\begin{pmatrix} 14 & -4 & -6 \\ 2 & 8 & 2 \\ -4 & 4 & 16 \end{pmatrix}$

Finding A^4

$A^3 = A^2 \cdot A = \begin{pmatrix} 25 & 39 & 17 \\ -61 & 125 & -61 \\ 39 & -39 & 47 \end{pmatrix}$

$A^4 = 11A^3 - 38A^2 + 40A$

$= \begin{pmatrix} 53 & 203 & 37 \\ -369 & 625 & -369 \\ 203 & -203 & 219 \end{pmatrix}$

Problem: Verify CHT for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$.

characteristic equation is

$\lambda^2 - 4\lambda - 5 = 0 \quad (1)$

By CHT, $A^2 - 4A - 5I = 0$

$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$

$A^2 - 4A - 5I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Hence verified

To find:

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I \quad (2)$

By dividing equations (2) by (1) we get:

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$= (A^2 - 4A - 5I)$

$(A^3 - 2A - 3I) + A + 5I$

$= 0 + A + 5I = \underline{A + 5I}$

which is a linear Polynomial in A.

NOTE:

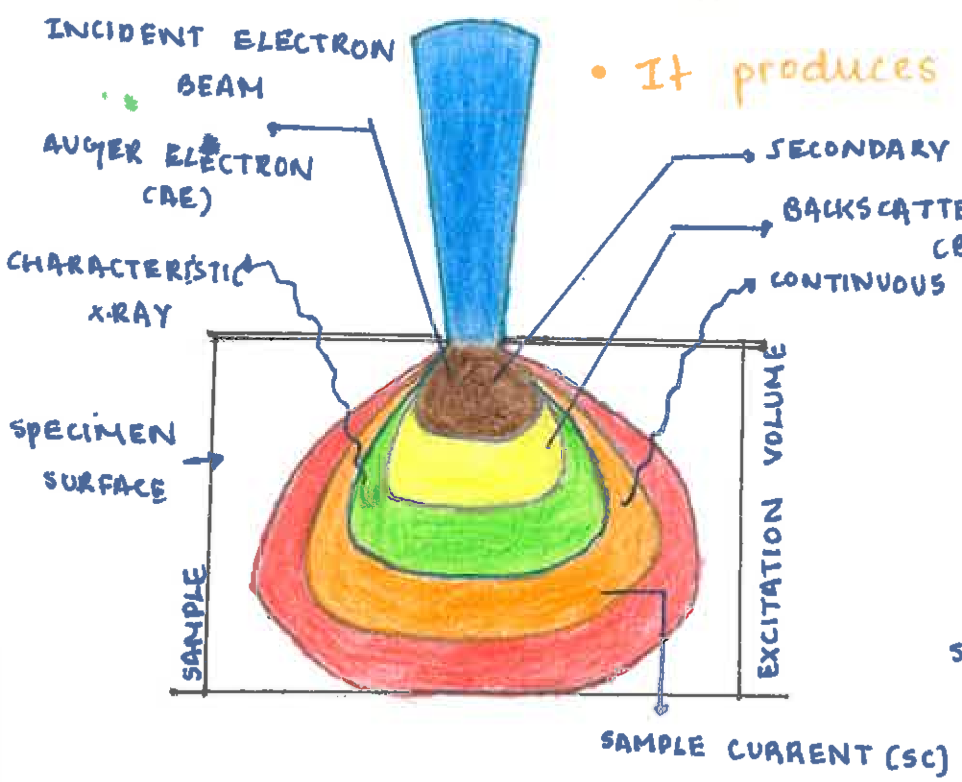
$\lambda^5 - 4\lambda^4 - 7\lambda^2 - \lambda - 10$

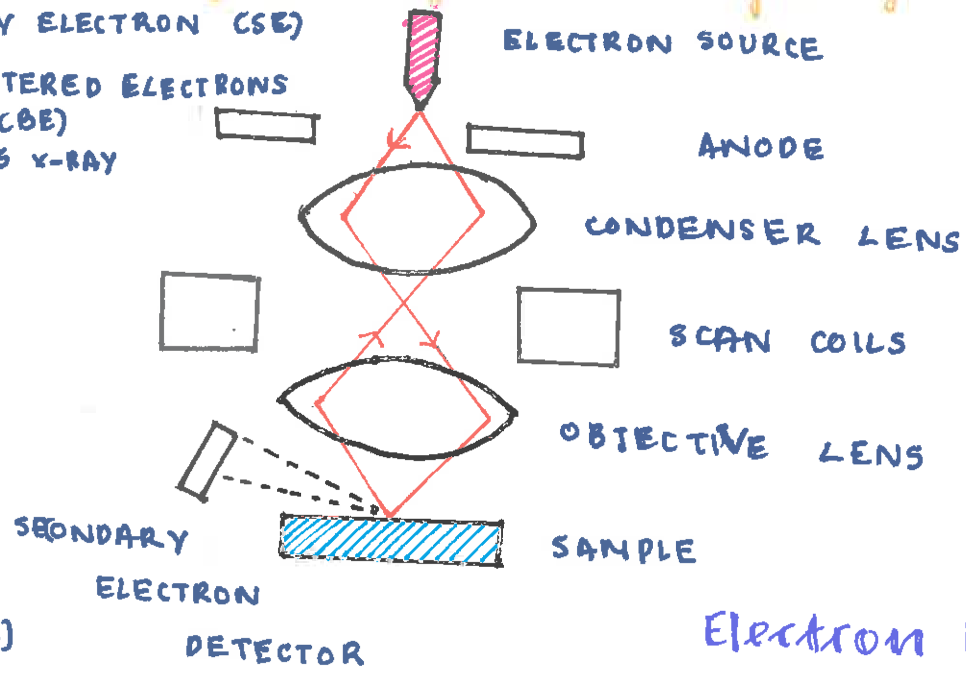
$= (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3)$

$+ \lambda + 5$.

Scanning Electron Microscope

- SEM, a analytical tool to analyse surface topography of nanostructured materials
- It produces a largely magnified image by using electron instead of light





Electron gun

- Thermionic emission (Tungsten)

Condenser lens

- Converge the electron path

Aperture

- Used to reduce and exclude extraneous electrons in the lenses

Electron Detector - X-ray signal - EDX

Advantages of SEM

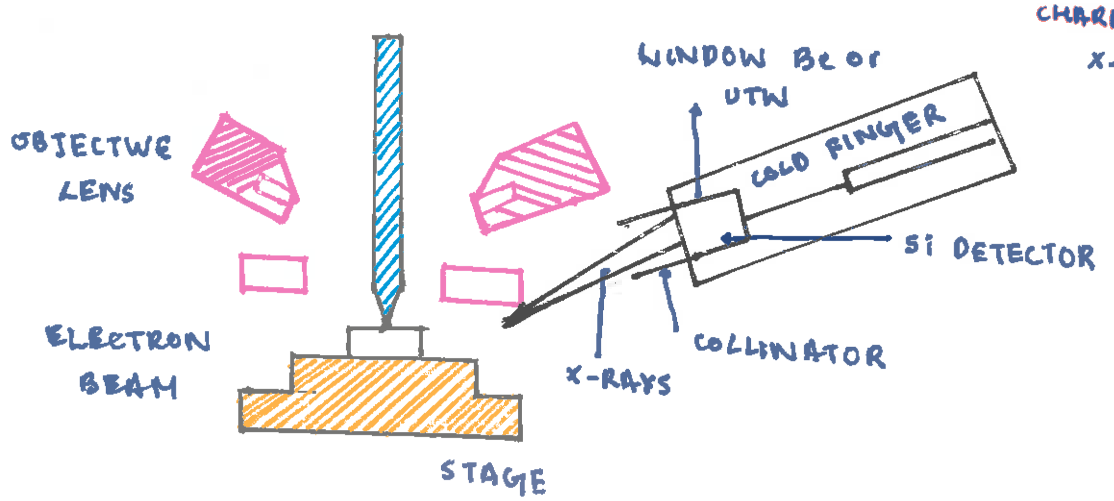
- It gives 3D and topographical image
- This instrument works very fast
- Most SEM sample require minimal preparation actions

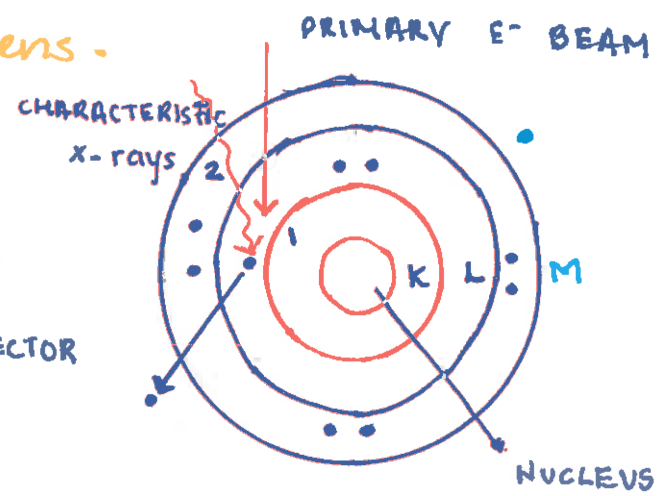
Disadvantages of SEM

- SEM's are expensive and large
- The preparation of samples results in artifacts.
- SEM's are limited to solid samples

Energy Dispersive X-Ray (XRD)

EDX is a technique of elemental analysis based on the generation of characteristic X-rays that reveals the presence of element present in the specimens.





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