Unit-2 : Electromagnetic Theory

Gauss Divergence Theorem (Green's theorem)

"The vector's outward flux through a closed surface is equal to the volume integral of the divergence over the area within the surface."

Or

"The surface integral of a vector field ($\vec{A}$) over a closed surface is equal to the volume integral of the divergence of the vector field ($\vec{A}$) over the volume (V) enclosed by the closed surface."

i.e

$$\boxed{\oint_S \vec{A}\cdot\vec{ds} = \int_V (\vec{\nabla}\cdot\vec{A})\, dV}$$

Stoke's Theorem

"The surface integral of the curl of a function over the surface bounded by a closed surface will be equal to the line integral of the particular vector function around it."

Or

"The line integral of a vector field ($\vec{A}$) around any closed ~~surface~~ curve is equal to the surface integral of the curl of vector $\vec{A}$ taken over any surface S of which the curve is a bounding edge.

i.e

$$\boxed{\oint_C \vec{A}\cdot\vec{dl} = \int_S (\vec{\nabla}\times\vec{A})\cdot\vec{ds}}$$

Displacement Current

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Concept: The concept of displacement current was first concise by Maxwell to explain the production of magnetic field in empty space.

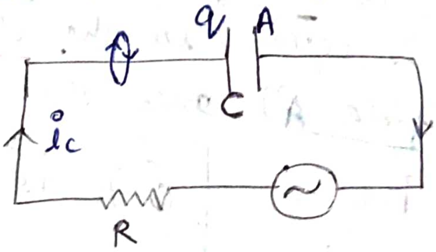
- We know that current carrying conductor or current element produces a magnetic field.
- Maxwell suggested that not only the current in the conductor produces the magnetic field but changing electric field in the region of space also produces a magnetic field. Hence it can be said that

changing electric field behaves as a current, which Maxwell named the displacement current.

Definition: Electric field changing w.r.t time, producing the same magnetic effects/field as that produce by conduction current is called displacement current.

It is given by $\boxed{i_d = \epsilon_0 \frac{d\phi_E}{dt}}$ where ϕ_E = electric flux

Proof: Let there be a parallel plate capacitor connected to an ac generator as shown in the figure.



The current in the circuit is $i_c = \frac{dq}{dt}$

We know that $|\vec{D}| = D = \sigma = \frac{q}{A}$

[where, $\vec{D} = \epsilon\vec{E}$, Electric Displacement vector
$\vec{D} = K\epsilon_0\vec{E}$
for free space $K = 1$ (dielectric) $\therefore \vec{D} = \epsilon_0\vec{E}$]

$\Rightarrow q = DA \qquad \therefore i_c = \frac{d\,DA}{dt} \quad \Rightarrow i_c = A\frac{dD}{dt}$

Maxwell suggested that the displacement current should be given as

$$i_d = A\frac{dD}{dt} \ (= i_c) \quad \text{or,} \quad i_d = A\frac{d(\epsilon_0 E)}{dt} = \epsilon_0 \frac{d(EA)}{dt}$$

$$\boxed{i_d = \epsilon_0 \frac{d\phi_E}{dt}}$$

* Displacement current density J_0 is given as

$$\vec{J_0} = \frac{i_d}{A} \Rightarrow \boxed{\vec{J_0} = \frac{d\vec{D}}{dt}}$$

or, $\boxed{\vec{J_0} = \epsilon_0 \frac{d\vec{E}}{dt}}$

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Equation of Continuity

"The total current flowing out of the system of volume must be equal to the rate at which the charge decreases within the volume."

It is given by $\boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0}$ where $\vec{J}$ = current density, ρ = volume charge density

* It is a consequence of the law of conservation of charges.

Let their be a charged region of volume 'V' bounded by a surface 'S'. Then the current flowing out of it is equal to the rate of decrease of charge inside it.

i.e $\quad i = -\dfrac{dq}{dt}$ —— (1)

If $\vec{J}$ is the current density, then

$$i = \int \vec{J}.\vec{ds} \quad \text{—— (2)}$$

And if 'ρ' is the volume charge density

$$q = \int \rho\, dV \quad \text{—— (3)}$$

placing the values from eq^n (2) & (3) in eq^n (1)

$$\Rightarrow \int_S \vec{J}.\vec{ds} = -\frac{\partial}{\partial t}\left[\int_V \rho\, dV\right]$$

$$\int_S \vec{J}.\vec{ds} = -\int_V \left(\frac{\partial \rho}{\partial t}\right) dV$$

Applying Gauss divergence theorem in LHS

$$\Rightarrow \int_V \vec{\nabla}.\vec{J}\, \vec{dV} = \int_V \left(-\frac{\partial \rho}{\partial t}\right) dV$$

Or, $\quad \vec{\nabla}.\vec{J} = -\dfrac{\partial \rho}{\partial t} \quad$ or, $\quad \boxed{\vec{\nabla}.\vec{J} + \dfrac{\partial \rho}{\partial t} = 0}$

* for static fields
  ρ = constant $\Rightarrow \dfrac{\partial \rho}{\partial t} = 0$

  $\therefore$ eq^n of Continuity gives $\boxed{\vec{\nabla}.\vec{J} = 0}$

  for static fields current flux through the closed surface is zero.
  Or, there is no 'source' or 'sink' present.

Maxwell's 1st. eq^n : Gauss Law in Electrostatic

"Net electric flux diverging out of a closed surface is $1/\epsilon_0$ times the total charge enclosed inside the surface."

i.e $\quad \phi_E = \int_S \vec{E}.\vec{ds} = \dfrac{1}{\epsilon_0}$ [Total charge enclosed]

$\Rightarrow \int \vec{E}.\vec{ds} = \dfrac{1}{\epsilon_0} \times q \quad$ { if q = total charge

$\Rightarrow \int_S (\epsilon_0 \vec{E}).\vec{ds} = q \quad \Rightarrow \boxed{\int_S \vec{D}.\vec{ds} = q}$ —— (1) $\quad \because \vec{D} = \epsilon_0 \vec{E}$

This is Maxwell's 1st eq in <u>integral form</u>

if 'ρ' is the volume charge density, then

$$q = \int_V \rho \, dV$$ then eq① becomes

$$\therefore \quad \int_S \vec{D} \cdot \vec{ds} = \int_V \rho \, dV$$

Applying Gauss divergence theorem in LHS, we get

$$\Rightarrow \int_V \vec{\nabla} \cdot \vec{D} \, dV = \int \rho \, dV \quad \Rightarrow \quad \boxed{\vec{\nabla} \cdot \vec{D} = \rho}$$

This Maxwell's 1st eqn in <u>Differential form</u>

* <u>Converting Diff. to integral form</u>

Maxwell 1st eqn in diff. form is $\vec{\nabla} \cdot \vec{D} = \rho$

Let us find volume, integration on both sides

i.e. $$\int \vec{\nabla} \cdot \vec{D} \, dV = \int \rho \, dV$$

Applying GDT in LHS

$$\int \vec{D} \cdot \vec{ds} = \int \rho \, dV$$

$$\Rightarrow \quad \boxed{\int \vec{D} \cdot \vec{ds} = q}$$

<u>Maxwell's 2nd eqn : Gauss law in Magnetism</u>

"<u>Net magnetic flux</u> <u>diverging out</u> of a <u>closed</u> surface is <u>always zero</u>."

i.e. $$\boxed{\int_S \vec{B} \cdot \vec{ds} = 0} = \phi_B$$

This is Maxwell's 2nd eqn in <u>integral form</u>.

Applying GDT in LHS, we get

$$\int_V (\vec{\nabla} \cdot \vec{B}) \, dV = 0$$

$$\Rightarrow \quad \boxed{\nabla \cdot B = 0}$$

This is Maxwell's 2nd eqn in Diff. form.

* <u>Convert Diff. form in integral form</u>

Maxwell's 2nd eqn in diff. form is

$$\vec{\nabla} \cdot \vec{B} = 0$$

Let us find Volume integration on both sides

i.e $\int(\vec{\nabla}\cdot\vec{B})\, dV = 0$

apply GDT in LHS

$$\boxed{\int_S \vec{B}\cdot\vec{ds} = 0}$$

Maxwell's 3rd eq : Universal Law of E.M Induction

" The net emf produced in a coil is directly proportional / equal to the rate of change in magnetic flux linked with it [Faraday's Law]"

"This emf opposes the very cause of its existance [Lenz's law]."

i.e $e = -\frac{d\phi_B}{dt}$ —— ① where, e = emf, ϕ_B = magnetic flux

Now, we know that emf is related with the electric field as under.

$$e = \int_C \vec{E}\cdot\vec{dl} \quad \text{—— ②}$$

& magnetic flux is given by

$$\phi_B = \int_S \vec{B}\cdot\vec{ds} \quad \text{—— ③}$$

placing the values from eqn's ② & ③ in eqn ① we get

$$\int_C \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\left[\int_S \vec{B}\cdot\vec{ds}\right]$$

$$\boxed{\int_C \vec{E}\cdot\vec{dl} = \int_S \left[\frac{-\partial \vec{B}}{\partial t}\right]\cdot\vec{ds}}$$

This is Maxwell's 3rd eqn in Integral form.

Now, let us apply Stokes theorem in LHS, we get

$$\Rightarrow \int_S (\vec{\nabla}\times\vec{E})\cdot\vec{ds} = \int_S \left[-\frac{\partial \vec{B}}{\partial t}\right]\cdot\vec{ds}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

This is Maxwell's 3rd eqn in differential form.

* Converting Diff. form into integral form

we have $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Let us find Surface int. on both sides

$$\int_S (\vec{\nabla}\times\vec{E})\cdot\vec{ds} = \int \left[\frac{-\partial \vec{B}}{\partial t}\right]\cdot\vec{ds}$$

applying stokes theorem in LHS

$$\boxed{\int \vec{E}\cdot\vec{dl} = \int\left[-\frac{\partial \vec{B}}{\partial t}\right]\cdot ds}$$

✲ Maxwell's 4th eqn : Modified Ampere's Law

Ampere's Law: "The line integral of magnetic field over the complete magnetic loop is equal to the net current threading through it".

i.e $\int_C \vec{B}\cdot\vec{dl} = \mu_0 i$

$$\Rightarrow \int_C \vec{H}\cdot\vec{dl} = i \quad \text{——} \quad (1) \qquad \left(\because \vec{H} = \frac{\vec{B}}{\mu_0}\right)$$

if $\vec{J}$ is the current density then $\quad i = \int_S \vec{J}\cdot\vec{ds}$

placing this eqn value in eqn (1), we get

$$\int_C \vec{H}\cdot\vec{dl} = \int_S \vec{J}\cdot\vec{ds}$$

Now applying stokes theorem in LHS,

$$\int_S (\vec{\nabla}\times\vec{H})\cdot\vec{ds} = \int_S \vec{J}\cdot\vec{ds}$$

$\Rightarrow \boxed{\vec{\nabla}\times\vec{H} = \vec{J}}$ This is Ampere's law in Differential form.

To investigate, let us find divergence on both sides.

i.e $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot\vec{J}$

In LHS we have scalar Triple product (STP) with two vector identical

$\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = 0$

$\Rightarrow \therefore \vec{\nabla}\cdot\vec{J} = 0$

or, $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \rho = \text{constant}$ (from eqn of continuity)

- This shows that Ampere's Law was applicable to static field only.

This is the reason maxwell modified Ampere's law to make it applicable for time varying fields as well.

Maxwell incorporated a time-dependent term $\vec{J_D}$ in Ampere's law as under.

i.e $\vec{\nabla}\times\vec{H} = \vec{J} + \vec{J_D}$

To investigate let us find divergence again

$$\Rightarrow \quad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\vec{J}_D$$

$$\Rightarrow \quad 0 = \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\vec{J}_D$$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{J}_D = -\vec{\nabla}\cdot\vec{J} = -\left(-\frac{\partial \rho}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) \qquad \left\{ \vec{\nabla}\cdot\vec{D} = \rho \right. \quad \text{Maxwell's 1st eq}^n$$

$$\vec{\nabla}\cdot\vec{J}_D = \vec{\nabla}\cdot\left(\frac{\partial \vec{D}}{\partial t}\right) \quad \Rightarrow \quad \vec{J}_D = \frac{\partial \vec{D}}{\partial t} \left\{ \begin{array}{l}\text{displacement current} \\ \text{densit}\end{array}\right.$$

$\therefore$ Modified Ampere's Law is

$$\boxed{\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$ Maxwell's 4th eq^n in differential form

* Converting Diff. form into integral form

we have $\quad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

let us find surface integral throughout

$$\int_S (\vec{\nabla}\times\vec{H})\cdot d\vec{s} = \int_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right]\cdot d\vec{s}$$

Applying Stokes theorem in LHS

$$\boxed{\oint_C \vec{H}\cdot d\vec{l} = \int_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right]\cdot d\vec{s}}$$

$$\left\{ \begin{array}{l} \oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s} + \int_S \vec{J}_D\cdot d\vec{s} \\ \\ \oint_C \vec{H}\cdot d\vec{l} = i_c + i_D \end{array}\right.$$

EM Waves in Free Space

Maxwell eq^{n}s are —

1. $\vec{\nabla}\cdot\vec{D} = \rho$
2. $\vec{\nabla}\cdot\vec{B} = 0$
3. $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$
4. $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

where, $\vec{D} = \epsilon\vec{E}$

$\vec{B} = \mu\vec{H}$

$\vec{J} = \sigma\vec{E}$

Free space is characterized by $\rho = 0$, $\sigma = 0$, $\epsilon = \epsilon_0$, $\mu = \mu_0$

$\therefore$ Maxwell's eqns for free space are –

$$\vec{\nabla}\cdot\vec{E} = 0 \quad — (1)$$

$$\vec{\nabla}\cdot\vec{H} = 0 \quad — (2)$$

$$\vec{\nabla}\times\vec{E} = -\mu_0 \frac{d\vec{H}}{dt} \quad — (3)$$

$$\vec{\nabla}\times\vec{H} = \epsilon_0 \frac{d\vec{E}}{dt} \quad — (4)$$

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* EM wave eqn

Let us find curl of eqn (3) i.e $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left[-\mu_0 \frac{\partial\vec{H}}{\partial t}\right]$

$$\Rightarrow (\vec{\nabla}\cdot\vec{E})\vec{\nabla} - (\vec{\nabla}\cdot\vec{\nabla})\vec{E} = -\mu_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

Using eqn (1) in LHS & eq (4) in RHS

$$-\nabla^2\vec{E} = -\mu_0 \frac{\partial}{\partial t}\left[\epsilon_0 \frac{\partial\vec{E}}{\partial t}\right]$$

$$-\nabla^2\vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2\vec{E}}{\partial t^2} \Rightarrow \boxed{\nabla^2\vec{E} - \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = 0} \quad — (5)$$

Similarly, solving eqn (4) as above, we get

$$\boxed{\nabla^2\vec{H} - \mu_0\epsilon_0\frac{\partial^2\vec{H}}{\partial t^2} = 0} \quad — (6)$$

Equation (5) & (6) are the E.M. Wave equations for free space

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* Wave speed

We know that the general wave eqn is given as :–

$$\boxed{\nabla^2\vec{u} - \frac{1}{v^2}\frac{\partial^2\vec{u}}{\partial t^2} = 0}$$

where u = wave function
v = wave speed

Comparing the general wave eqn with eqns (5) & (6)

$$\frac{1}{v^2} = \mu_0\epsilon_0 \Rightarrow v = \sqrt{\frac{1}{\mu_0\epsilon_0}} = \sqrt{\frac{1}{(4\pi\times10^{-7})\epsilon_0}}$$

$$v = \sqrt{10^7\times\frac{1}{4\pi\epsilon_0}} = \sqrt{10^7\times9\times10^9} = \sqrt{9\times10^{16}} = 3\times10^8 \text{ m/sec}$$

$$\Rightarrow \boxed{v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c}$$

Bohr's Quantization Rule

Of all possible circular orbits allowed by the classical theory, the electrons are permitted to circulate only in those orbits in which the angular momentum of an electron is an integral multiple of $\frac{h}{2\pi}$, where h is Plank's constant.

Therefore, for any permitted orbit,

$$\boldsymbol{L = mvr = n\frac{h}{2\pi}}$$

Where L, m, and v are the angular momentum, mass and the speed of the electron respectively. r is the radius of the permitted orbit and n is positive integer called principal quantum number.

The above equation is Bohr's famous quantum condition.

The de-Broglie wavelength λ associated with electron of mass m at velocity v, is: $\lambda = \frac{h}{p} = \frac{h}{mv}$

If the electron is confined to move in a circular orbit of radius r, then net path covered in one complete cycle is 2πr. According to Bohr' rule, only those orbits are allowed for which

$$2\pi r = n\lambda$$

$$2\pi r = n\frac{h}{mv}$$

$$mvr = n\frac{h}{2\pi}$$

$$\boxed{\boldsymbol{L = n\frac{h}{2\pi}}}$$

Hence, Angular momentum of electrons in an orbit in the atom is quantised.

This is a direct consequence of wave nature of the electron.

Maxwell's Equations

Name of the Law	Differential Form	Integral Form
1. Gauss Law (Electrostatics)	$\text{div}\,\vec{E} = \frac{\rho}{\varepsilon_0}$ or $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$	$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$ or $\oint \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\int \rho\, dV$
2. Gauss Law (Magnetism)	$\text{div}\,\vec{B} = 0$ or $\vec{\nabla}\cdot\vec{B} = 0$	$\oint \vec{B}\cdot d\vec{S} = 0$
3. Faraday's Law of EM Induc.	$\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ or $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial \phi_B}{\partial t}$
4. Ampere's Circuital Law	$\text{curl}\,\vec{H} = \vec{J} + \left(\frac{\partial \vec{D}}{\partial t}.\right)$ or $\vec{\nabla}\times\vec{H} = \vec{J} + \left(\frac{\partial \vec{D}}{\partial t}.\right)$	$\oint \vec{H}\cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{S}$

where

ρ is the charge density.

$\boxed{\vec{D} = \varepsilon_0 \vec{E},}$ electric displacement vector, ε_0 is the permittivity of the free space and $\vec{E}$ is the electric field strength.

$\boxed{\vec{B} = \mu_0 \vec{H}}$ where μ_0 is the magnetic permeability of free space and $\vec{H}$ is the magnetic field intensity.

$\boxed{\vec{J} = \sigma \vec{E}\,,}$ is the current density depending upon the conductivity σ and electric field $\vec{E}$.

Now for free space $\rho = 0\ ,\ \sigma = 0\ ,$
$permittivity = \varepsilon_o\ \&\ permeability = \mu_o$

Physical Significance of Maxwell's Equations

Maxwell's First Electromagnetic Equation

Because of time independence, *Maxwell's first electromagnetic equation* is a steady-state equation. It represents the Gauss' law in electrostatics which states that the electric flux through any closed hypothetical surface is equal to $1/\varepsilon_0$ times the total charge enclosed by the surface.

Maxwell's Second Electromagnetic Equation

Maxwell's second electromagnetic equation represents Gauss' law in magnetostatics. It states that the net magnetic flux through any closed surface is zero (i.e., the number of magnetic lines of flux entering any region is equal to the lines of flux leaving it). It also explains that no isolated magnetic pole exists.

Maxwell's Third Electromagnetic Equation

Maxwell's third electromagnetic equation represents Faraday's law in electromagnetic induction. It states that an electric field is induced in the form of close lines when magnetic flux (or lines of magnetic force) changes through an open surface. The line integral of induced electric field around a close path is equal to the negative rate of change of magnetic flux.

Maxwell's Fourth Electromagnetic Equation

Maxwell's fourth electromagnetic equation represents the modified form of Ampere's circuital law which states that a changing electric field produces a magnetic field and an electric field can also be produced by changing magnetic field. Therefore, *Maxwell's fourth electromagnetic equation* gives the new concept of generation of magnetic field by displacement current.

*Mode of Propagation (Transversal mode)

We have the wave eq^n as under –

$$\nabla^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\& \quad \nabla^2 \vec{H} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

The general solutions of each would be given as

$$\vec{E} = \vec{E_0}\, e^{i(\vec{K}.\vec{r} - \omega t)} \quad \text{——} \quad (7)$$

$$\vec{H} = \vec{H_0}\, e^{i(\vec{K}.\vec{r} - \omega t)} \quad \text{——} \quad (8)$$

where, $\vec{E_0} = E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k}$

$\vec{H_0} = H_{0x}\hat{i} + H_{0y}\hat{j} + H_{0z}\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{K} = K_x\hat{i} + K_y\hat{j} + K_z\hat{k}$

$= K\hat{n} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi\nu}{c}\hat{n} = \frac{\omega}{c}\hat{n}$

From eq^n (1), we have

$\vec{\nabla}.\vec{E} = 0$ (Maxwell's 1^{st} eq^n gives)

$$LHS = \vec{\nabla}.\vec{E}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . \left[\vec{E_0}\, e^{i(\vec{K}.\vec{r} - \omega t)}\right]$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . \left[(E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k}) . e^{i(\vec{K}.\vec{r} - \omega t)}\right]$$

$$= \frac{\partial}{\partial x}\left[E_{0x} . e^{i(\vec{K}.\vec{r} - \omega t)}\right] + \frac{\partial}{\partial y}\left[E_{0y} e^{i(\vec{K}.\vec{r} - \omega t)}\right] + \frac{\partial}{\partial z}\left[E_{0z} e^{i(\vec{K}.\vec{r} - \omega t)}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[E_{0x}\, e^{i(\vec{K}.\vec{r} - \omega t)}\right]$$

$$= \sum E_{0x}\, e^{i(\vec{K}.\vec{r} - \omega t)} \frac{\partial}{\partial x}\left[i(xK_x + yK_y + zK_z) - i\omega t\right]$$

$$= \sum E_{0x}\, e^{i(\vec{K}.\vec{r} - \omega t)} (iK_x)$$

$$= i(K_x E_{0x} + K_y E_{0y} + K_z E_{0z})\, e^{i(\vec{K}.\vec{r} - \omega t)}$$

$$= i(\vec{K}.\vec{E_0})\, e^{i(\vec{K}.\vec{r} - \omega t)}$$

$$= i\,\vec{K}.\vec{E}$$

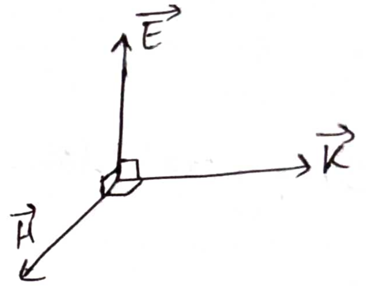
$$\Rightarrow \quad \vec{\nabla}.\vec{E} = i\,\vec{K}.\vec{E} = 0$$

$\Rightarrow \vec{K}\cdot\vec{E} = 0$ ——— (9) $\qquad \Rightarrow \vec{K} \perp \vec{E}$

Similarly, solving Maxwell's 2^{nd} eqn as above, we obtain

$$\vec{\nabla}\cdot\vec{H} = i\vec{K}\cdot\vec{H} = 0$$

$\Rightarrow \vec{K}\cdot\vec{H} = 0$ ——— (10) $\qquad \Rightarrow \vec{K} \perp \vec{H}$



eqn (9) & (10) proves that EM wave propagate Transversely in free space.

* Orthogonality of $\vec{E}$, $\vec{H}$ & $\vec{K}$

We have proved mathematically that $\vec{\nabla}\cdot\vec{E} = i\vec{K}\cdot\vec{E}$ & $\vec{\nabla}\cdot\vec{H} = i\vec{K}\cdot\vec{H}$

Hence we can also prove that $\vec{\nabla}\times\vec{E} = i\vec{K}\times\vec{E}$.

& $\vec{\nabla}\times\vec{H} = i\vec{K}\times\vec{H}$

Comparing this with Maxwell's 3rd & 4th eqn, we can write

$$\vec{\nabla}\times\vec{E} = i\vec{K}\times\vec{E} = -\mu_0\frac{\partial\vec{H}}{\partial t}$$

$$\Rightarrow i\vec{K}\times\vec{E} = -\mu_0\frac{\partial}{\partial t}\left[H_0\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}\right]$$

$$i\vec{K}\times\vec{E} = -\mu_0 H_0\, e^{i(\vec{K}\cdot\vec{r}-\omega t)}(-i\omega)$$

$$\cancel{i}\vec{K}\times\vec{E} = \cancel{i}\mu_0\omega\vec{H} \quad \Rightarrow \quad \vec{H} = \frac{1}{\mu_0\omega}(\vec{K}\times\vec{E}) \text{ ——— (11)}$$

Considering eqn (9), (10) & (11) together they show that

$$\boxed{\vec{K}\perp\vec{E}\perp\vec{H}}$$

* Wave Impedance

Nu.

(opposition offered by the medium to the wave propagation)

From eqn (11), we have

$$\vec{H} = \frac{1}{\mu_0\omega}(\vec{K}\times\vec{E}) = \frac{K}{\mu_0\omega}(\hat{n}\times\vec{E}) \qquad \{\because \vec{K} = K\hat{n}$$

$$= \frac{\cancel{\omega}/c}{\mu_0\cancel{\omega}}(\hat{n}\times\vec{E}) = \frac{1}{\mu_0 c}(\hat{n}\times\vec{E})$$

Considering magnitude only

i.e $H = \dfrac{1}{\mu_0 c}E$

$$\Rightarrow \quad \frac{E}{H} = \mu_0 c$$

$$= \mu_0 \times \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}} \text{ ohm}$$

$$\frac{E}{H} \simeq 376.6 \text{ Ohm}$$

$$\boxed{Z_0 = \frac{E}{H} = 376.7\,\Omega \simeq 377\,\Omega} \quad \text{—— (12)}$$

Z_0 = wave impedance

- Ratio of E & H is real & +ve which shows that $\vec{E}$ & $\vec{H}$ propagate in phase through free space.

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* Energy Density/Densities of $\vec{E}$ & $\vec{H}$

If U_E and U_M are the energy densities of Electric & Magnetic fields, then

$$U_E = \frac{1}{2}\epsilon_0 E^2 \qquad \& \qquad U_M = \frac{1}{2}\mu_0 H^2$$

$$\frac{U_E}{U_M} = \frac{\cancel{\frac{1}{2}}\epsilon_0 E^2}{\cancel{\frac{1}{2}}\mu_0 H^2} = \frac{\epsilon_0}{\mu_0}\left(\frac{E}{H}\right)^2 = \frac{\epsilon_0}{\mu_0} \times \frac{\mu_0}{\epsilon_0} = 1 \Rightarrow \boxed{U_E = U_M}$$

Then, $U_{AV} = U_E + U_M = \epsilon_0 E^2 = \mu_0 H^2$

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* Poynting Vector

"The net energy flowing per unit time per unit cross-sectional area in the form of E-M wave is called poynting vector."

$$\vec{S} = \vec{E} \times \vec{H} \qquad J/sec/m^2 \text{ or Watt}/m^2$$

$$\therefore \quad \vec{S} = \vec{E} \times \vec{H}$$

We know that

$$\vec{H} = \frac{1}{\mu_0 c}(\hat{n} \times \vec{E})$$

placing this in the above eqn

$$\vec{S} = \vec{E} \times \left[\frac{1}{\mu_0 c}(\hat{n} \times \vec{E})\right]$$

$$= \frac{1}{\mu_0 c}\left[\vec{E} \times (\hat{n} \times \vec{E})\right] = \frac{1}{\mu_0 c}\left[(\vec{E}\cdot\vec{E})\hat{n} - (\vec{E}\cdot\hat{n})\vec{E}\right] = \frac{1}{\mu_0 c}E^2\hat{n}$$

$$\boxed{\vec{S} = \frac{E^2}{\mu_0 c}\hat{n}} \Rightarrow S = \frac{E^2}{\mu_0 c}$$

Average over one complete cycle is

$$\langle \vec{S} \rangle = \left\langle \frac{E^2}{\mu_0 c}\hat{n} \right\rangle$$

$$= \frac{1}{\mu_0 c}\langle E^2 \rangle \hat{n} = \frac{1}{\mu_0 c} \times \frac{E_0^2}{2} \times \hat{n}$$

$$\boxed{\langle \vec{S} \rangle = \frac{E_{rms}^2}{\mu_0 c}\hat{n}}$$

$$\left\{ \begin{array}{l} \because E_0 = E_{rms} \times \sqrt{2} \\ E = E_{rms} = \frac{E_0}{\sqrt{2}} \end{array} \right.$$

EM Wave propagation in Dielectric Medium

Maxwell's eq^n in general form are

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

where

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{J} = \sigma \vec{E}$$

For dielectric medium $\rho = 0$, $\sigma = 0$, $\mu = \mu$, $\epsilon = \epsilon$

Maxwell's eq^ns are

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \text{--- (1)}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \quad \text{--- (2)}$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \text{--- (3)}$$

$$\vec{\nabla} \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{--- (4)}$$

EM wave eq^n

Let us find curl of eq^n (3)

i.e $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left[-\mu \frac{\partial \vec{H}}{\partial t}\right]$

$$\Rightarrow (\vec{\nabla} \cdot \vec{E})\vec{\nabla} - (\vec{\nabla} \cdot \vec{\nabla})\vec{E} = -\mu \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$$

Using eq^n (1) in LHS & eq^n (4) in RHS

$$\Rightarrow -\nabla^2 \vec{E} = -\mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \Rightarrow \boxed{\nabla^2 \vec{E} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \quad \text{--- (5)}$$

Similarly solving eqⁿ (4) as above

$$\boxed{\nabla^2 \vec{H} - \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0} \quad —— (6)$$

Eqⁿs (5) & (6) are the EM eqⁿs for Dielectric medium.

* **Wave Speed**

General wave eqⁿ is given as

$$\nabla^2 \vec{u} - \frac{1}{v^2}\frac{\partial^2 \vec{u}}{\partial t^2} = 0$$

where $\vec{u}$ = wave function
v = wave speed.

Comparing eqⁿs (5) & (6) with the above, we get

$$\frac{1}{v^2} = \mu\epsilon$$

$$\Rightarrow \quad v = \sqrt{\frac{1}{\mu\epsilon}} = \sqrt{\frac{1}{(\mu_r\mu_0)(\epsilon_r\epsilon_0)}} = \sqrt{\frac{1}{\mu_0\epsilon_0}} \times \sqrt{\frac{1}{\mu_r\epsilon_r}} \Rightarrow \boxed{v = \frac{c}{\sqrt{\mu_r\epsilon_r}}}$$

Refractive Index is given as

$$R.I = \frac{c}{v} = \frac{\cancel{c}}{\cancel{c}/\sqrt{\mu_r\epsilon_r}} \Rightarrow \boxed{RI = \sqrt{\mu_r\epsilon_r}}$$

For non-magnetic.

$$\mu_r = 1$$

$$\therefore \quad RI = \sqrt{\epsilon_r}$$

EM Wave Propagation in conducting Medium

Maxwell's eqⁿ in general form are

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho \\ \vec{\nabla}\cdot\vec{B} &= 0 \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \end{aligned}\right\}$$

where
$\vec{D} = \epsilon\vec{E}$
$\vec{B} = \mu\vec{H}$
$\vec{J} = \sigma\vec{E}$

For conducting medium $\rho = 0$, $\sigma \neq 0$, $\mu = \mu$, $\epsilon = \epsilon$

Maxwell's eqⁿs are

$$\vec{\nabla}\cdot\vec{E} = 0 \quad —— (1)$$

$$\vec{\nabla}\cdot\vec{H} = 0 \quad —— (2)$$

$$\vec{\nabla}\times\vec{E} = -\mu\frac{\partial \vec{H}}{\partial t} \quad —— (3)$$

$$\vec{\nabla}\times\vec{H} = \sigma\vec{E} + \epsilon\frac{\partial \vec{E}}{\partial t} \quad —— (4)$$

* EM Wave Equations

Let us find curl of eqⁿ (3)

i.e $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left[-\mu \frac{\partial \vec{H}}{\partial t}\right]$

$\Rightarrow (\vec{\nabla}.\vec{E})\vec{\nabla} - (\vec{\nabla}.\vec{\nabla})\vec{E} = -\mu \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H})$

Using eqⁿ (1) in LHS & eqⁿ (4) in RHS

$\Rightarrow -\nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$

$\Rightarrow \boxed{\nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t} = 0}$ — (5)

Similarly, solving eqⁿ (4) as above

$\Rightarrow \boxed{\nabla^2 \vec{H} - \mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0}$ — (6)

eqⁿ (5) & (6) are the EM wave eqⁿ for conducting medium.

where, $\mu\sigma$ = Diffusivity of medium (μ → permeability of medium) (σ → electrical conductivity)

& $\sqrt{\frac{1}{\mu\epsilon}}$ = Wave speed

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* (8) Nu'O Skin Depth / Penetration Depth (δ)

"It is the depth/distance travelled by the EM wave inside a medium where the amplitude of electric field reduces to 1/e times its original value."

* For good conductors

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \sqrt{\frac{1}{\pi f \mu\sigma}}$$

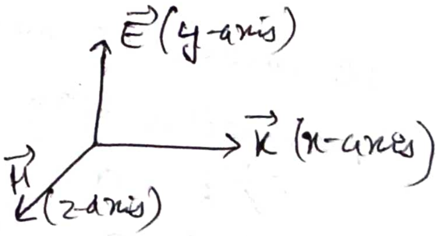
* For bad conductors

$$\delta = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$$

Derivation :- EM wave eqⁿs in conducting medium are –

$$\left.\begin{aligned} \nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} &= 0 \\ \& \quad \nabla^2 \vec{H} - \mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} &= 0 \end{aligned}\right\} \quad (1)$$

If we suppose that $\vec{K}$ is along x-axis,
$\vec{E}$ is along y-axis,
$\vec{H}$ is along z-axis.



Then eqⁿ (1) can be written as

$$\frac{\partial^2 E_y}{\partial x^2} - \mu\sigma \frac{\partial \vec{E_y}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{E_y}}{\partial t^2} = 0 \quad \text{—— (2)}$$

$$\frac{\partial^2 \vec{H_z}}{\partial x^2} - \mu\sigma \frac{\partial \vec{H_z}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{H_z}}{\partial t^2} = 0 \quad \text{—— (3)}$$

The general solution of eqⁿs (2) & (3) are

$$\vec{E_y} = \vec{E_{oy}}\, e^{i(xK_x - \omega t)} \quad \text{—— (4)}$$

$$\vec{H_z} = \vec{H_{oz}}\, e^{i(xK_x - \omega t)} \quad \text{—— (5)}$$

If we place the value from eqⁿ (4) in eqⁿ (2) & solve, we obtain

$$K_x^2 = \mu\epsilon\omega^2 + i\mu\omega\sigma \quad \text{—— (6)}$$

This shows that K_x is a complex quantity and has a form

$$\boxed{K_x = \alpha + i\beta} \quad \text{—— (7)}$$

where,

$$\alpha = \omega \times \sqrt{\frac{\mu\epsilon}{2}\left[\left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)^{1/2} + 1\right]}$$

$$\beta = \omega \times \sqrt{\frac{\mu\epsilon}{2}\left[\left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)^{1/2} - 1\right]}$$

$x \rightarrow$ distance inside the medium.

Now placing the values from eqⁿ (7) in eqⁿ (4) we can write it as

$$\vec{E_y} = \vec{E_{oy}}\, e^{i[x(\alpha + i\beta) - \omega t]} = \vec{E_{oy}}\, e^{[-\beta x + i(\alpha x - \omega t)]}$$

$$= \vec{E_{oy}}\, e^{(i\alpha x + i^2\beta x - i\omega t)}$$

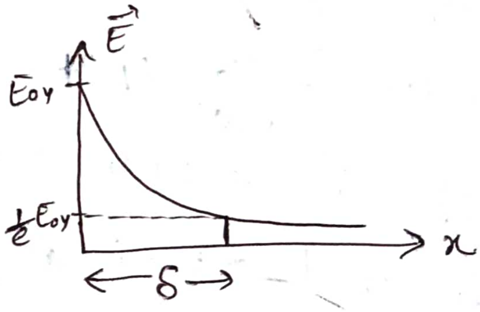
$$= \vec{E_{oy}}$$

$$\vec{E_y} = \left[\vec{E_{oy}}\, e^{-\beta x}\right] e^{i(\alpha x - \omega t)}$$

This shows that the amplitude of electric field reduces exponentially with distance along x-axis.

∴ By definition, at Skin Depth

i.e at $x = \delta$



Amplitude $= \frac{1}{e}\vec{E_{oy}}$

$$\Rightarrow \cancel{\vec{E_{oy}}}\, e^{-\beta\delta} = e^{-1}\,\cancel{\vec{E_{oy}}} \Rightarrow \beta\delta = 1$$

$$\boxed{\delta = \frac{1}{\beta}}$$

Case I: For Good Conductors

$$\sigma >> \omega\epsilon$$

$$\therefore \beta = \omega \times \sqrt{\frac{\mu\epsilon}{2} \times \frac{\sigma}{\omega\epsilon}} = \sqrt{\omega^{\cancel{2}} \times \frac{\mu\cancel{\epsilon}}{2} \times \frac{\sigma}{\cancel{\omega}\cancel{\epsilon}}} = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\therefore \text{skin depth, } \boxed{\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\omega\mu\sigma}}}$$

Case-II : for bad conductors

$\sigma << \omega\epsilon$

$$\beta = \omega \times \sqrt{\frac{\mu\epsilon}{2}\left[\cancel{1}+\frac{\sigma^2}{2\omega^2\epsilon^2}-\cancel{1}\right]}$$ {using Binomial expansion

$$= \sqrt{\omega^2\frac{\mu\epsilon}{2}\times\frac{\sigma^2}{2\omega^2\epsilon^2}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$$

$\therefore$ skin depth, $\boxed{\delta = \frac{1}{\beta} = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}}$

Nu. Poynting Theorem (Work-Energy Theorem)

" Work-done on the charges by the electromagnetic forces is equal to the decrease in energy stored in the fields and less than the energy that follows out through the surface."

i.e $$\boxed{\int_V (\vec{E}\cdot\vec{J})\,dV = -\int_V\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2+\frac{1}{2}\epsilon E^2\right)\right]dV - \oint_S (\vec{E}\times\vec{H})\cdot d\vec{S}}$$

This is mathematical statement of Poynting Theorem.

Significance of the terms —

1. The term $\boxed{\int_V(\vec{E}\cdot\vec{J})\,dV}$ represents the workdone per unit time on the charge by electromagnetic field.
2. The term $\boxed{-\int_V\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2+\frac{1}{2}\epsilon E^2\right)\right]dV}$ represents the rate of decrease of stored energy in the electric & magnetic fields in the volume V.
3. The term $\boxed{-\oint(\vec{E}\times\vec{H})\cdot d\vec{s}}$ represents the rate of flow of energy through the surface S enclosing the volume V.

Proof:- We know that Maxwell's 3rd & 4th eqn are —

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{—(1)} \quad \& \quad \vec{\nabla}\times\vec{H} = \vec{J}+\frac{\partial\vec{D}}{\partial t} \quad \text{—(2)}$$

Let us find scalar product of $\vec{H}$ with eqn (1) & $\vec{E}$ with eqn (2).

i.e $$\vec{H}\cdot(\vec{\nabla}\times\vec{E}) = \vec{H}\cdot\left(-\frac{\partial\vec{H}}{\partial t}\mu\right) \quad \text{—(3)}$$

& $$\vec{E}\cdot(\vec{\nabla}\times\vec{H}) = \vec{E}\cdot\vec{J} + \vec{E}\cdot\left(\epsilon\frac{\partial\vec{E}}{\partial t}\right) \quad \text{—(4)}$$

Subtracting eqn (4) from eqn (3)

$$\vec{H}.(\vec{\nabla}\times\vec{E}) - \vec{E}.(\vec{\nabla}\times\vec{H}) = -\mu\left(\vec{H}.\frac{\partial\vec{H}}{\partial t}\right) - \epsilon\left[\vec{E}.\frac{\partial\vec{E}}{\partial t}\right] - \vec{E}.\vec{J}$$

applying the property of STP in LHS, we get

$$\vec{\nabla}.(\vec{E}\times\vec{H}) = -\vec{E}.\vec{J} - \mu\left(\vec{H}.\frac{\partial\vec{H}}{\partial t}\right) - \epsilon\left(\vec{E}.\frac{\partial\vec{E}}{\partial t}\right) \quad \text{—— (5)}$$

If we evaluate as under –

$$\frac{\partial(E^2)}{\partial t} = \frac{\partial}{\partial t}(\vec{E}.\vec{E}) = \vec{E}.\frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{E}}{\partial t}.\vec{E} = 2\left(\vec{E}.\frac{\partial\vec{E}}{\partial t}\right)$$

$$\Rightarrow \quad \vec{E}.\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\frac{\partial(E^2)}{\partial t}$$

Similarly, $\vec{H}.\frac{\partial\vec{H}}{\partial t} = \frac{1}{2}\frac{\partial(H^2)}{\partial t}$

placing these values in eqn (5)

$$\nabla.(\vec{E}\times\vec{H}) = -\vec{E}.\vec{J} - \frac{\mu}{2}\frac{\partial(H^2)}{\partial t} - \frac{\epsilon}{2}\frac{\partial(E^2)}{\partial t}$$

$$= -\vec{E}.\vec{J} - \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right]$$

Let us find volume integration on both sides

$$\int_V [\vec{\nabla}.(\vec{E}\times\vec{H})].dV = -\int_V(\vec{E}.\vec{J})\,dV - \int_V\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right)\right]dV$$

applying GDT in LHS, we get

$$\int_S(\vec{E}\times\vec{H}).d\vec{s} = -\int_V(\vec{E}.\vec{J})\,dV - \int_V\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right)\right]dV$$

Hence proved

E & H = strength, Average value, Intensity, RMS value

E_0 & H_0 = Peak value, Maximum value, Amplitude