Stochastic Calculus

Kas Wei

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Chapter 1

Probabilities

1.1 Event Sets

<u>DEFINITION</u>. A collection (or call it a set) $\mathcal{F}$ of subsets of Ω is called a σ-algebra if it satisfies:

1. contains the empty set: $\emptyset \in \mathcal{F}$;
2. is closed under **countable** unions: $A_1, A_2, ..., \in \mathcal{F} \implies \cup_i A_i \in \mathcal{F}$;
3. is closed under complements: $A \in \mathcal{F} \implies A^C \in \mathcal{F}$;

It is trivial to know:

$$\begin{aligned} A_1, A_2, ... \in \mathcal{F} &\implies \cup_i A_i^C \in \mathcal{F} \\ &\implies (\cup_i A_i^C)^C \in \mathcal{F} \\ &\implies \cap_i A_i \in \mathcal{F} \end{aligned}$$

<u>TRIVAL σ-ALGEBRA</u>. $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is a σ-algebra.
<u>COLLECTION OF ALL SUBSETS</u>. $\mathcal{F} = 2^\Omega$ is a σ-algebra.
<u>DEFINITION</u>. Let $\mathcal{G}$ be a collection of subsets of Ω. The σ-algebra generated by $\mathcal{G}$: $\sigma(\mathcal{G})$ is the **smallest** σ-algebra that contains $\mathcal{G}$.
<u>DEFINITION</u>. A pair is called a *measure space* $(\Omega, \mathcal{F})$ if Ω is the sample space and $\mathcal{F}$ is a σ-algebra of subsets.

1.2 Probability

<u>DEFINITION</u>. A **function** $\mathbb{P}$ defined on $(\Omega, \mathcal{F})$: $\mathbb{P} : \mathcal{F} \to [0, 1]$ is called a *probability measure* if:

1. $\mathbb{P}(\Omega) = 1$

2. Only for *countable* unions: if $A_i \cap A_j = \emptyset$ for $i \neq j \implies \mathbb{P}(\cup_i A_i) = \sum_i \mathbb{P}(A_i)$

The countable sample space is easy to handle with, take $\mathcal{F} = 2^\Omega$ and some $\mathbb{P}$ to assign each event to [0, 1]. But the uncountable space could be more delicate to solve.

1.3 Infinite Spaces

1.3.1 Uniform Lebesgue Measure on (0, 1)

Define a *Lebesgue measure* μ, then we can determine the probability that ω falls within an **open interval**:

$$\mathbb{P}(\{\omega : \omega \in (a, b)\}) = \mu((a, b)) := b - a, 0 < a \leq b < 1$$

In this case we notice that $2^\Omega = 2^{(0,1)}$ is not a σ-algebra of (0, 1). Then we introduce the *Borel σ-algebra* on (0, 1) to make an appropriate sample space for our experiment:

$$\mathcal{B}((0, 1)) := \sigma(\mathcal{O}) \text{ where } \mathcal{O} = \{A \subseteq (0, 1) : A = (a, b), 0 < a \leq b < 1\}$$

This note will not discuss much on the *Borel set*, it is about creating some subsets of open sets in Ω.

1.3.2 Infinite Sequence of Coin Tosses

Let $\omega = \omega_1\omega_2...\omega_i...\omega_n$ where $\omega_i \in \{H, T\}$. If I have known ω_1, ω_2, I can tell you if ω belongs to each of the sets in $\mathcal{F}_2$: all possible cases for two tossing. When n becomes very large, then we have:

$$\mathcal{F} = \sigma(\mathcal{F}_\infty),\ \mathcal{F}_\infty = \cup_\infty \mathcal{F}_n$$

But things (or call them sets) like "sequences for which x percent of coin tosses are heads" are not in $\mathcal{F}_\infty$, they are actually in $\mathcal{F}$.
Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, then if a set $A \in \mathcal{F}$ s.t. $\mathbb{P}(A) = 1$, then the event A occurs $\mathbb{P}$ **almost surely** (i.e. $\mathbb{P}$-a.s.).

Chapter 2

Information and Conditioning

2.1 Information and σ-Algebras

<u>Definition</u> 2.1.1. Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is a