Distributions

- Discrete Theoretical Distributions
  - (i) Uniform
- Continuous Theoretical Distributions

Theoretical Probability Distributions →

There are two types of theoretical probability distributions:

(i) Discrete theoretical Distributions

(ii) Continuous theoretical Distributions.

- **Discrete Probability Distribution →**

The distribution which arises out of a discrete random variable is called discrete probability distribution.

- **Continuous Probability Distribution →**

The distribution which arises out of a continuous random variable is called continuous probability distribution.

Discrete

Distribution function →

The distribution f^n $F(x)$ of the discrete random variable X, is defined as

$$F(X) = P(X < x) = \sum_{i=1}^{n} p(x_i)$$

where $x_1 \le x, x_2 \le x, \ldots, x_n \le x$

Note → (i) The mean value (μ) of the probability distribution of the variable X, is known as expectation of X, $\mu = E(X) = \sum x_i p(x_i)$

(ii) Mean deviation about mean $= \sum |x_i - \mu| \, p(x_i)$

(iii) $\text{Variance}(X) = \sigma^2 = \sum (x_i - \mu)^2 p(x_i)$

(iv) r^{th} moment about the mean μ, is defined as

$$\mu_r = \sum (X-\mu)^r p(x_i)$$

Ques: Find the probability distribution of the number of green balls drawn when three balls are drawn one by one, without replacement, from a bag containing three green and five white balls.

Sol^n: Let X be the random variable which represents the no. of green balls drawn when three balls are drawn without replacement.

$\therefore$ $P(X=0) = P(\text{no green ball is drawn})$

$= P(W, W, W) = \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} = \frac{5}{28}$

$P(X=1) = P(\text{one green ball is drawn})$

$= P(G, W, W) + P(W, G, W) + P(W, W, G)$

$= \frac{3}{8} \cdot \frac{5}{7} \cdot \frac{4}{6} + \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} + \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} = \frac{15}{28}$

$P(X=2) = P(\text{two green balls are drawn})$

$= \frac{3}{8} \cdot \frac{2}{7} \cdot \frac{5}{6} + \frac{3}{8} \cdot \frac{5}{7} \cdot \frac{2}{6} + \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} = \frac{15}{56}$

$P(X=3) = P(\text{three green balls are drawn})$

$= P(G, G, G) = \frac{3}{8} \cdot \frac{2}{7} \cdot \frac{1}{6} = \frac{1}{56}$

So, probability distribution is

X	0	1	2	3
P(X)	5/28	15/28	15/56	1/56

* Binomial Probability Distribution:

Consider a sequence of 'n' Bernoulli trials such that

(i) trials are independent

(ii) each trial results in only two possible outcomes, labeled as success and failure

(iii) probability of success in each trial remains constant.

→ Probability Mass function (P.M.F.) →

If in each trial, all trials being independent, the probability of occurrence of an event, called "success", is 'p' and of non-occurrence of the event, called "failure" is 'q' $(=1-p)$, then the probability of r successes is

$$P(r) = {}^nC_r \, p^r q^{n-r}$$

→ Constants of Binomial Distribution →

(i) First Moment or Mean →

$$\mu = \sum_{r=0}^{n} x_r \, p(x_r)$$

where $\sum_{x=0}^{n} p(x) = 1$

$$\mu = \sum_{0}^{n} x \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{0}^{n} x \cdot \frac{n!}{x! \, (n-x)!} p^x q^{n-x}$$

$$= \sum_{1}^{n} \cancel{x} \, \frac{n \, (n-1)!}{\cancel{x} \, (x-1)! \, (n-x)!} \, p \cdot p^{x-1} q^{n-x}$$

$$= np \sum_{1}^{n} \frac{(n-1)!}{(x-1)! \, (n-x)!} p^{x-1} q^{n-x}$$

$$= np \times 1 = np$$ [$\because$ Sum of all probabilities of the distribution $= 1$]

(ii) Second Moment or Variance (σ^2):-

$$\mu_2 = \sigma^2 = \sum (x_i - \mu)^2 p(x_i)$$

$$= \sum_{0}^{n} (x^2 + \mu^2 - 2x\mu) \, p(x)$$

$$= \sum_{0}^{n} x^2 p(x) + \mu^2 \sum_{0}^{n} p(x) - 2\mu \sum_{0}^{n} x \, p(x)$$

$$= \sum_{0}^{n} x^2 \, {}^nC_x \, p^x q^{n-x} + (np)^2 - 2np \, (np)$$

[$\because \sum_{0}^{n} p(x) = 1$

and $\sum_{0}^{n} x \, p(x) = np$]

$$\mu_2 = \sigma^2 = \sum^{n} x^2 \, {}^{n}C_x \, p^x q^{n-x} - (np)^2$$

$$= \sum_{0}^{n} x^2 \frac{n!}{x!\,(n-x)!} p^x q^{n-x} - (np)^2$$

$$= \sum_{1}^{n} \cancel{x^2} \frac{n\,(n-1)!}{\cancel{x}\,(x-1)!\,(n-x)!} p\,p^{x-1} q^{n-x} - (np)^2$$

$$= np \sum_{1}^{n} \frac{x\,(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} q^{n-x} - (np)^2$$

$$= np \sum_{1}^{n} \frac{(x-1+1)\,(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} q^{n-x} - (np)^2$$

$$= (np) \sum_{1}^{n} \frac{(x-1)\,(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} q^{n-x} + (np) \sum_{1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} q^{n-x} - (np)^2$$

$$= (np) \sum_{2}^{n} \frac{\cancel{(x-1)}\,(n-1)\,(n-2)!}{\cancel{(x-1)}\,(x-2)!\,(n-x)!} p^{x-2}\cdot p\, q^{n-x}$$

$$\cancel{+(np)}$$

$$+ (np) \sum_{1}^{n-1} {}^{n-1}C_{x-1}\, p^{x-1} q^{n-x} - (np)^2$$

$$= np^2(n-1) \underbrace{\sum_{2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{n-x}}_{=1} + (np) \underbrace{\sum_{1}^{n-1} {}^{n-1}C_{x-1}\, p^{x-1} q^{n-x}}_{=1} - (np)^2$$

$= np^2(n-1) + np - n^2p^2$

$= n^2\cancel{p^2} - np^2 + np - \cancel{n^2p^2}$

$= \cancel{np(p-1)}\ np(1-p)$

$= npq \qquad [\because q = \cancel{p-1}\ 1-p]$

(iii) Same as Third Moment $\mu_3 = npq(q-p)$

(iv) Fourth Moment $\mu_4 = npq\,[1+3(n-2)pq]$

(v) Measure of skewness (γ_1) →

Since, $\gamma_1 = \sqrt{\beta_1}$

where $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{n^2p^2q^2(q-p)^2}{n^3p^3q^3}$

$= \dfrac{(q-p)^2}{npq}$

(vi) Kurtosis (β_2) →

$\text{Kurtosis} = \beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{npq\,[1+3(n-2)pq]}{n^2p^2q^2}$

$= \dfrac{npq}{n^2p^2q^2} + \dfrac{3np^2q^2(n-2)}{n^2p^2q^2}$

$$= \frac{1}{npq} + \frac{3n^2\cancel{p^2}q^2}{n^2\cancel{p^2}q^2} - \frac{6np^2q^2}{n^2p^2q^2}$$

$$= 3 + \frac{1}{npq} - \frac{6pq}{npq}$$

$$\boxed{\beta_2 = 3 + \frac{(1-6pq)}{npq}}$$

Ques 10 A coin is tossed six times. Calculate the probability of obtaining four or more heads

Solⁿ Here $n=6$, $p=\frac{1}{2}$, $q=\frac{1}{2}$

the probability of getting 4 heads $= P(4) = {}^6C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2$

$= 0.234$

the probability of getting 5 heads $= P(5) = {}^6C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^1 = 0.095$

and the probability of getting 6 heads $= P(6)$

$= {}^6C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^0 = 0.016$

∴ Probability of getting 4 or more heads

$= P(4)+P(5)+P(6) = 0.345$

Que - If the sum of the mean and variance of a Binomial distribution of 5 trials is $\frac{9}{5}$, find $P(X \geq 1)$

Sol^n Since, mean $= np$

variance $= npq = np(1-p)$ $\quad [\because q = 1-p]$

given no. of trials, $n = 5$

So,
$$np + npq = \frac{9}{5}$$

$$5p + 5p(1-p) = \frac{9}{5}$$

$$25p^2 - 50p + 9 = 0$$

$$\Rightarrow \boxed{p = \frac{1}{5}} \quad \text{then} \quad q = 1 - \frac{1}{5} = \frac{4}{5}$$

Now $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^5C_0 \, p^0 \, q^{5-0}$$

$$= 1 - \left(\frac{4}{5}\right)^5 = 1 - (0.8)^5$$

$$= 0.6724.$$

Ques:- A die is tossed thrice getting '5' or '6' on the die in a toss is taken as success. Find the mean and variance of no. of success

Solⁿ Given, $n = 3$

Probability of success (p) = probability of getting '5' or '6'

$$= \frac{2}{6} = \frac{1}{3}$$

So, Probability of failure $(q) = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

Now, Mean $= np = 3 \times \frac{1}{3} = 1$

variance $= npq = \cancel{3} \times \frac{1}{\cancel{3}} \times \frac{2}{3} = \frac{2}{3}$.

* **Poisson's Distribution** →

This distribution is the limiting form of the binomial distribution with constant mean and follows following conditions:

(i) n, the no. of trials are indefinitely large, i.e, $n \to \infty$

(ii) p, the probability of success for each trial is indefinitely small, i.e, $p \to 0$

(iii) the mean $np = \lambda$ (constant and finite)

Thus, $p = \frac{\lambda}{n}$, $q = 1 - \frac{\lambda}{n}$, where λ is a positive real no. — (I)

→ **Probability Mass Function** →

Since, ~~P(r)~~ Probability of r successes is

$$P(r) = {}^nC_r \, p^r q^{n-r}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} p^r q^{n-r}$$

$$= \frac{np(np-p)(np-2p)\ldots(np-pr+p)(1-p)^{n-r}}{r!}$$

[since, $q = 1-p$]

$$= \frac{\lambda(\lambda-p)(\lambda-2p)\ldots\ldots(\lambda-pr+p)}{r!}\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-r}$$

[$\because$ from eq^n (1)] [$\because$ $np = \lambda$]
[$p = \frac{\lambda}{n}$]

$$= \frac{\lambda\left(\lambda-\frac{\lambda}{n}\right)\left(\lambda-\frac{2\lambda}{n}\right)\ldots\left(\lambda-\frac{r\lambda}{n}+\frac{\lambda}{n}\right)}{r!}\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-r}$$

since, $n \to \infty$

$$\lim_{n\to\infty} P(r) = \frac{\lambda(\lambda-0)(\lambda-0)\ldots\ldots(\lambda-0)}{r!}(1-0)^{-r}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}$$

$$= \frac{\lambda^{r}}{r!}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}$$

$\boxed{\cancel{P(r) = \frac{\lambda^{n}\lambda^{r}}{r!}}}$ $P(r) = \frac{\lambda^{r}}{r!}e^{-\lambda}$

where $r = 0, 1, 2, \ldots$

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Constants of The Poisson's Distribution

(i) First Moment or Mean (μ)

$$\mu = \sum r \, p(r)$$

$$= \sum_{0}^{\infty} r \cdot \frac{e^{-\lambda} \lambda^{r}}{r!} = \sum_{r=1}^{\infty} \frac{e^{-\lambda} \lambda^{r-1} \lambda}{(r-1)!}$$

$$= \lambda e^{-\lambda} \sum_{r=1}^{\infty} \frac{\lambda^{r-1}}{(r-1)!}$$

$$= \lambda e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right]$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

So, $\boxed{\text{Mean } (\mu) = \lambda}$

(ii) Second Moment or Variance (σ^2)

$$\text{Variance} = \sigma^2 = \cancel{\sum x^2 p(x,\lambda)} \sum (r-\mu)^2 p(r)$$

$$= \sum_{r=0}^{\infty} (r-\lambda)^2 \cdot \frac{\lambda^{r} e^{-\lambda}}{r!}$$

$$= e^{-\lambda} \sum_{r=0}^{\infty} (r^2 + \lambda^2 - 2\lambda r) \frac{\lambda^{r}}{r!}$$

$$\sigma^2 = e^{-\lambda} \sum_{r=0}^{\infty} \left[\frac{r^2 \lambda^r}{\cancel{r}(r-1)!} + \frac{\lambda^2 \lambda^r}{r!} - \frac{2\lambda \cancel{r} \lambda^r}{\cancel{r}(r-1)!} \right]$$

$$= e^{-\lambda} \sum_{r=0}^{\infty} \left[\frac{(r-1+1)\lambda^r}{(r-1)!} + \lambda^2 \frac{\lambda^r}{r!} - 2\lambda^2 \frac{\lambda^{r-1}}{(r-1)!} \right]$$

$$= e^{-\lambda} \sum_{r=0}^{\infty} \left[\frac{\cancel{(r-1)}\lambda^r}{\cancel{(r-1)}(r-2)!} + \frac{\lambda^r}{(r-1)!} + \lambda^2 \cdot \frac{\lambda^r}{r!} - \frac{2\lambda^2 \lambda^{r-1}}{(r-1)!} \right]$$

$$= e^{-\lambda} \sum_{r=0}^{\infty} \left[\lambda^2 \cdot \frac{\lambda^{r-2}}{(r-2)!} + \lambda \cdot \frac{\lambda^{r-1}}{(r-1)!} + \lambda^2 \cdot \frac{\lambda^r}{r!} - 2\lambda^2 \cdot \frac{\lambda^{r-1}}{(r-1)!} \right]$$

$$= e^{-\lambda} \sum \left[\cancel{\lambda^2 e^{\lambda}} + \lambda e^{\lambda} + \cancel{\lambda^2 e^{\lambda}} - \cancel{2\lambda^2 e^{\lambda}} \right]$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

So, $\boxed{\text{Variance } (\sigma^2) = \lambda}$

So, Standard Deviation $= \sqrt{\lambda}$ $\quad \left[\because SD = \sqrt{\sigma^2} \right]$

Same as

(iii) Third Moment $\mu_3 = \lambda$

(iv) Fourth Moment, $\mu_4 = \lambda + 3\lambda^2$

(vi) Kurtosis → Kurtosis $\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\lambda + 3\lambda^2}{\lambda^2}$

$$= \left(3 + \frac{1}{\lambda}\right)$$

(iii) <u>Coefficient of Skewness →</u>

$$\gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{1}{\lambda}}$$

where $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda}$

<u>Ques</u>- In a factory manufacturing razor blades there is a small ~~change~~ chance of 1/50 for any blade to be defective. The blades are placed in packets of 10 blades. Using Poisson distribution, calculate the approximate number of packets containing not more than 2 defective blades in a consignment of 10,000 packets.

Let $N = 10,000$

<u>Sol</u> Here $n = 10$, $p = \frac{1}{50}$ (probability of success) (chance of defective)

Now, mean $= np = 10 \times \frac{1}{50} = 0.2 = (\lambda)$

Since, $P(x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$$P(x) = \frac{e^{-0.2} (0.2)^x}{x!}$$

Probability ~~no~~ packet having no defective blade

$= P(x=0)$

$= \frac{e^{-0.2}(0.2)^0}{0!} = 0.8187$

So, No. of packets having no defective blade $= N \times p(x=0)$

$= 10,000 \times 0.8187$

$= 8187.0$

Next, No. of packets having one defective blade

$= N \times p(x=1)$

$= 10,000 \times \frac{e^{-0.2}(0.2)^1}{1!} = 1637.4$

~~and~~ and, the no. of packets having two defective blades

$= N \times P(x=2)$

$= 10,000 \times \frac{e^{-0.2}(0.2)^2}{2!} = 163.74$

∴ The approximate no. of packets containing not more than two ~~variables~~ defective blades in a consignment of 10,000 packets

$= 10,000 - (8187 + 1637.4 + 163.74)$

$= 11.84 \simeq 12$ packets.

Ques:- If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 1000 individuals more than two will have bad reactions.

Sol^n Given $p = 0.001$, $n = 1000$
mean $= np = 1000 \times 0.001 = 1 = (\lambda)$

$$P(X=0,1,2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!}$$

$$= e^{-\lambda}\left[1 + 1 + \frac{1}{2}\right] \quad [\because \lambda = 1]$$

$$= \frac{5}{2}e^{-1} = \frac{2.5}{2.7183} = 0.912$$

Hence, $P(X>2) = 1 - 0.912 = 0.088$ (approx)

* Multinomial Distribution :-

If the possible outcomes of a random experiment are of k types, say $E_1, E_2, \ldots \ldots E_k$ with constant probabilities $p_1, p_2, \ldots p_k$ respectively, then the probability that in 'n' independent trials E_1 occurs x_1 times, E_2 occurs x_2 times, ... E_k occurs x_k times is given by

$$= \frac{n!}{x_1!\, x_2! \ldots \ldots x_k!} (p_1)^{x_1} \cdot (p_2)^{x_2} \ldots \ldots (p_k)^{x_k} \quad \text{—— (I)}$$

where $x_1 + x_2 + x_3 + \ldots\ldots + x_k = n$

$p_1 + p_2 + p_3 + \ldots\ldots + p_k = 1$

The discrete probability distribution defined by the above probability f^n is called multinomial distribution.

The binomial distribution is a special case of multinomial distribution when $k = 2$.

Ques: A die is rolled 6 times. Find the probability that each of the six faces appears exactly once.

Solⁿ: In a single throw of die, there are 6 possible outcomes: 1, 2, 3, 4, 5, 6 with probability $\frac{1}{6}$ for each (the die is assumed to be unbiased)

and 6 independent trials are made because the die is rolled 6 times, Hence,

$$p_1 = \frac{1}{6},\ p_2 = \frac{1}{6},\ p_3 = \frac{1}{6},\ p_4 = \frac{1}{6},\ p_5 = \frac{1}{6},\ p_6 = \frac{1}{6}$$

$k = 6$ and $n = 6$.

We have to find the probability that

$x_1 = 1,\ x_2 = 1,\ x_3 = 1,\ x_4 = 1,\ x_5 = 1,\ x_6 = 1$

Substituting the values in eqⁿ (I),

$$= \frac{6!}{1!\,1!\,1!\ldots 1!} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$$

$$= \frac{5!}{6^5} = \frac{5}{324}$$

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Continuous Probability Distribution :-

The distributions defined by the continuous variable like heights or weights, are continuous distribution.
~~In continuous distributions, in place of finding the probability that the variable x equals~~

In continuous distributions, we find the probability of x falling in certain interval.

- **Probability Density Function :-**

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\, dx.$$

It is also called cumulative distribution f^n.

where $\int_{-\infty}^{\infty} f(x)\, dx = 1$ and $f(x) \ge 0$

- **Mathematical Expectation In a Continuous Distribution :-**

If X is a continuous random variable with probability density f^n $f(x)$, then

(i) Mean or Expected Value of X →.

$$\text{Mean } (\mu) = \text{Expected Value} = \int_{-\infty}^{x} x f(x)\, dx$$

such that $\int_{-\infty}^{\infty} f(x)\, dx = 1$

(ii) Variance →.

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

(iii) r^{th} Moment about Mean →.

$$\mu_r = \int_{-\infty}^{\infty} (x-\mu)^r f(x)\, dx.$$

(iiii) r^{th} Moment about Origin →.

$$\mu'_r = \int_{-\infty}^{\infty} (x-0)^r f(x)\, dx$$

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(ii) Exponential Distribution →

A random variable X is said to have an exponential distribution with parameter $c>0$, if its probability density f^n is

$$f(x,c) = \begin{cases} c e^{-cx}, & 0<x<\infty,\ c>0 \\ 0, & x<0 \end{cases}$$

- Constants of Distribution →

(A) ~~Moments About Origin~~ →

(i) Mean or first Moment about Origin →

$$\text{Mean} = \mu_1' = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^{\infty} x c e^{-cx}\, dx$$

$$= c\left[\left(\frac{x e^{-cx}}{-c}\right)_0^{\infty} - \int_0^{\infty} \frac{e^{-cx}}{-c}\, dx\right]$$

$$= \left[x e^{-cx}\right]_0^{\infty} + \int_0^{\infty} e^{-cx}\, dx$$

$$= 0 + \left[\frac{e^{-cx}}{-c}\right]_0^{\infty}$$

$$= -\frac{1}{c}\left[e^{-\infty} - e^{0}\right] = \frac{1}{c}$$

(ii) Second Moment About Origin :-

$$\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^{\infty} x^2 c e^{-cx}\, dx$$

$$= c\left[\left(\frac{x^2 e^{-cx}}{-c}\right)_0^{\infty} - \int_0^{\infty} \frac{2x e^{-cx}}{-c}\, dx\right]$$

$$= 0 + 2\int_0^{\infty} x e^{-cx}\, dx$$

$$= 2\left[\left(\frac{x e^{-cx}}{-c}\right)_0^{\infty} - \int_0^{\infty} \frac{e^{-cx}}{-c}\, dx\right]$$

$$= \frac{-2}{c}\left[0 - \left(\frac{e^{-cx}}{-c}\right)_0^{\infty}\right]$$

$$= -\frac{2}{c}\cdot\frac{1}{c}\left(e^{-cx}\right)_0^{\infty}$$

$$= -\frac{2}{c^2}\left[e^{-\infty} - e^0\right] = -\frac{2}{c^2}(0-1)$$

$$\mu_2' = \frac{2}{c^2}$$

Same as,

(iii) Third Moment about Origin →

$$\mu_3' = \frac{6}{c^3}$$

(iv) Fourth Moment About Origin →

$$\mu_4' = \frac{24}{c^4}$$

(B) Moments About Mean →.

(1) First Moment About Mean →

$$\mu_1 = \int_{-\infty}^{\infty} \left(x - \frac{1}{c}\right) f(x)\,dx \qquad \left[\because \mu_1' = \frac{1}{c}\right]$$

$\therefore$ Mean $= \frac{1}{c}$

$$= \int_0^{\infty} \left(x - \frac{1}{c}\right) c e^{-cx}\,dx$$

$$= \int_0^{\infty} x c e^{-cx}\,dx - \int_0^{\infty} e^{-cx}\,dx$$

$$= \frac{1}{c} - \left[\frac{e^{-cx}}{-c}\right]_0^{\infty}$$

$$= \frac{1}{c} - \frac{1}{c} = 0.$$

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$\Rightarrow \quad \mu_1 = 0$

(ii) Variance or Second Moment About Mean :-

$$\mu_2 = \int_{-\infty}^{\infty} \left(x - \frac{1}{c}\right)^2 f(x)\, dx$$

$$= \frac{1}{c^2}$$

So, variance, $(\sigma^2) - \frac{1}{c^2}$

then standard deviation, $\sigma = \frac{1}{c}$

(iii) Third Moment About Mean :-

$$\mu_3 = \int_{\infty}^{\infty} \left(x - \frac{1}{c}\right)^3 f(x)\, dx$$

$$= \frac{2}{c^3}$$

(iv) Fourth Moment About Mean :-

$$\mu_4 = \int_{-\infty}^{\infty} \left(x - \frac{1}{c}\right)^4 f(x)\, dx$$

$$= \frac{9}{c^4}$$

(C) Coefficient of Kurtosis →

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\left(\frac{9}{c^4}\right)}{\left(\frac{1}{c^2}\right)^2} = 9$$

(D) Coefficient of Skewness →

$$\gamma_1 = \sqrt{\beta_1}.$$

where $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\left(\frac{2}{c^3}\right)^2}{\left(\frac{1}{c^2}\right)^3} = 4$

• Constants of Gamma Distributions :-

Gamma Distribution :-

This is the generalization of exponential distribution.

~~A probability~~ continuous random variable X with probability density f^n $f(x)$, for some $\alpha > 0$, $\beta > 0$,

the two parametric pdf.

$$f(x) = \begin{cases} \frac{\beta^{\alpha}}{\Gamma\alpha} x^{\alpha-1} e^{-\beta x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$$

If we put $\beta = 1$,

then p.d.f. of standard Gamma distribution be,

$$f(x) = \begin{cases} \frac{1}{\Gamma\alpha} x^{\alpha-1} e^{-x}, & x \geq 0 \\ 0, & \text{else.} \end{cases}$$

→ **Note:-** We will use p.d.f. of standard gamma p.d.f. = probability density function

distribution.

where $\Gamma\alpha = \int_0^\infty e^{-x} \cdot x^{\alpha-1} dx$

<u>Note</u> :- (i) for any $\alpha > 1$, $\Gamma\alpha = (\alpha-1)\Gamma(\alpha-1)$

(ii) for any +ve integer n,
$\Gamma n = (n-1)!$

eg $\Gamma 7 = (7-1)! = 6! = 720$

(iii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

∴ <u>Constants of Gamma Distribution</u> →.

(A) <u>Moments About Origin</u> →.

(i) <u>First Moment About Origin or Mean</u> →.

$$Mean = \mu_1' = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^\infty x \frac{1}{\Gamma\alpha} x^{\alpha-1} e^{-x} dx$$

$$\mu_1' = \frac{1}{\Gamma\alpha}\int_0^\infty x^{\alpha-1+1}\, e^{-x}\, dx$$

$$= \frac{1}{\Gamma\alpha}\int_0^\infty x^{(\alpha+1)-1}\, e^{-x}\, dx$$

$$= \frac{1}{\Gamma\alpha}\,\Gamma(\alpha+1) \qquad \left[\because \Gamma\alpha = \int_0^\infty e^{\alpha-1}\, e^{-x}\, dx\right]$$

$$= \frac{1}{\cancel{\Gamma\alpha}}\,\alpha\,.\,\cancel{\Gamma\alpha} \qquad \left[\because \Gamma(\alpha+1) = \alpha\,\Gamma\alpha\right]$$

$$= \alpha.$$

(ii) Second Moment about Origin →

$$\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^\infty x^2 . \frac{1}{\Gamma\alpha}\, x^{\alpha-1}\, e^{-x}\, dx$$

$$= \frac{1}{\Gamma\alpha}\int_0^\infty x^{\alpha+1}\, e^{-x}\, dx$$

$$= \frac{1}{\Gamma\alpha}\int_0^\infty x^{(\alpha+2)-1}\, e^{-x}\, dx$$

$$\mu_2' = \frac{1}{\Gamma\alpha} \cdot \Gamma\alpha+2$$

$$= \frac{1}{\Gamma\alpha} (\alpha+1)\,\Gamma\alpha+1$$

$$= \frac{1}{\cancel{\Gamma\alpha}} (\alpha+1)\,(\cancel{\alpha+1})\,\alpha\,\cancel{\Gamma\alpha}$$

$$= \alpha(\alpha+1)$$

~~Same as~~

(iii) <u>Third Moment about Origin</u> →

$$\mu_3' = \int_{\infty}^{\infty} x^3 f(x)\,dx$$

$$= \int_0^{\infty} x^3 \cdot \frac{1}{\Gamma\alpha} x^{\alpha-1} e^{-x}\,dx$$

$$= \frac{1}{\Gamma\alpha}\int_0^{\infty} x^{(\alpha+3)-1} e^{-x}\,dx$$

$$= \frac{1}{\Gamma\alpha}\,\Gamma\alpha+3 = \frac{1}{\cancel{\Gamma\alpha}}\,\cancel{\alpha+3}\,(\alpha+2)(\alpha+1)\,\alpha\,\cancel{\Gamma\alpha}$$

$$= \cancel{\alpha+3}\,(\alpha+2)(\alpha+1)\alpha = \alpha(\alpha+1)(\alpha+2)$$

Similarly, we can find fourth moment about origin.

(B) Moments About Mean :-

(i) First Moment About Mean :-

$$\mu_1 = \int_{-\infty}^{\infty} (x-\mu_1')^1 f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-\alpha) f(x)\,dx \qquad [\because \text{Mean} = \alpha]$$

$$= \int_0^{\infty} (x-\alpha) \frac{1}{\Gamma\alpha} x^{\alpha-1} e^{-x}\,dx$$

$$= \frac{1}{\Gamma\alpha}\left[\int_0^{\infty} x^{\alpha} e^{-x}\,dx - \alpha\int_0^{\infty} x^{\alpha-1} e^{-x}\,dx\right]$$

$$= \frac{1}{\Gamma\alpha}\left[\int_0^{\infty} x^{(\alpha+1)-1} e^{-x}\,dx - \alpha\int_0^{\infty} x^{\alpha-1} e^{-x}\,dx\right]$$

$$= \frac{1}{\Gamma\alpha}\left[\Gamma(\alpha+1) - \alpha\,\Gamma\alpha\right]$$

$$= \frac{1}{\Gamma\alpha}\left[\alpha\Gamma\alpha - \alpha\Gamma\alpha\right] = 0.$$

$$[\because \Gamma(\alpha+1) = \alpha\Gamma\alpha]$$

So, $\mu_1 = 0$

(ii) Second Moment About Mean or Variance :-

$$\mu_2 \text{ or } \sigma^2 = \int_{-\infty}^{\infty} (x-\mu_1')^2 f(x)\, dx$$

$$= \int_0^{\infty} (x-\alpha)^2 \frac{1}{\Gamma\alpha} x^{\alpha-1} e^{-x}\, dx$$

$$= \frac{1}{\Gamma\alpha} \int_0^{\infty} (x^2+\alpha^2-2\alpha x)\, x^{\alpha-1} e^{-x}\, dx$$

$$= \frac{1}{\Gamma\alpha}\left[\int_0^{\infty} x^{\alpha+1} e^{-x} dx + \alpha^2 \int_0^{\infty} x^{\alpha-1} e^{-x} dx - 2\alpha \int_0^{\infty} x^{\alpha} e^{-x} dx\right]$$

$$= \frac{1}{\Gamma\alpha}\left[\int_0^{\infty} x^{(\alpha+2)-1} e^{-x} dx + \alpha^2 \int_0^{\infty} x^{\alpha-1} e^{-x} dx - 2\alpha \int_0^{\infty} x^{(\alpha+1)-1} e^{-x} dx\right]$$

$$= \frac{1}{\Gamma\alpha}\left[\Gamma\alpha+2 + \alpha^2\, \Gamma\alpha - 2\alpha\, \Gamma\alpha+1\right]$$

$$= \frac{1}{\Gamma\alpha}\left[\cancel{(\alpha+2)}(\alpha+1)\,\alpha.\,\Gamma\alpha + \alpha^2.\,\Gamma\alpha - 2\alpha^2.\,\Gamma\alpha\right]$$

$$= \cancel{(\alpha+2)}(\alpha+1)\,\alpha + \alpha^2 - 2\alpha^2$$

$$= \cancel{(\alpha+2)}(\alpha^2+\alpha) - \alpha^2 = \cancel{\alpha^2} + \alpha - \cancel{\alpha^2}$$

$$= \cancel{\alpha^3+\alpha^2+2\alpha^2+2\alpha-\alpha^2} = \alpha$$

$$\text{Variance } (\sigma^2) = \alpha$$

So, Standard Deviation $= \sqrt{\alpha}$

$= \sqrt{\alpha}$

Same as we can find third and fourth moment about mean.

(iii) Third Moment about mean →

$$\mu_3 = 2\alpha$$

(iv) Fourth Moment about mean →

$$\mu_4 = 3\alpha^2 + 6\alpha$$

(C) Coefficient of Kurtosis →

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\alpha^2 + 6\alpha}{\alpha^2} = \frac{3\alpha(\alpha+2)}{\alpha^2}$$

$$= \frac{3}{\alpha}(\alpha+2)$$

(D) Coefficient of Skewness →

$$\gamma_1 = \sqrt{\beta_1}$$

where $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(2\alpha)^3}{\alpha^3} = 8$.

Ques:- Show that the f^n $f(x)$, defined as

$$f(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

is a probability density f^n and find the probability that the variate X having $f(x)$ as density f^n will lie in the interval $(1,2)$. Also find the p.d.f. $F(2)$.

Ans. $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_{0}^{\infty} = 1$$

$\Rightarrow$ Given f^n $f(x)$ is a probability density f^n.

- Probability of X in the interval $(1,2)$

$$P(1 \leq X \leq 2) = \int_{1}^{2} f(x)\,dx = \int_{1}^{2} e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_{1}^{2} = e^{-1} - e^{-2} = 0.233$$

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- $F(2) = \int_{-\infty}^{2} f(x)\,dx$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx$$

$$= 0 + \int_{0}^{2} e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_{0}^{2} = 1 - e^{-2} = 0.865.$$

Normal Distribution:-

Normal distribution is the limiting form of the binomial distribution when $n \to \infty$ but neither p nor q is very small. Let define a variable z as

$$z = \frac{x-\mu}{\sigma} = \frac{x-np}{\sqrt{npq}}$$

where 'x' is a continuous random variable with two parameters, mean ($\mu = np$) and standard deviation ($\sigma = \sqrt{npq}$).

• Probability Density Function:-

The continuous random variable X is said to have a normal distribution, if its p.d.f. is defined as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty \le X \le \infty, \quad \sigma > 0$$

where μ and σ are parameters of the normal distribution.

above p.d.f can be written as,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2}$$

where $z = \left(\frac{x-\mu}{\sigma}\right)$

Constants of Normal Distribution →

~~(A) Moments about Origin →~~

(i) First Moment about origin or Mean →

$$\mu_1' (\text{mean}) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, x\, e^{-(x-\mu)^2/2\sigma^2}\, dx$$

Put $\frac{x-\mu}{\sigma} = z$ then $dx = \sigma\, dz$ and

$x = \sigma z + \mu$.

So,

$$\mu_1' (\text{Mean}) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot (\mu + \sigma z)\, e^{-\frac{1}{2}z^2}\, \sigma\, dz$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz$$

$$= \frac{\mu}{\sqrt{2\pi}} 2\int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz + 0$$

$\left[\because z e^{-\frac{1}{2}z^2} \text{ is odd } f^n \text{ and } e^{-\frac{1}{2}z^2} \text{ is even } f^n\right]$

$$= \sqrt{\frac{2}{\pi}}\, \mu \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz$$

$$= \sqrt{\frac{2}{\pi}}\, \mu \cdot \frac{\sqrt{\pi}}{\sqrt{2}} = \mu.$$

$\therefore$ The mean of the normal distribution is μ.

(ii) Variance of the Normal Distribution →.

$$\text{Variance} = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx$$

Put $\frac{x-\mu}{\sigma} = z$ then $dx = \sigma\, dz$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-\frac{1}{2}z^2} \sigma \, dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} \, dz$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-\frac{1}{2}z^2} \, dz$$

[$\because$ $z^2 e^{-\frac{1}{2}z^2}$ is even f^n]

Put $\frac{z^2}{2} = t$ than $z\,dz = dt$

$\Rightarrow z = \sqrt{2t}$

So, Variance $= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} 2t \, e^{-t} \frac{dt}{\sqrt{2t}}$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} \sqrt{2t} \, e^{-t} \, dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t^{1/2} e^{-t} \, dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t^{3/2 - 1} e^{-t} \, dt$$

$$= \frac{2}{\sqrt{\pi}} \sigma^2 \Gamma\left(\frac{3}{2}\right)$$ [It form gamma f^n]

$$= \frac{2}{\sqrt{\pi}} \sigma^2 \Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{\cancel{2}}{\sqrt{\pi}} \sigma^2 \frac{1}{\cancel{2}} \Gamma\frac{1}{2}$$ [$\because \Gamma n+1 = n \Gamma n$]

$$= \frac{\sigma^2}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}}$$ [$\because \Gamma \frac{1}{2} = \sqrt{\pi}$]

$$= \sigma^2.$$

So, variance of normal distribution is σ^2.

(iii) <u>Mode of Normal Distribution</u> →.

Mode is the value of x at which $f(x)$ has maximum value.

Now, the probability density f^n of 'x' is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq \infty$$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left\{ -\frac{1}{2} \cancel{2}\left(\frac{x-\mu}{\sigma}\right) \right\}$$

$$= \frac{-1}{\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot (x-\mu)$$

for maxima and minima,

$$f'(x) = 0$$

$$\frac{-1}{\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} (x-\mu) = 0$$

$$\Rightarrow \boxed{x = \mu}$$

Now, $f''(x) = \frac{-1}{\sigma^2\sqrt{2\pi}}$ [at $x = \mu$]

Since, at $x = \mu$, $f''(x)$ is negative.

So, $f(x)$ attains maximum value at $x = \mu$.

So, the mode of the normal distribution is 'μ'.

(iv.) Median of the Normal Distribution →.

Let 'M' is the median of normal distribution then

$$\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2}$$

Now, take. $\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2}$

$$\Rightarrow \int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{M} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{\mu}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

(I)

Let $\int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

Put $\frac{x-\mu}{\sigma} = z \Rightarrow dx = \sigma dz$

Also limits are when $x = -\infty$, then $z = -\infty$
Also when $x = \mu$ then $z = 0$

$$\int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2}\, \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}z^2} \, dz \quad \text{[by symmetry Property]}$$

$$\therefore \int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}z^2} \, dz$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{2}} = \frac{1}{2} \quad \text{(II)}$$

Now, from eqⁿ (I) & (II),

$$\frac{1}{2} + \int_{\mu}^{M} f(x)\, dx = \frac{1}{2}$$

$$\Rightarrow \int_{\mu}^{M} f(x)\, dx = 0$$

- This implies that $\mu = M$

Hence, the median of the normal distribution is 'μ'.

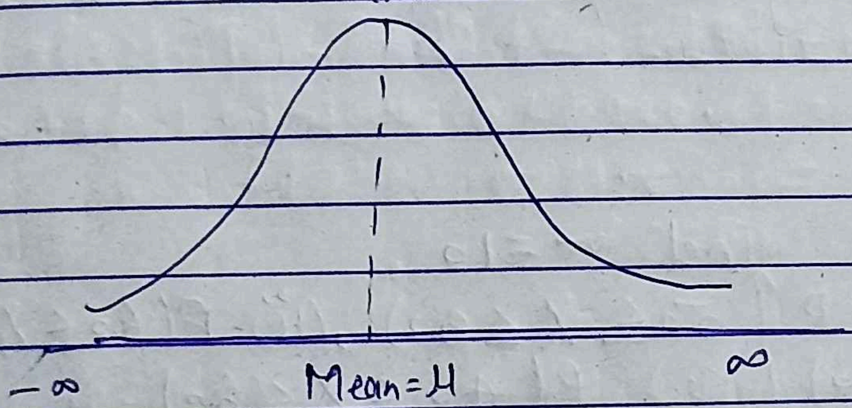
- **Note** for the normal distribution, the mean, median and mode are equal, i.e,

Mean = Mode = Median = μ

Properties of the Normal Distribution →

(i) The normal probability curve with mean μ and standard deviation 's' is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty \le X \le \infty$$



(ii) The curve is bell shaped and symmetrical about the line $x = \mu$.

(iii) Mean, Median, mode of the normal distribution coincide and the normal distribution is unimodal.

(iv) $f(x)$ decreases rapidly as x increases.

(v) X-axis is an asymptote to the curve (the tangent to the curve at ∞ is X-axis).

(vi) The maximum probability occurs at the point $x = \mu$ and is $\frac{1}{\sigma\sqrt{2\pi}}$.

(vii) Mean deviation about mean = $\frac{4}{5}\sigma$.

(viii) Since, f(x), being the probability, can never be negative, so that no portion of the curve lies below the x-axis.

(ix.) A linear fn of independent normal variates is also normal variate.

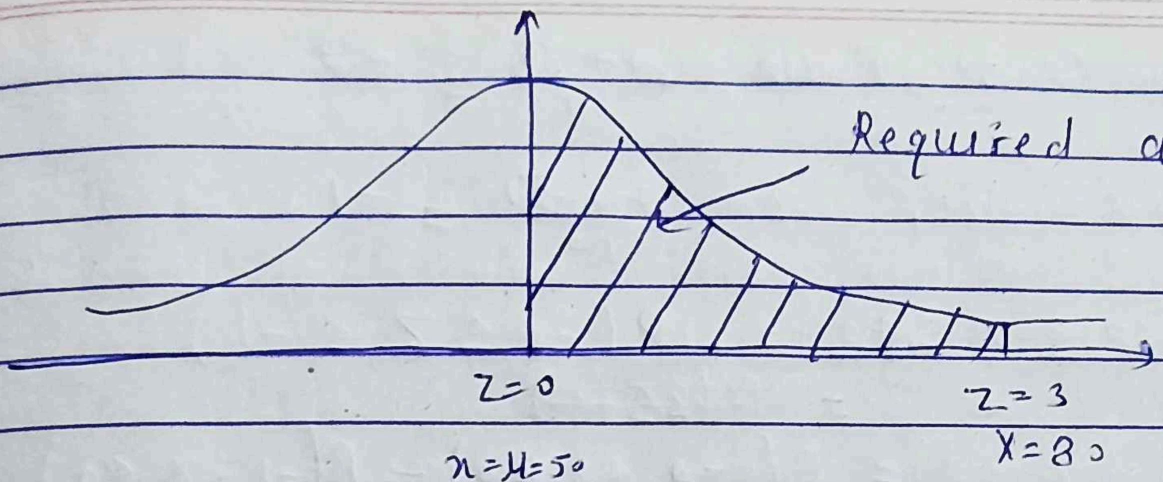
Ques:- If $\mu = 50$ and $\sigma = 10$,
find (i) $P(50 \le X \le 80)$, (ii) $P(60 < X \le 70)$
(iii) $P(30 \le X \le 40)$ (iv.) $P(40 \le X < 60)$
Use table: Area under the normal curve.

Soln Standard normal variate

$$z = \frac{X-\mu}{\sigma} = \frac{X-50}{10}$$

(i) $z = \frac{50-50}{10} = 0$ when $X = 50$ | and $z = \frac{80-50}{10} = 3$ when $X = 80$.

Hence, $P(50 \le X \le 80) = P(0 \le z \le 3)$
$= P(z \le 3) - P(z \ge 0)$ ~~= 0.4987~~ [by using table]
$= 0.9987 - 0.5000 = 0.497$.



(ii) when $X = 60$, $z = \frac{60-50}{10} = 1$

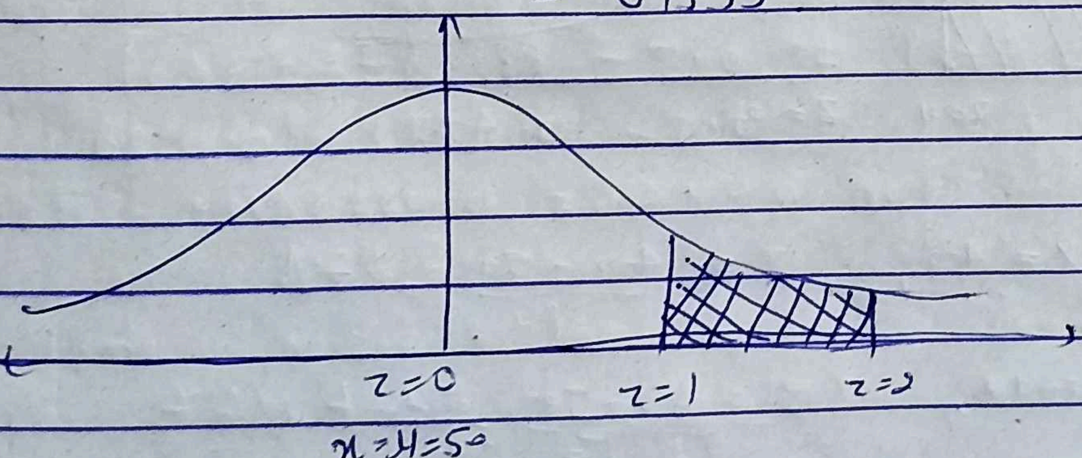
when $X = 70$, $z = \frac{70-50}{10} = 2$

So, $P(60 < X < 70) = P(1 \leq Z \leq 2)$

$= P(0 \leq Z \leq 2) - P(0 \leq Z \leq 1)$

$= 0.4772 - 0.3413$ [by using table]

$= 0.1359$



(iii) when $X = 30, \quad Z = \frac{30-50}{10} = -2$

when $X = 40, \quad Z = \frac{40-50}{10} = -1$

So, $P(30 \le X \le 40) = P(-2 \le Z \le -1)$

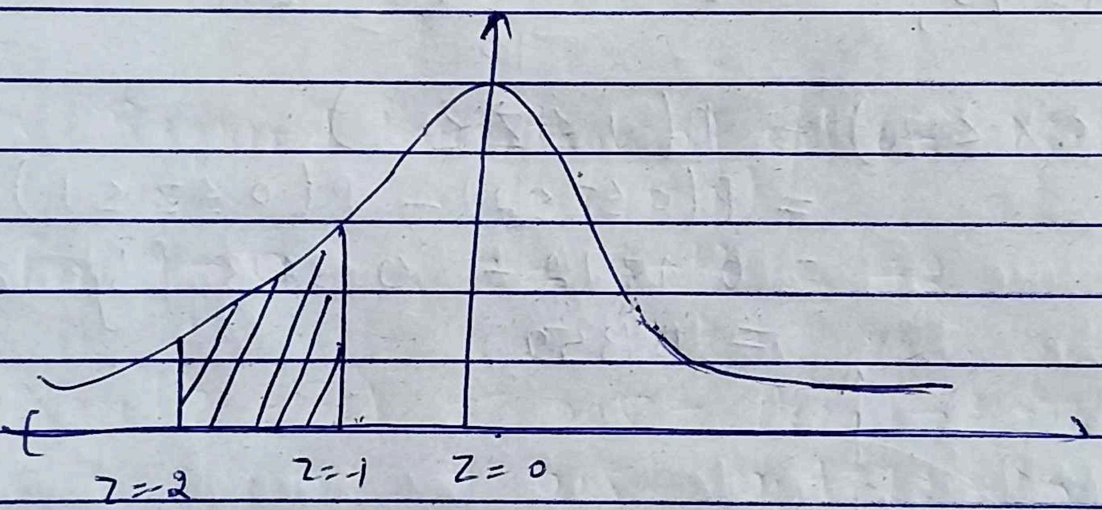
$= \cancel{P(Z \le -1)}$

$= P(-2 \le Z \le 0) - P(-1 \le Z \le 0)$

$= [P(-2 \le Z) + P(Z \le 0)] - [P(-1 \le Z) + P(Z \le 0)]$

$= [0.0228 + 0.5000] - [0.1587 + 0.500]$

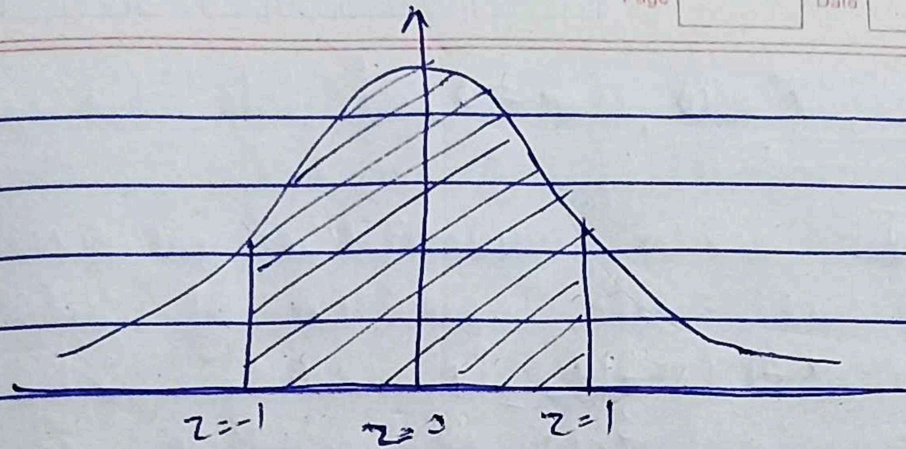
$= 0.1359$



(iv) when $X = 40, \quad Z = \frac{40-50}{10} = -1$

when $X = 60, \quad Z = \frac{60-50}{10} = +1$

$P(40 \le X \le 60) = P(-1 \le Z \le 1)$



$P(40 < X \leq 60) = P(-1 \leq Z \leq 1)$

$= 2\,P(0 \leq Z \leq 1)$

$= 2\,[P(Z \leq 1) - P(Z \geq 0)]$

$= 2\,[0.8413 - 0.5000]$

$= 2 \times 0.3413$

$= \cancel{0.6826}\ \ 0.6826$

Ques - A sample of 100 dry cells are tested to find the length of life, produced the following results:

$\bar{x} = 12$ hours, $\sigma = 3$ hours.

Assuming the data to be normally distributed, what percentage of cells are expected to have life:

(i) more than 15 hours

(ii) less than 6 hours

(iii) between 10 and 14 hours.

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Solⁿ $n = 100$, $\bar{x} = 12$, $\sigma = 3$,

$\therefore z = \dfrac{x - \bar{x}}{\sigma}$

(i) for $x = 15$, $z = \dfrac{15-12}{3} = 1$

So, $\text{Area}(z \geq 1) = 0.5 - \text{Area}(0 < z < 1)$
$= 0.5 - [\text{Area}(z < 1) - \text{Area}(z = 0)]$
$= 0.5 - [0.8413 - 0.5]$
$= 0.5 - [0.3413]$
$= 0.1587$

$\therefore$ Required Percentage of cells $= 15.87\%$

(ii) for $x = 6$, $z = \dfrac{6-12}{3} = -2$

$\text{Area}(z \leq -2) = 0.5 - \text{Area}(-2 \leq z < 0)$
$= 0.5 - 0.4772 = 0.0228$

$\therefore$ % cells for $0 < x < 6 = 0.0228 \times 100$
$= 2.28\%$

(iii) for $x = 10$, $z = \dfrac{10-12}{3} = -\dfrac{2}{3} = -0.67$

for $x = 14$, $z = \dfrac{14-12}{3} = \dfrac{2}{3} = 0.67$

$\text{Area}(-0.67 < z < 0.67)$
$= 2 \times \text{Area}(0 < z \leq 0.67)$

$= 2\,[\,\text{Area}\,(z \le 0.67) - \text{Area}\,(z \geq 0)\,]$

$= 2\,[\,0.7486 - 0.5000\,]$

$= 2 \times 0.2486$

$= 0.4972$

So, % of cells expected to have life b/w 10 and 14 hours $= 0.4972 \times 100$

$= 4.972\%$

Note :- Moment Generating function for all distribution, do yourself

Rank Correlation →

It is useful when quantitative measures for certain factors, like evaluation of leadership ability or the judgement in beauty contest, can't be fixed, but the individuals in the group can be arranged in order, thereby obtaining the rank for each in the group.

Spearman's correlation coefficient, denoted by 'r', is defined as

$$r = 1 - \left[\frac{6 \sum d^2}{N(N^2-1)}\right]$$

where 'd' represents the difference of ranks b/w paired items in two series.

Ques- Two ladies Deepti and Nancy, were asked to rank lipsticks from 7 known companies. The ranks given by them are as follows:

Lipstick Companies	A	B	C	D	E	F	G
Rank X by Deepti	2	1	4	3	5	7	6
Rank Y by Nancy	1	3	2	4	5	6	7

Sol^n

X R_1	Y R_2	d $= [R_1 - R_2]$	d^2
2	1	1	1
1	3	-2	4
4	2	2	4
3	4	-1	1
5	5	0	0
7	6	1	1
6	7	-1	1
		Total	$\Sigma d^2 = 12$

$N = 7$

$$r = 1 - \left[\frac{6 \Sigma d^2}{N(N^2-1)}\right] = \left[1 - \frac{6 \times 12}{7(49-1)}\right]$$

$$r = \left[1 - \frac{\overset{3}{\cancel{6}} \times \cancel{12}}{7 \times \cancel{48}\ \cancel{4}_{2}}\right] = 1 - \frac{3}{14} = \frac{11}{14}$$

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Que From the data given below, calculate the coefficient of rank correlation b/w X and Y.

X	78	89	97	69	59	79	68	57
Y	125	137	156	112	107	136	123	108

Solⁿ Here $N = 8$

X	Y	Rank in X (R_1)	Rank in Y (R_2)	$d = (R_1 - R_2)$	d^2
78	125	4	4	0	0
89	137	2	2	0	0
97	156	1	1	0	0
69	112	5	6	-1	1
59	107	7	8	-1	1
79	136	3	3	0	0
68	123	6	5	-1	1
57	108	8	7	1	1
					$\Sigma d^2 = 4$

Required coefficient of correlation,

$$r = \left[1 - \frac{6\Sigma d^2}{N(N^2-1)}\right] = \cancel{\frac{6 \times 4}{8 \times 63}} \quad \cancel{\frac{1}{21}}$$

$$= \left(1 - \frac{6 \times 4}{8 \times 63}\right) = 1 - \frac{1}{21} = \frac{20}{21} = 0.952$$