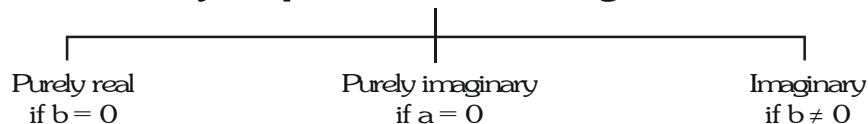


# COMPLEX NUMBER

## 1. DEFINITION :

Complex numbers are defined as expressions of the form  $a + ib$  where  $a, b \in \mathbb{R}$  &  $i = \sqrt{-1}$ . It is denoted by  $z$  i.e.  $z = a + ib$ . 'a' is called real part of  $z$  ( $\text{Re } z$ ) and 'b' is called imaginary part of  $z$  ( $\text{Im } z$ ).

**Every Complex Number Can Be Regarded As**



**Note :**

- (i) The set  $\mathbb{R}$  of real numbers is a proper subset of the Complex Numbers. Hence the Complex Number system is  $\mathbb{N} \subset \mathbb{W} \subset \mathbb{I} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ .
- (ii) Zero is both purely real as well as purely imaginary but not imaginary.
- (iii)  $i = \sqrt{-1}$  is called the imaginary unit. Also  $i = -1$  ;  $i^3 = -i$  ;  $i^4 = 1$  etc.

In general  $i^{4n} = 1$ ,  $i^{4n+1} = i$ ,  $i^{4n+2} = -1$ ,  $i^{4n+3} = -i$ , where  $n \in \mathbb{I}$

- (iv)  $\sqrt{a} \sqrt{b} = \sqrt{ab}$  only if atleast one of either  $a$  or  $b$  is non-negative.

**Illustration 1 :** The value of  $i^{57} + 1/i^{125}$  is :-

- (A) 0 (B)  $-2i$  (C)  $2i$  (D) 2

**Solution :**  $i^{57} + 1/i^{125} = i^{56} \cdot i + \frac{1}{i^{124} \cdot i}$

$$= (i^4)^{14} i + \frac{1}{(i^4)^{31} i}$$

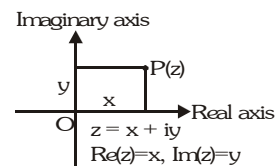
$$= i + \frac{1}{i} = i + \frac{i}{i^2} = i - i = 0$$

**Ans. (A)**

## 2. ARGAND DIAGRAM :

Master Argand had done a systematic study on complex numbers and represented every complex number  $z = x + iy$  as a set of ordered pair  $(x, y)$  on a plane called complex plane (Argand Diagram) containing two perpendicular axes. Horizontal axis is known as Real axis & vertical axis is known as Imaginary axis.

All complex numbers lying on the real axis are called as purely real and those lying on imaginary axis as purely imaginary.



## 3. ALGEBRAIC OPERATIONS :

Fundamental operations with complex numbers :

- (a) Addition  $(a + bi) + (c + di) = (a + c) + (b + d)i$
- (b) Subtraction  $(a + bi) - (c + di) = (a - c) + (b - d)i$
- (c) Multiplication  $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
- (d) Division  $\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$

**Note :**

- (i) The algebraic operations on complex numbers are similar to those on real numbers treating  $i$  as a polynomial.
- (ii) Inequalities in complex numbers (non-real) are not defined. There is no validity if we say that complex number (non-real) is positive or negative.  
 e.g.  $z > 0$ ,  $4 + 2i < 2 + 4i$  are meaningless.
- (iii) In real numbers, if  $a^2 + b^2 = 0$ , then  $a = 0 = b$  but in complex numbers,  $z_1^2 + z_2^2 = 0$  does not imply  $z_1 = z_2 = 0$ .

**Illustration 2 :**  $\frac{3 + 2i \sin \theta}{1 - 2i \sin \theta}$  will be purely imaginary, if  $\theta =$

- (A)  $2n\pi \pm \frac{\pi}{3}$ ,  $n \in I$       (B)  $n\pi + \frac{\pi}{3}$ ,  $n \in I$       (C)  $n\pi \pm \frac{\pi}{3}$ ,  $n \in I$       (D) none of these

**Solution :**  $\frac{3 + 2i \sin \theta}{1 - 2i \sin \theta}$  will be purely imaginary, if the real part vanishes, i.e.,

$$\frac{(3 + 2i \sin \theta)}{(1 - 2i \sin \theta)} \times \frac{(1 + 2i \sin \theta)}{(1 + 2i \sin \theta)} = \frac{(3 - 4 \sin^2 \theta) + i(8 \sin \theta)}{(1 + 4 \sin^2 \theta)}$$

$$\frac{3 - 4 \sin^2 \theta}{1 + 4 \sin^2 \theta} = 0 \quad \Rightarrow \quad 3 - 4 \sin^2 \theta = 0 \quad (\text{only if } \theta \text{ be real})$$

$$\Rightarrow \sin^2 \theta = \left( \frac{\sqrt{3}}{2} \right)^2 = \left( \sin \frac{\pi}{3} \right)^2$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in I$$

**Ans. (C)****Do yourself - 1 :**

- (i) Determine least positive value of  $n$  for which  $\left( \frac{1+i}{1-i} \right)^n = 1$
- (ii) Find the value of the sum  $\sum_{n=1}^5 (i^n + i^{n+2})$ , where  $i = \sqrt{-1}$ .

**3. EQUALITY IN COMPLEX NUMBER :**

Two complex numbers  $z_1 = a_1 + ib_1$  &  $z_2 = a_2 + ib_2$  are equal if and only if their real & imaginary parts are respectively equal.

**Illustration 3 :** The values of  $x$  and  $y$  satisfying the equation  $\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$  are

- (A)  $x = -1$ ,  $y = 3$       (B)  $x = 3$ ,  $y = -1$       (C)  $x = 0$ ,  $y = 1$       (D)  $x = 1$ ,  $y = 0$

**Solution :**  $\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i \Rightarrow (4+2i)x + (9-7i)y - 3i - 3 = 10i$

Equating real and imaginary parts, we get  $2x - 7y = 13$  and  $4x + 9y = 3$ .

Hence  $x = 3$  and  $y = -1$ .

**Ans. (B)**

**Illustration 4 :** Find the square root of  $7 + 24i$ .

**Solution :** Let  $\sqrt{7+24i} = a + ib$   
 Squaring  $a^2 - b^2 + 2iab = 7 + 24i$   
 Compare real & imaginary parts  $a^2 - b^2 = 7$  &  $2ab = 24$   
 By solving these two equations  
 We get  $a = \pm 4$ ,  $b = \pm 3$   
 $\sqrt{7+24i} = \pm(4 + 3i)$

**Illustration 5 :** If  $x = -5 + 2\sqrt{-4}$ , find the value of  $x^4 + 9x^3 + 35x^2 - x + 4$ .

**Solution :** We have,  $x = -5 + 2\sqrt{-4}$   
 $\Rightarrow x + 5 = 4i \Rightarrow (x + 5)^2 = 16i^2$   
 $\Rightarrow x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$   
 Now,  
 $x^4 + 9x^3 + 35x^2 - x + 4$   
 $\Rightarrow x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$   
 $\Rightarrow x^2(0) - x(0) + 4(0) - 160 \Rightarrow -160$

**Ans.**

**Do yourself - 2 :**

- (i) Find the value of  $x^3 + 7x^2 - x + 16$ , where  $x = 1 + 2i$ .
- (ii) If  $a + ib = \frac{c+i}{c-i}$ , where  $c$  is a real number, then prove that :  $a^2 + b^2 = 1$  and  $\frac{b}{a} = \frac{2c}{c^2 - 1}$ .
- (iii) Find square root of  $-15 - 8i$

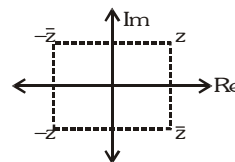
#### 4. THREE IMPORTANT TERMS : CONJUGATE/MODULUS/ARGUMENT :

##### (a) CONJUGATE COMPLEX :

If  $z = a + ib$  then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by  $\bar{z}$ . i.e.  $\bar{z} = a - ib$ .

**Note that :**

- (i)  $z + \bar{z} = 2 \operatorname{Re}(z)$   
 (ii)  $z - \bar{z} = 2i \operatorname{Im}(z)$   
 (iii)  $z\bar{z} = a^2 + b^2$ , which is purely real  
 (iv) If  $z$  is purely real, then  $z - \bar{z} = 0$   
 (v) If  $z$  is purely imaginary, then  $z + \bar{z} = 0$   
 (vi) If  $z$  lies in the 1<sup>st</sup> quadrant, then  $\bar{z}$  lies in the 4<sup>th</sup> quadrant and  $-\bar{z}$  lies in the 2<sup>nd</sup> quadrant.



##### (b) Modulus :

If  $P$  denotes complex number  $z = x + iy$ , then the length  $OP$  is called modulus of complex number  $z$ . It is denoted by  $|z|$ .

$$OP = |z| = \sqrt{x^2 + y^2}$$

Geometrically  $|z|$  represents the distance of point  $P$  from origin. ( $|z| \geq 0$ )

**Note :** Unlike real numbers,  $|z| = \begin{cases} z & \text{if } z > 0 \\ -z & \text{if } z < 0 \end{cases}$

is not correct.

(c) **Argument or Amplitude :**

If  $P$  denotes complex number  $z = x + iy$  and if  $OP$  makes an angle  $\theta$  with real axis, then  $\theta$  is called one of the arguments of  $z$ .

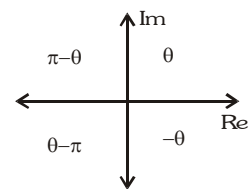
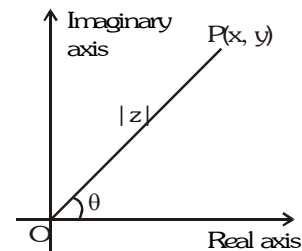
$$\theta = \tan^{-1} \frac{y}{x} \text{ (angle made by } OP \text{ with positive real axis)}$$

**Note :**

- (i) Argument of a complex number is a many valued function. If  $\theta$  is the argument of a complex number, then  $2n\pi + \theta$  ;  $n \in I$  will also be the argument of that complex number. Any two arguments of a complex number differ by  $2n\pi$ .
- (ii) The unique value of  $\theta$  such that  $-\pi < \theta \leq \pi$  is called **Amplitude (principal value of the argument)**.
- (iii) Principal argument of a complex number  $z = x + iy$  can be found out using method given below :

(a) Find  $\theta = \tan^{-1} \left| \frac{y}{x} \right|$  such that  $\theta \in \left( 0, \frac{\pi}{2} \right)$ .

(b) Use given figure to find out the principal argument according as the point lies in respective quadrant.



(iv) Unless otherwise stated,  $\text{amp } z$  implies principal value of the argument.

(v) The unique value of  $\theta = \tan^{-1} \frac{y}{x}$  such that  $0 < \theta \leq 2\pi$  is called **least positive argument**.

(vi) If  $z = 0$ ,  $\arg(z)$  is not defined

(vii) If  $z$  is real & negative,  $\arg(z) = \pi$ .

(viii) If  $z$  is real & positive,  $\arg(z) = 0$

(ix) If  $\theta = \frac{\pi}{2}$ ,  $z$  lies on the positive side of imaginary axis.

(x) If  $\theta = -\frac{\pi}{2}$ ,  $z$  lies on the negative side of imaginary axis.

By specifying the modulus & argument a complex number is defined completely. Argument impart direction & modulus impart distance from origin.

For the complex number  $0 + 0i$  the argument is not defined and this is the only complex number which is given by its modulus only.

**Illustration 6 :** Find the modulus, argument, principal value of argument, least positive argument of complex numbers (a)  $1 + i\sqrt{3}$  (b)  $-1 + i\sqrt{3}$  (c)  $1 - i\sqrt{3}$  (d)  $-1 - i\sqrt{3}$

**Solution :**

(a) For  $z = 1 + i\sqrt{3}$

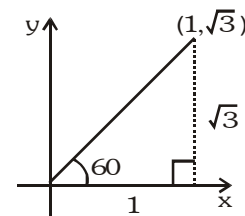
$$|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\arg(z) = 2n\pi + \frac{\pi}{3}, \quad n \in I$$

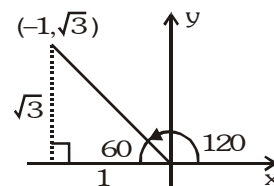
$$\text{Least positive argument is } \frac{\pi}{3}$$

If the point is lying in first or second quadrant then  $\text{amp}(z)$  is taken in anticlockwise direction.

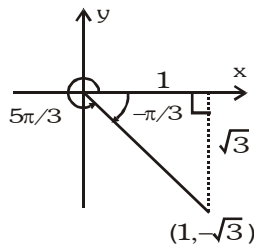
$$\text{In this case } \text{amp}(z) = \frac{\pi}{3}$$



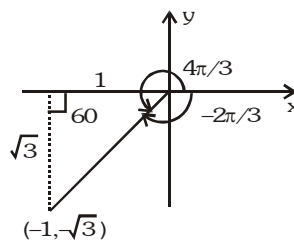
(b) For  $z = -1 + i\sqrt{3}$   
 $|z| = 2$   
 $\arg(z) = 2n\pi + \frac{2\pi}{3}, n \in \mathbb{I}$   
 Least positive argument =  $\frac{2\pi}{3}$   
 $\text{amp}(z) = \frac{2\pi}{3}$



(c) For  $z = 1 - i\sqrt{3}$   
 $|z| = 2$   
 $\arg(z) = 2n\pi - \frac{\pi}{3}, n \in \mathbb{I}$   
 Least positive argument =  $\frac{5\pi}{3}$   
 If the point lies in third or fourth quadrant then consider  $\text{amp}(z)$  in clockwise direction.  
 In this case  $\text{amp}(z) = -\frac{\pi}{3}$



(d) For  $z = -1 - i\sqrt{3}$   
 $|z| = 2$   
 $\arg(z) = 2n\pi - \frac{2\pi}{3}, n \in \mathbb{I}$   
 Least positive argument =  $\frac{4\pi}{3}$   
 $\text{amp}(z) = -\frac{2\pi}{3}$



**Illustration 7 :** Find modulus and argument for  $z = 1 - \sin \alpha + i \cos \alpha, \alpha \in (0, 2\pi)$

**Solution :**  $|z| = \sqrt{(1 - \sin \alpha)^2 + (\cos \alpha)^2} = \sqrt{2 - 2 \sin \alpha} = \sqrt{2} \left| \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right|$

Case (i) For  $\alpha \in \left(0, \frac{\pi}{2}\right)$ ,  $z$  will lie in I quadrant.

$$\text{amp}(z) = \tan^{-1} \frac{\cos \alpha}{1 - \sin \alpha} \Rightarrow \text{amp}(z) = \tan^{-1} \frac{\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}}{\left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}\right)^2} = \tan^{-1} \frac{\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}}$$

$$\Rightarrow \arg z = \tan^{-1} \tan \left( \frac{\pi}{4} + \frac{\alpha}{2} \right)$$

$$\text{Since } \frac{\pi}{4} + \frac{\alpha}{2} \in \left( \frac{\pi}{4}, \frac{\pi}{2} \right)$$

$$\therefore \text{amp}(z) = \left( \frac{\pi}{4} + \frac{\alpha}{2} \right), |z| = \sqrt{2} \left( \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right)$$

Case (ii) at  $\alpha = \frac{\pi}{2}$ :  $z = 0 + 0i$   
 $|z| = 0$   
 $\text{amp}(z)$  is not defined.

Case (iii) For  $\alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ ,  $z$  will lie in IV quadrant

$$\text{so amp } (z) = -\tan^{-1} \tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)$$

$$\text{Since } \frac{\alpha}{2} + \frac{\pi}{4} \in \left(\frac{\pi}{2}, \pi\right)$$

$$\therefore \text{amp } (z) = -\left(\frac{\alpha}{2} + \frac{\pi}{4} - \pi\right) = \frac{3\pi}{4} - \frac{\alpha}{2}, \quad |z| = \sqrt{2} \left(\sin \frac{\alpha}{2} - \cos \frac{\alpha}{2}\right)$$

Case (iv) at  $\alpha = \frac{3\pi}{2}$ :  $z = 2 + 0i$

$$|z| = 2$$

$$\text{amp } (z) = 0$$

Case (v) For  $\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$

$z$  will lie in I quadrant

$$\arg(z) = \tan^{-1} \tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)$$

$$\text{Since } \frac{\alpha}{2} + \frac{\pi}{4} \in \left(\pi, \frac{5\pi}{4}\right)$$

$$\therefore \arg z = \frac{\alpha}{2} + \frac{\pi}{4} - \pi = \frac{\alpha}{2} - \frac{3\pi}{4}, \quad |z| = \sqrt{2} \left(\sin \frac{\alpha}{2} - \cos \frac{\alpha}{2}\right)$$

### Do yourself - 3 :

Find the modulus and amplitude of following complex numbers :

(i) $-2 + 2\sqrt{3}i$	(ii) $-\sqrt{3} - i$	(iii) $-2i$	(iv) $\frac{1+2i}{1-3i}$	(v) $\frac{2+6\sqrt{3}i}{5+\sqrt{3}i}$
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## 5. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

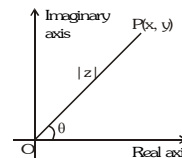
### (a) Cartesian Form (Geometrical Representation) :

Every complex number  $z = x + iy$  can be represented by a point on the cartesian plane known as complex plane by the ordered pair  $(x, y)$ . There exists a one-one correspondence between the points of the plane and the members of the set of complex numbers.

$$\text{For } z = x + iy; \quad |z| = \sqrt{x^2 + y^2}; \quad \bar{z} = x - iy \quad \text{and} \quad \theta = \tan^{-1} \frac{y}{x}$$

**Note :**

- (i) Distance between the two complex numbers  $z_1$  &  $z_2$  is given by  $|z_1 - z_2|$ .
- (ii)  $|z - z_0| = r$ , represents a circle, whose centre is  $z_0$  and radius is  $r$ .



**Illustration 8 :** Find the locus of :

$$(a) |z - 1|^2 + |z + 1|^2 = 4 \quad (b) \operatorname{Re}(z^2) = 0$$

**Solution :**

$$(a) \text{ Let } z = x + iy$$

$$\Rightarrow (|x + iy - 1|)^2 + (|x + iy + 1|)^2 = 4$$

$$\Rightarrow (x - 1)^2 + y^2 + (x + 1)^2 + y^2 = 4$$

$$\Rightarrow x^2 - 2x + 1 + y^2 + x^2 + 2x + 1 + y^2 = 4 \Rightarrow x^2 + y^2 = 1$$

Above represents a circle on complex plane with center at origin and radius unity.

- (b) Let  $z = x + iy$   
 $\Rightarrow z^2 = x^2 - y^2 + 2xyi$   
 $\therefore \operatorname{Re}(z^2) = 0$   
 $\Rightarrow x^2 - y^2 = 0 \Rightarrow y = \pm x$   
 Thus  $\operatorname{Re}(z^2) = 0$  represents a pair of straight lines passing through origin.

**Illustration 9 :** If  $z$  is a complex number such that  $z^2 = (\bar{z})^2$ , then

- (A)  $z$  is purely real (B)  $z$  is purely imaginary  
 (C) either  $z$  is purely real or purely imaginary (D) none of these

**Solution :** Let  $z = x + iy$ , then its conjugate  $\bar{z} = x - iy$

$$\text{Given that } z^2 = (\bar{z})^2 \Rightarrow x^2 - y^2 + 2ixy = x^2 - y^2 - 2ixy \Rightarrow 4ixy = 0$$

If  $x \neq 0$  then  $y = 0$  and if  $y \neq 0$  then  $x = 0$ .

**Ans. (C)**

**Illustration 10 :** Among the complex number  $z$  which satisfies  $|z - 25i| \leq 15$ , find the complex numbers  $z$  having

- (a) least positive argument (b) maximum positive argument  
 (c) least modulus (d) maximum modulus

**Solution :** The complex numbers  $z$  satisfying the condition

$$|z - 25i| \leq 15$$

are represented by the points inside and on the circle of radius 15 and centre at the point  $C(0, 25)$ .

The complex number having least positive argument and maximum positive arguments in this region are the points of contact of tangents drawn from origin to the circle

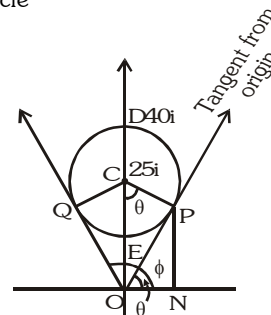
Here  $\theta$  = least positive argument

and  $\phi$  = maximum positive argument

$$\therefore \text{In } \triangle OCP, OP = \sqrt{(OC)^2 - (CP)^2} = \sqrt{(25)^2 - (15)^2} = 20$$

$$\text{and } \sin \theta = \frac{OP}{OC} = \frac{20}{25} = \frac{4}{5}$$

$$\therefore \tan \theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right)$$



Thus, complex number at  $P$  has modulus 20 and argument  $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$$\therefore z_p = 20(\cos \theta + i \sin \theta) = 20\left(\frac{3}{5} + i \frac{4}{5}\right)$$

$$\therefore z_p = 12 + 16i$$

$$\text{Similarly } z_q = -12 + 16i$$

From the figure,  $E$  is the point with least modulus and  $D$  is the point with maximum modulus.

$$\text{Hence, } z_E = \overline{OE} = \overline{OC} - \overline{EC} = 25i - 15i = 10i$$

$$\text{and } z_D = \overline{OD} = \overline{OC} + \overline{CD} = 25i + 15i = 40i$$

**Do yourself - 4 :**

- (i) Find the distance between two complex numbers  $z_1 = 2 + 3i$  &  $z_2 = 7 - 9i$  on the complex plane.  
 (ii) Find the locus of  $|z - 2 - 3i| = 1$ .  
 (iii) If  $z$  is a complex number, then  $z^2 + \bar{z}^2 = 2$  represents -  
 (A) a circle (B) a straight line (C) a hyperbola (D) an ellipse

**(c) Trigonometric / Polar Representation :**

$z = r(\cos \theta + i \sin \theta)$  where  $|z| = r$  ;  $\arg z = \theta$  ;  $\bar{z} = r(\cos \theta - i \sin \theta)$

**Note :**  $\cos \theta + i \sin \theta$  is also written as  $\text{CiS } \theta$ .

**Euler's formula :**

The formula  $e^{ix} = \cos x + i \sin x$  is called Euler's formula.

It was introduced by Euler in 1748, and is used as a method of expressing complex numbers.

Also  $\cos x = \frac{e^{ix} + e^{-ix}}{2}$  &  $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$  are known as Euler's identities.

**(d) Exponential Representation :**

Let  $z$  be a complex number such that  $|z| = r$  &  $\arg z = \theta$ , then  $z = r.e^{i\theta}$

**Illustration 11 :** Express the following complex numbers in polar and exponential form :

$$(i) \frac{1+3i}{1-2i} \quad (ii) \frac{i-1}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}}$$

**Solution :** (i) Let  $z = \frac{1+3i}{1-2i} = \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i} = -1 + i$

$$|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\tan \alpha = \left| \frac{1}{-1} \right| = 1 = \tan \frac{\pi}{4} \Rightarrow \alpha = \frac{\pi}{4}$$

$\therefore \text{Re}(z) < 0$  and  $\text{Im}(z) > 0 \Rightarrow z$  lies in second quadrant.

$$\therefore \theta = \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\text{Hence Polar form is } z = \sqrt{2} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

$$\text{and exponential form is } z = \sqrt{2} e^{3\pi/4}$$

$$(ii) \text{ Let } z = \frac{i-1}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} = \frac{i-1}{\frac{1}{2} + \frac{i\sqrt{3}}{2}} = \frac{2(i-1)}{(1+i\sqrt{3})}$$

$$\Rightarrow z = \frac{2(i-1)}{(1+i\sqrt{3})} \times \frac{(1-i\sqrt{3})}{(1-i\sqrt{3})} \Rightarrow z = \left( \frac{\sqrt{3}-1}{2} \right) + i \left( \frac{\sqrt{3}+1}{2} \right)$$

$\therefore \text{Re}(z) > 0$  and  $\text{Im}(z) > 0 \Rightarrow z$  lies in first quadrant.

$$\therefore |z| = \sqrt{\left( \frac{\sqrt{3}-1}{2} \right)^2 + \left( \frac{\sqrt{3}+1}{2} \right)^2} = \sqrt{\frac{2(3+1)}{4}} = \sqrt{2}$$

$$\tan \theta = \left| \frac{\sqrt{3}+1}{\sqrt{3}-1} \right| = \tan \frac{5\pi}{12} \Rightarrow \alpha = \frac{5\pi}{12}$$

$$\text{Hence Polar form is } z = \sqrt{2} \left( \cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right)$$

$$\text{and exponential form is } z = \sqrt{2} e^{5\pi/12}$$



(A)  $-1$                       (B)  $1$                       (C)  $0$                       (D)  $\infty$

**Ans. (A)**

(i)  $-2 + 2i$       (ii)  $-1 - \sqrt{3}i$       (iii)  $\frac{(1+7i)}{(2-i)^2}$       (iv)  $(1 - \cos\theta + i\sin\theta), \theta \in (0, \pi)$

(a)  $z + \bar{z} = 2 \operatorname{Re}(z)$       (b)  $z - \bar{z} = 2i \operatorname{Im}(z)$       (c)  $\overline{\bar{z}} = z$

(d)  $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$       (e)  $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(f)  $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ . In general  $\overline{z_1 z_2 \dots z_n} = \bar{z}_1 \cdot \bar{z}_2 \dots \bar{z}_n$

(g)  $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$  ;  $z_2 \neq 0$       (h) If  $f(\alpha + i\beta) = x + iy \Rightarrow f(\alpha - i\beta) = x - iy$

(a)  $|z| \geq 0$  (b)  $|z| \geq \operatorname{Re}(z)$  (c)  $|z| \geq \operatorname{Im}(z)$

(d)  $|z| = |\bar{z}| = |-z| = |-\bar{z}|$  (e)  $z\bar{z} = |z|^2$

(f)  $|z_1 z_2| = |z_1| \cdot |z_2|$ . In general  $|z_1 z_2 \dots z_n| = |z_1| \cdot |z_2| \dots |z_n|$

(g)  $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \quad z_2 \neq 0$

(h)  $|z^n| = |z|^n, \quad n \in \mathbb{I}$

(i)  $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \bar{z}_2)$

(j)  $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\alpha - \beta)$ , where  $\alpha, \beta$  are  $\arg(z_1), \arg(z_2)$  respectively.

(k)  $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

(l)  $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$  [Triangle Inequality]

(m)  $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$  [Triangle Inequality]

## 8. IMPORTANT PROPERTIES OF AMPLITUDE :

(a)  $\text{amp}(z_1 \cdot z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi ; k \in \mathbb{I}$

(b)  $\text{amp} \left( \frac{z_1}{z_2} \right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi ; k \in \mathbb{I}$

(c)  $\text{amp}(z^n) = n \text{amp}(z) + 2k\pi ; n, k \in \mathbb{I}$

where proper value of  $k$  must be chosen so that RHS lies in  $(-\pi, \pi]$ .

**Illustration 13 :** Find  $\text{amp } z$  and  $|z|$  if  $z = \left[ \frac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)} \right]^2$ .

**Solution :**  $\text{amp } z = 2 \left[ \text{amp}(3+4i) + \text{amp}(1+i) + \text{amp}(1+\sqrt{3}i) - \text{amp}(1-i) - \text{amp}(4-3i) - \text{amp}(2i) \right] + 2k\pi$   
 where  $k \in \mathbb{I}$  and  $k$  chosen so that  $\text{amp } z$  lies in  $(-\pi, \pi]$ .

$$\Rightarrow \text{amp } z = 2 \left[ \tan^{-1} \frac{4}{3} + \frac{\pi}{4} + \frac{\pi}{3} - \left( -\frac{\pi}{4} \right) - \tan^{-1} \left( -\frac{3}{4} \right) - \frac{\pi}{2} \right] + 2k\pi$$

$$\Rightarrow \text{amp } z = 2 \left[ \tan^{-1} \frac{4}{3} + \cot^{-1} \frac{4}{3} + \frac{\pi}{3} \right] + 2k\pi \Rightarrow \text{amp } z = 2 \left[ \frac{\pi}{2} + \frac{\pi}{3} \right] + 2k\pi$$

$$\Rightarrow \text{amp } z = -\frac{\pi}{3} \quad [\text{at } k = -1]$$

Ans.

Also,

$$|z| = \left| \frac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)} \right|^2 \Rightarrow |z| = \left( \frac{|3+4i||1+i||1+\sqrt{3}i|}{|1-i||4-3i||2i|} \right)^2$$

$$\Rightarrow |z| = \left( \frac{5 \times \sqrt{2} \times 2}{\sqrt{2} \times 5 \times 2} \right)^2 = 1$$

Ans.

**Aliter**

$$z = \left[ \frac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)} \right]^2 \Rightarrow z = \left[ -\frac{\sqrt{3}+i}{2} \right]^2 \Rightarrow z = \frac{2-2\sqrt{3}i}{4} = \frac{1}{2} - \frac{\sqrt{3}i}{2}$$

Hence  $|z| = 1$ ,  $\text{amp}(z) = -\frac{\pi}{3}$ .

**Illustration 14 :** If  $\left| \frac{z-i}{z+i} \right| = 1$ , then locus of  $z$  is -

(A) x-axis

(B) y-axis

(C)  $x = 1$ (D)  $y = 1$ 

**Solution :** We have,  $\left| \frac{z-i}{z+i} \right| = 1 \Rightarrow \left| \frac{x+i(y-1)}{x+i(y+1)} \right| = 1$

$$\Rightarrow \frac{|x+i(y-1)|^2}{|x+i(y+1)|^2} = 1 \Rightarrow x^2 + (y-1)^2 = x^2 + (y+1)^2 \Rightarrow 4y = 0; y = 0, \text{ which is x-axis}$$

Ans. (A)

**Illustration 15 :** If  $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$  then  $\left( \frac{z_1}{z_2} \right)$  is -

(A) zero or purely imaginary

(B) purely imaginary

(C) purely real

(D) none of these

**Solution :** Here let  $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ ,  $|z_1| = r_1$

$$z_2 = r_2 (\cos \theta_2 + i \sin \theta_2), |z_2| = r_2$$

$$\begin{aligned} \therefore |z_1 + z_2|^2 &= |(r_1 \cos \theta_1 + r_2 \cos \theta_2) + i(r_1 \sin \theta_1 + r_2 \sin \theta_2)|^2 \\ &= r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2) = |z_1|^2 + |z_2|^2 \text{ if } \cos(\theta_1 - \theta_2) = 0 \end{aligned}$$

$$\therefore \theta_1 - \theta_2 = \pm \frac{\pi}{2}$$

$$\Rightarrow \text{amp}(z_1) - \text{amp}(z_2) = \pm \frac{\pi}{2} \Rightarrow \text{amp}\left(\frac{z_1}{z_2}\right) = \pm \frac{\pi}{2} \Rightarrow \frac{z_1}{z_2} \text{ is purely imaginary} \quad \text{Ans. (B)}$$

**Illustration 16 :**  $z_1$  and  $z_2$  are two complex numbers such that  $\frac{z_1 - 2z_2}{2 - z_1 z_2}$  is unimodular (whose modulus is one), while

$z_2$  is not unimodular. Find  $|z_1|$ .

**Solution :** Here  $\left| \frac{z_1 - 2z_2}{2 - z_1 z_2} \right| = 1 \Rightarrow \left| \frac{z_1 - 2z_2}{2 - z_1 z_2} \right| = 1$

$$\Rightarrow |z_1 - 2z_2| = |2 - z_1 z_2| \Rightarrow |z_1 - 2z_2|^2 = |2 - z_1 z_2|^2$$

$$\Rightarrow (z_1 - 2z_2)(\overline{z_1 - 2z_2}) = (2 - z_1 z_2)(\overline{2 - z_1 z_2})$$

$$\Rightarrow (z_1 - 2z_2)(\overline{z_1} - 2\overline{z_2}) = (2 - z_1 z_2)(2 - \overline{z_1} \overline{z_2})$$

$$\Rightarrow z_1 \overline{z_1} - 2z_1 \overline{z_2} - 2z_2 \overline{z_1} + 4z_2 \overline{z_2} = 4 - 2\overline{z_1} z_2 - 2z_1 \overline{z_2} + z_1 \overline{z_1} z_2 \overline{z_2}$$

$$\Rightarrow |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2 |z_2|^2 \Rightarrow |z_1|^2 - |z_1|^2 |z_2|^2 + 4|z_2|^2 - 4 = 0$$

$$\Rightarrow (|z_1|^2 - 4)(1 - |z_2|^2) = 0$$

But  $|z_2| \neq 1$  (given)

$$\therefore |z_1|^2 = 4$$

Hence,  $|z_1| = 2$ .

**Illustration 17 :** The locus of the complex number  $z$  in argand plane satisfying the inequality

$$\log_{1/2} \left( \frac{|z-1|+4}{3|z-1|-2} \right) > 1 \quad \left( \text{where } |z-1| \neq \frac{2}{3} \right) \text{ is -}$$

(A) a circle (B) interior of a circle (C) exterior of a circle (D) none of these

**Solution :** We have,  $\log_{1/2} \left( \frac{|z-1|+4}{3|z-1|-2} \right) > 1 = \log_{1/2} \left( \frac{1}{2} \right)$

$$\Rightarrow \frac{|z-1|+4}{3|z-1|-2} < \frac{1}{2} \quad \left[ \because \log_a x \text{ is a decreasing function if } a < 1 \right]$$

$$\Rightarrow 2|z-1|+8 < 3|z-1|-2 \quad \text{as } |z-1| > 2/3$$

$$\Rightarrow |z-1| > 10$$

which is exterior of a circle.

**Ans. (C)**

**Illustration 18 :** If  $\left| z - \frac{4}{z} \right| = 2$ , then the greatest value of  $|z|$  is -

(A)  $1 + \sqrt{2}$  (B)  $2 + \sqrt{2}$  (C)  $\sqrt{3} + 1$  (D)  $\sqrt{5} + 1$

**Solution :** We have  $|z| = \left| z - \frac{4}{z} + \frac{4}{z} \right| \leq \left| z - \frac{4}{z} \right| + \left| \frac{4}{z} \right| = 2 + \frac{4}{|z|}$

$$\Rightarrow |z|^2 \leq 2|z| + 4 \Rightarrow (|z| - 1)^2 \leq 5$$

$$\Rightarrow |z| - 1 \leq \sqrt{5} \Rightarrow |z| \leq \sqrt{5} + 1$$

Therefore, the greatest value of  $|z|$  is  $\sqrt{5} + 1$ .

**Ans. (D)**

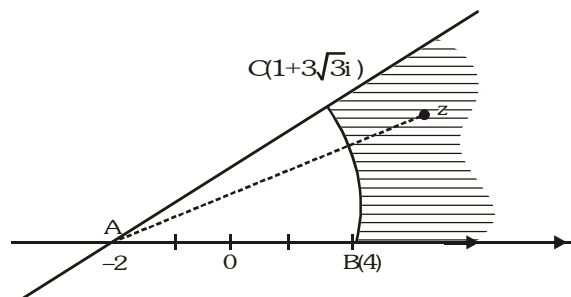
**Illustration 19 :** Shaded region is given by -

(A)  $|z + 2| \geq 6, 0 \leq \arg(z) \leq \frac{\pi}{6}$

(B)  $|z + 2| \geq 6, 0 \leq \arg(z) \leq \frac{\pi}{3}$

(C)  $|z + 2| \leq 6, 0 \leq \arg(z) \leq \frac{\pi}{2}$

(D) None of these



**Solution :** Note that  $AB = 6$  and  $1 + 3\sqrt{3}i = -2 + 3 + 3\sqrt{3}i = -2 + 6\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -2 + 6\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

$$\therefore \angle BAC = \frac{\pi}{3}$$

Thus, shaded region is given by  $|z + 2| \geq 6$  and  $0 \leq \arg(z + 2) \leq \frac{\pi}{3}$

**Ans. (C)**

**Do yourself - 6 :**

(i) The inequality  $|z - 4| < |z - 2|$  represents region given by -

(A)  $\operatorname{Re}(z) > 0$

(B)  $\operatorname{Re}(z) < 0$

(C)  $\operatorname{Re}(z) > 3$

(D) none

(ii) If  $z = re^{i\theta}$ , then the value of  $|e^z|$  is equal to -

(A)  $e^{-r \cos \theta}$

(B)  $e^{r \cos \theta}$

(C)  $e^{r \sin \theta}$

(D)  $e^{-r \sin \theta}$

## 9. SECTION FORMULA AND COORDINATES OF ORTHOCENTRE, CENTROID, CIRCUMCENTRE, INCENTRE OF A TRIANGLE :

If  $z_1$  &  $z_2$  are two complex numbers then the complex number  $z = \frac{nz_1 + mz_2}{m + n}$  divides the join of  $z_1$  &  $z_2$  in the ratio  $m : n$ .

**Note :**

(i) If  $a, b, c$  are three real numbers such that  $az_1 + bz_2 + cz_3 = 0$ ; where  $a + b + c = 0$  and  $a, b, c$  are not all simultaneously zero, then the complex numbers  $z_1, z_2$  &  $z_3$  are collinear.

(ii) If the vertices  $A, B, C$  of a triangle represent the complex numbers  $z_1, z_2, z_3$  respectively, then :

- Centroid of the  $\Delta ABC = \frac{z_1 + z_2 + z_3}{3}$

- Orthocentre of the  $\Delta ABC =$

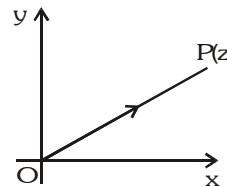
$$\frac{(a \sec A)z_1 + (b \sec B)z_2 + (c \sec C)z_3}{a \sec A + b \sec B + c \sec C} \quad \text{or} \quad \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$$

- Incentre of the  $\Delta ABC = \frac{(az_1 + bz_2 + cz_3)}{(a + b + c)}$

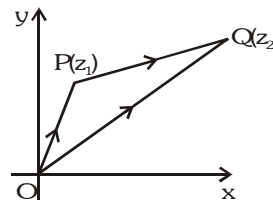
- Circumcentre of the  $\Delta ABC = \frac{(z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C)}{(\sin 2A + \sin 2B + \sin 2C)}$

## 10. VECTORIAL REPRESENTATION OF A COMPLEX NUMBER :

- (a) In complex number every point can be represented in terms of position vector. If the point P represents the complex number  $z$  then,  $\vec{OP} = z$  &  $|\vec{OP}| = |z|$ .

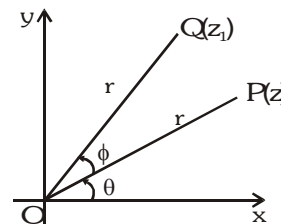


- (b) If  $P(z_1)$  &  $Q(z_2)$  be two complex numbers on argand plane then  $\vec{PQ}$  represents complex number  $z_2 - z_1$ .



**Note :**

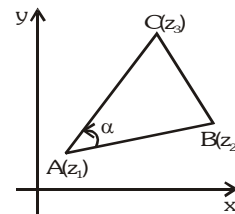
- (i) If  $\vec{OP} = z = r e^{i\theta}$  then  $\vec{OQ} = z_1 = r e^{i(\theta + \phi)} = z \cdot e^{i\phi}$ . If  $\vec{OP}$  and  $\vec{OQ}$  are of unequal magnitude then  $\vec{OQ} = \vec{OP} e^{i\phi}$  i.e.  $\frac{z_1}{|z_1|} = \frac{z}{|z|} e^{i\phi}$



- (ii) In general, if  $z_1, z_2, z_3$  be the three vertices of  $\Delta ABC$  then

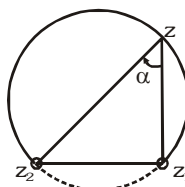
$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\alpha}. \text{ Here } \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \alpha.$$

- (iii) Note that the locus of  $z$  satisfying  $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$  is:



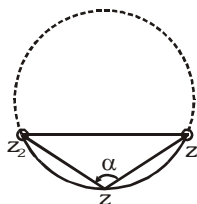
Case (a)  $0 < \alpha < \pi/2$

Locus is major arc of circle as shown excluding  $z_1$  &  $z_2$



Case (b)  $\frac{\pi}{2} < \alpha < \pi$

Locus is minor arc of circle as shown excluding  $z_1$  &  $z_2$



- (iv) If A, B, C & D are four points representing the complex numbers

$$z_1, z_2, z_3 \text{ \& } z_4 \text{ then } AB \parallel CD \text{ if } \frac{z_4 - z_3}{z_2 - z_1} \text{ is purely real ;}$$

$$AB \perp CD \text{ if } \frac{z_4 - z_3}{z_2 - z_1} \text{ is purely imaginary.}$$

- (v) If  $z_1, z_2, z_3$  are the vertices of an equilateral triangle where  $z_0$  is its circumcentre then

$$(1) \quad z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0 \quad (2) \quad z_1^2 + z_2^2 + z_3^2 = 3 z_0^2$$

**Illustration 20:** Complex numbers  $z_1, z_2, z_3$  are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that  $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$ .

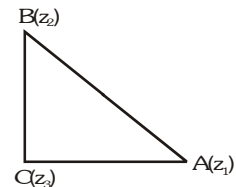
**Solution :** In the isosceles triangle ABC,  $AC = BC$  and  $BC \perp AC$ . It means that AC is rotated through angle  $\pi/2$  to occupy the position BC.

$$\text{Hence we have, } \frac{z_2 - z_3}{z_1 - z_3} = e^{+i\pi/2} = +i \Rightarrow z_2 - z_3 = +i(z_1 - z_3)$$

$$\Rightarrow z_2^2 + z_3^2 - 2z_2z_3 = -(z_1^2 + z_3^2 - 2z_1z_3)$$

$$\begin{aligned} \Rightarrow z_1^2 + z_2^2 - 2z_1z_2 &= 2z_1z_3 + 2z_2z_3 - 2z_1z_2 - 2z_3^2 \\ &= 2(z_1 - z_3)(z_3 - z_2) \end{aligned}$$

$$\Rightarrow (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$



**Illustration 21:** If the vertices of a square ABCD are  $z_1, z_2, z_3$  &  $z_4$  then find  $z_3$  &  $z_4$  in terms of  $z_1$  &  $z_2$ .

**Solution :** Using vector rotation at angle A

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\frac{\pi}{4}}$$

$$\therefore |z_3 - z_1| = AC \text{ and } |z_2 - z_1| = AB$$

$$\text{Also } AC = \sqrt{2} AB$$

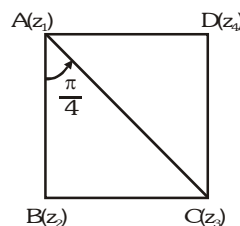
$$\therefore |z_3 - z_1| = \sqrt{2} |z_2 - z_1|$$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$\Rightarrow z_3 - z_1 = (z_2 - z_1)(1 + i)$$

$$\Rightarrow z_3 = z_1 + (z_2 - z_1)(1 + i)$$

$$\text{Similarly } z_4 = z_2 + (1 + i)(z_1 - z_2)$$



**Illustration 22 :** Plot the region represented by  $\frac{\pi}{3} \leq \arg\left(\frac{z+1}{z-1}\right) \leq \frac{2\pi}{3}$  in the Argand plane.

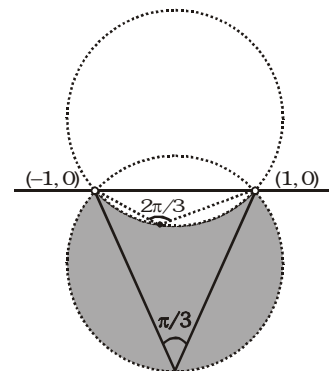
**Solution :** Let us take  $\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$ , clearly  $z$  lies on the minor arc of the circle passing through  $(1, 0)$  and  $(-1, 0)$ . Similarly,

$\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{3}$  means that ' $z$ ' is lying on the major arc of the

circle passing through  $(1, 0)$  and  $(-1, 0)$ . Now if we take any point in the region included between two arcs say  $P_1(z_1)$  we get

$$\frac{\pi}{3} \leq \arg\left(\frac{z+1}{z-1}\right) \leq \frac{2\pi}{3}$$

Thus  $\frac{\pi}{3} \leq \arg\left(\frac{z+1}{z-1}\right) \leq \frac{2\pi}{3}$  represents the shaded region (excluding points  $(1, 0)$  and  $(-1, 0)$ ).



**Do yourself - 7 :**

- (i) A complex number  $z = 3 + 4i$  is rotated about another fixed complex number  $z_1 = 1 + 2i$  in anticlockwise direction by  $45^\circ$  angle. Find the complex number represented by new position of  $z$  in argand plane.
- (ii) If  $A, B, C$  are three points in argand plane representing the complex number  $z_1, z_2, z_3$  such that  $z_1 = \frac{\lambda z_2 + z_3}{\lambda + 1}$ , where  $\lambda \in \mathbb{R}$ , then find the distance of point  $A$  from the line joining points  $B$  and  $C$ .
- (iii) If  $A(z_1), B(z_2), C(z_3)$  are vertices of  $\triangle ABC$  in which  $\angle ABC = \frac{\pi}{4}$  and  $\frac{AB}{BC} = \sqrt{2}$ , then find  $z_2$  in terms of  $z_1$  and  $z_3$ .
- (iv) If  $a$  &  $b$  are real numbers between 0 and 1 such that the points  $z_1 = a + i, z_2 = 1 + bi$  and  $z_3 = 0$  form an equilateral triangle then  $a$  and  $b$  are equal to :-  
 (A)  $a = b = 1/2$       (B)  $a = b = 2 - \sqrt{3}$       (C)  $a = b = -2 + \sqrt{3}$       (D)  $a = b = \sqrt{2} - 1$
- (v) If  $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ , find locus of  $z$ .

**11. DE'MOIVRE'S THEOREM :**

The value of  $(\cos\theta + i\sin\theta)^n$  is  $\cos n\theta + i\sin n\theta$  if 'n' is integer & it is one of the values of  $(\cos\theta + i\sin\theta)^n$  if  $n$  is a rational number of the form  $p/q$ , where  $p$  &  $q$  are co-prime.

**Note :** Continued product of the roots of a complex quantity should be determined by using theory of equations.

**Illustration 23:** If  $\cos\alpha + \cos\beta + \cos\gamma = 0$  and also  $\sin\alpha + \sin\beta + \sin\gamma = 0$ , then prove that

- (a)  $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$   
 (b)  $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$   
 (c)  $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

**Solution :**

Let  $z_1 = \cos\alpha + i\sin\alpha, z_2 = \cos\beta + i\sin\beta$  &  $z_3 = \cos\gamma + i\sin\gamma$ .  
 $\therefore z_1 + z_2 + z_3 = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$   
 $= 0 + i \cdot 0 = 0 \dots\dots\dots (i)$

(a) Also  $\frac{1}{z_1} = (\cos\alpha + i\sin\alpha)^{-1} = \cos\alpha - i\sin\alpha$

$$\frac{1}{z_2} = \cos\beta - i\sin\beta, \frac{1}{z_3} = \cos\gamma - i\sin\gamma$$

$$\therefore \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma) \dots\dots\dots (ii)$$

$$= 0 - i \cdot 0 = 0$$

$$\text{Now } z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$$

$$= 0 - 2z_1z_2z_3\left(\frac{1}{z_3} + \frac{1}{z_1} + \frac{1}{z_2}\right) = 0 - 2z_1z_2z_3 \cdot 0 = 0 \quad \{\text{using (i) and (ii)}\}$$

$$\text{or } (\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$$

$$\text{or } \cos 2\alpha + i\sin 2\alpha + \cos 2\beta + i\sin 2\beta + \cos 2\gamma + i\sin 2\gamma = 0 + i \cdot 0$$

Equating real and imaginary parts on both sides,

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0 \text{ and } \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$$

(b) If  $z_1 + z_2 + z_3 = 0$  then  $z_1^3 + z_2^3 + z_3^3 = 3z_1z_2z_3$

$$\therefore (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$$

$$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$

or  $\cos 3\alpha + i\sin 3\alpha + \cos 3\beta + i\sin 3\beta + \cos 3\gamma + i\sin 3\gamma$

$$= 3\{\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)\}$$

Equating imaginary parts on both sides,  $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

(c) Equating real parts on both sides,  $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

### Do yourself - 8 :

(i) If  $z_r = \cos \frac{2r\pi}{5} + i\sin \frac{2r\pi}{5}$ ,  $r = 0, 1, 3, 4, \dots$ , then  $z_1 z_2 z_3 z_4 z_5$  is equal to -

(A) -1

(B) 0

(C) 1

(D) none of these

(ii) If  $(x - 1)^4 - 16 = 0$ , then the sum of nonreal complex values of  $x$  is -

(A) 2

(B) 0

(C) 4

(D) none of these

(iii) If  $(\sqrt{3} - i)^n = 2^n$ ,  $n \in \mathbb{Z}$ , then  $n$  is a multiple of -

(A) 6

(B) 10

(C) 9

(D) 12

### 12. CUBE ROOT OF UNITY :

(a) The cube roots of unity are  $1$ ,  $\frac{-1 + i\sqrt{3}}{2}(\omega)$ ,  $\frac{-1 - i\sqrt{3}}{2}(\omega^2)$ .

(b) If  $\omega$  is one of the imaginary cube roots of unity then  $1 + \omega + \omega^2 = 0$ . In general  $1 + \omega^r + \omega^{2r} = 0$ ; where  $r \in \mathbb{I}$  but is not the multiple of 3 &  $1 + \omega^r + \omega^{2r} = 3$  if  $r = 3\lambda$ ;  $\lambda \in \mathbb{I}$

(c) In polar form the cube roots of unity are :

$$1 = \cos 0 + i\sin 0; \quad \omega = \cos \frac{2\pi}{3} + i\sin \frac{2\pi}{3}, \quad \omega^2 = \cos \frac{4\pi}{3} + i\sin \frac{4\pi}{3}$$

(d) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.

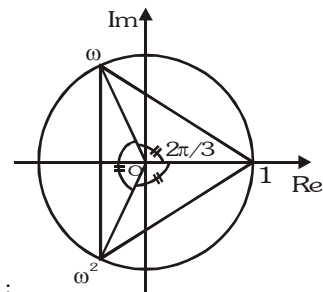
(e) The following factorisation should be remembered :

(a, b, c  $\in \mathbb{R}$  &  $\omega$  is the cube root of unity)

$$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b); \quad x^2 + x + 1 = (x - \omega)(x - \omega^2);$$

$$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b);$$

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$$



**Illustration 24 :** If  $\alpha$  &  $\beta$  are imaginary cube roots of unity then  $\alpha^n + \beta^n$  is equal to -

(A)  $2\cos \frac{2n\pi}{3}$

(B)  $\cos \frac{2n\pi}{3}$

(C)  $2i \sin \frac{2n\pi}{3}$

(D)  $i \sin \frac{2n\pi}{3}$

**Solution :**

$$\alpha = \frac{\cos 2\pi}{3} + \frac{i\sin 2\pi}{3}$$

$$\beta = \frac{\cos 2\pi}{3} - \frac{i\sin 2\pi}{3}$$

$$\alpha^n + \beta^n = \left( \frac{\cos 2\pi}{3} + \frac{i\sin 2\pi}{3} \right)^n + \left( \frac{\cos 2\pi}{3} - \frac{i\sin 2\pi}{3} \right)^n$$

$$= \left( \frac{\cos 2n\pi}{3} + \frac{i\sin 2n\pi}{3} \right) + \left( \frac{\cos 2n\pi}{3} - i\sin \left( \frac{2n\pi}{3} \right) \right) = 2\cos \left( \frac{2n\pi}{3} \right)$$

**Ans. (A)**



**Illustration 25** : If  $\alpha, \beta, \gamma$  are roots of  $x^3 - 3x^2 + 3x + 7 = 0$  (and  $\omega$  is imaginary cube root of unity), then find the

value of  $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1}$ .

**Solution :**

We have  $x^3 - 3x^2 + 3x + 7 = 0$

$$\therefore (x - 1)^3 + 8 = 0$$

$$\therefore (x - 1)^3 = (-2)^3$$

$$\Rightarrow \left( \frac{x-1}{-2} \right)^3 = 1 \Rightarrow \frac{x-1}{-2} = (1)^{1/3} = 1, \omega, \omega^2 \text{ (cube roots of unity)}$$

$$\therefore \quad x = -1, 1 - 2\omega, 1 - 2\omega^2$$

Here  $\alpha = -1$ ,  $\beta = 1 - 2\omega$ ,  $\gamma = 1 - 2\omega^2$

$$\therefore \alpha - 1 = -2, \beta - 1 = -2\omega, \gamma - 1 = -2\omega^2$$

$$\text{Then } \frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = \left( \frac{-2}{-2\omega} \right) + \left( \frac{-2\omega}{-2\omega^2} \right) + \left( \frac{-2\omega^2}{-2} \right) = \frac{1}{\omega} + \frac{1}{\omega} + \omega^2 = \omega^2 + \omega^2 + \omega^2$$

Therefore  $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = 3\omega^2$ .

**Ans.**

Do yourself - 9 :

(i) If  $\omega$  is an imaginary cube root of unity, then  $(1 + \omega - \omega^2)^2$  equals : -

(A)  $\omega$

(B)  $-4\omega$

(C)  $\omega^2$

(D)  $4\omega$

(ii) If  $\omega$  is a non real cube root of unity, then the expression  $(1 - \omega)(1 - \omega^2)(1 + \omega^4)(1 + \omega^8)$  is equal to : -

(A) 0

(B) 3

(C) 1

(D) 2

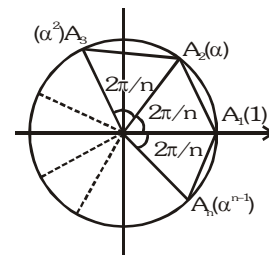
### 13. $n^{\text{th}}$ ROOTS OF UNITY :

If  $1, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-1}$  are the  $n, n^{\text{th}}$  root of unity then :

(a) They are in G.P. with common ratio  $e^{i(2\pi/n)}$

(b) Their arguments are in A.P. with common difference  $\frac{2\pi}{n}$

(c) The points represented by  $n$ ,  $n^{\text{th}}$  roots of unity are located at the vertices of a regular polygon of  $n$  sides inscribed in a unit circle having center at origin, one vertex being on positive real axis.



(d)  $1^p + \alpha_1^p + \alpha_2^p + \dots + \alpha_{n-1}^p = 0$  if  $p$  is not an integral multiple of  $n$   
 $= n$  if  $p$  is an integral multiple of  $n$

(e)  $(1 - \alpha_1)(1 - \alpha_2) \dots (1 - \alpha_{n-1}) = n$

(f)  $(1 + \alpha_1)(1 + \alpha_2) \dots (1 + \alpha_{n-1}) = 0$  if  $n$  is even and  
 $= 1$  if  $n$  is odd.

(g)  $1, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-1} = 1$  or  $-1$  according as  $n$  is odd or even.

**Illustration 26:** Find the value  $\sum_{k=1}^6 \left( \sin \frac{2\pi k}{7} - \cos \frac{2\pi k}{7} \right)$

**Solution :** 
$$\sum_{k=1}^6 \left( \sin \frac{2\pi k}{7} \right) - \sum_{k=1}^6 \left( \cos \frac{2\pi k}{7} \right) = \sum_{k=1}^6 \sin \frac{2\pi k}{7} - \sum_{k=0}^6 \cos \frac{2\pi k}{7} + 1$$

$$= \sum_{k=0}^6 (\text{Sum of imaginary part of seven seventh roots of unity})$$

$$-\sum_{k=0}^6 (\text{Sum of real part of seven seventh roots of unity}) + 1 = 0 - 0 + 1 = 1$$

14. THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED :

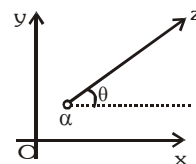
$$(a) \quad \cos \theta + \cos 2\theta + \cos 3\theta + \dots + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\frac{n+1}{2}\theta\right)$$

$$(b) \quad \sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\frac{n+1}{2}\theta\right)$$

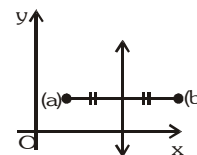
Note : If  $\theta = (2\pi/n)$  then the sum of the above series vanishes.

15. STRAIGHT LINES & CIRCLES IN TERMS OF COMPLEX NUMBERS :

- (a)  $\arg(z-\alpha) = \theta$  is a ray emanating from the complex point  $\alpha$  and inclined at an angle  $\theta$  to the x-axis.

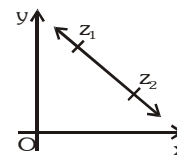


- (b)  $|z-a| = |z-b|$  is the perpendicular bisector of the segment joining a & b.



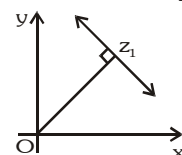
- (c) The equation of a line joining  $z_1$  &  $z_2$  is given by ;

$$z = z_1 + t(z_2 - z_1) \text{ where } t \text{ is a parameter.}$$



- (d)  $z = z_1(1+it)$  where  $t$  is a real parameter, is a line through the point  $z_1$  &

perpendicular to  $z_1$ .

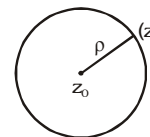


- (e) The equation of a line passing through  $z_1$  &  $z_2$  can be expressed in the determinant form as

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0. \text{ This is also the condition for three complex numbers to be collinear.}$$

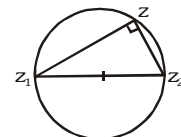
- (f) Complex equation of a straight line through two given points  $z_1$  &  $z_2$  can be written as  $z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + (z_1\bar{z}_2 - \bar{z}_1z_2) = 0$ , which on manipulating takes the form as  $\bar{\alpha}z + \alpha\bar{z} + r = 0$  where  $r$  is real and  $\alpha$  is a non zero complex constant.

- (g) The equation of circle having centre  $z_0$  & radius  $\rho$  is :  $|z - z_0| = \rho$  or  $z\bar{z} - z_0\bar{z} - \bar{z}_0z + \bar{z}_0z_0 - \rho^2 = 0$  which is of the form  $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ ,  $r$  is real centre =  $-\alpha$  & radius =  $\sqrt{\alpha\bar{\alpha} - r}$ . Circle will be real if  $\alpha\bar{\alpha} - r \geq 0$ .



- (h)  $\arg\left(\frac{z - z_2}{z - z_1}\right) = \pm \frac{\pi}{2}$  or  $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

this equation represents the circle described on the line segment joining  $z_1$  &  $z_2$  as diameter.



- (i) Condition for four given points  $z_1, z_2, z_3$  &  $z_4$  to be concyclic is, the number  $\frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1}$  is real. Hence the equation of a circle through 3 non collinear points  $z_1, z_2$  &  $z_3$  can be taken as

$$\frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} \text{ is real } \Rightarrow \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \frac{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}$$

**Miscellaneous Illustration :**

**Illustration 27:** If  $z$  is a point on the Argand plane such that  $|z - 1| = 1$ , then  $\frac{z-2}{z}$  is equal to -

- (A)  $\tan(\arg z)$  (B)  $\cot(\arg z)$  (C)  $i \tan(\arg z)$  (D) none of these

**Solution :**

Since  $|z - 1| = 1$ ,

$$\therefore \text{let } z - 1 = \cos \theta + i \sin \theta$$

Then,  $z - 2 = \cos \theta + i \sin \theta - 1$

$$= -2 \sin^2 \frac{\theta}{2} + 2i \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2i \sin \frac{\theta}{2} \left( \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \dots (i)$$

and  $z = 1 + \cos \theta + i \sin \theta$

$$= 2 \cos^2 \frac{\theta}{2} + 2i \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \cos \frac{\theta}{2} \left( \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \dots (ii)$$

From (i) and (ii), we get  $\frac{z-2}{z} = i \tan \frac{\theta}{2} = i \tan(\arg z) \left( \because \arg z = \frac{\theta}{2} \text{ from (ii)} \right)$

**Ans. (C)**

**Illustration 28:** Let  $a$  be a complex number such that  $|a| < 1$  and  $z_1, z_2, \dots, z_n$  be the vertices of a polygon such that  $z_k = 1 + a + a^2 + \dots + a^k$ , then show that vertices of the polygon lie within the circle

$$\left| z - \frac{1}{1-a} \right| = \frac{1}{|1-a|}.$$

**Solution :**

We have,  $z_k = 1 + a + a^2 + \dots + a^k = \frac{1-a^{k+1}}{1-a}$

$$\Rightarrow z_k - \frac{1}{1-a} = \frac{-a^{k+1}}{1-a} \Rightarrow \left| z_k - \frac{1}{1-a} \right| = \frac{|a|^{k+1}}{|1-a|} < \frac{1}{|1-a|} \quad (\because |a| < 1)$$

$\therefore$  Vertices of the polygon  $z_1, z_2, \dots, z_n$  lie within the circle  $\left| z - \frac{1}{1-a} \right| = \frac{1}{|1-a|}$

**Illustration 29 :** If  $z_1$  and  $z_2$  are two complex numbers and  $C > 0$ , then prove that

$$|z_1 + z_2|^2 \leq (1+C)|z_1|^2 + (1+C^{-1})|z_2|^2$$

**Solution :**

We have to prove that :  $|z_1 + z_2|^2 \leq (1+C)|z_1|^2 + (1+C^{-1})|z_2|^2$

$$\text{i.e. } |z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + \bar{z}_1 z_2 \leq (1+C)|z_1|^2 + (1+C^{-1})|z_2|^2$$

$$\text{or } z_1 \bar{z}_2 + \bar{z}_1 z_2 \leq C|z_1|^2 + C^{-1}|z_2|^2$$

$$\text{or } C|z_1|^2 + \frac{1}{C}|z_2|^2 - z_1 \bar{z}_2 - \bar{z}_1 z_2 \geq 0 \quad (\text{using } \operatorname{Re}(z_1 \bar{z}_2) \leq |z_1 \bar{z}_2|)$$

$$\text{or } \left( \sqrt{C}|z_1| - \frac{1}{\sqrt{C}}|z_2| \right)^2 \geq 0 \quad \text{which is always true.}$$

**Illustration 30 :** If  $\theta \in [\pi/6, \pi/3]$ ,  $i = 1, 2, 3, 4, 5$  and  $z^4 \cos \theta_1 + z^3 \cos \theta_2 + z^2 \cos \theta_3 + z \cos \theta_4 + \cos \theta_5 = 2\sqrt{3}$ , then

show that  $|z| > \frac{3}{4}$

**Solution :**

$$\text{Given that } \cos \theta_1 \cdot z^4 + \cos \theta_2 \cdot z^3 + \cos \theta_3 \cdot z^2 + \cos \theta_4 \cdot z + \cos \theta_5 = 2\sqrt{3}$$

$$\text{or } |\cos \theta_1 \cdot z^4 + \cos \theta_2 \cdot z^3 + \cos \theta_3 \cdot z^2 + \cos \theta_4 \cdot z + \cos \theta_5| = 2\sqrt{3}$$

$$2\sqrt{3} \leq |\cos \theta_1 \cdot z^4| + |\cos \theta_2 \cdot z^3| + |\cos \theta_3 \cdot z^2| + |\cos \theta_4 \cdot z| + |\cos \theta_5|$$

$$\therefore \theta_i \in [\pi/6, \pi/3]$$

$$\therefore \frac{1}{2} \leq \cos \theta_i \leq \frac{\sqrt{3}}{2}$$

$$2\sqrt{3} \leq \frac{\sqrt{3}}{2}|z|^4 + \frac{\sqrt{3}}{2}|z|^3 + \frac{\sqrt{3}}{2}|z|^2 + \frac{\sqrt{3}}{2}|z| + \frac{\sqrt{3}}{2}$$

$$\Rightarrow 3 \leq |z|^4 + |z|^3 + |z|^2 + |z|$$

$$\Rightarrow 3 < |z| + |z|^2 + |z|^3 + |z|^4 + |z|^5 + \dots \infty$$

$$\Rightarrow 3 < \frac{|z|}{1-|z|} \Rightarrow 3 - 3|z| < |z|$$

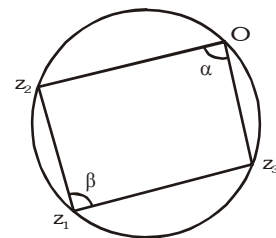
$$\Rightarrow 4|z| > 3 \quad \therefore |z| > \frac{3}{4}$$

**Illustration 31** : If  $z_1, z_2, z_3$  are complex numbers such that  $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$ , show that the points represented by  $z_1, z_2, z_3$  lie on a circle passing through the origin.

**Solution :** We have,  $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} \Rightarrow \frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1} \Rightarrow \frac{z_2 - z_1}{z_1 z_2} = \frac{z_1 - z_3}{z_1 z_3}$

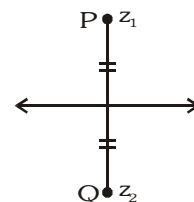
$$\Rightarrow \frac{z_2 - z_1}{z_3 - z_1} = \frac{-z_2}{z_3} \Rightarrow \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(\frac{-z_2}{z_3}\right)$$

$$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \pi + \arg\left(\frac{z_2}{z_3}\right) \Rightarrow \text{or } \beta = \pi - \arg\frac{z_3}{z_2} = \pi - \alpha = \alpha + \beta = \pi$$



Thus the sum of a pair of opposite angle of a quadrilateral is  $180^\circ$ . Hence, the points  $O, z_1, z_2$  and  $z_3$  are the vertices of a cyclic quadrilateral i.e. lie on a circle.

**Illustration 32** : Two given points  $P$  &  $Q$  are the reflection points w.r.t. a given straight line if the given line is the right bisector of the segment  $PQ$ . Prove that the two points denoted by the complex numbers  $z_1$  &  $z_2$  will be the reflection points for the straight line  $\bar{\alpha}z + \alpha\bar{z} + r = 0$  if and only if ;  $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$ , where  $r$  is real and  $\alpha$  is non zero complex constant.



**Solution :** Let  $P(z_1)$  is the reflection point of  $Q(z_2)$  then the perpendicular bisector of  $z_1$  &  $z_2$  must be the line  $\bar{\alpha}z + \alpha\bar{z} + r = 0$  ..... (i)

Now perpendicular bisector of  $z_1$  &  $z_2$  is,  $|z - z_1| = |z - z_2|$

$$\text{or } (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2) \\ -z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1 = -z\bar{z}_2 - z_2\bar{z} + z_2\bar{z}_2 \quad (z\bar{z} \text{ cancels on either side})$$

$$\text{or } (\bar{z}_2 - \bar{z}_1)z + (z_2 - z_1)\bar{z} + z_1\bar{z}_1 - z_2\bar{z}_2 = 0 \quad \dots\dots\dots \text{(ii)}$$

$$\text{Comparing (i) \& (ii) } \frac{\bar{\alpha}}{\bar{z}_2 - \bar{z}_1} = \frac{\alpha}{z_2 - z_1} = \frac{r}{z_1\bar{z}_1 - z_2\bar{z}_2} = \lambda$$

$$\therefore \bar{\alpha} = \lambda(\bar{z}_2 - \bar{z}_1) \quad \dots\dots\dots \text{(iii)} \quad \alpha = \lambda(z_2 - z_1) \quad \dots\dots\dots \text{(iv)}$$

$$r = \lambda(z_1\bar{z}_1 - z_2\bar{z}_2) \quad \dots\dots\dots \text{(v)}$$

Multiplying (iii) by  $z_1$ ; (iv) by  $\bar{z}_2$  and adding

$$\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$$

Note that we could also multiply (iii) by  $z_2$  & (iv) by  $\bar{z}_1$  & add to get the same result.

Hence  $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$

Again, let  $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$  is true w.r.t. the line  $\bar{\alpha}z + \alpha\bar{z} + r = 0$ .

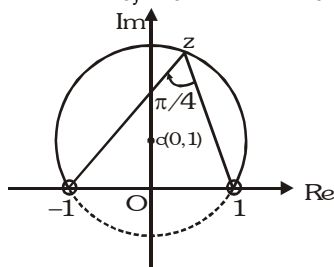
Subtracting  $\bar{\alpha}(z - z_1) + \alpha(\bar{z} - \bar{z}_2) = 0$

or  $|(z - z_1)||\bar{\alpha}| = |\alpha| |(\bar{z} - \bar{z}_2)|$  or  $|z - z_1| = |\bar{z} - \bar{z}_2| = |z - z_2|$

Hence 'z' lies on the perpendicular bisector of joins of  $z_1$  &  $z_2$ .

### ANSWERS FOR DO YOURSELF

- 1 : (i)  $n = 4$  (ii) 0
- 2 : (i)  $-17 + 24i$  (iii)  $\pm(1 - 4i)$
- 3 : (i)  $|z| = 4$ ;  $\text{amp}(z) = \frac{2\pi}{3}$  (ii)  $|z| = 2$ ;  $\text{amp}(z) = -\frac{5\pi}{6}$  (iii)  $|z| = 2$ ;  $\text{amp}(z) = -\frac{\pi}{2}$
- (iv)  $|z| = \frac{1}{\sqrt{2}}$ ;  $\text{amp}(z) = \frac{3\pi}{4}$  (v)  $|z| = 2$ ;  $\text{amp}(z) = \frac{\pi}{3}$
- 4 : (i) 13 units (ii) locus is a circle on complex plane with center at (2,3) and radius 1 unit. (iii) C
- 5 : (i)  $2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ ;  $2\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$  (ii)  $2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ ;  $2e^{i\left(\frac{4\pi}{3}\right)}$
- (iii)  $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ ;  $\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$  (iv)  $2\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)$ ;  $2\sin\left(\frac{\theta}{2}\right)e^{i\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}$
- 6 : (i) C (ii) D
- 7 : (i)  $1 + (2 + 2\sqrt{2})i$  (ii) 0 (iii)  $z_2 = z_3 + i(z_1 - z_3)$  (iv) B
- (v) Locus is all the points on the major arc of circle as shown excluding points 1 & -1.

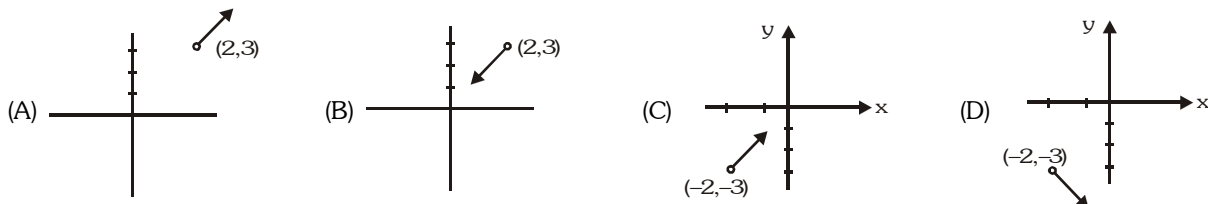


- 8 : (i) C (ii) A (iii) D
- 9 : (i) D (ii) B

**EXERCISE - 01****CHECK YOUR GRASP****SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)**

- The value of the sum  $\sum_{n=1}^{13} (i^n + i^{n+1})$ , where  $i = \sqrt{-1}$ , equals [JEE 98]  
 (A)  $i$  (B)  $i - 1$  (C)  $-i$  (D)  $0$
- The sequence  $S = i + 2i^2 + 3i^3 + \dots$  upto 100 terms simplifies to where  $i = \sqrt{-1}$  -  
 (A)  $50(1 - i)$  (B)  $25i$  (C)  $25(1 + i)$  (D)  $100(1 - i)$
- Let  $i = \sqrt{-1}$ . The product of the real part of the roots of  $z^2 - z = 5 - 5i$  is -  
 (A)  $-25$  (B)  $-6$  (C)  $-5$  (D)  $25$
- If  $z_1 = \frac{1}{a+i}$ ,  $a \neq 0$  and  $z_2 = \frac{1}{1+bi}$ ,  $b \neq 0$  are such that  $z_1 = \bar{z}_2$  then -  
 (A)  $a = 1, b = 1$  (B)  $a = 1, b = -1$  (C)  $a = -1, b = 1$  (D)  $a = -1, b = -1$
- The inequality  $|z - 4| < |z - 2|$  represents the following region -  
 (A)  $\operatorname{Re}(z) > 0$  (B)  $\operatorname{Re}(z) < 0$  (C)  $\operatorname{Re}(z) > 2$  (D) none of these
- If  $(1 + i)(1 + 2i)(1 + 3i) \dots (1 + ni) = \alpha + i\beta$  then  $2 \cdot 5 \cdot 10 \dots (1 + n^2) =$   
 (A)  $\alpha - i\beta$  (B)  $\alpha^2 - \beta^2$  (C)  $\alpha^2 + \beta^2$  (D) none of these
- In the quadratic equation  $x^2 + (p + iq)x + 3i = 0$ ,  $p$  &  $q$  are real. If the sum of the squares of the roots is 8 then :  
 (A)  $p = 3, q = -1$  (B)  $p = -3, q = -1$   
 (C)  $p = 3, q = 1$  or  $p = -3, q = -1$  (D)  $p = -3, q = 1$
- The curve represented by  $\operatorname{Re}(z^2) = 4$  is -  
 (A) a parabola (B) an ellipse  
 (C) a circle (D) a rectangular hyperbola
- Real part of  $e^{e^{i\theta}}$  is -  
 (A)  $e^{\cos \theta} [\cos (\sin \theta)]$  (B)  $e^{\cos \theta} [\cos (\cos \theta)]$  (C)  $e^{\sin \theta} [\sin (\cos \theta)]$  (D)  $e^{\sin \theta} [\sin (\sin \theta)]$
- Let  $z$  and  $\omega$  are two non-zero complex numbers such that  $|z| = |\omega|$  and  $\arg z + \arg \omega = \pi$ , then  $z$  equal to -  
 (A)  $\omega$  (B)  $-\omega$  (C)  $\bar{\omega}$  (D)  $-\bar{\omega}$
- Number of values of  $x$  (real or complex) simultaneously satisfying the system of equations  
 $1 + z + z^2 + z^3 + \dots + z^{17} = 0$  and  $1 + z + z^2 + z^3 + \dots + z^{13} = 0$  is -  
 (A) 1 (B) 2 (C) 3 (D) 4
- If  $|z_1| = 1$ ,  $|z_2| = 2$ ,  $|z_3| = 3$  and  $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$  then the value of  $|z_1 + z_2 + z_3|$  is equal to -  
 (A) 2 (B) 3 (C) 4 (D) 6
- A point 'z' moves on the curve  $|z - 4 - 3i| = 2$  in an argand plane. The maximum and minimum values of  $|z|$  are -  
 (A) 2, 1 (B) 6, 5 (C) 4, 3 (D) 7, 3
- The set of points on the complex plane such that  $z^2 + z + 1$  is real and positive (where  $z = x + iy$ ,  $x, y \in \mathbb{R}$ ) is -  
 (A) Complete real axis only  
 (B) Complete real axis or all points on the line  $2x + 1 = 0$   
 (C) Complete real axis or a line segment joining points  $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$  &  $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$  excluding both.  
 (D) Complete real axis or set of points lying inside the rectangle formed by the lines.  
 $2x + 1 = 0$ ;  $2x - 1 = 0$ ;  $2y - \sqrt{3} = 0$  &  $2y + \sqrt{3} = 0$

15. If  $\omega$  is an imaginary cube root of unity, then  $(1 + \omega - \omega^2)^7$  equals [JEE 98]  
 (A)  $128\omega$  (B)  $-128\omega$  (C)  $128\omega^2$  (D)  $-128\omega^2$
16. If  $i = \sqrt{-1}$ , then  $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$  is equal to : [JEE 99]  
 (A)  $1 - i\sqrt{3}$  (B)  $-1 + i\sqrt{3}$  (C)  $i\sqrt{3}$  (D)  $-i\sqrt{3}$
17. The set of points on an Argand diagram which satisfy both  $|z| \leq 4$  &  $\text{Arg } z = \frac{\pi}{3}$  are lying on -  
 (A) a circle & a line (B) a radius of a circle (C) a sector of a circle (D) an infinite part line
18. If  $\text{Arg}(z - 2 - 3i) = \frac{\pi}{4}$ , then the locus of  $z$  is -



19. The origin and the roots of the equation  $z^2 + pz + q = 0$  form an equilateral triangle if -  
 (A)  $p^2 = 2q$  (B)  $p^2 = q$  (C)  $p^2 = 3q$  (D)  $q^2 = 3p$
20. Points  $z_1$  &  $z_2$  are adjacent vertices of a regular octagon. The vertex  $z_3$  adjacent to  $z_2$  ( $z_3 \neq z_1$ ) can be represented by -  
 (A)  $z_2 + \frac{1}{\sqrt{2}}(1 \pm i)(z_1 + z_2)$  (B)  $z_2 + \frac{1}{\sqrt{2}}(-1 \pm i)(z_1 - z_2)$   
 (C)  $z_2 + \frac{1}{\sqrt{2}}(-1 \pm i)(z_2 - z_1)$  (D) none of these
21.  $\left[\frac{-1+i\sqrt{3}}{2}\right]^6 + \left[\frac{-1-i\sqrt{3}}{2}\right]^6 + \left[\frac{-1+i\sqrt{3}}{2}\right]^5 + \left[\frac{-1-i\sqrt{3}}{2}\right]^5$  is equal to -  
 (A) 1 (B) -1 (C) 2 (D) none of these
22. If  $z$  and  $\omega$  are two non-zero complex numbers such that  $|z\omega| = 1$ , and  $\text{Arg}(z) - \text{Arg}(\omega) = \pi/2$ , then  $\bar{z}\omega$  is equal to -  
 (A) 1 (B) -1 (C)  $i$  (D)  $-i$

**SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)**

23. For two complex numbers  $z_1$  and  $z_2$  :  $(az_1 + b\bar{z}_1)(cz_2 + d\bar{z}_2) = (cz_1 + d\bar{z}_1)(az_2 + b\bar{z}_2)$  if  $(a, b, c, d \in \mathbb{R})$  -  
 (A)  $\frac{a}{b} = \frac{c}{d}$  (B)  $\frac{a}{d} = \frac{b}{c}$  (C)  $|z_1| = |z_2|$  (D)  $\arg(z_1) = \arg(z_2)$
24. Which of the following, loci of  $z$  on the complex plane represents a pair of straight lines ?  
 (A)  $\text{Re}(z^2) = 0$  (B)  $\text{Im}(z^2) = 0$  (C)  $|z| + z = 0$  (D)  $|z - 1| = |z - i|$
25. If the complex numbers  $z_1, z_2, z_3$  represents vertices of an equilateral triangle such that  $|z_1| = |z_2| = |z_3|$ , then which of following is correct ?  
 (A)  $z_1 + z_2 + z_3 \neq 0$  (B)  $\text{Re}(z_1 + z_2 + z_3) = 0$  (C)  $\text{Im}(z_1 + z_2 + z_3) = 0$  (D)  $z_1 + z_2 + z_3 = 0$
26. If  $S$  be the set of real values of  $x$  satisfying the inequality  $1 - \log_2 \frac{|x+1+2i|-2}{\sqrt{2}-1} \geq 0$ , then  $S$  contains -  
 (A)  $[-3, -1]$  (B)  $(-1, 1]$  (C)  $[-2, 2]$  (D)  $[-3, 1]$

27. If  $\arg(z_1 z_2) = 0$  and  $|z_1| = |z_2| = 1$ , then :-

- (A)  $z_1 + z_2 = 0$  (B)  $z_1 z_2 = 1$  (C)  $z_1 = \bar{z}_2$  (D) none of these

28. If the vertices of an equilateral triangle are situated at  $z=0, z=z_1, z=z_2$ , then which of the following is/are true -

- (A)  $|z_1| = |z_2|$  (B)  $|z_1 - z_2| = |z_1|$   
(C)  $|z_1 + z_2| = |z_1| + |z_2|$  (D)  $|\arg z_1 - \arg z_2| = \pi/3$

29. Value(s) of  $(-i)^{1/3}$  is/are -

- (A)  $\frac{\sqrt{3}-i}{2}$  (B)  $\frac{\sqrt{3}+i}{2}$  (C)  $\frac{-\sqrt{3}-i}{2}$  (D)  $\frac{-\sqrt{3}+i}{2}$

30. If centre of square ABCD is at  $z=0$ . If affix of vertex A is  $z_1$ , centroid of triangle ABC is/are -

- (A)  $\frac{z_1}{3}(\cos \pi + i \sin \pi)$  (B)  $4 \left[ \left( \cos \frac{\pi}{2} \right) - i \left( \sin \frac{\pi}{2} \right) \right]$   
(C)  $\frac{z_1}{3} \left[ \left( \cos \frac{\pi}{2} \right) + i \left( \sin \frac{\pi}{2} \right) \right]$  (D)  $\frac{z_1}{3} \left[ \left( \cos \frac{\pi}{2} \right) - i \left( \sin \frac{\pi}{2} \right) \right]$

31. If  $\omega$  is an imaginary cube root of unity, then a root of equation  $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+2 \end{vmatrix} = 0$ , can be :-

- (A)  $x = 1$  (B)  $x = \omega$  (C)  $x = \omega^2$  (D)  $x = 0$

CHECK YOUR GRASP					ANSWER KEY			EXERCISE-1		
Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	A	B	B	D	C	C	D	A	D
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	A	A	D	B	D	C	C	A	C	B
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	A	D	A,D	A,B	B,C,D	A,B	B,C	A,B,D	A,C	C,D
Que.	31									
Ans.	D									



**EXERCISE - 02**

**BRAIN TEASERS**

**SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)**

- On the argand plane, let  $\alpha = -2 + 3z$ ,  $\beta = -2 - 3z$  &  $|z| = 1$ . Then the correct statement is -  
 (A)  $\alpha$  moves on the circle, centre at  $(-2,0)$  and radius 3  
 (B)  $\alpha$  &  $\beta$  describe the same locus  
 (C)  $\alpha$  &  $\beta$  move on different circles  
 (D)  $\alpha - \beta$  moves on a circle concentric with  $|z|=1$
- The value of  $i^n + i^{-n}$ , for  $i = \sqrt{-1}$  and  $n \in \mathbb{I}$  is -  
 (A)  $\frac{2^n}{(1-i)^{2n}} + \frac{(1+i)^{2n}}{2^n}$  (B)  $\frac{(1+i)^{2n}}{2^n} + \frac{(1-i)^{2n}}{2^n}$  (C)  $\frac{(1+i)^{2n}}{2^n} - \frac{2^n}{(1-i)^{2n}}$  (D)  $\frac{2^n}{(1+i)^{2n}} + \frac{2^n}{(1-i)^{2n}}$
- The common roots of the equations  $z^3 + (1+i)z^2 + (1+i)z + i = 0$ , (where  $i = \sqrt{-1}$ ) and  $z^{1993} + z^{1994} + 1 = 0$  are -  
 (where  $\omega$  denotes the complex cube root of unity)  
 (A) 1 (B)  $\omega$  (C)  $\omega^2$  (D)  $\omega^{981}$
- If  $x_r = \text{CiS}\left(\frac{\pi}{2^r}\right)$  for  $1 \leq r \leq n$ ;  $r, n \in \mathbb{N}$  then -  
 (A)  $\lim_{n \rightarrow \infty} \text{Re}\left(\prod_{r=1}^n x_r\right) = -1$  (B)  $\lim_{n \rightarrow \infty} \text{Re}\left(\prod_{r=1}^n x_r\right) = 0$  (C)  $\lim_{n \rightarrow \infty} \text{Im}\left(\prod_{r=1}^n x_r\right) = 1$  (D)  $\lim_{n \rightarrow \infty} \text{Im}\left(\prod_{r=1}^n x_r\right) = 0$
- Let  $z_1, z_2$  be two complex numbers represented by points on the circle  $|z_1| = 1$  and  $|z_2| = 2$  respectively, then -  
 (A)  $\max |2z_1 + z_2| = 4$  (B)  $\min |z_1 - z_2| = 1$  (C)  $\left|z_2 + \frac{1}{z_1}\right| \leq 3$  (D) none of these
- If  $\alpha, \beta$  be any two complex numbers such that  $\left|\frac{\alpha - \beta}{1 - \bar{\alpha}\beta}\right| = 1$ , then which of the following may be true -  
 (A)  $|\alpha| = 1$  (B)  $|\beta| = 1$  (C)  $\alpha = e^{i\theta}, \theta \in \mathbb{R}$  (D)  $\beta = e^{i\theta}, \theta \in \mathbb{R}$
- Let  $z, \omega z$  and  $z + \omega z$  represent three vertices of  $\Delta ABC$ , where  $\omega$  is cube root unity, then -  
 (A) centroid of  $\Delta ABC$  is  $\frac{2}{3}(z + \omega z)$  (B) orthocenter of  $\Delta ABC$  is  $\frac{2}{3}(z + \omega z)$   
 (C)  $ABC$  is an obtuse angled triangle (D)  $ABC$  is an acute angled triangle
- Which of the following complex numbers lies along the angle bisectors of the line -  
 $L_1 : z = (1 + 3\lambda) + i(1 + 4\lambda)$   
 $L_2 : z = (1 + 3\mu) + i(1 - 4\mu)$   
 (A)  $\frac{11}{5} + i$  (B)  $11 + 5i$  (C)  $1 - \frac{3i}{5}$  (D)  $5 - 3i$
- Let  $z$  and  $\omega$  are two complex numbers such that  $|z| \leq 1$ ,  $|\omega| \leq 1$  and  $|z + i\omega| = |z - i\bar{\omega}| = 2$ , then  $z$  equals -  
 (A) 1 or  $i$  (B)  $i$  or  $-i$  (C) 1 or  $-1$  (D)  $i$  or  $-1$
- If  $g(x)$  and  $h(x)$  are two polynomials such that the polynomial  $P(x) = g(x^3) + xh(x^3)$  is divisible by  $x^2 + x + 1$ , then -  
 (A)  $g(1) = h(1) = 0$  (B)  $g(1) = h(1) \neq 0$  (C)  $g(1) = -h(1)$  (D)  $g(1) + h(1) = 0$

BRAIN TEASERS				ANSWER KEY				EXERCISE-2		
Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A,B,D	B,D	B,C	A,D	A,B,C	A,B,C,D	A,C	A,C	C	A,C,D

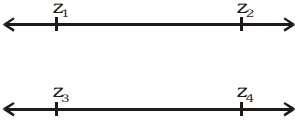
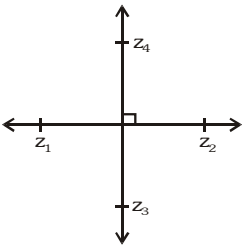
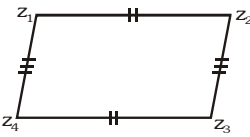
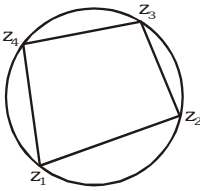
**EXERCISE - 03****MISCELLANEOUS TYPE QUESTIONS****MATCH THE COLUMN**

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

1.	Column-I	Column-II
(A)	If $z$ be the complex number such that $\left z + \frac{1}{z}\right  = 2$ then minimum value of $\frac{ z }{\tan \frac{\pi}{8}}$ is	(p) 0
(B)	$ z  = 1$ & $z^{2n+1} \neq 0$ then $\frac{z^n}{z^{2n}+1} - \frac{\bar{z}^n}{\bar{z}^{2n}+1}$ is equal to	(q) 3
(C)	If $8iz^3 + 12z^2 - 18z + 27i = 0$ then $2 z  =$	(r) 11
(D)	If $z_1, z_2, z_3, z_4$ are the roots of equation $z^4 + z^3 + z^2 + z + 1 = 0$ , then $\prod_{i=1}^4 (z_i + 2)$ is	(s) 1

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

2. Match the figure in column-I with corresponding expression -

	Column-I	Column-II
(A)	 <p>two parallel lines</p>	(p) $\frac{z_4 - z_3}{z_2 - z_1} + \frac{\bar{z}_4 - \bar{z}_3}{\bar{z}_2 - \bar{z}_1} = 0$
(B)	 <p>two perpendicular lines</p>	(q) $\frac{z_2 - z_1}{z_4 - z_3} = \frac{\bar{z}_2 - \bar{z}_1}{\bar{z}_4 - \bar{z}_3}$
(C)	 <p>a parallelogram</p>	(r) $\frac{z_4 - z_1}{z_2 - z_1} \cdot \frac{z_2 - z_3}{z_4 - z_3} = \frac{\bar{z}_4 - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \cdot \frac{\bar{z}_2 - \bar{z}_3}{\bar{z}_4 - \bar{z}_3}$
(D)		(s) $z_1 + z_3 = z_2 + z_4$

## ASSERTION & REASON

These questions contains, Statement I (assertion) and Statement II (reason).

- (A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.  
 (B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.  
 (C) Statement-I is true, Statement-II is false.  
 (D) Statement-I is false, Statement-II is true.

1. **Statement-I** : There are exactly two complex numbers which satisfy the complex equations  $|z - 4 - 5i| = 4$  and  $\text{Arg}(z - 3 - 4i) = \frac{\pi}{4}$  simultaneously.

**Because**

**Statement-II** : A line cuts the circle in atmost two points.

- (A) A (B) B (C) C (D) D

2. Let  $z_1, z_2, z_3$  satisfy  $\left| \frac{z+2}{z-1} \right| = 2$  and  $z_0 = 2$ . Consider least positive arguments wherever required.

**Statement-1** :  $2 \arg \left( \frac{z_1 - z_3}{z_2 - z_3} \right) = \arg \left( \frac{z_1 - z_0}{z_2 - z_0} \right)$ .

and

**Statement-2** :  $z_1, z_2, z_3$  satisfy  $|z - z_0| = 2$ .

- (A) A (B) B (C) C (D) D

3. **Statement-I** : If  $z = i + 2i^2 + 3i^3 + \dots + 32i^{32}$ , then  $z, \bar{z}, -z$  &  $-\bar{z}$  forms the vertices of square on argand plane.

**Because**

**Statement-II** :  $z, \bar{z}, -z, -\bar{z}$  are situated at the same distance from the origin on argand plane.

- (A) A (B) B (C) C (D) D

4. **Statement-I** : If  $z_1 = 9 + 5i$  and  $z_2 = 3 + 5i$  and if  $\arg \left( \frac{z - z_1}{z - z_2} \right) = \frac{\pi}{4}$  then  $|z - 6 - 8i| = 3\sqrt{2}$

**Because**

**Statement-II** : If  $z$  lies on circle having  $z_1$  &  $z_2$  as diameter then  $\arg \left( \frac{z - z_1}{z - z_2} \right) = \frac{\pi}{4}$ .

- (A) A (B) B (C) C (D) D

5. **Statement-1** : Let  $z_1, z_2, z_3$  be three complex numbers such that  $|3z_1 + 1| = |3z_2 + 1| = |3z_3 + 1|$  and  $1 + z_1 + z_2 + z_3 = 0$ , then  $z_1, z_2, z_3$  will represent vertices of an equilateral triangle on the complex plane.

and

**Statement-2** :  $z_1, z_2, z_3$  represent vertices of an equilateral triangle if  $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ .

- (A) A (B) B (C) C (D) D

## COMPREHENSION BASED QUESTIONS

### Comprehension # 1 :

Let  $z$  be any complex number. To factorise the expression of the form  $z^n - 1$ , we consider the equation  $z^n = 1$ . This equation is solved using De moiver's theorem. Let  $1, \alpha_1, \alpha_2, \dots, \alpha_{n-1}$  be the roots of this equation, then  $z^n - 1 = (z - 1)(z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_{n-1})$ . This method can be generalised to factorize any expression of the form  $z^n - k^n$ .

for example,  $z^7 + 1 = \prod_{m=0}^6 \left( z - \text{cis} \left( \frac{2m\pi}{7} + \frac{\pi}{7} \right) \right)$

This can be further simplified as

$$z^7 + 1 = (z + 1) \left( z^2 - 2z \cos \frac{\pi}{7} + 1 \right) \left( z^2 - 2z \cos \frac{3\pi}{7} + 1 \right) \left( z^2 - 2z \cos \frac{5\pi}{7} + 1 \right) \dots \dots \dots (i)$$

These factorisations are useful in proving different trigonometric identities e.g. in equation (i) if we put  $z = i$ , then equation (i) becomes

$$(1-i) = (i+1) \left(-2i \cos \frac{\pi}{7}\right) \left(-2i \cos \frac{3\pi}{7}\right) \left(-2i \cos \frac{5\pi}{7}\right)$$

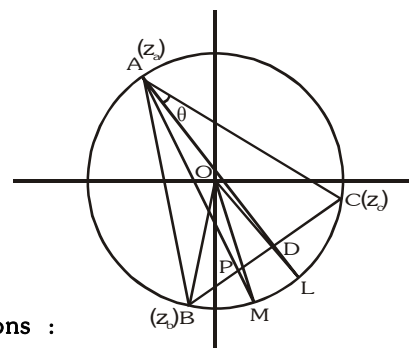
$$\text{i.e. } \cos \frac{\pi}{7} \cos \frac{3\pi}{7} \cos \frac{5\pi}{7} = -\frac{1}{8}$$

On the basis of above information, answer the following questions :

- If the expression  $z^5 - 32$  can be factorised into linear and quadratic factors over real coefficients as  $(z^5 - 32) = (z-2)(z^2 - pz + 4)(z^2 - qz + 4)$ , where  $p > q$ , then the value of  $p^2 - 2q$  -  
 (A) 8 (B) 4 (C) -4 (D) -8
- By using the factorisation for  $z^5 + 1$ , the value of  $4 \sin \frac{\pi}{10} \cos \frac{\pi}{5}$  comes out to be -  
 (A) 4 (B) 1/4 (C) 1 (D) -1
- If  $(z^{2n+1} - 1) = (z-1)(z^2 - p_1z + 1) \dots (z^2 - p_nz + 1)$  where  $n \in \mathbb{N}$  &  $p_1, p_2, \dots, p_n$  are real numbers then  $p_1 + p_2 + \dots + p_n =$   
 (A) -1 (B) 0 (C)  $\tan(\pi/2n)$  (D) none of these

### Comprehension # 2 :

In the figure  $|z| = r$  is circumcircle of  $\triangle ABC$ . D, E & F are the middle points of the sides BC, CA & AB respectively, AD produced to meet the circle at L. If  $\angle CAD = \theta$ ,  $AD = x$ ,  $BD = y$  and altitude of  $\triangle ABC$  from A meet the circle  $|z| = r$  at M,  $z_a, z_b$  &  $z_c$  are affixes of vertices A, B & C respectively.



On the basis of above information, answer the following questions :

- Area of the  $\triangle ABC$  is equal to -  
 (A)  $xy \cos(\theta + C)$  (B)  $(x+y) \sin \theta$  (C)  $xy \sin(\theta + C)$  (D)  $\frac{1}{2} xy \sin(\theta + C)$
- Affix of M is -  
 (A)  $2z_b e^{i2B}$  (B)  $z_b e^{i(\pi-2B)}$  (C)  $z_b e^{iB}$  (D)  $2z_b e^{iB}$
- Affix of L is -  
 (A)  $z_b e^{i(2A-2\theta)}$  (B)  $2z_b e^{i(2A-2\theta)}$  (C)  $z_b e^{i(A-\theta)}$  (D)  $2z_b e^{i(A-\theta)}$

### MISCELLANEOUS TYPE QUESTION

### ANSWER KEY

### EXERCISE-3

#### Match the Column

1. (A)  $\rightarrow$  (s), (B)  $\rightarrow$  (p), (C)  $\rightarrow$  (q), (D)  $\rightarrow$  (r)

2. (A)  $\rightarrow$  (q), (B)  $\rightarrow$  (p), (C)  $\rightarrow$  (q, s), (D)  $\rightarrow$  (r)

#### Assertion & Reason

1. D

2. A

3. B

4. C

5. B

#### Comprehension Based Questions

Comprehension # 1 : 1. A

2. C

3. A

Comprehension # 2 : 1. C

2. B

3. A

## EXERCISE - 04 [A]

## CONCEPTUAL SUBJECTIVE EXERCISE

1. Find the modulus, argument and the principal argument of the complex numbers.

(a)  $z = 1 + \cos \frac{10\pi}{9} + i \sin \left( \frac{10\pi}{9} \right)$  (b)  $(\tan 1 - i)^2$

(c)  $z = \frac{\sqrt{5+12i} + \sqrt{5-12i}}{\sqrt{5+12i} - \sqrt{5-12i}}$

2. Given that  $x, y \in \mathbb{R}$ , solve :  $4x + 3xy + (2xy - 3x)i = 4y - (x^2/2) + (3xy - 2y)i$

3. Let  $z_1$  and  $z_2$  be two complex numbers such that  $\left| \frac{z_1 - 2z_2}{2 - z_1 \bar{z}_2} \right| = 1$  and  $|z_2| \neq 1$ , find  $|z_1|$ .

4. If  $iz^3 + z^2 - z + i = 0$ , then prove that  $|z|=1$ .

5. If A, B and C are the angle of a triangle D =  $\begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$  where  $i = \sqrt{-1}$ , then find the value of D.

6. For complex numbers  $z$  &  $\omega$ , prove that,  $|z|^2 \omega - |\omega|^2 z = z - \omega$  if and only if,  $z = \omega$  or  $z\bar{\omega} = 1$

7. Let  $z_1, z_2$  be complex numbers with  $|z_1|=|z_2|=1$ , prove that  $|z_1 + 1| + |z_2 + 1| + |z_1 z_2 + 1| \geq 2$

8. Interpret the following locii in  $z \in \mathbb{C}$ .

(a)  $1 < |z - 2i| < 3$

(b)  $\operatorname{Re} \left( \frac{z+2i}{iz+2} \right) \leq 4$  ( $z \neq 2i$ )

(c)  $\operatorname{Arg}(z+i) - \operatorname{Arg}(z-i) = \pi/2$

(d)  $\operatorname{Arg}(z-a) = \pi/3$  where  $a = 3 + 4i$ .

9. Let  $A = \{a \in \mathbb{R} \mid \text{the equation } (1+2i)x^3 - 2(3+i)x^2 + (5-4i)x + 2a^2 = 0\}$  has at least one real root. Find the value of  $\sum_{a \in A} a^2$ .

10. ABCD is a rhombus in the Argand plane. If the affixes of the vertices be  $z_1, z_2, z_3, z_4$  and taken in anti-clockwise sense and  $\angle CBA = \pi/3$ , show that

(a)  $2z_2 = z_1(1 + i\sqrt{3}) + z_3(1 - i\sqrt{3})$  & (b)  $2z_4 = z_1(1 - i\sqrt{3}) + z_3(1 + i\sqrt{3})$

11. P is a point on the Argand plane. On the circle with OP as diameter two points Q & R are taken such that  $\angle POQ = \angle QOR = \theta$ . If 'O' is the origin & P, Q & R are represented by the complex numbers  $Z_1, Z_2$  &  $Z_3$  respectively, show that :  $Z_2^2 \cos 2\theta = Z_1 \cdot Z_3 \cos^2 \theta$ .

12. Let  $A \equiv z_1$  ;  $B \equiv z_2$  ;  $C \equiv z_3$  are three complex numbers denoting the vertices of an acute angled triangle.

If the origin 'O' is the orthocentre of the triangle, then prove that  $z_1 \bar{z}_2 + \bar{z}_1 z_2 = z_2 \bar{z}_3 + \bar{z}_2 z_3 = z_3 \bar{z}_1 + \bar{z}_3 z_1$ .

13. (a) If  $\omega$  is an imaginary cube root of unity then prove that :

$(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8) \dots$  to  $2n$  factors  $= 2^{2n}$

(b) If  $\omega$  is a complex cube root of unity, find the value of ;

$(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \dots$  to  $n$  factors.

14. If the biquadratic  $x^4 + ax^3 + bx^2 + cx + d = 0$  ( $a, b, c, d \in \mathbb{R}$ ) has 4 non real roots, two with sum  $3 + 4i$  and the other two with product  $13 + i$ . Find the value of 'b'.
15. If  $x = 1 + i\sqrt{3}$  ;  $y = 1 - i\sqrt{3}$  &  $z = 2$  , then prove that  $x^p + y^p = z^p$  for every prime  $p > 3$ .

CONCEPTUAL	SUBJECTIVE	EXERCISE	ANSWER KEY	EXERCISE-4(A)
<p>1. (a) Principal <math>\text{Arg } z = -\frac{4\pi}{9}</math> ; <math> z  = 2 \cos \frac{4\pi}{9}</math> ; <math>\text{Arg } z = 2k\pi - \frac{4\pi}{9}</math> <math>k \in \mathbb{I}</math></p> <p>(b) Modulus = <math>\sec^2 1</math>, <math>\text{Arg } z = 2n\pi + (2 - \pi)</math>, Principal <math>\text{Arg } z = (2 - \pi)</math></p> <p>(c) Principal value of <math>\text{Arg } z = -\frac{\pi}{2}</math> &amp; <math> z  = \frac{3}{2}</math>, <math>\text{Arg } z = 2n\pi - \frac{\pi}{2}</math>, <math>n \in \mathbb{I}</math></p> <p>Principal value of <math>\text{Arg } z = \frac{\pi}{2}</math> &amp; <math> z  = \frac{2}{3}</math>, <math>\text{Arg } z = 2n\pi + \frac{\pi}{2}</math>, <math>n \in \mathbb{I}</math></p> <p>2. <math>x = K</math>, <math>y = \frac{3K}{2}</math> <math>K \in \mathbb{R}</math>      3. 2      5. -4</p> <p>8. (a) The region between the concentric circles with centre at (0, 2) &amp; radii 1 &amp; 3 units</p> <p>(b) region outside or on the circle with centre <math>\frac{1}{2} + 2i</math> and radius <math>\frac{1}{2}</math></p> <p>(c) semi circle (in the 1st &amp; 4th quadrant) <math>x + y = 1</math></p> <p>(d) a ray emanating from the point <math>(3 + 4i)</math> directed away from the origin &amp; having equation <math>\sqrt{3}x - y + 4 - 3\sqrt{3} = 0</math></p> <p>9. 18      13. (b) one if n is even ; <math>-\infty</math> if n is odd      14. 51</p>				

**EXERCISE - 04 [B]**

**BRAIN STORMING SUBJECTIVE EXERCISE**

- (a) Let  $z = x + iy$  be a complex number, where  $x$  and  $y$  are real numbers. Let  $A$  and  $B$  be the sets defined by  $A = \{z \mid |z| \leq 2\}$  and  $B = \{z \mid (1-i)z + (1+i)\bar{z} \geq 4\}$ . Find the area of the region  $A \cap B$ .

(b) For all real numbers  $x$ , let the mapping  $f(x) = \frac{1}{x-i}$ , where  $i = \sqrt{-1}$ . If there exist real numbers  $a, b, c$  and  $d$  for which  $f(a), f(b), f(c)$  and  $f(d)$  form a square on the complex plane. Find the area of the square.
- If  $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$  ; where  $p, q, r$  are the moduli of non-zero complex numbers  $u, v, w$  respectively, prove that,  $\arg \frac{w}{v} = \arg \left( \frac{w-u}{v-u} \right)^2$ .
- For  $x \in (0, \pi/2)$  and  $\sin x = \frac{1}{3}$ , if  $\sum_{n=0}^{\infty} \frac{\sin(nx)}{3^n} = \frac{a+b\sqrt{b}}{c}$  then find the value of  $(a+b+c)$ , where  $a, b, c$  are positive integers. (You may use the fact that  $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ )
- If  $z_1, z_2$  are the roots of the equation  $az^2 + bz + c = 0$ , with  $a, b, c > 0$ ;  $2b^2 > 4ac > b^2$ ;  $z_1 \in$  third quadrant;  $z_2 \in$  second quadrant in the argand's plane then, show that

$$\arg \left( \frac{z_1}{z_2} \right) = 2 \cos^{-1} \left( \frac{b^2}{4ac} \right)^{1/2}$$
- If  $Z_r, r = 1, 2, 3, \dots, 2m, m \in \mathbb{N}$  are the roots of the equation  $Z^{2m} + Z^{2m-1} + Z^{2m-2} + \dots + Z + 1 = 0$  then prove that  $\sum_{r=1}^{2m} \frac{1}{Z_r - 1} = -m$
- If  $(1+x)^n = C_0 + C_1x + \dots + C_nx^n$  ( $n \in \mathbb{N}$ ), prove that :

  - $C_0 + C_4 + C_8 + \dots = \frac{1}{2} \left[ 2^{n-1} + 2^{n/2} \cos \frac{n\pi}{4} \right]$
  - $C_1 + C_5 + C_9 + \dots = \frac{1}{2} \left[ 2^{n-1} + 2^{n/2} \sin \frac{n\pi}{4} \right]$
  - $C_2 + C_6 + C_{10} + \dots = \frac{1}{2} \left[ 2^{n-1} - 2^{n/2} \cos \frac{n\pi}{4} \right]$
  - $C_3 + C_7 + C_{11} + \dots = \frac{1}{2} \left[ 2^{n-1} - 2^{n/2} \sin \frac{n\pi}{4} \right]$
  - $C_0 + C_3 + C_6 + C_9 + \dots = \frac{1}{3} \left[ 2^n + 2 \cos \frac{n\pi}{3} \right]$
- Prove that : (a)  $\cos x + {}^nC_1 \cos 2x + {}^nC_2 \cos 3x + \dots + {}^nC_n \cos (n+1)x = 2^n \cdot \cos^n \frac{x}{2} \cdot \cos \left( \frac{n+2}{2} x \right)$

(b)  $\sin x + {}^nC_1 \sin 2x + {}^nC_2 \sin 3x + \dots + {}^nC_n \sin (n+1)x = 2^n \cdot \cos^n \frac{x}{2} \cdot \sin \left( \frac{n+2}{2} x \right)$
- The points  $A, B, C$  depict the complex numbers  $z_1, z_2, z_3$  respectively on a complex plane & the angle  $B$  &  $C$  of the triangle  $ABC$  are each equal to  $\frac{1}{2}(\pi - \alpha)$ . Show that :  $(z_2 - z_3)^2 = 4(z_3 - z_1)(z_1 - z_2) \sin^2 \frac{\alpha}{2}$
- Evaluate :  $\sum_{p=1}^{32} (3p+2) \left( \sum_{q=1}^{10} \left( \sin \frac{2q\pi}{11} - i \cos \frac{2q\pi}{11} \right) \right)^p$ .
- Let  $a, b, c$  be distinct complex numbers such that  $\frac{a}{1-b} = \frac{b}{1-c} = \frac{c}{1-a} = k$ . Find the value of  $k$ .

BRAIN STORMING SUBJECTIVE EXERCISE			ANSWER KEY	EXERCISE-4(B)	
1.	(a) $\pi - 2$	(b) $1/2$	3. 41	9. $48(1-i)$	10. $-\omega$ or $-\omega^2$

**EXERCISE - 05 [A]****JEE-[MAIN] : PREVIOUS YEAR QUESTIONS**

1. The inequality  $|z - 4| < |z - 2|$  represents the following region [AIEEE-2002]
  - (1)  $\operatorname{Re}(z) > 0$
  - (2)  $\operatorname{Re}(z) < 0$
  - (3)  $\operatorname{Re}(z) > 2$
  - (4) none of these
2. Let  $z$  and  $\omega$  are two non-zero complex numbers such that  $|z| = |\omega|$  and  $\arg z + \arg \omega = \pi$ , then  $z$  equal to [AIEEE-2002]
  - (1)  $\omega$
  - (2)  $-\omega$
  - (3)  $\bar{\omega}$
  - (4)  $-\bar{\omega}$
3. Let  $z_1$  and  $z_2$  be two roots of the equation  $z^2 + az + b = 0$ ,  $z$  being complex, Further, assume that the origin  $z_3$ ,  $z_1$  and  $z_2$  form an equilateral triangle. then- [AIEEE-2003]
  - (1)  $a^2 = b$
  - (2)  $a^2 = 2b$
  - (3)  $a^2 = 3b$
  - (4)  $a^2 = 4b$
4. If  $z$  and  $\omega$  are two non-zero complex numbers such that  $|z\omega| = 1$ , and  $\operatorname{Arg}(z) - \operatorname{Arg}(\omega) = \pi/2$ , then  $\bar{z}\omega$  is equal to [AIEEE-2003]
  - (1) 1
  - (2) -1
  - (3) i
  - (4) -i
5. If  $\left(\frac{1+i}{1-i}\right)^x = 1$ , then [AIEEE-2003]
  - (1)  $x = 4n$ , where  $n$  is any positive integer
  - (2)  $x = 2n$ , where  $n$  is any positive integer
  - (3)  $x = 4n + 1$ , where  $n$  is any positive integer
  - (4)  $x = 2n + 1$ , where  $n$  is any positive integer
6. Let  $z, w$  be complex numbers such that  $\bar{z} + i\bar{w} = 0$  and  $\arg zw = \pi$ . Then  $\arg z$  equals [AIEEE-2004]
  - (1)  $\pi/4$
  - (2)  $\pi/2$
  - (3)  $3\pi/4$
  - (4)  $5\pi/4$
7. If  $|z^2 - 1| = |z|^2 + 1$ , then  $z$  lies on [AIEEE-2004]
  - (1) the real axis
  - (2) the imaginary axis
  - (3) a circle
  - (4) an ellipse
8. If  $z = x - iy$  and  $z^{1/3} = p + iq$ , then  $\frac{\left(\frac{x}{p} + \frac{y}{q}\right)}{(p^2 + q^2)}$  is equal to- [AIEEE-2004]
  - (1) 1
  - (2) -1
  - (3) 2
  - (4) -2
9. If  $z_1$  and  $z_2$  are two non zero complex numbers such that  $|z_1 + z_2| = |z_1| + |z_2|$  then  $\arg z_1 - \arg z_2$  is equal to- [AIEEE-2005]
  - (1)  $-\pi$
  - (2)  $\frac{\pi}{2}$
  - (3)  $-\frac{\pi}{2}$
  - (4) 0
10. If  $w = \frac{z}{z - \frac{1}{3}i}$  and  $|w| = 1$  then  $z$  lies on [AIEEE-2005]
  - (1) a circle
  - (2) an ellipse
  - (3) a parabola
  - (4) a straight line
11. If  $|z + 4| \leq 3$ , then the maximum value of  $|z + 1|$  is- [AIEEE-2007]
  - (1) 4
  - (2) 10
  - (3) 6
  - (4) 0
12. The conjugate of a complex number is  $\frac{1}{i-1}$ , then that complex number is- [AIEEE-2008]
  - (1)  $\frac{-1}{i-1}$
  - (2)  $\frac{1}{i+1}$
  - (3)  $\frac{-1}{i+1}$
  - (4)  $\frac{1}{i-1}$
13. If  $\left|Z - \frac{4}{Z}\right| = 2$ , then the maximum value of  $|Z|$  is equal to :- [AIEEE-2009]
  - (1) 2
  - (2)  $2 + \sqrt{2}$
  - (3)  $\sqrt{3} + 1$
  - (4)  $\sqrt{5} + 1$



14. The number of complex numbers  $z$  such that  $|z - 1| = |z + 1| = |z - i|$  equals :- [AIEEE-2010]  
 (1) 0 (2) 1 (3) 2 (4)  $\infty$
15. Let  $\alpha, \beta$  be real and  $z$  be a complex number. If  $z^2 + \alpha z + \beta = 0$  has two distinct roots on the line  $\operatorname{Re} z = 1$ , then it is necessary that :- [AIEEE-2011]  
 (1)  $|\beta| = 1$  (2)  $\beta \in (1, \infty)$  (3)  $\beta \in (0, 1)$  (4)  $\beta \in (-1, 0)$
16. If  $\omega (\neq 1)$  is a cube root of unity, and  $(1 + \omega)^7 = A + B\omega$ . Then  $(A, B)$  equals :- [AIEEE-2011]  
 (1) (1, 0) (2) (-1, 1) (3) (0, 1) (4) (1, 1)
17. If  $z \neq 1$  and  $\frac{z^2}{z - 1}$  is real, then the point represented by the complex number  $z$  lies : [AIEEE-2012]  
 (1) on the imaginary axis.  
 (2) either on the real axis or on a circle passing through the origin.  
 (3) on a circle with centre at the origin.  
 (4) either on the real axis or on a circle not passing through the origin.
18. If  $z$  is a complex number of unit modulus and argument  $\theta$ , then  $\arg\left(\frac{1+z}{1+\bar{z}}\right)$  equals [JEE (Main)-2013]  
 (1)  $-\theta$  (2)  $\frac{\pi}{2} - \theta$  (3)  $\theta$  (4)  $\pi - \theta$

**PREVIOUS YEARS QUESTIONS**
**ANSWER KEY**
**EXERCISE-5 [A]**

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans	4	4	3	4	1	3	2	4	4	4	3	3	4	2	2
Que.	16	17	18												
Ans	4	2	3												

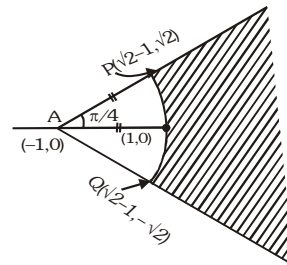
**EXERCISE - 05 [B]****JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS**

1. (a) If  $z_1, z_2, z_3$  are complex numbers such that  $|z_1| = |z_2| = |z_3| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| = 1$  then  $|z_1 + z_2 + z_3|$  is -  
 (A) equal to 1 (B) less than 1 (C) greater than 3 (D) equal to 3  
 (b) If  $\arg(z) < 0$ , then  $\arg(-z) - \arg(z) =$  [JEE 2000 Screening] 1+1M out of 35  
 (A)  $\pi$  (B)  $-\pi$  (C)  $-\frac{\pi}{2}$  (D)  $\frac{\pi}{2}$
2. (a) The complex numbers  $z_1, z_2$  and  $z_3$  satisfying  $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$  are the vertices of a triangle which is -  
 (A) of area zero (B) right-angled isosceles (C) equilateral (D) obtuse-angled isosceles  
 (b) Let  $z_1$  and  $z_2$  be  $n$ th roots of unity which subtend a right angle at the origin. Then  $n$  must be of the form  
 (A)  $4k + 1$  (B)  $4k + 2$  (C)  $4k + 3$  (D)  $4k$   
 [JEE 2001 (Screening) 1+1M out of 35]
3. (a) Let  $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ . Then the value of the determinant  $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 - \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$  is - [JEE 02 (Screening) 3M]  
 (A)  $3\omega$  (B)  $3\omega(\omega - 1)$  (C)  $3\omega^2$  (D)  $3\omega(1 - \omega)$   
 (b) For all complex numbers  $z_1, z_2$  satisfying  $|z_1| = 12$  and  $|z_2 - 3 - 4i| = 5$ , the minimum value of  $|z_1 - z_2|$  is [JEE 02 (Screening) 3M]  
 (A) 0 (B) 2 (C) 7 (D) 17  
 (c) Let a complex number  $\alpha, \alpha \neq 1$ , be a root of the equation  $z^{p+q} - z^p - z^q + 1 = 0$  where  $p, q$  are distinct primes. Show that either  $1 + \alpha + \alpha^2 + \dots + \alpha^{p-1} = 0$  or  $1 + \alpha + \alpha^2 + \dots + \alpha^{q-1} = 0$ , but not both together.  
 [JEE 02 (Mains) 5M]
4. If  $|z| = 1$  and  $\omega = \frac{z-1}{z+1}$  (where  $z \neq -1$ ), then  $\operatorname{Re}(\omega)$  equals - [JEE 03 (Screening) 3M]  
 (A) 0 (B)  $-\frac{1}{|z+1|^2}$  (C)  $\left| \frac{z}{z+1} \right| \cdot \frac{1}{|z+1|^2}$  (D)  $\frac{\sqrt{2}}{|z+1|^2}$
5. If  $z_1$  and  $z_2$  are two complex numbers such that  $|z_1| < 1$  and  $|z_2| > 1$  then show that  $\left| \frac{1 - z_1 \bar{z}_2}{z_1 - z_2} \right| < 1$   
 [JEE 03 (Mains) 2M out of 60]
6. Show that there exists no complex number  $z$  such that  $|z| < \frac{1}{3}$  and  $\sum_{r=1}^n a_r z^r = 1$   
 where  $|a_i| < 2$  for  $i = 1, 2, \dots, n$ . [JEE 03 (Mains) 2M out of 60]
7. The least positive value of 'n' for which  $(1 + \omega^2)^n = (1 + \omega^4)^n$ , where  $\omega$  is a non real cube root of unity is -  
 (A) 2 (B) 3 (C) 6 (D) 4  
 [JEE 04 (screening) 3M]
8. Find the centre and radius formed by all the points represented by  $z = x + iy$  satisfying the relation  $\frac{|z - \alpha|}{|z - \beta|} = K$  ( $K \neq 1$ ) where  $\alpha$  &  $\beta$  are constant complex numbers, given by  $\alpha = \alpha_1 + i\alpha_2$  &  $\beta = \beta_1 + i\beta_2$   
 [JEE 04 (Mains) (2 out of 60)]
9. If  $a, b, c$  are integers not all equal and  $\omega$  is cube root of unity ( $\omega \neq 1$ ) then the minimum value of  $|a + b\omega + c\omega^2|$  is - [JEE 05 (screening) 3M]  
 (A) 0 (B) 1 (C)  $\frac{\sqrt{3}}{2}$  (D)  $\frac{1}{2}$

10. Area of shaded region belongs to -

[JEE 05 (screening) 3M]

- (A)  $z : |z + 1| > 2, |\arg(z + 1)| < \pi/4$   
 (B)  $z : |z - 1| > 2, |\arg(z - 1)| < \pi/4$   
 (C)  $z : |z + 1| < 2, |\arg(z + 1)| < \pi/2$   
 (D)  $z : |z - 1| < 2, |\arg(z - 1)| < \pi/2$



11. If one of the vertices of the square circumscribing the circle  $|z - 1| = \sqrt{2}$  is  $2 + \sqrt{3}i$ . Find the other vertices of square.  
 [JEE 05 (Mains) 4 out of 60]

12. If  $w = \alpha + i\beta$  where  $\beta \neq 0$  and  $z \neq 1$ , satisfies the condition that  $\frac{w - \bar{w}z}{1 - z}$  is purely real, then the set of values of  $z$  is -  
 [JEE 06, 3M]

- (A)  $\{z : |z| = 1\}$  (B)  $\{z : z = \bar{z}\}$  (C)  $\{z : z \neq 1\}$  (D)  $\{z : |z| = 1, z \neq 1\}$

13. A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. From there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. Then the position of P in the Argand plane is :  
 [JEE 07, 3M]

- (A)  $3e^{i\pi/4} + 4i$  (B)  $(3 - 4i)e^{i\pi/4}$  (C)  $(4 + 3i)e^{i\pi/4}$  (D)  $(3 + 4i)e^{i\pi/4}$

14. If  $|z| = 1$  and  $z \neq \pm 1$ , then all the values of  $\frac{z}{1 - z^2}$  lie on :  
 [JEE 07, 3M]

- (A) a line not passing through the origin (B)  $|z| = \sqrt{2}$   
 (C) the x-axis (D) the y-axis

**Comprehension (for 15 to 17) :**

Let A, B, C be three sets of complex numbers as defined below

[JEE 2008, 4M, -1M]

$$A = \{z : \operatorname{Im} z \geq 1\}$$

$$B = \{z : |z - 2 - i| = 3\}$$

$$C = \{z : \operatorname{Re}((1 - i)z) = \sqrt{2}\}$$

15. The number of elements in the set  $A \cap B \cap C$  is -

- (A) 0 (B) 1 (C) 2 (D)  $\infty$

16. Let  $z$  be any point in  $A \cap B \cap C$ . Then  $|z + 1 - i|^2 + |z - 5 - i|^2$  lies between -

- (A) 25 and 29 (B) 30 and 34 (C) 35 and 39 (D) 40 and 44

17. Let  $z$  be any point in  $A \cap B \cap C$  and let  $\omega$  be any point satisfying  $|\omega - 2 - i| < 3$ . Then,  $|z| - |\omega| + 3$  lies between -

- (A) -6 and 3 (B) -3 and 6 (C) -6 and 6 (D) -3 and 9

18. A particle P starts from the point  $z_0 = 1 + 2i$ , where  $i = \sqrt{-1}$ . It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point  $z_1$ . From  $z_1$  the particle moves

$\sqrt{2}$  units in the direction of the vector  $\vec{i} + \vec{j}$  and then it moves through an angle  $\frac{\pi}{2}$  in anticlockwise direction on a circle with centre at origin, to reach a point  $z_2$ . The point  $z_2$  is given by - [JEE 2008, 3M, -1M]

- (A)  $6 + 7i$  (B)  $-7 + 6i$  (C)  $7 + 6i$  (D)  $-6 + 7i$

19. Let  $z = \cos \theta + i \sin \theta$ . Then the value of  $\sum_{m=1}^{15} \operatorname{Im}(z^{2m-1})$  at  $\theta = 2^\circ$  is -  
 [JEE 2009, 3M, -1M]

- (A)  $\frac{1}{\sin 2^\circ}$  (B)  $\frac{1}{3 \sin 2^\circ}$  (C)  $\frac{1}{2 \sin 2^\circ}$  (D)  $\frac{1}{4 \sin 2^\circ}$

20. Let  $z = x + iy$  be a complex number where  $x$  and  $y$  are integers. Then the area of the rectangle whose vertices are the roots of the equation  $z\bar{z}^3 + \bar{z}z^3 = 350$  is -  
 [JEE 2009, 3M, -1M]

- (A) 48 (B) 32 (C) 40 (D) 80

21. Match the conics in **Column I** with the statements/ expressions in **Column II**.

[JEE 2009, 8M]

**Column I**

**Column II**

- (A) Circle  
(B) Parabola  
(C) Ellipse  
(D) Hyperbola

- (P) The locus of the point  $(h, k)$  for which the line  $hx + ky = 1$  touches the circle  $x^2 + y^2 = 4$   
(Q) Points  $z$  in the complex plane satisfying  $|z + 2| - |z - 2| = \pm 3$   
(R) Points of the conic have parametric representation  

$$x = \sqrt{3} \left( \frac{1-t^2}{1+t^2} \right), y = \frac{2t}{1+t^2}$$
  
(S) The eccentricity of the conic lies in the interval  $1 \leq x < \infty$   
(T) Points  $z$  in the complex plane satisfying  $\operatorname{Re}(z + 1)^2 = |z|^2 + 1$

22. Let  $z_1$  and  $z_2$  be two distinct complex numbers and let  $z = (1 - t)z_1 + tz_2$  for some real number  $t$  with  $0 < t < 1$ . If  $\operatorname{Arg}(w)$  denotes the principal argument of a nonzero complex number  $w$ , then

- (A)  $|z - z_1| + |z - z_2| = |z_1 - z_2|$  (B)  $\operatorname{Arg}(z - z_1) = \operatorname{Arg}(z - z_2)$  [JEE 10, 3M]  
(C)  $\left| \frac{z - z_1}{z_2 - z_1} \cdot \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \right| = 0$  (D)  $\operatorname{Arg}(z - z_1) = \operatorname{Arg}(z_2 - z_1)$

23. Let  $\omega$  be the complex number  $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ . Then the number of distinct complex numbers  $z$  satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$
 is equal to [JEE 10, 3M]

24. Match the statements in **Column-I** with those in **Column-II**.

[JEE 10, 8M]

[Note : Here  $z$  takes values in the complex plane and  $\operatorname{Im} z$  and  $\operatorname{Re} z$  denote, respectively, the imaginary part and the real part of  $z$ .]

**Column I**

**Column II**

- (A) The set of points  $z$  satisfying  $|z - i||z| = |z + i||z|$  is contained in or equal to (p) an ellipse with eccentricity  $\frac{4}{5}$   
(B) The set of points  $z$  satisfying  $|z + 4| + |z - 4| = 10$  is contained in or equal to (q) the set of points  $z$  satisfying  $\operatorname{Im} z = 0$   
(C) If  $|w| = 2$ , then the set of points  $z = w - \frac{1}{w}$  is contained in or equal to (t) the set of points  $z$  satisfying  $|\operatorname{Im} z| \leq 1$   
(D) If  $|w| = 1$ , then the set of points  $z = w + \frac{1}{w}$  is contained in or equal to (s) the set of points  $z$  satisfying  $|\operatorname{Re} z| \leq 2$   
(t) the set of points  $z$  satisfying  $|z| \leq 3$

25. **Comprehension (3 questions together)**

Let  $a, b$  and  $c$  be three real numbers satisfying

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \dots(E)$$

(i) If the point  $P(a, b, c)$ , with reference to (E), lies on the plane  $2x + y + z = 1$ , then the value of  $7a + b + c$  is

- (A) 0 (B) 12 (C) 7 (D) 6

- (ii) Let  $\omega$  be a solution of  $x^3 - 1 = 0$  with  $\text{Im}(\omega) > 0$ . If  $a = 2$  with  $b$  and  $c$  satisfying (E), then the value of  $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$  is equal to -  
 (A) -2 (B) 2 (C) 3 (D) -3

- (iii) Let  $b = 6$ , with  $a$  and  $c$  satisfying (E). If  $\alpha$  and  $\beta$  are the roots of the quadratic equation  $ax^2 + bx + c = 0$ , then  $\sum_{n=0}^{\infty} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right)^n$  is -  
 (A) 6 (B) 7 (C)  $\frac{6}{7}$  (D)  $\infty$

[JEE 2011, 3+3+3]

26. If  $z$  is any complex number satisfying  $|z - 3 - 2i| \leq 2$ , then the minimum value of  $|2z - 6 + 5i|$  is  
 [JEE 2011, 4M]

27. Let  $\omega = e^{i\pi/3}$ , and  $a, b, c, x, y, z$  be non-zero complex numbers such that  
 $a + b + c = x$   
 $a + b\omega + c\omega^2 = y$   
 $a + b\omega^2 + c\omega = z$ .

Then the value of  $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$  is

[JEE 2011, 4M]

28. Match the statements given in **Column I** with the values given in **Column II**

- (A) If  $\vec{a} = \vec{j} + \sqrt{3}\vec{k}$ ,  $\vec{b} = -\vec{j} + \sqrt{3}\vec{k}$  and  $\vec{c} = 2\sqrt{3}\vec{k}$  form a triangle, then the internal angle of the triangle between  $\vec{a}$  and  $\vec{b}$  is

(p)  $\frac{\pi}{6}$

- (B) If  $\int_a^b (f(x) - 3x)dx = a^2 - b^2$ , then the value of  $f\left(\frac{\pi}{6}\right)$  is

(q)  $\frac{2\pi}{3}$

- (C) The value of  $\frac{\pi^2}{\ln 3} \int_{7/6}^{5/6} \sec(\pi x)dx$  is

(r)  $\frac{\pi}{3}$

- (D) The maximum value of  $\left| \text{Arg} \left( \frac{1}{1-z} \right) \right|$  for

(s)  $\pi$

$|z| = 1, z \neq 1$  is given by

(t)  $\frac{\pi}{2}$

[JEE 2011, 2+2+2+2M]

29. Match the statements given in **Column I** with the intervals/union of intervals given in **Column II**

- (A) The set  $\left\{ \text{Re} \left( \frac{2iz}{1-z^2} \right) : z \text{ is a complex number, } |z| = 1, z \neq \pm 1 \right\}$  is

(p)  $(-\infty, -1) \cup (1, \infty)$

- (B) The domain of the function  $f(x) = \sin^{-1} \left( \frac{8(3)^{x-2}}{1-3^{2(x-1)}} \right)$  is

(q)  $(-\infty, 0) \cup (0, \infty)$

- (C) If  $f(\theta) = \begin{vmatrix} 1 & \tan \theta & 1 \\ -\tan \theta & 1 & \tan \theta \\ -1 & -\tan \theta & 1 \end{vmatrix}$ , then the set  $\left\{ f(\theta) : 0 \leq \theta < \frac{\pi}{2} \right\}$  is

(r)  $[2, \infty)$

- (D) If  $f(x) = x^{3/2}(3x - 10)$ ,  $x \geq 0$ , then  $f(x)$  is increasing in

(s)  $(-\infty, -1] \cup [1, \infty)$

(t)  $(-\infty, 0] \cup [2, \infty)$

[JEE 2011, 2+2+2+2M]

30. Let  $z$  be a complex number such that the imaginary part of  $z$  is nonzero and  $a = z^2 + z + 1$  is real. Then  $a$  **cannot** take the value -  
[JEE 2012, 3M, -1M]
- (A)  $-1$  (B)  $\frac{1}{3}$  (C)  $\frac{1}{2}$  (D)  $\frac{3}{4}$
31. Let complex numbers  $\alpha$  and  $\frac{1}{\alpha}$  lie on circles  $(x - x_0)^2 + (y - y_0)^2 = r^2$  and  $(x - x_0)^2 + (y - y_0)^2 = 4r^2$  respectively. If  $z_0 = x_0 + iy_0$  satisfies the equation  $2|z_0|^2 = r^2 + 2$ , then  $|\alpha| =$  [JEE(Advanced) 2013, 2M]
- (A)  $\frac{1}{\sqrt{2}}$  (B)  $\frac{1}{2}$  (C)  $\frac{1}{\sqrt{7}}$  (D)  $\frac{1}{3}$
32. Let  $\omega$  be a complex cube root of unity with  $\omega \neq 1$  and  $P = [p_{ij}]$  be a  $n \times n$  matrix with  $p_{ij} = \omega^{i+j}$ . Then  $P^2 \neq 0$ , when  $n =$  [JEE(Advanced) 2013, 3, (-1)]
- (A) 57 (B) 55 (C) 58 (D) 56
33. Let  $w = \frac{\sqrt{3} + i}{2}$  and  $P = \{w^n : n = 1, 2, 3, \dots\}$ . Further  $H_1 = \left\{z \in \mathbb{C} : \operatorname{Re} z > \frac{1}{2}\right\}$  and  $H_2 = \left\{z \in \mathbb{C} : \operatorname{Re} z < \frac{-1}{2}\right\}$ , where  $\mathbb{C}$  is the set of all complex numbers. If  $z_1 \in P \cap H_1$ ,  $z_2 \in P \cap H_2$  and  $O$  represents the origin, then  $\angle z_1 O z_2 =$  [JEE-Advanced 2013, 4, (-1)]
- (A)  $\frac{\pi}{2}$  (B)  $\frac{\pi}{6}$  (C)  $\frac{2\pi}{3}$  (D)  $\frac{5\pi}{6}$

Paragraph for Question 34 and 35

Let  $S = S_1 \cap S_2 \cap S_3$ , where  $S_1 = \{z \in \mathbb{C} : |z| < 4\}$ ,  $S_2 = \left\{z \in \mathbb{C} : \operatorname{Im} \left[ \frac{z-1+\sqrt{3}i}{1-\sqrt{3}i} \right] > 0 \right\}$  and  $S_3 = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ .

34.  $\min_{z \in S} |1 - 3i - z| =$  [JEE(Advanced) 2013, 3, (-1)]
- (A)  $\frac{2-\sqrt{3}}{2}$  (B)  $\frac{2+\sqrt{3}}{2}$  (C)  $\frac{3-\sqrt{3}}{2}$  (D)  $\frac{3+\sqrt{3}}{2}$
35. Area of  $S =$  [JEE(Advanced) 2013, 3, (-1)]
- (A)  $\frac{10\pi}{3}$  (B)  $\frac{20\pi}{3}$  (C)  $\frac{16\pi}{3}$  (D)  $\frac{32\pi}{3}$

PREVIOUS YEARS QUESTIONS		ANSWER KEY		EXERCISE-5 [B]	
1. (a) A (b) A	2. (a) C, (b) D	3. (a) B; (b) B	4. A	7. B	
8. $\frac{\alpha - k^2\beta}{1 - k^2}$ & $\left  \frac{1}{k^2 - 1} \right  \sqrt{(\alpha - k^2\beta)^2 - (k^2 \beta ^2 -  \alpha ^2)(k^2 - 1)}$			9. B	10. A	
11. $(-\sqrt{3}i)$ , $(1 - \sqrt{3}) + i$ and $(1 + \sqrt{3}) - i$	12. D	13. D	14. D	15. B	
16. C	17. D	18. D	19. D	20. A	21. $A \rightarrow (P)$ ; $B \rightarrow (S, T)$ ; $C \rightarrow (R)$ ; $D \rightarrow (Q, S)$
22. A, C, D	23. 1	24. (A) $\rightarrow (q, r)$ , (B) $\rightarrow (p)$ , (C) $\rightarrow (p, s, t)$ , (D) $\rightarrow (q, r, s, t)$	25. (i) D, (ii) A, (iii) B		
26. 5	27. Bonus	28. (A) $\rightarrow (q)$ ; (B) $\rightarrow (p)$ or $(p, q, r, s, t)$ ; (C) $\rightarrow (s)$ ; (D) $\rightarrow (t)$			
29. (A) $\rightarrow (s)$ ; (B) $\rightarrow (t)$ ; (C) $\rightarrow (r)$ ; (D) $\rightarrow (r)$	30. D	31. C	32. B, C, D	33. C, D	
34. C	35. B				