Indian Institute of Information Technology Guwahati
Department of Science and Mathematics
MA204: Mathematics IV
Practice Problem 1: Complex Analysis

In addition to the problems mentioned in the lecture slides, try to solve the following problems.

1. Find argument and modulus of the following function:
   (a) $\tan\alpha - i$
   (b) $1 - \cos\alpha + i\sin\alpha$
   (c) $1 + \sin\alpha - i\cos\alpha$

2. Find locus of z if it satisfy the the following equation:
   (a) $z^{\frac{1}{3}} = a$ for some $a \in \mathbb{C}$.
   (b) $\mid \frac{z-a}{z-b} \mid = k =$ constant for some $a, b \in \mathbb{C}$.
   (c) $|z-a| + |z-b| = 2|a-b|$ for some $a, b \in \mathbb{R} - \{0\}$

3. If $|z_1 + z_2| = |z_1| + |z_2|$, then find a relation between two non-zero complex numbers z_1 and z_2.

4. If $|z_1| = 1$ or $|z_2| = 1$, but not both, then show that $|\frac{z_1 - z_2}{1 - z_1\bar{z}_2}| = 1$. What would happen if $|z_1| = |z_2| = 1$?

5. If z and w are in $\mathbb{C}$ such that $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(w) > 0$, show that $|\frac{z-w}{z-\bar{w}}| < 1$.

6. If z lies on a unite circle centered at the origin. Show that $|\mathrm{Re}(2 + \bar{z} + z^2)| \leq 4$.

7. If $x_r = \cos(\frac{\pi}{2^r}) + i\sin(\frac{\pi}{2^r})$, then find the value of $\lim\limits_{n\to\infty} x_1 x_2 x_3 \cdots x_n$.

8. Express the following equations in terms of z:
   (a) the hyperbola $x^2 - y^2 = 1$.
   (b) $x^2 - y^2 - 2y + i(2x - 2xy)$.

9. For each of the following subsets of $\mathbb{C}$, determine whether it is open, closed or neither. Justify your answers.
   (a) $A_1 = \{z \in \mathbb{C} : \mathrm{Re}(z) = 1 \text{ and } \mathrm{Im}(z) \neq 4\}$
   (b) $A_2 = B_1(1) \cup B_{\frac{1}{2}}(2) \cup B_{\frac{1}{3}}(3)$
   (c) $A_3 = \{z \in \mathbb{C} : |\frac{z-1}{z+1}| = 2\}$

10. For each of the following subsets of $\mathbb{C}$, determine their interior, exterior and boundary:
    (a) $S_1 = \{z \in \mathbb{C} : |z| < 1 \text{ and } \mathrm{Im}(z) \neq 0\} \cup \{z \in \mathbb{C} : |z| > 1 \text{ and } \mathrm{Im}(z) = 0\}$

(b) $S_2 = \{r(\cos(1/n) + \sin(1/n)) \in \mathbb{C} : r > 0, n \in \mathbb{N}\} \cup \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$

11. Examine if the $\lim_{z\to 0}(\frac{\bar{z}}{z})^2$ exists.

12. Check differentiability of the function $f(z) = x^3 + i(1-y)^3$ in the complex plan. Find the corresponding derivative.

13. Show that real and imarginary parts of the function $f(z) = \sqrt{|\mathrm{Re}(z)\mathrm{Im}(z)|}$ satify the C-R equation. Is the function differentiable at $z = 0$? Is the function analytic at 0?

14. Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there do not exist any point c on the line joining z_1 and z_2 such that $\frac{f(z_1)-f(z_2)}{z_1-z2} = f'(c)$.

15. If $f(z)$ is a real valued function in a domain $D \subseteq \mathbb{C}$, then show that either $f'(z) = 0$ or $f'(z)$ does not exist in D.

16. Let $f : D \to \mathbb{C}$ be a differentiable function such that, for all $z, w \in \mathbb{C}$, $f(z) = f(w)$ whenever $|z| = |w|$. Prove that f is a constant function.

17. Let $f = u + iv$ be an analytic function defined on the whole of $\mathbb{C}$. If $u(x, y) = \phi(x)$ and $v(x, y) = \chi(y)$. Prove that, for all $z \in \mathbb{C}$, $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

18. Find a function $u(r, \theta)$ such that $f(z) = u(r, \theta) + i(r^2 \cos 2\theta - r\cos\theta + 2)$ is analytic.

19. Find values of the constants a, b, c, and d such that the function $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$ is analytic.

20. Find harmonic conjugate of the following functions, and then express $f(z)$ in terms of z:
    (a) $\log(x^2 + y^2) + x - 2y$
    (b) $e^{-2xy}\sin(x^2 - y^2)$

21. Show that $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})|f(z)|^2 = 4|f'(z)|^2$.

22. Find all values of z which satisfy the following:
    (a) $\sin z = 1 + i$
    (b) $e^{2iz} = -2$
    (c) $\sin z = \cosh z$

23. Evaluate the following values:
    (a) $8^{\frac{1}{3}}$
    (b) $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$
    (c) $\sin^{-1}(z)$ in terms of z
    (d) $\log(\mathrm{Log}((1+i)^i))$
    (e) $e^{(5+3i)^2}$
    (f) $\frac{(1+i)^{1-i}}{(1-i)^{1+i}}$