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Filters in topology optimization

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SUMMARY

In this article, a modified ('filtered') version of the minimum compliance topology optimization problem is studied. The direct dependence of the material properties on its pointwise density is replaced by a regularization of the density field by the mean of a convolution operator. In this setting it is possible to establish the existence of solutions. Moreover, convergence of an approximation by means of finite elements can be obtained. This is illustrated through some numerical experiments. The 'filtering' technique is also shown to cope with two important numerical problems in topology optimization, *checkerboards* and *mesh dependent* designs. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: topology optimization; regularization method; convolution; finite element approximation; existence of solutions

1. INTRODUCTION

Topology optimization problems in mechanics, electro-magnetics and multi-physics settings are well known to be ill-posed in many typical problem settings, if one seeks, without restriction, an optimal distribution of void and material with prescribed volume (see, e.g. References [1–5]). This shows up through the possibility of building non-convergent minimizing sequences for the considered problem, the limiting solution achieved as a *micro perforated* material. A consequence of this non-existence is that even if each discretization (by, e.g. finite elements) of the unrestricted problem is well-posed, these solutions do not converge to a macroscopic design when the discretization parameter tends to 0, and smaller and smaller patterns are exhibited. This phenomenon is often also refered to in the literature as *mesh dependency*.

The methods currently used to provide solutions to these non-existence issues and associated numerical side-effects can be sorted in three categories. In the *homogenization* method, one enlarges the set of admissible domains with the limits (in the sense of homogenization) of any distribution of void and material and characterize these limits as *micro structures* that depend on a number of parameters (see, for example, Reference [6] or [7]). In another approach, one adds extra constraints on the set of admissible designs in order to ensure existence, for

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example by imposing an upper bound on the perimeter of the resulting design. The 'void and material problem' is then solved directly (see Reference [8] for the existence result and Reference [9] for a numerical implementation).

The third approach to continuum topology design in a well-posed setting, can be referred to as a 'filtered, penalized artificial material' method. This is the type of setting studied here, in a structural optimization context. In this method one defines at each point of the domain a *density of material* ρ that varies continuously between 1 and 0, with density 1 characterizing the material and 0 the void (no material). The elastic properties for intermediate densities are expressed in terms of the function ρ, for example, using a simple power-law interpolation which, through optimization, is known to lead to designs without intermediate densities. Even though such a method is often labeled as an *artificial power law* method, one can actually give a physical interpretation in terms of sub-optimal, isotropic micro-structures (see Reference [10]). In order to ensure existence of solutions for this method, one is normally also in this case required to add some extra constraints on the admissible designs, i.e., on the admissible densities ρ. This can take the form of a constraint on the total variation of the density (that can be interpreted as the perimeter, in case the design is a bi-level function whose level sets are regular enough) or on the gradient of the density (see References [11, 12] for the former method, and Reference [13] for the latter). An alternative to these approaches has been proposed in References [14, 15] and consists of a filtering technique implemented in the optimization algorithm. This gives mesh-independent designs at a moderate computational cost.

Filtering techniques is the subject of the present study, implemented in our case in a form where filtering is performed on the interpolation of the elastic properties, along the lines proposed in Reference [16]. Here we prove existence of solutions and show that one obtains, with a fixed filter function, convergence of finite element discretized forms of the problem. Numerical experiments are also reported, and the possibilities and complications of using alternatives such as design spaces of filtered density functions are also discussed. The method is applicable to a range of problems and application areas, but the scope of this paper is to focus on a simple two-dimensional minimum compliance optimization problem and to give a full mathematical justification of the filtering technique in this setting.

The use of filters in numerical methods in order to ensure regularity or existence of solutions to a problem has been used for many years in various domains of applications. The basic idea is to replace a (possibly) non-regular function by its *regularization* obtained by the convolution with a smooth function.

An overview of the paper is as follows. In Section 2, we fix notations and define the *filtered* version of the minimum compliance topology optimization problem, depending on a *filter* function, F, inspired by Bruns and Tortorelli [16] and Sigmund [14, 15]. In Section 3, it is shown that if the filter function F verifies some (weak) regularity assumptions, the filtered topology optimization problem admits at least one, possibly non-unique, solution. An approximation result in terms of finite elements is then given in Section 4. The numerical implementation is detailed in Section 5.1 and illustrated by some numerical experiments in Section 5.2. Finally, some further extensions of the method are discussed in Section 6.

2. NOTATIONS AND STATEMENT OF THE PROBLEM

In the following we will define a filtered version of the minimum compliance topology design problem based on the power-law interpolation approach. The technique is to replace the dependence of the elastic properties on the density of material by a dependence of a filtered version of the density function. This means that rapid variations in material properties are *not* allowed by the problem statement. Loosely speaking, this ensures existence of solutions. However, the proof involves some technicalities.

In the standard framework of material distribution methods for topology design, we work, throughout this paper, in a fixed domain $\Omega \subset \mathbb{R}^2$, and the optimal design is generated referring to this 'ground-structure'. Here, this domain is a Lipschitz bounded and open domain. We denote by $\|\cdot\|_{(m,p,\Omega)}$ and $|\cdot|_{(m,p,\Omega)}$ the usual norms and semi-norms over the Sobolev space $W^{m,p}(\Omega;\mathbb{R}^2)$ and for the sake of simplicity abbreviate the notation for $\|\cdot\|_{(m,2,\Omega)}$ and $|\cdot|_{(m,2,\Omega)}$ with $\|\cdot\|_m$ and $|\cdot|_m$ (we refer to Reference [17] for actual definitions and properties of the Sobolev spaces).

What will be called a *filter of characteristic radius* $R>0$, is a function F verifying the following properties:

$$\begin{aligned}
&F \in W^{1,\infty}(\mathbb{R}^2)\\
&\operatorname{Supp} F \subset B_R\\
&F \geqslant 0 \quad \text{a.e. in } B_R\\
&\int_{B_R} F\,\mathrm{d}x = 1
\end{aligned}$$

where B_R denotes the open ball of centre 0 and radius R.

The filtering operation is achieved by mean of the convolution product of the filter and the density

$$(F * \rho)(x) = \int_{\mathbb{R}^2} F(x-y)\rho(y)\,\mathrm{d}y$$

Loosely speaking, we replace at each point the density field by a weighted average of its values. One consequence of this operation is that the filtered density is then a *smooth* and *differentiable* function, among other properties [see e.g. Reference 18, IV 6, p. 66]. Remark that this definition requires to extend the density field ρ to the whole space $\mathbb{R}^2$. Some ways to address this technicality are detailed in Section 5.

In the following we will thus work with a parametrization of design through a density function ρ, while the material properties of the equilibrium equation will depend on the filtered density $F*\rho$. Thus, the 'filtered' version of the *minimum compliance* topology optimization problem (MC^F) in structural optimization is defined as (for further details on the minimum compliance topology optimization problem, refer to Reference [20])

$$(\mathrm{MC}^F): \quad \begin{cases} \inf\limits_{\rho\in\mathscr{H}} l(u)\\ \text{subject to}\\ (u,\rho)\in\mathscr{E}^F(\Omega) \end{cases}$$

where

$$l(u) = \int_\Omega f u\,\mathrm{d}x$$

is the compliance of the design given by the *density field* ρ subject to the body load $f\in L^2(\Omega)$ and where $\mathscr{E}^F$ denotes the set of densities and related displacements under the given load, the exponent F signifying that the displacement is computed from a filtered version of the density. In order to rigorously define the set of admissible couples $(u,\rho)\in\mathscr{E}^F$, one needs some extra definitions: by $\mathscr{U}$, one denotes the set of *kinematically admissible displacements*, i.e.

$$\mathscr{U} := \{u \in W^{1,2}(\Omega),\ u=0 \text{ on } \partial\Omega_D\}$$

while $\mathscr{H}$ is the *space of feasible designs*:

$$\mathscr{H} := \left\{\rho \in L^\infty(\mathbb{R}^2)\cap L^1(\mathbb{R}^2),\ 0<\underline{\rho}\leqslant\rho\leqslant 1 \text{ a.e. on } \Omega,\ \int_{\mathbb{R}^2}\rho\,\mathrm{d}x\leqslant V\right\}$$

for any given $0<\underline{\rho}$ and $0<V$.

A displacement field $u\in\mathscr{U}$ is said to verify the *equilibrium condition* for the density $\rho\in\mathscr{H}$ if one has

$$\int_\Omega (F*\rho)^p E\varepsilon(u):\varepsilon(v)\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x \quad \forall v\in\mathscr{U} \tag{1}$$

where we use the *filtered* density. In the equilibrium condition $e(u)$ is the *symmetrized gradient* of u ($\varepsilon(u)=(\nabla u+\nabla u^{\mathrm{T}})/2$), E is the fourth-order *elastic properties* tensor for a given material and the duality product ':' is the scalar product of two matrices (in other words, one has $E\varepsilon(u):\varepsilon(v)=E_{ijkl}(\partial u_i/\partial x_j)\partial v_k/\partial x_l$ where the implicit summation convention is used).

It is now possible to define the space $\mathscr{E}^F$ of admissible couples (u,ρ) for the problem (MC^F) by

$$\mathscr{E}^F(\Omega) := \{(u,\rho)\in\mathscr{U}\times\mathscr{H} \text{ satisfying (1)}\}$$

Remark that we in (1) have modelled the stiffness of intermediates densities as $(F*\rho)^p$ in accordance with the so-called power-law method in topology design of penalizing intermediates densities, as described in the introduction. We thus refer to (MC^F) as the filtered problem with penalization p ($p>1$).

Remark 1. Bicontinuity and ellipticity: From the definition of $\mathscr{H}$ and the properties of the filter, there exists two constants M and α depending only on Ω such that for each $\rho\in\mathscr{H}$, the following inequalities hold:

$$\int_\Omega (F*\rho)^p E\varepsilon(u):\varepsilon(v)\,\mathrm{d}x \leqslant M\|u\|_1\|v\|_1,\ \forall(u,v)\in\mathscr{U}\times\mathscr{U} \tag{2}$$

and

$$\int_\Omega (F*\rho)^p E\varepsilon(u):\varepsilon(u)\,\mathrm{d}x \geqslant \alpha\|u\|_1^2\ \forall u\in\mathscr{U} \tag{3}$$

Remark 2. Existence of the equilibrium configuration: From the previous remark it follows that for a given density field $\rho\in\mathscr{H}$, there exist a unique displacement field $u\in\mathscr{U}$ such that $(u,\rho)\in\mathscr{E}^F$.

3. EXISTENCE OF SOLUTIONS TO (MC^F)

The method used to prove existence of solutions to (MC^F) is the classical direct method of variational calculus and optimal control theory. Here the smoothing effect of the filter plays a central role.

First, we rewrite problem (MC^F) in a more typical optimization setting, that is to minimize the compliance of a structure subject to some load, for any admissible deformed configuration:

$$\inf_{u\in\mathscr{U}^*} l(u) \tag{4}$$

where

$$\mathscr{U}^* := \{u\in\mathscr{U} \mid \exists\rho\in\mathscr{H} : (u,\rho)\in\mathscr{E}^F\} \tag{5}$$

Then let $u_k\in\mathscr{U}^*$ be a minimizing sequence for $l(\cdot)$ and $\rho_k\in\mathscr{H}$, be a sequence of material densities where each ρ_k is associated with each u_k through the equilibrium condition written as (5). Then, Remark 1 implies that the sequence u_k is uniformly bounded in $W^{1,2}(\Omega)$ which in turns implies that there exists $u\in\mathscr{U}$ such that

$$u_k \rightharpoonup u \quad \text{in } W^{1,2}(\Omega) \quad \text{when } k\to+\infty \tag{6}$$

For the corresponding densities, the Dunford–Pettis criterion (cf. Reference [18, IV 29, p. 76]), gives us that there exists a density field $\rho\in\mathscr{H}$ such that

$$\rho_k \rightharpoonup \rho \quad \text{in } L^1_{\mathrm{loc}}(\Omega) \quad \text{when } k\to+\infty \tag{7}$$

Since the filter satisfies that $F\in W^{1,\infty}(\mathbb{R}^2)$, this implies that

$$F*\rho_k \longrightarrow F*\rho \quad \text{in } L^1(\Omega) \quad \text{when } k\to\infty \tag{8}$$

Using now Egoroff's theorem, one has that for any positive real number δ, there exists a subset $\Omega_\delta\subset\Omega$ such that $|\Omega_\delta|\leqslant\delta$ and

$$F*\rho_k \longrightarrow F*\rho \quad \text{uniformly in } \Omega\backslash\Omega_\delta \quad \text{when } k\to\infty \tag{9}$$

i.e. the filter F assures strong convergence of the stiffness appearing in the equilibrium equation. Then, combining (6) and (9), one can conclude that for each such δ:

$$(F*\rho_k)^p E\varepsilon(u_k) \rightharpoonup (F*\rho)^p E\varepsilon(u) \quad \text{in } W^{1,2}(\Omega\backslash\Omega_\delta) \tag{10}$$

Thus, the displacement u is related to the density ρ through equilibrium under the given load, that is

$$u\in\mathscr{U}^*$$

Furthermore, since u_k weakly converges in $W^{1,2}$ to u and since u_k is a minimizing sequence for $l(\cdot)$, one can conclude that

$$l(u_k) \longrightarrow l(u) = \inf_{v\in\mathscr{U}} l(v)$$

so that

$$(u,\rho) \text{ solves } (\mathrm{MC}^F)$$

4. FINITE ELEMENT APPROXIMATION

In this section, we will study the convergence of solutions to the finite element discretized version of the continuum problem defined in the previous section. In order to state the finite element approximation results, one needs first to define the discretized version of the sets defined in Section 2, and we here choose to consider low-order approximations.

By Ω_h, one partitions Ω into open triangles and quadrangles such that the radius of the included ball in each element is smaller than h, a given positive *discretization parameter*.

The discretized equivalent of $\mathscr{U}$ used here is then

$$\mathscr{U}_h := \{u \in \mathscr{C}^0(\Omega_h);\, u_{|_e} \in \mathbb{P}_1(e)\ \forall e \in \Omega_h,\ u = 0 \text{ on } \partial\Omega_D\}$$

where $\mathbb{P}_1(e)$ is the set of polynomials of total degree less than 1 if e is a triangle and the set of polynomials of degree less than 1 in each variable if e is a quadrangle. The discretization operator between $\mathscr{U}$ and $\mathscr{U}_h$ is the usual first order Lagrange operator and will be further denoted by π_h. It is well known (see Reference [20]), that there exists a constant C such that

$$\forall u \in \mathscr{U},\ \|u - \pi_h(u)\|_1 \leqslant Ch|u|_1 \tag{11}$$

which together with the density of $\mathscr{C}^\infty(\Omega)$ functions in $\mathscr{U}$ implies that

$$\forall u \in \mathscr{U},\ \pi_h(u) \to u \quad \text{in } W^{1,2}(\Omega) \quad \text{when } h \to 0^+ \tag{12}$$

The discretization of the design space is here chosen to be given by

$$\mathscr{H}_h := \left\{\rho;\, \rho_{|_e} \in \mathbb{P}_0(e),\, \underline{\rho} \leqslant \rho_{|_e} \leqslant 1\ \ \forall e \in \Omega_h,\, \int_\Omega \rho \leqslant V\right\}$$

where $\mathbb{P}_0$ is the set of functions that are constant on each element $e \in \Omega_h$ and the associated projection operator is defined by

$$\Pi_h : \mathscr{U} \longrightarrow \mathscr{U}_h,\ \ \rho \to \rho_h \quad \text{such that } \rho_{h_{|_e}} = \frac{1}{|e|}\int_e \rho(x)\,\mathrm{d}x \tag{13}$$

In order to define the discretized version of the elasticity operator, one needs to give sense to the convolution of the function of $\mathscr{H}_h$ by the filter F. One can here choose to extend the domain Ω in a open set $\tilde{\Omega}$ such that $\Omega_R := \Omega \cup \{x \in \mathbb{R}^2;\ \operatorname{dist}(x,\Omega) \leqslant R\} \subset \tilde{\Omega}$, and then extend each density field $\rho_h \in \mathscr{H}_h$ with value outside Ω and compute the restriction of $F * \rho$ to Ω. For the sake of simplicity, we will denote this operation by the operator $*_\Omega$. Remark that for each function $\rho \in \mathscr{H}$ such that $\rho = 0$ on $\mathbb{R}^2 \backslash \bar{\Omega}$, one has $F * \rho = F *_\Omega \rho$ in Ω.

In order to simplify notation we define for each $\rho \in \mathscr{H}_h$, $(u,v) \in \mathscr{U}_h^2$

$$a_\rho(u,v) := \sum_{e \in \Omega_h} \int_e [\Pi_h(F *_\Omega \rho)]^p E\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x$$

The discrete equivalent of the equilibrium condition (1) is given for each $(\rho_h, u_h) \in \mathscr{H}_h \times \mathscr{U}_h$ by

$$\sum_{e \in \Omega_h} \int_e [\Pi_h(F *_\Omega \rho_h)]^p E\varepsilon(u_h) : \varepsilon(v_h)\,\mathrm{d}x = \int_\Omega \pi_h(f) u_h\,\mathrm{d}x \quad \forall v_h \in \mathscr{U}_h \tag{14}$$

and this permits one to define the equivalent of the set $\mathscr{E}^F$ in the present discrete setting as:

$$\mathscr{E}_h^F(\Omega) = \{(u,\rho) \in \mathscr{U}_h \times \mathscr{H}_h \text{ such that (14) is satisfied}\}$$

so that the discretized version of the filtered minimum compliance problem becomes

$$(\mathrm{MC}_h^F): \quad \begin{cases} \inf\limits_{\rho \in \mathscr{H}_h} \displaystyle\int_\Omega \pi_h(f) v_h\,\mathrm{d}x \\ \text{subject to} \\ (u,\rho) \in \mathscr{E}_h^F(\Omega) \end{cases}$$

Remark 3. The equivalents of Remarks 1 and 2 also hold in the discrete setting.

After these definitions, it is possible to state the following approximation result.

4.1. *Finite element approximation of* (MC^F)

Set $F\in W^{1,\infty}(B_R)$ and let $(u_h^*,\rho_h^*)\in\mathscr{E}_h^F(\Omega)$ be a sequence of solutions of (MC_h^F). Then, there exists an element (u^*,ρ^*) of $\mathscr{E}^F(\Omega)$ and a subsequence $(u_{h_k}^*,\rho_{h_k}^*)$ of (u_h^*,ρ_h^*) such that when $h\to 0^+$,

$$u_{h_k}^*\longrightarrow u^*\quad\text{in } W^{1,2}(\Omega)$$

and

$$\rho_{h_k}^*\rightharpoonup\rho^*\quad\text{in } L^q(\Omega)\ \forall 0\leqslant q<\infty$$

Moreover, (u^*,ρ^*) solves (MC^F).

Proof. The proof of this statement consists of three steps. First, we establish the weak-$W^{1,2}\times L^q$ compactness of $\mathscr{E}_h^F$ and show that the weak-limit of a bounded sequence in $\mathscr{E}_h^F$ lies in $\mathscr{E}^F$. Then, we show that the limit of the sequence (u_h^*,ρ_h^*) is a solution of (MC^F) and finally the strong convergence of the sequence u_h^* is proven.

Remark first that the discrete approximations of the sets $\mathscr{U}$ and $\mathscr{H}$ are internal. Thus, the compactness results stated in Section 3 implies that for each pair $(u_h,\rho_h)\in\mathscr{E}_h^F$ uniformly bounded in $W^{1,2}(\Omega)\times L^q(\Omega)$, there exist two functions $(u,\rho)\in\mathscr{U}\times\mathscr{H}$ such that up to a subsequence extraction,

$$u_h\rightharpoonup u\quad\text{in } W^{1,2}(\Omega)\tag{15}$$

$$\rho_h\rightharpoonup\rho\quad\text{in } L^q(\Omega),\ 1\leqslant q<\infty\tag{16}$$

The second step of the proof is to check whether the limit point (u,ρ) satisfies the equilibrium condition (1). For that purpose, let (u_h,ρ_h) be in $\mathscr{E}_h^F$, let (u,ρ) satisfy (15) and (16), let v be an arbitrary element of $\mathscr{U}$, and set $v_h{:=}\pi_h(v)$. Then, since $(u_h,\rho_h)\in\mathscr{E}_h^F$, one has that

$$a_{\rho_h}(u_h,v_h-v)+a_{\rho_h}(u_h,v)=\int_\Omega \pi_h(f)v_h\,\mathrm{d}x\tag{17}$$

From (12), one has that $v_h{:=}\pi_h(v)$ converges to v and $\pi_h(f)$ converges to f strongly in $W^{1,2}(\Omega)$ so that

$$\int_\Omega \pi_h(f)v_h\,\mathrm{d}x\longrightarrow\int_\Omega fv\,\mathrm{d}x\tag{18}$$

From (16) we obtain that $F*_\Omega\rho_h\longrightarrow F*_\Omega\rho$ uniformly on Ω. Extending then ρ_h and ρ with value 0 outside Ω, one has that

$$F*_\Omega\rho_h\longrightarrow F*\rho\quad\text{uniformly on } \mathbb{R}^2$$

leading to

$$\Pi_h(F*_\Omega\rho_h)\longrightarrow F*\rho\quad\text{uniformly on } \mathbb{R}^2$$

This together with (15), (16), (18) and (17) implies that

$$\int_\Omega (F*\rho)^p E\varepsilon(u):\varepsilon(v)\,\mathrm{d}x=\int_\Omega fv\,\mathrm{d}x\tag{19}$$

By the arbitrariness of v, we can conclude that

$$(u,\rho)\in\mathscr{E}^F \tag{20}$$

ending the first part of the proof.

Let us now show that the limit (u^*,ρ^*) of the sequence of solutions to (MC_h^F), (u_h^*,ρ_h^*), converges to a solution of (MC^F).

Set $\rho_h:=\Pi_h(\rho)$ and let u_h satisfy that $(u_h,\rho_h)\in\mathscr{E}_h^F$. From the previous step, one has that

$$\rho_h\rightharpoonup\rho \quad \text{in } L^q(\Omega),\ 1\leqslant q<\infty$$

and that there exists $\bar{u}\in\mathscr{U}$ such that

$$u_h\rightharpoonup\bar{u} \quad \text{in } W^{1,2}(\Omega)$$

Using the previous step, we have that $(\bar{u},\rho)\in\mathscr{E}^F$ which, according to Remark 2 implies that $\bar{u}=u$. Finally, since (u^*,ρ^*) solves (MC_h^F), one has that

$$\int_\Omega \pi_h(f)u_h^*\,\mathrm{d}x\leqslant\int_\Omega \pi_h(f)u_h\,\mathrm{d}x$$

Thus, from the $W^{1,2}$-weak convergence of the u_h and u_h^*, the $L^2(\Omega)$-strong convergence of $\pi_h(f)$ to f, we see that

$$\int_\Omega f u^*\,\mathrm{d}x\leqslant\int_\Omega f u\,\mathrm{d}x$$

which concludes the second step.

Let us now end the proof by showing that the optimal displacement field for the discrete problem actually strongly converges in $W^{1,2}$ to that of (MC^F): From the discrete equivalent of Remark 1, one has that

$$\begin{aligned}
\sum_{e\in\Omega_h}\|u_h^*-u^*\|_{1,2,e} &\leqslant a_{\rho_h^*}(u_h^*-u^*,u_h^*-u^*)\\
&\leqslant a_{\rho_h^*}(u_h^*-u^*,-u^*)+a_{\rho_h^*}(u_h^*,u_h^*)+a_{\rho_h^*}(u^*,u_h^*)\\
&\leqslant a_{\rho_h^*}(u_h^*-u^*,-u^*)+\int_\Omega \pi_h(f)u_h\,\mathrm{d}x+a_{\rho_h^*}(u^*,u_h^*)
\end{aligned}$$

According to the previous steps, on has that $u_h^*\rightharpoonup u^*$ in $W^{1,2}(\Omega)$ which in turn implies that

$$\int_\Omega \pi_h(f)u_h^*\,\mathrm{d}x\longrightarrow\int_\Omega f u^*\,\mathrm{d}x$$

Moreover, recalling that $F*_\Omega\rho_h^*\to F*\rho^*$ uniformly in Ω, we have that

$$a_{\rho_h^*}(u_h^*-u^*,-u^*)\longrightarrow 0$$

and

$$a_{\rho_h^*}(u^*,u_h^*)\longrightarrow a_{\rho^*}(u^*,u^*)=\int_\Omega f u^*\,\mathrm{d}x$$

The previous three inequalities then permit a conclusion of the proof as we can state that

$$u_h^* \longrightarrow u^* \quad \text{in } W^{1,2}(\Omega) \qquad \square$$

5. NUMERICAL IMPLEMENTATION

In this section, we discuss various aspects of the computational implementation of the discrete version of the filtered minimum compliance design problem. As the problem setting is elsewise fairly well-known, we concentrate here on new aspects arising due to the use of filters.

The first difficulty for a computational implementation of the discrete setting is the numerical computation of the convolution operator on a bounded domain. Indeed, in a rigorous setting, one should store the values of the density field *outside* the computational domain since the value of the filtered density 'close' to the boundary of the domain relies on that outside. Three different method have been studied.

The first method is the one explained in the previous section: Denoting by ρ_e the value on each element e of the discrete field ρ and $V(e)$ the set of the elements of Ω_h whose distance from element e is less than R and c_e the co-ordinates of the centre of element e, one sets

$$(F *_\Omega \rho)_e := \sum_{i \in V(e)} \left(\rho_i \int_i F(x - c_e)\,\mathrm{d}x \right)$$

The drawback of this definition is that a *smoothing* effect happens around the boundary of the domain (i.e. the density ρ cannot take the value 1 at the edges of the domain).

Also, a second method similar to that used in Reference [14] has been used. On each element e, one 'renormalizes' the convolution by dividing the previously shown computation by the integral of the filter function over $V(e)$:

$$(F *_\Omega \rho)_e := \frac{\sum_{i \in V(e)} \left(\rho_i \int_i F(x - c_e)\,\mathrm{d}x \right)}{\sum_{i \in V(e)} \int_i F(x - c_e)\,\mathrm{d}x}$$

The effect of this method is, contrary to the first method, to 'force' the density to take high values on the edges of the domain. Also, it increases the requirements on computational time and storage space (but only moderately).

The method that has been used in the numerical implementation is to extend the density ρ to the whole space $\mathbb{R}^2$ by the mean of symmetries and translations and to compute the convolution of the extended density by the filter function. Of course, this does not require the storage of the values of the extended function. Denoting the extended density by $\bar{\rho}$, and by $\bar{V}(e)$ the equivalent of $V(e)$ for $\bar{\rho}$, one defines our discrete convolution by

$$(F *_\Omega \rho)_e := \sum_{i \in \bar{V}(e)} \left(\bar{\rho}_i \int_i F(x - c_e)\,\mathrm{d}x \right)$$

The numerical method used in the actual implementation of the optimization scheme is that of the optimality criterion, as detailed in [14, 19] and that is not repeated here. The main difference in taking the filtering step into account is in the computation of the sensitivity of the compliance with respect to a small design change. It is detailed here below only in the case of the third discretization of the convolution proposed earlier.

5.1. Sensitivity analysis

Let us write the discretized problem in matrix form and use that (MC^F) is equivalent to the following problem:

$$\begin{aligned}&\text{find } (u,\rho)\in\mathscr{U}\times\mathscr{H} \text{ minimizing}\\&\quad\langle K(\rho)u,u\rangle\\&\text{such that}\\&\quad K(\rho)u=P\end{aligned}\tag{21}$$

where $K(\rho)$ denotes the stiffness matrix associated with the density ρ, through the filter. It is well known that the sensitivity of the compliance Pu with respect to a small design change $\delta\rho$ is equal to

$$\frac{\partial(Fu)}{\partial\rho_e}(\delta\rho)=-\left\langle\frac{\partial K(\rho)}{\partial\rho_e}(\delta\rho)u,u\right\rangle$$

where one needs to compute the sensitivity of the stiffness matrix.

The matrix K can be rewritten as a weighted sum of the *local element stiffness matrices* K_i as

$$K(\rho):=\sum_{i\in\Omega_h}\left(\sum_{j\in\bar V(i)}F_{i,j}\bar\rho_j\right)_i^p K_i$$

where

$$F_{i,j}:=\int_j F(x-c_i)\,\mathrm{d}x$$

Remarking that the contribution of element i in the first sum depends on ρ_e if and only if $i\in\bar V(e)$, one has

$$\begin{aligned}\frac{\partial K(\rho)}{\partial\rho_e}(\delta\rho)&=\sum_{i\in\bar V(e)}\frac{\partial}{\partial\rho_e}\left\{\left(\sum_{j\in\bar V(i)}F_{i,j}\bar\rho_j\right)_i^p\delta\rho\right\}K_i\\&=p\sum_{i\in\bar V(e)}\left\{\left(\sum_{j\in\bar V(i)}F_{i,j}\,\bar\rho_j\right)_i^{p-1}F_{i,e}K_i\right\}\delta\rho\end{aligned}$$

Denoting the discretized convolution by $*_h$, this can be rewritten as

$$\frac{\partial K(\rho)}{\partial\rho_e}(\delta\rho)=pF*_h(F*_h\rho^{p-1}K)\delta\rho$$

Thus, the sensitivity of the compliance with respect to a small design change $\delta\rho$ is given by

$$\frac{\partial(Fu)}{\partial\rho_e}(\delta\rho)=-\,p\langle F*_h(F*_h\rho^{p-1}K)\delta\rho u,u\rangle\tag{22}$$

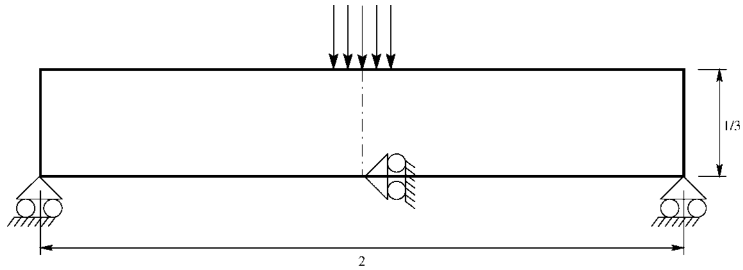

Figure 1. A symmetrical beam: geometry and loading.

Note that this expression retains some similarities with the filtered gradients, as used in [14, 15]. Here however, we have worked with the filter appearing directly in the statement of the optimization problem.

5.2. *Numerical experiments*

We here illustrate some basic features of the 'mesh independency' of discrete solutions assured by the the existence result and the associated convergence of finite element solutions. The numerical experiments are based on the same geometry and loading for various discretization and filter parameters.

For this, we consider a symmetrical beam, loaded with a single force applied in the middle of its upper part. Its lower extremities are free to move in the x-direction and fixed in the y-direction (see Figure 1). According to the symmetry hypothesis, the computations are performed in only one-half of the domain. The parameters used for all of the following simulations are $\underline{\rho}=0.001$, $p=3$. The volume constraint is set so that $\int_\Omega \rho \, \mathrm{d}x \leqslant 0.5|\Omega|$ and the Poisson ratio is $v=0.3$. The discrete convolution is realized by means of the periodic extension method detailed in Section 5.1. The optimality criterion iterations are performed until the L^∞ error between successive designs (i.e. ρ) is lower than 0.01 (i.e. the maximal relative change in all elements is less than 0.01). This strict requirement typically requires around 1000–10 000 iterations of the optimality criterion algorithm, while for practical purposes the more normal use of approximately 100–500 iterations suffice.

The filter function used in the following numerical experiments is a radially linear 'hat' function defined by

$$F(x,y)=\frac{3}{\pi R^2}\max\left(0, 1-\frac{\sqrt{x^2+y^2}}{R}\right)$$

Figure 2 shows the results for a *fixed filter* ($R=0.015$) and several discretization levels. The left row shows plots of the density field ρ while the right hand row shows the 'effective' density, $(F*\rho)^p$, i.e., the material stiffness on which the compliance effectively depends on (further discussions on mechanical interpretations of both fields can be found in Section 6 below).

In Figures 2(c) and 2(d), the finite element discretization is chosen so that its elements are three times wider than they are tall. This underlines the independence of the numerical results in relation to the discretization size and its type. In Figures 2(a) and 2(b), the support of the filter is smaller than the grid size, so the filtering has no effect. However, as soon as the

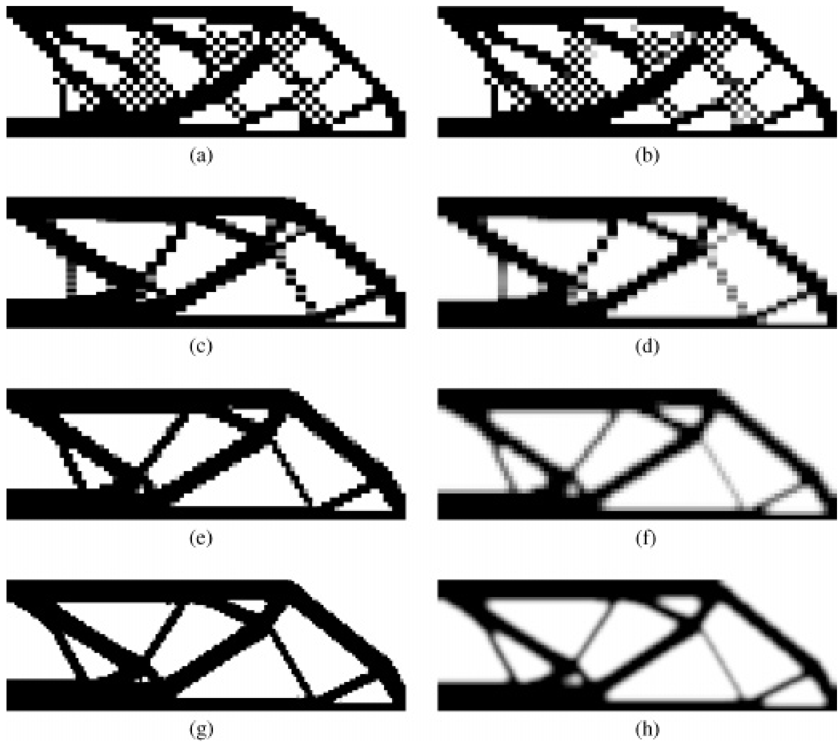

Figure 2. Symmetrical beam, filter radius $R = 1.5\mathrm{E}-2$: (a) ρ, 60×20 elements; (b) $(F*\rho)^p$, 60×20 elements; (c) ρ, 40×40 elements; (d) $(F*\rho)^p$, 40×40 elements; (e) ρ, 90×30 elements; (f) $(F*\rho)^p$, 90×30 elements; (g) ρ, 150×50 elements; and (h) $(F*\rho)^p$, 150×50 elements.

resolution of the grid increases, the *checkerboards* disappear and the actual computed topology does not change when one refines the mesh or when one changes the shape of the elements. It is also noticed that while the *effective density* is relatively smooth, the unfiltered field gives a purely '*black and white*' design, even for 'relatively small' values of the penalization exponent, p (a value of p as low as $p = 1.5$ seems to be enough to get well separated void and material designs).

Figure 3 shows computational results for the same problem, but with a different characteristic radius for the filter. Its aim is to illustrate the fact that a change on the filter induces a change of design. Also, convergence of discretized solutions only holds when the filter is fixed. In Table I, one shows the compliance of the optimal design, as well as the number of iterations performed before stopping, for several discretizations, and two different filters characteristic length. The unusually low compliance of the coarser discretization can be explained by artificially high stiffness arising from checkerboards patterns and these compliance values should thus not be considered as admissible values (see, e.g. Reference [11] or [22]). Also, the compliance of the second set of computations (Figures 3) is bigger than that of the first simulations set (Figures 2). This is due to the fact that the characteristic radius of the filter is bigger in the second set, so that the space of feasible effective designs is smaller (see Section 6 for more details on this argument).

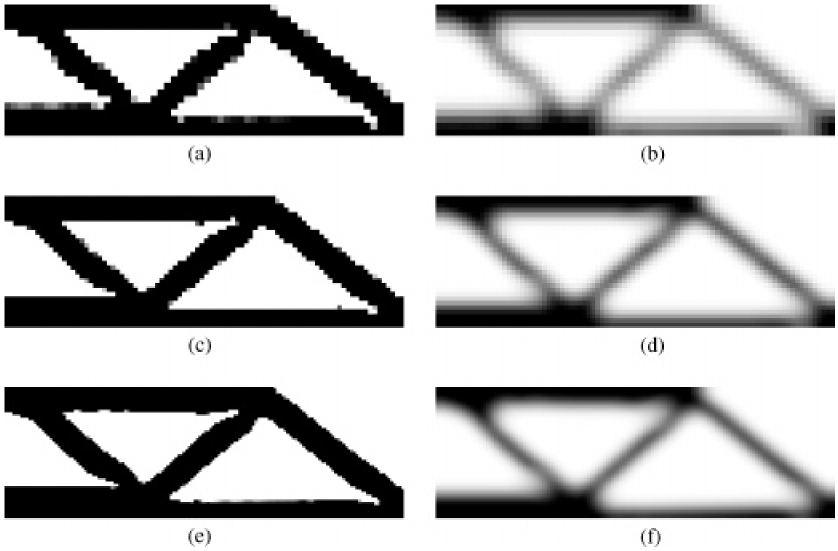

Figure 3. Symmetrical beam, filter radius $R = 1.5\text{E} - 2$: (a) ρ, 60×20 elements; (b) $(F * \rho)^p$, 60×20 elements; (c) ρ, 90×30 elements; (d) $(F * \rho)^p$, 90×30 elements; (e) ρ, 150×50 elements; and (f) $(F * \rho)^p$, 150×50 elements.

Table I. Symmetrical beam, numerical results.

Discretization	R	Compliance	# iterations
60×20	$1.5\text{E} - 2$ (1×1 element)	223.42	773
40×40	$1.5\text{E} - 2$ (1×3 elements)	221.66	405
90×30	$1.5\text{E} - 2$ (3×3 elements)	232.32	1584
105×35	$1.5\text{E} - 2$ (3×3 elements)	229.15	806
150×50	$1.5\text{E} - 2$ (5×5 elements)	232.52	3792
195×65	$1.5\text{E} - 2$ (5×5 elements)	224.49	9937
60×20	$5.0\text{E} - 2$ (7×7 elements)	309.10	3900
90×30	$5.0\text{E} - 2$ (9×9 elements)	283.52	5005
150×50	$5.0\text{E} - 2$ (15×15 elements)	281.65	10 000

6. EXTENSIONS OF THE PROPOSED METHOD

In the following we will attempt to generalize the previous results and will present some possible further extensions of the filtering methods, both in a theoretical and a numerical setting.

For the sake of simplicity, the problem has been stated in a two-dimensional setting but the generalization of the results from Sections 3 and 4 to the three-dimensional case is straightforward.

One drawback of the proposed model is that it is not entirely clear how one should mechanically interpret the computational results. The *mechanical properties* of the computed design are those of $(F*\rho)^p$ while the design constraints are enforced on the ρ field. A similar interpretation is, in essence, required for an unfiltered power-law model, where one needs to distinguish between ρ, used for the volume evaluation and ρ^p, used for the stiffness evaluation. However, in the latter case any density field ρ that only attains values 0 and 1 remains unchanged, and one can, for p large enough, give an interpretation of the combination of a density ρ and a stiffness proportional to ρ^p in terms of sub-optimal, isotropic micro-structures (see Reference [10]).

However, one can formulate another variant of the *filtered minimum compliance* problem where the filter defines a *natural design domain*, and where the 'density' of the volume evaluation is also the density appearing in the stiffness evaluation.

First, one has to extend the set $\mathscr{H}$ by removing the pointwize constraints on the density field and define

$$\mathscr{H}^R := \{\mu \in \mathscr{M}^+(\mathbb{R}^2) \mid \mu(\Omega) \leqslant V, \mu(U) = 0 \ \forall U \subset \mathbb{R}^2 \backslash \tilde{\Omega}\} \tag{23}$$

where $\mathscr{M}^+(\mathbb{R}^2)$ is the set of non-negative Radon measures on $\mathbb{R}^2$. Then, for each $\mu \in \mathscr{M}^+(\mathbb{R}^2)$ and each filter F, $F*\mu$ is defined for each $x \in \mathbb{R}^2$ by

$$(F*\mu)(x) := \int_{\mathbb{R}^2} F(x-y)\,\mathrm{d}\mu(y) \tag{24}$$

With this at hand, one can redefine the minimum compliance problem to the form

$$(\mathrm{MC}_\mu^F): \quad \begin{cases} \inf\limits_{\mu \in \mathscr{H}^R} l(u) \\ \text{subject to} \\ 0 < \underline{\rho} \leqslant (F*\mu) \leqslant 1 \ \text{ a.e. on } \Omega \\ \displaystyle\int_\Omega (F*\mu)^p E\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x \quad \forall v \in \mathscr{U} \end{cases}$$

Here the same methods as used in Section 3 can be applied to establish the existence of solutions.

Remark that by enforcing the pointwise constraint not on the density field but on the filtered densities one has to consider radon measures as the natural density functions, which is rather problematic in view of an actual numerical implementation. The reason for this is that the Dunford–Pettis criterion does not apply anymore (see Equation (7)), so that a bounded sequence of densities does not necessarily weakly-converge to a L^1 function but rather to a radon measure. Moreover, the equivalent of the local constraints on the density $(0 < \underline{\rho} \leqslant (F*\mu) \leqslant 1)$ is no longer in the form of a 'box constraint', further complicating the optimization algorithm. Also, since the equivalent μ of the density field ρ is no longer to be found in a space of *functions* but of *measures*, it has no pointwise value, which complicates its mechanical interpretation. Remark however than one can give a sense to $(F*\mu)^p$ in terms of *effective properties*, as in the homogenization framework.

These difficulties can be hidden by rewriting (MC_μ^F) as a minimization problem over a set of density fields which are given as filtered functions:

$$(\mathrm{MC}_\eta^F): \quad \begin{cases} \inf\limits_{\eta} l(u) \\ \text{subject to} \\ 0 < \underline{\rho} \leqslant \eta \leqslant 1 \ \text{ a.e. on } \Omega \\ \exists \mu \in \mathscr{H}^R \ \text{ such that } \ \eta = F * \mu \\ \displaystyle\int_\Omega \eta^p E \varepsilon(u) : \varepsilon(v)\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x \quad \forall v \in \mathscr{U} \end{cases}$$

Thus the parametrization of design is now implicit and reflects a wish to work only with functions with certain regularity, as controlled by the filter. The existence of solution to this problem is a consequence of the existence of solutions to (MC_μ^F) and the numerical difficulty arises now from the fact that the set of admissible densities is not explicitly given. Indeed, in an analogy with signal processing, the *deconvolution constraint* ($\exists\, \mu \in \mathscr{H}^R$ such that $\eta = F * \mu$) can be seen as a constraint in *frequency space*. Thus, this last formulation is particularly well suited for meshless methods where the density field is no longer approximated by the mean of finite elements but by wavelet decomposition or Fourier series (see, for example, Reference [23]). Furthermore, by considering filters bounded in the frequency plane, one forces the admissible densities to also admit a bounded support in frequency space, and this could eventually simplify the approximation method and the actual implementation.

Associating a filter with a specific design space raises a new problem. That is, to characterize *a priori* the properties of a design by means of the properties of a filter. It is obvious that the filtered density inherits regularity from the differentiability of the considered filter function. Also, in case $F \in W^{1,\infty}(B_R)$, one has a natural bound on the gradient of $F * \rho$ and then on the size of the smallest pattern to appear in the optimal design. This criterion gives a first rule of thumb on the way to fix the characteristic radius of the filter. The proposed method would benefit from further studies on *filter design*, i.e. can one by appropriate filters control various geometric features of the optimal design, such as for example curvature or minimum width of each bar. However, such a characterization requires another theoretical study, as well as an intensive numerical experiments campaign which are still to be achieved.

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