MARKOV CHAINS

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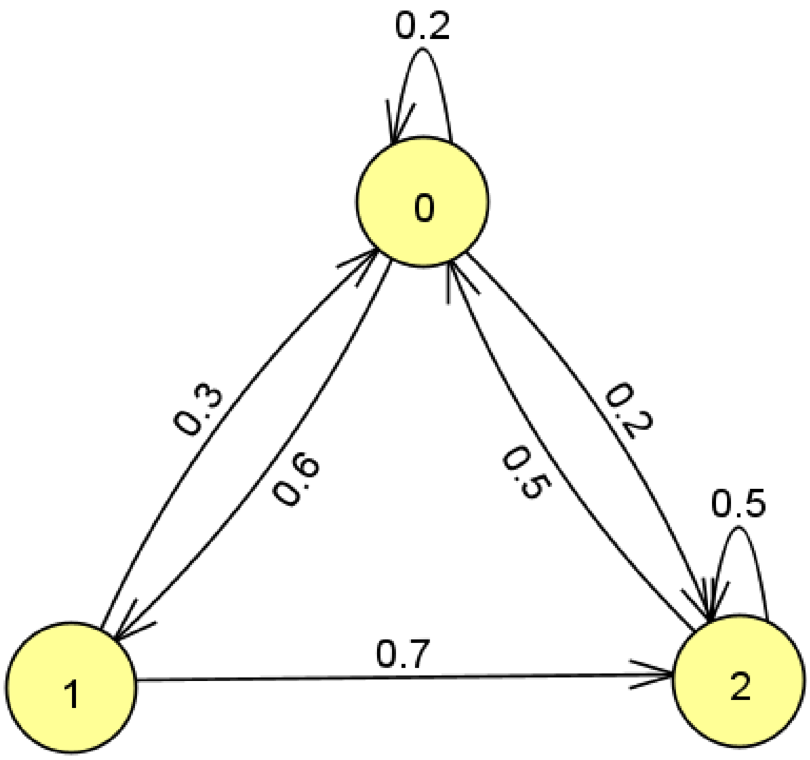
What is a Markov Chain?

A stochastic model (in other terms a transition diagram) describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event.

For Example, a basic Markov Chain is shown below:



The above Markov Chain can be represented as an adjacency Matrix A.

$$A = \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

Where is a Markov Chain used?

Markov chains is used in many fields like Statistics, Biology, Economics, Physics and Machine Learning.

Which Application of Linear Algebra is used in Markov Chains?

Eigen Values and Eigen Vectors.

Basic Idea (Abstract)

- Markov Chains is used in many fields.
- Here we are trying to find these things:
  - If started from a given state after n given transitions what is the probability that we stay in each state.
  - Also find the probability that we stay in each state after infinite (sufficiently long) transitions.
    - Will it reach a stationary state, or no?
    - Will the final state probability matrix depend on initial state?
- We will be answering these questions.

Steps of Execution:

1. Let us assume we are starting at state 1.

2. We define a matrix π_0 =[0 1 0].

3. We multiply the matrix with A and repeat it in a loop.

  1. $\pi_0 A = \pi_1$

  2. $\pi_1 A = \pi_2$

  3. $\pi_2 A = \pi_3$

  4. …..

  5. $\pi_n A = \pi_n$ (Stationary Position)

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.3 & 0 & 0.7 \end{bmatrix}$$

$$\begin{bmatrix} 0.3 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.41 & 0.18 & 0.41 \end{bmatrix}$$

$$\begin{bmatrix} 0.41 & 0.18 & 0.41 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.34 & 0.25 & 0.41 \end{bmatrix}$$

4. **Here,** π_0=[0 1 0], $\pi_1 = [0.3\ 0\ 0.7]$,
$\pi_2 = [0.41\ 0.18\ 0.41], \pi_3 = [0.34\ 0.25\ 0.41]$

$$\pi_n A = \pi_n$$

5. We recognize this equation with eigen value equation $Ax = \lambda x$.
6. So, using similarity we need solutions of x where x is a probability vector so sum of all elements is x must be 1.
7. By solving eigen vector equation we get the value of $\pi = [0.34 \quad 0.25 \quad 0.41]$.
8. So, we can say that there will be a stationary state attained by the Markov chain after n cycles as n tends to ∞.
9. As there will be state attained and the final value of π did not depend on any of the previous value of π we can say that there will be no effect of starting state to get the final stationary state.
10. Hence, I conclude that this Markov Chain will come to a stationary state π when we start from any state.
11. Yeah!! The questions asked above had been answered but there is a question still!!! **What is the probability that we be in state j when started from state i after n cycles???** Well, this is to be discussed now!!

1. Let us denote the probability to reach state j from state i after n cycles as P_{ij}(n).
2. Initially, let us start with n value equal to 1.
   a. From state 0 to state 2 the probability is equal to 0.2 means we can say that $P_{ij}(1) = A_{ij}$ for a given value of i and j.
3. Now Let is take value of n to be 2.
   a. From state 0 to state 2 in 2 cycles there are 3 possible ways:
      i. 0->1->2 P=0.6*0.7=0.42
      ii. 0->0->2 P=0.2*0.2=0.04
      iii. 0->2->2 P=0.2*0.5=0.10
   b. Total Probability $P_{02}(2)$=0.56.
   c. If observed... the probability is equal to product of 1st row with 3rd column respectively. Hence if we compute A^2 we get

$$A^2 = \begin{bmatrix} 0.32 & 0.12 & 0.56 \\ 0.41 & 0.18 & 0.41 \\ 0.35 & 0.3 & 0.35 \end{bmatrix}$$

   d. We can say that $P_{ij}(2) = A^2{}_{ij}$.
4. Now from above assumptions/examples we can conclude that $P_{ij}(n) = A^n{}_{ij}$.
5. Therefore, I can say that the probability to reach state j from state I in exactly n cycles is just the element in i^{th}row and j^{th}column of A^n.

CONCLUSION

- For a given Markov Chain (a transition diagram or an adjacency matrix) there exists a stationary state after n states as n tends to (inf) only if the sum of elements in eigen vector of Ax is 1.
- If the sum of elements in eigen vector is 1 then that eigen vector itself is the stationary state, the Markov chain reaches eventually.
- The probability to reach state j starting from state i exactly after n cycles is the element in i^{th}row and j^{th}column of A^n.