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Fundamentals of Linear Algebra

Mathematics for AI

August 21, 2022

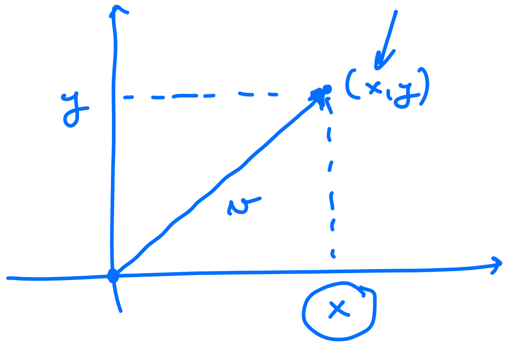
Our Goal

1. linear combinations of vectors;
2. vector space;
3. dot product;
4. orthogonal projections.

$\mathbb{R}$

$(\cdot\, , \,\cdot)$

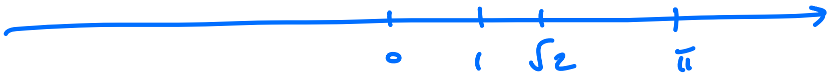
$x \qquad y$



$$v = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

Vectors in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$

$\mathbb{R}$



$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3$$

$$\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$$

Vector Operations

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \qquad a \in \mathbb{R}$$

1. multiplication of a vector $\mathbf{v}$ by a scalar a:

$$a\mathbf{v} = a \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} av_1 \\ \vdots \\ av_n \end{bmatrix};$$

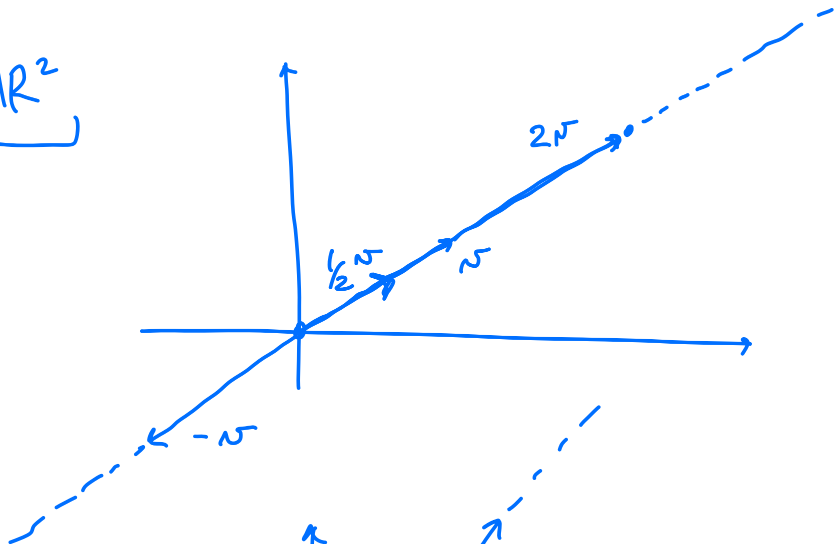
2. sum of $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}.$$

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n \qquad \underbrace{a \in \mathbb{R}}$$

$$a v \equiv \begin{bmatrix} a v_1 \\ \vdots \\ a v_n \end{bmatrix} \in \mathbb{R}^n$$

$\underbrace{\mathbb{R}^2}$

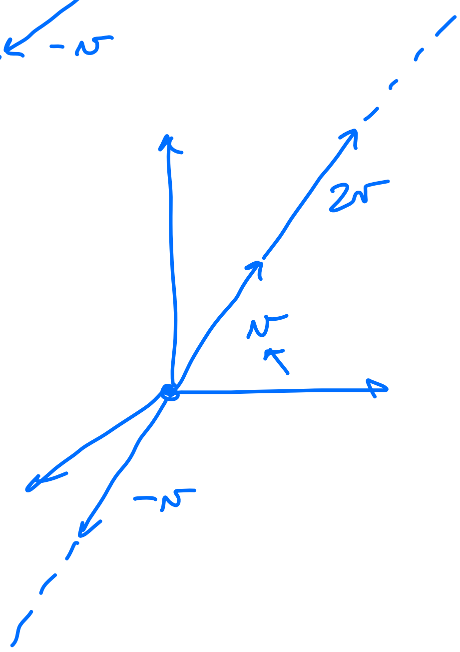


$a = 1$, $1v = v$

$a = 2$, $2v$

$a v$

$\mathbb{R}^3$



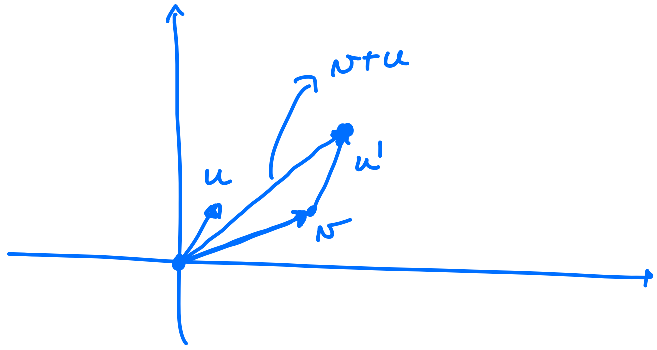
$a v$

$a = 2 \quad 2v$

$a = -1 \quad -v$

$v, u \in \mathbb{R}^n$ $\qquad$ $v + u = \begin{bmatrix} v_1 + u_1 \\ \vdots \\ v_n + u_n \end{bmatrix}$

$\mathbb{R}^2$



Linear Combinations

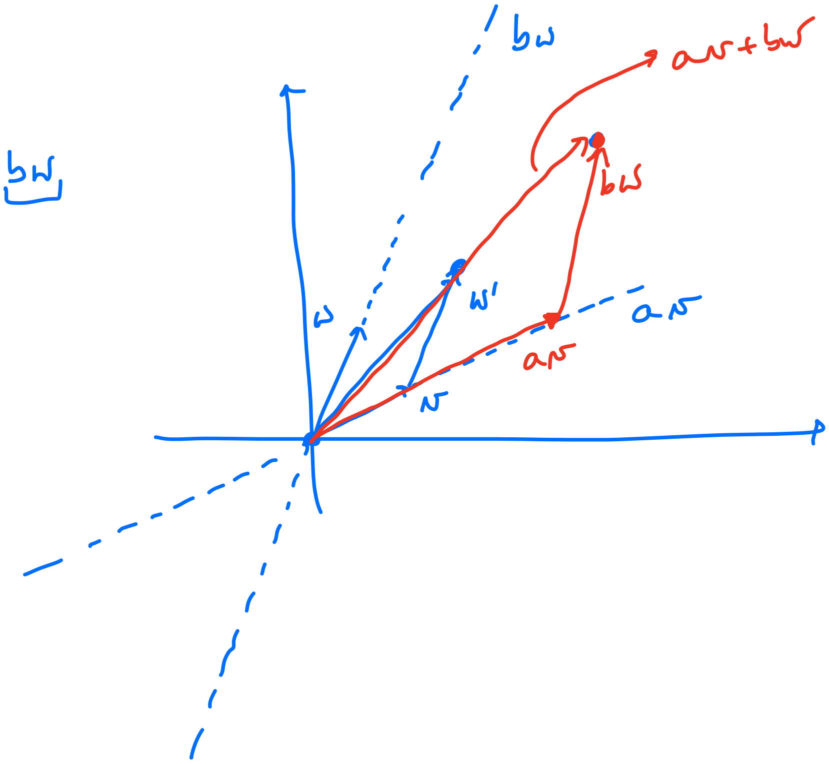
$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \qquad a, b \in \mathbb{R}$$

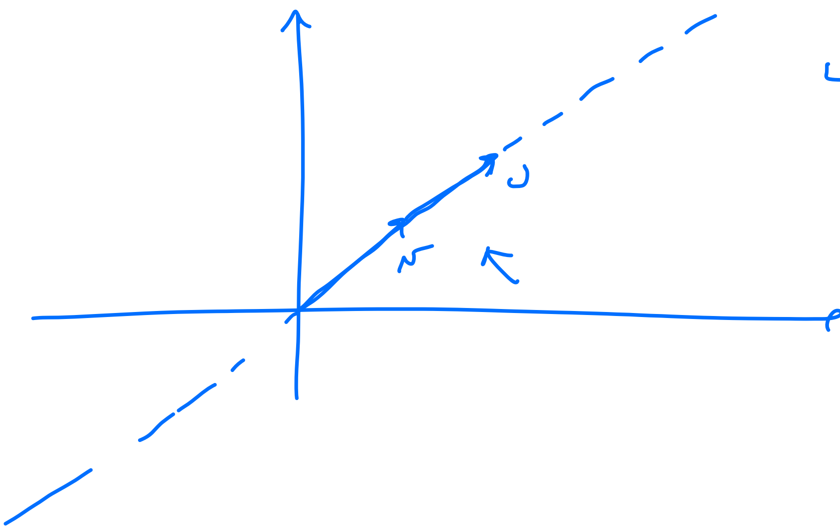
$a\mathbf{v} + b\mathbf{w}$: linear combination of v and w with coefficients a and b

example: express $\mathbf{u} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

$\mathbb{R}^2$

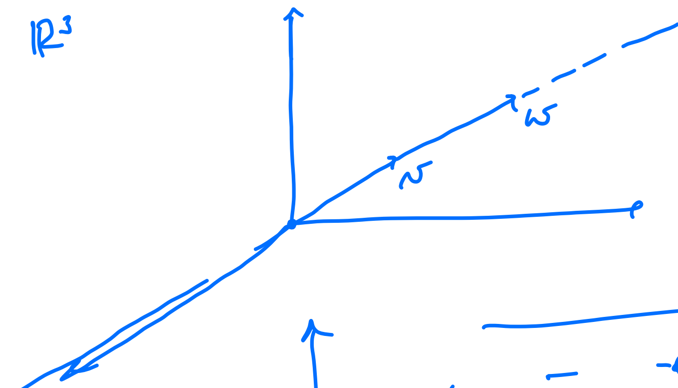
$\underbrace{av} + \underbrace{bw}$





$\underbrace{av + bw \,?}$

$\mathbb{R}^3$

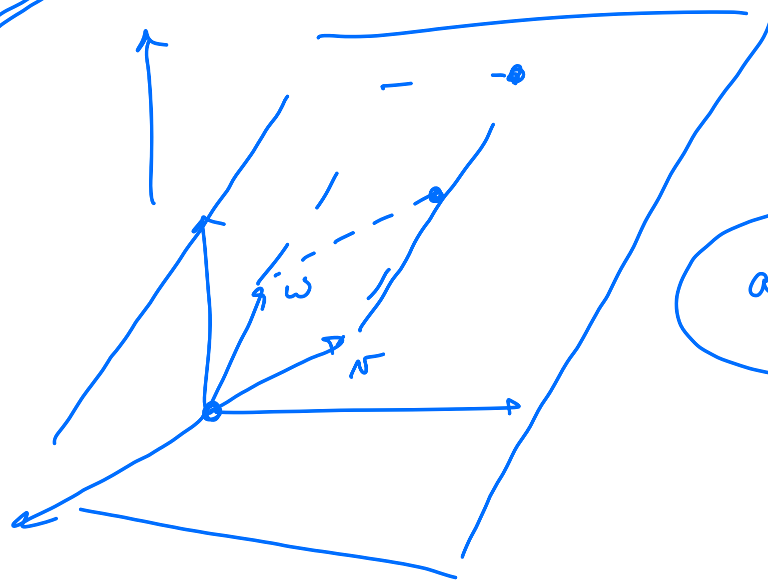


$v, w \in \mathbb{R}^3$

$a, b \in \mathbb{R}$

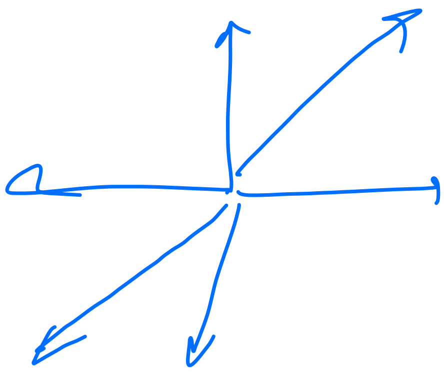
$av + bw$?

$\mathbb{R}^3$



$av + bw$?

$\mathbb{R}^4$



The Important Questions and Pictures

$$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \qquad a, b, c \in \mathbb{R}$$

1. what is the picture of all combinations $a\mathbf{u}$?
2. what is the picture of all combinations $a\mathbf{u} + b\mathbf{v}$?
3. what is the picture of all combinations $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$?

Vector Spaces

V is a vector space in $\mathbb{R}^n$ if it is a subset of $\mathbb{R}^n$ closed under linear combinations.

examples:

1. $\mathbb{R}^2$
2. $\mathbb{R}^3$

examples about vector spaces

(1) $\mathbb{R}$

- $v, w \in \mathbb{R}$ $\qquad v + w \in \mathbb{R}$? ✓
- $\underbrace{a \in \mathbb{R}}$ $\qquad \underbrace{a v \in \mathbb{R} ?}$ ✓

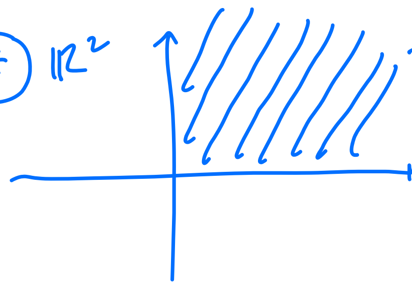
(2) $[1, 2] \in \mathbb{R}$ $\qquad 2 + 2 = 4 \notin [1, 2] \leftarrow$

$\Rightarrow [1,2]$ is not a vector space

(3) $\mathbb{R}^2$ $\qquad \mathbb{R}^2$ is a vector space

$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $\qquad v + u = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \end{bmatrix} \in \mathbb{R}^2$

(4) $\mathbb{R}^2$



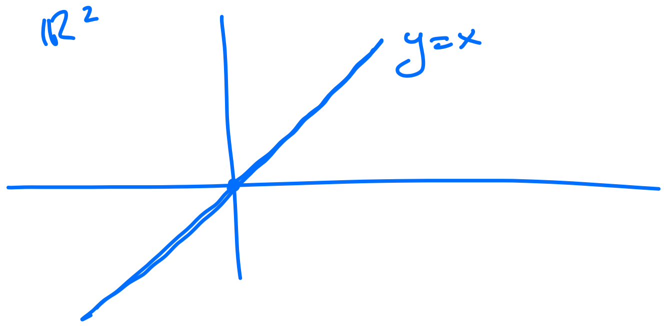
$\rightarrow$ is the 1st quadrant of $\mathbb{R}^2$ a vector space?

$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad -1 \cdot v = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin Q1$

(5) $\underbrace{\mathbb{R}^2 \setminus \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}}$ $\qquad$ not a vector space

(6) $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ $\qquad$ is a vector space!

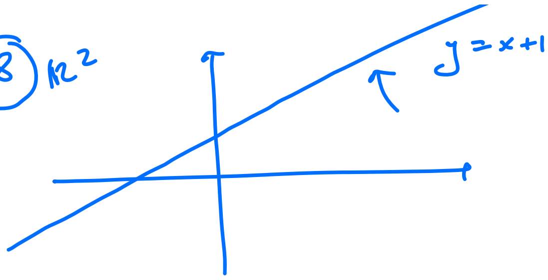
(7) $\mathbb{R}^2$



$\underbrace{\left\{ v \in \mathbb{R}^2 : v = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix} \right\}}$

$v = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_1 \end{bmatrix}$; $\underbrace{v + u =}$ $\begin{bmatrix} v_1 + u_1 \\ v_1 + u_1 \end{bmatrix}$

8 $\mathbb{R}^2$



$\left\{ v \in \mathbb{R}^2 : v = \begin{bmatrix} x \\ x+1 \end{bmatrix} \right\}$

Vector Spaces: Zero Vector

A vector space always contains the zero vector.

Subspaces of $\mathbb{R}^n$

Subspaces of $\mathbb{R}^n$ are vector spaces in $\mathbb{R}^n$.

The subspaces of $\mathbb{R}^2$:

1. $\mathbb{R}^2$
2. lines through the zero vector
3. the zero vector

The subspaces of $\mathbb{R}^3$:

1. $\mathbb{R}^3$
2. planes through the zero vector
3. lines through the zero vector
4. the zero vector

What are all the subspaces in $\mathbb{R}^2$?

- $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$
- any line through the origin
- $\mathbb{R}^2$

What are all the subspaces in $\mathbb{R}^3$?

- $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$
- any line through the origin
- all planes through the origin
- $\mathbb{R}^3$

Span

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of vectors in $\mathbb{R}^m$. The span of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is the set of all the linear combinations of the vectors.

Linear Independence

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of vectors in $\mathbb{R}^m$. The vectors are linearly dependent if there exists

$$a_1, a_2, \ldots a_n \in \mathbb{R},$$

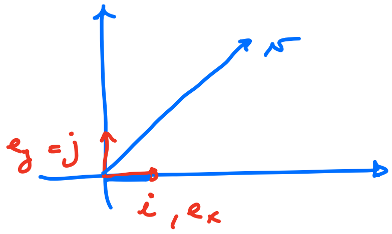
not all zero, such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = 0.$$

If $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = 0$ can only be verified for $a_1 = a_2 = \ldots = a_n = 0$, then, the vectors are linearly independent.

Bases :

① $\mathbb{R}^2$



$i = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad j = \begin{bmatrix}0\\1\end{bmatrix}$

$\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$

$$v = \begin{bmatrix}v_1\\v_2\end{bmatrix} = \overset{a}{v_1}\overset{i}{\begin{bmatrix}1\\0\end{bmatrix}} + \overset{b}{v_2}\overset{j}{\begin{bmatrix}0\\1\end{bmatrix}} =$$

② $\left\{ \begin{bmatrix}1\\0\end{bmatrix} \right\}$ is enough to span $\mathbb{R}^2$

③ $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$

$$\begin{bmatrix}x\\1\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}2\\2\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$$

④ $\left\{ \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$

$$\boxed{av + bw = 0} \longrightarrow av = -bw$$

$$v = -\frac{b}{a} w \qquad a \neq 0$$

Basis

A basis for a vector space is a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d$ in the vector space with two properties:

1. they are linearly independent;
2. they span the space.

A basis is not unique.

The Dot Product

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$$

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$$

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$$

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \ldots + v_n w_n = \sum_{i=1}^{n} v_i w_i$$

$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad v \cdot w = 1.3 + 2.4 = 3 + 8 = 11 \in \mathbb{R}$

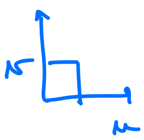
$$\boxed{Ax = b}$$

$A_{m \times n}$

$$A = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

m rows
n column

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$



Length and Unit Vector

$$\mathbf{v} \in \mathbb{R}^n$$

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

1. the unit vector $\mathbf{u}$ with the same direction as $\mathbf{v}$ is

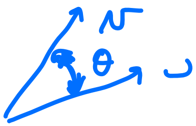
$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||};$$

2.

$$||\mathbf{v}|| = 0 \qquad \Longleftrightarrow \qquad \mathbf{v} = \mathbf{0}.$$

The Angle Between Two Vectors

$\boxed{\mathbf{v}, \mathbf{w} \in \mathbb{R}^n}$



- cosine formula:

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| \, ||\mathbf{w}|| \cos \theta$$

- $\mathbf{v} \cdot \mathbf{w} = 0 \qquad \Longleftrightarrow \qquad \mathbf{v} \perp \mathbf{w}$

1) $v \cdot w = 0 \Rightarrow \underbrace{\|v\|}_{>0} \underbrace{\|w\|}_{>0} \cos\theta = 0 \Rightarrow \cos\theta = 0$

$v \neq 0$
$w \neq 0$

$\Rightarrow \theta = \pi/2$

2) $\theta = \pi/2 \Rightarrow v \cdot w = \|v\| \|w\| \, 0 = 0$

Example

Let $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

1. What space do $\mathbf{u}$ and $\mathbf{v}$ belong to?
2. Are $\mathbf{u}$ and $\mathbf{v}$ perpendicular?
3. What is the length of $\mathbf{v}$?
4. What is the unit vector in the direction of $\mathbf{v}$?

Matrix

A matrix $A_{m\times b}$ is an ordered collection of numbers arranged in a $m \times n$ rectangular format:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix},$$

- m is the number of rows;
- n is the number of columns.

Matrix Addition

Let A and B be 2 matrices of the same size $m \times n$. Then, their sum $A + B$ is defined as

$$\begin{aligned} A + B &= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix}. \end{aligned}$$

Note that $A + B = B + A$.

Scalar Multiplication of a Matrix

Let A be a $m \times n$ matrix and $k \in \mathbb{R}$. Then, kA is defined as

$$kA = \begin{bmatrix} ka_{11} & \dots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \dots & ka_{mn} \end{bmatrix}.$$

System of Linear Equations: Matrices

$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ x_2 + 3x_3 = 1 \end{cases} \longrightarrow \underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\underbrace{x_1 - x_2 + 2x_3 = 1} \quad \text{and} \quad x_2 + 3x_3 = 1$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} \dots & r_1 & \dots \\ \dots & r_2 & \dots \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \vdots \\ x \\ \vdots \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b}.$$

Matrix Multiplication: Keypoints

- $A_{m \times n} B_{n \times \ell} = C_{m \times \ell}$;
- the product of matrices is not commutative: $AB \neq BA$.

Matrix Multiplication: Linear Combination of Vectors

The product of two matrices arises naturally from the linear combination of vectors: if

$$u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \qquad v = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \qquad w = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

their linear combination $au + bv + cw$, $a, b, c \in \mathbb{R}$ can be written as

$$a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ 2a + 3b \\ 4b + 5c \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{x}.$$

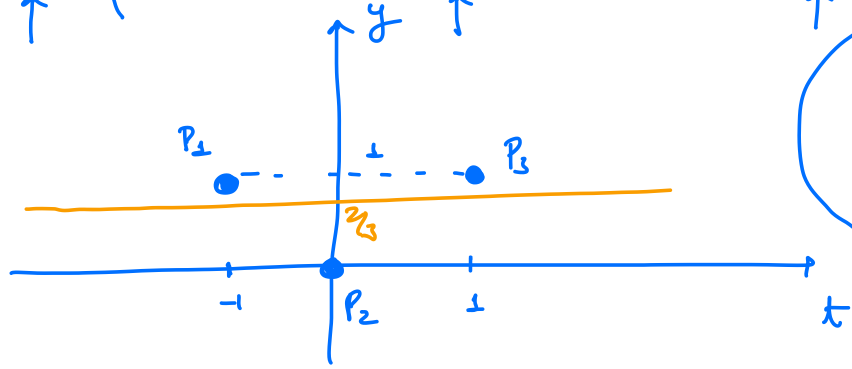
Ax: a Linear Combination of the Columns of A

Solving

$$Ax = b$$

is finding the linear combination of the columns of A that yields b.

P_1 $(-1, 1)$ $\quad$ P_2 $(0, 0)$ $\quad$ P_3 $(1, 1)$



$$y = C + Dt$$

P_1: $\quad C + D(-1) = 1$

P_2: $\quad C = 0$

P_3: $\quad C + D(1) = 1$

$$\Rightarrow \begin{cases} C - D = 1 \\ C = 0 \\ C + D = 1 \end{cases}$$

$Ax = b$ $\quad$ $x = \begin{bmatrix} C \\ D \end{bmatrix}$ $\quad$ $b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\overbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} C \\ D \end{bmatrix}}^{x} = \overbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}^{b}$$

$$\boxed{Ax = b}$$

$$A\hat{x} = p$$

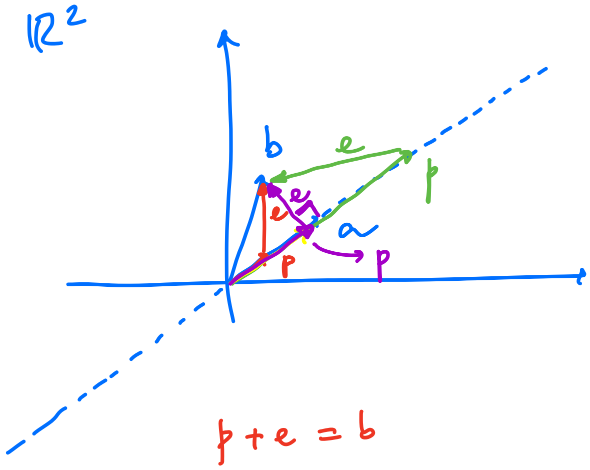
Strang, Intro to LA ← videos on youtube

$$\left(\boxed{\begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = b} \quad \boxed{\underbrace{x_1 \begin{bmatrix} | \\ c_1 \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ c_2 \\ | \end{bmatrix}} = b} \right)$$

$$Ax = b$$

$a, b \in \mathbb{R}^2$

$\mathbb{R}^2$



$p + e = b$

$p + e = b$

$x \in \mathbb{R}$

$$\boxed{a x = b} \leftarrow \text{no solution}$$

$p \leftarrow$ vector along the line spanned by a

$$\boxed{a \hat{x} = p} \leftarrow \text{has a solution}$$

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\underbrace{a \cdot e} = 0$$

$$a \cdot e = \underbrace{a_1 e_1 + a_2 e_2} = 0$$

$$\underbrace{\begin{bmatrix} a_1 & a_2 \end{bmatrix}}_{a^T} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0$$

$$a^T e = 0 \leftarrow \hat{x}\,?$$

$e = b - p$

$$a^T (b - p) = 0$$

$$\boxed{a^T (b - a\hat{x}) = 0} \quad \text{normal equations}$$

$$a^T b = a^T a \hat{x} \rightarrow \boxed{\hat{x} = \frac{a^T b}{a^T a}}$$

Orthogonal Projections

What if $Ax = b$ has no solution? For example, given data points

$$\begin{array}{ccc} t_1 \quad y_1 & t_2 \quad y_2 & t_3 \quad y_3 \\ (-1, 1) & (0, 0) & (1, 1) \end{array}$$

what is the best straight line $\boxed{y = C + Dt}$?

$$\boxed{y = C + Dt}$$

I solve for an approximation $A\hat{x}$ of x:

$$A\hat{x} = p,$$

in which p is the projection of b onto the set of all the linear combinations of the columns of A.

Projection of a Vector onto a Line

1. a line spanned by a;
2. the projection p of b onto the line is $p = \hat{x}a$, $\hat{x} \in \mathbb{R}$
3. $\hat{x}$?

$$a^T(b - a\hat{x}) = 0 \qquad \hat{x} = \frac{a^T b}{a^T a}$$

$$p = a\frac{a^T b}{a^T a}$$

what happens to p

- if b is doubled?
- if a is doubled?

Projection of a Vector onto a Subspace

1. the subspace spanned by the columns of A;
2. the projection p of b onto the subspace is $p = A\hat{x}$, $\hat{x} \in \mathbb{R}^n$
3. $\hat{x}$?

$$\boxed{A^T(b - A\hat{x}) = 0} \qquad \boxed{A^TA\hat{x} = A^Tb}$$

1: little example → $\boxed{a^T(b - a\hat{x}) = 0}$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$y = \hat{C} + \hat{D} t$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} \hat{C} &= 2/3 \\ \hat{D} &= 0 \end{aligned}$$

$$y = 2/3 + 0t$$

Orthogonal Projections

What is the best fitting line $y = C + Dt$ through

$$(-1, 1) \qquad (0, 0) \qquad (1, 1)?$$

Coffee

?