Mathematics for AI

Problem Set: Linear Algebra

August 20, 2022

1. Without calculating the value of the angle θ between $v = [2 \quad 1 \quad -1]^T$ and $w = [3 \quad -4 \quad 1]^T$, explain whether $0 \leq \theta < \pi/2$, or $\theta = \pi/2$ or $\pi/2 < \theta \leq \pi$.

2. Let L be the z-axis and P the xy-plane in $\mathbb{R}^3$.
   a) is the union of P with L a vector space in $\mathbb{R}^3$?
   b) is the intersection of P with L a vector space in $\mathbb{R}^3$?

3. Consider the 3 points of data
$$(t_1, b_1) = (0, 6), \quad (t_2, b_2) = (1, 0), \quad (t_3, b_3) = (2, 0)$$
which we want to fit to a parabola $b = C + Dt + Et^2$ using the least squares approximation.
   a) Write the system of equations $Ax = b$ we would like to solve and identify A, x and b.
   b) What is the projection matrix P corresponding to this least squares fitting?
   c) What is the error vector corresponding to this least squares fitting? Justify.

4. The least-square linear fit to three points $(0, b_1)$, $(1, b_2)$ and $(2, b_3)$ is $C + Dt$ for $C = 1$ and $D = -2$. That is, the fit is the line $1 - 2t$. In this question, the goal is to work backwards from this fit to the unknown values $b = (b_1 \quad b_2 \quad b_3)^T$ at the coordinates $t = 0, 1, 2$.
   a) Write down the explicit equations that b must satisfy for $1 - 2t$ to be the least-squares linear fit.
   b) If all the points fall exactly on the line $1 - 2t$, what are the components of b? Check that this b satisfies the normal equations.
   c) More generally, if all the points $(a_1, b_1), \ldots, (a_n, b_n)$ fall exactly on any straight line, then b is a linear combination of which vectors?

Some of the following problems are chosen from Strang's book: Introduction to Linear Algebra, 5th International Edition, 2016, Author: Gilbert Strang, Wellesley-Cambridge Press.

SOLUTIONS:

1. We look at the dot product of the 2 vectors: $v^T w = 6 - 4 - 1 = 3 > 0$. From the cosine formula, this implies that $\cos\theta > 0$, and therefore, from the properties of the cosine function, we know that $0 < \theta < \pi/2$.

2. a) We use contradiction to show that the union of P with L is not a vector space: the sum of any non-zero vector on the z axis with any non-zero vector in P is a vector that does not belong to the union. For example, $[0 \quad 0 \quad 1]^T + [1 \quad 1 \quad 0]^T = [1 \quad 1 \quad 1]^T$ does not belong to either L or P, and thus it does not belong to their union. Therefore, the union of P with L is not a vector space.

   b) The intersection of L with P is the zero vector, which is a vector space. The answer is yes.

3. a) If the parabola goes through the 3 given points, then the following three equations would be satisfied:

$$6 = C, \qquad 0 = C + D + E, \qquad 0 = C + 2D + 4E,$$

   which can easily be written in matrix form as

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \\ E \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}}_{b}.$$

   b) The projection matrix is the 3×3 identity matrix. Matrix A has 3 linearly independent columns which span the whole $\mathbb{R}^3$. Therefore, we are projecting b onto $\mathbb{R}^3$. The projection p is b and the projection matrix is the identity matrix.

   c) The error vector is $e = b - p = b - b = 0$, because $p = b$.

4. a) The system that we would solve if the line passed exactly through all of the points is

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

   Since the line may not pass through all the points this system may have no solution, and instead we find the least-square solution by solving the normal equations:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

which is

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} b_1 + b_2 + b_3 \\ b_2 + 2b_3 \end{bmatrix}$$

with solution $C = 1$ and $D = -2$. So the components of b have to verify

$$b_1 + b_2 + b_3 = -3 \qquad b_2 + 2b_3 = -7.$$

b) If b falls exactly on the line:

$$b_1 = 1 - 2 \quad 0 = 1, \qquad b_2 = 1 - 2 \quad 1 = -1 \qquad b_3 = 1 - 2 \quad 2 = -3.$$

c) If all points fall exactly on any straight line, the system

$$\begin{bmatrix} 1 & a_1 \\ 1 & a_2 \\ \vdots & \vdots \\ 1 & a_n \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

would have a solution, that is, b is a linear combination of the columns of

$$\begin{bmatrix} 1 & a_1 \\ 1 & a_2 \\ \vdots & \vdots \\ 1 & a_n \end{bmatrix}.$$