Fundamentals of Calculus

Mathematics for AI

August 21, 2022

Our Goal

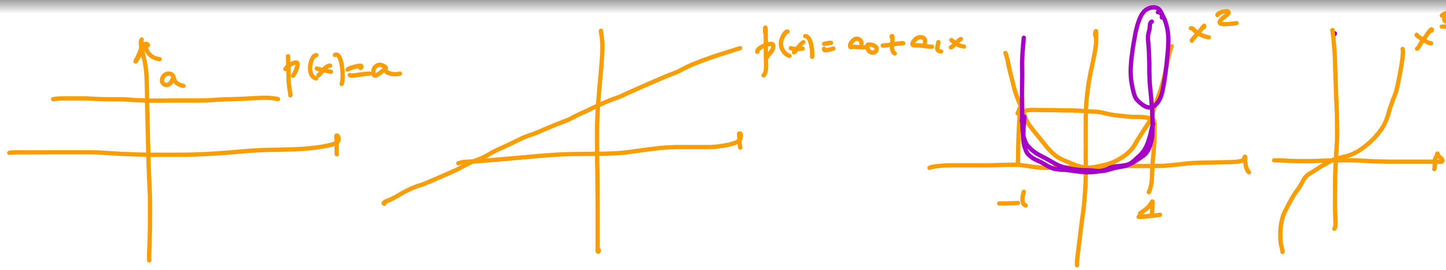
1. composition of functions;
2. behavior of a function from its derivative;
3. linear and quadratic approximations;
4. functions of more than one variable.

$$x \in \mathbb{R} \longrightarrow \boxed{\underline{f}} \longrightarrow f(x) \in \mathbb{R}$$

Function

A function f is a rule that assigns to each element $x \in D_f$ exactly one value, $f(x)$.

Polynomials



Polynomial

A function P is a polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where n is a nonnegative integer and the coefficients $a_n, \ldots, a_0$ are constants. The domain of any polynomial is the real line.

$f(x) = x^{-1} = \frac{1}{x}$

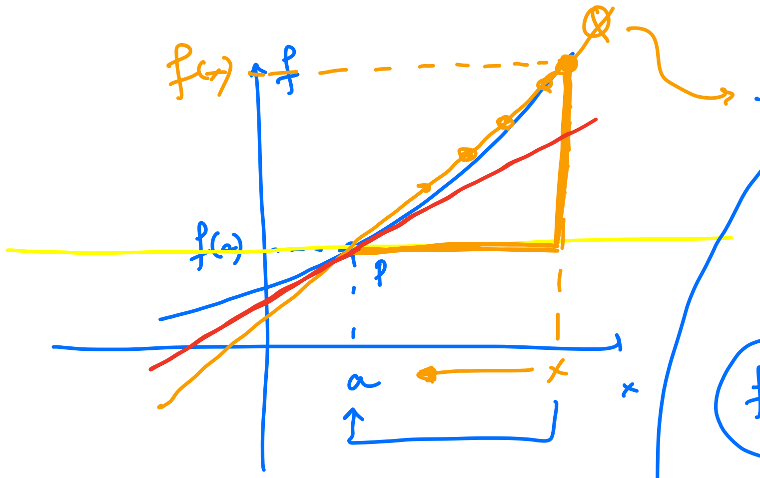
$h(x) = x^{1/2} = \sqrt{x}$

Approximation of a Function

Question

How to approximate a function?

1. P is $(a, f(a))$ and Q is $(x, f(x))$;
2. slope of secant line through P and Q is $\frac{\Delta f}{\Delta x} = \frac{f(x)-f(a)}{x-a}$;
3. consider the $\frac{f(x)-f(a)}{x-a}$ as $x \to a$.b



$$y - y_0 = m(x - x_0)$$

$$\underbrace{y - f(a)}_{y_0} = m(x - \underbrace{a}_{x_0})$$

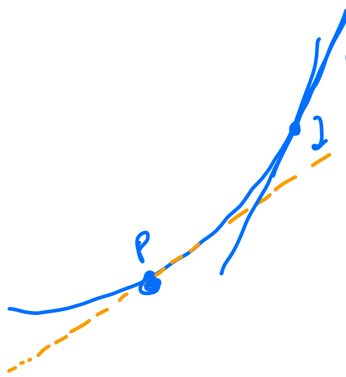
$$m = \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \boxed{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

$P\ (a, f(a))$

$$\boxed{y - f(a) = m(x - a)}$$ secant

$$\boxed{y - f(a) = f'(a)(x - a)}$$ tangent line



$$\boxed{y - f(a) = f'(a)(x - a)}$$

$$\boxed{l(x) = f'(a)(x - a) + f(a)}$$

the linear approx to the function at $(a, f(a))$

quadratic approximation to $f(x)$ at

$$\boxed{g(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2}$$

Limit

$$f(x) = \frac{\sin x}{x}$$

x	$\frac{\sin x}{x}$	x	$\frac{\sin x}{x}$
1	0.841470985	-1	0.841470985
0.5	0.958851077	-0.5	0.958851077
0.1	0.998334166	-0.1	0.998334166
0.05	0.999583385	-0.05	0.999583385
0.01	0.999983333	-0.01	0.999983333
0.005	0.999995833	-0.005	0.999995833
0.001	0.999999833	-0.001	0.999999833
$x \to 0^+$	$f(x) \to 1$	$x \to 0^-$	$f(x) \to 1$

Limit

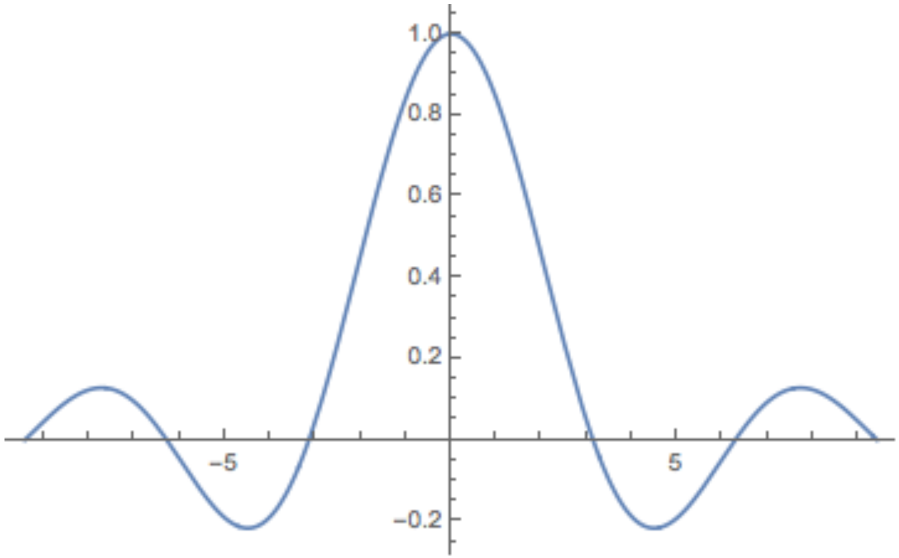


Figure: Graph of $f(x) = \frac{\sin x}{x}$

Limit

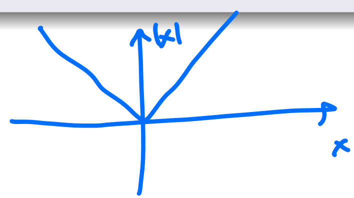
Limit

Let $f(x)$ be defined for all x near c but not necessarily at c. The limit of $f(x)$ as x approaches c is L,

$$\lim_{x \to c} f(x) = L,$$

if $|f(x) - L|$ becomes arbitrarily small when x is sufficiently close to c but not equal to c.

Given $f(x) = |x|$, what is $\lim_{x \to 0} |x|$?

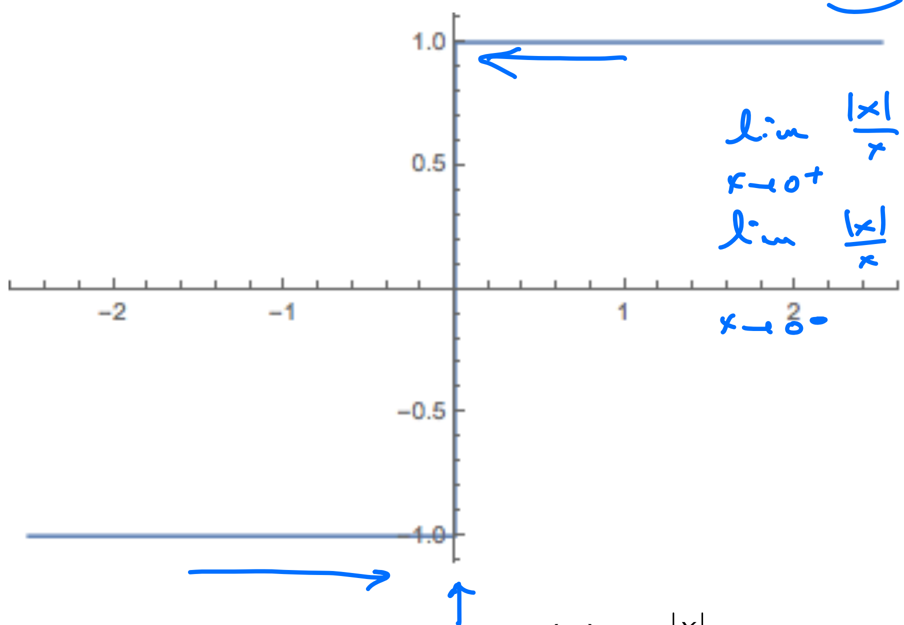


One-Sided Limits

$f(x)=\frac{|x|}{x}$

$\lim_{x\to 0}\frac{|x|}{x}=?$

$\frac{|x|}{x}=\begin{cases}1 & x>0\\ -1 & x<0\end{cases}$



$\lim_{x\to 0^+}\frac{|x|}{x}=1$

$\lim_{x\to 0^-}\frac{|x|}{x}=-1$

$\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Figure: Graph of $f(x)=\frac{|x|}{x}$

Derivative

Question

How to approximate a function?

1. P is $(a, f(a))$ and Q is $(x, f(x))$;
2. slope of secant line through P and Q is $\frac{\Delta f}{\Delta x} = \frac{f(x)-f(a)}{x-a}$;
3. consider the limit $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.

Derivative

If the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

and f is **differentiable** at $x = a$.

Tangent Line

Assume f is differentiable at $x = a$. The **tangent line** to the graph of f at point $P = (a, f(a))$ is the line through P and slope $f'(a)$:

$$y - f(a) = f'(a)(x - a).$$

What is the equation of the tangent line to the graph of $f(x) = x^2$ at $x = 0$?

Linear Approximation

Linear Approximation

Assume $f'(a)$ exists. The linear approximation of f at $(a, f(a))$ is

$$\ell(x) = f(a) + f'(a)(x - a).$$

What is the linear approximation of $f(x) = x^2$ at $x = 5$?

Quadratic Approximation

Assume $f'(a)$ and $f''(a)$ exist. The quadratic approximation of f at $(a, f(a))$ is

$$q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

Taylor Polynomials

Taylor Polynomials

Assume the first n derivatives of f exist at $x = a$. The nth order polynomial approximation of f at point $(a, f(a))$ is

$$p_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

Question

How to predict the graph of $f = x^3 - 12x$ based on f' and f''?

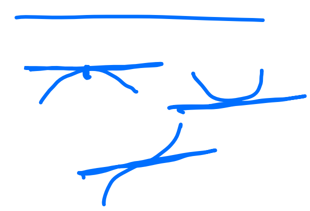
① $f(x) = x(x^2 - 12) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{12} = \pm 2\sqrt{3}$

② $f'(x) = \underbrace{3x^2 - 12} = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

③ $f''(x) = 6x$

		$-2\sqrt{3}$		-2		0		$+2$		$+2\sqrt{3}$	
$f(x)$	inc	0	inc		dec	0	dec		inc	0	inc
$f'(x)$	+		+	0	−		−	0	+		+
$f''(x)$	∩ −		∩ −		∩ −		∪ +		∪ +		∪ +

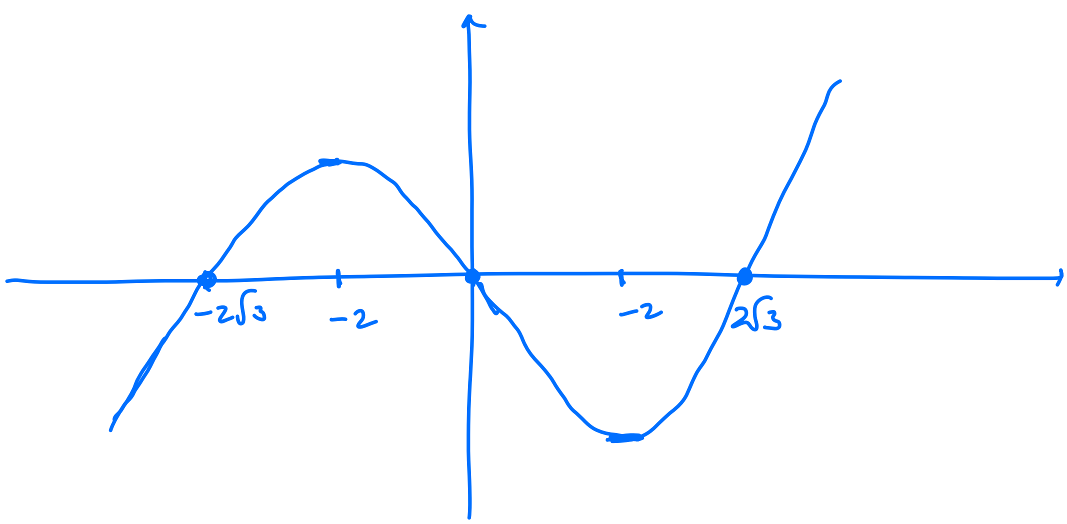
$$f'(a) = \frac{\Delta f}{\Delta x} = \lim_{x \to a} \frac{\overbrace{f(x) - f(a)}^{\Delta f}}{\underbrace{x - a}_{\Delta x}}$$



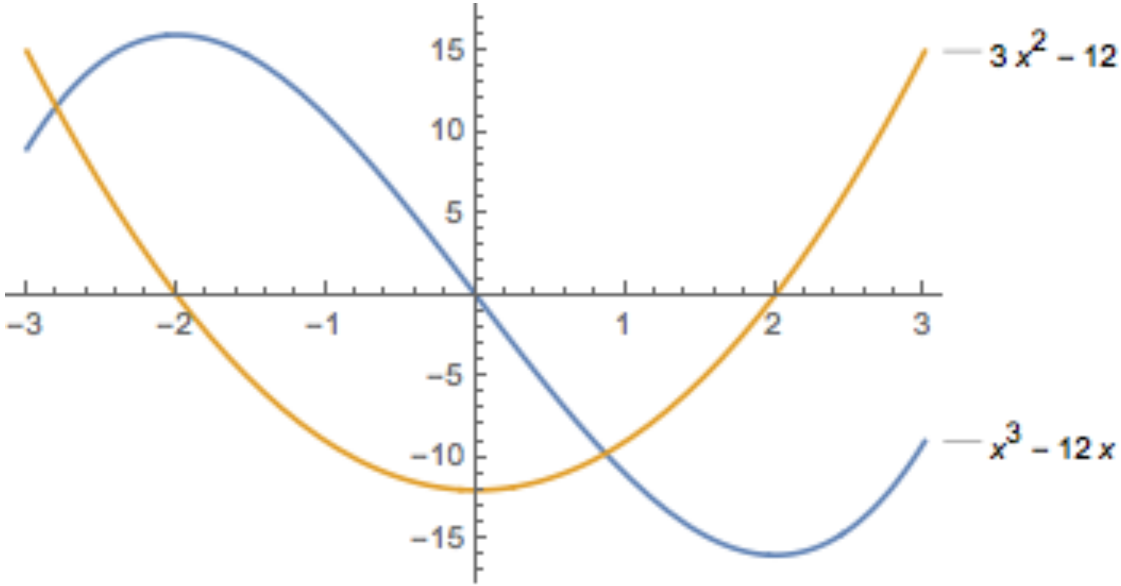
$f' > 0 \Rightarrow f$ is increasing

$f' < 0 \Rightarrow f$ is decreasing

$f' = 0$



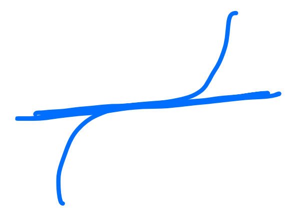
Behavior of f from f'



Behavior of f from f'

1. if a is a local maximizer or minimizer, then $f'(a) = 0$;
2. if $f' > 0$, then f increases;
3. if $f' < 0$, then f decreases;
4. if $f'' > 0$, then f is concave up;
5. if $f'' < 0$, then f is concave down.

a is max / min $\Longrightarrow$ $f'(a)=0$



Composition of Functions

Given two functions f and g, the composite function $f \circ g$ is defined as

$$(f \circ g)(x) = f(g(x)).$$

$$\xrightarrow{x} \boxed{g} \xrightarrow{g(x)} \boxed{f} \longrightarrow \underbrace{f(g(x))} = (f \circ g)(x)$$

Given $f(x) = x^2$ and $g(x) = x - 3$, determine $f \circ g$ and $g \circ f$.

$$\rightarrow (f \circ g)(x) = f(g(x)) = f(\underbrace{x-3}) = (x-3)^2 \leftarrow$$

$$(g \circ f)(x) = g(\underbrace{x^2}) = x^2 - 3 \leftarrow$$

Question

What is the derivative of $F(x) = \sqrt{x^2 + 1}$?

$f(x) = \sqrt{x}$; $f'(x) = \frac{1}{2} x^{-1/2}$ $\leftarrow$ $F'(x) = ?$

$g(x) = x^2+1$; $g'(x) = 2x$ $\leftarrow$

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $\boxed{F = f \circ g}$ is differentiable at x and F' is

$$\boxed{F'(x) = f'(g(x))g'(x).}$$

$$F'(x) = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

Moving Forward

....

Real Function of Two Variables

$$f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$$

$$f = f(x, y)$$

1. linear approximation?
2. maxima and minima?

Partial Derivatives

$$f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$$

$$f_x \equiv \frac{\partial f}{\partial x}$$

$$f_y \equiv \frac{\partial f}{\partial y}$$

What are f_x and f_y for $f(x, y) = x^3 y^5$?

$f_x = 3x^2 y^5$

$f_y = 5y^4 x^3$

The Gradient

$$f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$$

$$\nabla f = \langle f_x, f_y \rangle$$

Maxima and Minima

Maxima and Minima

If (a, b) is a local maximizer or minimizer, then $\nabla f(a, b) = 0$.

Linear Approximation of $g : D_g \subseteq \mathbb{R} \to \mathbb{R}$

Assume $g'(a)$ exists. The linear approximation of g at $(a, g(a))$ is

$$\boxed{\ell(x) = g(a) + g'(a)(x - a).}$$

Linear Approximation of $f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$

Assume f is differentiable at (a, b). The linear approximation of f at $(a, b, f(a, b))$ is

$$\boxed{L(x, y) = f(a, b) + f_x(a, b)(x - a) + \underbrace{f_y(a, b)(y - b)}.}$$

Coffee Break

alexandra.gomes@kaust.edu.sa

?