Mathematics for AI

Problem Set: Calculus

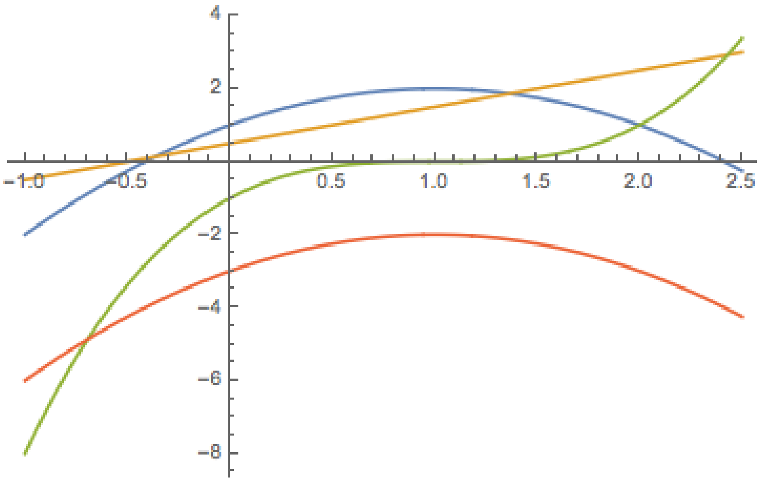
August 20, 2022

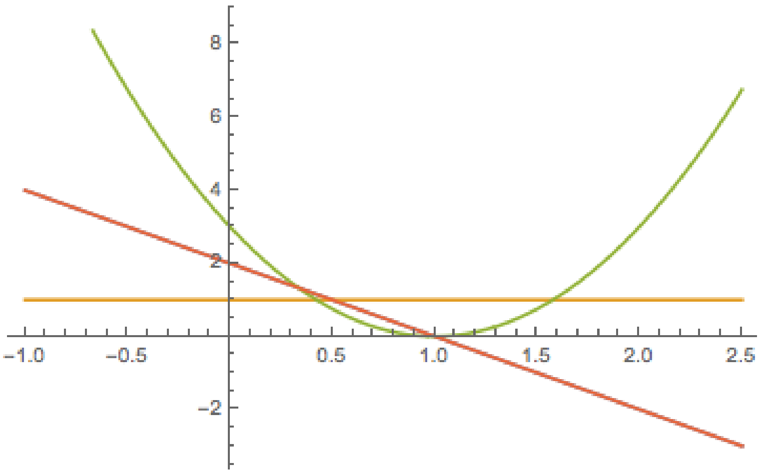
Some of the following problems are chosen from Dennis G. Zill, Advanced Engineering Mathematics, 6th ed., Jones & Bartlett Learning.

1. Find all the values of x where the tangent lines to $y = x^3$ and $y = x^4$ are parallel.
2. Find the equation of the tangent line of $y = e^{x+2}$ at $x = -1$.
3. Find the Taylor polynomial of the kth order, p_k, of $\ln x$ at $x = 1$. Draw the graphs of $\ln x, p_1$ and p_2, and note that, in a neighborhood of $x = 1$, the graph of p_2 is closer to the graph of $\ln x$ than that of p_1.
4. Use the chain rule to find the derivative of $f \circ g$ for $f(u) = \sin u$ and $g(x) = 2x + 1$.
5. What is the least squares best fitting line $y = \hat{C} + \hat{D}t$ through

$$(-1, 1) \qquad (0, 0) \qquad (1, 1)?$$

   Minimize the sum of the squares of the errors at $t = -1, 0, 1$.
6. Match the functions in the graphs of the first figure with their derivatives in the second figure:





SOLUTIONS:

1. We look for the tangent lines that have the same slope. The slope of the tangent line to the graph of a function at a point is the value of the derivative of the function. As such, we need only to equate the derivatives of x^3 and x^4:

$$3x^2 = 4x^3$$

   to find $x = 0$ or $x = 3/4$.

2. The equation for the line of slope m through the point (x_0, y_0) is

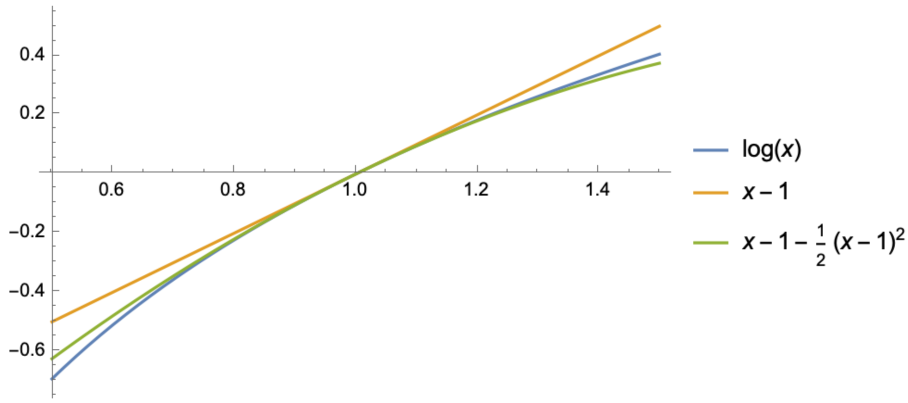
$$y - y_0 = m(x - x_0).$$

   In our case, $x_0 = -1$ and $y_0 = e$. The slope is the derivative of $y = e^{x+2}$ at $x = x_0 = -1$: $y' = e^{x+2}$ evaluated at -1 is e. Finally, the equation of the requested line is

$$y - e = e(x+1), \qquad y = e(x+2).$$

3. The Taylor poly of $\ln x$ at $x = 1$ is

$$p_k(x) = \sum_{n=1}^{k} \frac{(-1)^{n-1}}{n}(x-1)^n.$$



4. The chain rule gives the derivative of $f \circ g$ as

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

In our case, $f' = \cos u$ and $g' = 2$. Then,

$$(f \circ g)'(x) = (\sin(2x+1))' = \cos(2x+1)2 = 2\cos(2x+1).$$

5. We want to find $\hat{C}, \hat{D}$ such that the error is minimized in the least squares sense. At each given value of t, t_i, the square of the error, e_i^2, is

$$e_i^2 = (y_i - \hat{C} - \hat{D}t_i)^2.$$

As such, we wish to minimize

$$f(\hat{C}, \hat{D}) = \sum_{i=1}^{3} e_i^2 = (1 - \hat{C} + \hat{D})^2 + \hat{C}^2 + (1 - \hat{C} - \hat{D})^2 = 2 - 4\hat{C} + 3\hat{C}^2 + 2\hat{D}^2.$$

This is a quadratic function of both $\hat{C}$ and $\hat{D}$, with positive coefficients of both $\hat{C}^2$ and $\hat{D}^2$. As such, the minimum is found by setting the derivative of f with respect to both $\hat{C}$ and $\hat{D}$ to zero:

$$\frac{\partial f}{\partial \hat{C}} = -4 + 6\hat{C} = 0, \qquad \frac{\partial f}{\partial \hat{D}} = 4\hat{D} = 0.$$

These are the normal equations. Note that these are linear in both $\hat{C}$ and $\hat{D}$. Solving, yields $\hat{C} = 2/3$ and $\hat{D} = 0$.

6. Yellow to yellow, green to green and the functions blue and red to the derivative in the red.