Fundamentals of Linear Algebra

Mathematics for AI

November 27, 2022

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Our Goal

1. Linear Combinations of Vectors and Vector Spaces
2. The Dot Product
3. Matrices
4. Orthogonal Projections

Vectors in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3$$

$$\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$$

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Vector Operations

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \qquad a \in \mathbb{R}$$

1. Multiplication of a vector $\mathbf{v}$ by a scalar a:

$$a\mathbf{v} = a \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} av_1 \\ \vdots \\ av_n \end{bmatrix};$$

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2. Sum of $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}.$$

Linear Combinations

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \qquad a, b \in \mathbb{R}$$

$$\boxed{a\mathbf{v} + b\mathbf{w}\text{: linear combination of } v \text{ and } w \text{ with coefficients } a \text{ and } b}$$

Example

1. $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ is a linear combination of $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
2. Express $\mathbf{u} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

The Important Questions and Pictures

$$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \qquad a, b, c \in \mathbb{R}$$

1. What is the picture of all combinations $a\mathbf{u}$?
2. What is the picture of all combinations $a\mathbf{u} + b\mathbf{v}$?
3. What is the picture of all combinations $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$?

Vector Spaces

V is a vector space in $\mathbb{R}^n$ if it is a subset of $\mathbb{R}^n$ that contains the vectors and their linear combinations.

A vector space always contains the zero vector.

Example

1. $\mathbb{R}^2$

What are all the vector spaces in $\mathbb{R}^2$?

The subspaces of $\mathbb{R}^2$:

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3. $\mathbb{R}^2$

Span

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. The span of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is the set of all the linear combinations of the vectors.

$$\text{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \{a_1 v_1 + a_2 v_2 + \ldots a_r v_r\}$$

Example

$$\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} =$$

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Example

$$\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \mathbb{R}^2$$

Linear Independence

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. The vectors are linearly dependent if one of the $v_i's$ can be expressed as a linear combination of the others. Otherwise they are linearly independent.

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Example

1. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$ are linearly dependent.
2. $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -3 \end{bmatrix}$ are linearly independent.

Basis

A basis for a vector space is a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in the vector space with two properties:

1. the vectors are linearly independent;
2. the vectors span the space.

A basis is not unique.

The Dot Product

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$$

$$\boxed{\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2}$$

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$$

$$\boxed{\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \ldots + v_n w_n = \sum_{i=1}^{n} v_i w_i}$$

The Angle Between Two Vectors

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$$

- $||\mathbf{v}|| = \sqrt{v.v}$
- Cosine formula:
$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| ||\mathbf{w}|| \cos\theta$$
- $\mathbf{v} \cdot \mathbf{w} = 0 \quad \iff \quad \mathbf{v} \perp \mathbf{w}$

Let $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

1. What space do **u** and **v** belong to?
2. Are **u** and **v** perpendicular?

Matrix

A matrix $A_{m \times n}$ is an ordered collection of numbers arranged in a $m \times n$ rectangular format:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

- m is the number of rows;
- n is the number of columns.
- $m \times n$ is the size of the matrix.

Matrix Addition

Let A and B be 2 matrices of the same size $m \times n$. Then, their sum $A + B$ is defined as

$$\begin{aligned} A + B &= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix}. \end{aligned}$$

Note that $A + B = B + A$.

Scalar Multiplication of a Matrix

Let A be a $m \times n$ matrix and $k \in \mathbb{R}$. Then, kA is defined as

$$kA = \begin{bmatrix} ka_{11} & \dots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \dots & ka_{mn} \end{bmatrix}.$$

Matrix Multiplication

$$A = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}_{2\times 3} \text{ and } B = \begin{bmatrix} s_1 & t_1 \\ s_2 & t_2 \\ s_3 & t_3 \end{bmatrix}_{3\times 2}$$

$$AB = \begin{bmatrix} us & ut \\ vs & vt \end{bmatrix}_{2\times 2}$$

Matrix Multiplication: Linear Combination of Vectors

If

$$u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \qquad v = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \qquad w = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

their linear combination $au + bv + cw$, $a, b, c \in \mathbb{R}$ can be written as

$$a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ 2a + 3b \\ 4b + 5c \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{x}.$$

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Ax: a Linear Combination of the Columns of A

Solving

$$Ax = b$$

is finding the linear combination of the columns of A that yields b.

System of Linear Equations: Matrices

$$\begin{cases} x_1 - x_2 + 2x_3 & = 1 \\ \qquad\; x_2 + 3x_3 & = 1 \end{cases}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} \dots & r_1 & \dots \\ \dots & r_2 & \dots \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \vdots \\ x \\ \vdots \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b}.$$

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Linear Systems in Two Unknowns

Usually this system is represented as:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{cases} \iff \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{b}$$

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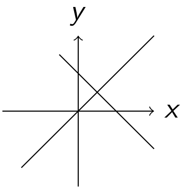
The system has at least one solution or no solution.

When the system has a solution then we can write b as a linear combinations of the column vectors of A

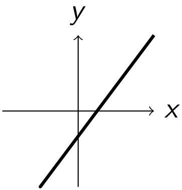
Type of Solutions

$$\begin{cases} x - y &= 0 \\ x + y &= 1 \end{cases}$$



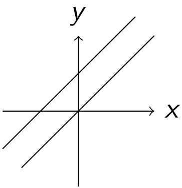
Unique solution

$$\begin{cases} x - y &= -1 \\ 2x - 2y &= -2 \end{cases}$$



Infinitely many solutions

$$\begin{cases} x - y &= -1 \\ x - y &= 0 \end{cases}$$



No solution

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For example, given data points

$$(-1,-1) \qquad (0,-3) \qquad (1,1)$$

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We solve for an approximation $\hat{x}$ of x:

$$A\hat{x} = p,$$

in which p is the projection of b onto the set of all the linear combinations of the columns of A.

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Projection of a Vector onto a Line

1. a line spanned by a;
2. the projection p of b onto the line is $p = \hat{x}a$, $\hat{x} \in \mathbb{R}$
3. $\hat{x}$?

$$a^T(b - a\hat{x}) = 0 \qquad \hat{x} = \frac{a^T b}{a^T a}$$

$$p = a\frac{a^T b}{a^T a}$$

what happens to p

- if b is doubled?
- if a is doubled?

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Projection of a Vector onto a Subspace

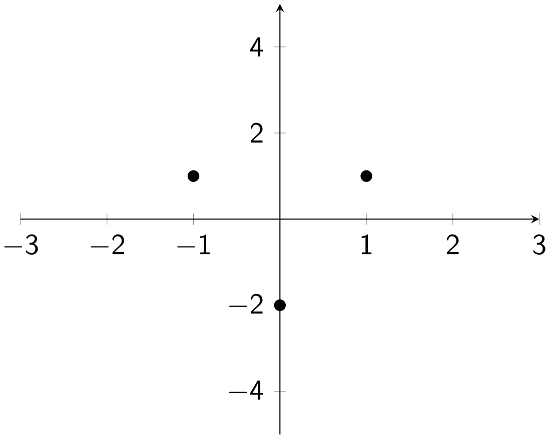
1. the subspace spanned by the columns of A;
2. the projection p of b onto the subspace is $p = A\hat{x}$, $\hat{x} \in \mathbb{R}^n$
3. $\hat{x}$?

$$A^T(b - A\hat{x}) = 0 \qquad A^T A\hat{x} = A^T b$$

Example

What is the best fitting line $y = C + Dt$ through

$$(-1,1) \qquad (0,-2) \qquad (1,1)?$$



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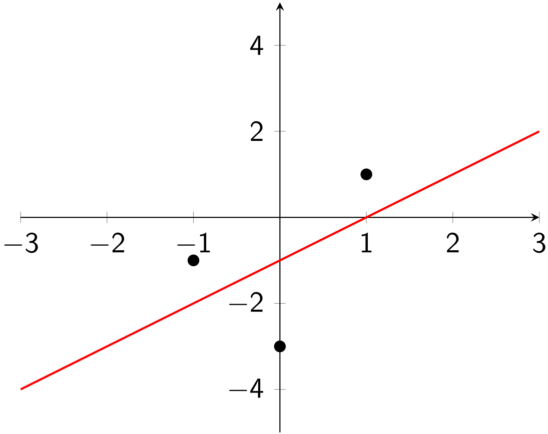

Coffee

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