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Fundamentals of Linear Algebra

Mathematics for AI

November 27, 2022

King Abdullah University of Science and Technology

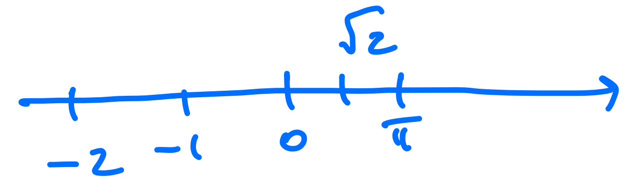
KAUST ACADEMY

Our Goal

1. Linear Combinations of Vectors and Vector Spaces
2. The Dot Product
3. Matrices
4. Orthogonal Projections

Vectors in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$

$\mathbb{R} \leftarrow$ set of real numbers



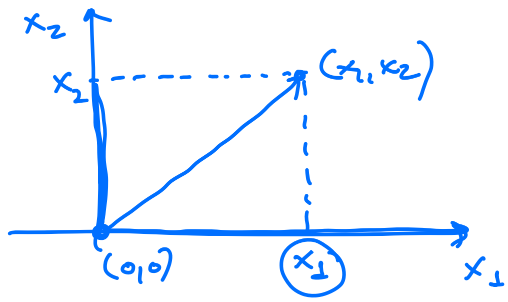
$(x_1, x_2) \leftarrow$ pair of real numbers

$(x_1, x_2) \in \mathbb{R}^2$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3$$

$$\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$$

$\mathbb{R}^2$



$$v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

Vector Operations

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \qquad a \in \mathbb{R}$$

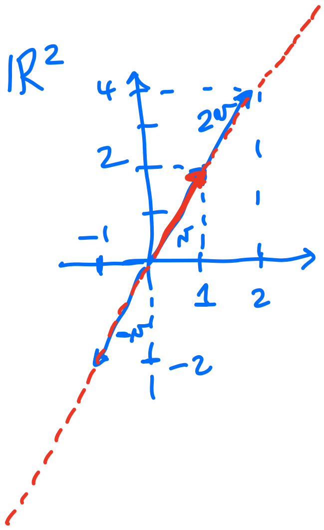
1. Multiplication of a vector $\mathbf{v}$ by a scalar a:

$$a\mathbf{v} = a \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} av_1 \\ \vdots \\ av_n \end{bmatrix};$$

$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad a = 2 \qquad av = \begin{bmatrix} 2.1 \\ 2.2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$a = -1 \qquad av = \begin{bmatrix} (-1).1 \\ (-1)\,2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} =$

$\mathbb{R}^2$



Vector Operations

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \qquad a \in \mathbb{R}$$

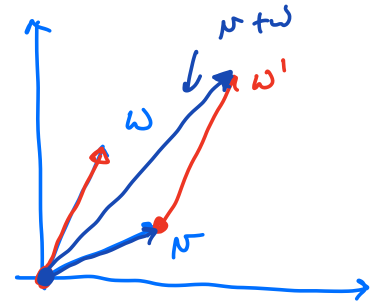
1. Multiplication of a vector $\mathbf{v}$ by a scalar a:

$$a\mathbf{v} = a \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} av_1 \\ \vdots \\ av_n \end{bmatrix};$$

2. Sum of $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}.$$

$\mathbb{R}^2$



$v + w$

$$\overbrace{\begin{bmatrix}1\\2\end{bmatrix}}^{v} + \overbrace{\begin{bmatrix}-1\\0\end{bmatrix}}^{w} = \begin{bmatrix}1+(-1)\\2+0\end{bmatrix} = \overbrace{\begin{bmatrix}0\\2\end{bmatrix}}^{v+w}$$

Linear Combinations

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \qquad a, b \in \mathbb{R}$$

$$\boxed{a\mathbf{v} + b\mathbf{w}\text{: linear combination of } v \text{ and } w \text{ with coefficients } a \text{ and } b}$$

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \qquad a, b \in \mathbb{R}$$

$$\boxed{a\mathbf{v} + b\mathbf{w}\text{: linear combination of } v \text{ and } w \text{ with coefficients } a \text{ and } b}$$

$$u = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example

1. $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ is a linear combination of $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
2. Express $\mathbf{u} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

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$$u = \begin{bmatrix}4\\4\end{bmatrix} = a\underbrace{\begin{bmatrix}6\\2\end{bmatrix}}_{v} + b\underbrace{\begin{bmatrix}2\\4\end{bmatrix}}_{w}$$

a? b?

$$\begin{cases} a6 + b2 = 4 \\ a2 + b4 = 4 \end{cases} \longrightarrow \underbrace{\begin{bmatrix}6 & 2\\2 & 4\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}a\\b\end{bmatrix}}_{x} = \underbrace{\begin{bmatrix}4\\4\end{bmatrix}}_{b}$$

$$x = A^{-1}b$$

$$= \underbrace{\frac{1}{\det A}\begin{bmatrix}4 & -2\\-2 & 6\end{bmatrix}}_{A^{-1}}\begin{bmatrix}4\\4\end{bmatrix}$$

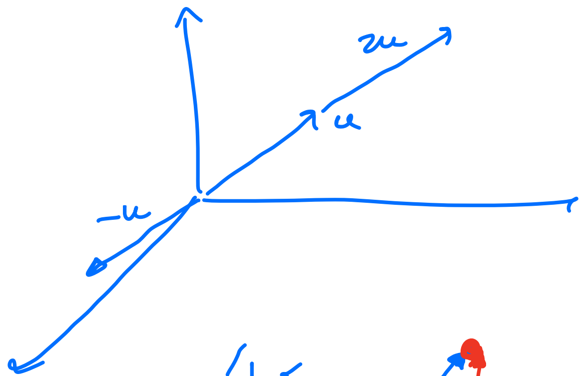
$$= \frac{1}{}$$

The Important Questions and Pictures

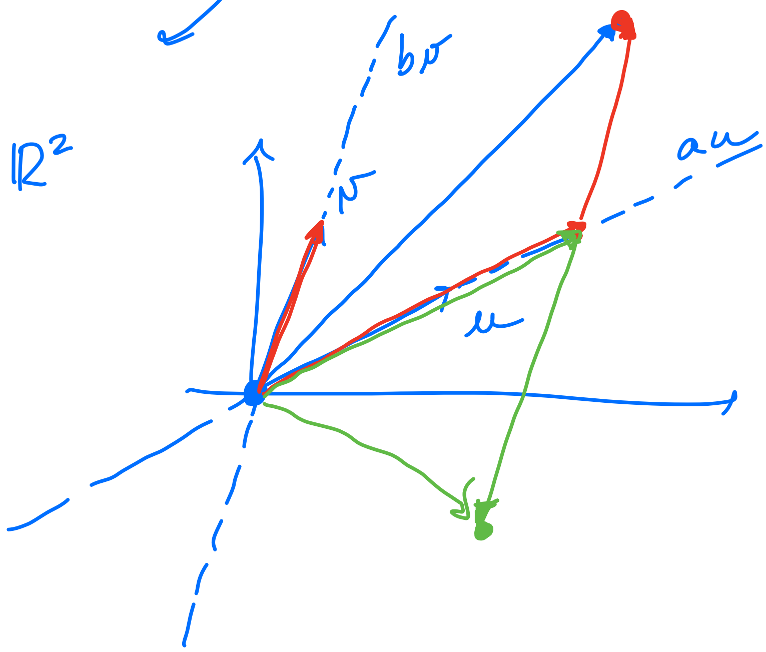
$$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \qquad a, b, c \in \mathbb{R}$$

1. What is the picture of all combinations $a\mathbf{u}$?
2. What is the picture of all combinations $a\mathbf{u} + b\mathbf{v}$?
3. What is the picture of all combinations $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$?

$\mathbb{R}^3$

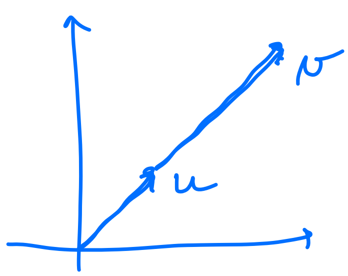


$\mathbb{R}^2$



$$u = v = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies au + bv = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mathbb{R}^2$



$au + bv$?

Vector Spaces

V is a vector space in $\mathbb{R}^n$ if it is a subset of $\mathbb{R}^n$ that contains the vectors and their linear combinations.

A vector space always contains the zero vector.

Example

1. $\mathbb{R}^2$

Examples of vector spaces:

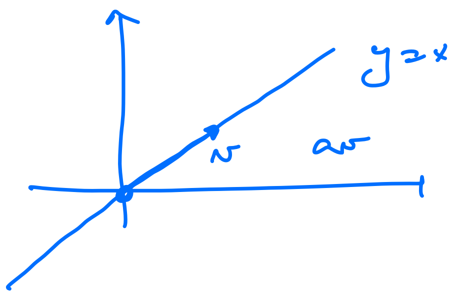
→ $[0, 1] \in \mathbb{R}$ ← not a vector space

→ $\mathbb{R}$? yes, vector space

→ $\{0\} \in \mathbb{R}$ yes, vector space

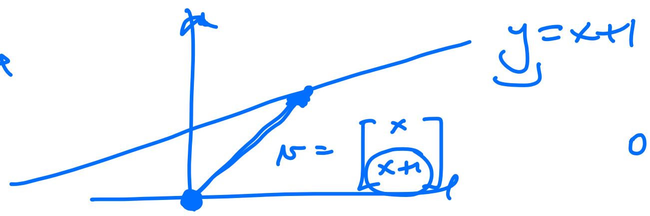
→ $\mathbb{R}^2$? yes, vector space.

→



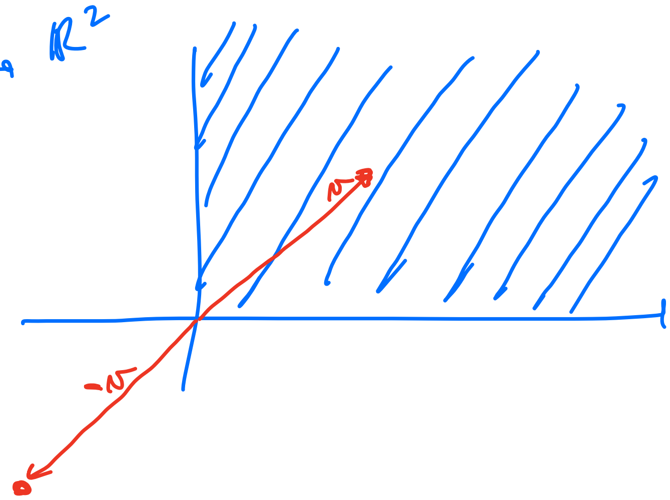
yes, vector space

→



$0v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

→ $\mathbb{R}^2$



$Q_1 = \{ (x,y) \in \mathbb{R}^2;\ x \geq 0 \wedge y \geq 0 \}$

What are all the vector spaces in $\mathbb{R}^2$?

The subspaces of $\mathbb{R}^2$:

What are all the vector spaces in $\mathbb{R}^2$?

The subspaces of $\mathbb{R}^2$:

1. the zero vector

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The subspaces of $\mathbb{R}^2$:

1. the zero vector
2. lines through the zero vector

What are all the vector spaces in $\mathbb{R}^2$?

The subspaces of $\mathbb{R}^2$:

1. the zero vector
2. lines through the zero vector
3. $\mathbb{R}^2$

Span

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. The span of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is the set of all the linear combinations of the vectors.

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \{a_1 v_1 + a_2 v_2 + \ldots a_r v_r\}$$

Example

$$\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} =$$

$$a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

Span

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. The span of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is the set of all the linear combinations of the vectors.

$$\text{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\} = \{a_1 v_1 + a_2 v_2 + \dots a_r v_r\}$$

Example

$$\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \mathbb{R}^2$$

Linear Independence

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. The vectors are linearly dependent if one of the $v_i's$ can be expressed as a linear combination of the others. Otherwise they are linearly independent.

Linear Independence

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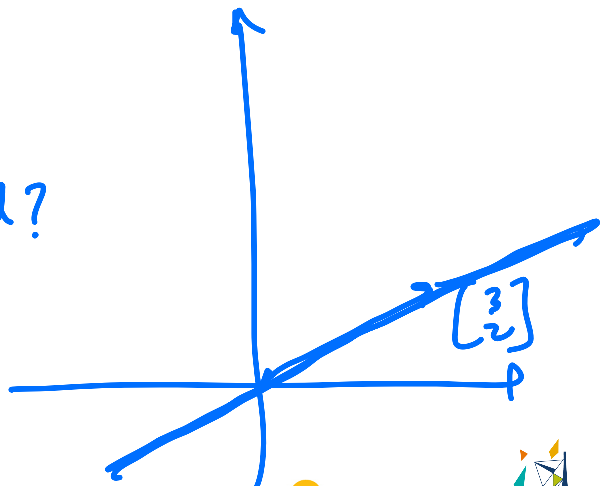
Example

1. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$ are linearly dependent.

$\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$ are they lin dep, or ind?

$\begin{bmatrix} 6 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow$ lin. dep.



Linear Independence

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Example

1. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$ are linearly dependent.
2. $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -3 \end{bmatrix}$ are linearly independent.

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Basis

A basis for a vector space is a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in the vector space with two properties:

1. the vectors are linearly independent;
2. the vectors span the space.

A basis is not unique.

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

$\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \; : \; \begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The Dot Product

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$$

$$\boxed{\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2}$$

$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$$

$$\boxed{\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + \ldots + v_n w_n = \sum_{i=1}^{n} v_i w_i} \in \mathbb{R}$$

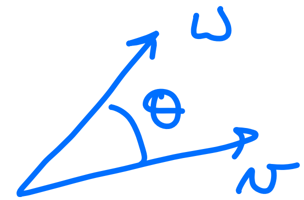
$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad v \cdot w = 1 \times 2 + 2 \times 4 = 2 + 8 = 10$$

The Angle Between Two Vectors

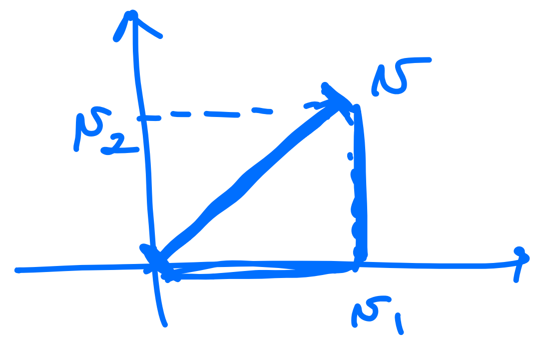
$$\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$$

- $||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$
- Cosine formula:

$$\boxed{\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\,||\mathbf{w}||\cos\theta}$$



- $\mathbf{v} \cdot \mathbf{w} = 0 \quad \Longleftrightarrow \quad \mathbf{v} \perp \mathbf{w}$



$$||v||^2 = v_1^2 + v_2^2$$

$$||v|| = \sqrt{v_1^2 + v_2^2} = \sqrt{v \cdot v}$$

Let $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

1. What space do $\mathbf{u}$ and $\mathbf{v}$ belong to?
2. Are $\mathbf{u}$ and $\mathbf{v}$ perpendicular?

$u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

1. $\mathbb{R}^2$

2. $u \cdot v = 4(-1) + 2 \cdot 2 = -4 + 4 = 0$, ye

Matrix

A matrix $A_{m\times n}$ is an ordered collection of numbers arranged in a $m \times n$ rectangular format:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

- m is the number of rows;
- n is the number of columns.
- $m \times n$ is the size of the matrix.

Matrix Addition

Let A and B be 2 matrices of the same size $m \times n$. Then, their sum $A + B$ is defined as

$$\begin{aligned} A + B &= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix}. \end{aligned}$$

Note that $A + B = B + A$.

Scalar Multiplication of a Matrix

Let A be a $m \times n$ matrix and $k \in \mathbb{R}$. Then, kA is defined as

$$kA = \begin{bmatrix} ka_{11} & \dots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \dots & ka_{mn} \end{bmatrix}.$$

Matrix Multiplication

$$A = \begin{array}{c} u \\ \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}_{2\times 3} \\ v \end{array} \quad \text{and} \quad B = \begin{array}{c} \begin{array}{cc} s & t \end{array} \\ \begin{bmatrix} s_1 & t_1 \\ s_2 & t_2 \\ s_3 & t_3 \end{bmatrix}_{3\times 2} \end{array}$$

$$AB = \begin{bmatrix} \mathbf{u}\cdot\mathbf{s} & \mathbf{u}\cdot\mathbf{t} \\ \mathbf{v}\cdot\mathbf{s} & \mathbf{v}\cdot\mathbf{t} \end{bmatrix}_{2\times 2}$$

$$\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} s_1 & t_1 \\ s_2 & t_2 \\ s_3 & t_3 \end{bmatrix} = \begin{bmatrix} \boxed{u\cdot s} & \boxed{u\cdot t} \\ \boxed{v\cdot s} & \boxed{v\cdot t} \end{bmatrix}$$

$AB \neq BA.$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1.1+2.2+3.3 & 1.0+2.2+3.1 \\ & \end{bmatrix}$$

If

$$u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \qquad v = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \qquad w = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

their linear combination $au + bv + cw$, $a, b, c \in \mathbb{R}$ can be written as

$$a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} a \\ 2a + 3b \\ 4b + 5c \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{x}.$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{x} = a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$$Ax = b \iff \underbrace{\begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{} = \boxed{x_1\begin{bmatrix} | \\ c_1 \\ | \end{bmatrix} + x_2\begin{bmatrix} | \\ c_2 \\ | \end{bmatrix} + x_3\begin{bmatrix} | \\ c_3 \\ | \end{bmatrix}} = \begin{bmatrix} | \\ b \\ | \end{bmatrix}$$

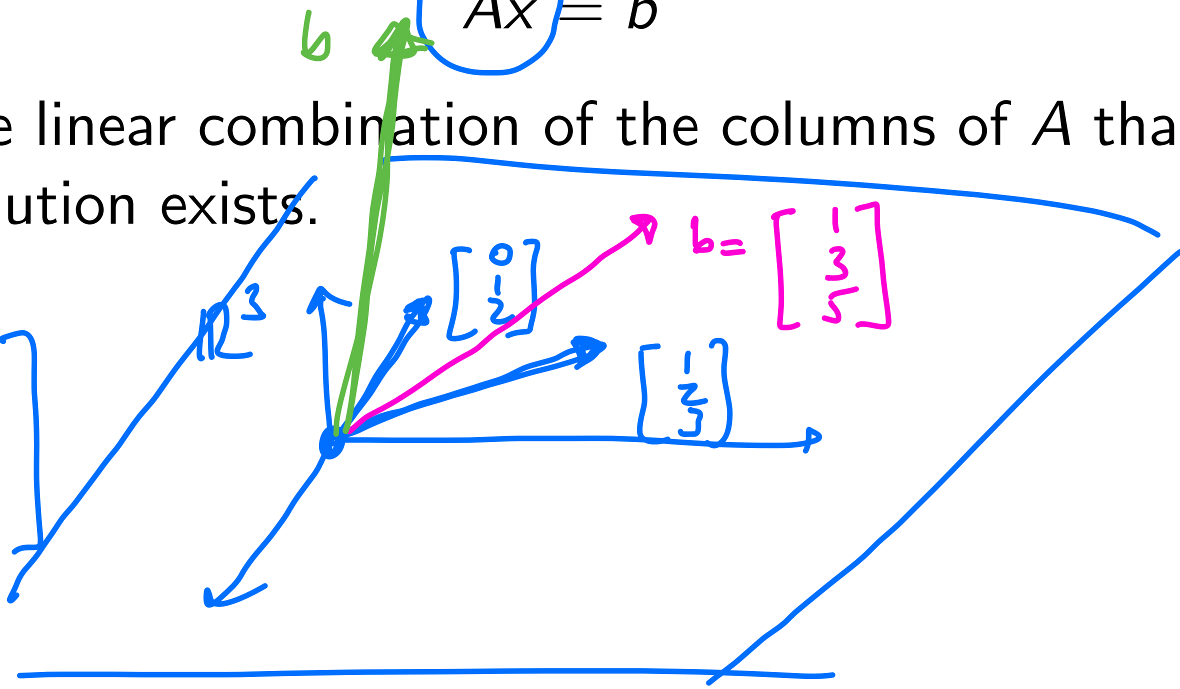
linear comb. of the columns of A

Solving

$$Ax = b$$

is finding the linear combination of the columns of A that yields b when the solution exists.

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$$



System of Linear Equations: Matrices

$$\begin{cases} x_1 - x_2 + 2x_3 &= 1 \\ \quad\quad x_2 + 3x_3 &= 1 \end{cases}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} \cdots & r_1 & \cdots \\ \cdots & r_2 & \cdots \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \vdots \\ x \\ \vdots \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b}.$$

Linear Systems in Two Unknowns

Usually this system is represented as:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{cases} \iff \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{b}$$

The system has at least one solution or no solution.

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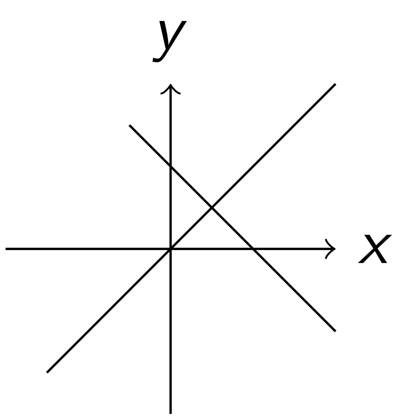
The system has at least one solution or no solution.

When the system has a solution then we can write b as a linear combinations of the column vectors of A

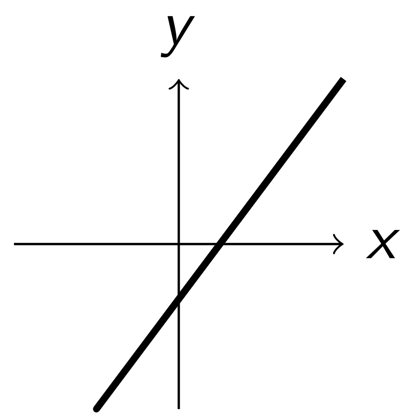
Type of Solutions

$$\begin{cases} x - y &= 0 \\ x + y &= 1 \end{cases}$$



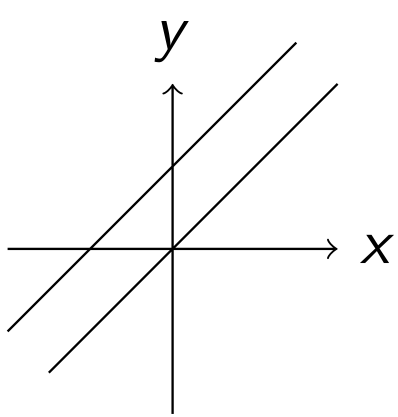
Unique solution

$$\begin{cases} x - y &= -1 \\ 2x - 2y &= -2 \end{cases}$$



Infinitely many solutions

$$\begin{cases} x - y &= -1 \\ x - y &= 0 \end{cases}$$



No solution

Orthogonal Projections

What if $Ax = b$ has no solution?

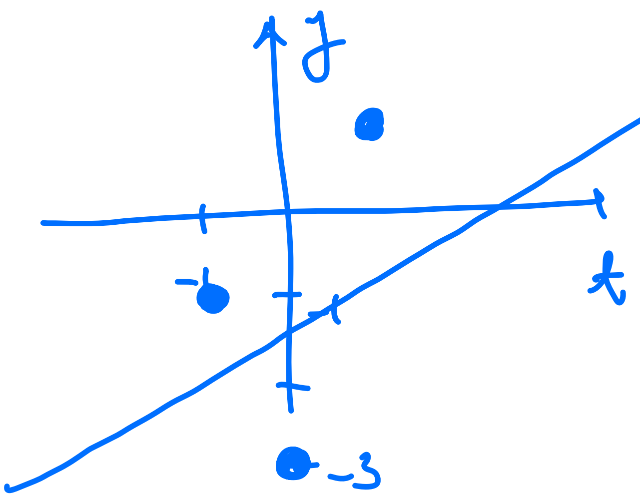
What if $Ax = b$ has no solution?

For example, given data points

$$(\overset{t_1}{-1}, \overset{y_1}{-1}) \qquad (\overset{t_2}{0}, \overset{y_2}{-3}) \qquad (\overset{t_3}{1}, \overset{y_3}{1})$$

Can we fit the three points on a line?

What is the best straight line $\boxed{y = C + Dt}$?



$$\begin{cases} y_1 = C + Dt_1 \\ y_2 = C + Dt_2 \\ y_3 = C + Dt_3 \end{cases}$$

$$\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$$

What if $Ax = b$ has no solution?

For example, given data points

$$(-1,-1) \qquad (0,-3) \qquad (1,1)$$

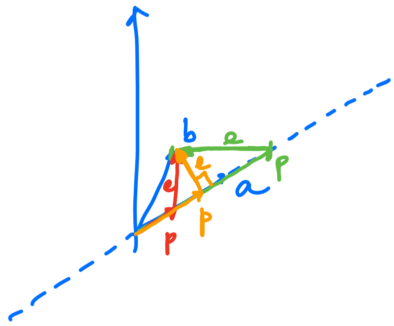
Can we fit the three points on a line?
What is the best straight line $y = \underbrace{C + Dt}$?

We solve for an approximation $\hat{x}$ of x:

$$A\hat{x} = p,$$

in which p is the projection of b onto the set of all the linear combinations of the columns of A.

$\mathbb{R}^2$



$a \in \mathbb{R}^2$, $b \in \mathbb{R}^2$

$\boxed{ax = b}$?

$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

x does not exist

$b = p + e \Rightarrow e = b - p$

min $\|e\|$

$\hat{x} = ?$

$\boxed{a\hat{x} = p}$

$a^T e = 0$

$a \cdot e = 0$

$[a_1 \; a_2] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0$

$a^T e = 0$

$a^T (b - p) = 0$

$a^T (b - a\hat{x}) = 0$

$\boxed{a^T b = a^T a \hat{x}}$ ← normal equations

$\hat{x} = \dfrac{a^T b}{a^T a}$

Projection of a Vector onto a Line

1. a line spanned by a;
2. the projection p of b onto the line is $p = \hat{x}a$, $\hat{x} \in \mathbb{R}$
3. $\hat{x}$?

$$a^T(b - a\hat{x}) = 0 \qquad \hat{x} = \frac{a^T b}{a^T a}$$

$$p = a\frac{a^T b}{a^T a}$$

what happens to p

- if b is doubled?
- if a is doubled?

Projection of a Vector onto a Subspace

1. the subspace spanned by the columns of A;
2. the projection p of b onto the subspace is $p = A\hat{x}$, $\hat{x} \in \mathbb{R}^n$
3. $\hat{x}$?

$$\boxed{a^T a \hat{x} = a^T b}$$

$$A^T(b - A\hat{x}) = 0 \qquad \boxed{A^T A\hat{x} = A^T b}$$

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c \\ d \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}}_{b}$$

$$\underbrace{\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}}_{A}}\ \underbrace{\begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix}} = \overbrace{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}}^{A^T} \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} \cancel{-1} - 3 \cancel{+1} \\ 1 + 0 + 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} 3\hat{C} = -3 \\ 2\hat{D} = 2 \end{cases} \Rightarrow \begin{cases} \hat{C} = -1 \\ \hat{D} = 1 \end{cases}$$

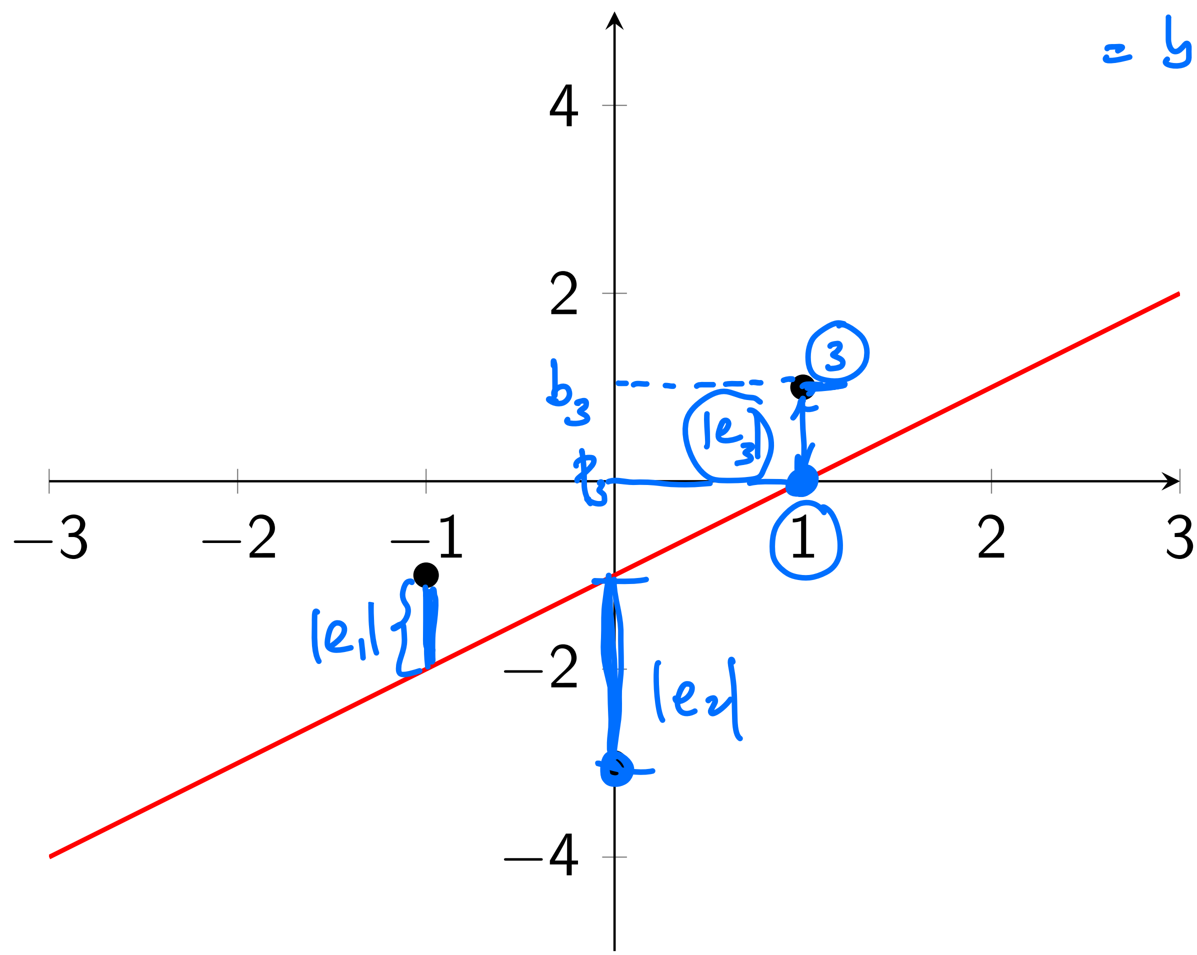
$$y = \hat{C} + \hat{D}t = -1 + 1t = -1 + t$$

Example

What is the best fitting line $y = C + Dt$ through

$$(-1,-1) \qquad (0,-3) \qquad (1,1)?$$

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = b - p = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$



$y = t - 1$

$$\boxed{\min \|e\|} = \min_{\hat{C},\hat{D}} \|e\|^2$$

$$= \min_{\hat{C},\hat{D}} e_1^2 + e_2^2 + e_3^2$$

Coffee

?

Example

What is the best fitting line $y = C + Dt$ through

$$(-1, 1) \qquad (0, -2) \qquad (1, 1)?$$