Mathematics for AI

Problem Set: Linear Algebra

November 27, 2022

1. Without calculating the value of the angle θ between $v = [2 \quad 1 \quad -1]^T$ and $w = [3 \quad -4 \quad 1]^T$, explain whether $0 \leq \theta < \pi/2$, or $\theta = \pi/2$ or $\pi/2 < \theta \leq \pi$.

2. Let L be the z-axis and P the xy-plane in $\mathbb{R}^3$.
   a) is the union of P with L a vector space in $\mathbb{R}^3$?
   b) is the intersection of P with L a vector space in $\mathbb{R}^3$?

3. Consider the 3 points of data

$$(t_1, b_1) = (0, 6), \quad (t_2, b_2) = (1, 0), \quad (t_3, b_3) = (2, 0)$$

   which we want to fit to a parabola $b = C + Dt + Et^2$ using the least squares approximation.
   a) Write the system of equations $Ax = b$ we would like to solve and identify A, x and b.
   b) What is the projection matrix P, such that the projection vector p is $p = Pb$, corresponding to this least squares fitting?
   c) What is the error vector corresponding to this least squares fitting? Justify.

4. The least-square linear fit to three points $(0, b_1)$, $(1, b_2)$ and $(2, b_3)$ is $C + Dt$ for $C = 1$ and $D = -2$. That is, the fit is the line $1 - 2t$. In this question, the goal is to work backwards from this fit to the unknown values $b = (b_1 \quad b_2 \quad b_3)^T$ at the coordinates $t = 0, 1, 2$.
   a) Write down the explicit equations that b must satisfy for $1 - 2t$ to be the least-squares linear fit.
   b) If all the points fall exactly on the line $1 - 2t$, what are the components of b? Check that this b satisfies the normal equations.

c) More generally, if all the points $(a_1, b_1), \ldots, (a_n, b_n)$ fall exactly on any straight line, then b is a linear combination of which vectors?

Some of the following problems are chosen from Strang's book: Introduction to Linear Algebra, 5th International Edition, 2016, Author: Gilbert Strang, Wellesley-Cambridge Press.