Fundamentals of Calculus

Mathematics for AI

November 27, 2022

King Abdullah University of Science and Technology

Our Goal

1. Linear and quadratic approximations.
2. Behavior of a function from its derivative.
3. Composition of functions.
4. Functions of more than one variable.

Real Function of a Real Variable

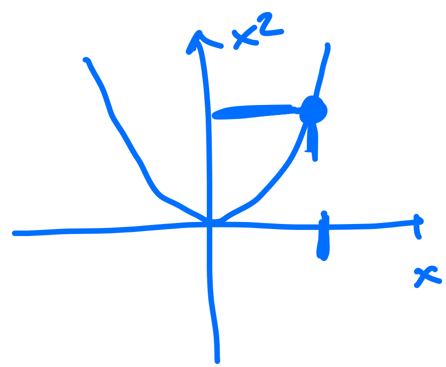
Function

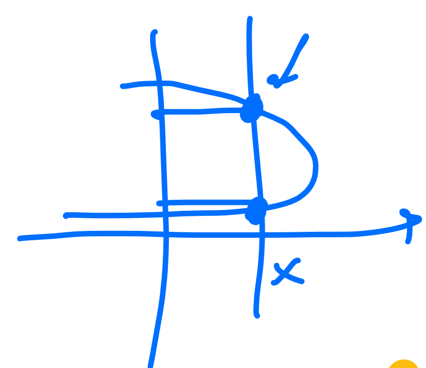
A function f is a rule that assigns to each element $x \in D_f$ exactly one value, $f(x)$.

$$x \in D_f \longrightarrow \boxed{f} \longrightarrow f(x)$$

$f(x) = x^2$

$f(2) = 2^2 = 4$





Polynomial

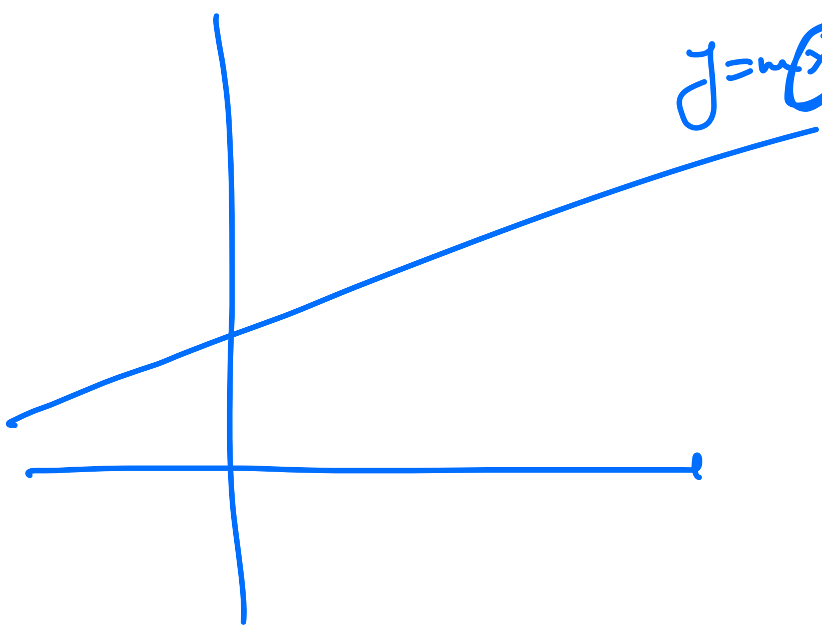
A function P is a polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where n is a nonnegative integer and the coefficients $a_n, \ldots, a_0$ are constants. The domain of any polynomial is the real line.

$n=1$

$y = mx + b$



$f(x) = x^{1/2} = \sqrt{x} \leftarrow$ not a poly

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Why do we need polynomials?

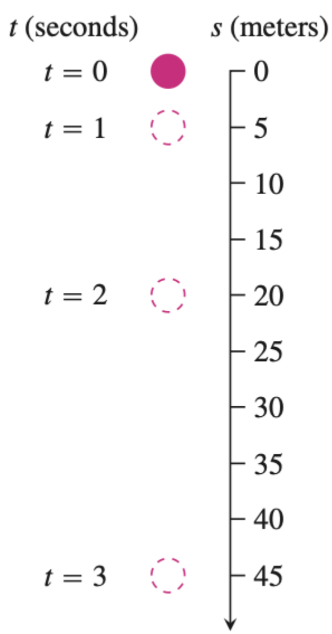
1. They are easy to compute
2. We need them to approximate complex functions
3. Let us first understand the notion of rate of change

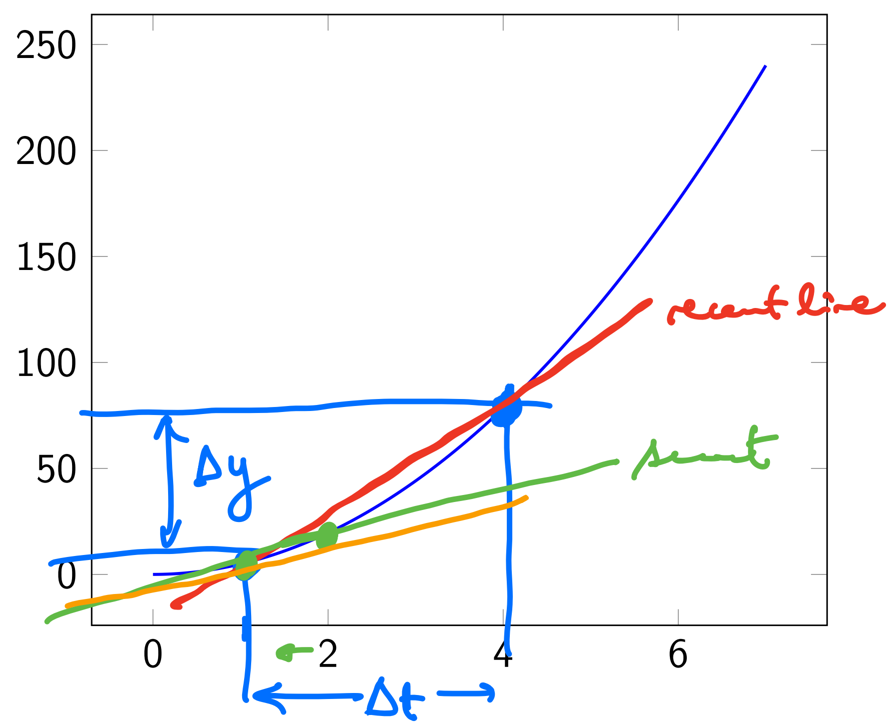
Example

A rock falls down from the top of a tall cliff, and the distance from top is given as a function of time t:

$$y = f(t) = 4.9t^2$$

What is its average speed between second 1 and second 4?





$$as = \frac{\Delta y}{\Delta t}$$

Rate of Change

1. The rate of change between $P(a, f(a))$ and $Q(b, f(b))$ is

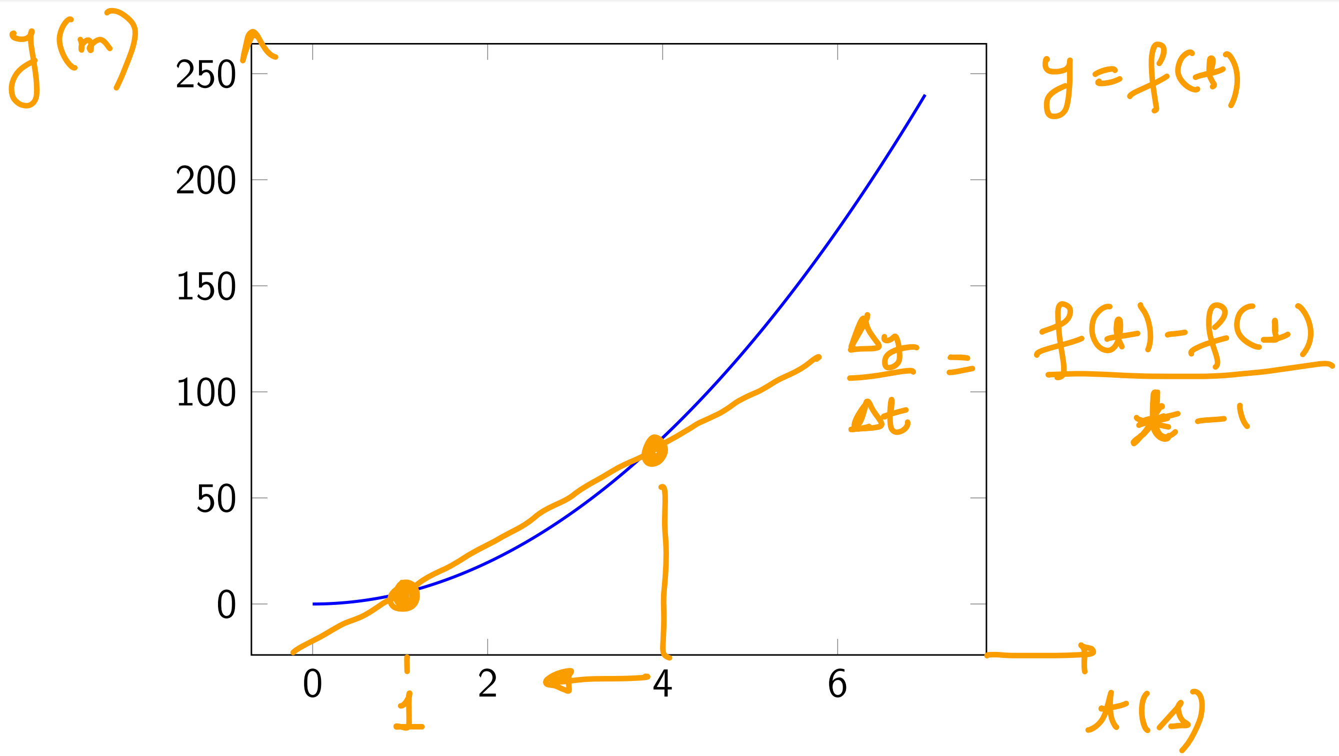
$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

2. Slope of secant line through P is $(a, f(a))$ and Q is $(b, f(b))$ is

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Question

What happens if we want to know the speed at a given time t? Say $t = 1$?



Derivative

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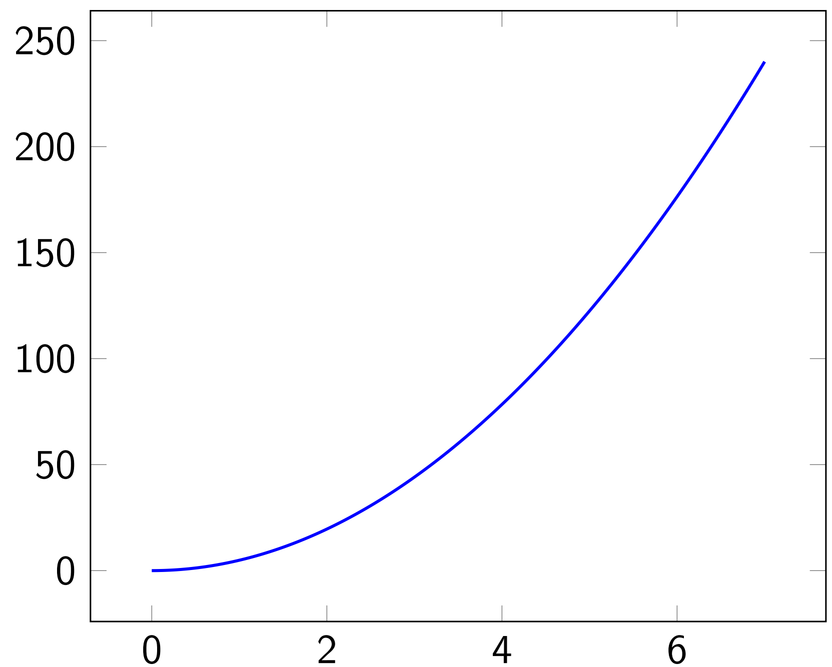
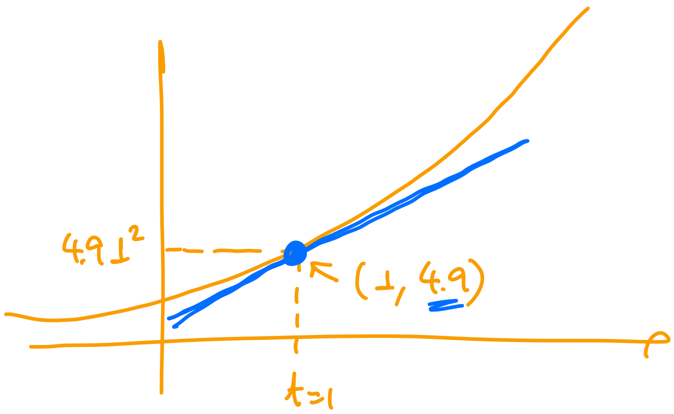


$$f'(1) = \lim_{t \to 1} \frac{f(t) - f(1)}{t - 1}$$

$\frac{df}{dt}(1)$



$f(t) = 4.9t^2$

$f'(t) = 2 \times 4.9\, t$

$= 9.81\, t$

$\underbrace{f'(t=1)} = \underline{9.81}$ m/s

find the equation to the tangent line:

m ← slope

(x_0, y_0)

$y - y_0 = m(x - x_0)$

$y - 4.9 = 9.81(x - 1)$

$y = 9.81(x-1) + 4.9$

Derivative

If the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

and f is **differentiable** at $x = a$.

Example

Find the derivative of

1. $f(x) = x^2$ $\quad f'(x) = 2x$
2. $f(x) = x^3 + \sin x$ $\quad : f'(x) = 3x^2 + \cos x$

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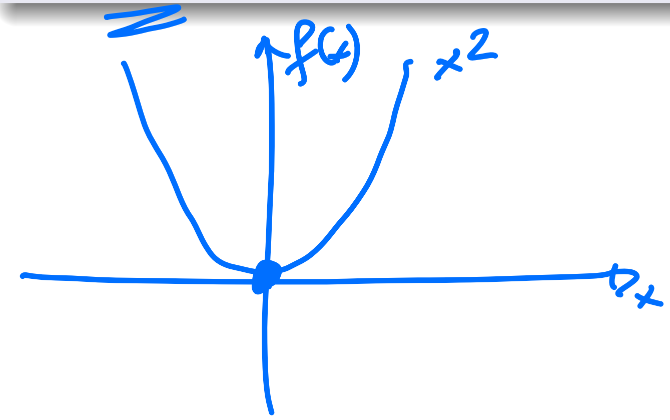
Tangent Line

Tangent Line

Assume f is differentiable at $x = a$. The **tangent line** to the graph of f at point $P = (a, f(a))$ is the line through P and slope $f'(a)$:

$$\boxed{y - f(a) = f'(a)(x - a).}$$

What is the equation of the tangent line to the graph of $f(x) = x^2$ at $x = 0$?



$y = 0$?

$f'(x) = 2x$

$$\boxed{y - f(0) = f'(0)(x - 0)}$$

$y - 0 = 0\,x$

$\Rightarrow y = 0$

Linear Approximation

Assume $f'(a)$ exists. The linear approximation of f at $(a, f(a))$ is

$$\boxed{f(x) \approx f(a) + f'(a)(x-a).}$$

What is the linear approximation of $f(x) = \sqrt{x}$ at $a = 4$?
Can we find an approximation of $\sqrt{3}$?

$f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2} x^{1/2 - 1} = \frac{1}{2\sqrt{x}}$

$\sqrt{3} = 1.732$

$\ell(x) = f(4) + f'(4)(x-4) = 2 + \frac{1}{2 \cdot 2}(x-4) = \boxed{2 + \frac{1}{4}(x-4)}$

linear approx to f at $a = 4$

$\sqrt{3} \simeq 2 + 1/4(3-4) = 2 - 1/4 = 1.75$

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Quadratic Approximation

Quadratic Approximation

Assume $f'(a)$ and $f''(a)$ exist. The quadratic approximation of f at $(a, f(a))$ is

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.$$

What is the quadratic approximation of $f(x) = \sqrt{x}$ at $a = 4$? Can we find an approximation of $\sqrt{3}$?

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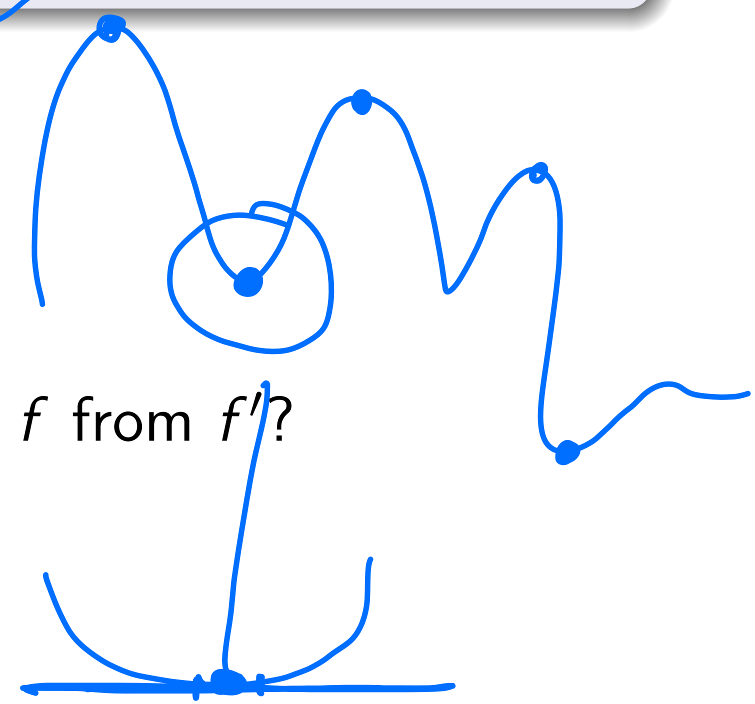
Behavior of f from f' and f''

Question

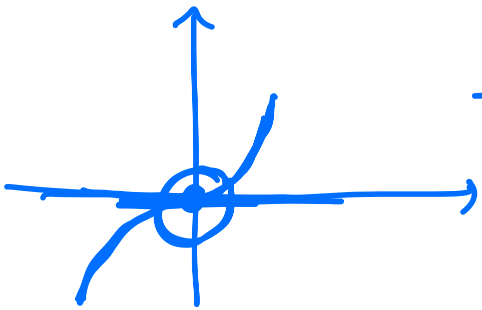
How to predict the graph of $f(x) = x^3 - 12x$ based on f'?

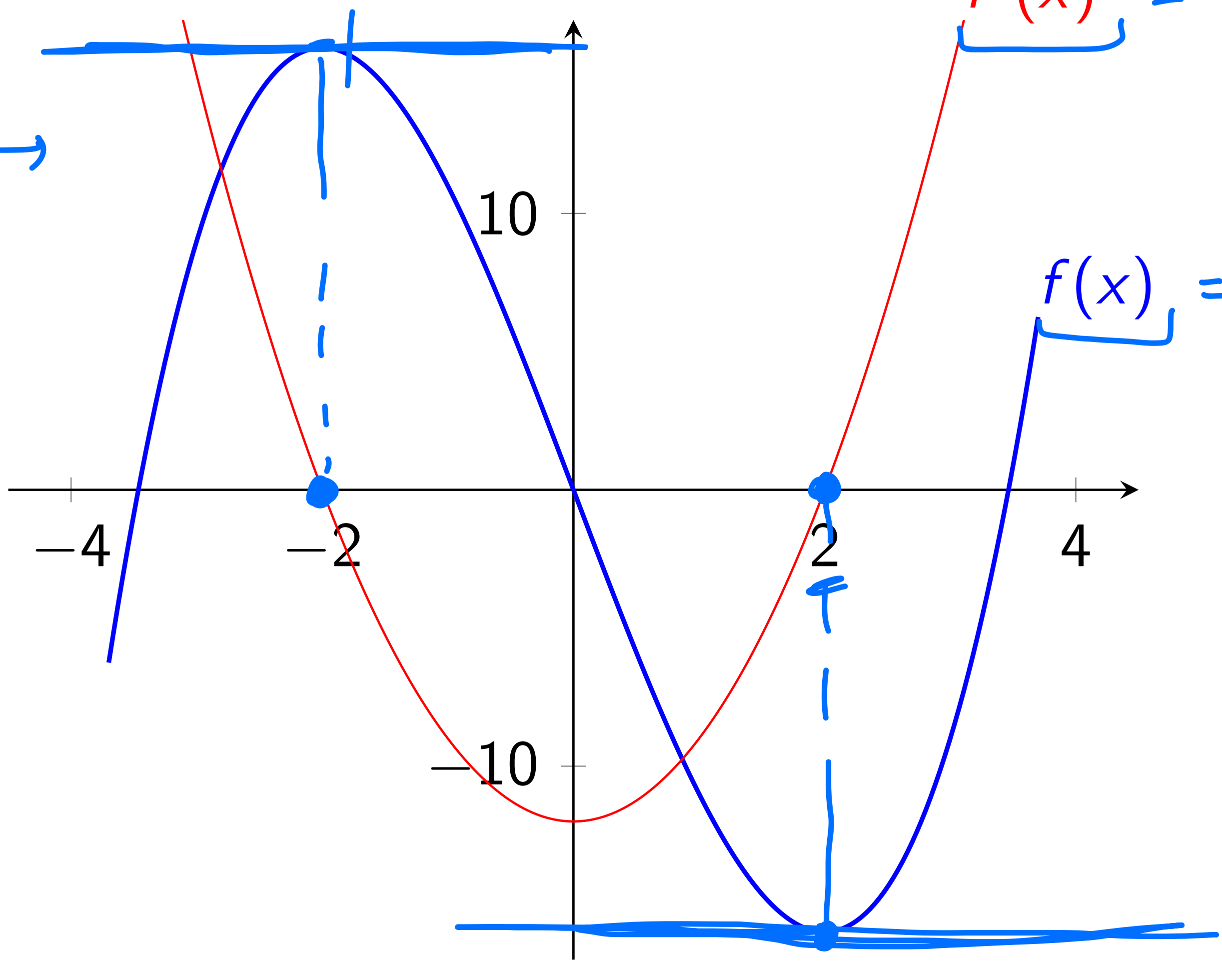
How to predict the behavior of f from f'?



Behavior of $f = x^3 - 12x$ from f'

1- $\boxed{\text{If } a \text{ is a local maximizer or minimizer, then } f'(a) = 0;}$





$\underbrace{f'(x)} = 3x^2 - 12$

$3x^2 - 12 = 0$

$x^2 = 4$

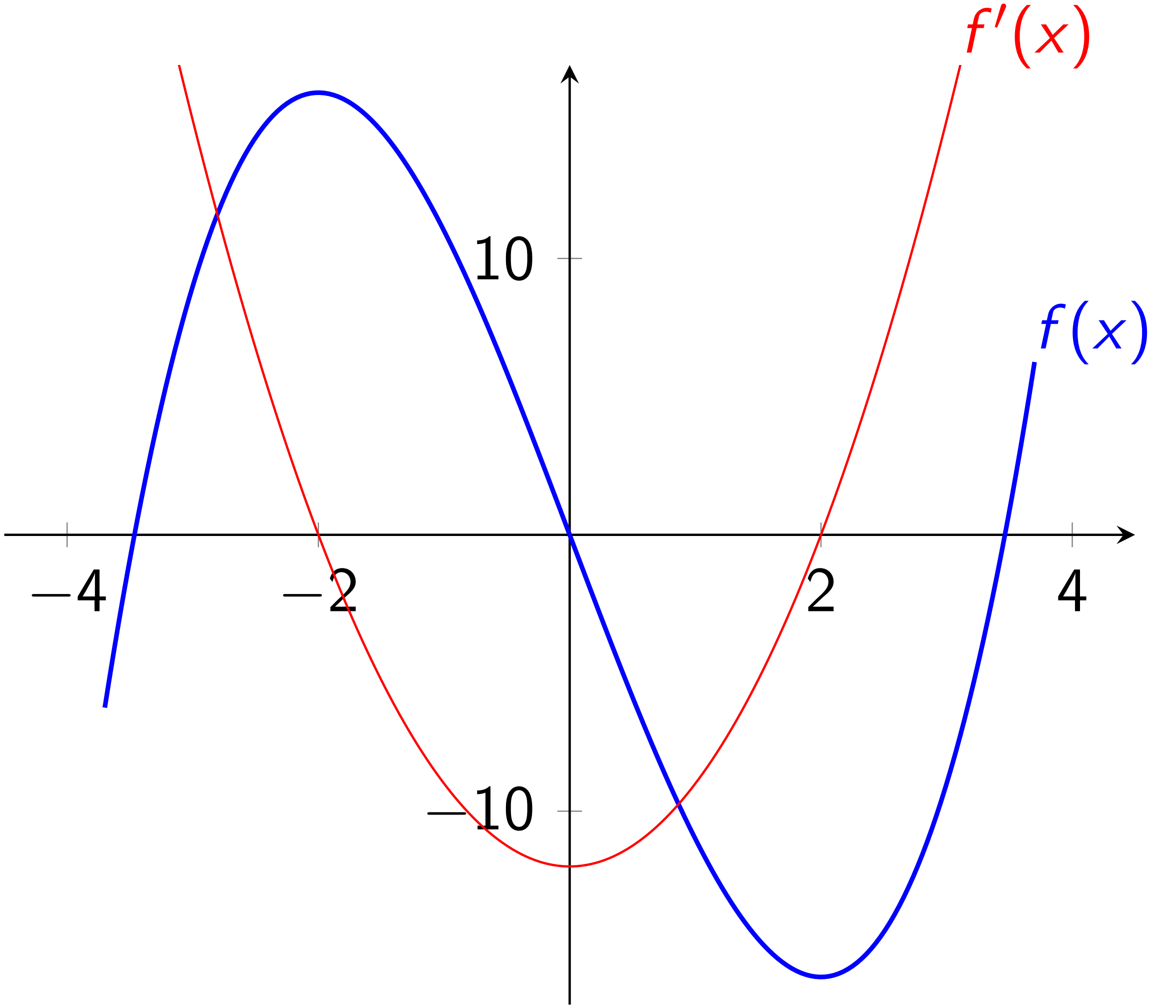
$x = \pm 2$

$\underbrace{f(x)} = x^3 - 12x$

Behavior of $f = x^3 - 12x$ from f'

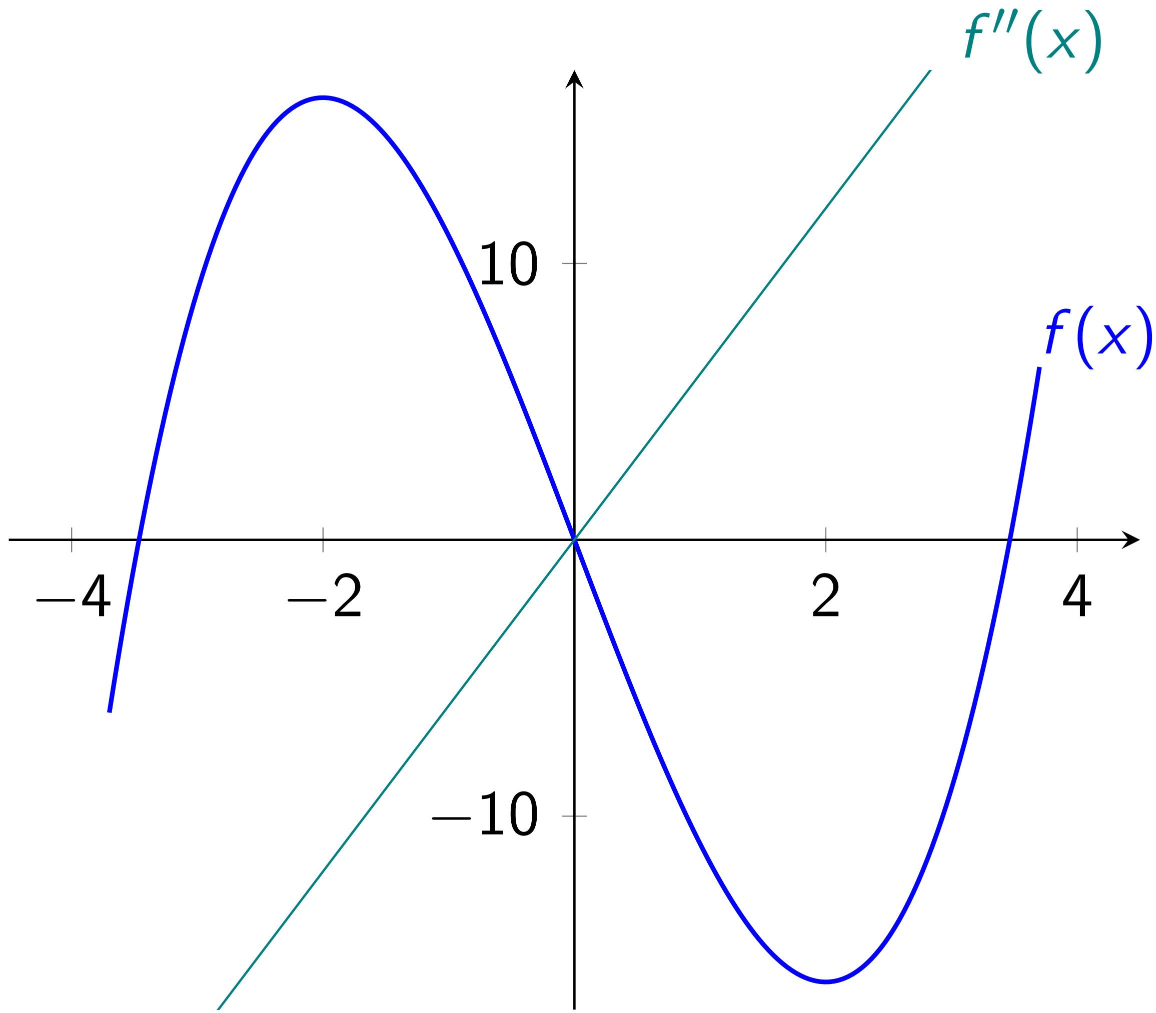
2- If $f' > 0$, then f increases;

3- If $f' < 0$, then f decreases;



Behavior of $f = x^3 - 12x$ from f''

1. if $f'' > 0$, then f is concave up;
2. if $f'' < 0$, then f is concave down.

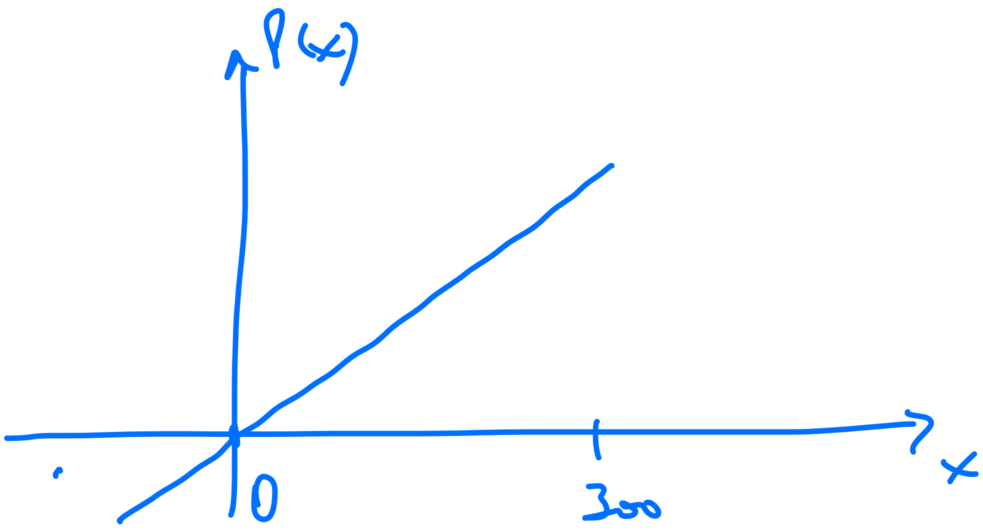


Example

A company has 300 apartments to rent. The monthly profit when renting x apartments, in dollars, is given by,

$$P(x) = -4x^2 + 1600x - 4000$$

How many apartments should they rent in order to maximize their profit?



Composition of Functions

Given two functions f and g, the composite function $f \circ g$ is defined as

$$(f \circ g)(x) = f(g(x)).$$

Given $f(x) = x^2$ and $g(x) = x - 3$, determine $f \circ g$ and $g \circ f$.

$(f \circ g)(x) = f(g(x)) = f(x-3) = (x-3)^2$

$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$

The Chain Rule

Question

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by

$$F(t) = \sqrt{t^2 + 1}$$

Find the speed of the object as a function of t.

$$F(t) = \sqrt{t^2+1}$$

$$f(t) = \sqrt{t} \longrightarrow f'(t) = \frac{1}{2\sqrt{t}}$$

$$g(t) = t^2 - 1 \longrightarrow g'(t) = 2t$$

$$F(t) = \boxed{(f \circ g)(t)} = f(g(t)) = f(t^2-1) = \sqrt{t^2-1}$$

$$F'(t) = f'(g(t))\, g'(t) = \underbrace{f'(\boxed{t^2-1})}_{f'(g(t))} \cdot \overbrace{2t}^{g'} =$$

chain rule

$$= \underbrace{\frac{1}{2\sqrt{t^2-1}}}_{f'(t^2-1)} \cdot 2t = \frac{t}{\sqrt{t^2-1}}$$

Question

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by

$$F(t) = \sqrt{t^2 + 1}$$

Find the speed of the object as a function of t.

The Chain Rule

If g is differentiable at t and f is differentiable at $g(t)$, then the composite function $F = f \circ g$ is differentiable at t and F' is

$$F'(t) = f'(g(t))\, g'(t).$$

$F'(t) =$

Example

The temperature of a hood of car with a running engine is given by:

$$f(x, y) = -x^2 - y^2 + 27$$

Real Function of Two Variables

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Real Function of Two Variables

$$f : \underline{D_f \subseteq \mathbb{R}^2} \to \mathbb{R}$$

$$f = f(x, y)$$

$$\underbrace{f(x,y) = -x^2 - y^2 + 27}\in \mathbb{R}$$

Example

The temperature of a hood of car with a running engine is given by:

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Real Function of Two Variables

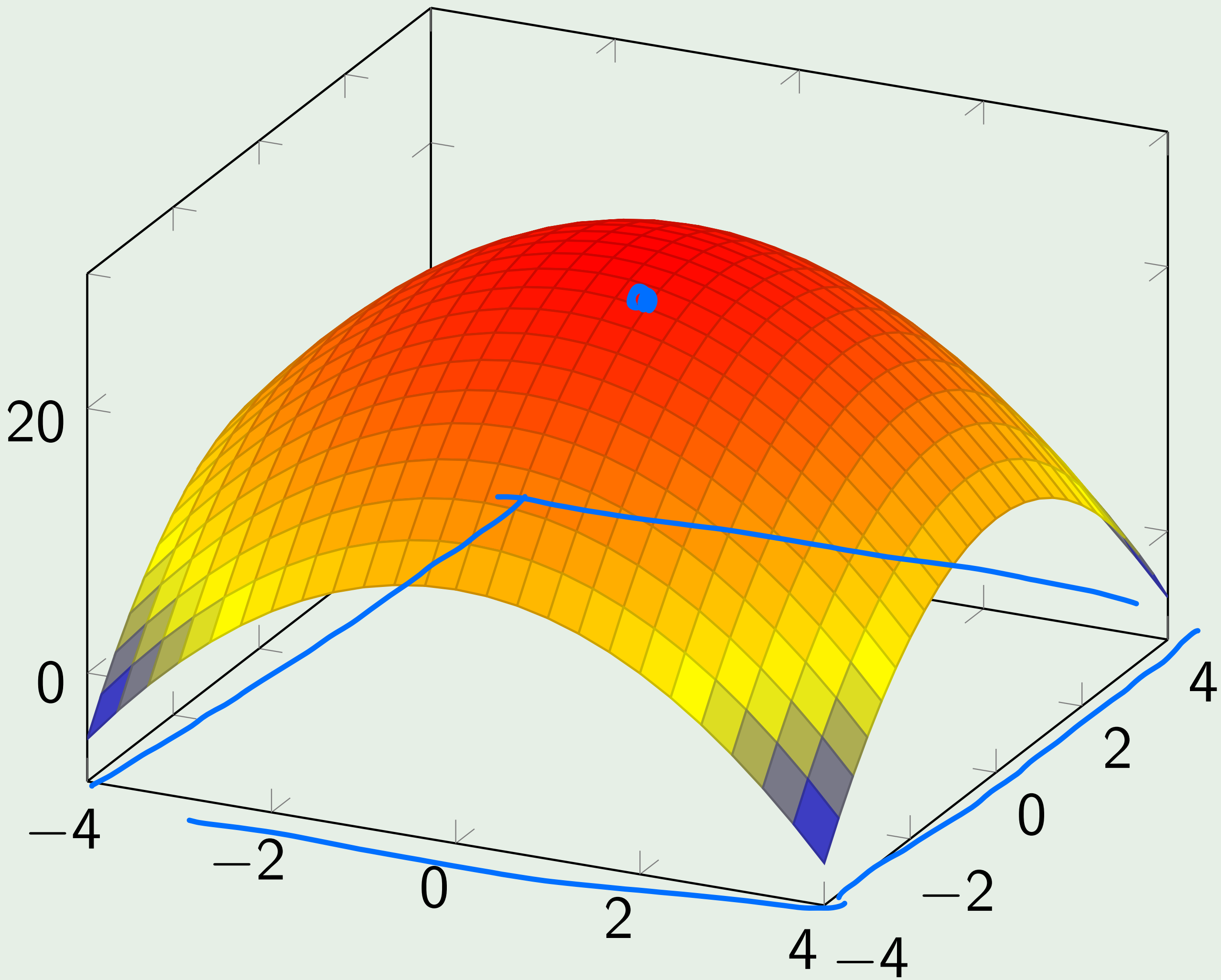
$$f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$$

$$f = f(x, y)$$

1. Linear approximation?
2. Maxima and minima?

Example

$f(x, y) = -x^2 - y^2 + 27$



Partial Derivatives

$$f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$$

$g(x) : \mathbb{R} \to \mathbb{R}$

$g'(x) = \frac{dg}{dx}$

$$f_x \equiv \frac{\partial f}{\partial x}$$

$$f_y \equiv \frac{\partial f}{\partial y}$$

What are f_x and f_y for $f(x, y) = -x^2 - y^2 + 27$?

$f_x = \frac{\partial f}{\partial x} = -2x$; $f_y = \frac{\partial f}{\partial y} = -2y$

How about f_x and f_y when $f(x, y) = x^3y^5$? $\longrightarrow$ $f_x = 3x^2y^5$

The Gradient

The Gradient

$$f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$$

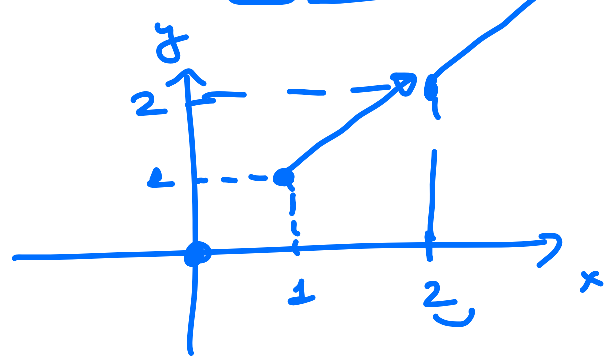
$$\nabla f = \langle f_x, f_y \rangle$$

$f(x,y) = x^2 + y^2$

$f_x = 2x$

$f_y = 2y$

$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle$



Linear Approximation

Linear Approximation of $g : D_g \subseteq \mathbb{R} \to \mathbb{R}$

Assume $g'(a)$ exists. The linear approximation of g at $(a, g(a))$ is

$$g(x) \approx g(a) + g'(a)(x - a).$$

Linear Approximation of $f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$

Assume f is differentiable at (a, b). The linear approximation of f at $(a, b, f(a, b))$ is

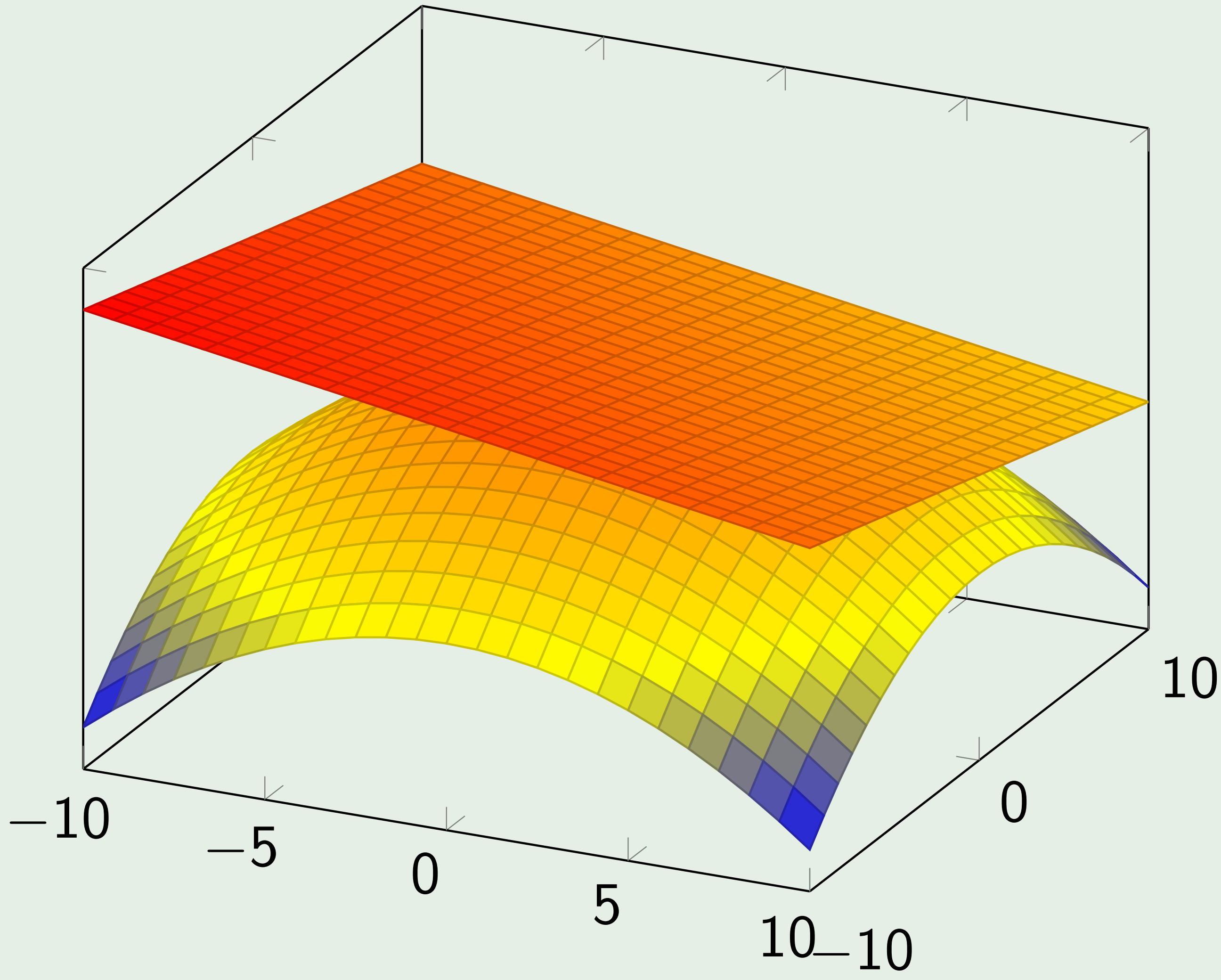
$$\boxed{f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}$$

$$f(x, y) \approx f(a, b) + \begin{bmatrix} f_x(a, b) & f_y(a, b) \end{bmatrix} \begin{bmatrix} x - a \\ y - b \end{bmatrix}$$

Graph of $f(x,y) = -x^2 - y^2 + 27$

Example

$f(x,y) = -x^2 - y^2 + 27$ then $f(x,y) \approx 35 - 4x - 4y$ at $(2,2)$ (locally)



Interpreting the gradient

$$\nabla f = \langle f_x, f_y \rangle$$

1. If (a, b) is a local maximizer or minimizer $\Rightarrow \nabla f(a, b) = 0$.

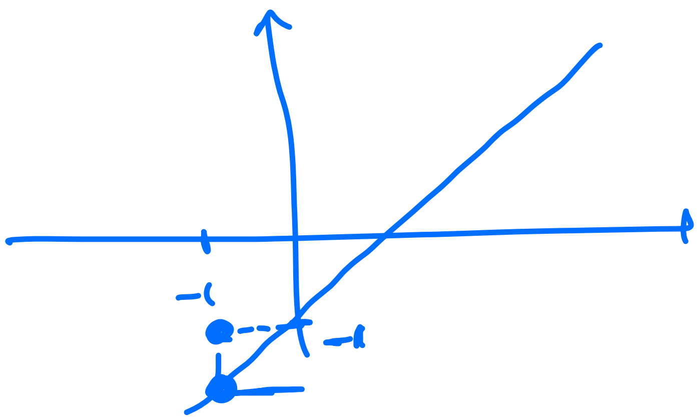
2. The gradient points in the direction of the greatest increase of f, that is, the direction of steepest ascent.

$$(\underbrace{-1}_{t_1}, \underbrace{-1}_{y_1}) \;①, \quad (\underbrace{0}_{t_2}, \underbrace{-3}_{y_2}) \;②, \quad (\underbrace{1}_{t_3}, \underbrace{1}_{y_3}) \;③ \qquad y(t) = \hat{C} + \hat{D} t$$

$$\begin{array}{l} ① \\ ② \\ ③ \end{array} \begin{cases} \hat{C} + \hat{D} t_1 = y_1 \\ \hat{C} + \hat{D} t_2 = y_2 \\ \hat{C} + \hat{D} t_3 = y_3 \end{cases} \Rightarrow \begin{cases} \hat{C} + \hat{D}(-1) = -1 \\ \hat{C} + \hat{D}\, 0 = -3 \\ \hat{C} + \hat{D}(1) = 1 \end{cases}$$

Coffee?

$$\min_{\hat{C}, \hat{D}} \boxed{\|e\|^2} \qquad e = b - p$$



$$\to f(\hat{C}, \hat{D}) = \underbrace{e_1}^2 + \underbrace{e_2}^2 + \underbrace{e_3}^2 \qquad e_1 = y_1 - (\hat{C} + \hat{D} t_1)$$

$$= [y_1 - (\hat{C} + \hat{D} t_1)]^2 + [y_2 - (\hat{C} + \hat{D} t_2)]^2 + [y_3 - (\hat{C} + \hat{D} t_3)]^2$$

$$f(\hat{C}, \hat{D}) = [-1 - (\hat{C} - \hat{D})]^2 + [-3 - \hat{C}]^2 + [1 - (\hat{C} + \hat{D})]^2$$

Graph of $f(x, y) = -x^2 - y^2 + 27$

min $f(\hat{c}, \hat{b})$?

$$\nabla f = \left\langle \frac{\partial f}{\partial \hat{c}}, \frac{\partial f}{\partial \hat{b}} \right\rangle = \langle 0, 0 \rangle \rightarrow \begin{cases} \frac{\partial f}{\partial \hat{c}} = 0 \\ \frac{\partial f}{\partial \hat{b}} = 0 \end{cases}$$