Fundamentals of Calculus

Mathematics for AI

November 27, 2022

King Abdullah University of Science and Technology

Our Goal

1. Linear and quadratic approximations.
2. Behavior of a function from its derivative.
3. Composition of functions.
4. Functions of more than one variable.

Real Function of a Real Variable

Function

A function f is a rule that assigns to each element $x \in D_f$ exactly one value, $f(x)$.

Polynomial

A function P is a polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where n is a nonnegative integer and the coefficients $a_n, \ldots, a_0$ are constants. The domain of any polynomial is the real line.

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Why do we need polynomials?

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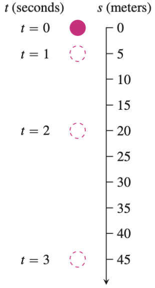
2. We need them to approximate complex functions
3. Let us first understand the notion of rate of change

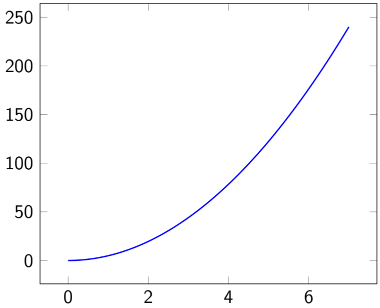
Example

A rock falls down from the top of a tall cliff, and the distance from top is given as a function of time t:

$$y = f(t) = 4.9t^2$$

What is its average speed between second 1 and second 4?





Rate of Change

1. The rate of change between $P(a, f(a))$ and $Q(b, f(b))$ is
$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$
2. Slope of secant line through P is $(a, f(a))$ and Q is $(b, f(b))$ is
$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

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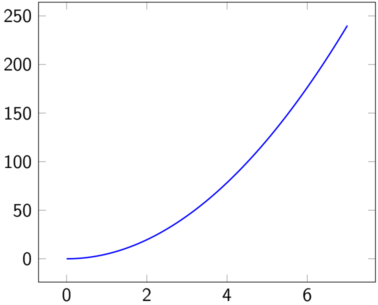
Question

What happens if we want to know the speed at a given time t? Say $t = 1$?



Derivative

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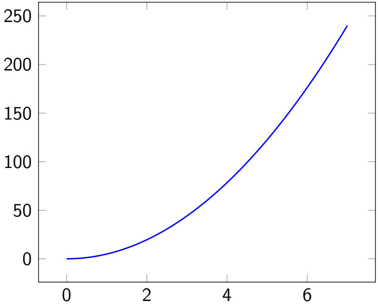


$$f'(1) = \lim_{t \to 1} \frac{f(t) - f(1)}{t - 1}$$

Derivative

Derivative

If the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

and f is **differentiable** at $x = a$.

Example

Find the derivative of

1. $f(x) = x^2$
2. $f(x) = x^3 + \sin x$

Tangent Line

Tangent Line

Assume f is differentiable at $x = a$. The **tangent line** to the graph of f at point $P = (a, f(a))$ is the line through P and slope $f'(a)$:

$$y - f(a) = f'(a)(x - a).$$

What is the equation of the tangent line to the graph of $f(x) = x^2$ at $x = 0$?

Linear Approximation

Linear Approximation

Assume $f'(a)$ exists. The linear approximation of f at $(a, f(a))$ is

$$\ell(x) = f(a) + f'(a)(x - a).$$

What is the linear approximation of $f(x) = \sqrt{x}$ at $a = 4$?
Can we find an approximation of $\sqrt{3}$?

Quadratic Approximation

Assume $f'(a)$ and $f''(a)$ exist. The quadratic approximation of f at $(a, f(a))$ is

$$q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

What is the quadratic approximation of $f(x) = \sqrt{x}$ at $a = 4$? Can we find an approximation of $\sqrt{3}$?

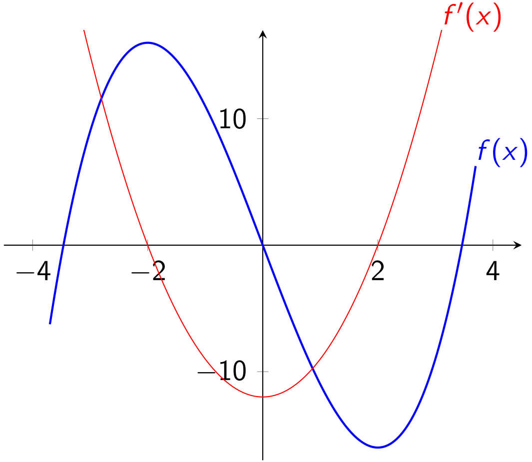
Question

How to predict the graph of $f(x) = x^3 - 12x$ based on f'?

How to predict the behavior of f from f'?

Behavior of $f = x^3 - 12x$ from f'

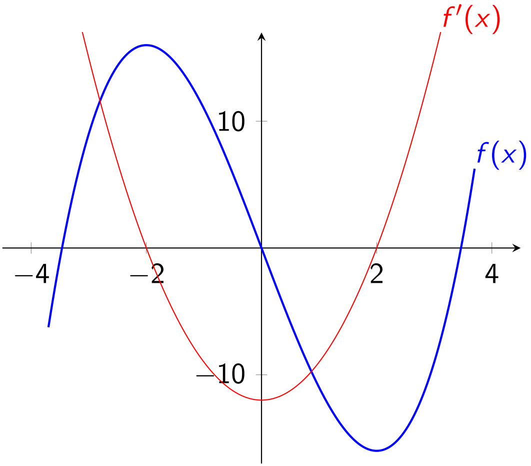
1- If a is a local maximizer or minimizer, then $f'(a) = 0$;



Behavior of $f = x^3 - 12x$ from f'

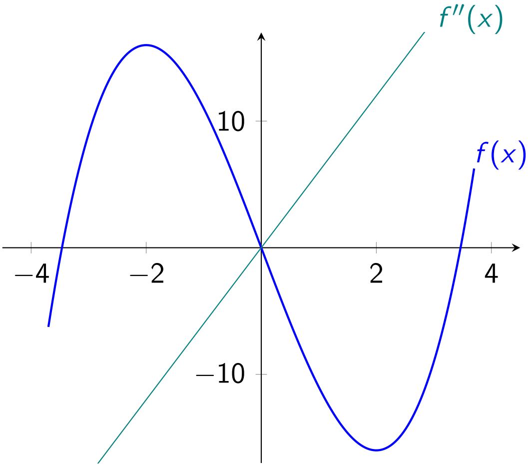
2- If $f' > 0$, then f increases;

3- If $f' < 0$, then f decreases;



Behavior of $f = x^3 - 12x$ from f''

1. if $f'' > 0$, then f is concave up;
2. if $f'' < 0$, then f is concave down.



Example

A company has 300 apartments to rent. The monthly profit when renting x apartments, in dollars, is given by,

$$P(x) = -4x^2 + 1600x - 4000$$

How many apartments should they rent in order to maximize their profit?

Composition of Functions

Given two functions f and g, the composite function $f \circ g$ is defined as

$$(f \circ g)(x) = f(g(x)).$$

Given $f(x) = x^2$ and $g(x) = x - 3$, determine $f \circ g$ and $g \circ f$.

Question

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by

$$F(t) = \sqrt{t^2 + 1}$$

Find the velocity of the object as a function of t.

The Chain Rule

Question

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by

$$F(t) = \sqrt{t^2 + 1}$$

Find the velocity of the object as a function of t.

The Chain Rule

If g is differentiable at t and f is differentiable at $g(t)$, then the composite function $F = f \circ g$ is differentiable at t and F' is

$$F'(t) = f'(g(t))\, g'(t).$$

Real Function of Two Variables

Example

The temperature of a hood of car with a running engine is given by:

$$f(x, y) = -x^2 - y^2 + 27$$

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Real Function of Two Variables

$$f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$$

$$f = f(x, y)$$

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Real Function of Two Variables

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1. Linear approximation?
2. Maxima and minima?

Partial Derivatives

Partial Derivatives

$$f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$$

$$f_x \equiv \frac{\partial f}{\partial x}$$

$$f_y \equiv \frac{\partial f}{\partial y}$$

What are f_x and f_y for $f(x, y) = -x^2 - y^2 + 27$?

How about f_x and f_y when $f(x, y) = x^3 y^5$?

The Gradient

$$f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$$

$$\nabla f = \langle f_x, f_y \rangle$$

Linear Approximation

Linear Approximation of $g : D_g \subseteq \mathbb{R} \to \mathbb{R}$

Assume $g'(a)$ exists. The linear approximation of g at $(a, g(a))$ is

$$\ell(x) = g(a) + g'(a)(x - a).$$

Linear Approximation of $f : D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$

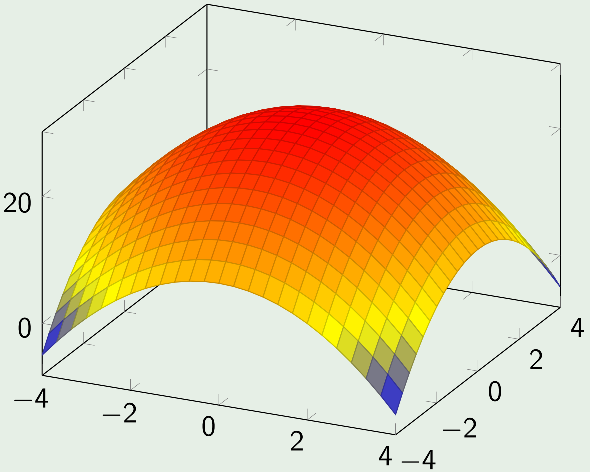
Assume f is differentiable at (a, b). The linear approximation of f at $(a, b, f(a, b))$ is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$L(x, y) = f(a, b) + [f_x(a, b) \quad f_y(a, b)] \begin{bmatrix} x - a \\ y - b \end{bmatrix}$$

Example

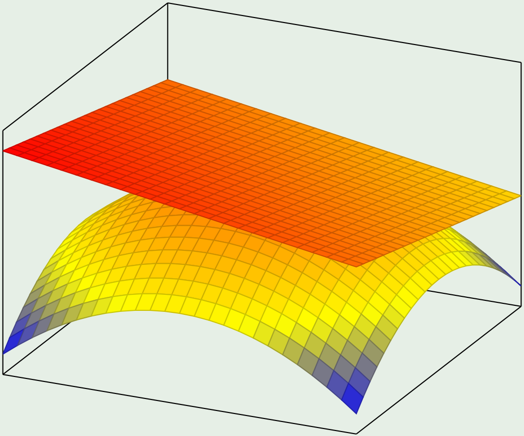
$f(x, y) = -x^2 - y^2 + 27$ then $\nabla f = \langle -2x, -2y \rangle$



Graph of $f(x, y) = -x^2 - y^2 + 27$

Example

$f(x, y) = -x^2 - y^2 + 27$ then $L(x, y) = 35 - 4x - 4y$ at $(2, 2)$



Interpreting the gradient

$$\nabla f = \langle f_x, f_y \rangle$$

1. If (a, b) is a local maximizer or minimizer $\Rightarrow \nabla f(a, b) = 0$.

2. The gradient points in the direction of the greatest increase of f, that is, the direction of steepest ascent.

Coffee?

Graph of $f(x, y) = -x^2 - y^2 + 27$