SMA 431 Differential Geometry(Methods of calculus applied to Geometry of Curved Spaces)

Differential Geometry (Kenyatta University)

KENYATTA UNIVERSITY
INSTITUTE OF OPEN LEARNING
SMA 431
DIFFERENTIAL GEOMETRY

BY

DR. D. SENGOTTAIYAN

B. Ed., B. A. (HONS).,
M. A., M. Sc., Ph. D.,
SENIOR LECTURER IN MATHEMATICS
KENYATTA UNIVERSITY, NAIROBI

DEPARTMENT OF MATHEMATICS
SCHOOL OF PURE AND APPLIED SCIENCES
KENYATTA UNIVERSITY

PREFACE

This module is designed primarily to provide the readers with the best preparation possible for the Differential Geometry (SMA 431) in its present form this module has developed from the courses given by the author over the last **thirty two** years in various universities to the audience of Mathematicians, Physicists and Engineer in the university of Madras, Kenyatta University, University of Nairobi and Jomo Kenyatta University of Agriculture and Technology.

This module, Differential Geometry is compiled from the Author's Advanced Differential Geometry, Oxford Publications, London and Nairobi. Most of the theory and problems are freely taken from the Author's Book for which the author has sole Copyright.

It is hoped that it will be of great interest to students of pure and Applied Mathematicians and Engineers following Differential Geometry.

Each lesson begins with a brief statement of definitions principles and important Theorems followed by a set of solved problems.

The author is pleased to acknowledge Dr L. O. Odongo B.Ed (Hons) M.Sc (UON) and M.Sc (Canada), Ph.D.(Canada) who has encouraged me to write this module in a short time.

SENGOTTAIYAN
Kenyatta University

Course outline

SMA 431: DIFFERENTIAL GEOMETRY

Vector algebra, triple products and vector quantities, vector functions of real variables, Concepts of curves: Curvative, torsion and general theory of curves. Concepts and general theory of surfaces. First and second fundamental forms of surfaces.

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Lesson 1

Vector Algebra

1.1 Introduction

Physical quantities are mainly divided into two categories, namely scalar (magnitudes only) and vector (magnitudes and directions) quantities. For example the mass of the book is 1.5 kg. This is a scalar and has no direction. The velocity of the car is 20m/s in the northern direction. This is a vector since it has not only magnitude 20m/s but also the direction is towards North. We shall study more about vectors in this Lesson.

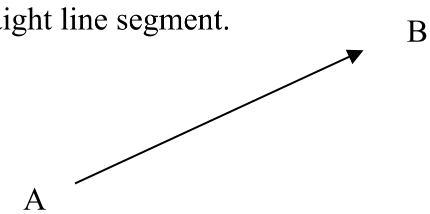
1.2 Objectives of the Lesson

By the end of this Lesson you will be able to define

- a vector
- the components of a vector
- position vector
- addition and subtraction of two or more vectors
- the dot product or scalar product of two vectors
- the cross product or vector product of two vectors
- to find the angle between two vectors
- to apply the above in solving problems

1.3 Definition of a vector

A vector is a quantity having both magnitude and direction. Hence it can be represented as a directed straight line segment.



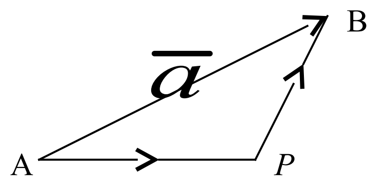
The length of the line AB denotes the magnitude of the vector and the direction of the line denotes the direction in which the vector quantity acts.

The vector AB is denoted by $\overline{AB}$ or with a single letter $\overline{a}$.

Negative of vector $\overline{AB}$ or - $\overline{AB}$.

The direction of BA is opposite of the direction of AB. Hence $\overline{BA} = - \overline{AB}$

1.4 Components of a vector

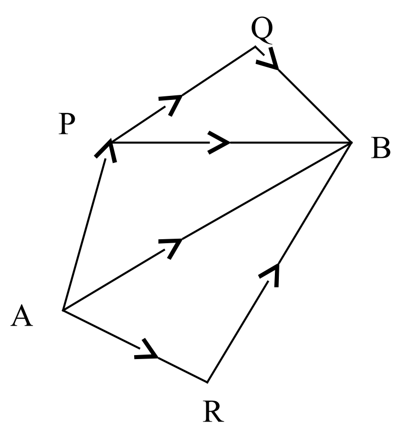


$\overline{AB}$ can be interpreted as "the effect of going from A to B in a straight line". The effect of going from A to B is the same as the effect of going from A to P in a straight line and then from P to B in another straight line.

Hence we write $\overline{AB} = \overline{AP} + \overline{PB}$

Here $\overline{AP}$ and $\overline{PB}$ are called the components of the vector $\overline{AB}$.

For example consider the figure given below



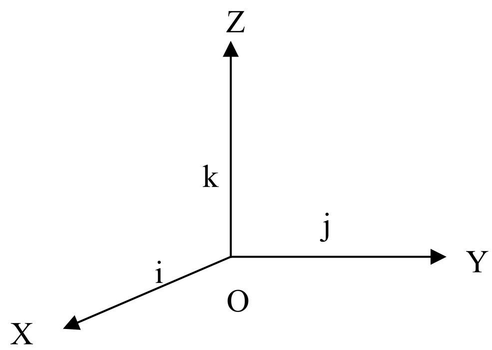
We can represent $\overline{AB}$ in different ways.

$\overline{AB} = \overline{AP} + \overline{PB}.$ ($\overline{AP}$ and $\overline{PQ}$ are components of $\overline{\text{AB}}$)

$\overline{AB} = \overline{AP} + \overline{PQ} + \overline{QB}.$ ($\overline{AP}$, $\overline{PQ}$ $\overline{\text{QB}}$ are components of $\overline{\text{AB}}$)

$\overline{AB} = \overline{AQ} + \overline{QB}$

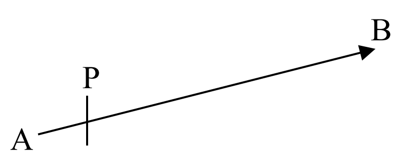
1.5 Unit vectors in the direction of the axes of the Rectangular coordinate system.



Consider the rectangular coordinate system OXYZ. A vector of unit length in the directions of x, y and z axes are represented by $\overline{i}$, $\overline{j}$ and $\overline{k}$ respectively. They are written generally without the vector symbol as *i, j* and *k* and called **unit vectors** in the direction of *x, y* and *z* axes respectively.

1.6 Unit vector in the direction of $\overline{AB}$ or $\overline{a}$

A unit vector in the direction of $\overline{AB}$ is a vector having unit magnitude in the direction of $\overline{AB}$

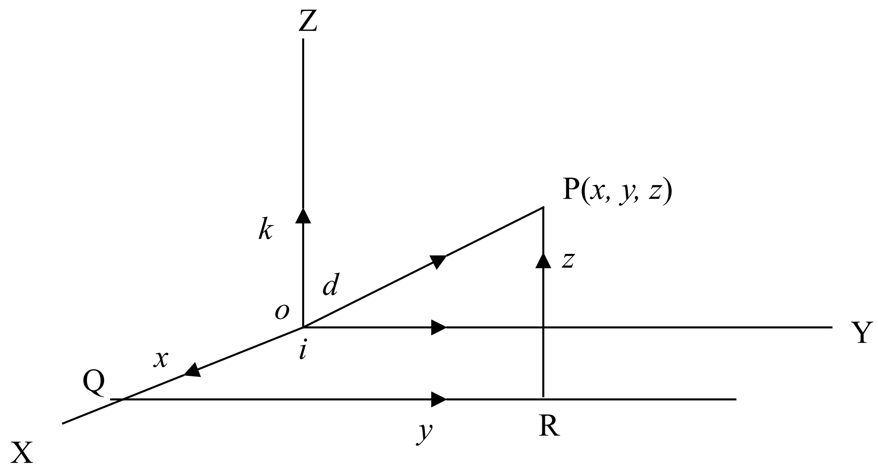


In the above figure $\overline{AB}$ is any vector. Let the length of $\overline{AP}$ = 1 unit then $\overline{AP}$ is called a unit vector in the direction of $\overline{AB}$.

Unit vector $\overline{AP} = + \dfrac{\overline{AB}}{\text{Total length of } \overline{AB}} = \dfrac{\overline{AB}}{\left|\overline{AB}\right|}$

1.7 Representation of a vector using i, j and k.

Let (x, y, z) be the coordinates of a point P in a rectangular coordinate system OXYZ.



Join $\overline{OP}$. Let $\overline{OP} = \overline{p}$

$$\overline{OP} = \overline{OQ} + \overline{QR} + \overline{RP}$$

$$\overline{p} = xi + yj + zk$$

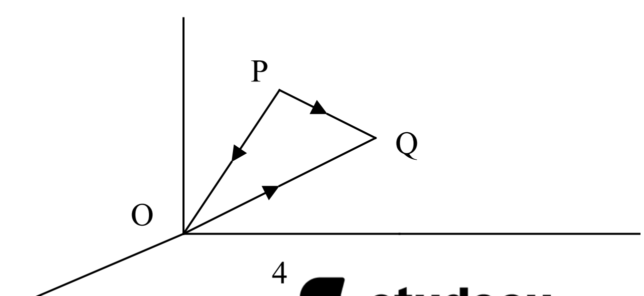
Representation of a vector

If the point P has coordinates (x, y, z) in a rectangular coordinate system OXYZ, then $\overline{OP} = xi + yj + zk$ where $i, j,$ and k are the unit vectors in the direction of X, Y and Z axes respectively.

1.8 Vector joining two points

If P is the point (x_1, y_1, z_1) and Q in the point (x_2, y_2, z_2) then $\overline{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

Proof



Let O be the origin

$\overline{PQ} = \overline{PO} + \overline{OQ}$

$= -(x_1 i + y_1 j + z_1 k) + (x_2 i + y_2 j + z_2 k)$

$\overline{PQ} = (x_2 - x_1) i + (y_2 + y_1) j + (z_2 + z_1) k$

Row vector and column vector

We have seen that a vector $\overline{OP}$ can be written as $\overline{OP} = xi + yj + zk$. Here OP is written as a row and so $\overline{OP}$ is written as a **row vector.** $\overline{OP} = xi + yz + zk$ can also be written as a column as $\overline{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Here *i, j* and *k* are understood. This form of the vector is called **a column vector**. For example,

$3i + 4j + 5k = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ written as a row vector and also as a column vector respectively.

1.9 Magnitude/length/modules of a vector

Let $\overline{a} = xi + yj + zk$ be any vector having particular magnitude and direction. The modulus of the vector $\overline{a}$ is the magnitude of the vector $\overline{a}$. Modulus/length of $\overline{a} = \sqrt{x^2 + y^2 + z^2}$. It is written as $|\overline{a}|$.

Thus if $\overline{a} = xi + yj + zk$ **then** $|\overline{a}| = \sqrt{x^2 + y^2 + z^2}$ **.**

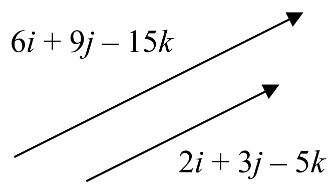
1.10 Parallel vectors

Two vectors $\overline{a}$ and $\overline{b}$ are said to be parallel if one is the scalar multiple of the other or if $\overline{a} = k\overline{b}$ where *k* is a real number.

Example 1
Show that $2i + 3j - 5k$ is parallel to $6i + 9j - 15k$.

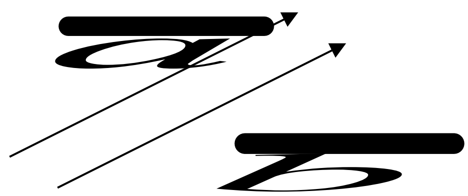
Solution
$6i + 9j - 5k = 3(2i + 3j - 15k)$

Hence the vector $6i + 9j - 10k$ is parallel to $2i + 3j - 5k$ (Direction same)



1.11 Equal vectors

Two vectors $\overline{a}$ and $\overline{b}$ are said to be equal if they have same magnitude and same direction



1.12 Algebra of vectors

Addition and subtraction of vectors.

Vector addition and subtraction are the same as ordinary algebraic addition and subtraction. For example;

$$3x + 4y - 5z + 7x - 2y + 3z$$

$$= (3x + 7x) + (4y - 2y) - (5z - 3z)$$

$$= 10x + 2y - 2z.$$

In the same way,
If $\overline{a} = 3i + 4j - 5k$ and $\overline{b} = 7i + 2j - 3k$

$$\overline{a}+\overline{b}=3i+4j-5k+7i-2j+3k$$
$$=(3i+7i)+(4j-2j)+(-5k+3k)$$
$$=10i+2j-2k.$$

Example 2

If $\overline{a}=6i+2j-5k$, $\overline{b}=8i+7j-k$ and $\overline{c}=3i-j+4k$ then

$$\overline{a}-\overline{b}+\overline{c}=(6i+2j-5k)-(8i+7j-k)+(3i-j+4k)$$
$$=-i+8j-2k.$$

1.13 Multiplication of a vector by a scalar

If $\overline{a}=a_1i+a_2j+a_3k$ then $m\overline{a}$ is defined as $ma_1i+ma_2j+ma_3k$. Where m is a constant.

Thus $3(5i-6j+7k)=15i-18j+21k$.

Example 3

If $\overline{a}=3i+4j+5k$, $\overline{b}=2i-j+3k$ and $\overline{c}=8i+2j-k$ find,

i) $a+b-c$

ii) $2\overline{a}+3\overline{b}+5\overline{c}$

Solution

i) $a+b-c=(3i+4j+5k)+(2i-j+3k)-(8i+2j-k)$

$$=-3i+j+9k$$

ii) $2\overline{a}+3\overline{b}+5\overline{c}=2(3i+4j+5k)+3(2i-j+3k)+5(8i+2j-k)$

$$=6i+8j+10k+6i-3j+9k+40i+10j-5k$$
$$=52i+15j+14k$$

Product of two vectors

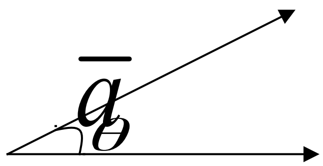
There are two kinds of product of two vectors. They are

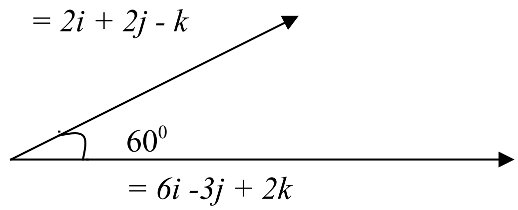
i) Dot product or scalar product of two vectors and

ii) Cross product or the vector product of two vectors

1.14 Dot product of two vectors

If $\overline{a}$ is one vector and $\overline{b}$ is a second vector then the **dot product** of $\overline{a}$ and $\overline{b}$ is written as $\overline{a}\,.\,\overline{b}$. (We put a dot in between $\overline{a}$ and $\overline{b}$). $\overline{a}\,.\,\overline{b}$ is defined as the product of $|\overline{a}|$, $|\overline{b}|$ and **cosine of the angle** between the two vectors.

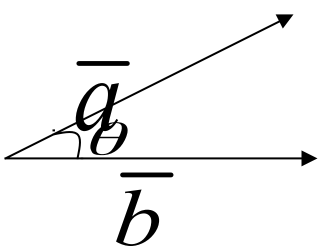


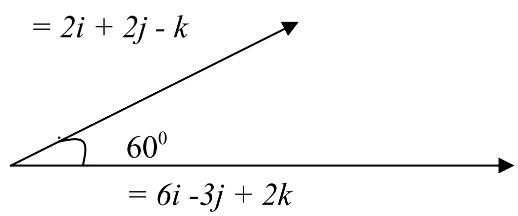


Thus $\overline{a}\,.\,\overline{b} = |\overline{a}|\;|\overline{b}|\cos\theta$.

The dot product is a scalar. So the dot product is also called as **SCALAR PRODUCT** and the angle between $\overline{a}$ and $\overline{b}$ is 60^0.

Example 4
If $\overline{a} = 2i + 2j - k$ and $\overline{b} = 6i - 3j + 2k$ find $\overline{a}\,.\,\overline{b}$





Solution
Let $\overline{a} = 2i + 2j - k$ then $|\overline{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

Let $\overline{b} = 6i - 3j + 2k$ then $\overline{b} = \sqrt{6^2 + 3^2 + 2^2} = 7$

$\cos\ 60^0 = \frac{1}{2}$. Then by definition,

$\overline{a}\,.\,\overline{b} = |\overline{a}|\;|\overline{b}|\cos\theta$

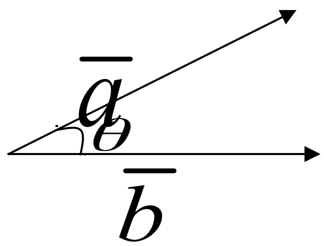
$$=(3)(7)\left(\frac{1}{2}\right)$$

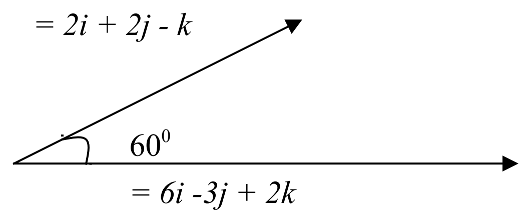
$$=10.5$$

1. Can you prove that $\bar{a}.\bar{b}=0$ then $\theta=90^0$?
2. Can you prove that $\bar{a}.\bar{b}=\bar{b}.\bar{a}$?

1.15 Cross product of two vectors

If $\bar{a}$ is one vector and $\bar{b}$ is a second vector then the **cross product** of two vectors is written as $\bar{a}\times\bar{b}$ (we put a cross in between $\bar{a}$ and $\bar{b}$) is defined as the product of $|\bar{a}|$, $|\bar{b}|$ and **sine of the angle** between the two vectors $\bar{a}$ and $\bar{b}$





Thus

$\bar{a}\times\bar{b}=|\bar{a}|\ |\bar{b}|\sin\theta\bar{n}$ where $\bar{n}$ is a unit vector perpendicular to both $\bar{a}$ and $\bar{b}$

The cross product is a vector and cross product is also called as **vector product**.

Three important properties of unit vectors *i*, *j*, and *k*

(i). $i\,.\,i=j\,.\,j=k\,.\,k=1$

(ii). $i\,.\,j=j\,.\,k=k\,.\,i=0$

(iii). $i\times i=j\times j=k\times k=0$

(iv). $i\times j=k\,,\ j\times k=i\,,\ k\times i=j$

Proof

By definition of dot product since $\theta=0$

$i\,.\,i=|i|\,|i|\cos\theta=(1)(1)(1)=1$

Similarly $j\,.\,j=1,\quad k\,.\,k=1$

$i\,.\,j=|i|\,|j|\cos 90=0$

Similarly $j\,.\,k=0,\quad k\,.\,i=0$

$i \times i = |i|\,|i| \sin 0\, \overline{n} = 0$
where n is a vector perpendicular to *k, i,*
Similarly $j \times j = 0 \qquad k \times k = 0$

$i \times j = |i|\,|j| \sin 90\,(\overline{n}) = 0$ where $\overline{n}$ is a unit vector perpendicular *i* and *j*
= (1) (1) (1) R
= k

In the same way

$j \times k = i, \qquad k \times i = j$ but $j \times i = -k, \qquad k \times j = -i, \qquad i \times k = -j$

1.16 Second definition for $\overline{a}\,.\,\overline{b}$ and a x b

when $\overline{a} = a_1 i + a_2 j + a_3 k$

and $\quad \overline{b} = b_1 i + b_2 j + b_3 k$

$$\overline{a}\,.\,\overline{b} = (a_1 i + a_2 j + a_3 k)\,.\,(b_1 i + b_2 j + b_3 k)$$

$$= a_1 b_1 i\,.\,i + a_1 b_2 i\,.\,j + a_1 b_3 i\,.\,k + a_2 b_1 j\,.\,i + a_2 b_2 j\,.\,j + a_2 b_3 j\,.\,k + a_3 b_1 k\,.\,i + a_3 b_2 k\,.\,j + a_3 b_3 k\,.\,k$$

$$= a_1 b_1 (1) + a_1 b_2 (1) + a_3^2 b_3 (1)$$

Since $i\,.\,i = 1 = j\,.\,j = k\,.\,k$

and $\quad i\,.\,j = i\,.\,k = k\,.\,j$ are all 0.

$$\overline{a}\,.\,\overline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

This result is generally written as,

$$\overline{a}\,.\,\overline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example 5
If $\quad \overline{a} = 2i + 3j + 4k$ and
$\overline{b} = 5i - 6j + 7k$
evaluate $\overline{a}\,.\,\overline{b}$

Solution

$$\bar{a}\cdot\bar{b}=\begin{pmatrix}2\\3\\4\end{pmatrix}\cdot\begin{pmatrix}5\\-6\\7\end{pmatrix}=10-18+28=20$$

1.17 Second formula for $\bar{a}\times\bar{b}$

If $\bar{a}=a_1i+a_2j+a_3k$ **and** $\bar{b}=b_1i+b_2j+b_3k$

then $\bar{a}\times\bar{b}=\begin{vmatrix}i & j & k\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

Proof

$\bar{a}\times\bar{b}=(a_1i+a_2j+a_3k)\times(b_1i+b_2j+b_3k)$

Opening the bracket we have

$\bar{a}\times\bar{b}=a_1b_1i\times i+a_1b_2i\times j+a_1b_3i\times k+a_2b_1j\times i+a_2b_2j\times j+a_2b_3j\times k+a_3b_1k\times i+a_3b_2k\times j+a_3b_3k\times k$

Using the fact $i\times i=j\times j=k\times k=0$

and $\quad i\times j=k,\quad j\times k=i,\quad k\times i=j$

$\quad j\times i=-1,\quad k\times j=-i,\quad i\times k=-j$

we have,

$$\bar{a}\times\bar{b}=i(a_2b_3-a_3b_2)+j(a_3b_1-a_1b_3)+k(a_1b_2-a_2b_1)$$

$$=\begin{vmatrix}i & j & k\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Example 6

If $\quad \bar{a}=2i+3j+4k$ and

$\quad \bar{b}=5i-6j+7k$

evaluate $\bar{a}\times\bar{b}$

Solution

By definition

$$\bar{a}\times\bar{b}=\begin{vmatrix}i & j & k\\ 2 & 3 & 4\\ 5 & -6 & 7\end{vmatrix}$$

$$=(21i-12k+20j)-(15k+24i+14j)$$
$$=i(21-24)+j(20-14)+k(-12-15)$$
$$=-3i+6j-27k$$
$$=3(-i+2j-9k)$$

Example 7
Using the formula
$$\bar{a}\,.\,\bar{b}=|a|\,|b|\cos\theta=a_1b_1+a_2b_2+a_3b_3$$
calculate the angle between the two vectors
$\bar{a}=2i+2j-k$ and $\bar{b}=6i-3j+2k$

Solution
Using the two formulae,
$\bar{a}\,.\,\bar{b}=|a|\,|b|\cos\theta=a_1b_1+a_2b_2+a_3b_3$ we have
$$\bar{a}\,.\,\bar{b}=|a|\,|b|\cos\theta$$
$$=\sqrt{2^2+2^2+1^2}\quad\sqrt{6^2+3^2+2^2}\cos\theta=\sqrt{9}\quad\sqrt{49}\cos\theta=21\cos\theta$$
Also $\bar{a}\,.\,\bar{b}=\begin{pmatrix}2\\2\\-1\end{pmatrix}.\begin{pmatrix}6\\-3\\2\end{pmatrix}=12-6-2=4$
Equating the two results we have,
$21\cos\theta=4$
or $\cos\theta=\dfrac{4}{21}=0.1905$
or $\theta=79$ approximately.

Example 8
Show that the vectors
$\bar{a}=3i-2j-k$ and $\bar{b}=2i+j-4k$ are perpendicular.

Solution
Let $\bar{a}=3i-2j-k$ and $\bar{b}=2i+j-4k$. Then
$$\bar{a}\,.\,\bar{b}=\begin{pmatrix}3\\-2\\1\end{pmatrix}.\begin{pmatrix}2\\1\\4\end{pmatrix}=6-2-4=0$$
or $|\bar{a}|\,.\,|\bar{b}|\cos\theta=0$ and $|\bar{a}|=\sqrt{14}\quad|\bar{b}|=21$ are not zero.
Hence $\cos\theta=0$ and $\theta=90^0$

Example 9

Determine the value of p so that $\overline{a} = 2i + 3j + k$ and $\overline{b} = ai - 2j - 2k$ are perpendicular.

Solution

If $\overline{a}$ and $\overline{b}$ are perpendicular $\overline{a} . \overline{b} = 0$ and conversely.

Now

$$\overline{a} . \overline{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = 0$$

or $2b - 6 - 2 = 0$

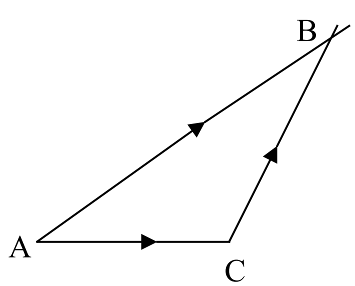
or $b = 4$

Exercise 1

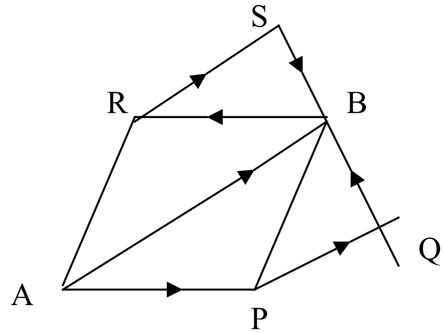
1. What is a vector?
2. If the coordinates of a point P is (3, 4), express $\overline{OP}$ using i and j

. what is the length $\overline{OP}$.

3. State two components of $\overline{AB}$ in the following figure:



4. Express $\overline{AB}$ using any three components



5. State two conditions for the two vectors AB and CD be equal.

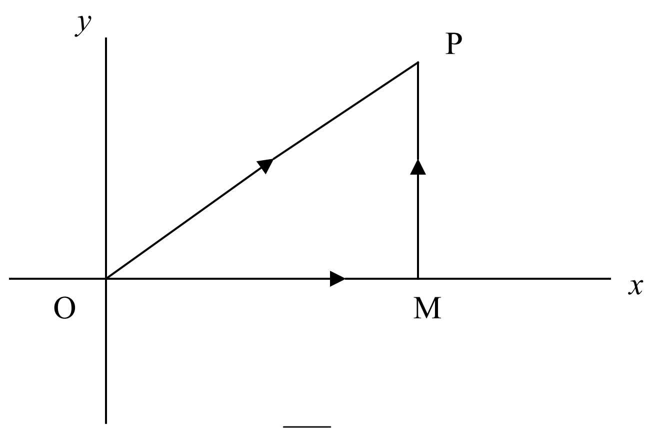
6. Simplify into a single vector

   a) $\overline{AB} + \overline{BC} + \overline{CF}$

   b) $\overline{LM} + \overline{MP} + \overline{PQ}$

   c) $\overline{PQ} + \overline{QR} + \overline{ST} + \overline{RS}$

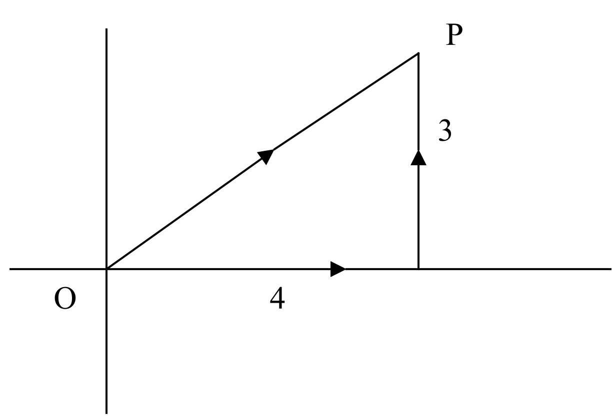
   (Note that the end of first vector is the same as the beginning of the next vector. Draw a figure and see)



7. Write down the vector $\overline{OP}$ in terms of the rectangular components parallel to x and the y axes. (See the figure above)

8. Explain the unit vectors *i, j and k*

9. Write down $\overline{OP}$ in terms of *i* and *j*



10. The coordinate of a point P in two dimension is (6, 8).

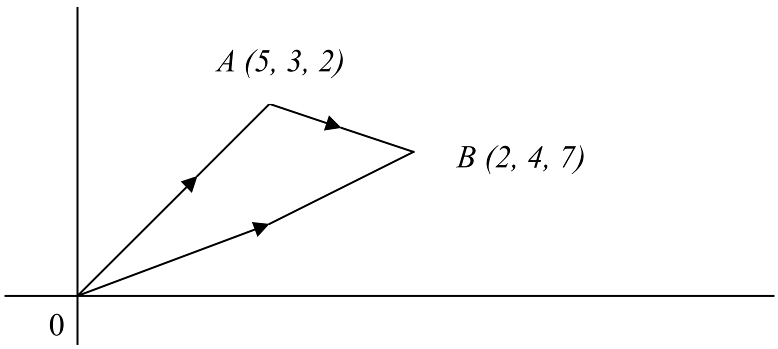
Write down $\overline{OP}$ as a row vector and also as a column vector.

11. The coordinate of a point P in three dimension is (3, 4, 5).

Write down $\overline{OP}$ as a

i). Row vector using *i, j and k*

ii). Column vector



12. If the coordinates of a point A are (5, 3, 2) write down the position vector $\overline{OA}$ and $\overline{AO}$. The coordinates of B are (2, 4, 7). Write down $\overline{OB}$ and $\overline{BO}$.

Writing $\overline{AB} = \overline{AO} + \overline{OB}$ find $\overline{AB}$. Hence find $\overline{BA}$.

13. If the coordinates of A are (x_1, y_1, z_1) and those of B are (x_2, y_2, z_2) show that

$$\overline{AB} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

14. Find the modulus of the vectors:

a) $-2i - 2j + k$

b) $5i - 2j - 5k$

c) $3i - 2j + 6k$

d) $-5i + 12j$

15. Find a unit vector in the direction of.

a) $\left(\text{unit vector} = \dfrac{-5i+12j}{\sqrt{5^2+12^2}}\right)$

b) $8i - j - 4k$

c) $7i - 5j + k$

d) $3i - 2j + 6k$

16. Find the vector AB when A is (1, 2, 5) and B is (2, 3, 4). Hence find the distance AB.

17. Find the distance between (3, -2, -4) and (6, 1, 0).

18. Find a vector of magnitude 3 units which is parallel to $i - 3j + 2k$.

19. Given that $\overline{a} = 2\lambda i + 3j - \lambda k$ and that $\left|\overline{a}\right| = 5$, find the possible values of λ.

20. Given that $\overline{b} = 3i - 2\lambda j + 5\lambda k$ and $\left|\overline{b}\right| = \sqrt{67}$ find the possible values of λ.

Dot product and cross product

21. Find $\overline{a}.\overline{b}$ when

i). $\overline{a} = 2i - j + 5k$ and $\overline{b} = -3i + 5j - 7k$

ii). $\overline{a} = -5i + 7j - 8k$ and $\overline{b} = 2i - 5j + 11k$

22. Write down $\overline{a}.\overline{b}$ in two ways and determine the angle between $\overline{a}$ and $\overline{b}$ in degrees (1 decimal place) where $\overline{a} = 2i - j - 2k$ and $\overline{b} = 3i + j - 5k$

23. Find the angle between the vectors $\overline{a} = 5i - 3j + 7k$ and $-3i - 4j + 2k$

24. a). What is the condition for the vectors

i). $\overline{a}$ and $\overline{b}$ perpendicular

ii). $\overline{a}$ and $\overline{b}$ are parallel

b). Given that $\overline{p} = 3i + 2j$ and $q = 2i + \lambda j$ determine the values of λ such that

i). $\overline{p}$ and $\overline{q}$ are at right angles

ii). $\overline{p}$ and $\overline{q}$ are parallel iii). The acute angle between $\overline{p}$ and $\overline{q}$ is 45^0

25. Find the values of λ for which the vectors $2\lambda i + \lambda j - 4k$ and $\lambda i - 2j + k$ are perpendicular.

26. Find a vector $\overline{p}$ which is perpendicular to both $\overline{a}$ and $\overline{b}$ where

$\overline{a} = i + 3j - k$

$\overline{b} = 3i - j - k$

27. Show that $-i + 6j - 9k$ is perpendicular to both the vectors $3i + 2j + k$ and $-3i + 4j + 3k$

28. Find the projection of $\overline{a} = i - 2j + k$ in the direction of $\overline{b} = 4i - 4j + 7k$. It is $\overline{a}.b_1$ where b_1 is the unit vector of $\overline{b}$

29. The vectors $\overline{u}$ and $\overline{v}$ are $i - 2j + 3k$ and $8i + 9j + 12k$ respectively.

30. Find a vector perpendicular to both $\overline{a} = 2i + 3j + 4k$ and $\overline{b} = -3i + j - k$.

31. Find the angle between the vectors $\overline{a}$ and $\overline{b}$ given that $|\overline{a}| = 3$, $|\overline{b}| = 5$ and $\overline{a} - \overline{b} = 7$.

32. If $\overline{a} = a_1 i + a_2 j + a_3 k$ and $\overline{b} = b_1 i + b_2 j + b_3 k$

i). Write down $\overline{a} \times \overline{b}$ in determinant form

ii). Show that

$$\overline{a} \times \overline{b} = i(a_2 b_3 - a_3 b_2) + j(a_3 b_1 - a_1 b_3) + k(a_1 b_2 - a_2 b_1)$$

33. If $\overline{a} = 2i - 3j - k$ and $\overline{b} = i + 4j - 2k$ determine

i). $\overline{a} \times \overline{b}$

ii). $\overline{b} \times \overline{a}$

34. Evaluate:

$i \times j,\quad j \times k$ and $k \times i$

$i \times i,\quad j \times j$ and $k \times k$

35. a). If $\overline{a} = 3i - j + 2k$ and $\overline{b} = 2i + j - k$ show that $\overline{a} \times \overline{b} = -i + 7j + 5k$

b). If $c = i - 2j + 2k$ find $(\bar{a} \times \bar{b}) \times \bar{c}$

36. Determine a unit vector perpendicular to the plane of $\bar{a} = 2i - 6j - 3k$ and $\bar{b} = 4i + 3j - k$.

37. Prove that the area of a parallelogram with sides $\bar{a}$ and $\bar{b}$ is $\bar{a} \times \bar{b}$ in magnitude.

38. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ represent the sides of a triangle show that

i). $a \times b = b \times c = c \times a = \dfrac{\text{area of triangle}}{2}$

ii). $ab \sin C = bc \sin A = ca \sin B$

iii). Hence show that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

39. Show that $\bar{a} = \frac{1}{3}(2i - 2j + k)$, $\bar{b} = \frac{1}{3}(i + 2j + 2k)$ and $\bar{c} = \frac{1}{3}(2i + j - 2k)$ are mutually orthogonal unit vectors.

Further reading

1.) Differential geometry of three dimensions

By Weatherburn,

Cambridge 1957

2.) Differential geometry

By Dr. Sengottaiyan

Oxford publications London. Nairobi

Lesson two

Triple products of vectors and vector quantities in Triple products

2.1 Introduction

You have learnt the dot and cross product of two vectors, in the previous Lesson. In this Lesson you will study the dot and cross product for three or more vectors. Not all the triple products are meaningful as we shall see in this Lesson

2.2 Objectives of the Lesson

By the end of this Lesson you should be able to

- define the triple product of vectors in different forms
- evaluate the dot and cross products of three vectors when the products are meaningful.
- find an expression for the volume of a tetrahedron whose edges are vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$
- to determine whether three given vectors are coplanar (lie on the same plane).

2.3 Vector triple products

Dot and cross products of three vectors $\bar{A}$, $\bar{B}$ and $\bar{C}$ may produce meaningful products of the form $(\bar{A}.\bar{B})C$, $\bar{A}.(\bar{B}\times\bar{C})$ and $A\times(B\times C)$. These products are called **triple products of vectors.**

The following laws are valid (you can verify).

i) $\bar{A}.(\bar{B}\times\bar{C}) = \bar{B}.(\bar{C}\times\bar{A}) = \bar{C}.(\bar{A}\times\bar{B})$

ii) $A.(A\times C) = 0$

iii) $A\times(B\times C) \neq (A\times B)\times C$

iv) $A\times(B\times C) = (A.C)B - (A.B)C$

v) $(A\times B)\times C = (A.C)B - (B.C)A$

vi) $\bar{A}.(\bar{B}+\bar{C}+\bar{D}) = \bar{A}.\bar{B} + \bar{A}.\bar{C} + \bar{A}.\bar{D}$

2.4 The Scalar triple product

The product $\overline{A}.(\overline{B}\times\overline{C})$ is called the **scalar triple product** and is denoted by [A B C].

2.5 The vector triple product

The product $\overline{A}\times(\overline{B}\times\overline{C})$ is called the vector triple product.

2.6 Theorem 1

If $\overline{a} = a_1 i + a_2 j + a_3 k$
$\overline{b} = b_1 i + b_2 j + b_3 k$
$\overline{c} = c_1 i + c_2 j + c_3 k$

Show that $\overline{a}.(\overline{b}\times\overline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Solution

We know that $\overline{b}\times\overline{c} = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Then $\overline{a}.(\overline{b}\times\overline{c}) = \overline{a}.\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= (a_1 i + a_2 j + a_3 k).(b_2c_3 - b_3c_2)\, i + (b_3c_1 - b_1c_3)\, j + (b_1c_2 - b_2c_1 k)$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_2 - b_2c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Since $i.i = j.j = k.k$ and $i.j = 0 = j.k = k.j$

2.7 Theorem 2

If $\overline{a} = a_1 i + a_2 j + a_3 k$

$\overline{b} = b_1 i + b_2 j + b_3 k$
$\overline{c} = c_1 i + c_2 j + c_3 k$

show that $\overline{a} \times \left(\overline{b} \times \overline{c} \right) = \overline{b}\left(\overline{a} . \overline{c} \right) - \overline{c}\left(\overline{a} . \overline{b} \right)$

Solution

$$\overline{a} \times \left(\overline{b} \times \overline{c} \right) = \left(a_1 i + a_2 j + a_3 k \right) \times \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \left(a_1 i + a_2 j + a_3 k \right) \times \left[\left(b_2 c_3 - b_3 c_2 \right) i + \left(b_3 c_1 - b_1 c_3 \right) j + \left(b_1 c_2 - b_2 c_1 \right) k \right]$$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_2 c_3 - b_3 c_2 & b_3 c_1 - b_1 c_3 & b_1 c_2 - b_2 c_1 \end{vmatrix}$$

$$= \left(a_2 b_1 c_2 - a_2 b_2 c_1 - a_3 b_3 c_1 + a_3 b_1 c_3 \right) i + \left(a_3 b_2 c_3 - a_3 b_3 c_2 - a_1 b_1 c_2 + a_1 b_2 c_1 \right) j \left(a_2 b_3 c_1 - a_1 b_1 c_3 - a_2 b_2 c_3 + a_2 b_3 c_2 \right) k$$

Also $\overline{b}\left(\overline{a} . \overline{c} \right) - \overline{c}\left(\overline{a} . \overline{b} \right)$

$$= \left(b_1 i + b_2 j + b_3 k \right)\left(a_1 c_1 + a_2 c_2 + a_3 c_3 \right) - \left(c_1 i + c_2 j + c_3 k \right)\left(a_1 b_1 + a_2 b_2 + a_3 b_3 \right)$$

$$= \left(a_2 b_1 c_2 - a_2 b_2 c_1 - a_3 b_3 c_1 + a_3 b_1 c_3 \right) i + \left(a_3 b_2 c_3 - a_3 b_3 c_2 - a_1 b_1 c_2 + a_1 b_2 c_1 \right) j \left(a_2 b_3 c_1 - a_1 b_1 c_3 - a_2 b_2 c_3 + a_2 b_3 c_2 \right) k$$

Hence $\overline{a} \times \left(\overline{b} \times \overline{c} \right) = \overline{b}\left(\overline{a} . \overline{c} \right) - \overline{c}\left(a . b \right)$

Example 1

Using the Theorems above show that

$a \times \left(b \times c \right) + b \times \left(c \times a \right) + c \times \left(a \times b \right) = 0$

Using theorem II we have

$\overline{a} \times \left(\overline{b} \times \overline{c} \right) + \overline{b} \times \left(\overline{c} \times \overline{a} \right) + \overline{c} \times \left(\overline{a} \times \overline{b} \right)$

$$= \overline{b}\left(\overline{a} . \overline{c} \right) - \overline{c}\left(\overline{a} . \overline{b} \right) + \overline{c}\left(\overline{b} . \overline{a} \right) - \overline{a}\left(\overline{b} . \overline{c} \right) + \overline{a}\left(\overline{c} . \overline{b} \right) - \overline{b}\left(\overline{c} . \overline{a} \right)$$

$= 0$

Since $\overline{a} . \overline{c} = \overline{c} . \overline{a}$, $\overline{a} . \overline{b} = \overline{b} . \overline{a}$ and $\overline{c} . \overline{b} = \overline{b} . \overline{c}$

Example 2

If $\overline{a} = 2i - 3j + 4k$, $\overline{b} = i - 2j + 3k$ and $\overline{c} = 2i + j + 2k$

Find

i) $\overline{b} \times \overline{c}$

ii) $\overline{a}.(\overline{b} \times \overline{c})$

iii) $\overline{a} \times (\overline{b} \times \overline{c})$

iv) $\overline{b}(\overline{a}.\overline{c}) - \overline{c}(\overline{a}.\overline{b})$

Solution

Let $\overline{a} = 2i - 3j + 4k$, $\overline{b} = i - 2j + 3k$ and $\overline{c} = 2i + j + 2k$

then

i) $(\overline{b} \times \overline{c}) = \begin{vmatrix} i & j & k \\ 1 & -2 & -3 \\ 2 & 1 & 2 \end{vmatrix}$

$= (-4i - 6j + k) - (2j - 3i - 4k)$

$= -i - 8j + 5k$ (2)

ii) $\overline{a}.(\overline{b} \times \overline{c}) = (2i - 3j + 4k).(-i - 8j + 5k)$

$= \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} . \begin{pmatrix} -1 \\ -8 \\ 5 \end{pmatrix}$ from 2

= -2 + 24 + 20

= 42

iii) $\overline{a} \times (\overline{b} \times \overline{c}) = (2i - 3j + 4k) \times (-i - 8j + 5k)$

$= \begin{vmatrix} i & j & k \\ 2 & -3 & 4 \\ -1 & -8 & 5 \end{vmatrix}$ from 2

$= (-15i - 4j - 16k) - (3k - 32i + 10j)$

$= 17i - 14j - 19k$ (3)

iv) $\overline{b}(\overline{a}.\overline{c}) - \overline{c}(\overline{a}.\overline{b}) = \overline{a} \times (\overline{b} \times \overline{c})$ by Theorem II

$= 17i - 14j - 19k$ using 3

(or do it independently)

$$\bar{b}(\bar{a}.\bar{c}) = (i - 2j - 3k)\left[(2i - 3j + 4k).(2i + j + 2k)\right]$$

$$= (i - 2j - 3k)\left[4 - 3 + 8\right]$$

$$= 9i - 18j - 27k$$

$$\bar{c}(\bar{a}.\bar{b}) = -(2i + j + 2k)\left[(2i - 3j + 4k).(i - 2j + 3k)\right]$$

$$= -(2i + j + 2k)\left[2 + 6 + 12\right]$$

$$= -(2i + j + 2k)(20)$$

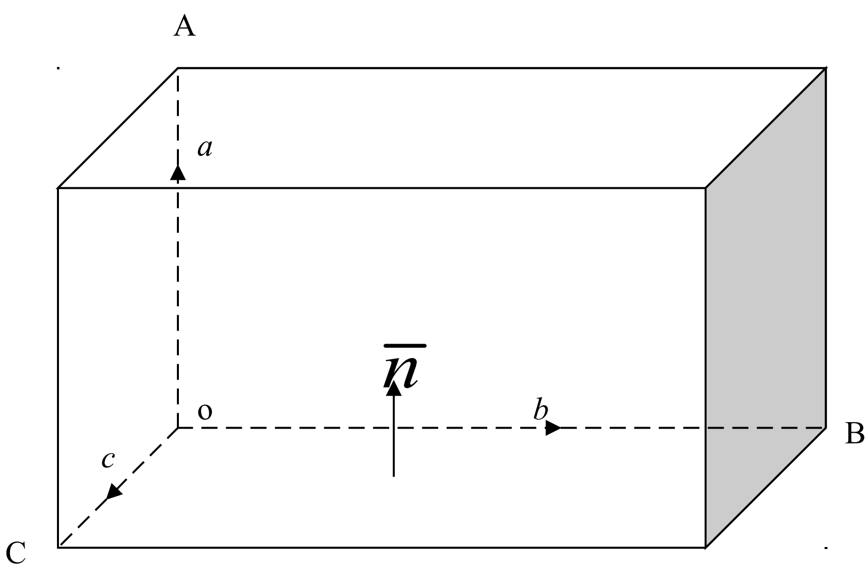
$$= -(40i + 20j + 40k)$$

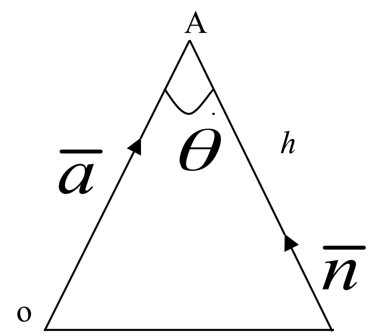
2.8 Prove that $\bar{a}.(\bar{b} \times \bar{c})$ is in absolute value equal to the volume of a parallelopiped with sides $\bar{a}$, $\bar{b}$ and $\bar{c}$.

Show also that the volume $= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

if $\bar{a} = a_1 i + a_2 j + a_3 k$

$\bar{b} = b_1 i + b_2 j + b_3 k$ and $\bar{c} = c_1 i + c_2 j + c_3 k$





$$\bar{a}.\bar{n} = |a||n|\cos\theta$$

$$= |\bar{a}|\cos\theta$$

$$= h$$

Since $\cos\theta = \dfrac{h}{|a|}$

Let $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ represent the three sides of the parallelepiped. Then OA, O:B and OC are not mutually perpendicular. It will be a cuboid if they are perpendicular.

Volume of parallelepiped = (height h) (base area of the parallelogram)

If |a| is the slant height and the unit vector $\overline{n}$ is perpendicular to the base then the perpendicular height $h = \overline{a}.\overline{n}$.

The base area = $|\overline{b}||\overline{c}|\sin\phi$ when ϕ is the angle between OB and OC. Then the volume of the parallelepiped is given by

$$v = (\overline{a}.\overline{n})\left[|\overline{b}||c|\sin\theta\right]$$

$$= \overline{a}.\left[|b||c|\sin\theta\right]\overline{n}$$

$$v = \overline{a}.(\overline{b}\times\overline{c})$$

If $\overline{a} = a_1 i + a_2 j + a_3 k$

$\overline{b} = b_1 i + b_2 j + b_3 k$

$\overline{c} = c_1 i + c_2 j + c_3 k$

Then

$$v = \overline{a}.(\overline{b}\times\overline{c})$$

$$= (a_1 i + a_2 j + a_3 k).\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.\begin{pmatrix} (b_2c_3 & - & b_3c_2) \\ (b_3c_1 & - & b_1c_3) \\ b_2c_1 & - & b_1c_2 \end{pmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_2c_1 - b_1c_2)$$

Thus $v = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

What is your conclusion if $v = 0$. Think!

Example 3

Find the volume of the parallelepiped whose edges are given by $\overline{a} = 2i - 3j + 4k$, $\overline{b} = i + 2j - k$ and $\overline{c} = 3i - j + 2k$

Solution

Volume of parallelepiped = $\overline{a}.(\overline{b} \times \overline{c})$ in absolute value

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

= 7 units

Example 4

Show that the three vectors $\overline{a} = 2i - j + k$, $\overline{b} = i + 2j - 3k$ and $\overline{c} = 3i - 4j + 5k$ are coplanar (or lie on the same plane).

Solution

If the volume of the parallelepiped formed by three vectors is zero then the three vectors lie on the same plane.

Now volume $V = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & -4 & 5 \end{vmatrix}$

= (20 + 9 – 4) – (6 + 24 – 5)

= 25 – 25 = 0

Hence the three vectors are coplanar.

Example 5

Determine the value of m such that the vectors $\overline{a} = 2i + j + 4k$, $\overline{b} = 3i + 2j + mk$ and $\overline{c} = i - 4j + 2k$ are coplanar.

Solution

If the three vectors are coplanar, the volume of the parallelepiped formed by these vectors must be zero.

Hence $V = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 2 & m \\ 1 & 4 & 2 \end{vmatrix} = 0$

or $(8 + m + 48) - (8 + 8m + 6) = 0$

or $56 + m - 14 - 8m = 0$

or $42 - 7m = 0$

Hence $m = 6$

Example 6

Show that $\bar{a}.(\bar{a} \times \bar{b}) = 0$

Solution

Let $\bar{a} = a_1 i + a_2 j + a_3 k$

$\bar{b} = b_1 i + b_2 j + b_3 k$

$\bar{c} = c_1 i + c_2 j + c_3 k$

$$\bar{a}.(\bar{a} \times \bar{b}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= 0$$

Since two rows of the determinant are identical.

2.9 Vector Quantities of Triple Products

Triple products may yield either scalar quantities or vector quantities. You have already seen that the dot product $\bar{a}.\bar{b}$ is a scalar but $\bar{a} \times \bar{b}$ is a vector.

Using this fact we shall analyse the triple products of vectors

i) Consider $\bar{a}.(\bar{b} \times \bar{c})$

Here $\overline{a}$ is a vector and $\overline{b} \times \overline{c}$ is a vector. Hence the product $\overline{a}.(\overline{b} \times \overline{c})$ is a scalar. We have seen that

$$\overline{a}.(\overline{b} \times \overline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ which is a scalar $\overline{a}.(\overline{b} \times \overline{c}) = (\overline{b} \times \overline{c}).\overline{a}$

Since dot products are commutative.

ii) Consider $\overline{a} \times (\overline{b} \times \overline{c})$

Here the cross product $\overline{b} \times \overline{c}$ is a vector and $\overline{a}$ is also a vector. Hence the vector (cross) product $\overline{a} \times (\overline{b} \times \overline{c})$ is a vector quantity. $\overline{a} \times (\overline{b} \times \overline{c})$ is the sum of two vectors $\overline{b}$ and $\overline{c}$ with coefficients $\overline{a}.\overline{c}$ and - $\overline{a}.\overline{b}$ respectively.

Thus $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a}.\overline{c})\overline{b} - (\overline{a}.\overline{b})\overline{c}$

Note that the coefficients of $\overline{b}$ and that of $\overline{c}$ are scalars ($\overline{a}.\overline{c}$) and – ($\overline{a}.\overline{b}$)

iii) Consider $\overline{a}.(\overline{b}.\overline{c})$

Here $\overline{a}$ is a vector and $\overline{b}.\overline{c}$ is a scalar the dot product of a vector and a scalar has no meaning.

iv) $\overline{a} \times (\overline{b}.\overline{c})$

Here $\overline{a}$ is a vector and $\overline{b}.\overline{c}$ is a scalar. Hence $\overline{a} \times (\overline{b}.\overline{c})$ has no meaning

Exercise 2

1. If $\overline{a} = 2i - 3j$, $\overline{b} = i + j - k$ and $\overline{c} = 3i - k$. Find the value of $\overline{a}.(\overline{b} \times \overline{c})$
2. If $\overline{a}$, $\overline{b}$ and $\overline{c}$ are any three vectors show that $\overline{a}.(\overline{a} \times \overline{c}) = 0$
3. a). Prove that $\overline{a} \times (\overline{b} \times \overline{c}) = \overline{b}(\overline{a}.\overline{c}) - \overline{c}(\overline{a}.\overline{b})$

b). Hence show that $\bar{a}\times(\bar{b}\times\bar{c})+\bar{b}\times(\bar{c}\times\bar{a})+c(a\times b)=0$

4. If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are any three vectors

   i) Show that $(\bar{a}\times\bar{b})\times\bar{c}=-\bar{c}\times(\bar{a}\times\bar{b})$

   ii) Prove that $a\times(b\times c)\neq(a\times b)\times c$

5. If $\bar{a}=i-2j-3k$

   $\bar{b}=2i+j-k$

   $\bar{c}=i+3j-2k$

Find

   i) $\left|(\bar{a}\times\bar{b})\times\bar{c}\right|$

   ii) $\bar{a}.(\bar{b}\times\bar{c})$

   iii) $(\bar{a}\times\bar{b})\times(\bar{b}\times\bar{c})$

   iv) $(\bar{a}\times\bar{b}).\bar{c}$

   v) $(\bar{a}\times\bar{b})(\bar{b}.\bar{c})$

6. Prove that $(\bar{a}\times\bar{b}).(\bar{c}\times\bar{a})+(\bar{b}\times\bar{c}).(\bar{a}\times\bar{d})+(\bar{c}\times\bar{a}).(\bar{b}\times\bar{d})=0$

7. a). Show that $\bar{a}.(\bar{b}\times\bar{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

   b). Hence show that $a.(b\times c)=b.(c\times a)=c.(a\times b)$

8. Show that $a.(b\times c)=(a\times b).c$

9. Find the volume of a parallelepiped whose edges are $\bar{a}=2i-3j$, $\bar{b}=i+j-k$, $\bar{c}=3i-k$.

10. Find the volume of the parallelepiped whose edges are represented by $\bar{a}=2i-3j+4k$, $\bar{b}=i+2j-k$, $\bar{c}=2i-j+2k$.

11. The vectors $\bar{a}=2i-j+k$, $\bar{b}=i+2j-3k$ and $\bar{c}=3i-pj+5k$ are coplanar. Find the constant p. (Hint: The volume of the parallelepiped = 0)

12. If $\bar{a} = i - 2j - 3k$, $\bar{b} = 2i + j - k$ and $\bar{c} = i + 3j - 3k$ are the edges of a parallelepiped determine its volume.

Summary of the Lesson

You have learnt the following from this Lesson:

i) $\bar{a}.(\bar{a} \times \bar{c}) = 0$

ii) $\bar{a}.(\bar{b} \times \bar{c}) = \bar{b}.(\bar{c} \times \bar{a}) = c.(\bar{a} \times \bar{b})$

iii) $\bar{a}.(\bar{b} \times \bar{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

where $\bar{a} = a_1 i + a_2 j + a_3 k$, $\bar{b} = b_1 i + b_2 j + b_3 k$ and $\bar{c} = c_1 i + c_2 j + c_3 k$

iv) If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are edges of a parallelepiped then its volume is given by

V = $\bar{a}.(\bar{b} \times \bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

If V = 0 then the three vectors are coplanar

v) $\bar{a}.(\bar{b} \times \bar{c})$ is a scalar where as

$\bar{a} \times (\bar{b} \times \bar{c})$ is a vector

$\bar{a}.(\bar{b} . \bar{c})$ has no meaning

Further Reading

Differential Geometry
By Dr. D. Sengottaiyan
Oxford Publications
London, Nairobi.

Lesson three

Equations of Straight Lines and Planes

3.1 Introduction

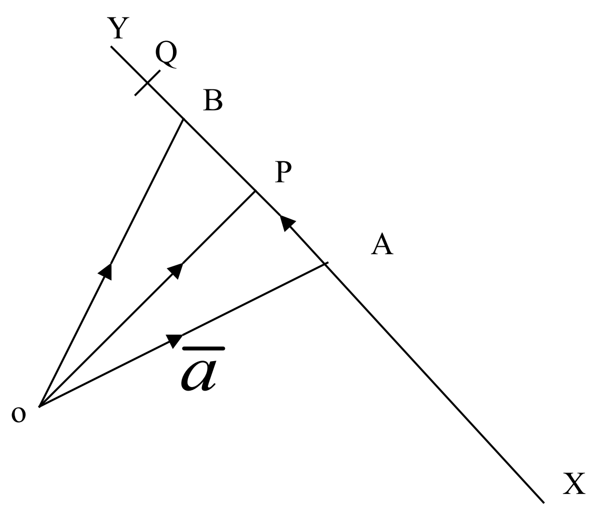
In this lesson you will study the nature of equations to the straight lines in three dimension. The vector Equation, the Cartesian equation and the parametric equation of the straight lines in space are derived. The vector equation and the rectangular Cartesian Equation of Planes are also studied.

3.2 Objectives of the lesson

By the end of this lesson you should be able to

- derive the equation of a space line in vector form, rectangular Cartesian form and in parametric form.
- derive the equation of planes in different forms.
- apply the equations to solve problems

3.3 Vector equation of a straight line when two points on the line are known



Consider any straight line X Y in space. Let A(x y z) and B(x, y, z) are given with respect to a rectangular system with origin O.

Let $\overline{OA} = \overline{a}$ and $\overline{OB} = \overline{b}$ so that $\overline{AB} = \overline{AO} + \overline{OB} = - \overline{a} + \overline{b} = \overline{b} - \overline{a}$. Consider a variable point P on the line XY. Let the variable vector $\overline{OP} = \overline{r}$.

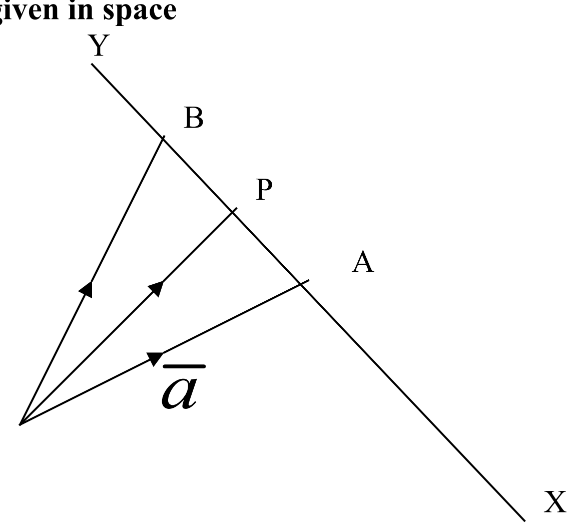
Now $\overline{OP} = \overline{OA} + \overline{AP}$

$= \overline{OA}$ + a fraction of $\overline{AB}$

Then $\overline{r} = \overline{a} + t\left(\overline{b} - \overline{a}\right)$

where t is a scalar $\lessgtr 1$. For example if P is taken at Q the $\overline{AP}$ is greater than $\overline{AB}$ then t is greater than 1.
Thus equation (1) represents the equation of a straight line when the coordinates of two points or two vectors are known on the line.

3.4 Cartesian equation of a straight line when two points $A(x_1 y_1 z_1)$ and $B(x_2 y_2 z_2)$ are given in space



Let $A(x_1 y_1 z_1)$ and $B(x_2 y_2 z_2)$ be two points. Let $P(x, y, z)$ be a variable point on the line. We have seen in the previous article that the vector equation of the line AB is given by,

$$\overline{r} = \overline{a} + t\left(\overline{b} - \overline{a}\right) \qquad (1)$$

where $\overline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\quad \overline{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\overline{b} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

Using Cartesian coordinates for $\overline{a}$ $\overline{b}$ and $\overline{OP}$ in (1) we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\left[\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right]$$

or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

or

$$\begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix} = t \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

where $\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$ is the direction vector of the line AB.

From (3) we have $x - x_1 = lt$, $y - y_1 = mt$ and $z - z_1 = nt$
or

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = t \qquad (4)$$

This is the Cartesian equation of the line AB.

3.5 Parametric equations of the line AB

From (4), $\frac{x - x_1}{l} = t$ gives

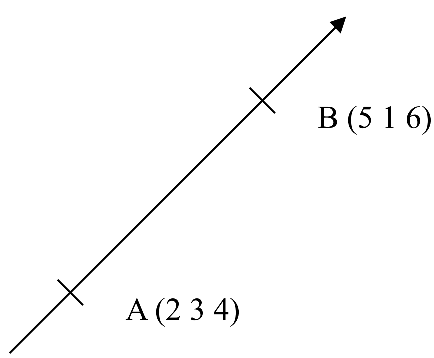
$$x = x_1 + lt$$

Similarly $$y = y_1 + mt$$

and $$z = z_1 + nt$$

are the parametric equations of the line AB, passing through the point (x_1, y_1, z_1) and having direction vector $li + mj + nk$

Example 1
A straight line passes through the points A(2,3,4) and B(5,1,6)



a). Write down the direction vector of $\overline{AB}$

b). State

i) the vector equation of the line AB.

ii) the Cartesian an equation of the line

iii) the parametric equation of the line AB

Solution

The direction vector is

$$\begin{pmatrix} 5 & - & 2 \\ 1 & - & 3 \\ 6 & - & 4 \end{pmatrix} \text{ or } \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \text{ or } \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$$

By formula the vector equation is $\overline{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 5 & - & 2 \\ 1 & - & 3 \\ 6 & - & 4 \end{pmatrix}$ or $\overline{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ the

Cartesian equation is given by,

$\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ each = t

Hence $\dfrac{x-2}{3} = \dfrac{y-3}{-2} = \dfrac{z-4}{2}$ is a Cartesian equation.

Instead of the point (2, 3, 4) we can also use the point (5, 1, 6). Then another form is given by $\dfrac{x-5}{3} = \dfrac{y-1}{-2} = \dfrac{z-6}{2}$.

The parametric equation is given by

$x = x_1 + lt$, $y = y_1 + mt$, $z = z_1 + nt$ or $x = 2 + 3t$, $y = 3 - 2t$, $z = 4 + 2t$ is the one form of the parametric equation.

Example 2

Find the parametric equations and a vector equation of a line whose Cartesian equations are given by

$\dfrac{x-1}{5} = \dfrac{y+1}{2} = \dfrac{z}{-5}$

Let $\lambda = \dfrac{x-1}{5} = \dfrac{y+1}{2} = \dfrac{z}{-5}$

Then $x = 5\lambda + 1$, $y = 2\lambda - 1$, $z = -5\lambda$ is the parametric equation of the line. The line passes through the point (1, -1, 0). So a vector equation is given by

$\overline{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 5 \\ 2 \\ -5 \end{pmatrix}$ or $\overline{r} = i - j + \lambda(5i + 2j - 5k)$.

Example 3
Find a vector equation for the line that passes through the points (3, 1, -2) and (4, 3, 4).

Solution

A direction vector of the line is $(4-3)i+(3-1)j+(4+2)k = i+2j+6k$ or $\begin{pmatrix}1\\2\\6\end{pmatrix}$

So a vector equation of the line is given by $y = 3i + j - 2k + \lambda(i+2j+6k)$.

Example 4
Find the acute angle between the lines
$\frac{x-1}{5} = \frac{y+2}{4} = \frac{z}{6}$ and $\frac{x+3}{2} = \frac{y-5}{7} = \frac{z+1}{-1}$
The first line has direction vector *5i + 4j + 6k* and the second line has *2i + 7j – k*. Taking the dot products

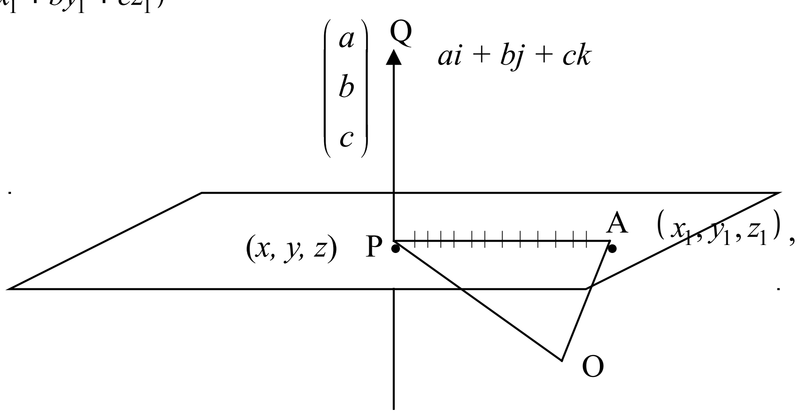
$$|5i+4j+6k|.|2i+7j-k|\cos\theta = \begin{pmatrix}5\\4\\6\end{pmatrix}.\begin{pmatrix}2\\7\\-1\end{pmatrix}$$

$$\sqrt{77}\quad\sqrt{54}\quad\cos\theta = 10+28-6$$
$$\sqrt{77}\quad\sqrt{54}\quad\cos\theta = 32$$
$$\cos\theta = \frac{32}{\sqrt{77}\quad\sqrt{54}} = 0.4963$$
$$\theta = 60.2$$

3.6 Equations of planes in space

If a plane is perpendicular to the vector $ai+bj+ck$ and is it passes through a point (x_1, y_1, z_1), then the equation for the plane is $ax+by+cz+d=0$ where $d = -(ax_1+by_1+cz_1)$

Proof



Let the perpendicular line to the plane have the direction vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and the point A (x_1, y_1, z_1) be on the plane. Consider any point $P(x, y, z)$ on the plane and draw the perpendicular line through the point P as in the figure.

The direction vector AP

If o is the origin, $\overline{AP} = \overline{AO} + \overline{OP}$ and the direction vector of $\overline{AP}$ is given by

$$\overline{AP} = \overline{AO} + \overline{OP} = \begin{pmatrix} -x_1 \\ -y_1 \\ -z_1 \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix}$$

Since $\overline{AP}$ is on the plane and $\overline{PQ}$ is perpendicular to the plane the dot product

$\overline{PQ} \cdot \overline{AP} = 0$ or $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix} = 0$

or $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

or $ax + by + cz = ax_1 + by_1 + cz_1$

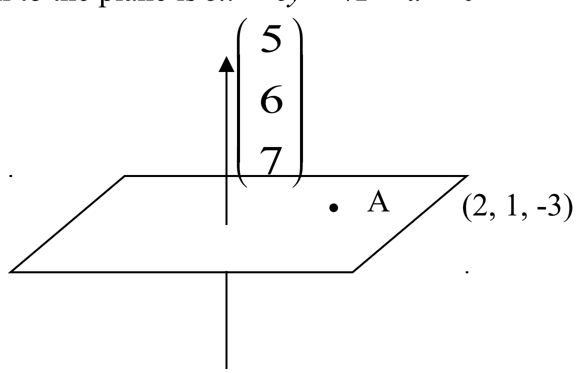
or $ax + by + cz + d = 0$ where $d = -(ax_1 + by_1 + cz_1)$.

Example 5

Find the equation of a plane passing through the point (2, 1, -3) given that it is perpendicular to the vector 5*i* + 6*j* + 7*k*.

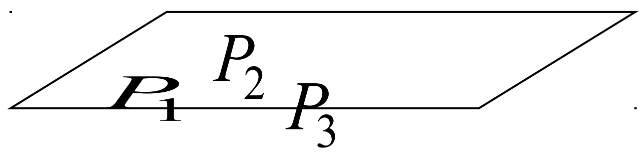
Solution

The equation to the plane is $5x + 6y + 7z + d = 0$ (1)



Since (2, 1, -3) is a point on the plane it should satisfy the equation (1). Hence 5 (2) + 6 (1) + 7(-3) + d = 0. Then d = 5. Thus 5x + 6y + 7z + 5 = 0 is the equation required.

3.7 Equation of a plane passing through three non-collinear points, $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$



Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$ be the three points on a plane. Let O be the origin and let P be any point on the plane.

Let $\overline{OP_1}$, $\overline{OP_2}$, $\overline{OP_3}$ and $\overline{OP}$ be the vectors $\overline{r_1}$, $\overline{r_2}$, $\overline{r_3}$ and $\overline{r}$ respectively.

Then $\overline{P_1P_2} = \overline{p_1o} + \overline{op_2} = \overline{r_2} - \overline{r_1}$ (1)

Similarly, $\overline{P_1P_3} = \overline{r_3} - \overline{r_1}$ (2)

And $P_1P = r - r_1$ (3)

There are three vectors $(\overline{r} - \overline{r_1})$, $(\overline{r_2} - \overline{r_1})$ and $(\overline{r_3} - \overline{r_1})$ which lie on the plane.

We have seen in lesson 2 that if $\overline{A}$, $\overline{B}$, $\overline{C}$ are three vectors in space the volume of the parallelopiped formed by the vectors $\overline{A}$, $\overline{B}$, and $\overline{C}$ is given by $V = \overline{A} \cdot (\overline{B} \times \overline{C})$.

Here the three vectors given in (1), (2) and (3) are all on the same plane and hence of the parallelopiped formed is zero. Thus

$$(\overline{r} - \overline{r_1}) \cdot \left[(\overline{r_2} - \overline{r_1}) \times (\overline{r_3} - \overline{r_1})\right] = 0 \qquad (4)$$

(4) is the vector equation of the plane passing through the points $P_1, P_2,$ and P_3. (4) is written in rectangular coordinates

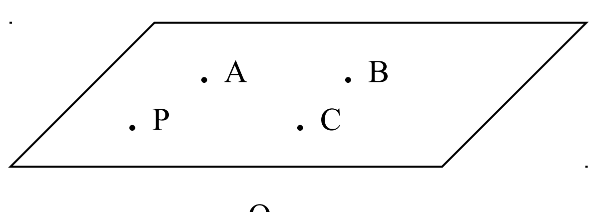
$$\left[(x - x_1)i + (y - y_1)j + (z - z_1)k\right] \cdot \left[(x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k\right] \times \left[(x_3 - x_1)i + (y_3 - y_1)j + (z_3 - z_1)k\right]$$

Which on simplification gives

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Example 6

Find the equation for the plane determined by the points A(2, -1, 1), B(3, 2, -1) and C(-1, 3, 2)



Solution
Let O be the origin and A, B, C and a variable point P be points on the plane.
The position vectors of A, B, C and P are

$\overline{OP} = \overline{r} = xi + yj + zk$

$\overline{OA} = \overline{r_1} = 2i - j + k$

$\overline{OB} = \overline{r_2} = 3i + 2j - k$

$\overline{OC} = \overline{r_3} = -i + 3j + 2k$

Hence $\overline{PA} = (\overline{r} - \overline{r_1})$, $\overline{BA} = (\overline{r_2} - \overline{r_1})$, $\overline{CA}(\overline{r_3} - \overline{r_1})$. Since they lie on a plane the volume V of the parallelepiped is zero. Hence

$(\overline{r} - \overline{r_1}) . \left[(\overline{r_2} - \overline{r_1}) \times (\overline{r_3} - \overline{r_1})\right] = 0$

or $\left[(x-2)i + (y+1)j + (z-1)k\right] . \left[(i + 3j - 2k) \times (-3i + 4j + k)\right] = 0$

or $\left[(x-2)i + (y+1)j + (z-1)k\right] . \left[11i + 5j + 13k\right] = 0$

or $11(x-2) + 5(y+1)j + 13(z-1) = 0$

or $11x + 5y + 13z - 30 = 0$ (5)

(5) is the equation for the plane passing through the three points.

Method II (using the formula)
By formula the equation to the plane is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

or $\begin{vmatrix} x-2 & y+1 & z-1 \\ 1 & 3 & -2 \\ -3 & 4 & 1 \end{vmatrix} = 0$. Expanding the determinant we have $11x + 5y + 13z - 30 = 0$

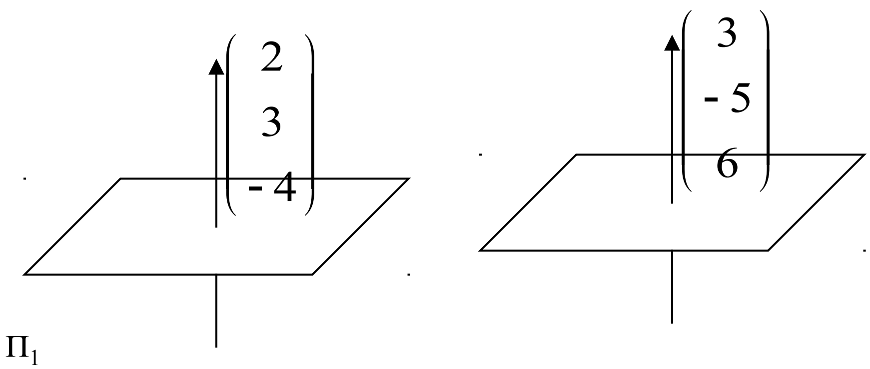
Example 7

The equation to two planes Π_1 and Π_2 are given by,

$2x + 3y - 4z + 1 = 0$ and $3x - 5y + 6z + 8 = 0$. Calculate the angle between the two planes.

Solution

The angle between two planes is same as the angle between their normals.



We must find the angle between the normal lines $2i+3j-4k$ and $3i-5j+6k$ using the dot products

$$|2i+3j-4k||3i-5j+6k|\cos\theta = \begin{pmatrix}2\\3\\-4\end{pmatrix}.\begin{pmatrix}3\\-5\\6\end{pmatrix}$$

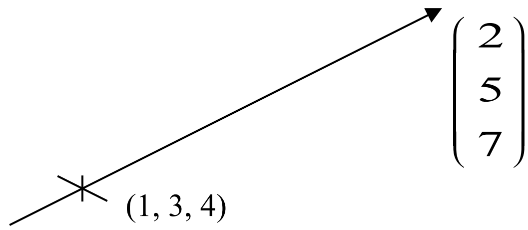
$$\sqrt{29}\quad\sqrt{70}\cos\theta = 6-15+24$$

$$\cos\theta = \frac{15}{\sqrt{29}\quad\sqrt{70}}.\text{ Hence } \theta = \cos^{-1}\left(\frac{15}{\sqrt{2030}}\right)$$

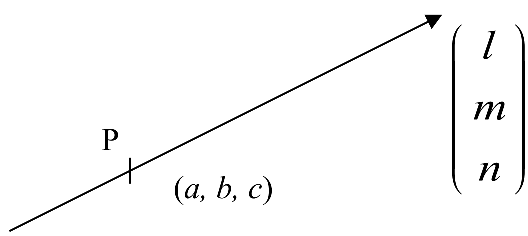
Exercise 3

1. State the equation of a straight line passing through the two points whose position vectors are $\overline{OA} = \overline{a}$ and $\overline{OB} = \overline{b}$ respectively
    i) In vector form.
    ii) If the coordinates of A and B are (x_1, y_1, z_1) and (x_2, y_2, z_2) state the equation of the line AB in rectangular Cartesian form.
    iii) Also in parametric form
2. A straight line passes through the points P(1, 0, 1) and Q(-1, -1, 0). Determine the equation of the line in
    i) Rectangular Cartesian form and

ii) Parametric form



3. Find the equation the line in Cartesian and parametric form if the direction vector is $2i+5j+7k$ given that it passes through the point (1, 3, 4).
4. Find an equation of a plane perpendicular to a given vector $\overline{a}$ and distance P from the origin.
5. Given that $\overline{a}=3i+j+2k$ and $\overline{b}=i-2j-4k$ are position vectors of points P and Q respectively, find an equation of the plane passing through Q and Perpendicular to line PQ in space.

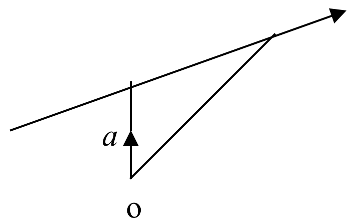


6. A straight line passes through the point P (*a, b, c*) and has direction vector $\begin{pmatrix}l\\m\\n\end{pmatrix}$ or $li+mj+nk$. State the equation of the line in
   i) Cartesian form
   ii) Parametric form
   iii) Vector form

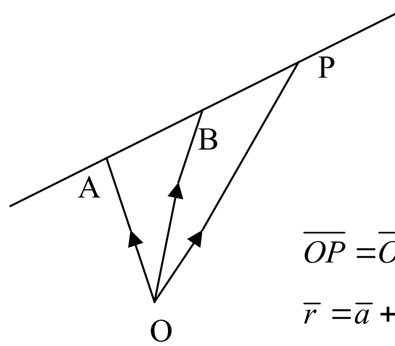
[Hint: Cartesian form: $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$; parametric form: Taking each of the above = λ and solving for *x, y, z* we have, $x=a+l\lambda,\quad y=b+m\lambda,\quad z=c+n\lambda$

vector form: $\bar{r}=\bar{a}+t\bar{b}$

$\bar{r}$ = (any one point) + t(difference of two points)



7. A straight line passes through the point (2, 3, 1). Its direction vector is $4i+5j-6k$. State the equation of the line in
   i) Cartesian form
   ii) Parametric form
   iii) Vector form
8. Find the equation of the straight line which passes through the points A and B where $\overline{OA}=a$ and $\overline{OB}=\bar{b}$



$\overline{OP}=\overline{OA}+\overline{AP}$

$\bar{r}=\bar{a}+\lambda(AB)$

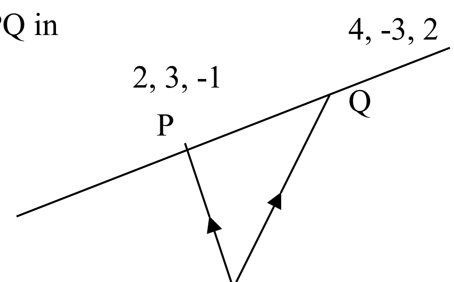
$\bar{r}=\bar{a}+\lambda(b-a)$

It may be written in other ways as: $\bar{r}=\bar{a}+\lambda(a-b)$

or $\bar{r}=\bar{b}+\lambda(a-b)$

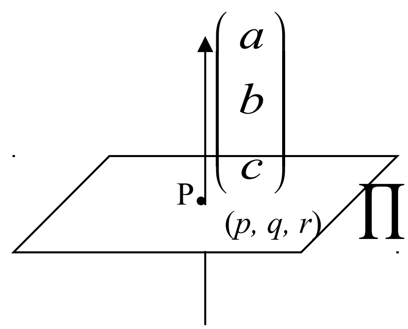
or $\bar{r}=\bar{b}+\lambda(\bar{b}-\bar{a})$

9. The position vectors of P and Q are given by $\bar{r}_1=2i+3j-k$ and $\bar{r}_2=4i-3j+2k$. Determine the equation of line PQ in
   i) Cartesian form
   ii) Parametric form
   iii) Vector form



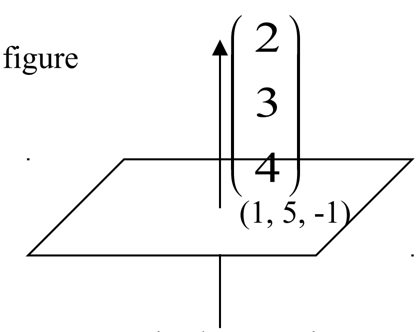
The direction vector is $\overline{PQ} = (4-2)i + (-3-3)j + (2+1)k$ or $\begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$

10. The plane II passes through a point (*p, q, r*). A normal to the plane has the direction vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ or $ai + bj + ck$.



Find the equation of the plane Π.
(Equation is $ax + by + cz + d = 0$ where $d = -(ap + bq + cr)$.

11. Find the equation of the plane in the given figure

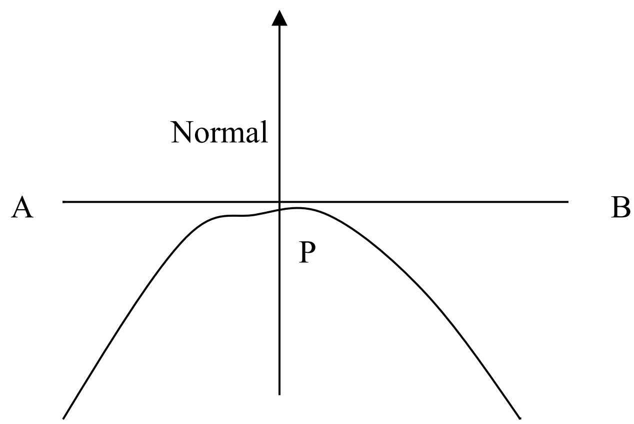


($2x + 3y + 4z + d = 0$. Substitute $x = 1$, $y = 5$, $z = -1$ in the equation to get $d = -13$. Hence $2x + 3y + 4z - 13 = 0$)

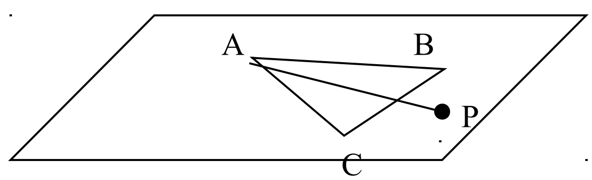
12. a). Find the equation for a perpendicular to the vector $\overline{a} = 2i + 3j + 6k$ and passing through the point where position vector is $\overline{b} = i + 5j + 3k$.

b). What is the perpendicular distance of the plane from the origin.

13. Find the equation of the tangent plane AB to a surface AB to a surface if the normal to the surface has the vector $2i + 3j + 4k$ and P is the point (4, 5, 6).



14. A straight line l has the equation $\frac{x-2}{3} = \frac{y-1}{4} = \frac{z+1}{5}$. Find the equation of a plane passing through (1, -1, 0) and perpendicular to the line l.



15. The coordinates of three points on a plane are A(2, -1, 1), B(3, 2, -1) and C(-1, 3, 2) as in the figure.
    a). Calculate: $\overline{AB}$ and AC
    b). Using the fact that the volume of the parallelepiped formed by $\overline{AB}$, $\overline{AC}$ and $\overline{AP}$ is zero, find the equation of the plane.
16. Find the equation for the plane passing through the points A(2, -1, 1), B(3, 2, -1) and C(-1, 3, 2).
17. The position vectors of the points $\overline{A}$ and $\overline{B}$ are $3i + j + 2k$ and $i - 2j - 4k$. Find an equation for the plane passing through B and Perpendicular to AB.

Summary

You have learnt the following from this lesson:

a). The equations of straight lines

i) $\bar{r} = \bar{a} + t(\bar{b} - \bar{a})$ if $\bar{a}$ and $\bar{b}$ are position vectors of two points on the line.

ii) $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is Cartesian equation of the line when (x_1, y_1, z_1) is one point on the line and $li + mj + nk$ is the direction vector of the line.

iii) $x = x_1 + lt$, $y = y_1 + mt$, $z = z_1 + nt$ are the parametric equation of a line where (x_1, y_1, z_1) is one point on the line and $li + mj + nk$ is the direction vector of the line and t is a parameter.

b). i). If the normal to a plane has the vector $ai + bj + ck$ and if the plane passes through a point (x_1, y_1, z_1) then the equation to the plane is $ax + by + cz + d = 0$ where $d = -(ax_1 + by_1 + cz_1)$.

ii). If (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are the three points on a plane then the equation of the plane is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

iii). If r_1, r_2 and r_3 are position vectors of three non collinear points on a plane, then the equation of the plane in vector form is given by $(\bar{r} . \bar{r}_1) . [(\bar{r}_2 . \bar{r}_1) \times (\bar{r}_3 . \bar{r}_1)] = 0$

Further reading

3.) Differential geometry of three dimensions

By Weatherburn,

Cambridge 1957

4.) Differential geometry

By Dr. Sengottaiyan

Oxford publications London. Nairobi

Lesson 4

Vector Functions of Real Variables and Scalar Point Functions

4.1 Introduction

In this lesson we shall introduce the concepts of vector functions of a single and several real variables and those of their limits and derivability. It will be seen that the treatment of fundamentals in this lesson is a foundation for the curves and surfaces of Differential Geometry.

4.2 Objectives of the lesson

By the end of this lesson you will be able to:

i. define the vector functions of a single and two real variables.
ii. define the derivatives of vector functions.
iii. determine the tangents and unit tangents to a curve represented by a vector function of one variable *t*.
iv. define the vector equation of a surface.
v. define the partial derivatives of a vector function of two variables and their meaning.
vi. define the meaning of $\dfrac{\partial \bar{r}}{\partial u}$, $\dfrac{\partial \bar{r}}{\partial v}$ and $\dfrac{\partial \bar{r}}{\partial u} \times \dfrac{\partial \bar{r}}{\partial v}$.
vii. define a scalar point function ϕ.
viii. define the differential operators $\nabla, \nabla\phi, \nabla.\bar{F}$ and $\nabla \times \bar{F}$
ix. determine the unit tangent, the equation for tangent plane and normal of a given point on the surface.

4.3 Vector functions of real variables

Consider a function, $\bar{F}(t) = if_1(t) + jf_2(t) + kf_3(t)$ where *t* is a real variable. Then $\bar{F}(t)$ is called a vector function of the real variable *t*.

Examples of vectors function of the real variable *t*

i. The velocity at any point in a moving fluid

ii. The gravitational force on a particle in space.

iii. The path of a moving particle:

$\bar{r} = if_1(t) + jf_2(t) + kf_3(t)$

4.4 Vector functions of two real variables u and v.

Consider a function,

$\bar{F}(u,v) = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$

where u and v are continuous real variables. Then $\bar{F}(u,v)$ is called a vector function of two real variables u and v.

Example 1

$\bar{r} = \bar{r}(u,v) = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$

is the vector equation of the surface in space or in three dimension. Here the real variables u and v are called the curvilinear coordinates. This form of a surface is called the Gaussian surface. For example,

$\bar{r} = ia\sin u\cos v + j\sin u\sin v + ka\cos u$

is a vector function of real variables u and v. This is the vector equation of a sphere of radius 'a'.

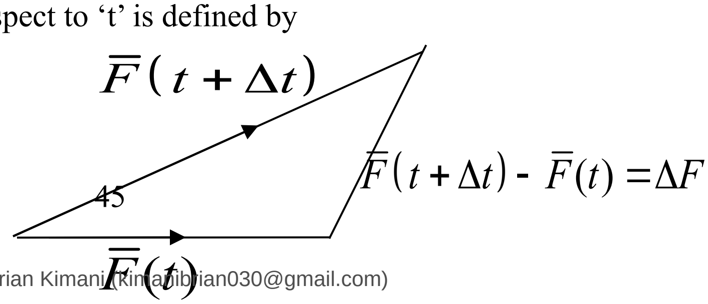
4.5 Derivatives of a vector function of a real variable t.

Let

$$\bar{F}(t) = if_1(t) + jf_2(t) + kf_3(t)$$

be a vector function whose values depend on a single scalar variable t'

Then the derivative of $\bar{F}(t)$ with respect to 't' is defined by



$$\frac{d\overline{F}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \overline{F}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\overline{F}(t + \Delta t) - \overline{F}(t)}{\Delta t}$$

if the limit exists.

Similarly $\frac{d^2\overline{F}}{dt^2}$ denotes the derivative of $\frac{d\overline{F}}{dt}$, provided it exists.

Example 2

a. Find the velocity and acceleration of a particle which moves along the curve $\overline{r} = 2i\cos 3t + 2j\cos 3t + 8kt$

b. Find also the magnitudes of the velocity and acceleration.

Solution

The position vector of the particle on the curve is given by,

$$\overline{r}(t) = 2\sin 3ti + 2\cos 3tj + 8tk$$

Velocity $\overline{v} = \frac{d\overline{v}}{dt} = 6\cos 3ti - 6\sin 3tj + 8k$

$$\overline{a} = \frac{d\overline{v}}{dt} = -18\sin 3ti - 18\cos 3tj$$

Magnitude of $\overline{v} = |\overline{v}|$

$$= \sqrt{36\cos^2 3t + 36\sin^2 3t + 64}$$

$$= \sqrt{36 + 64}$$

$= 10$ units

Magnitude of acceleration $= |\overline{a}|$

$$=\sqrt{18^2\sin^2 3t+18^2\cos^2 3t}$$
$$=18 \text{ units}$$

4.6 Differentiation of vector functions

If $\overline{a}$ and $\overline{b}$ are two vector functions of a scalar 't' then

i. $$\frac{d}{dt}\left(\overline{a}\pm\overline{b}\right)=\frac{d\overline{a}}{dt}\pm\frac{d\overline{b}}{dt}$$

ii. $$\frac{d}{dt}\left(\overline{a}.\overline{b}\right)=\overline{a}.\frac{d\overline{b}}{dt}+\frac{d\overline{a}}{dt}.\overline{b}$$

iii. $$\frac{d}{dt}\left(\overline{a}\times\overline{b}\right)=\overline{a}\times\frac{d\overline{b}}{dt}+\frac{d\overline{a}}{dt}\times\overline{b}$$

iv. If a is a continuous function of a scalar variable 's' and s is a continuous function of another scalar *'t'* then,

$$\frac{d\overline{a}}{ds}=\frac{d\overline{a}/dt}{ds/dt}$$

4.7 The Tangent and the unit tangent to a curve

$$\overline{r}=if_1(t)+jf_2(t)+kf_3(t)$$

If $\overline{r}=if_1(t)+jf_2(t)+kf_3(t)$ is a space curve then the rate of change of $\overline{r}$ with respect to *'t'* is defined as the general tangent to the space curve. Tangent $=d\overline{r}/dt$

The unit tangent $\overline{T}$ is given by $\overline{T}=\dfrac{d\overline{r}/dt}{\left|d\overline{r}/dt\right|}$

Example 3

a. Find the unit tangent vector to the curve at the point p on the curve where

$$x=1+t^2,\quad y=4t-3,\quad z=2t^2-6t$$

b. Determine the unit tangent at the point where $t = 2$.

Solution

a. Let the curve be defined by the vector, $\overline{r} = xi + yj + zk$.

Then $\overline{r} = (1+t^2)\,i + (4t-3)\,j + (2t^2-6t)\,k$. The tangent $= \frac{d\overline{r}}{dt} = 2ti - 4j + (4t-6)k$

Then the unit tangent $\overline{T} = \frac{d\overline{r}/dt}{|d\overline{r}/dt|}$

$$= \frac{2ti - 4j + (4t-6)k}{\sqrt{4t^2 + 16 + (4t-6)^2}}$$

$$= \frac{2ti - 4j + (4t-6)k}{2\sqrt{5t^2 - 12t + 13}}$$

$$= \frac{ti - 2j + (2t-3)k}{\sqrt{5t^2 - 12t + 13}}$$

b. If $t = 2$ then the unit tangent $\overline{T} = \frac{2i - j + k}{3}$

4.8 Vector Equation of a surface

If $\overline{r} = \overline{f}(u,v) = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$ be thought of as a moving point p, then the locus of *p*, for different pairs of values of *(u,v)* is a surface and we say that $\overline{r} = \overline{f}(u,v)$ is the parametric vectorial equation or simply the vector function of a surface.

4.9 Partial derivatives of $\overline{f}(u,v)$.

If $\overline{f}(u,v)$ is any vector function of two real variables *u* and *v* then

$$\lim_{\Delta u \to 0} \frac{f(u+\Delta u, v) - f(u,v)}{\Delta u},$$

if it exists is called the partial derivative of *u*. Thus

$$\frac{d\overline{r}}{du} = \lim_{\Delta u \to 0} \frac{f(u+\Delta u, v) - f(u,v)}{\Delta u}$$

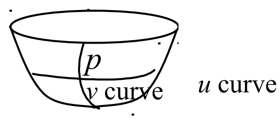
Similarly

$$\frac{d\overline{r}}{dv} = \lim_{\Delta v \to 0} \frac{f(u, u+\Delta v) - f(u,v)}{\Delta v}$$

is the partial derivative of $\overline{f}(u,v)$ with respect to *v*

4.10 Meaning of $\frac{d\overline{r}}{du}, \frac{d\overline{r}}{dv}$ and $\frac{d\overline{r}}{du} \times \frac{d\overline{r}}{dv}$

i. If $\overline{r} = \overline{f}(u,v) = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$ is any surface then $\frac{\partial \overline{r}}{\partial u}$ represents the tangent to the *u* curve at *p* on the surface. Hence it is a tangent to the surface also.



ii. In the same way $\frac{\partial \overline{r}}{\partial v}$ represents the tangent to the *v* curve and have the tangent to the surface at *p*.

iii. $\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v}$ represents a line perpendicular to both the tangents to *u* and *v* curves and hence it is a normal to the surface.

iv. Unit normal to the surface.

Since $\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v}$ is a normal to the surface

$\dfrac{\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v}}{\left|\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v}\right|}$ is the unit normal to the surface.

$\overline{r} = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$

4.11 Scalar point function ϕ

If a scalar $\phi(x,y,z)$ is defined corresponding to each point *(x,y,z)* of a region *R*, then ϕ is called a scalar point function.

For example the temperature of a heated body is a scalar point function since at different points (*x,y,z*) the temperature is defined. For example,

$$\phi(x,y,z) = x^2 + y^2 + z^2 - 25$$

is a scalar point function.

Example 4

Find the value of the scalar fields ϕ defined by

$$\phi(x, y, z) = 2yz^2 + 3xyz - z^2 + 15$$

at point (1, -1, -2)

Solution

Substituting $x = 1$, $y = -1$, $z = -2$ we have

$$\phi(x, y, z) = -8 + 6 - 4 + 15 = 9$$

Differential operators $\nabla\phi, \nabla.\bar{F}, \nabla\times\bar{F}$

4.12 Vector differential operator ∇

The notation ∇ is defined as,

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

∇can act on a scalar point function or on a vector point function which are differentiable.

4.13 If ϕ is differentiable scalar point function then expression for $\nabla\phi$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)\phi = i\frac{\partial}{\partial x}\phi + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$\nabla\phi$ is called the gradient of ϕ. It is abbreviated as grad ϕ

4.14 If $\bar{F}$ is a vector point function continuously differentiable in a given region R then

$\nabla.\bar{F}$ is defined by

$$\nabla.\bar{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right).(if_1 + jf_2 + kf_3)$$

$$=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$$

The dot product is a scalar and hence $\nabla.\overline{F}$ is also a scalar.

$\nabla.\overline{F}$ is called the divergence of $\overline{F}$.

4.15 Curl of a vector point function

If a vector point function is differentiable then the curl of $\overline{F}$ is defined by,

$$Curl\,\overline{F}=\nabla\times\overline{F}=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left(if_1+jf_2+kf_3\right)$$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$=i\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)+j\left(\frac{\partial f_1}{\partial z}-\frac{\partial f}{\partial x}\right)+k\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

Example 5

a. Find

   i. A unit normal to the surface

      $\phi(x,y,z)=xy-z^2=0$ at the point (1, 4, -2)

   ii. The equation to the normal line

   iii. The equation to the tangent plane at (1, 4, -2)

Solution

i. unit normal to a surface $\phi(x,y,z)=0$ is given by $\overline{n}=\frac{\nabla\phi}{|\nabla\phi|}$. Since $\nabla\varphi$ is a normal to the surface $\varphi=0$

Hence $\nabla\phi=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(xy-z^2)$

$$= i\frac{\partial}{\partial x}(xy - z^2) + j\frac{\partial}{\partial y}(xy - z^2) + k\frac{\partial}{\partial z}(xy - z^2)$$

$$= yi + xj + 2zk$$

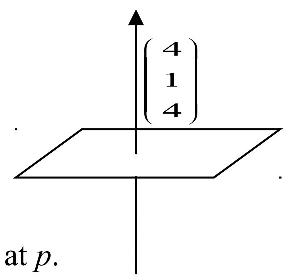
Unit normal $= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{yi + xj + 2zk}{\sqrt{y^2 + x^2 + 4z^2}}$.

Unit normal at the point (1, 4, -2) is given by

$$\overline{n} = \left.\frac{yi + xj + 2zk}{\sqrt{y^2 + x^2 + 4z^2}}\right|_{(1,4,-2)}$$

$$= \frac{4i + j - 4k}{\sqrt{33}}$$

ii. The equation to the normal line is $\dfrac{x-1}{4} = \dfrac{y-4}{1} = \dfrac{z-2}{4}$



iii. The tangent plane at (1, 4, -2) is perpendicular to the normal at *p*.

Hence the Cartesian equation to the tangent plane is

$4x + y - 4z + d = 0$ since it passes through point (1, 4, -2) we have

$4(1) + 4 + 4(-2) + d = 0$ or $d = 0$

Then the equation to the tangent plane is given by

$4x + y + 4z = 0$

In vector form the equation to the tangent plane is given by

$(\overline{r} - \overline{r}_0)\,.\nabla\phi = 0$ where $\overline{r}$ is the variable point on the plane $\overline{r}_0$ is the point (1, 4, -2) on the surface $\nabla\phi$ is the normal to the surface

To deduce the Cartesian Equation from the vector equation

or $(x-1)i + (y-4)j + (z+2)\,.(4i + j + 4k) = 0$

or $4(x-1) + (y-4) + 4(z+2) = 0$

or $4x + y + 4z - 4 - 4 + 8 = 0$

or $2x + y + z = 0$

Exercise 4

1. Give an example of a vector function of a single variable '*t*'
2. State an example of a vector function of two real variables u and v.
3. State the equation of a space curve in the form of a vector function of one real variable 't'
4. Write down a vector function of a surface in the form of two variables u and v.
5. Give an example of a scalar point function of three real variables *x, y* and *z*.
6. Find $\left[\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial u}\right] + \left[\frac{\partial \bar{r}}{\partial v} \times \frac{\partial \bar{r}}{\partial v}\right]$

   when $\bar{r} = ui + vj + \left(u^2 + v^2\right) k$
7. Find a unit tangent vector for the space curve whose vector equation is given by $\bar{r} = 12 \cos t\, i + 12 \sin t\, j + 5t\, k$
8. Find a unit normal $\bar{n}$ to the surface $\bar{r} = ui + vj + \left(u^2 + v^2\right) k$
9. If $\phi(x, y, z) = 0$ is the Cartesian equation of a surface. Interpret the meaning of the following:

   a). $\nabla\phi$ b). $\frac{\nabla\phi}{|\nabla\phi|}$

   c). $(\bar{r} - \bar{r}_0) . \nabla\phi$ when $\bar{r}$ is a variable point and $\bar{r}_0$ is a point on the surface.

Summary of the lesson

You have learnt the following from this lesson:

i) Definition of vector functions of a single and two real variables e.g.

$$\bar{r} = if_1(t) + jf_2(t) + kf_3(t)$$

$$\bar{r} = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$$

ii) Definition of the derivatives of a vector function $\bar{F}(t)$

$$\frac{d\bar{F}}{dt} = \lim_{\Delta t \to 0} \frac{\bar{F}(t + \Delta t) - \bar{F}(t)}{\Delta t}$$

iii) Determination of the tangent and unit tangent to a space curve $\bar{r} = \bar{F}(t)$.

$\frac{d\bar{r}}{dt}$ is the tangent and $\frac{d\bar{r}}{dt} \Big/ \left|\frac{d\bar{r}}{dt}\right|$ is the unit tangent to the space curve.

iv) Vector equation of a surface is given by;

$\bar{r} = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$

and $\frac{\partial \bar{r}}{\partial u}$ is the tangent to the u curve

and $\frac{\partial \bar{r}}{\partial v}$ is the tangent to the v curve at a point P.

v) The unit normal to a surface $\bar{r} = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$ is given by:

$$\frac{\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}}{\left|\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right|}$$

vi) The definition of a scalar point function: $\phi(x,y,z)$ is a scalar point function.

$\phi(x,y,z) = 0$ is the equation of a surface in Cartesian coordinates.

vii) The differential operators

a). $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

b). $\nabla\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$ and $\nabla\phi$ is a vector and it is called grade ϕ

c). If $\bar{F} = if_1 + jf_2 + kf_3$ then $\nabla.\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$\nabla.\bar{F}$ is a scalar and $\nabla.\bar{F}$ is called div $\bar{F}$

d). $\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$. It is called curl $\bar{F}$

viii) If $\phi(x,y,z) = 0$ is any surface then

a). A unit normal to the surface is given by

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

b). The equation to the tangent plane to the surface at the point $\bar{r}_0$ is given by

$(\bar{r} - \bar{r}_0) . \nabla\phi$

c). The equation to the normal line at (x_0, y_0, z_0) on the surface is given by,

$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$ where $li + mj + nk$ is the direction vector of the normal line.

Further reading

5.) Differential geometry of three dimensions

By Weatherburn,

Cambridge 1957

6.) Differential geometry

By Dr. Sengottaiyan

Oxford publications London. Nairobi

Lesson 5

Concept and Theory of Curves

5.1 Introduction

Differential Geometry involves a study of space curves and surfaces. A space curve is visualized very well from the following example:
Suppose a house – fly is sitting upon the book on your desk and you chase it. The fly moves here and there in space and at last it sits on the nose of a sleeping student at the back-bench. The path of the house-fly in space is a space curve or a three dimensional curve. In this lesson we shall study the concept and theory of space curves.

5.2 Objective of this lesson

By the end of this lesson you should be able to

- define a space curve in terms of a vector equation
- establish the fundamental formula for the expression $\frac{d\bar{r}}{ds}$
- define the unit tangent, curvature, radius of curvature, unit normal, Torsion and radius of torsion for a space curve.
- define and find the equations of osculating plane, normal plane and the rectifying plane at a point on the space curve.

5.3 Vector equation to a space curve

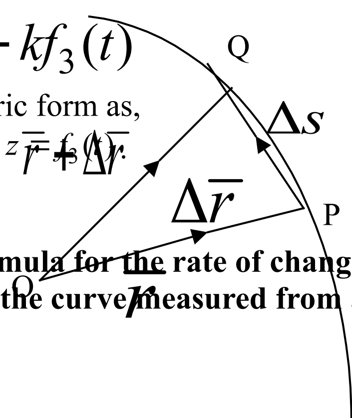
The equation to a space curve is generally given in terms of a vector point function with one parameter t as,

$$\bar{r} = if_1(t) + jf_2(t) + kf_3(t)$$

It is also given in the parametric form as,

$x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$.

5.4 The fundamental formula for the rate of change of the vector $\bar{r}$ with respect to the arc-length *s* of the curve measured from a fixed point.



Consider a point P on a space curve C having position vector $\overline{OP} = \bar{r}$. Let Q be a neighbouring pint on the curve with position vector $\overline{OQ} = \bar{r} + \Delta\bar{r}$. Then

$\overline{OP} + \overline{PQ} = \bar{r} + \Delta\bar{r}$ so that $\overline{PQ} = \Delta\bar{r}$ when Q approaches P, $\Delta\bar{r} \rightarrow \Delta s$

Thus $\lim_{\Delta t \rightarrow \Delta s} \frac{\Delta\bar{r}}{\Delta s} = \frac{d\bar{r}}{ds} = \frac{d\bar{r}/dt}{ds/dt}$

$\frac{ds}{dt}$ is a scalar where as $\frac{d\bar{r}}{dt}$

Since $\Delta\bar{r} \rightarrow \Delta s$ in magnitude in the limit $\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right|$

Then $$\boxed{\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right|}$$

This is fundamental identity in space curve.

Some important definitions and formulae

5.5 Unit tangent $\bar{T}$ and the formula for $\bar{T}$

The unit tangent $\bar{T}$ to a space curve is defined as the rate of change of $\bar{r}$ with respect to its arc – length s measured from a fixed point A on the curve.

Thus $$\boxed{\bar{T} = \frac{d\bar{r}}{dt}\left|\frac{d\bar{r}}{dt}\right|}$$

Formula for unit Tangent

$$\bar{T} = \frac{d\bar{r}}{ds} = \frac{d\bar{r}/dt}{ds/dt} = \frac{d\bar{r}/dt}{|d\bar{r}/dt|}$$

Definition and formulae for curvature K, radius of curvature ρ normal $\bar{N}$ and unit normal $\bar{n}$

5.6 Curvature k

The rate at which the unit tangent $\bar{T}$ changes with respect to its arc – length s is the measure of the curvature of the curve at a point on *C*.

Thus $\boxed{\bar{K} = \dfrac{|d\bar{T}/dt|}{|d\bar{r}/dt|}}$

Formula for the curvature K of a space curve. In Lesson one we proved that $\bar{a}.\dfrac{d\bar{a}}{dt}=0$ and hence $\bar{a}$ and $\dfrac{d\bar{a}}{dt}$ are perpendicular.

In the same way,

$T.\dfrac{d\bar{T}}{ds}=0$; hence $\bar{T}$ and $\dfrac{d\bar{T}}{ds}$ are perpendicular.

Then $\dfrac{d\bar{T}}{ds}$ is a vector in the direction of the normal to the curve.

$\dfrac{d\bar{T}}{ds}=\bar{N}$

Where $\bar{N}$ is a normal and $\bar{n}$ is the unit normal to the curve C.

Thus if $\dfrac{d\bar{T}}{ds}=\bar{N}=K\bar{n}$ where K is a constant then K is called the curvature of the curve at P.

Then $\left|\dfrac{d\bar{T}}{ds}\right| = |K\bar{n}| = K$

Thus $K = \left|\dfrac{d\bar{T}}{ds}\right|$

5.7 The reciprocal of curvature is called the radius of curvature ρ of the curve C at the point P.

$$\rho = \frac{1}{K}$$

Formulae for the curvature K, radius of curvature ρ and the unit normal $\bar{n}$

$$K = \left|\frac{d\bar{T}/dt}{d\bar{r}/dt}\right| \text{ since } \frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right|$$

$$\rho = \left|\frac{d\bar{r}/dt}{d\bar{T}/dt}\right|$$

Unit normal $\bar{n}$

$$\frac{d\bar{T}}{ds} = K\bar{n}$$

Then $\bar{n} = \frac{1}{K}\frac{d\bar{T}}{ds}$

$$\bar{n} = \frac{1}{K}\frac{d\bar{T}/dt}{|d\bar{r}/dt|} \text{ since } \frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right|$$

5.8 The unit binormal $\bar{b}$

The unit normal $\bar{b}$ is defined as $\bar{b} = \bar{T} \times \bar{n}$ where $\bar{T}$ is the unit tangent and $\bar{n}$ is the unit normal to a sace curve at a point P on it.

Thus $$\boxed{\bar{b} = \bar{T} \times \bar{n}}$$

Hence $\bar{b}$ is perpendicular to both $\bar{T}$ and $\bar{n}$. Thus $\bar{T}, \bar{n}, \bar{b}$ form a localized right handed rectangular coordinate system at any specified point on C

5.9 The Torsion τ and the radius of Torsion σ

We have seen that $\bar{A} \cdot \frac{d\bar{A}}{dt} = 0$

Thus $\bar{b} \cdot \frac{d\bar{b}}{ds} = 0$

Hence $\bar{b}$ is perpendicular to $\frac{d\bar{b}}{ds}$. Then $\frac{d\bar{b}}{ds}$ is a normal to the curve.

Let $\frac{d\bar{b}}{ds}$ = constant times $\bar{N}$

or $\frac{d\bar{b}}{ds} = -\tau\bar{n}$

where τ is a constant known as **the Torsion of the curve at P**

Formula for Torsion τ and the radius of Torsion σ

$$-\tau\bar{n} = \frac{d\bar{b}}{ds}$$

$$\tau = \frac{-1}{n}\frac{d\bar{b}}{ds}$$

or $\left|-\tau\bar{n}\right| = \left|\frac{d\bar{b}}{ds}\right|$

$$\tau = \left|\frac{d\bar{b}}{ds}\right| = \frac{\left|d\bar{b}/dt\right|}{\left|ds/dt\right|} = \frac{\left|d\bar{b}/dt\right|}{\left|d\bar{r}/dt\right|}$$

Thus $$\boxed{\tau = \frac{\left|d\bar{b}/dt\right|}{\left|d\bar{r}/dt\right|}}$$

Since $\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right|$

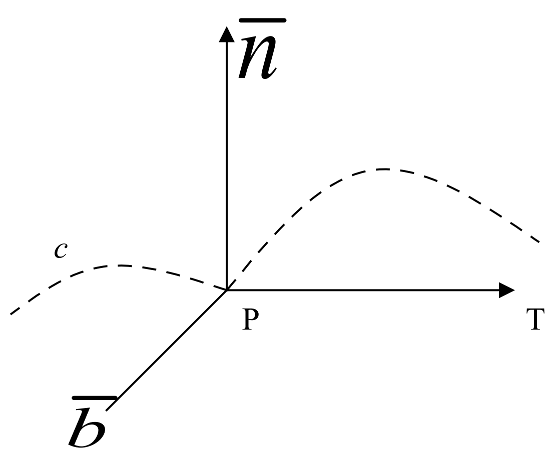
The reciprocal of τ is called the **Radius of Torsion** σ to the curve C

Thus $$\boxed{\sigma = \frac{\left|d\bar{r}/dt\right|}{\left|d\bar{b}/dt\right|}}$$

5.10 **Definition and the equation to**

i) Osculating plane
ii) Normal plane
iii) The rectifying plane

The unit tangent $\bar{T}$, the unit normal $\bar{n}$ and the unit binormal $\bar{b}$ form a localized rectangular coordinate system at any point on C



The osculating plane of a space curve.
The plane perpendicular to $\overline{b}$, (containing $\overline{T}$ and $\overline{n}$) is called the osculating plane to a curve at P.

The equation to the osculating plane
If the position vector of the point P on the curve is $\overline{r}_0$, the variable point on the osculating plane is $\overline{r}$ and $\overline{b}$ is the unit binormal at P on the curve then the equation to the osculating plane is given by,

$\overline{b} \cdot (\overline{r} - \overline{r}_0) = 0$ in vector form

The normal plane to a curve
The normal plane is the plane perpendicular to the unit Tangent $\overline{T}$ (and hence containing $\overline{n}$ and $\overline{b}$)
The equation to the normal plane is given by

$$\overline{T} \cdot (\overline{r} - \overline{r}_0) = 0$$

where $\overline{r}$ is the position vector of the variable point on the normal plane and $\overline{r}_0$ is the position vector of the point P on the curve C.

The rectifying plane to a curve C
The rectifying plane is the plane perpendicular to the normal $\overline{n}$ to a space curve (and hence it contains $\overline{b}$ and $\overline{T}$)
The equation to the **rectifying plane** is given by,

$$\overline{n} \cdot (\overline{r} - \overline{r}_0) = 0$$

where $\overline{n}$ is the unit normal $\overline{r}$ is the position vector of the variable point on the rectifying plane and $\overline{r}_0$ is the position vector of the point P on the curve.

5.11 Frenet – Serret formulae

There are three important formulae for a space curve. The derivative of $\overline{T}$, $\overline{n}$ and $\overline{b}$ with respect to the arc – length *s* of a curve at a point P are given by the following formula known as Frenet – Serret formulae
They are given by:

i) $\dfrac{d\overline{T}}{ds} = K\overline{n}$

ii) $\dfrac{d\overline{b}}{ds} = -\tau\overline{n}$ and

iii) $\dfrac{d\overline{n}}{ds} = \tau\overline{b} - K\overline{T}$

Proof

i) To prove that $\frac{d\overline{T}}{ds} = K\overline{n}$.

Let $\overline{T}$ be the unit tangent to a curve C $\overline{T}\,.\,\overline{T} = |\overline{T}|\,|\overline{T}|\cos 0 = 1$

$$\frac{d(\overline{T}\,.\,\overline{T})}{ds} = \overline{T}\,.\,\frac{d\overline{T}}{ds} + \frac{d\overline{T}}{ds}\,.\,\overline{T} = 0$$

Then $2\overline{T}\,.\,\frac{d\overline{T}}{ds} = 0$

$$\overline{T}\,.\,\frac{d\overline{T}}{ds} = 0$$

Hence $\overline{T}$ and $\frac{d\overline{T}}{ds}$ are perpendicular

or $\frac{d\overline{T}}{ds}$ is a normal.

Hence $\frac{d\overline{T}}{ds} = \overline{N}$

$$= K\overline{n}$$

$$= \overline{T} \times \frac{d\overline{n}}{ds} \text{ since } \overline{n} \times \overline{n} = |\overline{n}|\,|\overline{n}|\sin 0 = 0$$

ii) To prove $\frac{d\overline{n}}{ds} = \tau\overline{b} - K\overline{T}$

We have seen that $\overline{T}$, $\overline{n}$ and $\overline{b}$ form an orthogonal system, we have

$\overline{n} = \overline{b} \times \overline{T}$, $\overline{b} = \overline{T} \times \overline{n}$ and $\overline{T} = \overline{n} \times \overline{b}$

Then $\frac{d\overline{n}}{ds} = \overline{b} \times \frac{d\overline{T}}{ds} + \frac{d\overline{b}}{ds} \times \overline{T}$

$$= \overline{b} \times (K\overline{n} - \tau\overline{n}) \times \overline{T}$$

$$= K\overline{b} \times \overline{n} - \tau\overline{n} \times \overline{T}$$

$$= -K(\overline{n} \times \overline{b}) + \tau(\overline{T} \times \overline{n})$$

$$= -K\overline{T} + \overline{T}\overline{b} \text{ since } \overline{n} \times \overline{b} = \overline{T} \text{ and } \overline{T} \times \overline{n} = \overline{b}$$

$$= \tau\overline{b} - K\overline{T}$$

The Frenet – Serret formulae can be written in matrix form as,

$$\begin{pmatrix} \overline{T} \\ \overline{n} \\ \overline{b} \end{pmatrix} = \begin{pmatrix} 0 & K & 0 \\ -K & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \overline{T} \\ \overline{n} \\ \overline{b} \end{pmatrix}$$

$$= 2\overline{T} \cdot \frac{d\overline{T}}{ds} \text{ since } \overline{a} \cdot \overline{b} = \overline{b} \cdot \overline{a}$$

Then $\frac{d1}{ds} = 2\overline{T} \cdot \frac{d\overline{T}}{ds}$

or $\overline{T} \cdot \frac{d\overline{T}}{ds} = 0$

we have proved that if $\overline{a} \cdot \frac{d\overline{a}}{dt} = 0$ in vector Algebra then $\overline{a}$ and $\frac{d\overline{a}}{dt}$ are perpendicular.

Thus $\overline{T}$ and $\frac{d\overline{T}}{ds}$ are perpendicular

or $\frac{d\overline{T}}{ds}$ = a normal $\overline{N}$

or $\frac{d\overline{T}}{ds} = K\overline{n}$ where $\overline{n}$ is the unit normal to the curve C and K is the constant called the curvature of the curve.

Thus $\frac{d\overline{T}}{ds} = K\overline{n}$ (1)

iii) To prove $\frac{d\overline{b}}{ds} = -\tau\overline{n}$

Proof

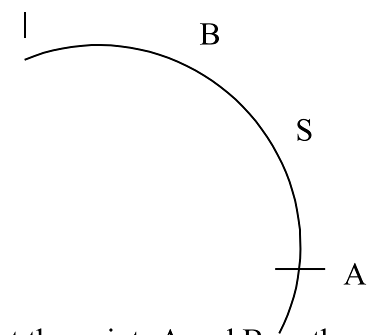
The unit normal $\overline{b} = \overline{T} \times \overline{n}$

Then $\frac{d\overline{b}}{ds} = \overline{T} \times \frac{d\overline{n}}{ds} + \frac{d\overline{T}}{ds} \times n$

$$= \overline{T} \times \frac{d\overline{n}}{ds} + K\overline{n} \times \overline{n} \text{ from (1)}$$

5.12 Length of the arc of a space curve from $t = a$, to $t = b$

Let $\bar{r} = \bar{F}(t)$ be a space curve



Let t = a and t – b represent the points A and B on the curve. The arch length

$$AB = \int_{t=a}^{b} ds$$

or $$S = \int_{a}^{b} \frac{ds}{dt} dt$$

$$S = \int_{a}^{b} \left| \frac{d\bar{r}}{dt} \right| dt$$

Example 1

Determine the length of the arc of the curve

$\bar{r} = e^t \cos t\, i + e^t \sin t\, j + e^t K$ from $t = 0$ to $t = \pi$

Solution

Let $\bar{r} = e^t \cos t\, i + e^t \sin t\, j + e^t K$ (1)

By formula,

$$S = \int_{0}^{\pi} \left| \frac{d\bar{r}}{dt} \right| dt$$

From (1) $\frac{d\bar{r}}{dt} = \left(e^t \cos t - e^t \sin t\right) i + \left(e^t \sin t - e^t \cos t\right) j + e^t K$

$$\left| \frac{d\bar{r}}{dt} \right| = \left[e^{2t}(-2\cos t \sin t + 1)\, i + e^{2t}(2\cos t \sin t + 1) + e^{2t} \right]^{\frac{1}{2}}$$

$$= \sqrt{3e^{2t}} = \sqrt{3} e^t$$

Thus $S = \int_0^{\pi} \left| \frac{d\bar{r}}{dt} \right| dt$

$$= \int_0^{\pi} \sqrt{3e^{2t} dt}$$

$$= \sqrt{3} e^t \Big|_{t=0}^{\pi}$$

$$= \sqrt{3}\left(e^{\pi} - 1\right)$$

Example 2

For a space curve

$\bar{r} = l\cos t\, i + l\sin t\, j + mtK$ where l and m are constants

determine the following

i) the unit tangent $\bar{T}$

ii) the curvature K and the radius of curvature ρ

iii) the unit normal $\bar{n}$

iv) the unit binormal $\bar{b}$

v) the Torsion τ and the radius of Torsion σ

Solution

Let $\bar{r} = l\cos t\, i + l\sin t\, j + mtK$ (1)

Then $\frac{d\bar{r}}{dt} = -l\sin t\, i + l\cos t\, j + mK$

$$\left| \frac{d\bar{r}}{dt} \right| = \left[l^2 \sin^2 t\, i + l^2 \cos^2 t\, j + m^2 \right]^{\frac{1}{2}}$$

$$= \sqrt{l^2 + m^2}$$

i) The unit tangent $\bar{T} = \frac{d\bar{r}/dt}{\left| d\bar{r}/dt \right|}$

$$= \frac{-l\sin t\, i + l\cos t\, j + mK}{\sqrt{l^2 + m^2}}$$

ii) The curvature K

From (3) $\frac{d\bar{T}}{dt} = \frac{1}{\sqrt{l^2 + m^2}}\left[-l\cos t\, i - l\sin t\, j + 0 \right]$

The curvature

$$K\bar{n} = \frac{d\bar{T}/dt}{|d\bar{r}/dt|} \qquad \text{(A)}$$

$$|K\bar{n}| = \frac{|d\bar{T}/dt|}{|d\bar{r}/dt|} = \frac{1}{\sqrt{l^2+m^2}} \frac{\left(l^2\cos^2 t + l^2\sin^2 t\right)^{\frac{1}{2}}}{\sqrt{l^2+m^2}} = \left(l^2\right)^{\frac{1}{2}}$$

$$K = \frac{l}{l^2+m^2}$$

$$\rho = \frac{l^2+m^2}{l}$$

$$K\bar{n} = \frac{d\bar{T}/dt}{|d\bar{r}/dt|} \qquad \text{from (A)}$$

Then $\bar{n} = \frac{1}{K} \cdot \frac{1}{\sqrt{l^2+m^2}} \cdot \frac{1}{\sqrt{l^2+m^2}}(-l\cos t\, i - l\sin t\, j)$

$$= \frac{l^2+m^2}{l} \cdot \frac{1}{l^2+m^2}(-l\cos t\, i - l\sin t\, j)$$

$$= -(\cos t\, i + \sin t\, j)$$

The binormal $\bar{b}$

$$\bar{b} = \bar{T} \times \bar{n} = \frac{1}{\sqrt{l^2+m^2}} \begin{vmatrix} i & j & k \\ -l\sin t & l\cos t & m \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

Then

$$\bar{b} = \frac{1}{\sqrt{l^2+m^2}}(m\sin t\, i - m\cos t\, j + lk)$$

$$\frac{d\bar{b}}{dt} = \frac{1}{\sqrt{l^2+m^2}}(m\cos t\, i + m\sin t\, j)$$

$$\frac{d\bar{b}}{ds} = \frac{d\bar{b}/dt}{ds/dt} = \frac{d\bar{b}/dt}{|d\bar{r}/dt|} = \frac{1}{\sqrt{l^2+m^2}} \frac{(m\cos t\, i + m\sin t\, j)}{\sqrt{l^2+m^2}}$$

$$= \frac{m\cos t\, i + m\sin t\, j}{l^2+m^2}$$

$$\frac{d\bar{b}}{ds} = -\tau\bar{n}$$

$$\left|\frac{d\bar{b}}{ds}\right| = \tau = \frac{\left[m^2\cos^2 t + m^2\sin^2 t\right]^{\frac{1}{2}}}{l^2+m^2}$$

$$\tau = \frac{m}{l^2+m^2}$$

$$\sigma = \frac{l^2 + m^2}{m}$$

Exercise 5

1. For a space curve

   $\bar{r} = 3\cos t\, i + 3\sin t\, j + 4tk$

   Find

   i) The unit tangent $\bar{T}$

   ii) The curvature K

   iii) The radius of curvature ρ

   iv) The unit normal $\bar{n}$

   v) The unit binormal $\bar{b}$

   vi) The torsion τ and

   vii) The radius of torsion σ

2. A space curve is given by

   $x = t - \frac{t^3}{3}, \qquad y = t^2, \qquad z = t + \frac{t^3}{3}$

   At the point t = 2, calculate

   i) The unit tangent $\bar{T}$

   ii) The curvature K

   iii) The radius of curvature ρ

   iv) The unit normal $\bar{n}$

   v) The unit binormal $\bar{b}$

   vi) The torsion τ and

   vii) The radius of torsion σ

3. A space curve is given by

   $\bar{r} = (t - \sin t)\, i + (1 - \cos t)\, j + 4\sin\left(\frac{t}{2}\right) K$

   Find K and τ for the space curve at the point where $t = \frac{\pi}{2}$

4. The equation of a space curve is given by

$$\bar{r} = 7\cos t\, i + 7\sin t\, j + \sqrt{15} t\, k$$

Calculate at $t = \frac{\pi}{2}$

i) The unit tangent $\bar{T}$

ii) The curvature K and the radius of curvature ρ

iii) The unit normal $\bar{n}$

iv) The unit binormal $\bar{b}$

v) The torsion τ and the radius of torsion σ

vi) The Cartesian equation to the osculating plane

5. Prove that

$$\frac{d\bar{r}}{ds} \cdot \frac{d^2\bar{r}}{ds^2} \times \frac{d^2\bar{r}}{ds^2} = \frac{\tau}{\rho^2}$$

6. Prove that the radius of curvature of the curve with parametirc equations

x = x(s), y = y(s), z = z(s) is given by

$$\rho = \left[\left(\frac{d^2x}{ds^2} \right)^2 + \left(\frac{d^2y}{ds^2} \right)^2 + \left(\frac{d^2z}{ds^2} \right)^2 \right]^{\frac{1}{2}}$$

7. Find K and τ for the space curve $x = t$, $y = t^2$ and $z = t^3$ called the twisted cubic

8. Show that he Frenet – Serret formulae can be written in the form;

$$\frac{d\bar{T}}{ds} = \bar{w} \times \bar{T}$$

$$\frac{d\bar{n}}{ds} = \bar{w} \times \bar{n}$$

$\frac{d\bar{b}}{ds} = \bar{w} \times \bar{b}$ and determine w

Summary of the lesson

You have learnt the following from this lesson:

i) A space curve C in defined in vector form as;

$$\bar{r} = if_1(t) + jf_2(t) + Kf_3(t)$$

or $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$

or $\bar{r} = xi + yj + zk$

or $\bar{r} = f(s)$ where s is the arc length

ii) The fundamental formulae $\dfrac{d\bar{r}}{ds} = \left|\dfrac{d\bar{r}}{dt}\right|$

B i). The unit tangent $\bar{T} = \dfrac{d\bar{r}}{ds} = \dfrac{d\bar{r}/dt}{ds/dt} = \dfrac{d\bar{r}/dt}{|d\bar{r}/dt|}$

ii). The curvature K is given by $K = \dfrac{|d\bar{T}/dt|}{|d\bar{r}/dt|}$

iii). The radius curvature is given by $\rho = \dfrac{|d\bar{r}/dt|}{|d\bar{T}/dt|}$

iv). The unit normal is given by $\bar{n} = \dfrac{1}{K}\dfrac{d\bar{T}/dt}{|d\bar{r}/dt|}$

v). The unit binormal $\bar{b}$ **is given by** $\bar{b} = \bar{T} \times \bar{n}$

vi). The Torsion τ **is given by** $\tau = \left|\dfrac{d\bar{b}}{ds}\right| = \dfrac{|d\bar{b}/dt|}{|d\bar{r}/dt|}$

vii). The radius of Torsion is given by $\sigma = \dfrac{1}{\tau} = \dfrac{|d\bar{r}/dt|}{|d\bar{b}/dt|}$

C. If $\bar{r}$ is a variable point, r_0 is the point on the curve at P, $\bar{b}$, $\bar{T}$ and $\bar{n}$ are the unit binormal, the unit tangent and the unit normal respectively then,

i) The osculating plane is given by $\bar{b}.(\bar{r} - \bar{r}_0) = 0$

ii) The normal plane is given by $\bar{T}.(\bar{r} - \bar{r}_0) = 0$

iii) The rectifying plane is given by $\bar{n}.(\bar{r} - \bar{r}_0) = 0$

The Frenet – Serret formulae

i) $$\frac{d\bar{T}}{ds} = K\bar{n}$$

ii) $$\frac{d\bar{b}}{ds} = -\tau\bar{n}$$

iii) $\frac{d\overline{n}}{ds} = \tau\overline{b} - K\overline{T}$

D. The length of the arc of a space curve $\overline{r} = if_1(t) + jf_2(t) + Kf_3(t)$ from $s = a$ to $s = b$ is given by,

$$s = \int_a^b ds = \int_a^b \frac{ds}{dt} dt = \int_a^b \left|\frac{d\overline{r}}{dt}\right| dt$$

Further reading

7.) Differential geometry of three dimensions
By Weatherburn,
Cambridge 1957

8.) Differential Geometry
By Dr. Sengottaiyan
Oxford publications London. Nairobi

Lesson 6

Concepts and General Theory of surfaces

6.1 Introduction

You have studied the concept of curves in the previous lesson. In this lesson you will study the concept of surfaces whose equations are vector point functions of two real variables *u* and *v*.

6.2 Objectives of the lesson

By the end of this lesson you should be able to

i) define different forms of equations to surfaces
ii) establish the tangents to the *u* curves, *v* curves and determine the equation to the unit normal to a surface at a general point.
iii) define the tangent planes, normal planes and unit normal
iv) determine the equation to the tangent plane and the equation to the normal line.

6.3 Equations of surfaces in different forms

General form of the equation of a surface in rectangular Cartesian form

A surface is defined as the locus of a point P(*x, y, z*) satisfying an equation of the form

$$\mathbf{F}(x, y, z) = 0 \tag{1}$$

This is a scalar point function and it is in the implicit form.

For example

$x^2 + y^2 + z^2 - 25 = 0$ is the equation of a sphere with centre origin and radius 5 units.

6.4 Monge's form of a surface

Sometimes the explicit form of a surface in Rectangular Cartesian form of a surface is preferable to implicit form.

$$z = f(x, y) \tag{2}$$

is the explicit form of a surfaces

This form (2) of the surface is called Monge's form of the surface.

6.5 Gaussian form of a surface

There is another useful form of the surface called the Gausssian form of the surface.

According to Gauss a surface is defined as the locus of a point whose coordinates are functions of two independent parameters *u* and *v*.

Here x is expresses as a function $f_1(u,v)$

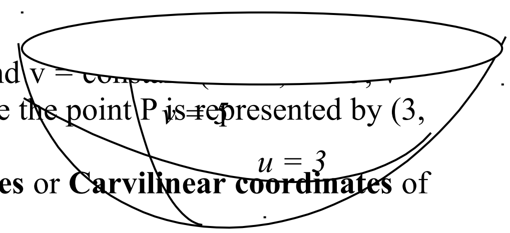
y is expressed as $f_2(u,v)$ and

z is expressed as $f_3(u,v)$.

Thus $\quad \bar{r} = xi + yi + zk$

$\quad$ or $\bar{r} = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$

Thus $\bar{r}$ is expressed as a vector point function



Here u = constant (u = 1, u = 2…..) is one parameter and v = con[illegible] 5…) is the second parameter of the surface. In the figure the point P is represented by (3, 5). It means u = 3 and v = 5 represents the point P.
The parameter *u* and *v* are called the **Surface coordinates** or **Carvilinear coordinates** of the moving point P. on the surface.

A space curve c is expressed as vector point function with only one parameter *t* as, $F(t) = i\ f_1(t) + j\ f_2(t) + k\ f_3(t)$.
In the same way a surface in space is expressed as a vector point function with two parameters *u* and *v* as

$$F(u,v) = i\ f_1(u,v) + j\ f_2(u,v) + k\ f_3(u,v) \qquad (3)$$

This form of the surface in vector form is called the **Gaussian form of the surface.**

For example the equation to a sphere in vector point function of the real variables *u* and *v* (Gaussian form) is given by,

$$F(u,v) = 2i\sin u \cos v + 2j\sin u \sin v + 2k\cos u$$

The radius of the sphere is 2 units

6.6 Expressions for

i). the tangent to the *u* curve
ii). the tangent to the *v* curve and

iii). the normal to a surface at a point P on the surface

Let $\overline{r} = i\ f_1(u,v) + j\ f_2(u,v) + k\ f_3(u,v)$ then the partial derivative,

$\frac{\partial \overline{r}}{\partial u}$ represents the tangent to the *u* curves at a point P on the surface.

$\frac{\partial \overline{r}}{\partial v}$ represents the tangent to the *v* curves at a point P on the surface.

$\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v}$ is a vector perpendicular to both the tangents for the *u* curve and the *v* curve at the point P on the surface.

Hence $\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v}$ represents the normal to the surface $\overline{r} = \overline{F}(u,v)$.

Example 1

Consider the surface $\overline{r} = ui + vj + \left(u^2 + v^2\right)k$

a) Find a vector in the direction of the tangent to the curve $u = u_0$ on the surface.
b) Find a vector in the direction of the tangent to the curve $v = v_0$ on the surface.
c) Determine a vector normal to the surface
d) What is the unit vector normal to the surface or the **unit normal vector**?
e) Find the equation to the tangent plane at the point P on the surface where $u = -1$, $v = 1$
f) Equation to the normal line at P

Solution

Let $\overline{r} = \overline{r}(u,v) = ui + vj + \left(u^2 + v^2\right)k$ be any surface.

a) $\frac{\partial \overline{r}}{\partial u} = i + oj + 2uk = i + 2uk$ is a vector in the direction of the tangent to the curve $u = u_0$ on the surface.

b) $\frac{\partial \overline{r}}{\partial v} = j + 2vk$ is a vector in the direction of the tangent to the curve $v = v_0$ on the surface.

c) Since $\frac{\partial \overline{r}}{\partial u}$ and $\frac{\partial \overline{r}}{\partial v}$ are two tangents to the curves $u = u_0$ and $v = v_0$ on the surface, the vector $\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v}$ is a vector normal to the surface.

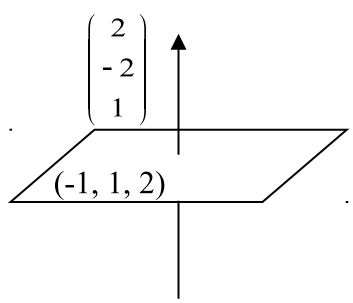
Then $$\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix}$$
$$= (0 + 0 + k) - (0 + 2ui + 2vj)$$
$$= -2ui - 2vj + k \text{ or } 2ui + 2vj - k$$

Hence $\overline{N} - 2ui - 2vj + k$ is the normal to the surface.

d) The unit normal to the surface
$$= \frac{-2ui - 2vj + k}{\sqrt{4u^2 + 4v^2 + 1}}$$

e) The equation to the Tangent plane at the point P on the surface

where u = -1 and v = 1



If u = -1 and v = 1 the point on the surface $\overline{r} = ui + vj + \left(u^2 + v^2\right)k$ is $\overline{r} = -i + j + 2k$. Hence P is the point (-1, 1, 2) in Cartesian form.

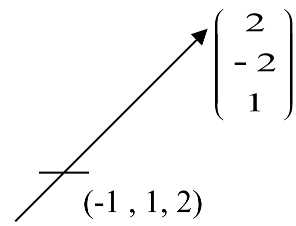
The direction vector of the normal at u = -1 and v = 1 on the surface is [2, -2, 1]

Hence the equation to the tangent plane whose normal $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is given by $2x - 2y + z + d = 0$ where d is to be found. Since the plane passes through the point (-1 , 1, 2), we substitute x = -1, y = 1, z= 2 in $2x - 2y + z + d = 0$. Then d = 2

Thus the equation to the tangent plane to the surface at the point (-1 , 1, 2) is given by $2x - 2y + z + 2 = 0$.

f) The equation to the normal line (in vector form) through the point (-1 , 1, 2) and having direction vector $2i - 2j + k$

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} . \begin{pmatrix} x & - & (-1) \\ y & - & 1 \\ z & - & 2 \end{pmatrix} = 0$$



or

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} . \begin{pmatrix} x & + & 1 \\ y & - & 1 \\ z & - & 1 \end{pmatrix} = 0$$

The equation to a line in Cartesian form is given by

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

Where $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is the direction vector of the line and (a, b, c) is a point on the line. Hence the equation to the line having direction vector $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and passing through the point (-1, 1, 2) is given by $\frac{x+1}{2} = \frac{y-1}{-2} = \frac{z-2}{1}$

The equation to the normal line in parametric form

Let $\frac{x+1}{2} = \frac{y-1}{-2} = \frac{z-2}{1} = \lambda$

Then

$$\frac{x+1}{2} = \lambda \quad \text{or} \quad x = -1 + 2\lambda$$

$$\frac{y-1}{-2} = \lambda \quad \text{or} \quad y = 1 + 2\lambda$$

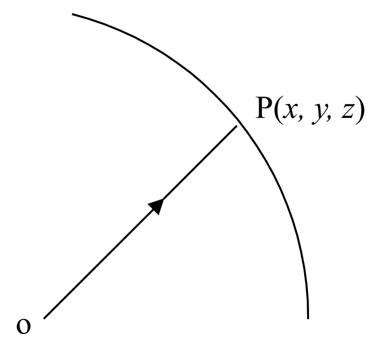
$$\frac{z-2}{1}=\lambda \quad \text{or} \quad z=2+\lambda$$ is the parametric equation of the normal.

Example 2

a) Show that the surface represented by $\bar{r}=ia\sin u\cos v+ja\sin u\sin v+ka\cos u$ is a sphere whose centre is the origin and the radius is a units.

b) Prove that the parametric curves on the surface form an orthogonal system for the surface $\bar{r}=ia\sin u\cos v+ja\sin u\sin v+ka\cos u$

Solution



Any point P on the surface is represented by $\bar{r}=\overline{OP}=xi+yj+zk$

Here $\bar{r}=ia\sin u\cos v+ja\sin u\sin v+ka\cos u$

Hence the coordinates are given in Cartesian form as

$x=a \quad \sin u \quad \cos v$

$y=a \quad \sin u \quad \cos v$

$z=a \quad \cos u$

Then $x^2+y^2+z^2=a^2\sin^2 u\cos^2 v+a^2\sin^2 u\sin^2 v+a^2\cos^2 u$

$$=a^2\sin^2 u\left(\cos^2 v+a^2\sin^2 v\right)+a^2\cos^2 u$$

or $x^2+y^2+z^2=a^2\sin^2 u+a^2\cos^2 u$

$$=a^2\left(\sin^2 u+\cos^2 u\right)$$

or $x^2+y^2+z^2=a^2$

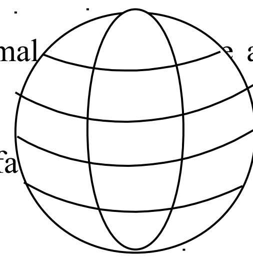
Just like $x^2+y^2=a^2$ is a circle in two D, three dimension $x^2+y^2+z^2=a^2$ is a sphere with centre origin and radius a units.

Exercise 6

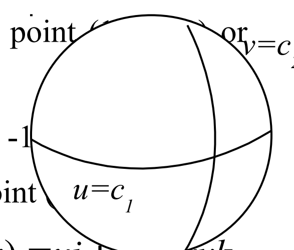
1. What are the parametric curves on the surface

$$\bar{r} = \bar{F}(u,v) = f_1(u,v)i + f_2(u,v)j + f_3(u,v)k$$

2. If $\bar{r} = \bar{F}(u,v)$ is a surface, what does $\frac{\partial \bar{r}}{\partial u}$ and $\frac{\partial \bar{r}}{\partial v}$ represent?

3. If $\bar{r} = \bar{F}(u,v)$, show that $\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}$ represent a normal  at the point P(u, v).
4. State the formula for the unit normal vector for the surfa
5. Define a tangent plane to a surface at the point P(u, v).

6. $z = x^2 + y^2$ is the equation of a surface in Monge's form. Find the equation of the same surface in (i) parametric form and (ii) vector form.

7. a). Find a normal to the surface $\bar{r} = ui + vj + (u^2 + v^2)$ at the point 

or $\bar{r} = i - j + 2k$.
   b). Determine the equation to the tangent plane to the surface at (1, -1
8. Find an equation to the tangent plane to the surface $z = xy$ at the point
   [Hint: if $x = u$, $y = v$ then $z = uv$. Hence the surface is $\bar{r} = \bar{F}(u,v) = ui + vj + uvk$.
   At the point $\bar{r} = 2i + 3j + 6k$, $u = 2$, $v = 3$]

9. Show that a unit normal to the surface $x = \frac{u+v}{\sqrt{2}}$, $y = \frac{u-v}{\sqrt{2}}$, $z = uv$ at P(u, v) is given by $\bar{n} = \frac{(u+v)i - (u-v)j - \sqrt{2}k}{\left[2(1+u^2+v^2)\right]^{\frac{1}{2}}}$

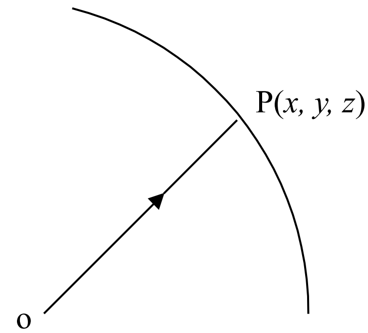
10. a). Show that the parametric equation, $x = 10 \sin u \cos v$, $y = 10 \sin u \sin v$, $z = 10 \cos u$ represents a sphere with centre origin and radius 10 units.

b). Prove that the parametric curves form an orthogonal system.

[Hint: $x^2 + y^2 + z^2 = a^2$ hence sphere. $\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v} = 0$. Hence orthogonal]

11. If $\overline{r} = ui + vj + \left(u^2 + v^2\right)k$

i). Find the position vector of a point and the Cartesian coordinates of P on the surface, at
$u = 2, v = 3$. When $u = 2, v = 3$
$\overline{r} = 2i + 3j + 13k$
Cartesian coordinates of P are (2, 3, 13).



ii). Find the tangent to the surface for the u system and v system

$\frac{\partial \overline{r}}{\partial u} = i + 2uk$ is the tangent to the u system

$\frac{\partial \overline{r}}{\partial v} = j + 2vk$ is the tangent to the v system

iii). Find the normal to the surface. Normal to the surface is: $\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v}$

$$\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2ui - 2vj + k$$

iv). Find the unit normal to the surface at $u = -1$, $v = 1$

$$\overline{n} = \frac{-2ui - 2vj + k}{\sqrt{4u^2 + 4v^2 + 1}}$$

If $u = -1$

$v = 1$ Normal is $\frac{2i - 2j + k}{3}$

v). The point on the surface $\overline{r} = ui + vj + \left(u^2 + v^2\right)k$ when $u = -1$, $v = 1$

Point P is $\overline{r} = -i + j + 2k$

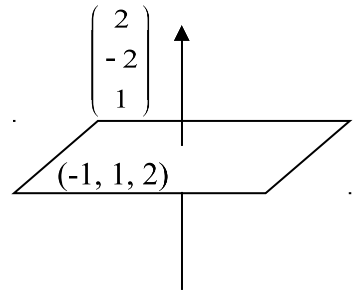
$= xi + yj + zk \Rightarrow (x, y, z) = (-1, 1, 2)$

vi). Find the equation to the tangent plane at (-1, 1, 2) on the surface:

$ax + by + cz + d = 0$

$2x - 2y + z + d = 0$; (-1, 1, 2) is a point

Therefore d = 2



Equation to the plane is 2x – 2y + z + d = 0

Vector form of the equation is

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x + 1 \\ y - 1 \\ z - 2 \end{pmatrix} = 0$$

Summary of the lesson

You have learnt the following from this lesson:

1. The equation to a surface in Rectangular Cartesian form is given by
   - i). F(x, y, z) = 0, (the implicit form)
   - ii). $z = f(x, y)$ the explicit form or Monge's form.
   - iii). $\bar{r} = xi + yj + zk$ (vector form)

     or $\bar{r} = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$ (vector point function Gaussian form)

     or $x = f_1(u,v)$, $y = f_2(u,v)$ and $z = f_3(u,v)$ (parametric form).
   - iv). If $\bar{r} = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$ then
     - a) $\frac{\partial \bar{r}}{\partial u}$ represents the tangent to the *u* curves.
     - b) $\frac{\partial \bar{r}}{\partial v}$ represents the tangent to the *v* curves.
     - c) $\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}$ is the normal to the surface.
     - d) The unit normal to the surface is given by $\bar{n} = \dfrac{\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}}{\left|\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right|}$
   - v). The tangent plane to the surface is the plane perpendicular to the normal line at the point on the surface.

Further Reading

1. Text Book of Differential Geometry
   By Dr. D. Sengottaiyan Ph. D
   Oxford Publications
   London.
   Nairobi.

2. Differential Geometry
   By Weatherbarn
   Cambridge 1959

Lesson 7

First and Second Fundamental Forms of a Surface

7.1 Introduction

A space curve is uniquely determined if the curvature and Torsion of the curve is known as functions of arc lengths. In the same way a surface can be uniquely determined if two local univariant quantities are known. These two local invariant quantities are called **the first and the second fundamental forms of the surface**. The first fundamental form is denoted by E, F, and G and second fundamental form is denoted by L, M and N.

In this lesson we shall define the two forms of a surface and calculate them for a given surface.

7.2 Objectives of the lesson

By the end of this lesson you will be able to:

ii. Define the **first fundamental magnitude and the first fundamental form** of a surface. $\overline{r} = \overline{r}(u,r)$

iii. Define the second **fundamental magnitude and the second fundamental form** of a surface $\overline{r} = \overline{r}(u,r)$.

iv. Establish the expression for the arc length of a curve on a surface.

v. Apply the above in solving problems

7.3 The first fundamental form of a surface.

Let P be any point on the surface,

$\overline{r} = \overline{F}(u,v) = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$

The coordinates of P are a function of the arc lengths s measured from a fixed point A.

Now $d\overline{r} = \frac{d\overline{r}}{du}du + \frac{d\overline{r}}{dv}dv$ (Since $\overline{r}$ is a function of u and v) (1)

Then $\frac{d\overline{r}}{ds}=\frac{d\overline{r}}{du}+\frac{d\overline{r}}{dv}\frac{dv}{ds}$ (2)

where $\frac{d\overline{r}}{ds}$ is a unit tangent.

Now $\frac{d\overline{r}}{ds}.\frac{d\overline{r}}{ds}=\left|\frac{d\overline{r}}{ds}\right|\left|\frac{d\overline{r}}{ds}\right|\cos\theta$

$$1=\left(\frac{d\overline{r}}{ds}\right)^2 \text{ in magnitude}$$

$$=\left(\frac{d\overline{r}}{du}\frac{du}{ds}+\frac{d\overline{r}}{dv}\frac{dv}{ds}\right)^2 \text{ from (2)}$$

$$=\left(\frac{d\overline{r}}{du}\right)^2\left(\frac{du}{ds}\right)^2+2\frac{\partial\overline{r}}{\partial u}\frac{\partial\overline{r}}{\partial v}\frac{du}{ds}\frac{dv}{ds}+\left(\frac{d\overline{r}}{dv}\right)^2\left(\frac{dv}{ds}\right)^2$$

Then $(\partial s)^2=\frac{\partial\overline{r}}{\partial u}.\frac{\partial\overline{r}}{\partial u}(du)^2+2\frac{\partial\overline{r}}{\partial u}.\frac{\partial\overline{r}}{\partial v}(du)(dv)+\frac{\partial\overline{r}}{\partial v}.\frac{\partial\overline{r}}{\partial v}(dv)^2$

$$=E(du)^2+2Fdudv+G(dv)^2 \text{ say,}$$

where $E=\frac{\partial\overline{r}}{\partial u}.\frac{\partial\overline{r}}{\partial u}$

$$F=\frac{\partial\overline{r}}{\partial u}.\frac{\partial\overline{r}}{\partial v}$$

$$G=\frac{\partial\overline{r}}{\partial v}.\frac{\partial\overline{r}}{\partial v}$$

E, F, and G are called the first fundamental magnitudes of a surface.

Example 1

Consider the surface

$$\overline{r}=i(u,v)+j(uv-v)+k(uv-u)$$

Calculate the first Quadratic differential form in general and also when $u=-1$ and $v=1$.

Solution

Let $\overline{r}=i(uv)+j(uv-v)+k(uv-u)$

$$\frac{\partial \overline{r}}{\partial u} = iv + jv + k(v-1)$$

$$\frac{\partial \overline{r}}{\partial v} = iu + j(u-1) + ku$$

$$E = \frac{\partial \overline{r}}{\partial u} \cdot \frac{\partial \overline{r}}{\partial u} = \begin{pmatrix} v \\ v \\ v-1 \end{pmatrix} \cdot \begin{pmatrix} v \\ v \\ v-1 \end{pmatrix} = 2v^2 + (v-1)^2$$

$$F = \frac{\partial \overline{r}}{\partial u} \cdot \frac{\partial \overline{r}}{\partial v} = \begin{pmatrix} v \\ v \\ v-1 \end{pmatrix} \cdot \begin{pmatrix} u \\ u-1 \\ u \end{pmatrix} = 3uv - u - v$$

$$G = \frac{\partial \overline{r}}{\partial v} \cdot \frac{\partial \overline{r}}{\partial v} = \begin{pmatrix} u \\ u-1 \\ u \end{pmatrix} \cdot \begin{pmatrix} u \\ u-1 \\ u \end{pmatrix} = 2u^2 + (u-1)^2$$

If $u = -1$ and $v = 1$ we have

$$E = 2v^2 + (v-1)^2 = 2(1)^2 + (0)^2 = 2$$
$$F = 3uv - u - v = -3 + 1 - 1 = -3$$
$$G = 2u^2 + (u-1)^2 = 2 + (-2)^2 = 6$$

Example 2

a. Find the first fundamental form on the surface,

$$\overline{r} = (u+v)i + (u-v)j + avk$$

b. Calculate the fundamental coefficients or the fundamental magnitudes E, F and G at the point $u = 1, v - 1$.

c. Write down the First Fundamental form of the surface

7.4 Arc length of a curve on the surface

The arc length of a curve on the surface

$$\overline{r} = \phi(u,v) = if_1(u,v) + jf_2(u,v) + kf_3(u,v)$$

is given by

$$\int_a^b \left[E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2 \right]^{\frac{1}{2}} dt$$

Proof

By definition,

$$s = \int_a^b ds = \int_a^b \frac{ds}{dt} dt$$

or $s = \int_a^b \left|\frac{d\overline{r}}{dt}\right| dt = \int_a^b \left(\frac{d\overline{r}}{dt}.\frac{d\overline{r}}{dt}\right)^{\frac{1}{2}} dt$

$$= \int_a^b \left[\left(\frac{d\overline{r}}{du}\frac{du}{dt} + \frac{d\overline{r}}{dv}\frac{dv}{dt}\right).\left(\frac{d\overline{r}}{du}\frac{du}{dt} + \frac{d\overline{r}}{dv}\frac{dv}{dt}\right) \right]^{\frac{1}{2}} dt$$

$$= \int_a^b \left[\left(\frac{\partial \overline{r}}{\partial u}.\frac{\partial \overline{r}}{\partial u}\right)\left(\frac{du}{dt}\right)^2 + 2\left(\frac{\partial \overline{r}}{\partial u}.\frac{\partial \overline{r}}{\partial v}\right)\frac{du}{dt}\frac{dv}{dt} + \left(\frac{\partial \overline{r}}{\partial v}.\frac{\partial \overline{r}}{\partial v}\right)\left(\frac{dv}{dt}\right)^2 \right]^{\frac{1}{2}} dt$$

$$= \int_a^b \left[E\left(\frac{\partial u}{\partial t}\right)^2 + 2F\frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + G\left(\frac{\partial v}{\partial t}\right)^2 \right]^{\frac{1}{2}} dt$$

= integral of the square root of the first fundamental form.

Example 3

Find the length of the arc of the curve

$$\overline{r} = \cos t i + e^t \sin t j + e^t k$$

From $0 \le t \le \pi$

Solution

The length of the arc s is given by

$$s = \int_{t=0}^{\pi} \left|\frac{d\overline{r}}{dt}\right| dt$$

$$= \int_0^{\pi} \left|\left(-e^t \sin t + e^t \cos t\right) i + \left(e^t \cos t + e^t \sin t\right) j + e^t k\right| dt$$

$$=\int_0^{\pi}\left[\left(-e^t\sin t+e^t\cos t\right)^2+\left(e^t\cos t+e^t\sin t\right)^2+\left(e^t\right)^2\right]^{\frac{1}{2}}dt$$

$$=\int_0^{\pi}\left[e^{2t}\sin t+e^{2t}\cos^2 t-2e^{2t}\sin t\cos t+e^{2t}\cos^2 t+e^{2t}\sin^2 t+2e^{2t}\sin t\cos t+e^{2t}\right]^{\frac{1}{2}}dt$$

$$=\int_0^{\pi}\left[e^{2t}(1)+e^{2t}(1)+e^{2t}\right]^{\frac{1}{2}}dt$$

$$=\int_0^{\pi}\left[3e^{2t}\right]^{\frac{1}{2}}dt$$

$$=\sqrt{3}\int_0^{\pi}e^t dt$$

$$=\left[\sqrt{3}\;\;e^t\right]_0^{\pi}$$

$$=\sqrt{3}\left(\;e^{\pi}-1\right)$$

7.5 The second fundamental form of a surface

Let $\overline{r}=if_1(u,v)+jf_2(u,v)+kf_3(u,v)$ be any surface.

$\frac{\partial\overline{r}}{\partial u}$ is the tangent to u curves on the surface.

$\frac{\partial\overline{r}}{\partial v}$ is the tangent to v curves on the surface.

Hence $\frac{\partial\overline{r}}{\partial u}\times\frac{\partial\overline{r}}{\partial v}$ is normal to the surface and the unit normal $\overline{n}$ to the surface is given

by $\overline{n}=\dfrac{\frac{\partial\overline{r}}{\partial u}\times\frac{\partial\overline{r}}{\partial v}}{\left|\frac{\partial\overline{r}}{\partial u}\times\frac{\partial\overline{r}}{\partial v}\right|}$ ……………………………1

Now $\overline{n}$ and $\overline{r}$ are functions of u and v.

Hence $d\overline{n} = \frac{\partial \overline{n}}{\partial u} du + \frac{\partial \overline{n}}{\partial v} dv$

And $d\overline{r} = \frac{\partial \overline{r}}{\partial u} du + \frac{\partial \overline{r}}{\partial v} dv$

Now $\left(\frac{\partial \overline{r}}{\partial u} du + \frac{\partial \overline{r}}{\partial v} dv \right) . \left(\frac{\partial \overline{n}}{\partial u} du + \frac{\partial \overline{n}}{\partial v} dv \right)$

$= \left(\frac{\partial \overline{r}}{\partial u} . \frac{\partial \overline{n}}{\partial u} \right) du^2 + \left(\frac{\partial \overline{r}}{\partial u} . \frac{\partial \overline{n}}{\partial v} \right) dudv + \left(\frac{\partial \overline{r}}{\partial v} . \frac{\partial \overline{n}}{\partial u} \right) dudv + \left(\frac{\partial \overline{r}}{\partial v} . \frac{\partial n}{\partial v} \right) dv^2$ …………………..2

Let us call (2) as $= Ldu^2 + 2Mdudv + Ndv^2$

Then, $L = \frac{\partial \overline{r}}{\partial u} . \frac{\partial \overline{n}}{\partial u}$

$M = \frac{\partial \overline{r}}{\partial u} . \frac{\partial \overline{n}}{\partial v} + \frac{\partial \overline{r}}{\partial v} . \frac{\partial \overline{n}}{\partial u}$

$N = \frac{\partial \overline{r}}{\partial v} . \frac{\partial n}{\partial v}$

The quantity of $L(du)^2 + 2Mdudv + N(dv)^2$ is called the second fundamental form of the surface. L, M, N, all called the second fundamental coefficients of the surface.

7.6 Another form of the second fundamental form.

$\frac{\partial \overline{r}}{\partial u}$ is the tangent and $\overline{n}$ is a normal hence $n.\frac{\partial \overline{r}}{\partial u} = 0$. Similarly $n.\frac{\partial \overline{r}}{\partial v} = 0$

$\frac{\partial}{\partial u} \left(\frac{\partial \overline{r}}{\partial u} . \overline{n} \right) = \frac{\partial^2 r}{\partial u^2} . \overline{n} + \frac{\partial \overline{r}}{\partial u} . \frac{\partial \overline{n}}{\partial u} = 0$

$\frac{\partial}{\partial u} \left(\frac{\partial \overline{r}}{\partial v} . \overline{n} \right) = \frac{\partial^2 r}{\partial v^2} . \overline{n} + \frac{\partial \overline{r}}{\partial v} . \frac{\partial \overline{n}}{\partial v} = 0$

Similar result hold for $\frac{\partial \overline{r}}{\partial v}$

Using these relations we have; $L = \overline{n} . \frac{\partial^2 r}{\partial u^2}$

$$M = \overline{n}.\frac{\partial^2 r}{\partial u \partial v}$$

$$N = \overline{n}.\frac{\partial^2 r}{\partial v^2}$$

These forms of L,M and N are more useful to solve problems.

Example 4

Find the second fundamental form on the surface $\overline{r} = ui + vj + (u^2 - v^2)k$

Solution

Let $\overline{r} = ui + vj + (u^2 - v^2)k$ then; $\frac{\partial \overline{r}}{\partial u} = i + 2uk$ and $\frac{\partial^2 \overline{r}}{\partial u^2} = 2k$

$$\frac{\partial \overline{r}}{\partial v} = j - 2vk \text{ and } \frac{\partial^2 \overline{r}}{\partial v^2} = -2k$$

$$\frac{\partial^2 \overline{r}}{\partial u \partial v} = 0$$

Hence the second fundamental coefficients are given by;

$$L = \overline{n}.\frac{\partial^2 \overline{r}}{\partial u^2} = \frac{(-2ui + 2vj + k)}{\sqrt{4u^2 + 4v^2 + 1}}.2k$$

$$= \frac{2}{(4u^2 + 4v^2 + 1)^{\frac{1}{2}}} \qquad 1$$

$$M = \overline{n}.\frac{\partial^2 \overline{r}}{\partial u \partial v} = \frac{(-2ui + 2vj + k).0}{(4u^2 + 4v^2 + 1)^{\frac{1}{2}}} \qquad 2$$

$$N = \overline{n}.\frac{\partial^2 \overline{r}}{\partial v^2} = \frac{(-2ui + 2vj + k).(-2k)}{(4u^2 + 4v^2 + 1)^{\frac{1}{2}}}$$

$$= \frac{-2}{(4u^2 + 4v^2 + 1)^{\frac{1}{2}}} \qquad .3$$

Hence the second fundamental form is given by,

$II = L(du)^2 + 2Mdudv + N(dv)^2$ where L, M and N are given from (1), (2) and (3)

Or $II = \dfrac{2(du)^2 - 2(dv)^2}{\left(4u^2 + 4v^2 + 1\right)^{\frac{1}{2}}}$

Exercise 7

1. Find the first fundamental form of the surface

   $\overline{r} = (uv)i + (u - v)j + uvk$

   $\left[Ans.E = 2 + v^2, F = uv, G = 2 + u^2, I = (2 + v)^2 du^2 + 2uvdudv + (2 + u^2)dv^2 \right]$

2. Calculate the first fundamental form of the surfaces

   $\overline{r} = iu\cos v + ju\sin v + ku$

   $\left[Ans: E = u^2, F = 0, G = 2 \right]$

3. Find the fundamental magnitudes of the first and the second order of the surface;

   $\overline{r} = iu + 2jv + 2kuv$

4. determine the first and the second fundamental magnitudes for the surface whose parametric equations are:

   $x = u\cos v, y = u\sin v, z = c\ln\left(u + \sqrt{u^2 - c^2}\right)$

5. show that the asymptotic lines of the hyperboloid,

   $x = a\cos u\sec v, y = b\sin u\sec v, z = c\tan u$

Summary of the lesson

You have learnt the following from this lesson:

i.) The first fundamental magnitudes E, F and G are given by:

$$E = \frac{\partial \overline{r}}{\partial u} \cdot \frac{\partial \overline{r}}{\partial u}$$

$$F = \frac{\partial \overline{r}}{\partial u} \cdot \frac{\partial \overline{r}}{\partial v}$$

$$G = \frac{\partial \overline{r}}{\partial v} \cdot \frac{\partial \overline{r}}{\partial v}$$

ii.) The expression $E(du)^2 + 2Fdudv + G(dv)^2$ is called the first fundamental form of a surface $\overline{r} = \overline{r}(u,v)$

iii.) The second fundamental magnitude L, M, N are given by:

$$L = \overline{n}.\frac{\partial^2 r}{\partial u^2}$$

$$M = \overline{n}.\frac{\partial^2 r}{\partial u \partial v}$$

$$N = \overline{n}.\frac{\partial^2 r}{\partial v^2}$$

And the second fundamental form of the surface is given by :

$$L(du)^2 + 2Mdudv + N(dv)^2$$

Arc length of a curve on the surface $\overline{r} = \overline{r}(u,v)$ is given by

$$s = \int_a^b ds = \int_a^b \frac{ds}{dt} dt = \int \left| \frac{d\overline{r}}{dt} \right| dt$$

$$= \int_a^b \left[E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}.\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2 \right]^{\frac{1}{2}} dt$$

Further reading

1.) Differential geometry of three dimensions

By Weatherburn,

Cambridge 1957

2.) Differential geometry

By Dr. Sengottaiyan

Oxford publications London. Nairobi