NUMERICAL SERIES

Contents

Context

To solve a numerical problem, such as an equation or finding a function's optimum, a common method consists in searching the solution with successive increments: we start with an initial guess u_0 of the solution, then we enter into a loop which, at each iteration, increments this guess with a value u_n. This process results in a sequence (S_n):

$$S_n = u_0 + u_1 + \cdots + u_n$$

which is expected to get close to the solution. Because of the computer's physical limitations, we have to stop the process at a finite value of n. Still, it remains useful to study the behavior of the sequence (S_n) as n tends to $+\infty$.

The notion of series extends the notion of sum to cases where there are an infinite number of terms. Its aim is to give a meaning to an expression such as

$$S = \sum_{n=0}^{+\infty} u_n$$

where the terms u_n are elements of a given set for which an addition exists. Then the basic idea is to define the infinite sum as the limit, if it exists, of the sequence (S_n).

Basically, a series $\sum u_n$ is just the sequence (S_n). We will see that, in reverse order, any sequence can be seen as a series. The methods specific to numerical series can hence be used to study any numerical sequence.

In a series, the terms u_n can of course be real or complex numbers (in which case we use the term ***numerical series***), but they may also be, in general cases, elements of any vector space, provided that it is possible to define a convergence in this space. An interesting and difficult case is when the terms u_n are functions from an interval $I \subset \mathbb{R}$ to $\mathbb{R}$.

In this worksheet, we will focus on numerical series. Further in the semester, we will work with a particular type of series of functions, the «power series».

Series appear in many situations:

- in probability, if the set Ω of possible outcomes of a random experiment is infinite but can be indexed by $\mathbb{N}$, the sum of each elementary probability (which results in 1) is the limit of a numerical series. The probability of an infinite event, or the expectation and the variance of a random variable, are also infinite sums;
- the price of a financial asset (a stock or a bond) is, at least in theory, the infinite sum of the future incomes (the dividends for a stock), each income being counted with an interest rate;
- many functions can be expressed as the sum of a series, for example a power series or a Fourier series (an infinite sum of sine and cosine functions);

Learning outcomes

Here are the skills that you are asked to develop. You will be evaluated on them:

- understanding the object «numerical series» and its relation to the object «numerical sequence»;
- analyzing the nature (convergent or divergent) of a numerical series, given its general term (u_n).

Each section focuses on some particular aspects of these skills.

Numerical series in general

- understanding the object «numerical series»;
- switching from a numerical series to a numerical sequence;

- switching from a numerical sequence to a numerical series.

Numerical series whose general term has a constant sign

- knowledge of the reference series;
- understanding the comparisons $o(\cdots)$ and $\sim$: the so-called «negligeable terms» are not zeros, they must be properly addressed;
- being able to determine the nature of a numerical series by comparison with a reference series;
- being aware of the reasoning mistakes when misusing these comparison criteria.

Other numerical series

- understanding the limitations of applying the classical criteria;
- knowing how to return to the classical criteria;
- being able to detect alternating series;
- being aware of the strange behavior of semi-convergent series.

1 Numerical series in general

1.1 Summary

Suppose that you compute S_n with successive increments: for any $n \in \mathbb{N}$,

$$S_{n+1} = S_n + u_{n+1}$$

Then you must see clearly the difference between the sequence (u_n) of the general terms, and the sequence (S_n) of the partial sums.

This section focuses on this difference: the convergences of these sequences are not equivalent. Furthermore, any sequence (u_n) can be seen as the sequence of the partial sums of a numerical series, but the latter is not the series $\sum u_n$.

1.2 Available MiMos (in French)

Those of you who can understand French can see the following MiMo:
Module 3: Généralités sur les séries numériques.

1.3 Exercises

Exercise 1.1
Let $q \in \mathbb{R}$ and let $\sum u_n$ be the numerical series defined by its general term $u_n = q^n$.
Discuss the nature of the series depending on q.

Exercise 1.2

1. Explain why, when the general term u_n of a series is positive, the following properties are equivalent:

   « $\sum u_n$ convergent » and « (S_n) bounded above ».

2. What can be said when the general term is negative?

3. What can be said about the series $\sum(-1)^n$?

Exercise 1.3
Consider the numerical series $\sum \frac{1}{n}$.

1. Check that the general term (u_n) converges to 0.
2. Show that for any $n \in \mathbb{N}$, $S_{2n} - S_n \geqslant \frac{1}{2}$.
3. Deduce the nature of the series $\sum \frac{1}{n}$.

Exercise 1.4
Let $\sum u_n$ be the series whose general term is $u_n = \dfrac{1}{n(n+1)}$.

1. Show that for any $n \in \mathbb{N}^*$, $u_n = \dfrac{1}{n} - \dfrac{1}{n+1}$.
2. Deduce that the series is convergent and determine the value of its sum.

2 Numerical series whose general term has a constant sign

2.1 Summary

When the general term of a numerical series has a constant sign, ***and only in this case***, the nature of the series can be deduced from a comparison with a reference series. The reference series you are asked to know are the Riemann series and the geometric series.

Practically, for the comparisons:

- to compare with a Riemann series, do an asymptotic study of (u_n), for example a Taylor expansion, which will compare u_n to a monomial function of $\frac{1}{n}$;
- to compare with a geometric series, use d'Alembert's test or Cauchy's test.

2.2 Available MiMos (in French)

Module 4 : Séries à termes positifs (1/2). Comparison with one of the reference series.
Module 5 : Séries à termes positifs (2/2). In some situations, the comparison of the general term with that of a geometric series can be done using the d'Alembert test or the Cauchy test.

2.3 Exercises

Exercise 2.5
Determine the nature of the following series.

1. $\sum \sin\left(\dfrac{1}{n}\right)$
2. $\sum \left(\cos\left(\dfrac{1}{n}\right) - 1\right)$
3. $\sum \ln\left(\dfrac{n+1}{n}\right)$

Exercise 2.6

1. Let (u_n) be the sequence defined for any $n \geqslant 2$ by $u_n = \dfrac{1}{n} - \dfrac{1}{n\ln(n)}$
   (a) What is the nature of $\sum \dfrac{1}{n}$?
   (b) By comparing the partial sum of the series $\sum \dfrac{1}{n\ln(n)}$ with the integral
   $$\int_2^{n+1} \frac{1}{x\ln(x)}\,\mathrm{d}x$$
   show that this series diverges.
   (c) What is the nature of $\sum u_n$?
2. Let (u_n) and (v_n) be two sequences such that
   - (v_n) has a constant sign and, in $+\infty$, $v_n = o\left(\dfrac{1}{n}\right)$
   - for any $n \in \mathbb{N}^*$, $u_n = \dfrac{1}{n} + v_n$

   (a) What can be said about the nature of $\sum v_n$?
   (b) What can be said about the nature of $\sum u_n$?

Exercise 2.7

Let (u_n) be the sequence defined for any $n \geqslant 2$ by $u_n = \dfrac{(-1)^n}{n} - \dfrac{1}{n\ln(n)}$

1. Study of the series $\sum \dfrac{(-1)^n}{n}$.
   Let (S_n) denote the partial sum of this series: $S_n = -\dfrac{1}{1} + \dfrac{1}{2} - \dfrac{1}{3} + \cdots + \dfrac{(-1)^n}{n}$
   (a) Show that the sequences (S_{2n}) and (S_{2n+1}) are adjacent.
   (b) Deduce that the series $\sum \dfrac{(-1)^n}{n}$ converges.
2. Show that $u_n \sim \dfrac{(-1)^n}{n}$.
3. Do the series $\sum u_n$ and $\sum \dfrac{(-1)^n}{n}$ have the same nature?
   What can we say about the relevance of the comparison criteria in this case?
   Hint: use the property shown in exercise 2-6, question 1b.

Exercise 2.8

Study the nature of the series $\sum u_n$ in the following cases:

1. $u_n = \ln\left(\dfrac{n^2+2n+1}{n^2+2n}\right)$
2. $u_n = \left(\ln(n)\right)^{-\sqrt{n}}$

3. $u_n = \left(1 + \dfrac{1}{n}\right)^n - e$

4. $u_n = \sqrt{n^3 + n + 1} - \sqrt{n^3 + n - 1}$

5. $u_n = \sin\left(\dfrac{1}{n}\right) - \dfrac{1}{n}$

6. $u_n = \dfrac{2 \times 4 \times \cdots \times 2n}{(n!)^2}$

7. $u_n = \dfrac{(n!)^\alpha}{n^n}$ where $\alpha \in \mathbb{R}$

8. $u_n = \left(\dfrac{n}{n+a}\right)^{n^2}$ where $a \in \mathbb{R}$

9. $u_n = \dfrac{n^2}{2^{n^2}}$

10. $u_n = \dfrac{(n!)^2}{(2n)!}\, a^n$ where $a \in \mathbb{R}_+^*$

11. $u_n = \dfrac{n^{\ln(n)}}{\big(\ln(n)\big)^n}$

Exercise 2.9

Let $(u_n)_{n\in\mathbb{N}^*}$ be the sequence defined for any $n \in \mathbb{N}^*$ by

$$u_n = \ln\big((n-1)!\big) - \left(n - \frac{1}{2}\right)\ln(n) + n$$

1. Show that
$$u_{n+1} - u_n = 1 - \left(n + \frac{1}{2}\right)\ln\left(1 + \frac{1}{n}\right)$$

2. Show that
$$u_{n+1} - u_n \underset{+\infty}{\sim} -\frac{1}{12n^2}$$

3. Deduce that (u_n) is convergent. Let ℓ be its limit.

4. Show that
$$e^{u_n} = \frac{n!e^n}{n^n\sqrt{n}}$$
and deduce that
$$n! \underset{+\infty}{\sim} e^\ell n^n e^{-n}\sqrt{n}$$

3 Other numerical series

3.1 Summary

When the general term of a series does not have a constant sign, the comparison criteria are not valid. There are for example many cases when ***two series have a different nature while their general terms are equivalent***.

In fact, when the general term of a series does not have a constant sign, there is no rule leading to the divergence of the series. Yet, there are two particular cases where the convergence can be proven:

- if the series is alternating and complies with the conditions of the Leibniz rule;
- if the series is absolutely convergent.

Otherwise, to determine the nature of a series, we split its general term as the sum of the general terms of several series and we study each series separately, using different criteria. Several examples are given (see exercises 3-11 and further).

A series which is convergent but not absolutely convergent is said to be «semi-convergent». Then its sum has strange properties which are not intuitive and make it difficult to interpret (see the last exercise). In probability for example, when the series which defines the expectation of a random variable is semi-convergent, we say that the expectation does not exist.

3.2 Available MiMos (in French)

Module 6: Séries à termes quelconques.

3.3 Exercises

Exercise 3.10
Study the nature of the series $\sum u_n$ in the following cases:

1. $u_n = \dfrac{\sin(n)}{n^2}$
2. $u_n = \dfrac{(-1)^n}{n\ln(n)}$
3. $u_n = \dfrac{(-1)^n\ln(n)}{n}$
4. $u_n = \dfrac{n^4 2^{-n^2} + (-1)^n}{\sqrt{n}}$

Exercise 3.11
Let (u_n) be the sequence defined for any $n \geqslant 2$ by $u_n = \dfrac{(-1)^n}{n+(-1)^n}$.

1. Does the series $\sum u_n$ comply with the conditions of the Leibniz rule?
2. Determine $c \in \mathbb{R}$ such that, in the neighborhood of $+\infty$,

$$u_n = \frac{(-1)^n}{n} + \frac{c}{n^2} + o\left(\frac{1}{n^2}\right)$$

3. Deduce the nature of $\sum u_n$.

Exercise 3.12

Let $a \in \mathbb{R}_+^*$ and let (u_n) be the sequence defined for any $n \in \mathbb{N}^*$ by $u_n = \ln\left(1 + \frac{(-1)^n}{n^a}\right)$.

1. Determine $c \in \mathbb{R}$ such that, in the neighborhood of $+\infty$,
$$u_n = \frac{(-1)^n}{n^a} + \frac{c}{n^{2a}} + o\left(\frac{1}{n^{2a}}\right)$$
2. Discuss the nature of $\sum u_n$ depending on the value of a.

Exercise 3.13

The purpose of this exercise is to give the nature of the series whose general term is
$$u_n = (-1)^n n^\alpha \left(\ln\left(\frac{n+1}{n-1}\right)\right)^\beta$$
where $(\alpha, \beta) \in \mathbb{R}^2$ and $n \in \mathbb{N} - \{0, 1\}$.

1. Show that for any $n \in \mathbb{N} - \{0, 1\}$,
$$\ln\left(\frac{n+1}{n-1}\right) = \ln\left(1 + \frac{1}{n}\right) - \ln\left(1 - \frac{1}{n}\right)$$
2. Show that
$$\ln\left(\frac{n+1}{n-1}\right) = \frac{2}{n}\left(1 + \frac{1}{3n^2} + o\left(\frac{1}{n^2}\right)\right)$$
3. Deduce that
$$\left(\ln\left(\frac{n+1}{n-1}\right)\right)^\beta = \frac{2^\beta}{n^\beta}\left(1 + \frac{\beta}{3n^2} + o\left(\frac{1}{n^2}\right)\right)$$
and that
$$u_n = (-1)^n \frac{2^\beta}{n^{\beta-\alpha}}\left(1 + \frac{\beta}{3n^2} + o\left(\frac{1}{n^2}\right)\right)$$
4. Show that, if $\beta \leqslant \alpha$, the series $\sum u_n$ diverges.
5. Study of the case $\beta > \alpha$.
   a. Check that $u_n = v_n + w_n$ where
$$v_n = (-1)^n \frac{2^\beta}{n^{\beta-\alpha}} \quad \text{and} \quad w_n = (-1)^n \frac{\beta 2^\beta}{3n^{2+\beta-\alpha}} + o\left(\frac{1}{n^{2+\beta-\alpha}}\right).$$
   b. What is the nature of $\sum v_n$?
   c. What is the nature of $\sum w_n$?
   d. Deduce the nature of $\sum u_n$.

★ **Exercice 3.14** *Rearrangement of a series*

We know that, when adding a finite number of terms, the final result does not depend on the order of the terms in the sum. The purpose of this exercise is to show that this property is not always true for an infinite sum.

First, let us consider a finite sum

$$S = \sum_{k=1}^{n} u_k$$

Then for any bijection $\varphi : [\![1, n]\!] \longrightarrow [\![1, n]\!]$, the following relation holds:

$$S = \sum_{k=1}^{n} u_{\varphi(k)}$$

For example, if $n = 2$ and $\varphi : \left| \begin{array}{l} \varphi(1) = 2 \\ \varphi(2) = 1 \end{array} \right.$

then

$$\sum_{k=1}^{n} u_k = u_1 + u_2 \qquad \text{and} \qquad \sum_{k=1}^{2} u_{\varphi(k)} = u_2 + u_1$$

Both sums are equal because the addition commutes.

Consider now an infinite sum and a bijection $\varphi : \mathbb{N}^* \longrightarrow \mathbb{N}^*$. We wonder whether the result of the sum remains the same or not.

Let $\sum u_n$ be the semi-convergent series whose general term is $u_n = \dfrac{(-1)^{n+1}}{n}$, let (S_n) be the sequence of its partial sum and S its limit:

$$S = \sum_{n=1}^{+\infty} u_n$$

Choose for φ the following function:

$$\begin{array}{rccl} \varphi : & \mathbb{N}^* & \longrightarrow & \mathbb{N}^* \\ & n & \longmapsto & \left| \begin{array}{ll} 4k & \text{if } n = 3k \\ 2k+1 & \text{if } n = 3k+1 \\ 4k+2 & \text{if } n = 3k+2 \end{array} \right. \end{array}$$

Furthermore, let $\sum v_n$ be the series whose general term is $v_n = u_{\varphi(n)}$ and let (T_n) be the sequence of its partial sum.

1. Write in extension (using the «$\cdots$» notation) the partial sums S_n and T_n.
2. Let $k \in \mathbb{N}$. Express $v_{3k+1} + v_{3k+2} + v_{3k+3}$ in terms of u_{2k+1} and u_{2k+2}.
3. Deduce a relation between T_{3n+3} and S_{2n+2}.
4. Deduce that the series $\sum v_n$ is convergent, but that its sum T is different from S.
5. Check that φ is bijective.

Remark: such phenomena typically happen with ***semi-convergent*** series. Furthermore, in this case, Riemann's rearrangement theorem states that for any $\ell \in \mathbb{R} \cup \{-\infty, +\infty\}$, there exists a bijection $\varphi : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ such that

$$\lim_{n \to +\infty} \sum_{k=1}^{n} u_{\varphi(k)} = \ell$$

Anyway, we will accept without proof that, for an ***absolutely convergent*** series, the nature of the series and the value of its sum do not depend on the order of the terms.