Chapter 2

Vector spaces

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

2.1 Definitions

2.1.1 Vector spaces

Definition 1

E is a $\mathbb{K}$-vector space if E is equipped with a law of internal product denoted $+$ and of a law of external product (of $\mathbb{K} \times E$ in E) denoted . such that

- $(E, +)$ is a commutative group
- $\forall(\lambda, \mu) \in \mathbb{K}^2, \forall(x, y) \in E^2$
  - $(\lambda + \mu).x = \lambda.x + \mu.x$
  - $\lambda.(x + y) = \lambda.x + \lambda.y$
  - $\lambda.(\mu.x) = (\lambda\mu).x$
  - $1.x = x$

Examples

1. For all $n \in \mathbb{N}^*$, $\mathbb{R}^n$ is a $\mathbb{R}$-vector space.

2. The space of polynomials with real coefficients $\mathbb{R}[X]$ is a $\mathbb{R}$-vector space.

3. The space of functions from $\mathbb{R}$ to $\mathbb{R}$ is a $\mathbb{R}$-vector space.

4. The space of real sequences $\mathbb{R}^{\mathbb{N}}$ is a $\mathbb{R}$-vector space.

2.1.2 Vector subspaces

Definition 2

Let E be a $\mathbb{K}$-vector space. F is a vector subspace of E if

- $F \subset E$

- $F \neq \emptyset$

- $\forall (x, y) \in F^2, \forall (\lambda, \mu) \in \mathbb{K}^2,\ \lambda x + \mu y \in F$

Remark

To prove that a set is vectorial, we show in almost all the cases that it is a vector subspace of a classical vector space such as one of the examples above.

Examples

1. Let $n \in \mathbb{N}$. The space of polynomials of degree less than or equal to n with real coefficients is a $\mathbb{R}$-vector space as it is a vector subspace of the $\mathbb{R}$-vector space $\mathbb{R}[X]$.

2. The set of functions even from $\mathbb{R}$ to $\mathbb{R}$ is a $\mathbb{R}$-vector space as it is a vector subspace of the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{R}}$.

3. The set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is a $\mathbb{R}$-vector space as it is a vector subspace of the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{R}}$.

2.2 Operations

2.2.1 Intersection of vector subspaces

Proposition 1

Let E be a $\mathbb{K}$-vector space, F and G be two vector subspacess of E. Then $F \cap G$ is a vector subspace of E. More generally, if $(F_i)_{i \in I}$ is a family of vector subspaces of E, then $\underset{i \in I}{\cap} F_i$ is a vector subspace of E.

Remark

This proposition is essential for the notion of spanned vector subspace.

2.2.2 Sum of vector subspaces

Definition 3

Let E be a $\mathbb{K}$-vector space, F and G be two vector subspaces of E.
We define the set $F + G$ by

$$F + G = \left\{z \in E,\ \exists (x, y) \in F \times G,\ z = x + y\right\}$$

Proposition 2

Let E be a $\mathbb{K}$-vector space, F and G two vector subspaces of E. Then $F + G$ is a vector subspace of E.

2.3 Supplementary subspaces

2.3.1 Vector subspaces in direct sum

Definition 4

Let F and G be two vector subspaces of a $\mathbb{K}$-vector space E. We say that F and G are in direct sum if

$$\forall (x, y) \in F \times G : x + y = 0 \Longrightarrow x = y = 0$$

Example

In $\mathbb{R}^3$, the vector subspaces $\mathbb{R} \times \{0\} \times \{0\}$ and $\{0\} \times \mathbb{R} \times \{0\}$ are in direct sum.

Proposition 3

The following assertions are equivalent :

(i) F and G are in direct sum

(ii) $F \cap G = \{0\}$

(iii) $\forall x \in F + G,\ \exists!(y, z) \in F \times G,\ \ x = y + z$

2.3.2 Supplementary subspaces

Definition 5

Let F and G be two vector subspaces of a $\mathbb{K}$-vector space E. We say that F and G are supplementary in E and we write $E = F \oplus G$ if

$$F \cap G = \{0\} \quad \text{and} \quad E = F + G$$

Proposition 4

Let E be a $\mathbb{K}$-vector space, F and G two vector subspaces of E. Then

$$E = F \oplus G \Longleftrightarrow \forall x \in E,\ \exists!(y, z) \in F \times G, \quad x = y + z$$

Example

Let $E = \mathbb{R}^{\mathbb{R}}$, $P = \left\{f \in \mathbb{R}^{\mathbb{R}},\ f \text{ even}\right\}$ and $I = \left\{f \in \mathbb{R}^{\mathbb{R}},\ f \text{ odd}\right\}$. Then P and I are supplementary in E.

2.4 Spanned vector subspace

Definition 6

Let E be a $\mathbb{K}$-vector space and $X \subset E$. there exists a smallest vector subspace of E containing X. It is the intersection of all the vector subspaces of E containing X. It is called vector subspace of E spanned by X and noted $Span(X)$.

Proposition 5

Let E be a $\mathbb{K}$-vector space and $X = \{x_1, \dots, x_n\} \subset E$. Then

$$\text{Span}(X) = \left\{\lambda_1 x_1 + \cdots + \lambda_n x_n\, ;\, (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n\right\}$$

Examples

1. In the plan $E = \mathbb{R}^2$, the vector subspace spanned by a non null vector x of E is $\text{Span}\big(\{x\}\big) = \mathbb{R}x$ i.e. the vectorial line spanned par x.

2. In the plan $E = \mathbb{R}^2$, the vector subspace spanned by two non null and non collinear vectors of E is whole E.

2.5 Linear map

2.5.1 Definition

Definition 7

Let E, F be two $\mathbb{K}$-vector spaces and $f : E \to F$. We say that f is linear (and we write $f \in \mathscr{L}(E, F)$) if

$$\forall (x, y) \in E^2,\ \forall (\lambda, \mu) \in \mathbb{K}^2, \quad f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$$

Examples

1. $I : \begin{cases} C^0(\mathbb{R}, \mathbb{R}) & \longrightarrow \mathbb{R} \\ f & \longmapsto \displaystyle\int_a^b f(x)\mathrm{d}x \end{cases}$ is linear.

2. $D : \begin{cases} \mathbb{R}[X] & \longrightarrow \mathbb{R}[X] \\ P & \longmapsto P' \end{cases}$ is linear.

Proposition 6

1. $\mathscr{L}(E, F)$ is a $\mathbb{K}$-vector space

2. Let $f \in \mathscr{L}(E, F)$. Then

   a. $f(0) = 0$

   b. $\forall (x, y) \in E^2, \quad f(x - y) = f(x) - f(y)$

2.5.2 Kernel and image

Definition 8

Let $f \in \mathscr{L}(E, F)$.

We call kernel (resp. image) of f the set denoted $\mathrm{Ker}(f) = \big\{x \in E,\ f(x) = 0\big\}$ $\Big($resp. $\mathrm{Im}(f) = \big\{f(x);\ x \in E\big\}\Big)$.

Proposition 7

Let $f \in \mathscr{L}(E, F)$.

1. $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are respectively vector subspaces of E and F.
2. f injective $\Longleftrightarrow \mathrm{Ker}(f) = \{0\}$.
3. f surjective $\Longleftrightarrow \mathrm{Im}(f) = F$.

Example

Let us consider the linear map $f : \begin{cases} \mathbb{R}_n[X] & \longrightarrow \mathbb{R}_n[X] \\ P & \longmapsto P' \end{cases}$

then $\mathrm{Ker}(f) = \{$constant polynomials$\}$ and $\mathrm{Im}(f) = \mathbb{R}_{n-1}[X]$.

2.6 Basis and dimension

2.6.1 Linearly independent family, spanning family, basis

Definition 9

Let E be a $\mathbb{K}$-vector space and $L = \{x_1, \dots, x_n\} \subset E$. We say that L is a linearly independent family if

$$\forall (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \ : \ \lambda_1 x_1 + \cdots + \lambda_n x_n = 0 \Longrightarrow (\lambda_1, \dots, \lambda_n) = (0, \dots, 0)$$

.

Definition 10

Let E be a $\mathbb{K}$-vector space and $G = \{x_1, \ldots, x_n\} \subset E$. We say that G spans E (or that E is spanned by G or that G is a spanning part of E) if $E = \text{Vect}(G)$.

Definition 11

Let E be a $\mathbb{K}$-vector space and $B = \{x_1, \ldots, x_n\} \subset E$. We say that B is a basis of E if B is linearly independent and spans E.

Example

$B = \big(1, X-1, (X+1)^2\big)$ is a basis of $\mathbb{R}_2[X]$.

2.6.2 Dimension

Definition 12

Let E be a $\mathbb{K}$-vector space. We say that E is of finite dimension if it is spanned by a finite family of vectors of E.

Example

Let $n \in \mathbb{N}$. Then $\mathbb{R}_n[X]$ is of finite dimension as $\mathbb{R}_n[X] = \text{Vect}\big(1, X, X^2, \ldots, X^n\big)$.

Theorem 1

Let E be a $\mathbb{K}$-vector space of finite dimension. All the basis of E have same cardinal. This number is called dimension of E and denoted $\dim_{\mathbb{K}}(E)$ yet $\dim(E)$.

Examples

1. Let $n \in \mathbb{N}$. Then $\mathbb{R}_n[X]$ is of dimension $n+1$ as $B = \big(1, X, X^2, \ldots, X^n\big)$ is a basis of E and $\text{Card}(B) = n+1$.

2. $E = \mathbb{C}$ is a $\mathbb{C}$-vector space of dimension 1 as $B = \{1\}$ is a basis of E as a $\mathbb{C}$-vector space.

3. $F = \mathbb{C}$ is a $\mathbb{R}$-vector space of dimension 2 as $B = \{1, i\}$ is a basis of F as a $\mathbb{R}$-vector space.

2.6.3 The incomplete basis theorem and its corollaries

Theorem 2

Let E be a $\mathbb{K}$-vector space of finite dimension spanned by G. Let L be a linearly independent family of E. Then we can complete the family L by vectors taken in G to obtain a basis of E.

Proposition 8

Let E be a $\mathbb{K}$-vector space of finite dimension such that $E \neq \{0\}$. Then E admits at least one basis.

Proposition 9

Let E be a $\mathbb{K}$-vector space of finite dimension and F be a vector subspace of E.
Then F admits at least one supplementary subspace in E.

Proposition 10

Let E be a $\mathbb{K}$-vector space of finite dimension, F and G two vector subspaces supplementary in E, B a basis of F and B' a basis of G.
then $B \sqcup B'$ is a basis of E.

Proposition 11

$E = F \oplus G \Longrightarrow \dim(E) = \dim(F) + \dim(G)$

Proposition 12

Let E be a $\mathbb{K}$-vector space of finite dimension, F and G be two vector subspaces of E such that

$$F \subset G \quad \text{and} \quad \dim(F) = \dim(G)$$

Then $F = G$

Proposition 13

Let E and F be two $\mathbb{K}$-vector spaces of finite dimension. Then

$$f \in \mathscr{L}(E, F) \quad \text{bijective} \quad \Longrightarrow \quad \dim(E) = \dim(F)$$

2.6.4 Rank-nullity theorem

Theorem 3

Let $f \in \mathscr{L}(E, F)$ where E is of finite dimension. Then

$$\dim(E) = \dim\big(\mathrm{Ker}(f)\big) + \dim\big(\mathrm{Im}(f)\big)$$

Proposition 14

Let E be a $\mathbb{K}$-vector space of finite dimension, F and G be two vector subspaces of E. Then

$$\dim(F + G) = \dim(F) + \dim(G) - \dim(F \cap G)$$

2.7 Standard exercises

Exercise 1

Let E be a $\mathbb{K}$-vector space of finite dimension n which is even and $f \in \mathscr{L}(E)$. Prove that

$$\Big(f^2 = 0 \text{ and } \dim\big(\mathrm{Im}(f)\big) = \frac{n}{2}\Big) \Longleftrightarrow \big(\mathrm{Im}(f) = \mathrm{Ker}(f)\big)$$

Solution

$\boxed{\Longrightarrow}$ Let us prove that $\mathrm{Im}(f) \subset \mathrm{Ker}(f)$.

Let $y \in \mathrm{Im}(f)$. Then there exists $x \in E$ such that $y = f(x)$. Yet $f^2 = 0$ so $f(y) = f^2(x) = 0$ so $y \in \mathrm{Ker}(f)$.

On the other hand, using the Rank-nullity theorem, $\dim\big(\mathrm{Ker}(f)\big)+\dim\big(\mathrm{Im}(f)\big) = n$ or $\dim\big(\mathrm{Im}(f)\big) = \frac{n}{2}$ thus $\dim\big(\mathrm{Ker}(f)\big) = \frac{n}{2}$.
Hence $\dim\big(\mathrm{Im}(f)\big) = \dim\big(\mathrm{Ker}(f)\big)$.

As $\mathrm{Im}(f) \subset \mathrm{Ker}(f)$, we deduce that $\mathrm{Im}(f) = \mathrm{Ker}(f)$.

$\boxed{\Longleftarrow}$ Let us prove that $f^2 = 0$.

Let $x \in E$. Then $f(x) \in \mathrm{Im}(f)$ yet $\mathrm{Im}(f) = \mathrm{Ker}(f)$ so $f(x) \in \mathrm{Ker}(f)$ i.e. $f^2(x) = 0$.

On the other hand, using the Rank-nullity theorem, we have $\dim\big(\mathrm{Ker}(f)\big) + \dim\big(\mathrm{Im}(f)\big) = n$ yet $\mathrm{Im}(f) = \mathrm{Ker}(f)$

so $\dim\big(\mathrm{Im}(f)\big) = \frac{n}{2}$.

Exercise 2

Let E be a $\mathbb{R}$-vector space and $(f,g) \in \mathscr{L}(E) \times \mathscr{L}(E)$.

1. Prove that $\big(\mathrm{Ker}(g \circ f) = \mathrm{Ker}(f)\big) \iff \big(\mathrm{Ker}(g) \cap \mathrm{Im}(f) = \{0\}\big)$

2. Prove that $\big(\mathrm{Im}(g \circ f) = \mathrm{Im}(g)\big) \iff \big(\mathrm{Ker}(g) + \mathrm{Im}(f) = E\big)$

Solution

1. $\boxed{\Longrightarrow}$

   Let us assume $\mathrm{Ker}(g \circ f) = \mathrm{Ker}(f)$.

   Let $y \in \mathrm{Ker}(g) \cap \mathrm{Im}(f)$. Then there exists $x \in E$ such that $y = f(x)$ and $g(y) = 0$. Hence $g\big(f(x)\big) = 0$ i.e. $x \in \mathrm{Ker}(g \circ f)$. Yet

   $$\mathrm{Ker}(g \circ f) = \mathrm{Ker}(f)$$

   So $x \in \mathrm{Ker}(f)$ so $y = f(x) = 0$. Hence $\mathrm{Ker}(g) \cap \mathrm{Im}(f) = \{0\}$ (the other inclusion being trivial).

   $\boxed{\Longleftarrow}$

   Let us assume $\mathrm{Ker}(g) \cap \mathrm{Im}(f) = \{0\}$.

   Let us prove first that $\mathrm{Ker}(g \circ f) \subset \mathrm{Ker}(f)$.

   Let $x \in \mathrm{Ker}(g \circ f)$. Then $g\big(f(x)\big) = 0$. So $f(x) \in \mathrm{Ker}(g)$ and $f(x) \in \mathrm{Im}(f)$.

   Thus $f(x) \in \mathrm{Ker}(g) \cap \mathrm{Im}(f)$. Yet

   $$\mathrm{Ker}(g) \cap \mathrm{Im}(f) = \{0\}$$

   so $f(x) = 0$ i.e. $x \in \mathrm{Ker}(f)$.

   Let us prove now that $\mathrm{Ker}(f) \subset \mathrm{Ker}(g \circ f)$.

   Let $x \in \mathrm{Ker}(f)$. Then $f(x) = 0$. So $g\big(f(x)\big) = g(0) = 0$ so $x \in \mathrm{Ker}(g \circ f)$.

2. $\boxed{\Longrightarrow}$

   Let us assume $\mathrm{Im}(g \circ f) = \mathrm{Im}(g)$.

   Let us prove that $E \subset \mathrm{Ker}(g) + \mathrm{Im}(f)$.

Let $x \in E$. Then $g(x) \in \mathrm{Im}(g) = \mathrm{Im}(g \circ f)$. So there exists $z \in E$ such that $g(x) = g\big(f(z)\big)$.
Hence $x = x - f(z) \; + f(z)$ where $f(z) \in \mathrm{Im}(f)$ and $x - f(z) \in \mathrm{Ker}(g)$ as

$$g\big(x - f(z)\big) = g(x) - g\big(f(z)\big) = 0$$

So $x \in \mathrm{Ker}(g) + \mathrm{Im}(f)$. Hence $E = \mathrm{Ker}(g) + \mathrm{Im}(f)$ (the other inclusion being trivial).

$\boxed{\Longleftarrow}$

Let us assume $E = \mathrm{Ker}(g) + \mathrm{Im}(f)$.
The inclusion $\mathrm{Im}(g \circ f) \subset \mathrm{Im}(g)$ is trivial.
Let us prove that $\mathrm{Im}(g) \subset \mathrm{Im}(g \circ f)$.
Let $y \in \mathrm{Im}(g)$. Then there exists $x \in E$ such that $y = g(x)$. Yet

$$E = \mathrm{Ker}(g) + \mathrm{Im}(f)$$

so $x = x_1 + x_2$ where $(x_1, x_2) \in \mathrm{Ker}(g) \times \mathrm{Im}(f)$. Hence

$$y = g(x) = g(x_1 + x_2) = g(x_2)$$

Yet, $x_2 \in \mathrm{Im}(f)$ so there exists $z \in E$ such that $x_2 = f(z)$. Hence

$$y = g\big(f(z)\big)$$

So $y \in \mathrm{Im}(g \circ f)$. Thus $\mathrm{Im}(g) \subset \mathrm{Im}(g \circ f)$.

Exercise 3

Let E be a $\mathbb{R}$-vector space, $(f, g, h) \in \mathscr{L}(E) \times \mathscr{L}(E) \times \mathscr{L}(E)$. Prove that

$$\mathrm{Im}(h \circ f) \subset \mathrm{Im}(h \circ g) \Longleftrightarrow \mathrm{Im}(f) + \mathrm{Ker}(h) \subset \mathrm{Im}(g) + \mathrm{Ker}(h)$$

Solution

$\boxed{\Longrightarrow}$

Let us assume $\mathrm{Im}(h \circ f) \subset \mathrm{Im}(h \circ g)$.
Let $y \in \mathrm{Im}(f) + \mathrm{Ker}(h)$. Then there exists $(x, z) \in E \times \mathrm{Ker}(h)$ such that $y = f(x) + z$. So

$$h(y) = h\big(f(x) + z\big) = h\big(f(x)\big) \in \mathrm{Im}(h \circ f) \subset \mathrm{Im}(h \circ g)$$

Hence there exists $t \in E$ such that $h(y) = h\big(g(t)\big)$. So

$$y = g(t) + \ y - g(t) \in \mathrm{Im}(g) + \mathrm{Ker}(h)$$

as

$$h\big(y - g(t)\big) = h(y) - h\big(g(t)\big) = 0$$

Hence $\mathrm{Im}(f) + \mathrm{Ker}(h) \subset \mathrm{Im}(g) + \mathrm{Ker}(h)$.

$\boxed{\Longleftarrow}$

Let us assume $\mathrm{Im}(f) + \mathrm{Ker}(h) \subset \mathrm{Im}(g) + \mathrm{Ker}(h)$.
Let $y \in \mathrm{Im}(h \circ f)$. Then there exists $x \in E$ such that $y = h\big(f(x)\big)$. Yet

$$f(x) \in \mathrm{Im}(f) \subset \mathrm{Im}(f) + \mathrm{Ker}(h) \subset \mathrm{Im}(g) + \mathrm{Ker}(h)$$

Thus there exists $(u, v) \in E \times \mathrm{Ker}(h)$ such that $f(x) = g(u) + v$. Hence

$$y = h\big(g(u) + v\big) = h\big(g(u)\big) \in \mathrm{Im}(h \circ g)$$

Thus $\mathrm{Im}(h \circ f) \subset \mathrm{Im}(h \circ g)$.

Exercise 4

Let E be a $\mathbb{K}$-vector space and $(p, q) \in \mathscr{L}(E) \times \mathscr{L}(E)$ be such that $p^2 = p$ and $q^2 = q$ (i.e. p and q are projectors). We assume $p \neq 0$, $q \neq 0$ and $p \neq q$.

1. Let $\alpha \in \mathbb{R}$. Prove that $q = \alpha p \Rightarrow \alpha p = \alpha^2 p$.

2. Prove that (p, q) forms a linearly independent family in $\mathscr{L}(E)$.

Solution

1. Let us assume $q = \alpha p$. Then $\alpha p = q^2$ (as $q = q^2$) so $\alpha p = (\alpha p)^2$ (as $q = \alpha p$) i.e. $\alpha p = \alpha^2 p^2$
   Let $\alpha p = \alpha^2 p$ (as $p^2 = p$).

2. We use a proof by contradiction. We assume that (p, q) are linearly dependent i.e. p and q are collinear.
   Then $q = \alpha p$ where $\alpha \in \mathbb{R}$.

Using question 1., we have $\alpha p = \alpha^2 p$ Let $(\alpha^2 - \alpha)p = 0$ i.e. $\alpha(\alpha - 1)p = 0$ then $\alpha(\alpha - 1) = 0$ as $p \neq 0$. So $\alpha = 0$ or $\alpha = 1$.
Yet, if $\alpha = 0$ then $q = 0$ which contradicts the hypothesis. Similarly, if $\alpha = 1$ then $q = p$ which also contradicts the hypothesis.