Vector Spaces (again)

Revisions.

Vector Spaces.

1) Definitions

1.1) Internal and external product. and properties.

E is a $\mathbb{K}$-vector space if E is equipped

with a law of internal product denoted +

and external product such that:

- $\forall (\lambda,\mu) \in \mathbb{K}^2, \forall (x,y) \in E^2$
  - $(\lambda+\mu)x = \lambda x + \mu x$
  - $\lambda(x+y) = \lambda x + \lambda y$
  - $\lambda(\mu x) = (\lambda\mu)x$
  - $1 \cdot x = x$

⚠ 1.2) Vector subspaces.

Let E be a vector space,

F is a vector subspace of E if:

- $F \subset E$.
- $0_E \in F$
- $\forall (x,y) \in F^2, \forall (\lambda,\mu) \in \mathbb{K}^2,$

  $\lambda x + \mu y \in F.$

1.3) Direct sum.

Let E be a $\mathbb{K}$-vector space and F, G subspaces of E.

Normal sum: $F+G = \{ z \in F+G, \exists (x,y) \in F \times G, z = x+y \}$

A direct sum is $F+G$ such that the only element in common is 0.

Ex: $(x,0) + (0,y)$ is in direct sum.

1.3.1) Supplementary subspaces.

A subgenre of direct sum is supplementary subspaces:

$E = F \oplus G, \ \forall x \in E, \ \exists! (y,z) \in F \times G, \ x = y + z$

The direct sum $F + G$ creates an entire vector space where F and G can be seen as a coordinate system.

Ex: $(x, 0) \oplus (0, y) = \mathbb{R}^2$.

Any coordinate (x, y) is a combination of F and G which is unique.

⚠ 2) Spanned vector subspace

2.1) Definition.

$\mathrm{Span}(X) =$

Let $X = \{x_1, \dots, x_n\} \subset E$.

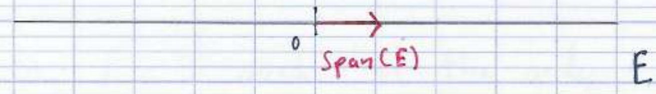
$\mathrm{Span}(X) = \{ \lambda_1 x_1 + \dots + \lambda_n x_n \,;\, (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \}$

If this is complicated (and it is),

Span(X) is the smallest set possible that can be used to express every element of X as a linear combination.

(read that n times)

Ex :



Linear Map.

1) Definition.

Let E and F be two vector spaces and

$f : E \to F$.

f linear iff :

$$\forall (x,y) \in E^2, \forall (\lambda, \mu) \in \mathbb{K}^2,$$

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$$

2) Kernel and image.

The kernel is the set of values in E such that:

$$\forall x \in \mathrm{Ker}(f), f(x) = 0_F.$$

The image is the set containing all the output from f.

3) Rank-nullity theorem.

The most useful theorem ever:

$$\dim(E) = \dim(\mathrm{Ker}(f)) + \dim(\mathrm{Im}(f)).$$

For a linear map $f : E \to F$, we have:

- $\dim(E) = \dim(F)$:

  Injective $\Leftrightarrow$ Surjective

- $\dim(E) > \dim(F)$

  It can't be injective

- $\dim(E) < \dim(F)$

  It can't be surjective

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Matrices and Linear Map.

1) Defining f.

Let f be a linear map $f : E \to F$.

We have two basis : One for E
One for F.

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of E.

and $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p\}$ a basis of F.

The set $\{f(e_1), \ldots, f(e_n)\}$ defines f.

$f(e_1)$ is expressed in the basis of F:

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cc} \left(\begin{array}{ccc} & & \\ & & \\ \end{array}\right) & \begin{array}{l} \leftarrow \text{Coef } \varepsilon_1 \\ \vdots \\ \leftarrow \text{Coef } \varepsilon_p \end{array} \\ \begin{array}{ccc} \uparrow & & \uparrow \\ f(e_1) & \cdots & f(e_n) \end{array} & \end{array}$$

Let $u \in E$ with coord X expressed with E.

$$u = x_1 e_1 + \ldots + x_n e_n$$

$f(u)$ with coord Y with F.

$$f(u) = y_1 \varepsilon_1 + \ldots + y_n \varepsilon_n.$$

Then $Y = AX$.

Examples.

1) Let f be a linear map :

$$f : \begin{array}{l} \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \\ (x,y) \longrightarrow (x+y,\ 2x+4y,\ -3y) \end{array}$$

$$A = \begin{array}{cc} \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & -3 \end{pmatrix} & \begin{array}{l} \leftarrow x+y \\ \leftarrow 2x+4y \\ \leftarrow 0x-3y. \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ f(e_1) & f(e_2) \end{array} & \end{array}$$

$$e_1 = (1,0),\ e_2 = (0,1).$$

2) Let g be a linear map.

$$g : \begin{array}{l} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ (x,y) \longrightarrow (-10x+8y,\ -12x+8y) \end{array}$$

Try this one yourself!

$$A = \begin{pmatrix} -10 & 8 \\ -12 & 8 \end{pmatrix}$$