1 Magnetic properties of electric currents

Certain materials, in particular so-called **ferromagnetic materials**, have the property of exerting an attraction on other bodies, for example iron. These materials are said to be **magnets** and can exert a magnetic force or interaction. This is a remote interaction. As for the electrostatic force and the gravitational interaction, this interaction can be modeled by the presence of a field : thus, **a magnet creates a magnetic field in the space which surrounds it and an object placed in this field can interact with it**.

We will see in that the electric and magnetic phenomena are intimately linked, to such an extent that they are in reality two facets of one and the same characteristic of the physical world around us : **electromagnetism.**

1.1 Magnetic interaction and magnetic field

The magnetic field which prevails in a region of space is represented at any point M of this region by a magnetic field vector whose direction and intensity vary with the position of this point M. In a region where the field vector is constant i.e same intensity and same direction at any point, we say that the magnetic field is uniform. The Earth's magnetic field, for example, is not uniform since, regardless of our position on the surface of the globe, the field vector is always directed towards the magnetic north pole. However, it can be considered locally as uniform, taking into account the dimensions of the planet.

A magnetic field can be created by a magnet or by moving electric charges (therefore electric currents). We will see time and again that electricity and magnetism are intertwined phenomena and that we often speaks of electromagnetism.

The magnetic field is generally denoted by $\vec{B}$ and expressed in **Tesla** (T). Bodies that undergo a magnetic action when placed in a magnetic field are ferrous metals, certain metal alloys and moving electrical charges (therefore conductors carrying currents).

1.2 Force de Laplace et force de Lorentz

The Laplace force is the force experienced by a moving electric charge q placed in a magnetic field. Let q be an electric charge located at a point M, animated by a vector speed $\vec{v}$ and let $\vec{B}$ be the magnetic field prevailing at point M. The charge q undergoes a force F_B known as Laplace such that :

$$\vec{F}_B = q\vec{v} \wedge \vec{B} \tag{1}$$

Taking into account the properties of the vector product, the magnetic force is orthogonal to the plane formed by the velocity vector and the magnetic field vector. Moreover, its intensity (modulus) is determined by the expression :

$$F_B = qvBsin\alpha \tag{2}$$

Where α represents the angle formed by the vectors $\vec{v}$ and $\vec{B}$. The direction of the magnetic force is determined by imagining the progression of a corkscrew turning from $\vec{v}$ to $\vec{B}$. Figure 1 illustrates these properties.

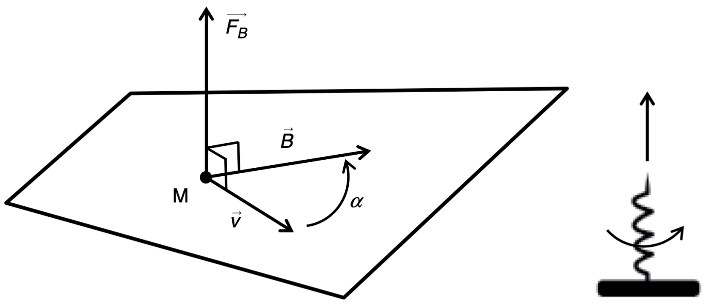

Figure 1 –

If the charge q is subjected to both an electric field $\vec{E}$ and a magnetic field $\vec{B}$, it undergoes not only the magnetic interaction (provided it is in motion), but also an electrostatic force (whether it is fixed or in motion). Under these conditions, the overall force undergone by the charge q therefore results from both the electrostatic interaction and the magnetic interaction. It is then called **Lorentz force** and therefore has the following expression :

$$\vec{F} = q\vec{E} + q\vec{v} \wedge \vec{B} = q(\vec{E} + \vec{v} \wedge \vec{B}) \tag{3}$$

1.3 Electric current

An electric current is a global movement of mobile charges in a conductor. Since moving charges are susceptible to magnetic interactions, it is clear that a conductor carrying a current will be the seat of magnetic forces. We are used to dealing with global movements of charges, which leads us to define the notion of electric current in a conductor.

Consider a rectilinear conductor of section S in which free electric charges are animated by an overall rectilinear movement of average velocity $\vec{v}$ (figure 2).

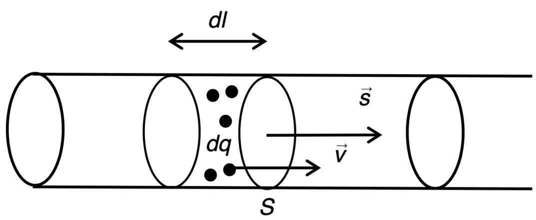

Figure 2 –

During a time dt, a charge dq contained in the volume element Sdl will traverse the section S of the conductor. By calling ρ the quantity of mobile charges per unit of volume, we have :

$$dq = \rho S dl \tag{4}$$

As the charges are animated with a velocity v, we have :

$$dl = vdt \tag{5}$$

We define the electric current i through the surface S by :

$$i = \frac{dq}{dt} \tag{6}$$

Or

$$i = \frac{\rho S dl}{dt} = \rho\vec{v}.\vec{S} = \vec{j}.\vec{S} \tag{7}$$

We define by $\vec{j} = \rho\vec{v}$ **the current density vector across the surface** S.

1.4 Magnetic field created by moving charges

Moving charges not only experience forces due to the presence of a magnetic field, they also create a magnetic field in their surroundings, proving, once again, that electricity and magnetism are still linked.

Thus, a point charge q placed in a vacuum at a point M and animated by a velocity vector $\vec{v}$, creates at any point N in space a magnetic field $\overrightarrow{B(M)}$ having as expression :

$$\overrightarrow{B(M)} = \frac{\mu_0}{4\pi} q \frac{\vec{v} \wedge \vec{u}}{r^2} \tag{8}$$

With :

$$r = MN \tag{9}$$

And $\vec{u}$ being defined as the unit vector of the line (MN) :

$$\vec{u} = \frac{\overrightarrow{MN}}{MN} \tag{10}$$

We have :

$$\frac{\mu_0}{4\pi} = 10^{-7}\ H \times m^{-1} \tag{11}$$

μ_0 is the magnetic permeability of vacuum.
We also sometimes note :

$$\overrightarrow{B(M)} = \frac{\mu_0}{4\pi} q \frac{\vec{v} \wedge \vec{MN}}{MN^3} \tag{12}$$

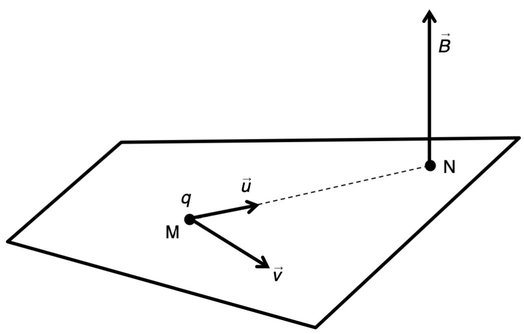

Figure 3 –

When the medium considered is not vacuum, we replace μ_0 by $\mu = \mu_r\mu_0$ where μ_r is the relative magnetic permeability of this medium, μ being its absolute permeability. Note that the magnetic permeability of air is very close to that of vacuum.

1.5 Field lines and magnetic field flux

1.5.1 Magnetic field lines

Lines of equal magnetic field magnitude created by any circuit or distribution are called magnetic field lines. The field vector is tangent to the field line at any point (figure 7.5). The magnetic field lines are always closed.

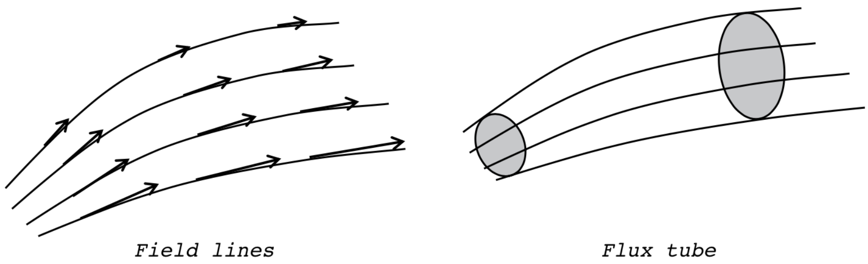

Figure 4 –

A set of field lines resting on a closed contour constitutes a flux tube. As the magnetic field lines are always closed, this set is sometimes referred to as a toric field tube.

1.5.2 Magnetic field flux through a surface

Let (S) be a surface crossed by magnetic field lines (figure 5). Let M a point on this surface and $\overrightarrow{dS}$ the surface element located around M.

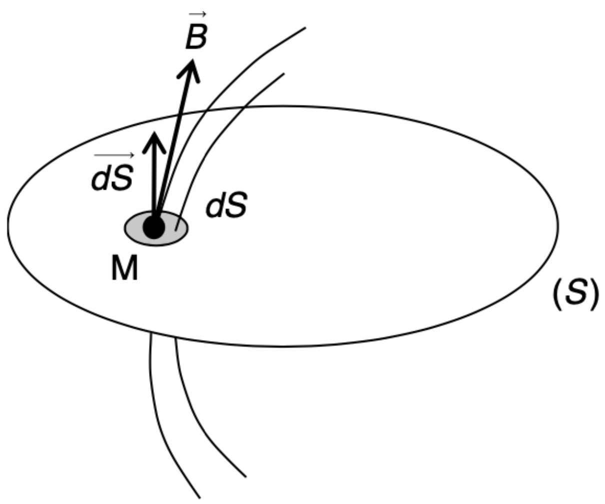

Figure 5 –

Let $\overrightarrow{B(M)}$ be the magnetic field vector at this point. We define the elementary flux of the magnetic field in M through the surface $\overrightarrow{dS}$ by :

$$d\phi = \overrightarrow{B(M)}.\overrightarrow{dS} \tag{13}$$

By integrating over the entire surface (S), we define the total flux of the magnetic field crossing this surface :

$$\phi = \iint_S \overrightarrow{B(M)}.\overrightarrow{dS} \tag{14}$$

The magnitude ϕ is also called magnetic flux and is expressed in **Webers** (Wb).

2 Magnetic field created by a current

In the previous section, we studied, in particular, the magnetic fields created by electric charges in motion. However, if there is a device characterized by movements of charges, it is indeed a conductor carrying a current. We will see, in this section, how an electric current creates a magnetic field and how to characterize this field. Like what we have studied in the case of static charges, we will also define the notion of magnetic field flux, a notion that allows us to easily calculate the magnetic fields induced by currents.

2.1 Biot-Savart law

In a filiform conductor (whose section is small compared to curvature radius), each element $\overrightarrow{dl}$ of the circuit creates at any point M in space an elementary magnetic field $\overrightarrow{dB}$ defined by Biot and Savart's law (figure 6) :

$$\overrightarrow{dB} = \frac{\mu_0}{4\pi}\frac{i\overrightarrow{dl} \wedge \vec{u}}{r^2} \tag{15}$$

In this expression, r represents the distance between the point M and the circuit element $\vec{dl}$ while $\vec{u}$ represents the unit vector of the line connecting $\overrightarrow{dl}$ and the point M.

Note : The quantity $i\overrightarrow{dl}$ is called the current element and can be noted $\overrightarrow{di}$.

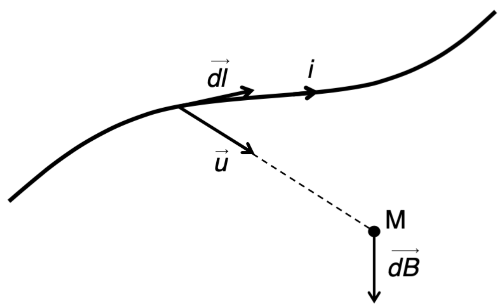

Figure 6 –

Therefore, magnetic field created at any point M in the vicinity of a circuit (C) traversed by a current i is obtained by integrating, that is to say by summing all the elementary fields $\overrightarrow{dB}$ created by each element $\overrightarrow{dl}$ of the circuit and defined by the Biot and Savart law :

$$\overrightarrow{B(M)} = \int_{(C)} \frac{\mu_0}{4\pi}\frac{i\overrightarrow{dl} \wedge \vec{u}}{r^2} \tag{16}$$

Similarly, the current being the same throughout the circuit, we have :

$$\overrightarrow{B(M)} = \frac{\mu_0 i}{4\pi}\int_{(C)} \frac{\overrightarrow{dl} \wedge \vec{u}}{r^2} \tag{17}$$

2.2 Field created by a rectilinear conductor

A filiform rectilinear conductor, of infinite length, traversed by a current i, creates at any point M located in its vicinity at a distance r from the conductor, a magnetic field B (M) of intensity :

$$B(M) = \frac{\mu_0 i}{2\pi r_0} \tag{18}$$

Since this field is both orthogonal to the conductor and to the normal $\vec{u}$ to the conductor oriented towards M, it follows that the field $\overrightarrow{B(M)}$ is oriented as shown in the figure 7.

2.3 Ampere's observer rule

Although the direction and the sense of $\overrightarrow{B(M)}$ are easily determined by the fact that this field is the result of a cross product, we sometimes use the so-called Ampère's observer rule to determine them.

An observer placed on the conductor so that it is traversed by the current i from the feet to the head and facing the point M, "sees" the magnetic field $\overrightarrow{B(M)}$ oriented towards its left (figure 7).

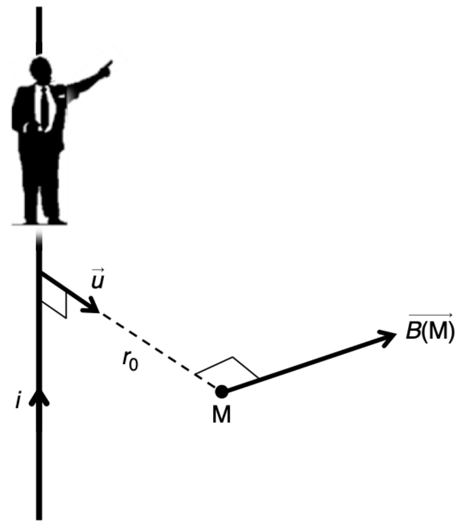

Figure 7 –

3 Current distribution

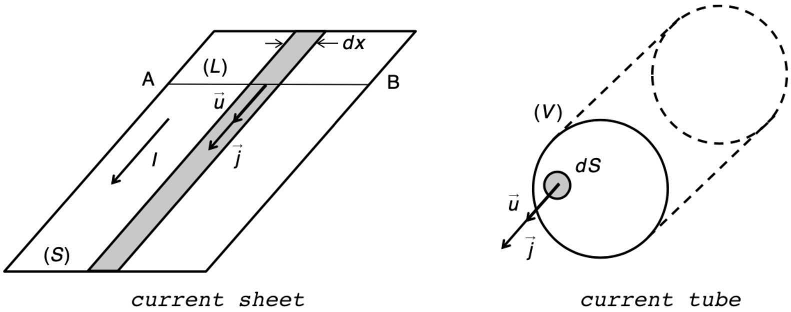

Figure 8 –

Electric currents do not flow only in wire-shaped conductors. They can circulate on the conductive surface of an object or in conductive objects in volume. We then speak respectively of current sheets or current tubes (figure 8).

For a current sheet, consider a line $(L) = AB$ of the conductive surface and suppose that the current crosses this line orthogonally. We define the current density $\vec{j}$ by :

$$I = \int_L \vec{j}.\vec{u}\ dx \tag{19}$$

where $\vec{u}$ is the unit vector orthogonal to (L) line.

In the case of a current tube, the current I passes through a section (S) of the conductor. We define the current density $\vec{j}$ by :

$$I = \int_S \vec{j}.\vec{u}\ dS \tag{20}$$

In both cases, Biot and Savart's law applies to calculate the magnetic field created by the current distribution at any point M in space (Figure 9).

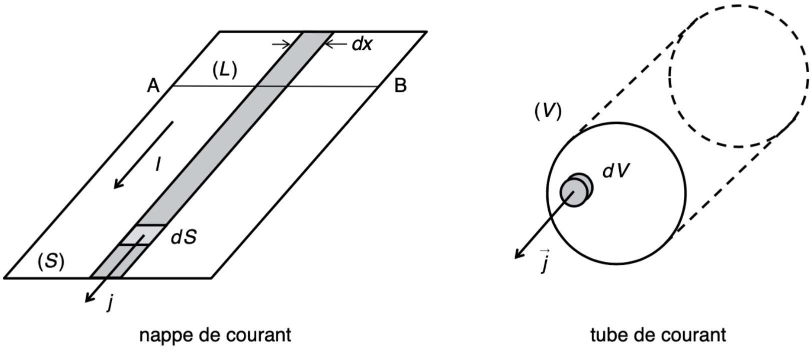

Figure 9 –

Thus, in the case of a current sheet, each surface element dS of the sheet separated by M by a distance r creates an elementary magnetic field defined by :

$$\overrightarrow{dB(M)} = \frac{\mu_0 i}{4\pi} \frac{\vec{j} dS \wedge \vec{u}}{r^2} \tag{21}$$

For a current tube, each element of volume dV of the conductor creates in M an elementary field current tube :

$$\overrightarrow{dB(M)} = \frac{\mu_0 i}{4\pi} \frac{\vec{j} dV \wedge \vec{u}}{r^2} \tag{22}$$

4 Ampère's circuital law

4.1 Magnetic field circulation

Consider a region of space where there is a magnetic field and two points A and B of this region. Let (L) be any path between these two points and M a point belonging to (L). We define the elementary circulation of the magnetic field in M by :

$$dC = \overrightarrow{B(M)}.\overrightarrow{dl} \tag{23}$$

In this expression, $\overrightarrow{dl}$ is an elementary displacement in M along the curve (L). The circulation of the magnetic field along the curve (L), from A to B, has the expression :

$$C = \int_L \overrightarrow{B(M)}.\overrightarrow{dl} \tag{24}$$

4.2 Ampère's circuital law

The circulation of the magnetic field along a closed contour (L) is equal to the product of μ_0 times the intensity of the current flowing through any surface that rests on this contour(figure 10).

$$\oint_L \vec{B}.\overrightarrow{dl} = \mu_0 I \tag{25}$$

This expression constitutes Ampère circuital law.

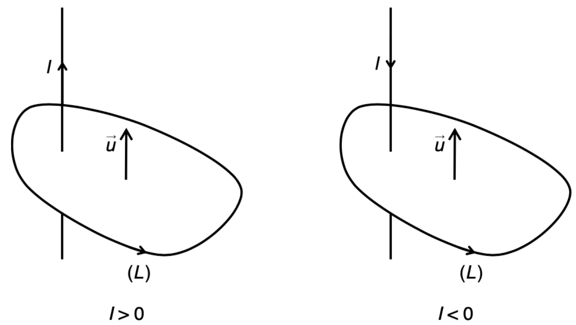

Figure 10 –

The normal $\vec{u}$ to the surface being oriented according to the direction of travel of the contour (figure 10 - think of the corkscrew rule), the current will be counted positively if it is directed in the same direction as $\vec{u}$ and negatively otherwise. If several conductors intersect the surface resting on the contour, the current I involved in Ampere's circuital law corresponds to the algebraic sum of the different currents in these conductors (figure 11).

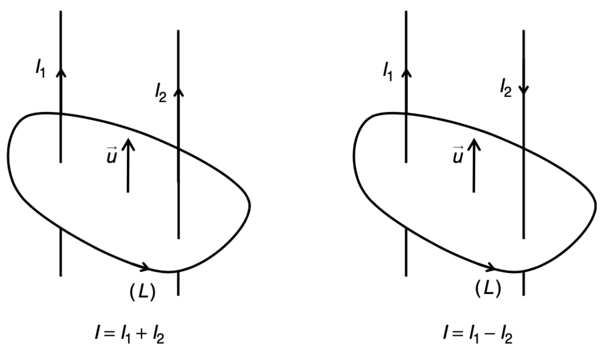

Figure 11 –

5 Actions of a magnetic field on conductors carrying a current

We studied in section 1 the magnetic force undergone by a moving charge in a magnetic field. We will now generalize the results obtained to mobile charges inside conductors, in other words to electric currents flowing in circuits.

5.1 Laplace force

Let us consider a filiform conductor traversed by a current I and placed in a magnetic field B (figure 12). Let $\overrightarrow{dl}$ be an element of this conductor located at a point M. This element is subjected to an elementary force known as Laplace whose expression is :

$$\overrightarrow{df} = I\,\overrightarrow{dl} \wedge \vec{B} \tag{26}$$

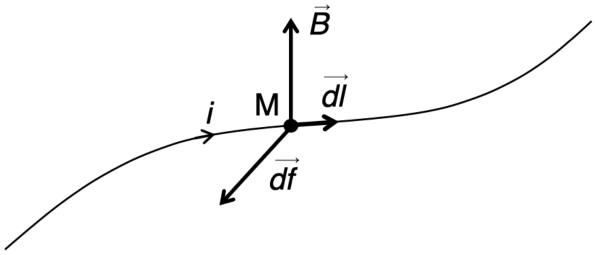

Figure 12 –

The direction and the intensity of the force $\overrightarrow{df}$ result from the properties of the cross product. This elementary force is therefore always orthogonal to $\overrightarrow{dl}$ and to the magnetic field vector.

You can also use the corkscrew rule by rotating from $\overrightarrow{dl}$ to $\vec{B}$. The progress of the corkscrew then indicates the direction of force. One can also use the rule of the observer of Ampère : an observer placed on the conductor so that the current crosses it from the feet to the head and looking in the same direction as the magnetic field, sees the magnetic force directed towards its left (figure 13).

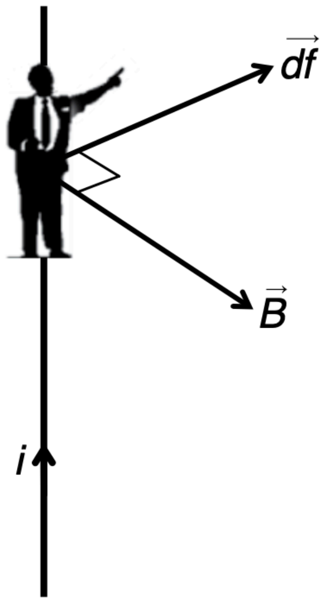

Figure 13 –

The total force acting on a conductor or a complete circuit (C) is obtained by integrating the elementary Laplace forces acting on each element $\overrightarrow{dl}$ of the conductor or circuit :

$$\overrightarrow{F} = \int_C \overrightarrow{df} = \int_C I\overrightarrow{dl} \wedge \vec{B} \tag{27}$$

The magnetic field vector can vary according to the position of $\overrightarrow{dl}$ and consequently, except for the case of a uniform field, it depends on the integration variable and cannot be taken out of the integral as a simple constant. On the other hand, if the current I is the same throughout the circuit, which is often the case, then I can be taken out of the integral as follows :

$$\overrightarrow{F} = \int_C \overrightarrow{df} = I \int_C \overrightarrow{dl} \wedge \vec{B} \tag{28}$$

5.2 Magnetic moment

We consider a filiform contour circuit (L) traversed by a current I. Let (S) be a surface based on this contour (L) (figure 14). We define the magnetic moment of the circuit by the expression :

$$\vec{M} = I \int_S \overrightarrow{dS} \tag{29}$$

or else

$$\vec{M} = \frac{I}{2} \int_C \overrightarrow{OP} \wedge \overrightarrow{dl} \tag{30}$$

In this expression, the point O denotes any origin chosen arbitrarily in the space.

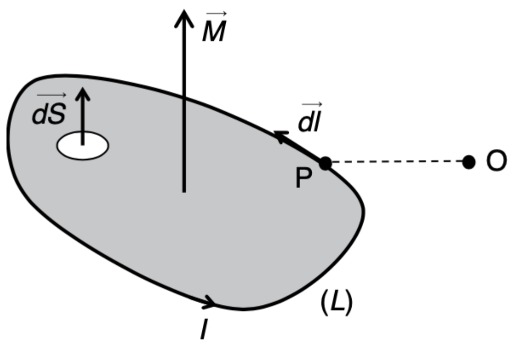

Figure 14 –

For a plane circuit, the magnetic moment is orthogonal to the surface resting on the contour of the circuit and oriented in the path direction. In this particular case, we have :

$$\vec{M} = I\vec{S} \tag{31}$$

5.3 Action of a uniform magnetic field on a plane circuit

A plane circuit traversed by a current I, of magnetic moment $\overrightarrow{M}$ and placed in a uniform magnetic field $\overrightarrow{B}$ (figure 15) is subjected to a couple of moments Γ such that :

$$\Gamma = \vec{M} \wedge \vec{B} \tag{32}$$

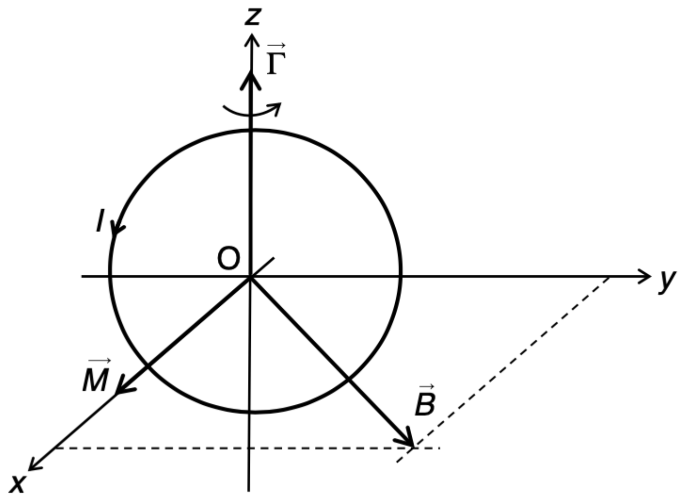

Figure 15 –

In figure 15, the plane circuit is contained in the plane Oyz. The magnetic moment of the circuit is therefore carried by Ox. The magnetic field vector was chosen to belong to the Oxy plane.

The vector Γ is carried by the axis orthogonal to the plane $(\vec{M}, \vec{B})$. The moment couple Γ therefore tends to rotate the plane circuit around its axis. Generally speaking, the moment of the couple Γ tends to bring the magnetic moment M back in line with the magnetic field $\vec{B}$.

In the case shown in Figure 15, the moment of the couple Γ tends to rotate the circuit an eighth of a turn to the right.