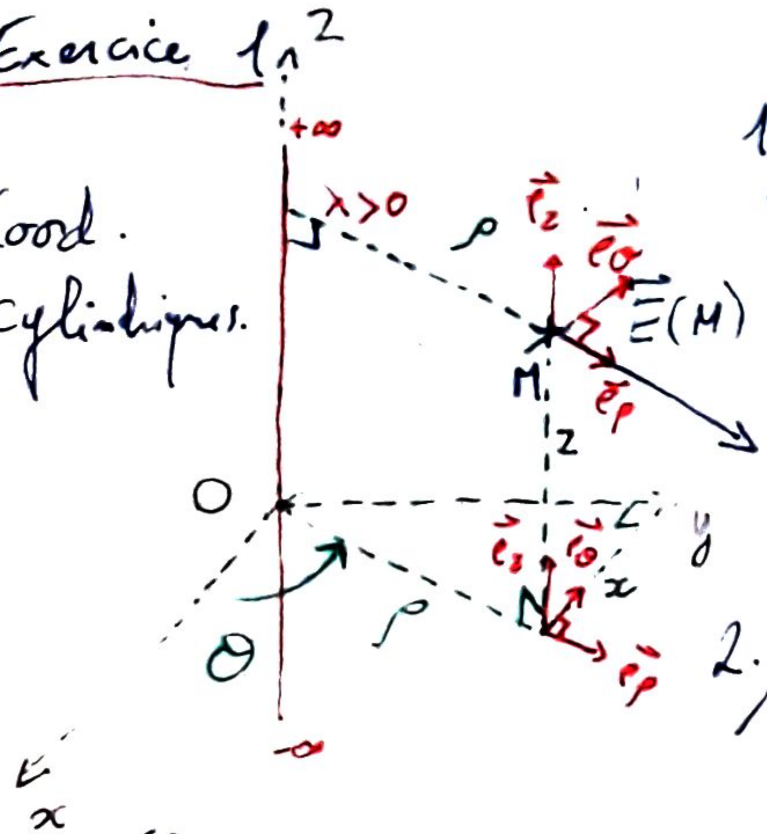
TD4 : Théorème de Gauss

Exercice 1



1./ $\boxed{\vec{E}(M) = E_\rho(\rho, \cancel{\theta}, \cancel{z})} \times \vec{e_\rho}$

Il y a invariance selon θ et z.

Le champ sera radial

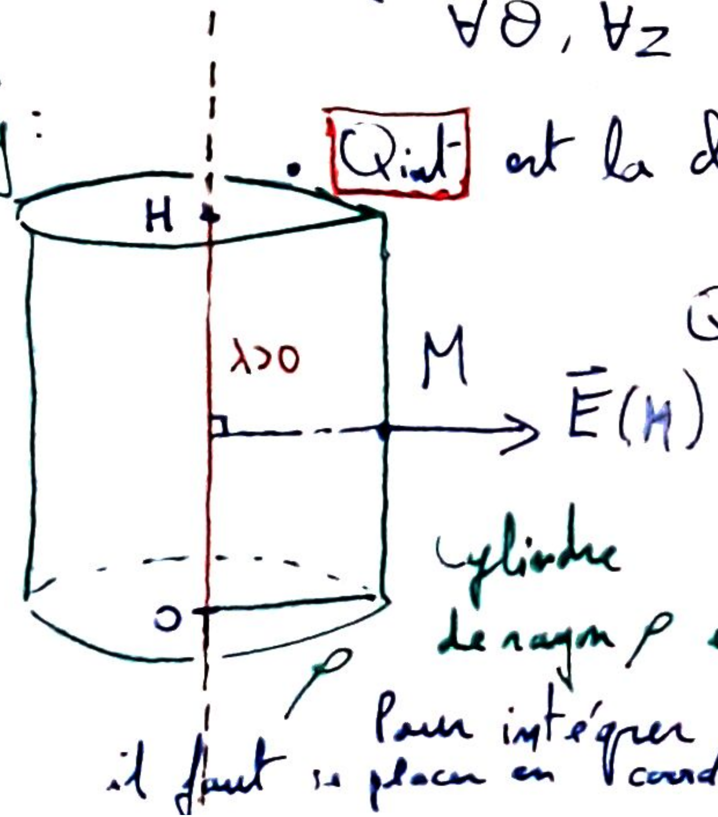
2./ <u>Théorème de Gauss :</u>

$$\oiint_{\Sigma_g} \vec{E}(M) \cdot \vec{dS} = \frac{Q_{int}}{\varepsilon_0}$$; <u>Il faut choisir la surface de Gauss Σ_g</u>

telle que : • $M \in \Sigma_g$ • Σ_g surface fermée.

• Σ_g doit conserver la symétrie du problème $\forall \theta, \forall z$ pour un $M(\rho)$ donné.

Σ_g :



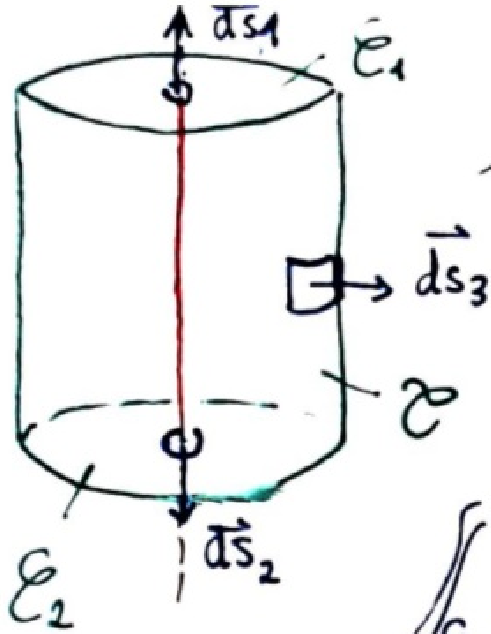
• $\boxed{Q_{int}}$ est la charge incluse dans Σ_g

$$Q_{int} = \int_0^H \lambda \, dz = \lambda [z]_0^H$$

$$\underline{Q_{int} = \lambda H}$$

cylindre de rayon ρ et de hauteur H.

Pour intégrer sur la surface de Gauss Σ_g il faut se placer en coordonnées cylindriques.

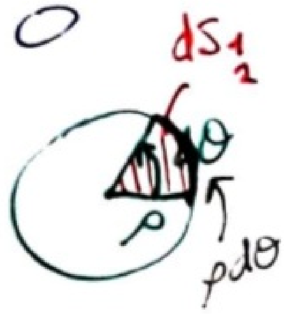


$$\oiint_{\Sigma_g} \vec{E}\cdot\overrightarrow{dS} = \iint_{\mathcal{E}_1} \vec{E}\cdot\overrightarrow{dS_1} + \iint_{\mathcal{E}_2} \vec{E}\cdot\overrightarrow{dS_2}$$

$$+ \iint_{\mathcal{C}} \vec{E}\cdot\overrightarrow{dS_3}$$

$\overrightarrow{dS_1} = dS_1 \cdot \vec{e_z}$

$$\iint_{\mathcal{E}_1} E(\rho)\underbrace{\vec{e_\rho}\,\overbrace{d\rho\,\rho\,d\theta}^{dS_1}\,\vec{e_z}}_{=0} = 0$$



$$\iint_{\mathcal{E}_2} E(\rho)\underbrace{\vec{e_\rho}\,\overbrace{d\rho\,\rho\,d\theta}^{dS_2}\,(-\vec{e_z})}_{=0} = 0$$

$$\iint_{\mathcal{C}} E(\rho)\underbrace{\vec{e_\rho}\cdot\overbrace{\rho\,d\theta\,dz}^{dS_3}\,\vec{e_\rho}}_{=1} = E(\rho)\cdot\rho\int_0^{2\pi} d\theta \int_0^{H} dz$$

$$= 2\pi H \cdot E(\rho) \quad \text{ainsi :}$$

$$\oiint_{\Sigma_g} \vec{E}\cdot\overrightarrow{dS} = \frac{Q_{int}}{\varepsilon_0} \Leftrightarrow 2\pi \cancel{H}\rho E(\rho) = \frac{\lambda \cancel{H}}{\varepsilon_0} \quad \text{d'où}$$

$$E(\rho) = \frac{\lambda}{2\pi\varepsilon_0\rho} \quad \text{et} \quad \boxed{\vec{E}(M) = \frac{\lambda}{2\pi\varepsilon_0\rho}\vec{e_\rho}}$$

Exercice 2 : $(r = \rho)$

1./ En coordonnées cylindriques :

$$\vec{E}(M_1) = E(\rho, \cancel{\theta}, \cancel{z})(-\vec{e_\rho}) = -E(\rho)\vec{e_\rho}$$

$$\vec{E}(M_2) = E(\rho, \cancel{\theta}, \cancel{z})\vec{e_\rho} = E(\rho)\vec{e_\rho}$$

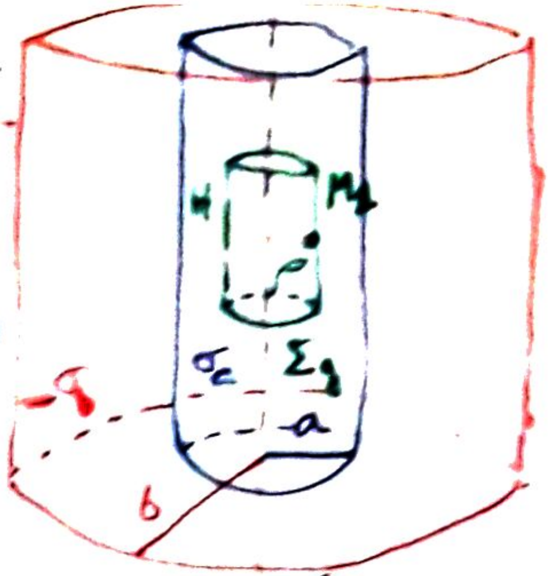
$$\vec{E}(M_3) = E(\rho, \cancel{\theta}, \cancel{z})(-\vec{e_\rho}) = -E(\rho)\vec{e_\rho}$$

2) • $\rho < a$

$Q = 2\pi a l \sigma_a$

$-Q = 2\pi b l (-\sigma_b)$

$(\sigma_a > 0, \sigma_b > 0)$



$Q_{int} = 0\ C$

Il n'y a pas de charge dans Σ_g.

$$\oiint_{\Sigma_g} \vec{E} \cdot \vec{dS} = 0$$

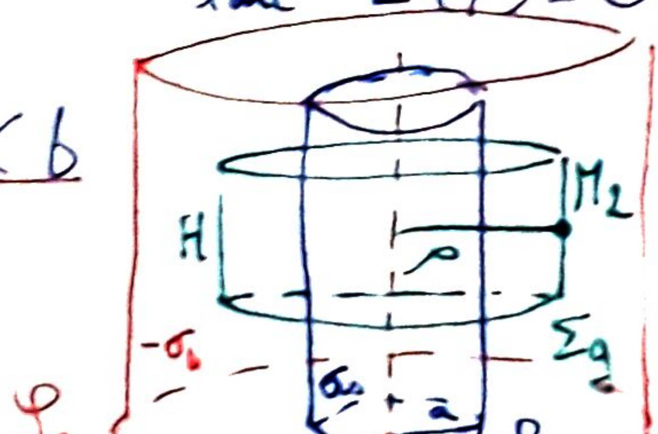
$$\iint_{\Sigma_1} E(\rho)\cancel{(-\vec{e}_\rho) \cdot dS\,\vec{e}_z} + \iint_{\Sigma_2} E(\rho)\cancel{(-\vec{e}_\rho) \cdot dS(-\vec{e}_z)}$$

$$+ \iint_{\Sigma} E(\rho)(-\vec{e}_\rho)\,\rho\, d\theta\, dz\, \vec{e}_\rho = 0 \Rightarrow$$

$$-\rho E(\rho) \int_0^{2\pi} d\theta \int_0^H dz = 0 \Rightarrow -2\pi H \rho E(\rho) = 0$$

donc $E(\rho) = 0$ et $\boxed{\vec{E}(M_1) = \vec{0}}$

• $a < \rho < b$



$Q_{int} = \iint_{v(\Sigma_g) \cap S_1} \sigma_a\, dS$

$Q_{int} = \iint_{\theta\, z} \sigma_a\, a\, d\theta\, dz$

$$Q_{int} = a \cdot \sigma_a \int_0^{2\pi} d\theta \int_0^H dz = 2\pi H a \sigma_a = \frac{\cancel{2\pi} H \cancel{a} Q}{\cancel{2\pi a} l}$$

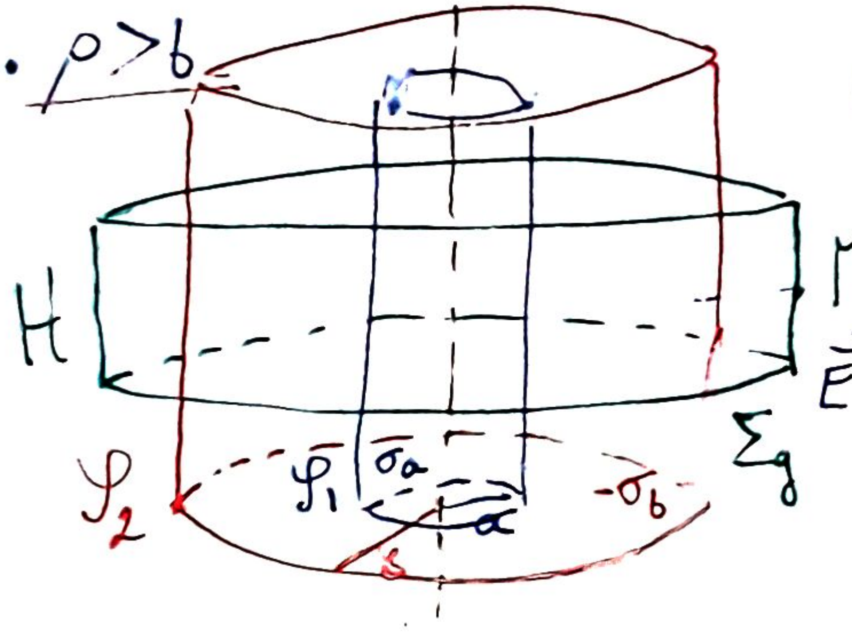
$Q_{int} = \dfrac{HQ}{l}$; $\oiint_{\Sigma_g} \vec{E} \cdot \vec{dS} = \iint E(\rho)\vec{e}_\rho\, dS \cdot \vec{e}_\rho$

$$= \iint_{\theta\, z} E(\rho)\,\rho\, d\theta\, dz = E(\rho)\,\rho \int_0^{\pi} d\theta \int_0^H dz = 2\pi \rho H E$$

d'où $\oiint_{\Sigma_g} \vec{E}\cdot\vec{ds} = \frac{Q_{int}}{\varepsilon_0} \Rightarrow 2\pi\rho H E(\rho) = \frac{HQ}{\varepsilon_0 \ell}$

$$\boxed{\vec{E}(\rho) = \frac{Q}{2\pi\varepsilon_0 \ell}\cdot\frac{1}{\rho}\,\vec{e_\rho}}$$

• $\rho > b$



$Q_{int} = \iint_{S(\Sigma_g)\cap\varphi_1} \sigma_a\, dS + \iint_{S(\Sigma_g)\cap\varphi_2} -\sigma_b\, dS$

$Q_{int} = 2\pi a H \sigma_a + 2\pi b H(-\sigma_b)$

$Q_{int} = \frac{2\pi a H Q}{2\pi a \ell} - \frac{2\pi b H Q}{2\pi b \ell} \Rightarrow Q_{int} = \frac{HQ}{\ell} - \frac{HQ}{\ell} = 0$

$Q_{int} = 0 \Rightarrow \boxed{\vec{E}(M_3) = \vec{0}}$

3/ $\vec{E} = -\vec{grad}\, V$; $\vec{grad}\, V = \vec{\nabla} V = \frac{\partial V}{\partial \rho}\vec{e_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \theta}\vec{u_\theta} + \frac{\partial V}{\partial z}\vec{e_z}$

si $\vec{E} = E(\rho)\vec{e_\rho} \Rightarrow V = V(\rho)$ donc $\vec{\nabla} V = \frac{\partial V}{\partial \rho}\vec{e_\rho}$

• $\rho < a$ $\quad \vec{E}(\rho<a) = E(\rho<a)\vec{e_\rho} = -\frac{\partial V(\rho)}{\partial \rho}\vec{e_\rho}$

d'où $E(\rho<a) = -\frac{dV(\rho<a)}{d\rho} \Rightarrow V(\rho<a) = -\int E(\rho<a)\, d\rho$

$V(\rho<a) = -\int 0\, d\rho \Rightarrow \underline{V(\rho<a) = \alpha}$

• $a < \rho < b$ $\quad V(a<\rho<b) = -\int \frac{Q}{2\pi\varepsilon_0 l}\frac{d\rho}{\rho} = \underbrace{-\frac{Q}{2\pi\varepsilon_0 l}\ln\rho + \beta}$

• $\rho > b$: $V(\rho>b) = -\int 0\, d\rho \Rightarrow \underline{V(\rho>b) = \gamma}$

En $\rho = a$ il y a "continuité" du champ et du potentiel :

$$\underset{\rho<a}{V(a)} = \underset{(a<\rho<b)}{V(a)} \Rightarrow \alpha = \frac{-Q}{2\pi\varepsilon_0 l}\ln a + \beta$$

$$\underset{a<\rho<b}{V(b)} = \underset{\rho>b}{V(b)} \Rightarrow \gamma = \frac{-Q}{2\pi\varepsilon_0 l}\ln b + \beta$$

$$\lim_{\rho\to+\infty} V(\rho>b) = 0 \Rightarrow \boxed{\gamma = 0} \text{ et } \boxed{\beta = \frac{Q}{2\pi\varepsilon_0 l}\ln b}$$

D'où $\boxed{\alpha = \frac{Q}{2\pi\varepsilon_0 l}\ln\left(\frac{b}{a}\right)}$ On déduit ainsi complètement $V(\rho)$:

$$\boxed{V(\rho) = \begin{cases} \frac{Q}{2\pi\varepsilon_0 l}\ln\left(\frac{b}{a}\right) & \text{si } \rho \leq a \\ \frac{Q}{2\pi\varepsilon_0 l}\ln\frac{b}{\rho} & \text{si } a \leq \rho \leq b \\ 0 & \text{si } \rho \geq b \end{cases}}$$

4./ $C = \dfrac{Q}{V(a)-V(b)} = \dfrac{\cancel{Q}}{\frac{\cancel{Q}}{2\pi\varepsilon_0 l}\ln\left(\frac{b}{a}\right) - 0}$

$$\boxed{C = \frac{2\pi\varepsilon_0 l}{\ln(b/a)}}$$

Exercice 3

1./ $\vec{E} = E(r;\cancel{\theta},\cancel{\varphi})\,\vec{e_r} \Rightarrow \boxed{\vec{E} = E(r)\,\vec{e_r}}$

2./ • $r<R$

$Q_{int} = 0$ donc

$$\boxed{\vec{E}(r<R) = \vec{0}}$$

• $r>R$

$Q_{int} = \iint_{V(\Sigma_g)\cap\mathcal{S}} \sigma\, dS$

$Q_{int} = \sigma . S$

$Q_{int} = 4\pi R^2 \sigma$

$$\oiint_{\Sigma_g} \vec{E}\cdot\vec{dS} = \oiint_{\Sigma_g} E(r)\vec{e_r}\cdot\underbrace{\vec{e_r}\,dS}_{\vec{dS}} = E(r)\times S_{\Sigma_g} = 4\pi r^2 E$$

d'où $\oiint_{\Sigma_g} \vec{E}\cdot\vec{dS} = \dfrac{Q_{int}}{\varepsilon_0} \Rightarrow \cancel{4\pi} r^2 E(r) = \cancel{4\pi} R^2 \sigma/\varepsilon_0$

$$\boxed{\vec{E}(r>R) = \frac{R^2\sigma}{r^2\varepsilon_0}\cdot\vec{e_r}}$$

3./ $\vec{E} = -\vec{\text{grad}}\,V$; $\vec{\text{grad}}\,V = \vec{\nabla}V = \begin{pmatrix} \partial V/\partial r \\ \cancel{\frac{1}{r}\frac{\partial V}{\partial \theta}} \\ \cancel{\frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}} \end{pmatrix}$

or $\vec{E} = E(r)\vec{e_r} \Rightarrow V = V(r,\cancel{\theta},\cancel{\varphi})$

$$E(r)\vec{e_r} = -\frac{\partial V}{\partial r}\vec{e_r} \Rightarrow E(r) = -\frac{dV(r)}{dr}$$ d'où

$$V(r) = -\int E(r)\,dr$$ ainsi :

- $V(r<R) = -\int 0\,dr \Rightarrow \underline{V(r<R) = \alpha}$
- $V(r>R) = -\int \frac{R^2}{\varepsilon_0 r^2}\sigma\,dr \Rightarrow V(r>R) = -\frac{R^2\sigma}{\varepsilon_0}\int\frac{dr}{r^2}$

$$\underline{V(r>R) = \frac{R^2\sigma}{\varepsilon_0 r} + \beta}$$

On trouve les constantes par continuité en $r=R$:

$$\alpha = \frac{R^2\sigma}{\varepsilon_0 R} + \beta \Rightarrow \alpha = \frac{R\sigma}{\varepsilon_0} + \beta$$

de plus : $\lim\limits_{r\to+\infty} V(r>R) = 0 \Rightarrow \underline{\beta = 0}$ d'où

$$\boxed{V(r\leq R) = \frac{R\sigma}{\varepsilon_0} \quad \text{et} \quad V(r\geq R) = \frac{R^2\sigma}{\varepsilon_0 r}}$$

4./ 1./ $\boxed{\vec{E} = E(r)\,\vec{e_r}}$

1.2) $\underline{\cdot\ r < R}$

$$Q_{int} = \iiint_{v(\Sigma_g)\cap v} \rho\, dV = \rho\, v(\Sigma_g)$$

$$\underline{Q_{int} = \frac{4}{3}\pi r^3 \rho}$$

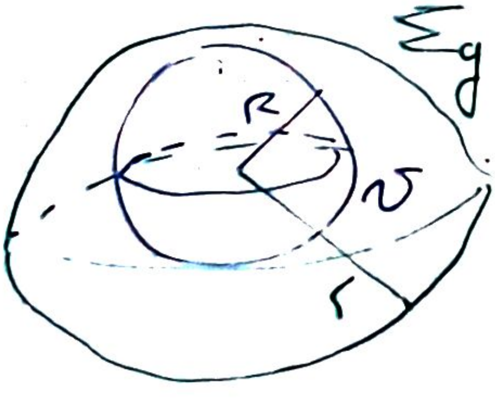
$$\oint_{\Sigma_g} \vec{E}\cdot\vec{dS} = \frac{Q_{int}}{\varepsilon_0} \Rightarrow \oint_{\Sigma_g} E(r)\,\underbrace{\vec{e_r}\cdot\vec{e_r}}_{=1}\,dS = \frac{4\pi\rho r^3}{3\varepsilon_0}$$

$$E(r)\times S_{\Sigma_g} = \frac{4\pi\rho r^3}{3\varepsilon_0} \Rightarrow E(r)\times 4\pi r^2 = \frac{4\pi\rho r^3}{3\varepsilon_0}$$

$$\boxed{\vec{E}(r<R) = \frac{\rho r}{3\varepsilon_0}\,\vec{e_r}}$$

$\underline{\cdot\ r > R}$

$$Q_{int} = \iiint_{v} \rho\, dV = \frac{4}{3}\pi R^3 \rho$$



$$\oint_{\Sigma_g} \vec{E}\cdot\vec{dS} = \oint_{\Sigma_g} E(r)\,\vec{e_r}\cdot\vec{e_r}\,dS$$

$$= E(r)\times 4\pi r^2 \quad \text{d'où}$$

$$\oint_{\Sigma_g} \vec{E}\cdot\vec{dS} = \frac{Q_{int}}{\varepsilon_0} \Rightarrow E(r)\,4\pi r^2 = \frac{4}{3\varepsilon_0}\pi R^3\rho \Rightarrow E(r) = \frac{R^3\rho}{[illegible]}\,\frac{1}{[illegible]}$$

$$\boxed{\vec{E}(r>R) = \frac{R^3 . \rho}{3\varepsilon_0} \frac{1}{r^2} \vec{e_r}}$$

4.3) $V(r<R) = -\int \frac{\rho r}{3\varepsilon_0} dr = -\frac{\rho}{6\varepsilon_0} . r^2 + \alpha$

$$V(r>R) = -\int \frac{R^3 \rho}{3\varepsilon_0} \frac{1}{r^2} = \frac{R^3 \rho}{3\varepsilon_0} \frac{1}{r} + \beta$$

$\lim\limits_{r \to +\infty} V(r>R) = 0 \Rightarrow \beta = 0$

Par continuité en $r = R$: $-\frac{\rho}{6\varepsilon_0} R^2 + \alpha = \frac{R^3 \rho}{3\varepsilon_0} \frac{1}{R}$

$\alpha = \frac{R^2 \rho}{3\varepsilon_0} + \frac{R^2 \rho}{6\varepsilon_0} \Rightarrow \alpha = \frac{R^2 \rho}{2\varepsilon_0}$ d'où

$$\boxed{V(r) = \begin{cases} -\frac{\rho}{6\varepsilon_0} \cdot r + \frac{R^2 \rho}{2\varepsilon_0} & \text{si } r \leq R \\ \frac{R^3 \rho}{3\varepsilon_0} \cdot \frac{1}{r} & \text{si } r \geq R \end{cases}}$$