Contents

Part I

Kinematics and dynamics

1

Coordinate systems

Contents

1 Cartesian coordinates

1.1 Basis

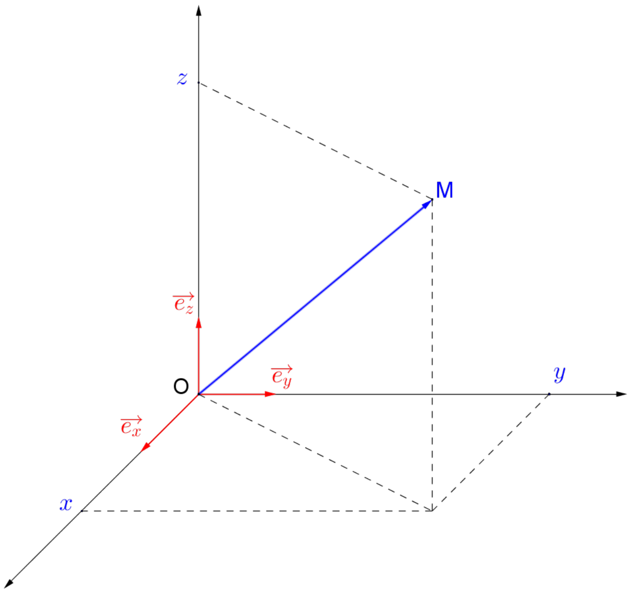
In cartesian coordinates, the basis vectors are $(\overrightarrow{e_x};\overrightarrow{e_y};\overrightarrow{e_z})$ and any vector $\overrightarrow{a}$ can be written as

$$\overrightarrow{a} = a_x\overrightarrow{e_x} + a_y\overrightarrow{e_y} + a_z\overrightarrow{e_z} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

and the coordinates of point M are

$$\overrightarrow{OM} = x\overrightarrow{e_x} + y\overrightarrow{e_y} + z\overrightarrow{e_z} \qquad M(x;y;z)$$

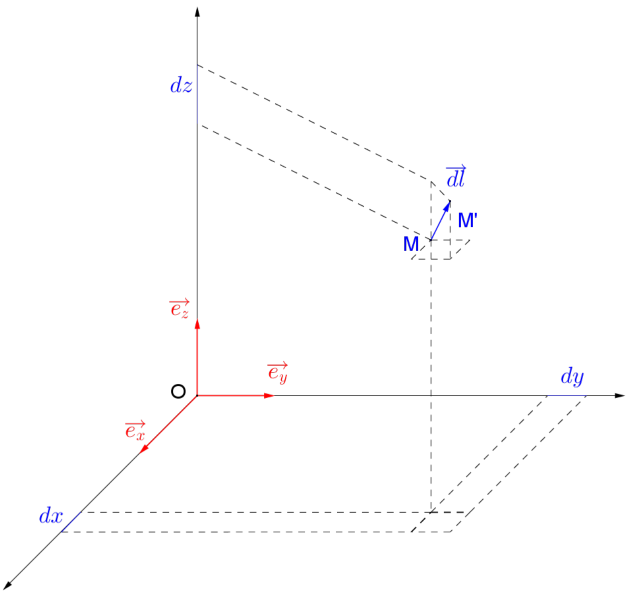
Variables are $(x;y;z)$, thus a function is generally expressed as $f(x,y,z)$.



1.2 Length element

When M moves to a very close M' point, the infinitesimal displacement is

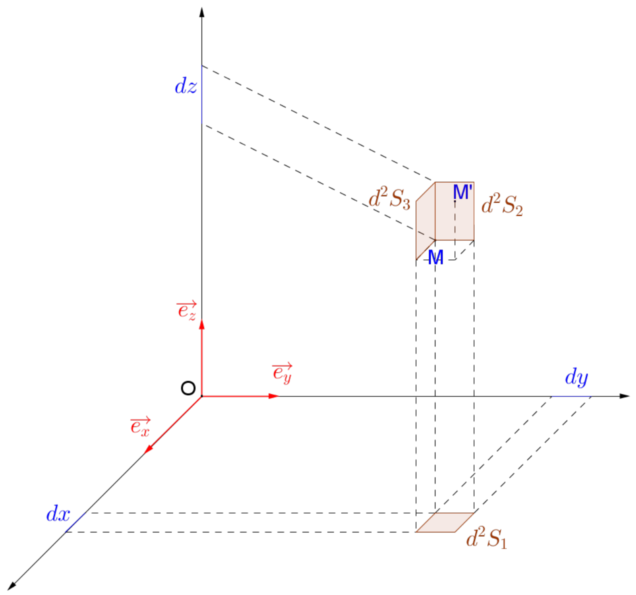
$$\overrightarrow{dl} = dx\overrightarrow{e_x} + dy\overrightarrow{e_y} + dz\overrightarrow{e_z} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$



1.3 Surface element

Infinitesimal surface elements d^2S are

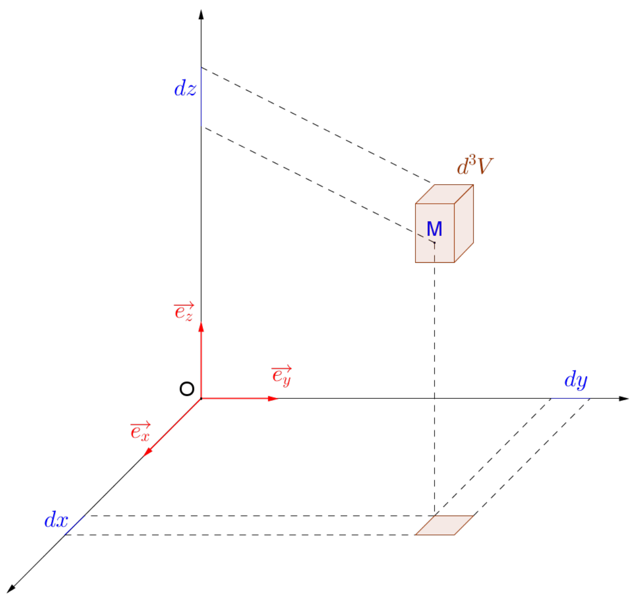
- in the (xOy) plane:
  $d^2S_1 = dx\,dy$;
- in the (yOz) plane:
  $d^2S_2 = dy\,dz$;
- in the (xOz) plane:
  $d^2S_3 = dx\,dz$.



1.4 Volume element

The infinitesimal volume element d^3V is

$$d^3V = dx\,dy\,dz$$



2 Polar coordinates

2.1 Basis

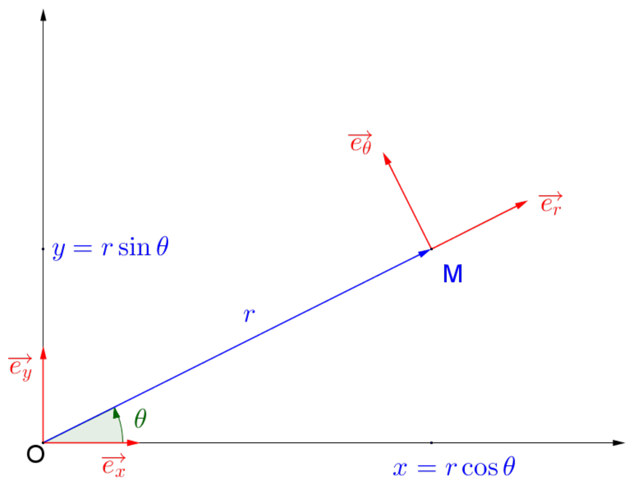
In polar coordinates, the basis vectors are $(\vec{e_r};\vec{e_\theta})$ with $\vec{e_r}$ a unitary radial vector (along the $\overrightarrow{OM}$ direction) and $\vec{e_\theta}$ a unitary tangential vector oriented by θ. Any vector $\vec{a}$ can be written as

$$\vec{a} = a_r\vec{e_r} + a_\theta\vec{e_\theta}$$

and the position of M is

$$\overrightarrow{OM} = r\vec{e_r} \qquad M(r;\theta)$$

Variables are $(r;\theta)$, thus a function is generally expressed as $f(r,\theta)$.

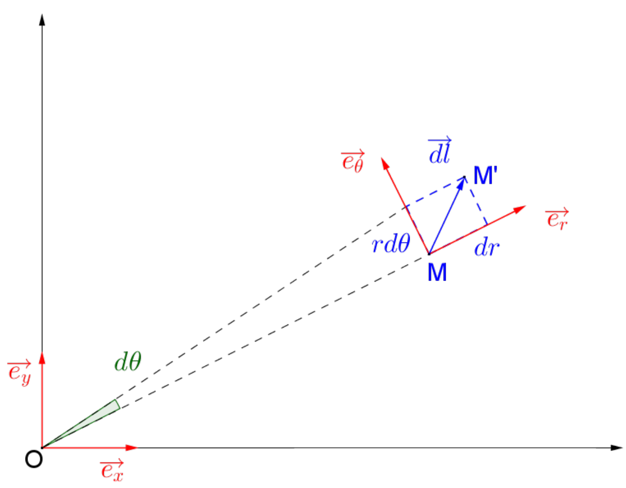


2.2 Length element

When M moves to a very close M' point, the infinitesimal displacement is

$$\vec{dl} = dr\vec{e_r} + r\,d\theta\vec{e_\theta} = \begin{pmatrix} dr \\ r\,d\theta \end{pmatrix}$$

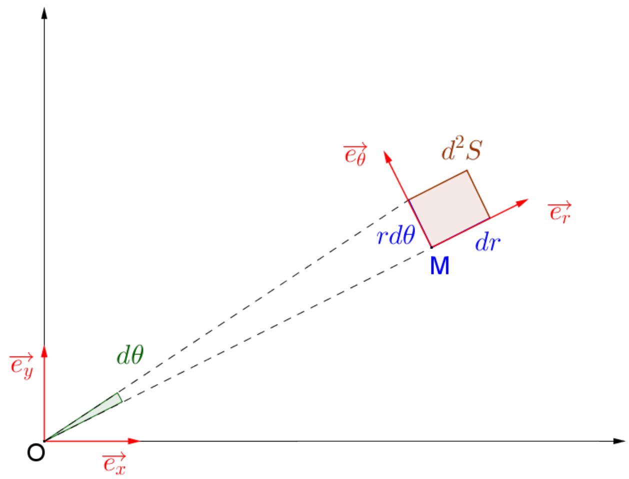
$r\,d\theta$ being the length of the small arc of radius r and intercepting the infinitesimal angle $d\theta$.



2.3 Surface element

Infinitesimal surface element d^2S is

$$d^2S = r\,dr\,d\theta$$



3 Cylindrical coordinates

3.1 Basis

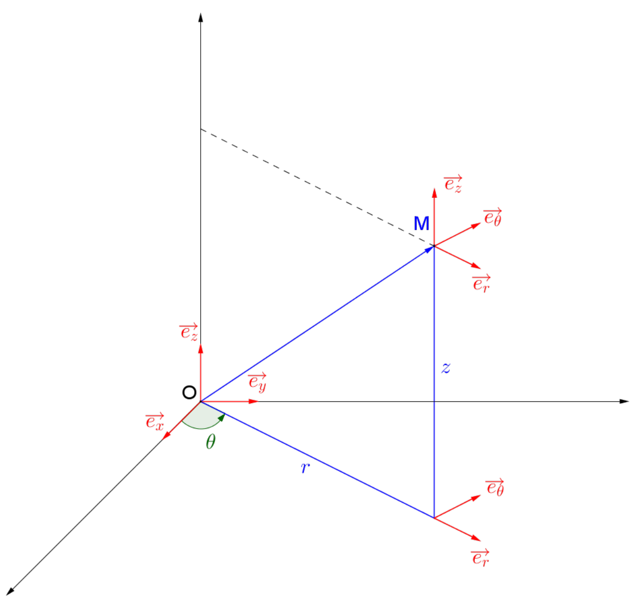
In cylindrical coordinates, the basis vectors are $(\overrightarrow{e_r};\overrightarrow{e_\theta};\overrightarrow{e_z})$, where $\overrightarrow{e_r}$ and $\overrightarrow{e_\theta}$ are the same as in polar coordinates. Any vector $\overrightarrow{a}$ can be written as

$$\overrightarrow{a} = a_r\overrightarrow{e_r} + a_\theta\overrightarrow{e_\theta} + a_z\overrightarrow{e_z} = \begin{pmatrix} a_r \\ a_\theta \\ a_z \end{pmatrix}$$

and the coordinates of point M are

$$\overrightarrow{OM} = r\overrightarrow{e_r} + z\overrightarrow{e_z} \qquad M(r;\theta;z)$$

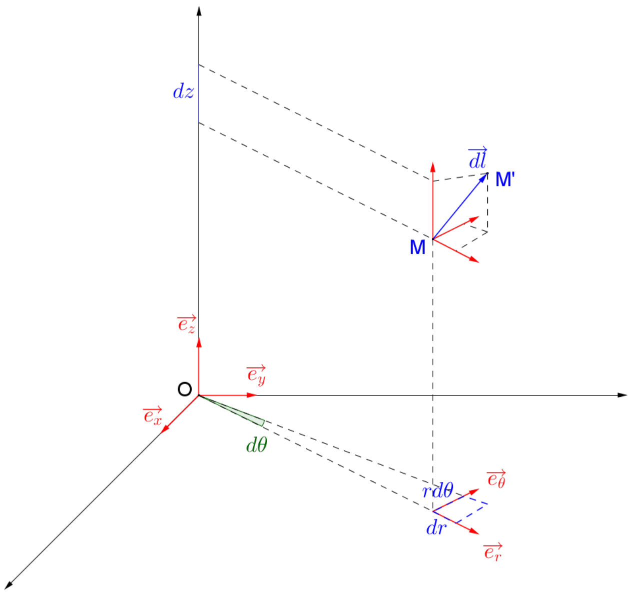
Variables are $(r;\theta;z)$, thus a function is generally expressed as $f(r,\theta,z)$.



3.2 Length element

When M moves to a very close M' point, the infinitesimal displacement is

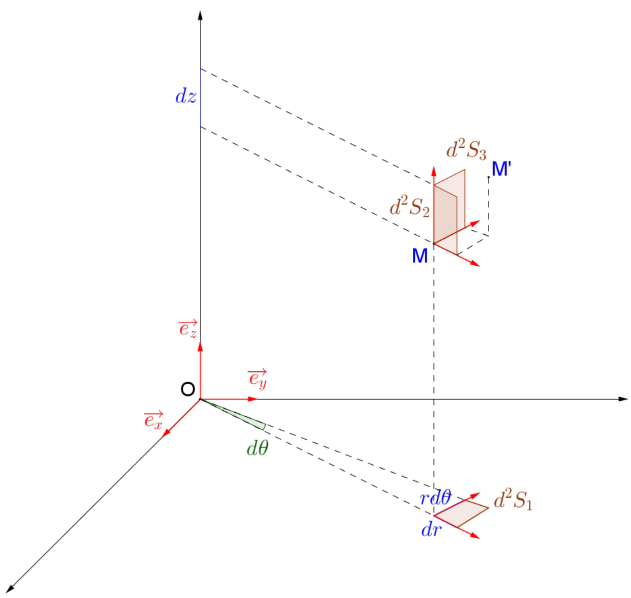
$$\overrightarrow{dl} = dr\overrightarrow{e_r} + r\,d\theta\overrightarrow{e_\theta} + dz\overrightarrow{e_z} = \begin{pmatrix} dr \\ r\,d\theta \\ dz \end{pmatrix}$$



3.3 Surface element

Infinitesimal surface elements d^2S are

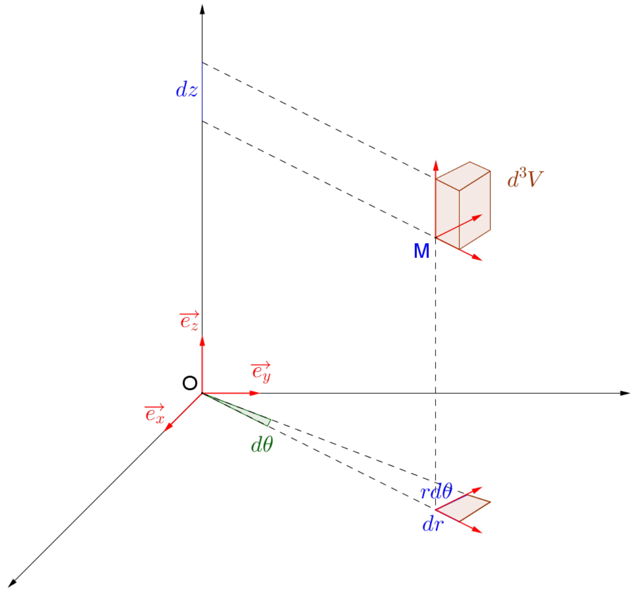
- in the (xOy) plane:
  $d^2S_1 = r\,dr\,d\theta$;
- in the $(M, \overrightarrow{e_r}, \overrightarrow{e_z})$ plane:
  $d^2S_2 = dr\,dz$;
- in the $(M, \overrightarrow{e_\theta}, \overrightarrow{e_z})$ plane:
  $d^2S_3 = r\,d\theta\,dz$.



3.4 Volume element

The infinitesimal volume element d^3V is

$$d^3V = r\,dr\,d\theta\,dz$$



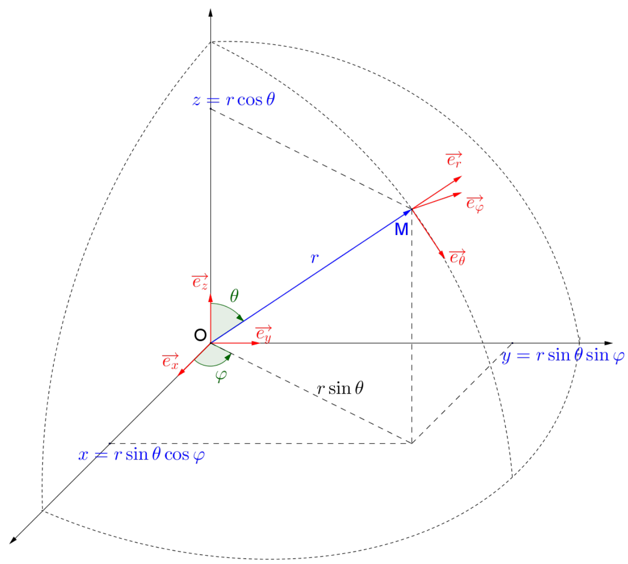
4 Spherical coordinates

4.1 Basis

In spherical coordinates, the basis vectors are $(\overrightarrow{e_r};\overrightarrow{e_\theta};\overrightarrow{e_\varphi})$, where

- $\overrightarrow{e_r}$ is along the $\overrightarrow{OM}$ direction (pointing outwards);
- $\overrightarrow{e_\theta}$ is in the $(OM,\overrightarrow{e_z})$ plane and is orthogonal to $\overrightarrow{e_r}$;
- $\overrightarrow{e_\varphi}$ is such as $(\overrightarrow{e_r};\overrightarrow{e_\theta};\overrightarrow{e_\varphi})$ is a direct trihedron[a].

Warning: $\overrightarrow{e_r}$ and $\overrightarrow{e_\theta}$ are *not the same* as in spherical coordinates.



[a] This means that $(\overrightarrow{e_r};\overrightarrow{e_\theta};\overrightarrow{e_\varphi})$, in this order, correspond to the first three fingers of your *right-hand*.

Any vector $\overrightarrow{a}$ can be written as

$$\overrightarrow{a} = a_r\overrightarrow{e_r} + a_\theta\overrightarrow{e_\theta} + a_\varphi\overrightarrow{e_\varphi} = \begin{pmatrix} a_r \\ a_\theta \\ a_\varphi \end{pmatrix}$$

and the coordinates of point M are

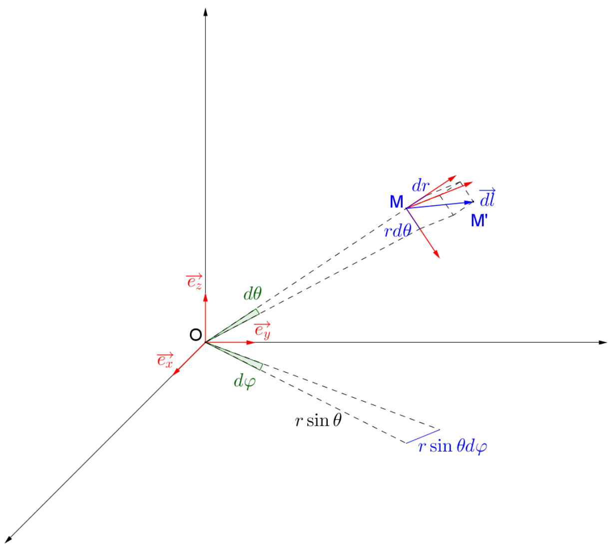
$$\overrightarrow{OM} = r\overrightarrow{e_r} \qquad M(r;\theta;\varphi)$$

Variables are $(r;\theta;\varphi)$, thus a function is generally expressed as $f(r,\theta,\varphi)$.

4.2 Length element

When M moves to a very close M' point, the infinitesimal displacement is

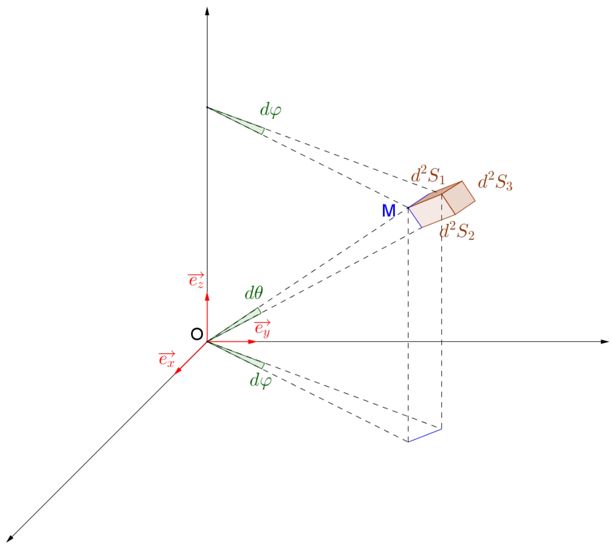
$$\overrightarrow{dl} = dr\overrightarrow{e_r} + r\,d\theta\overrightarrow{e_\theta} + r\sin\theta\;d\varphi\overrightarrow{e_\varphi}$$
$$= \begin{pmatrix} dr \\ r\,d\theta \\ r\sin\theta\;d\varphi \end{pmatrix}$$



4.3 Surface element

Infinitesimal surface elements d^2S are

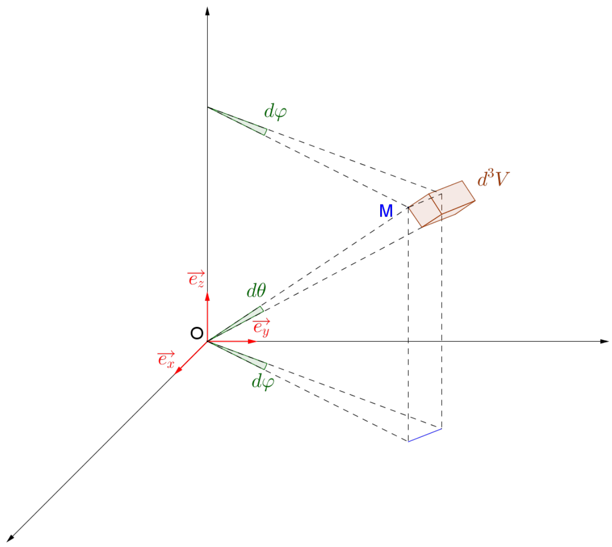
- in the $(M, \overrightarrow{e_r}, \overrightarrow{e_\varphi})$ plane: $d^2S_1 = r \sin\theta \; dr \; d\varphi$;
- in the $(M, \overrightarrow{e_\theta}, \overrightarrow{e_\varphi})$ plane: $d^2S_2 = r^2 \sin\theta \; d\theta \; d\phi$;
- in the $(M, \overrightarrow{e_r}, \overrightarrow{e_\theta})$ plane: $d^2S_3 = r \; dr \; d\theta$.



4.4 Volume element

The infinitesimal volume element d^3V is

$$d^3V = r^2 \sin\theta \; dr \; d\theta \; d\varphi$$



2 Kinematics

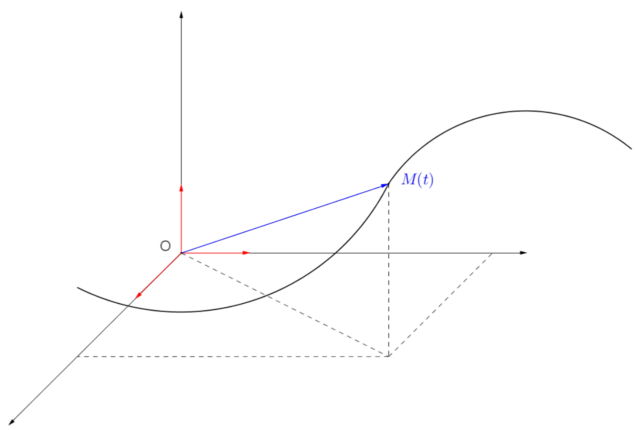
Contents

1 What kinematics deals with…

Kinematics is the description of motion in a given frame of reference (heliocentric, geocentric…), putting aside the study of its causes. In kinematics, we obtain parametric equations of motion giving the expressions of acceleration, speed and position as functions of time. From there, we can establish the equation for the trajectory.

2 Position

The position of point M in a given frame of reference is given by the position vector $\overrightarrow{OM}$, where O is the origin of the frame.
Position vector can be expressed in any coordinate system: cartesian, spherical… but we have to keep in mind that, while cartesian coordinates use *fixed* basis vectors – $\overrightarrow{e_x}, \overrightarrow{e_y}, \overrightarrow{e_z}$ point to fixed-directions – other coordinate systems use *moving* basis vectors – tied to the point M.
The trajectory of M is defined as the set of all positions occupied by M – the path followed by M.



In cartesian coordinates for instance:

- equations of motion are the components of $\overrightarrow{OM}(t)$

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

- the trajectory is an equation relating x, y and z, with no mention to time t

$$f(x,y,z) = 0$$

In polar coordinates, equations of motions and trajectory are

$$\begin{cases} r = r(t) \\ \theta = \theta(t) \end{cases} \qquad \text{and} \qquad f(r,\theta) = 0$$

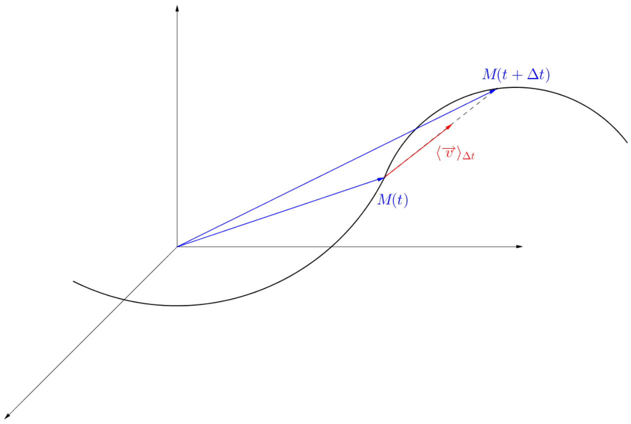
3 Velocity

3.1 Average velocity

Average speed of M is defined as the distance travelled by M divided by the time interval:

$$\begin{aligned} \langle \vec{v}(t) \rangle_{\Delta t} &= \frac{\overrightarrow{M(t)M(t+\Delta t)}}{\Delta t} \\ &= \frac{\overrightarrow{OM}(t+\Delta t) - \overrightarrow{OM}(t)}{\Delta t} \end{aligned}$$

with v in m.s^{-1}



3.2 Instantaneous velocity

Instantaneous velocity is the limit of the former expression when $\Delta t \to 0$, which mathematically corresponds to the derivative

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} \qquad \text{with } v \text{ in m.s}^{-1}$$

This vector is always tangent to the path.

3.3 Velocity in cartesian coordinates

In cartesian coordinates we have

$$\begin{aligned} \vec{v} &= \frac{d}{dt}(x\vec{e_x} + y\vec{e_y} + z\vec{e_z}) \\ &= \frac{dx}{dt}\vec{e_x} + x\frac{d\vec{e_x}}{dt} + \frac{dy}{dt}\vec{e_y} + y\frac{d\vec{e_y}}{dt} + \frac{dz}{dt}\vec{e_z} + z\frac{d\vec{e_z}}{dt} \\ &= \dot{x}\vec{e_x} + \dot{y}\vec{e_y} + \dot{z}\vec{e_z} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \end{aligned}$$

since basis vectors are independent of time – a dot corresponding to a time-derivative.

3.4 Time derivative of a rotating vector

In some coordinate systems, basis vectors are time-dependend: they rotated together with the M point they are tied to. Time-derivative of these vectors has then to be taken into account.

Consider for instance the polar coordinates basis $(\overrightarrow{e_r};\overrightarrow{e_\theta})$. We can decompose these vectors on the cartesian basis vectors

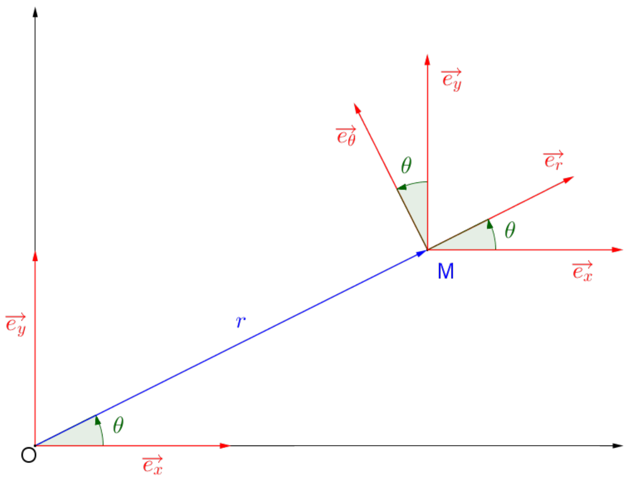
$$\overrightarrow{e_r} = \cos\theta\,\overrightarrow{e_x} + \sin\theta\,\overrightarrow{e_y}$$
$$\overrightarrow{e_\theta} = -\sin\theta\,\overrightarrow{e_x} + \cos\theta\,\overrightarrow{e_y}$$

and then

$$\begin{aligned}\frac{d}{dt}\overrightarrow{e_r} &= \frac{d}{dt}(\cos\theta\,\overrightarrow{e_x} + \sin\theta\,\overrightarrow{e_y}) \\ &= -\dot\theta\sin\theta\,\overrightarrow{e_x} + \dot\theta\cos\theta\,\overrightarrow{e_y} \\ &= \dot\theta\,\overrightarrow{e_\theta}\end{aligned}$$

Similarly, we get

$$\frac{d}{dt}\overrightarrow{e_\theta} = -\dot\theta\,\overrightarrow{e_r}$$



3.5 Velocity in cylindrical coordinates

Using the derivative of a rotating vector, velocity in cylindrical coordinates is

$$\begin{aligned}\overrightarrow{v} &= \frac{d}{dt}(r\overrightarrow{e_r} + z\overrightarrow{e_z}) = \frac{dr}{dt}\overrightarrow{e_r} + r\frac{d\overrightarrow{e_r}}{dt} + \frac{dz}{dt}\overrightarrow{e_z} \\ &= \dot r\overrightarrow{e_r} + r\dot\theta\overrightarrow{e_\theta} + \frac{dz}{dt}\overrightarrow{e_z} = \begin{pmatrix}\dot r\\ r\dot\theta\\ \dot z\end{pmatrix}\end{aligned}$$

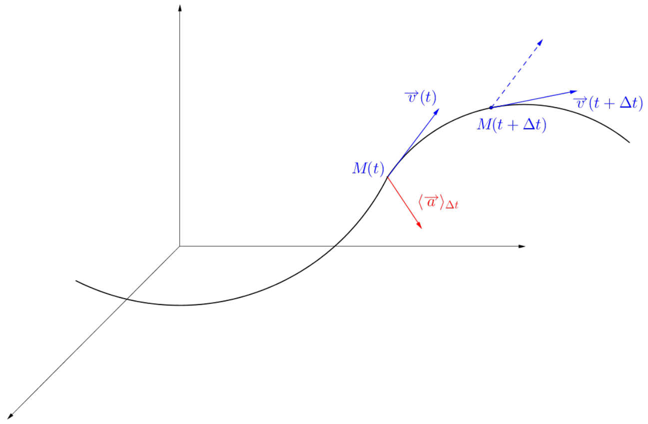
4 Acceleration

4.1 Average acceleration

Average acceleration of M is defined as the variation of its velocity divided by the time interval:

$$\langle \overrightarrow{a}(t) \rangle_{\Delta t} = \frac{\overrightarrow{v}(t+\Delta t) - \overrightarrow{v}(t)}{\Delta t}$$

with a in m.s^2



4.2 Instantaneous acceleration

In the limit of an infinitely small time interval, the instantaneous acceleration is

$$\overrightarrow{a}(t) = \frac{d\overrightarrow{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} \qquad \text{with } a \text{ in m.s}^{-2}$$

Here again, the complexity of the expression depends on the coordinate system.

4.3 Acceleration in cartesian coordinates

$$\overrightarrow{a} = \ddot{x}\overrightarrow{e_x} + \ddot{y}\overrightarrow{e_y} + \ddot{z}\overrightarrow{e_z} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}$$

4.4 Acceleration in cylindrical coordinates

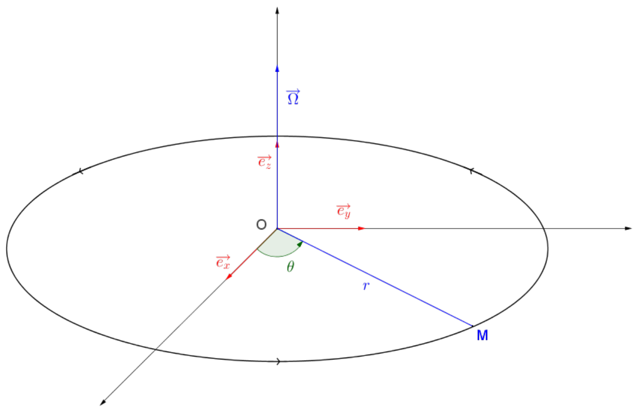
$$\begin{aligned} \overrightarrow{a} &= \frac{d}{dt}(\dot{r}\overrightarrow{e_r} + r\dot{\theta}\overrightarrow{e_\theta} + \dot{z}\overrightarrow{e_z}) \\ &= (\ddot{r} - r\dot{\theta}^2)\overrightarrow{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\overrightarrow{e_\theta} + \ddot{z}\overrightarrow{e_z} = \begin{pmatrix} \ddot{r} - r\dot{\theta}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} \\ \ddot{z} \end{pmatrix} \end{aligned}$$

5 Circular motion and Frenet-Serret coordinates

5.1 Position

Consider the circular motion of M around a (Δ) axis. The (Oz) axis is chosen so that it matches (Δ) and the motion lies in the (xOy) plane. M is then spotted by its polar coordinates $(r;\theta)$, with $r = cst$[a].

[a] The orientation of $\overrightarrow{e_z}$ and the sign of θ follow the "right-hand rule": if $\overrightarrow{e_z}$ is oriented as your right-thumb, your other fingers indicate the direction of the positive θ's.



5.2 Angular velocity

In addition to the usual velocity, we define the angular-speed vector

$$\overrightarrow{\Omega} = \dot{\theta}\overrightarrow{e_z}$$

where $\dot{\theta}$ is the angular speed in rad.s^{-1}.

5.3 Acceleration

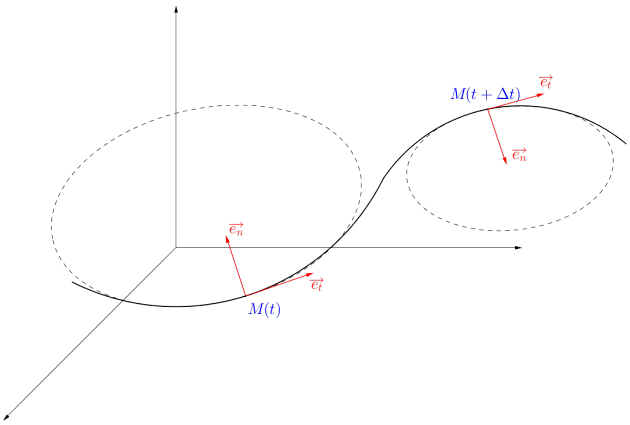
The expression of $\overrightarrow{a}$ in cylindrical coordinates with $r = cst$ and $z = 0$ yields

$$\overrightarrow{a} = -r\dot{\theta}^2\overrightarrow{e_r} + r\ddot{\theta}\overrightarrow{e_\theta}$$

6 Frenet-Serret coordinates

The Frenet frame of reference is a frame $(M;\overrightarrow{e_t};\overrightarrow{e_n};\overrightarrow{e_b})$ tied to the M point, where

- $\overrightarrow{e_t}$ is the unitary "tangential" vector, oriented along the path in the direction of the motion;
- $\overrightarrow{e_n}$ is the unitary "normal" vector, orthogonal to the path and pointing to the center of its curvature;
- $\overrightarrow{e_b}$ is the unitary "binormal" vector, defined in such a way that $(\overrightarrow{e_t};\overrightarrow{e_n};\overrightarrow{e_b})$ is a direct trihedron.



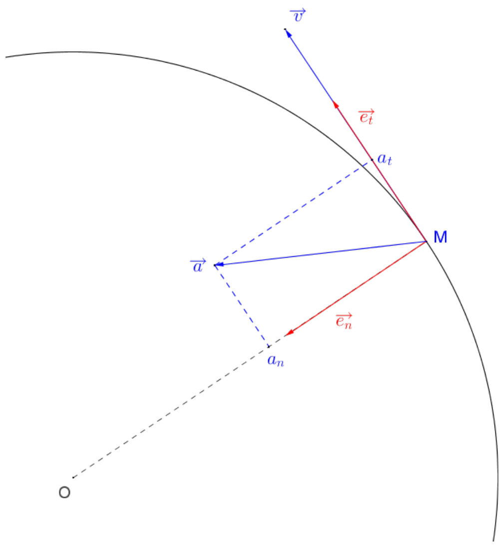
For a circular motion, the Frenet coordinates ressemble the polar coordinates, with $\overrightarrow{e_t} = \overrightarrow{e_\theta}$ and $\overrightarrow{e_n} = -\overrightarrow{e_r}$, so

$$\overrightarrow{v} = r\dot{\theta}\,\overrightarrow{e_t}$$

$$\overrightarrow{a} = r\ddot{\theta}\,\overrightarrow{e_t} + r\dot{\theta}^2\overrightarrow{e_n} = \frac{dv}{dt}\overrightarrow{e_t} + \frac{v^2}{r}\overrightarrow{e_n}$$

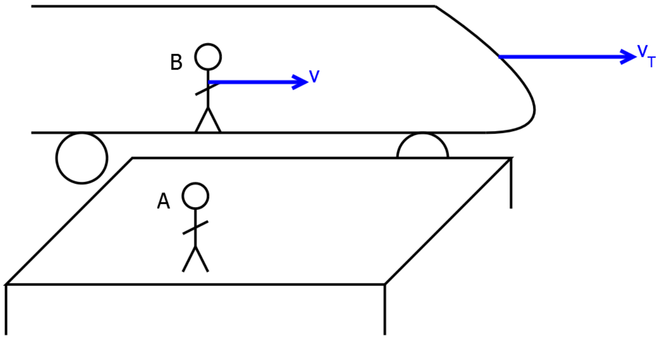
Note that the acceleration has 2 components

- the tangential acceleration $\overrightarrow{a_t} = \frac{dv}{dt}\overrightarrow{e_t}$, that vanishes in the case of a uniform motion;
- the normal acceleration $\overrightarrow{a_n} = \frac{v^2}{r}\overrightarrow{e_n}$, that points to the center of the curvature.



7 Velocity-addition formula for simple cases

The description of motion depends on the frame of reference. Let's consider for instance this figure where

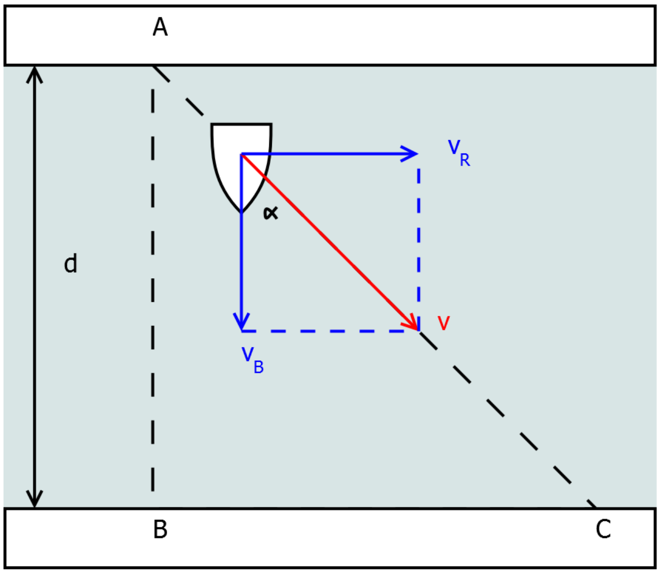


- A is standing still on the platform of a train station;
- B is walking with a speed $\vec{v}$ on a train, moving with a speed $\vec{v_T}$ relative to the platform;

The speed of B relative to the platform is then

$$\vec{v}_{B/A} = \vec{v}_{B/T} + \vec{v_T}$$

Another example: a boat tries to cross a river. The boat leaves the north-shore (point A) and aims the opposite shore (point B) at constant speed v_B relative to the river. The river is d-wide and flows from west to east at constant speed v_R relative to the ground. The boat eventually reaches the south-shore at C.
What is the distance between B and C?



We use $v_B = 5$ m.s^{-1}, $v_C = 10$ m.s^{-1} and $d = 50$ m.
The boat speed relative to the ground is $\vec{v} = \vec{v_B} + \vec{v_R}$, as shown on the figure. With $\alpha = (\vec{v_B}; \vec{v})$, we have

$$\tan\alpha = \frac{v_R}{v_B} = 2 = \frac{BC}{AB} \qquad BC = 2AB = 100 \text{ m}$$

8 Some vocabulary

8.1 Trajectories

- Linear (or rectilinear): the path is a straight line (curvature radius is infinite). Motion can be described using only one spatial dimension (x for instance) instead of vectors, $a_n = 0$ and $a = a_t = \dfrac{dv}{dt}$;
- Circular: the path lies in the plane and is a circle of constant radius;
- Helicoid: the path is an helix;
- Curvilinear: the path is a curve.

8.2 Motion type

- Uniform: $\overrightarrow{v}$ magnitude is a constant (but $\overrightarrow{v}$ can changes its direction), tangential acceleration is naught;
- Uniformy accelerated: tangential acceleration is a constant, motion is accelerated if $\overrightarrow{a_t}$ is oriented as the motion, decelerated if $\overrightarrow{a_t}$ is oriented in the opposite way of the motion;
- Sinusoidal: one coordinate of the position depends on time as a sine.
- Rigid body translation: velocity $\overrightarrow{v}$ is the same for all points of the solid;
- Rigid body rotation: all points of the solid have a circular motion.