EPITA

S1 2020

Mathematics

Intensive Maths Prep

(09.21.2020 - 10.01.2020)

INTENSIVE MATHS PREP

Contents

Context

Until now, some of you may have considered mathematics as a set of strange formulas that you must learn or hide in your pocket calculator, then recite during an exam in order to deduce new strange formulas, which may gain you a few points on your final mark. The most cynical people may even think that the main purpose of studying mathematics is to become a math teacher and hence to keep alive a tradition of sadism.

These points of view are not completely true.

Mathematics is above all ***the art of doing conjectures, of trying to prove or contradict them***, for the purpose of ***understanding better*** the phenomena we work with and of ***forecasting*** their behaviors. From this point of view, all of us, like Mr. Jourdain, speak mathematics[1] all our life without knowing it! Mathematics seems to be a science which relies more on reasoning than on experimentation. This aspect is due to the way it is formalized: we start with properties that we admit as true, then we transform and combine these properties gradually, remaining in the «true» area, until we reach a final property which is still true and solves the initial question. This is the purely technical point of view. But the latter should not mask the experimental point of view: ***it is important to test ideas***, to keep going on if the first reasoning does not work (figure 1) or if it does not immediately lead to the final result. It is important to focus on particular cases before solving the most general case. Sometimes, this will highlight defective reasoning or a counter example. And sometimes, this will help to understand the underlying architecture of the problem.

Figure 1: `The Shadoks, from Jacques Rouxel`

To go through all these steps of a mathematical thinking, it is important to have a good grasp of the math language and all its possibilities[2]: this will be the purpose of the first section, about logic.

Another aspect, which is often neglected, is the ability to represent problems and objects: graphically, geometrically, or with a short and clear statement. It is necessary to switch easily from one representation to another. One never knows which of them will stimulate the imagination for a more complex structure, for a solution or for a counter example highlighting defective reasoning. This way of thinking is also necessary for communication: an engineer is often in a situation to present his ideas to people who do not know anything about the topic he's working on. Then a clear and short graphic explanation is much more effective (figure 2). We will work with these graphic representations and imagine several ways to draw some typical problems. We will discuss their advantages and their draw-backs.

Finally, we will start working with basic examples of mathematical constructions. We will briefly

[1] as well as English or French, you can ask your English or TE teacher.

[2] « What looks good is clearly stated
and the words to say it flow with ease. »
One more question for your English or T.E. teacher:-p

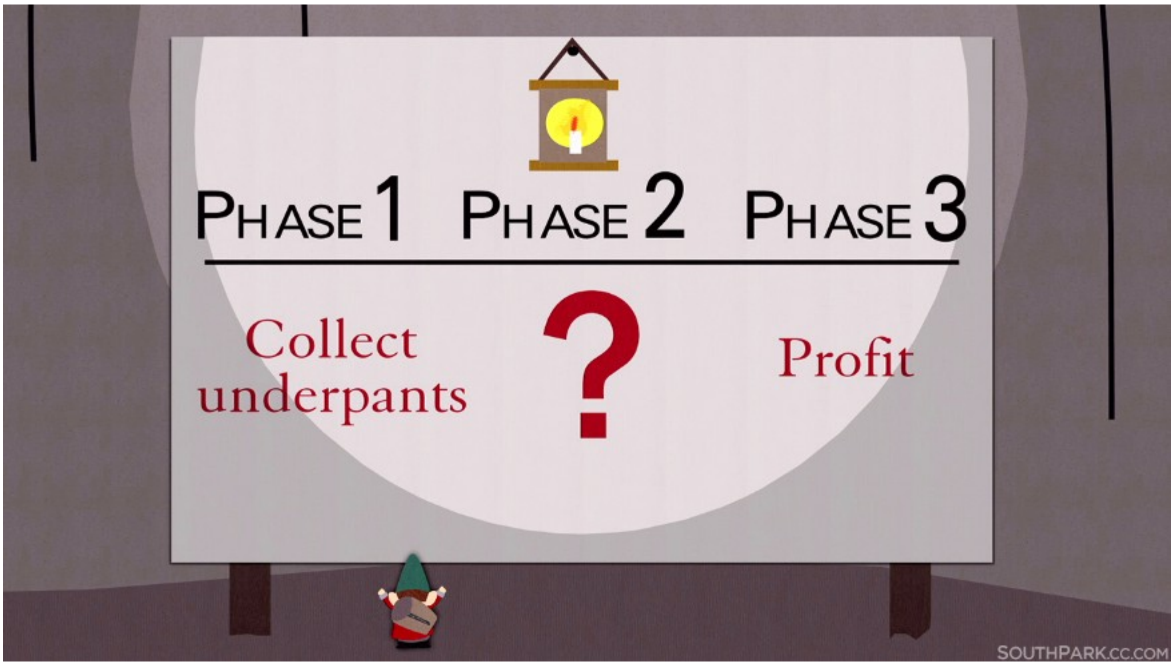
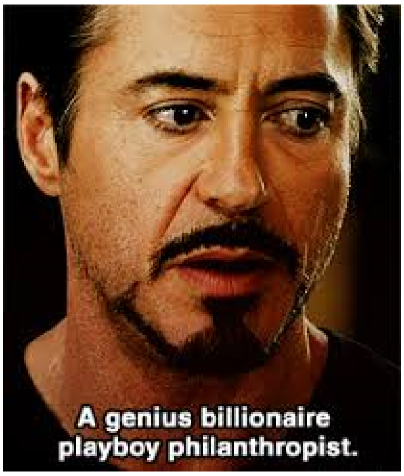


Figure 2: `A clear graphic can be much more effective to explain a reasoning... or to detect the mistakes in a plan which, at first, seemed to be perfect (SOUTH PARK: Gnomes business plan).`

introduce the vocabulary of sets and functions, echoing the statically typed language `CamL` that you are currently studying in algorithmics. This is just one out of the many connections that you will see between the math and algo classes. It will be frequent that a given question will be addressed from these complementary points of view. Indeed, what kind of engineer would be a computer scientist with no solid mathematical background?



Learning outcomes

The skills that we expect you to develop during this session are simple to state: we expect you to become familiar with mathematical vocabulary and reasoning. This will be the bases of all the forthcoming developments during your studies at EPITA. It is not a common chapter, it is the basic introduction to reasoning, building and representing mathematical objects.

Specifically, about logic, you will have to check that you are able to:

- translate properties into a mathematical language and, conversely, interpret expressions written in this language;

- use the logical connectors and understand their meanings;
- use different logical processes to prove some properties. For each process, you should understand why it indeed proves the property and have intuition about when it is efficient.

The second part, about sets, is complementary to the first one. You will have to:

- understand the different ways to define sets and be able to use them with a correct syntax;
- see the similarity between the logical connectors from the first part and their replications as set operators;
- count the elements in finite sets and handle sets cardinal numbers.

The third part focuses on functions and on their main properties. Check that you can:

- represent a function graphically, or draw a graph to fulfill a given property;
- determine the image or the inverse image of a set through a function;
- understand the properties of injectivity and surjectivity; determine whether or not a function satisfies one of these properties.

Furthermore, this part introduces the notion of relations between elements in a given set, such as equality, «less than or equal to» or set inclusion relations. Such relations can be defined as functions from $E \times E$ to the boolean set $\{true, false\}$. You will have to:

- be able to characterize some relations, identify them and find ways to represent them graphically;
- deduce from these representations some architectures or structures leading to fruitful analyses of the related questions.

Finally, this part about functions ends with real functions and integration. We provide a general method for the derivative of a composed function and we develop two new technical skills for the computation of an integral: integration by parts (IBP) and substitution. The purpose is to be able to determine the value of an integral:

- by finding a primitive;
- using an integration by parts;
- using a substitution;
- using a combination of the methods above.

In return, we will never ask you to recite by heart the resolution of some exercises done in class. Our purpose is not to fill your brain with formulas. We prefer to focus on your understanding of your knowledge, on your ability to apply it in different contexts. What you will have to do, at each chapter, is to know the basic definitions and then to check that you understand them, that you can use them to build your own algorithms leading to the exercises' solutions.

Part I
Logic

This part, about logic, is the beginning of ***formalization***. This consists in translating properties, from intuitions or spoken language, into mathematical objects. Then these objects can be manipulated and combined with some specified rules. To build this mathematical language, we need an alphabet (a set of characters), syntactic rules (different ways to combine the characters in order to express something) and some operations enabling one to alter the global meaning of the expressions.

Then this language will be used to lead different ways of reasoning, that is, proving that some properties are true or false.

1 Logic of propositions

1.1 Summary

The logic of propositions defines the syntactic rules of the mathematical language, that is, the different ways to combine basic propositions in order to produce a more complex one. This allows one to build very sophisticated sentences, using only basic elements and logical connectors. In a logical system accepting the law of excluded middle (we will remain in this framework), it is possible to determine the Boolean value (true or false) of any complex proposition as a function of the Boolean values of its basic propositions.

Furthermore, you will see in the second part of this work session, about sets, that all the logical connectors have a replication in terms of set operations.

1.2 Exercises

Exercise 1.1

1. Comparison of $A \vee (B \wedge C)$ with $(A \vee B) \wedge (A \vee C)$

   (a) For any real number x, consider the three following propositions:
   $A(x)$: "$x \leqslant 2$"; $B(x)$: "$x \geqslant -1$"; $C(x)$: "$x \leqslant 3$".
   Determine the domains of truth of $A \vee (B \wedge C)$ and $(A \vee B) \wedge (A \vee C)$.

   (b) Same question with the propositions:
   A : "The weather is sunny. " B : "I go to the beach."; C : "I go to the cinema." .

   (c) Using a double implication, prove the relation between
   $A \vee (B \wedge C)$ and $(A \vee B) \wedge (A \vee C)$.

2. Using a similar reasoning, compare $A \wedge (B \vee C)$ with $(A \wedge B) \vee (A \wedge C)$

Exercise 1.2

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Translate the following properties into mathematical language, using quantifiers and symbolic notations:

1. f is bounded.
2. f never cancels.
3. f is periodic.
4. f is increasing.
5. f is not the null function.
6. f is positive on $\mathbb{R}^+$ (two translations: a first one with a quantifier, a second one with an implication).

Exercise 1.3

1. Determine the meaning, in spoken language, of each of these properties. Then say whether the property is true or false and write its negation.

   a. $\exists y \in \mathbb{R} \quad \forall x \in \mathbb{R}^+ \quad x = y^2$

   b. $\forall x \in \mathbb{R}^+ \quad \exists y \in \mathbb{R} \quad x = y^2$

   c. $\forall (x, y) \in \mathbb{Q}^2 \quad [x < y \Longrightarrow \exists\, q \in \mathbb{Q},\ x < q < y]$

2. Write the negation of the following sentences. Keep in mind that the word «some» means «at least one».

   a. « This house is large ».

   b. « All the EPITA students will do the ING curriculum in the Paris area

   c. « Some students sleep in math class! ».

   d. « No criminal hypothesis can be discarded in this investigation ».

   e. « Some students will get below 10 on the maths test ».

   f. « If the winter is not too cold, I will save energy ».

   g. « Theo will join us if and only if I leave my house ». ».

2 Reasoning

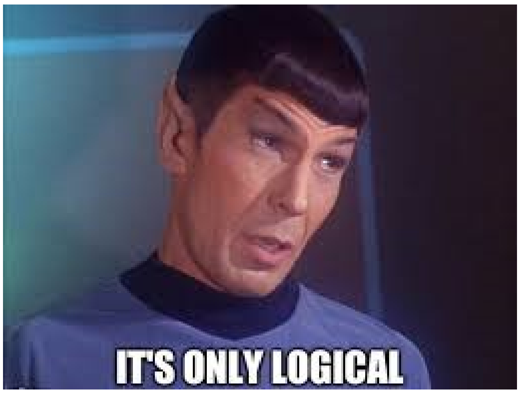
2.1 Summary

This section focuses on the different ways of reasoning which can be used to prove that a mathematical property is true. Most of the time, several ways may be considered to prove the property, so it is important to guess which one will be the easiest or the most convenient.

- The direct reasoning starts from the hypotheses and then uses a succession of implications to reach the desired property.
- The proof by contrapositive consists in proving the contrapositive of the desired property. Indeed, both properties are equivalent and, sometimes, the contrapositive is simpler to handle.
- The proof by contradiction consists in showing that, if the desired property was false, this would imply something that is obviously false. We can, therefore, deduce that the desired property is true.
- The proof by induction requires the proof of two sub-properties: first, the desired property is true in at least one particular case, and second, it propagates from this particular case to all other cases. We will mainly do proofs by induction on the set $\mathbb{N}$ of the natural numbers, but this reasoning can be extended to other sets. [3]

It is important to connect the latter reasoning with the recursive functions that you are studying in algorithmics. Proving a property by induction and writing a recursive function are very similar activities.

[3] Of course, it is easy to extend the reasoning to $\mathbb{Z}$; you will see later some induction proofs on more complex sets, using different propagation processes.



2.2 Exercises

Exercise 2.4

Let $n \in \mathbb{N}$. Using a proof by contrapositive, show that if n^2 is even, then n is even.

Exercise 2.5

Using the property from exercise 2-4, show that $\sqrt{2}$ is not a rational number.
Hint: you can use the fact that any positive rational number can be expressed as a ratio $\dfrac{p}{q}$, where p and q are positive natural numbers, and where these numbers are co-prime (that is to say, they have no co-divisor except ± 1).

Exercise 2.6

Prove by induction the following properties[4]: for any $n \in \mathbb{N}^*$,

1. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

2. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Exercise 2.7

1. Let $f : [0,2] \to [1,2]$ and (u_n) the sequence defined by: $\begin{cases} u_0 = 0,5 \\ \forall n \in \mathbb{N},\ u_{n+1} = f(u_n) \end{cases}$.

   Show by induction that: $\forall n \in \mathbb{N},\ u_n \in [0,2]$

2. Let a be a strictly positive real number. Show by induction that:

$$\forall k \in \mathbb{N}^*, \quad H_k : \text{« } a^k + k a^{k-1} \leqslant (a+1)^k \text{ »}$$

[4]These relations, especially the first two of them, are often used to determine the complexity of recursive algorithms.

Part II
Sets

The previous part has defined the mathematical language's syntax. The purpose of this new part is to add something which is required by any language: its vocabulary. The objects and relations that we define now will be used during the whole year to write mathematical properties.

Sets are the most basic type of mathematical construction. In fact, most of the mathematical environments we will work with will be ***algebraic structures***, that is to say, sets enhanced by some available operations[5] between elements. A set, then, is a universe in which we can develop reasoning about the operations. But to start with, we will handle sets for themselves.

The membership relation on a set is a way to characterize this set. This relation is a Boolean function which maps elements to a Boolean value: true if the element belongs to the set, false otherwise. So there is a strong similarity between sets and logic: all the operations over sets consist in combining the membership relations on the involved sets with some logical connectors.

Finally, we will define relations between elements within a set. We will focus on two types of relations: equivalence relations and order relations. They enable one to define some strong architectures in the set, which will facilitate the formalization and the understanding of properties. They are particular cases of graphs, whose nodes are the set's elements.

3 Sets: definitions and operations

3.1 Summary

To start with, we review the different ways to define a set: extensional or intentional definition. The latter consists in the formula defining the membership relation. The logical connectors, when applied to membership relations, can be extended to set operations: the logical negation results in the set complement, the logical disjunction in the set union, the logical conjunction in the set intersection. Thus, the computation rules about logical connectors and about set operations are the same.

Some more operations will be defined over sets. The «Cartesian product» of two sets results in a new set of a different type: it is a set of 2-tuples (ordered pairs).

The «power set» of a given set is the set of all its subsets. It is, therefore, a set of sets. We will use it when counting elements of subsets or in probability.

3.2 Exercises

Exercise 3.8
Consider the following sets and subsets:
Case 1 in $\mathbb{R}$: $A =]-\infty, 2[\,;\quad B =]-1, +\infty[\,;\quad C =]-\infty, 3[$
Case 2 in $E = \{\text{the class}\}$:
$A = \{\text{the boys}\}\,;\quad B = \{\text{the students wearing glasses}\}\,;\quad C = \{\text{the students being 1m75 tall or more}\}$

1. (a) Compare the sets $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ in both cases.
   (b) Prove formally the relation.
2. Same question for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$.

[5]For example, the set $\mathbb{N}$ of the natural numbers may be enhanced by the operations of addition, multiplication, concatenation of the numbers written in the base 10 system, and so forth. A structure is defined by the set and by the available operations.

Exercise 3.9

1. Represent graphically (using a Venn diagram) two sets A and B whose intersection may be empty or not.

   Display a partition of the universe related to this scheme.

2. For each of the following relations, highlight on your graph the left hand side and the right hand side sets. Deduce that the relation is true:

   a. $\overline{A \cup B} = \overline{A} \cap \overline{B}$
   b. $\overline{A \cap B} = \overline{A} \cup \overline{B}$
   c. $\overline{\overline{A}} = A$

   Do the connection with similar logical operations.

Exercise 3.10

Let E and F be the sets $E = \{1, 2, 3\}$ and $F = \{a, b\}$ where $(a, b) \notin \{1, 2, 3\}^2$.

1. Enumerate the sets $E \times F$ and $F \times E$. How many elements do they have? How many elements in both of them?

2. Enumerate $\mathscr{P}(E)$ and $\mathscr{P}(F)$. For an arbitrary finite set X, can you guess the number of elements in $\mathscr{P}(X)$ as a function of the number of elements in X?

4 Finite sets

4.1 Summary

Finite sets are the easiest ones to study: it is possible to do an exhaustive enumeration (brute force enumeration) of all the possible situations one might face. Many algorithmic questions lead to counting the elements in a set. The difficult points are, first to determine the appropriate set, second, to count the elements. This will be the purpose of the next chapter, after this work session.

When we cannot get an exact count, it can be useful to determine an order of magnitude (upper or lower bound) of the size of the set. For example, it may help to decide how many resources (in terms of processors or computation time) should be allocated to an algorithm's execution. This way of thinking is the starting point of many algorithmic discussions.

4.2 Exercises

Exercise 4.11

Referring to the modeling done on question 1 of exercise 3-9, express the cardinal number of $A \cup B$ in terms of those of the partition's sets, and then in terms of the cardinal numbers of A, B and $A \cap B$.

Part III
Functions

The rigorous definition of a function from an input set E to an output set F is a subset of $E \times F$ fulfilling some specific properties. Yet, it is more intuitive to see functions as processes which transform elements from E into elements from F, or as a mapping between elements from both sets. You have started, in `CamL`, to program objects called «functions»: they take an argument (an element from the input set) and return a result (an element from the output set).

5 Composition of functions

5.1 Summary

The composition of functions is a binary operation which is not commutative: the functions $f \circ g$ and $g \circ f$ are different, if they exist. Indeed, the composition of two functions requires a condition: the output set of the first one must be a part of the input set of the second one. When this condition is fulfilled, the composition consists in defining a new function as the successive applications of the initial functions.

5.2 Exercises

Exercise 5.12

1. Let f and g be the functions of $\mathbb{R}^{\mathbb{R}}$ defined for any $x \in \mathbb{R}$ by:

$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 1.$$

   Give explicit formulas for the functions $f \circ f, f \circ g, g \circ f$ and $g \circ g$.

2. In $\mathbb{R}^{\mathbb{R}}$, consider the functions $f : x \longmapsto 2x$, $g : x \longmapsto -2x$ and $h : x \longmapsto x^2$.

   Note that $f \neq g$. But check that $h \circ f = h \circ g$.

6 Injective and surjective functions

6.1 Summary

Injectivity and surjectivity are two properties about functions, but which are very different. The fact that we study these properties together should not lead you to believe that one is the negation of the other.

A function is said to be injective if no pair of input values can lead to the same output value. Thus, from the value of $f(x)$ we can deduce the value of x and the function is, hence, reversible. Injectivity is closely related to the input set: any function can be transformed into an injective function by restricting the input set to remove the duplicates, that is, by keeping for each image only one reverse image.

A function is said to be surjective if any element of the output set is the image of an input element. Then the whole output set is the image of the input set, it has no unused value. For example, this is a good property for a hash table, the aim of which is to spread the input values into many output cells. Surjectivity is related to the output set.

A function which is both injective and surjective is bijective. Such a function implements an invertible transformation between the input and the output sets. We will see that the existence of such a function implies a strong relation between these sets.

6.2 Exercises

Exercise 6.13

Discuss the injectivity and surjectivity of the function $\left\{\begin{array}{ccc} E & \longrightarrow & F \\ x & \longmapsto & x^2 \end{array}\right.$ in the following situations:

1. $E = \mathbb{R}$ and $F = \mathbb{R}$
2. $E = \mathbb{R}$ and $F = \mathbb{R}^+$
3. $E = \mathbb{R}^+$ and $F = \mathbb{R}$
4. $E = \mathbb{R}^+$ and $F = \mathbb{R}^+$

Exercise 6.14
Consider three sets E, F and G and two functions $f : E \longrightarrow F$ and $g : F \longrightarrow G$.

1. Show that if f and g are injective, then $g \circ f$ is injective.
2. Show that if f and g are surjective, then $g \circ f$ is surjective.
3. Show that if $g \circ f$ is injective, then f is injective.

   Using graphical representations of f and g, find an example where g is not injective while $g \circ f$ is injective.
4. Show that if $g \circ f$ is surjective, then g is surjective.

   Using graphical representations of f and g, find an example where f is not surjective while $g \circ f$ is surjective.

7 Image and reverse image

7.1 Summary

The notions of image and reverse image, which you have applied before to elements, can be extended to sets. Take care with the confusing notations: when A is a subset of the input set, $f(A)$ does not denote «the function f applied to the subset A», but «the set of all the values that we can get by applying f to the elements of A». Similarly, $f^{-1}(B)$ is defined for any subset B of the output set, even when the function f^{-1} does not exist. Furthermore, you will see in the exercise **??** that the notations f and f^{-1}, when applied to sets, cannot be composed as they can be when applied to elements.

7.2 Exercises

Exercise 7.15
For each of the following properties, find a function graphically from $[\![1,4]\!]$ to $[\![1,5]\!]$ such that the property is fulfilled.

1. $f([\![1,4]\!]) = \{1,4\}$
2. $f^{-1}(\{3,4\}) = \{1,2\}$
3. $f(\{1,2\}) = \{3,4\}$
4. $f^{-1}(\{2,3,4\}) = \{1\}$

Exercise 7.16
Consider in $\mathbb{R}^{\mathbb{R}}$ the three functions $f : x \longmapsto \mathrm{e}^x$, $g : x \longmapsto x^2$ and $h : x \longmapsto \sin(x)$.

1. Draw the graphs of these functions.

2. Determine:

   - $f(\mathbb{R})$ and $f^{-1}(\{0\})$;

   - $g([-1,4])$ and $g^{-1}([-1,4])$;

   - $h(\mathbb{R}), h([0,\pi/2]), h^{-1}([0,1]), h^{-1}([1,2])$ and $h^{-1}([3,4])$.

8 Relations

8.1 Summary

Relations between elements in a set E are Boolean functions defined on $E \times E$. They can be represented as oriented graphs. In this part, we will focus on two types of relations which fulfill some specific properties. Their handling is simplified consequently.

Equivalence relations split the set E into a partition of subsets, where the elements in each subset are mutually connected. The idea is to classify the elements in boxes, in such a way that all elements in a given box share a property defined by the relation. Such relations will be used in arithmetics (congruence) and in linear algebra (when defining matrix reductions).

Order relations enable one to compare the elements in a set. Some of them are «total order relations»: any pair of elements can be compared and a global ranking is possible. Other order relations are only «partial». Order relations are omnipresent, in mathematics as well as in algorithmics: comparison and sorting of elements, topology, flow graphs, optimization problems and so forth.

8.2 Exercises

Exercise 8.17
Let us define, on the set $[\![0,7]\!]$, the relation[6] R by:

$$\forall (x,y) \in [\![0,7]\!]^2,\ x\ R\ y \Longleftrightarrow 3 \text{ divides } x-y$$

1. Show that R is an equivalence relation.

2. Draw its graph. What do you notice?

Exercise 8.18

1. (a) Show that the $\leqslant$ relation is a total order relation over $[\![0,5]\!]$.
   (b) Draw its graph and its Hasse diagram.

2. (a) Show that the $\subset$ relation is a partial order relation over $\mathscr{P}(\{1,2,3\})$.
   (b) Draw its graph and its Hasse diagram.

[6]This relation is the congruence modulo 3.

Exercise 8.19
Using the Latin alphabet $E = \{a, b, \dots, z\}$, let us define the set E^* of all the words [7] writable with these letters, of any length.

For the following relations over E^*, say whether they are total order, partial order or equivalence relations (w and w' denote two arbitrary words):

1. $w \; R_1 \; w'$ if the words w and w' have the same length;
2. $w \; R_2 \; w'$ if w would be ranked before w' in a dictionary;
3. $w \; R_3 \; w'$ if w is a sub-word of w', that is, if we can find two words u and v (maybe empty) such that w' is the concatenation $u \cdot w \cdot v$.

   For example: **logic** is a sub-word of *psycho***logic***altrauma*[8].

9 Integration

9.1 Summary

For a pair $(a, b) \in \mathbb{R}^2$ such that $a \leqslant b$ and a continuous function $f : [a, b] \longrightarrow \mathbb{R}$, we define the integral

$$I = \int_a^b f(t)\,\mathrm{d}t$$

as the area between the lines $x = a$, $x = b$, $y = 0$ and the graph of the function f.

Furthermore, for any $x \in [a, b]$, we can define the function

$$F : x \longmapsto \int_a^x f(t)\,\mathrm{d}t$$

Then F is differentiable and satisfies to $F' = f$, that is, F is a primitive of the function f. For any other primitive F_2, we have

$$F_2(x) - F_2(a) = F(x) - \underbrace{F(a)}_{=0} = F(x)$$

This leads to:

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a) = F_2(b) - F_2(a)$$

Thus, the most common way of computing an integral, that you have seen before, is to find a primitive of f and to compute $F(b) - F(a)$.

However, when no formal primitive is available, there are two other methods: the integration by parts and the substitution. These two methods enable one to transform the initial integral into another one, for which we can expect to find a primitive.

9.2 Exercises

Exercise 9.20
Determine the derivative of the functions f, g and h (no need to discuss the definition domain).

$$f(x) = \sin^{42}\Big(\ln\big(x \sin(x)\big)\Big) \qquad g(x) = \sqrt{\ln\Big(e^{x^2} + 1\Big)} \qquad h(x) = \ln\Big(2^{\,x} - \sin\big(\sin(x)\big)\Big)$$

[7] That is, character strings.

[8] In English, we should write «psychological trauma», with a blank character. Note that the character strings that we use in the exercise do not need to be existing English words. They are just arbitrary sequences of letters.

Exercise 9.21
Using primitives, compute the following integrals.

1. $\displaystyle\int_0^2 3te^{-t^2}\ \mathrm{d}t.$
2. $\displaystyle\int_0^1 2t(2t^2+1)^4\ \mathrm{d}t.$
3. $\displaystyle\int_1^e \frac{(\ln(t))^2}{t}\ \mathrm{d}t.$
4. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan(x)\,\mathrm{d}x$
5. $\displaystyle\int_0^1 \frac{t^2+1}{2\sqrt{t^3+3t}}\ \mathrm{d}t.$
6. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\sqrt{\cos(x)}}\ \mathrm{d}x.$
7. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan(x)(1+\tan^2(x))\,\mathrm{d}x$

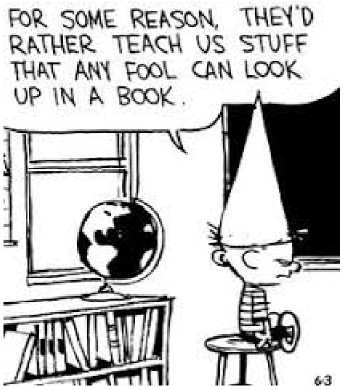
Exercise 9.22
Using integrations by parts (IBP), determine the following integrals.

1. $\displaystyle\int_0^1 (2x+3)e^{3x}\ \mathrm{d}x.$
2. $\displaystyle\int_0^{\frac{\pi}{2}} x\cos\left(\frac{x}{2}\right)\ \mathrm{d}x.$
3. $\displaystyle\int_1^e \ln(x)\ \mathrm{d}x.$
4. $\displaystyle\int_0^{\pi} x^2\sin(x)\ \mathrm{d}x.$
5. $\displaystyle\int_0^{\pi} e^x\sin(x)\ \mathrm{d}x.$
6. $\displaystyle\int_1^e \frac{\ln(x)}{x^2}\ \mathrm{d}x.$
7. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x}{\cos^2(x)}\ \mathrm{d}x.$

Exercise 9.23
Using substitutions, compute the following integrals:

1. $\displaystyle\int_0^4 e^{\sqrt{x}}\ \mathrm{d}x$ using the substitution $t=\sqrt{x}$
2. $\displaystyle\int_0^{\ln(\sqrt{3})} \frac{\mathrm{d}t}{e^t+e^{-t}}$ using the substitution $u=e^t$
3. $\displaystyle\int_1^e \frac{\ln(x)}{\sqrt{x}}\ \mathrm{d}x$ using the substitution $u=\ln x$

Bibliography



There are very few mathematical books with a point of view adapted to programmers. Fortunately, there are many more books written by computer scientists with a solid mathematical background. Here are some examples:

- *The Art of Computer Programming*, from Donald Knuth, is certainly the one containing the widest mathematical knowledge for a computer scientist; the author explains how mathematics can lead to efficient programming.
- *Introduction to Algorithms*, from Thomas E. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein, is more algorithmics oriented. Still, it includes many discussions highlighting the interest of the mathematical notions.

Nevertheless, you should feel free to look for references by yourself and there are many. For most of your forthcoming projects, a first step will consist in building an adapted bibliography: it is much more effective to take time to understand the problem and the existing solutions, than to start programming immediately and then to realize that the idea was not the good one.