LINEAR ALGEBRA

Contents

Context

To store data in the memory of a computer, a common method is to use a list of real numbers. The resulting list is a vector from $\mathbb{R}^n$, where n is the length of the list. Similarly, a signal is a function depending on time, it is hence a vector from $\mathbb{R}^{\mathbb{R}}$. In such cases, we need to develop reasoning with the whole vector: it is not fruitful to isolate each of its components. Thus, many real life situations involve vectors in spaces much greater than the physical space, $\mathbb{R}^2$ or $\mathbb{R}^3$. Here are some examples:

- When analyzing data, we have a large number of samples of a random vector. For example, some individuals in a population can be surveyed about their income, rents, food expenses, health status, and so on. Then we get, for each surveyed person, a vector whose components are all the numerical answers. It is a vector from $\mathbb{R}^n$ where the dimension n is most of the times greater than 3. Furthermore, the set of all these data can be put together into a matrix, each column being related to a specific person.

- In signal analysis, a vector is a real function depending on time. When the signal is sampled, it is a vector from $\mathbb{R}^n$ but some information is lost. It is hence possible to project the initial vector onto a finite dimensional subspace, but this projection requires choosing the subspace and understanding which information is saved, which information is lost.

In such situations, we need to work in vector spaces whose dimensions are far greater than 3.

Yet, a first understanding of linear algebra involves geometric representations in spaces of dimensions 2 or 3. This mode of thinking is essential: it is in this way that intuitions are formed that often remain valid in higher dimensions. But to move to higher dimensions, we must build some certitudes about what can be generalized to cases $n > 3$ and the right way to do it. This requires a certain formalism.

During the previous semester, we have started using jointly these two ways of thinking. We will go on in this direction this semester. The purpose is to link geometrical reasoning to formal reasoning: the latter can be extended to high dimensions.

Particular emphasis will be placed on the notion of ***basis of a vector space***. A seemingly inextricable question can become extremely simple if we represent vectors in a proper basis. We will see several examples.

Learning outcomes

Here are the skills that you are expected to develop, about which you will be evaluated:

- Understanding what a basis of a vector space is. Being able to represent vectors in different bases.

- Understanding the properties of linear maps and their relations with the matrices that represent the maps. You should be able to watch a matrix, to know where you can find information about the kernel and the image.

- Being able to simplify the study of an endomorphism, by choosing a basis in which it is easier to understand.

The sections from the document focus on specific part of these skills:

Sets of vectors, bases

- Understanding what a basis is.

- Being able to switch from one system of coordinates to the other, to represent a vector in several bases.

Linear maps

- Understanding the relation from a linear map to its matrix. This includes the ability to read the kernel and the image on the matrix.

- Being able to switch from a system a coordinates to the other, to operate the required transformation if we change the basis of the input and/or output space.

Understanding an endomorphism

- Understanding the notions of eigenvector and eigenbasis.

- Being able to say whether an eigenbasis exists or not, and to determine such a basis when it exists.

- Understanding geometrically what the reduction of an endomorphism is. Being able to exploit it in practical situations, in low or high dimension.

Part I
Revisions and skills enhancement

1 Sets of vectors, bases

1.1 Summary

The most immediate representation of a vector, in an n-dimensional vector space, is the n-tuple containing its components in the standard basis. However, this is not always the most efficient representation, both in terms of computing time and of understanding the information contained in the vector.

For example, when analyzing a sampled signal, one can use its ***discrete Fourier transform***. Then we can «forget» the n successive values of the signal (that is, its components in the standard basis) and replace them with the Fourier coefficients. The latter are the signal's coordinates in another basis. They contain all the information carried by the signal, but in another form, that of a frequency representation, which allows a particular analysis of the signal. Then it remains possible, if we need, to get the coordinates in the initial basis (that is, the successive values of the signal) from those in the new basis (that is, from the Fourier coefficients).

In general, changing the system of coordinates is like choosing a new basis and working with the coordinates of the vector in this new basis. Thus, you have to know how to go from one system of coordinates to another.

1.2 Exercises

Exercise 1.1

Let E be a vector space over $\mathbb{R}$ and $A = \{e_1, \ldots, e_n\} \subset E$. Let $F = \left\{ \sum_{i=1}^{n} \lambda_i e_i, (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \right\}$.

1. Show that F is a linear subspace of E.

2. Let G be another linear subspace of E such that $A \subset G$. Show that $F \subset G$.

Remark: F is denoted by $\mathrm{Span}(A)$. It is the smallest linear subspace of E containing A.

Exercise 1.2

Let E be a vector space over $\mathbb{R}$ and $\mathscr{B} = \{e_1, \ldots, e_n\} \subset E$. By definition, $\mathscr{B}$ is a basis of E if:

1. it is a spanning set of E;

2. this set is linearly independent.

Explain why the definition requires each of these conditions.

Exercise 1.3

Are the following families bases of $\mathbb{R}^3$? If they are not, transform them by removing and/or adding vectors until you get a basis of $\mathbb{R}^3$.

1. $A_1 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}$

2. $A_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

3. $A_3 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} \right\}$

★ **Exercice 1.4**

In the following case, is the set A linearly independent? If it is not, provide a linearly independent subset B such that $\mathrm{Span}(B) = \mathrm{Span}(A)$.

1. $E = \mathbb{R}[X]$ and $A = \left\{ (X-1)^2, (X+1)^2, X^2 \right\}$

2. $E = \mathbb{R}^{\mathbb{R}}$ and $A = \left\{ x \mapsto e^{2x}, x \mapsto x^2, x \mapsto x \right\}$

3. $E = \mathbb{R}^{\mathbb{R}}$ and $A = \left\{ x \mapsto e^x, x \mapsto e^{x+1}, x \mapsto e^{x+2} \right\}$

4. $E = \mathbb{R}^{\mathbb{R}}$ and $A = \left\{ x \mapsto \cos(2x), x \mapsto \cos^2(x), x \mapsto 1 \right\}$

Exercise 1.5   *Transition matrices*

1. Let $\mathscr{B}$ be the standard basis of $\mathbb{R}^2$ and $\mathscr{B}' = \{(1,1), (-1,2)\}$ a second basis.

   For an arbitrary vector $u = (x, y) \in \mathbb{R}^2$, let X and X' be the column matrices representing the coordinates of u in the bases $\mathscr{B}$ and $\mathscr{B}'$.

   Furthermore, the ***transition matrix*** from $\mathscr{B}$ to $\mathscr{B}'$ is $P = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$

   a. Represent graphically the vectors of $\mathscr{B}$, those of $\mathscr{B}'$, and an arbitrary vector u.

   b. Expressing u as a linear combination of the vectors from $\mathscr{B}'$, determine the matrix relation leading to X as a function of X'.

   c. Check that P is invertible and determine the matrix relation leading to X' as a function of X.

   d. Let L be the line whose equation is $x + 2y = 0$. Provide its equation in basis $\mathscr{B}'$.

2. Let $E = \mathbb{R}_2[X]$ and consider its bases $\mathscr{B} = (1, X, X^2)$ and $\mathscr{B}' = (1, (X+1), (X+1)^2)$.

   a. Determine the transition matrix P and its inverse P^{-1}.

   b. For any $Q \in E$, provide a matrix formula for the relation between the coordinates of Q in bases $\mathscr{B}$ and $\mathscr{B}'$.

2 Linear maps

2.1 Summary

A linear map conserves the linear relations, if any, between its input vectors. When the input space is finite dimensional, the linear map is totally determined by the images of the input basis vectors. If the output space is also finite dimensional, the linear map can be defined with a matrix, as soon as the input and output bases have been specified.

A matrix can also represent a family of vectors: then the successive columns of the matrix are the coordinates of the successive vectors. In statistics, for example, matrices are often used in this way: we study a random vector and each column is an observation of this vector. Then a statistical analysis can have the purpose to highlight linear relations between the vector's coordinates, that is, dependence relations between the matrix rows.

In any case, it is important to know how to watch a matrix. Determining its rank, image and kernel, enables one to study both a linear map and a statistical data set.

2.2 Exercises

Exercise 2.6

Let $n \in \mathbb{N}$ and $(x_0, x_1, \ldots x_n) \in \mathbb{R}^{n+1}$. Provide the matrices of the following linear maps, in the standard bases of the input and output spaces.

1. $f : \begin{vmatrix} \mathscr{M}_2(\mathbb{R}) & \longrightarrow & \mathbb{R}_2[X] \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & (a+d)X^2 + (b+c)X + (d-c) \end{vmatrix}$

2. $g : \begin{vmatrix} \mathbb{R}_n[X] & \longrightarrow & \mathbb{R} \\ P & \longmapsto & \int_0^1 P(x)\,\mathrm{d}x \end{vmatrix}$

3. $V : \begin{vmatrix} \mathbb{R}_n[X] & \longrightarrow & \mathbb{R}^{n+1} \\ P & \longmapsto & \begin{pmatrix} P(x_0) \\ \vdots \\ P(x_n) \end{pmatrix} \end{vmatrix}$

Exercise 2.7

1. Let $E = \mathbb{R}^3$, $\mathscr{B} = \{e_1, e_2, e_3\}$ its standard basis and consider $f \in \mathscr{L}(E)$ the linear map defined by its matrix in basis $\mathscr{B}$:

$$A = \mathrm{Mat}_{\mathscr{B}}(f) = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

The three columns of A are denoted by C_1, C_2 and C_3.

a. Let $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E$ and $v = f(u)$. Express v in terms of x, y, z, C_1, C_2 and C_3.

b. Is f surjective? Is $\{C_1, C_2, C_3\}$ a spanning set of $\mathbb{R}^3$?

c. Is f injective? Is $\{C_1, C_2, C_3\}$ linearly independent?

d. Is f bijective? Is $\{C_1, C_2, C_3\}$ a basis of $\mathbb{R}^3$?

2. Consider:

A ℝ-vs E of dimension n, together with a basis $\mathscr{B}_1 = \{e_1, \cdots, e_n\}$.
A ℝ-vs F of dimension p, together with a basis $\mathscr{B}_2 = \{\varepsilon_1, \cdots, \varepsilon_p\}$.
A linear map $f \in \mathscr{L}(E, F)$ defined by its matrix $A = \text{Mat}_{\mathscr{B}_1, \mathscr{B}_2}(f)$.
The columns of A are denoted by $C_1, \cdots, C_n$.

   a. Show that: f surjective $\Longleftrightarrow \{C_1, \cdots, C_n\}$ is a spanning set of $\mathbb{R}^p$.

   b. Show that: f injective $\Longleftrightarrow \{C_1, \cdots, C_n\}$ is linearly independent.

   c. Deduce that: f bijective $\Longleftrightarrow \{C_1, \cdots, C_n\}$ is a basis of $\mathbb{R}^p$.

   d. What can we say if $n < p$? And if $n > p$?

Exercise 2.8   *Rank of a linear map*
Let $a \in \mathscr{L}(\mathbb{R}^3, \mathbb{R}^2)$ be the linear map defined by its matrix in the standard bases

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

The rank of A is defined by $\text{rank}(A) = \text{rank}(a) = \dim\big(\text{Im}(a)\big)$.

1. Without considering the values of the matrix coefficients, show that $\text{rank}(A) \leqslant 2$. Then show that, in the general case, for any $(n, p) \in \mathbb{N}_*^2$,

$$\forall A \in \mathscr{M}_{np}(\mathbb{R}), \ \text{rank}(A) \leqslant \min(n, p)$$

2. Find a basis of $\text{Ker}(a)$.

3. Find a basis of $\text{Im}(a)$. Is the matrix A full rank?

4. Determine the rank of the following matrices:

$$B = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -3 & -4 \\ -1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

Exercise 2.9   *Determinant of a square matrix*
Compute the determinants of the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -3 \\ -1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -3 & -2 \\ -1 & 2 & 0 & 2 \end{pmatrix}$$

Exercise 2.10   *Change of bases*
1. Let $E = \mathbb{R}_2[X]$ and consider its bases $\mathscr{B} = (1, X, X^2)$ and $\mathscr{B}' = (1, (1 + X), (1 + X)^2)$. Let $f \in \mathscr{L}(E, \mathbb{R})$ be defined for any $P \in E$ by

$$f(P) = \int_0^3 \frac{P(x)}{\sqrt{1 + x}} \, dx$$

   a. Determine the matrix A' of f in basis $\mathscr{B}'$.

   b. Determine the transition matrix P from $\mathscr{B}$ to $\mathscr{B}'$.

   c. Deduce the matrix A of f in basis $\mathscr{B}$.

2. Let E and F be two vector spaces of dimensions n and p, $\mathscr{B}_1$ and $\mathscr{B}'_1$ two bases of E, $\mathscr{B}_2$ and $\mathscr{B}'_2$ two bases of F. Let P be the transition matrix from $\mathscr{B}_1$ to $\mathscr{B}'_1$ and Q the transition matrix from $\mathscr{B}_2$ to $\mathscr{B}'_2$. Finally, for $f \in \mathscr{L}(E, F)$, let us define

$$A = \mathrm{Mat}_{\mathscr{B}_1, \mathscr{B}_2}(f) \qquad \text{and} \qquad A' = \mathrm{Mat}_{\mathscr{B}'_1, \mathscr{B}'_2}(f)$$

   a. Determine the matrix relation between A' and A.

   b. Write this relation in the particular case where

$$E = F, \qquad \mathscr{B}_1 = \mathscr{B}_2 \qquad \text{and} \qquad \mathscr{B}'_1 = \mathscr{B}'_2$$

Exercise 2.11 *Projectors*

1. Let E be a $\mathbb{R}$-vs, F and G two supplementary subspaces in E.

   Thus, for any $w \in E$, there exists a unique $(u, v) \in F \times G$ such that $w = u + v$.

   Let $p : w \longmapsto v$.

   a. Check that $p \in \mathscr{L}(E)$ and that $p^2 = p$.

   b. Determine $\mathrm{Ker}(p)$ and $\mathrm{Im}(p)$.

   c. Let $q = id - p \in \mathscr{L}(E)$. Say what $q(w)$ is and check that $q^2 = q$.
      Determine $\mathrm{Ker}(q)$ and $\mathrm{Im}(q)$.

   d. Assume in this question that E is finite-dimensional. We build a basis of E by concatenating a basis of F and a basis of G.
      What are the matrices of p and q in this basis?

2. Let E be a $\mathbb{R}$-vs and p a projector, that is: $p \in \mathscr{L}(E)$ and $p^2 = p$.

   a. Let $w \in E$. Show that $w - p(w) \in \mathrm{Ker}(p)$.

   b. Show that $E = \mathrm{Ker}(p) \oplus \mathrm{Im}(p)$.

   c. Explain in a few words what the endomorphism p is.

   d. Let $q = id - p$. Show that q is also a projector. Determine $\mathrm{Ker}(q)$ and $\mathrm{Im}(q)$.

Exercise 2.12

Consider the endomorphisms p_1, p_2 and p_3 defined by the matrices A_1, A_2 and A_3 in the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$:

$$A_1 = \begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -2 & -2 & -2 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad A_3 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

For each of these endomorphisms:

1. Check that p is a projector.

2. Find a basis of $\mathrm{Ker}(p)$ and a basis of $\mathrm{Im}(p)$.

3. Let $\mathscr{B}$ be the concatenation of these two bases. It is a basis of E. Determine the matrices in $\mathscr{B}$ of p and of $q = id - p$.

Part II
Understanding an endomorphism

3   Diagonalizable endomorphism

3.1   Summary

An endomorphism, unlike other linear maps, has identical input and output spaces. When this space is high dimensional, we better understand the endomorphism if we can highlight smaller subspaces which are closed under the endomorphism. Reducing an endomorphism consists in finding such subspaces which, furthermore, are supplementary. Then we can define a basis of the whole space as the concatenation of the subspaces' bases. In this new basis, the matrix of the endomorphism contains diagonal blocks, all other coefficients being zeros. We say that it is a «block diagonal» matrix.

The ideal case is one where we get a basis in which the matrix is simply diagonal. Such a representation, when it exists, is obtained in a basis of eigenvectors: they are vectors that are collinear with their image by the endomorphism. Diagonalizing a matrix consists in highlighting a basis of eigenvectors.

Fortunately, this ideal case is the most frequent one. Most of the matrices are diagonalizable in $\mathbb{R}$ or in $\mathbb{C}$. This nice property justifies dealing with the diagonalizable matrices separately.

3.2   Exercises

Exercise 3.13   *Motivations*

We work in $\mathbb{R}^2$. Consider the standard basis $\mathscr{B} = \{(1,0),\,(0,1)\}$ and a second basis $\mathscr{B}' = \{(2,1),\,(1,2)\}$.

For any $a = (x,y) \in \mathbb{R}^2$, let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $U = \begin{pmatrix} u \\ v \end{pmatrix}$ be its coordinates in bases $\mathscr{B}$ and $\mathscr{B}'$.

Furthermore, let f be the endomorphism of $\mathbb{R}^2$ defined by its matrix in the standard basis

$$M = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

1. Matrix of f in basis $\mathscr{B}'$.

    a. Represent graphically $\mathscr{B}'$ and $f(\mathscr{B}')$. Find the matrix D of f in basis $\mathscr{B}'$.

    b. Using the transition matrix P from $\mathscr{B}$ to $\mathscr{B}'$, determine the matrix relation between M and D.

2. Let $(x_0, y_0) \in \mathbb{R}^2$ and consider the real sequences (x_n) and (y_n) defined by their initial values x_0 and y_0 and by the recursive relation:

$$\forall n \in \mathbb{N}, \quad \left| \begin{array}{lcl} x_{n+1} & = & 3x_n - 2y_n \\ y_{n+1} & = & 2x_n - 2y_n \end{array} \right.$$

    Furthermore, define $a_n = (x_n, y_n)$, then X_n and U_n the coordinates of a_n in bases $\mathscr{B}$ and $\mathscr{B}'$.

    a. Show that for any $n \in \mathbb{N}$, $a_{n+1} = f(a_n)$. Represent graphically the first three terms of (a_n) in the cases $a_0 = (2,1)$, $a_0 = (1,2)$ and $a_0 = (3,3)$.

    b. Let $n \in \mathbb{N}$. Express U_{n+1} as a function of U_n.

    c. Deduce U_n as a function of n, then X_n as a function of n.

    d. Check that the point $a_0 = (0,0)$ is a steady state. Is it stable?

3. Let $(x_0, y_0) \in \mathbb{R}^2$. We look for all the the differentiable real functions $t \mapsto x(t)$ and $t \mapsto y(t)$ such that $x(0) = x_0$, $y(0) = y_0$ and, for any $t \in \mathbb{R}$,

$$\left| \begin{array}{lcl} x'(t) & = & 3x(t) - 2y(t) \\ y'(t) & = & 2x(t) - 2y(t) \end{array} \right.$$

Define $a(t) = (x(t), y(t))$, then $X(t)$ and $U(t)$ the coordinates of $a(t)$ in bases $\mathscr{B}$ and $\mathscr{B}'$.

   a. Show that for any $t \in \mathbb{R}$, $a'(t) = f(a(t))$. Represent graphically the trajectory of $a(t)$ on a small interval $[0, \delta t]$ in the cases $a(0) = (2, 1)$, $a(0) = (1, 2)$ and $a(0) = (3, 3)$.

   b. Express $U'(t)$ as a function of $U(t)$.

   c. Deduce $U(t)$ as a function of t, then $X(t)$ as a function of t.

   d. Check that the point $a(0) = (0, 0)$ is a steady state. Is it stable?

Exercise 3.14

Are the following matrices diagonalizable in $\mathscr{M}_3(\mathbb{R})$?
When they are, determine the matrices P and D.

$$A_1 = \begin{pmatrix} -1 & -2 & -2 \\ -3 & -1 & -3 \\ 3 & 2 & 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 4 & 2 \\ 1 & -1 & -1 \\ 0 & 4 & 1 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{pmatrix}.$$
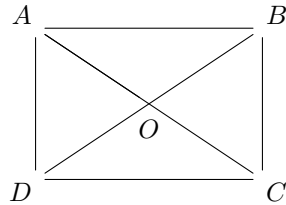
Exercise 3.15

Let $a \in \mathbb{R}$. Discuss the diagonalizability of the following matrices depending on a.
When they are diagonalizable, the eigenbasis is not required.

$$A_1 = \begin{pmatrix} -1 & a+1 & 0 \\ 1 & a & 1 \\ 3 & -a-1 & 2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & a & a \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}.$$

★ **Exercice 3.16** *Stochastic matrix*

A token moves randomly on the set $\{A, B, C, D, O\}$, where the points A, B, C, D are O the followings:

At iteration $n = 0$, the token is in A. Then at each iteration n, it moves randomly to one of the adjacent nodes:

- if it is in A, B, C or D, it goes to one of the three adjacent nodes with the probability $\frac{1}{3}$;

- if it is in O, it goes to one of the four adjacent nodes with the probability $\frac{1}{4}$.

Let us define the event $A_n =$ "the token is in A at iteration n" and, similarly, the events B_n, C_n, D_n and O_n. Furthermore, consider the vector

$$a_n = \Big(P(A_n), P(B_n), P(C_n), P(D_n), P(O_n) \Big)$$

and X_n the column matrix of the coordinates of a_n in the standard basis of $\mathbb{R}^5$.

   1. Determine $f \in \mathscr{L}(\mathbb{R}^5)$ such that, for any $n \in \mathbb{N}$, $a_{n+1} = f(a_n)$.
     Display the matrix A of f in the standard basis of $\mathbb{R}^5$.

2. We accept without proof that a diagonalization of A is given by

$$P = \frac{1}{4}\begin{pmatrix} 3 & 1 & 1 & 0 & 1 \\ 3 & -1 & 1 & 1 & 0 \\ 3 & 1 & 1 & 0 & -1 \\ 3 & -1 & 1 & -1 & 0 \\ 4 & 0 & -4 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad P^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -4 & 4 & -4 & 0 \\ 1 & 1 & 1 & 1 & -3 \\ 0 & 8 & 0 & -8 & 0 \\ 8 & 0 & -8 & 0 & 0 \end{pmatrix}$$

and we denote by $U_n = \begin{pmatrix} u_n \\ v_n \\ w_n \\ s_n \\ t_n \end{pmatrix}$ the coordinates of a_n in the eigenbasis.

   a. Express the first coordinate u_n in term of X_n. What is the relation between u_{n+1} and u_n? Deduce u_n as a function of n.

   b. Same question for the last two coordinates s_n and t_n. Deduce a relation between $P(B_n)$ and $P(D_n)$, then a relation between $P(A_n)$ and $P(C_n)$, which are satisfied for any $n \geqslant 1$.

   c. Determine $\lim\limits_{n \to +\infty} U_n$ and deduce $\lim\limits_{n \to +\infty} X_n$.

4 Non diagonalizable endomorphism

4.1 Summary

When an endomorphisme is not diagonalizable, it remains possible to do a reduction. We are no longer in the ideal case but, as long as the characteristic polynomial is split (which is always true in $\mathbb{C}$), we can get a basis in which the endomorphism has a block diagonal matrix. Furthermore, it is possible to get triangle blocks.

4.2 Exercises

Exercise 4.17 *Reduction of a non diagonalizable matrix*
Let f be the endomorphism of $\mathbb{R}^3$ whose matrix in the standard basis is

$$A = \begin{pmatrix} -1 & 3 & 0 \\ 1 & 2 & 1 \\ 3 & -3 & 2 \end{pmatrix}$$

1. Show that $P_A(X) = (-1 - X)(2 - X)^2$, and that A is not diagonalizable.

2. Determine a basis $\mathscr{B}_{-1} = \{u_1\}$ of E_{-1} and a basis $\mathscr{B}_2 = \{u_2\}$ of E_2.

3. Study of $\operatorname{Ker}\left((A - 2I)^2\right)$.

   a. Show that $E_2 \subset \operatorname{Ker}\left((A - 2I)^2\right)$.
   b. Complete the set $\{u_2\}$ to get a basis $\mathscr{B}'_2$ of $\operatorname{Ker}\left((A - 2I)^2\right)$.

4. Show that the concatenation of $\mathscr{B}_{-1}$ and $\mathscr{B}'_2$ is a basis of $\mathbb{R}^3$.

5. Determine the matrix of f in this basis.

6. *(**Bonus**)* Solve the differential system $X'(t) = AX(t)$.

★ **Exercice 4.18** *Cayley-Hamilton theorem*

Let $f_0 : x \mapsto e^x$, $f_1 : x \mapsto xe^x$ and $f_2 : x \mapsto x^2 e^x$ be three real functions.

Let us define the vector space $E = \text{Span}(f_0, f_1, f_2)$ and the linear map $\Delta : f \mapsto f'$.

1. Show that $\mathscr{B} = (f_0, f_1, f_2)$ is a basis of E.

2. Show that Δ is an endomorphism of E.

3. Determine the matrix A of Δ in the basis $\mathscr{B}$. Is this matrix diagonalizable?

4. Using Cayley-Hamilton theorem, determine A^n.

5. Deduce $f^{(n)}$ for any $f \in E$.