Improper integrals

(two weeks)

(from Monday, 13 January 2020 to Friday, 24 January 2020)

Introduction

The purpose of this chapter is to give sense to integrals such as $I_1 = \displaystyle\int_1^{+\infty} \frac{1}{t^2}\,\mathrm{d}t$ or $I_2 = \displaystyle\int_0^1 \frac{1}{\sqrt{t}}\,\mathrm{d}t$.

For the first one, the function $f(t) = \frac{1}{t^2}$ is defined on the whole integration interval $[1, +\infty[$. Yet, one of the bounds is infinite. We say that $+\infty$ is an *improper bound*. We will define I_1 as

$$I_1 = \lim_{x \to +\infty} \int_1^x \frac{1}{t^2}\,\mathrm{d}t$$

But first of all, we have to prove that the integral converges, that is, whether the limit exists and is not infinite.

For the second integral I_2, the function $g(t) = \frac{1}{\sqrt{t}}$ is defined on $]0, 1]$ but it is not defined at $t = 0$. Hence, 0 is an *improper bound*. Here also, we can define I_2 as

$$I_2 = \lim_{x \to 0} \int_x^1 \frac{1}{\sqrt{t}}\,\mathrm{d}t$$

But to start with, we have to test whether the integral converges, that is, whether the limit exists and is not infinite.

When an integral $\displaystyle\int_a^b f(t)\,\mathrm{d}t$ has two improper bounds, that is, f continuous on $]a, b[$ but being defined neither in a nor in b (or one of these bounds being infinite), then we pick an arbitrary $c \in]a, b[$ and we study separately $\displaystyle\int_a^c f(t)\,\mathrm{d}t$ and $\displaystyle\int_c^b f(t)\,\mathrm{d}t$.

If both converge, then $\displaystyle\int_a^b f(t)\,\mathrm{d}t$ converges and is equal to $\displaystyle\int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t$. If one of the two diverges, then $\displaystyle\int_a^b f(t)\,\mathrm{d}t$ diverges.

Take care of this common mistake : for any $x > 0$, $\displaystyle\int_{-x}^x t\,\mathrm{d}t = \left[\frac{t^2}{2}\right]_{-x}^x = \frac{x^2}{2} - \frac{(-x)^2}{2} = 0$. Hence, $\displaystyle\int_{-\infty}^{+\infty} t\,\mathrm{d}t$ converges and is equal to 0.
This way of reasoning is wrong : since $\displaystyle\int_0^{+\infty} t\,\mathrm{d}t$ diverges, the whole integral $\displaystyle\int_{-\infty}^{+\infty} t\,\mathrm{d}t$ diverges.

The exercises that may be asked will consist in answering one (or both) of these questions :
- Determine the nature (convergent or divergent) of a specific integral.
- If it converges, determine its value.

1 Case when f is positive (or has constant sign) : comparison criteria

Consider an integral $\displaystyle\int_a^b f(t)\,\mathrm{d}t$, where the improper bound is b and f is non negative on $[a, b[$. In this case, the function

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is increasing on $[a, b[$. Then, the question whether $F(x)$ has a finite limit as x approaches b can be solved in testing whether F is bounded above or not.

There is a similar result when a is the improper bound. Thus, the nature of $\int_a^b f(t)\,\mathrm{d}t$ can be deduced from a comparison of f with another function g if we know the nature of $\int_a^b g(t)\,\mathrm{d}t$.

Take care with this common mistake : if we study $\int_0^{+\infty} \sin(t)\,\mathrm{d}t$, we note that for any $x > 0$,

$$\int_0^x \sin(t)\,\mathrm{d}t = \Big[-\cos(t)\Big]_0^x = 1 - \cos(x) \Longrightarrow 0 \leqslant \int_0^x \sin(t)\,\mathrm{d}t \leqslant 2$$

Hence, the integral does not tend to ∞ and it converges.
This way of reasoning is wrong : $1 - \cos(x)$ as no limit as x approaches $+\infty$. It is not valid because the sin function does not have a constant sign.

1.1 Local comparison of functions

Let f and g be two functions, defined on $]a, b[$. We say that, as t approaches a :

1. $f(t) = o\big(g(t)\big)$ or $f(t) \ll g(t)$ or «f is negligible before g» if
$$\lim_{t \to a} \frac{f(t)}{g(t)} = 0$$
For example, as t approaches 0, $t^2 = o(t)$ and, as t approaches $+\infty$, $t = o\left(t^2\right)$.
2. $f(t) = O\big(g(t)\big)$ or «f is dominated by g» if
$$\frac{f(t)}{g(t)} \text{ is bounded (above and below)}$$
This is equivalent to :
$$\exists M \in \mathbb{R} \text{ such that } \forall t \in]a, b[\,, \big|f(t)\big| \leqslant M\big|g(t)\big|$$
3. $f(t) \sim g(t)$ or «f and g are equivalent» if
$$\lim_{t \to a} \frac{f(t)}{g(t)} = 1$$
This is equivalent to : $f(t) = g(t) + o\big(g(t)\big)$.
For example, as t approaches 0, $t + t^2 \sim t$ and as t approaches $+\infty$, $t + t^2 \sim t^2$.

We get such relations using the reference comparisons, which are valid for any $\alpha > 0$, $\beta > 0$ and $\gamma > 0$:

1. As t approaches $+\infty$, $\ln^\alpha(t) \ll t^\beta \ll e^{\gamma t}$.
2. As t approaches 0, $\ln^\alpha(t) \ll \dfrac{1}{t^\beta}$.

We can also get such relations using Taylor expansions. You have to know the following ones : as t approaches 0,

$$\begin{array}{|lcl}
e^t & = & 1 + t + \dfrac{t^2}{2!} + \cdots + \dfrac{t^n}{n!} + o\big(t^n\big) \\[2ex]
\ln(1+t) & = & t - \dfrac{t^2}{2} + \dfrac{t^3}{3} + \cdots + \dfrac{(-1)^{n+1} t^n}{n} + o\big(t^n\big)) \\[2ex]
(1+t)^\alpha & = & 1 + \alpha t + \dfrac{\alpha(\alpha-1)}{2!} t^2 + \dfrac{\alpha(\alpha-1)(\alpha-2)}{3!} t^3 + o\big(t^3\big) \\[2ex]
\sin(t) & = & t - \dfrac{t^3}{3!} + \dfrac{t^5}{5!} + o\big(t^5\big) \\[4ex]
\cos(t) & = & 1 - \dfrac{t^2}{2!} + \dfrac{t^4}{4!} + o\big(t^4\big)
\end{array}$$

For example, $\sin(t) \sim t$ because $\sin(t) = t + o(t)$.

1.2 Using the comparisons to determine the nature of an integral

There are two kind of reference integrals :

1. Riemann integrals :
$$\int_0^1 \frac{1}{t^\alpha}\,\mathrm{d}t \text{ converges } \Longleftrightarrow \alpha < 1 \quad \text{and} \quad \int_1^{+\infty} \frac{1}{t^\alpha}\,\mathrm{d}t \text{ converges } \Longleftrightarrow \alpha > 1$$
2. Exponential integrals :
$$\int_0^{+\infty} e^{-\alpha t}\,\mathrm{d}t \text{ converges } \Longleftrightarrow \alpha > 0$$

Now, the following criteria may be used to determine the nature of an integral, by comparing $f(t)$ with one of the reference functions $\frac{1}{t^\alpha}$ or $e^{-\alpha t}$.

Theorem : consider two *positive* function, both continuous on an interval $]a,b[$, and suppose that both integrals $\displaystyle\int_a^b f(t)\,\mathrm{d}t$ and $\displaystyle\int_a^b g(t)\,\mathrm{d}t$ have one improper bound. Then :

1. If $f(t) \leqslant g(t)$ (this includes $f(t) = o\big(g(t)\big)$ or $f(t) = o\big(g(t)\big)$ as t approaches the improper bound), then
$$\int_a^b g(t)\,\mathrm{d}t \text{ converges } \Longrightarrow \int_a^b f(t)\,\mathrm{d}t \text{ converges} \quad \text{and} \quad \int_a^b f(t)\,\mathrm{d}t \text{ diverges } \Longrightarrow \int_a^b g(t)\,\mathrm{d}t \text{ diverges}$$
2. If $f(t) \sim g(t)$ as t approaches the improper bound, then
$$\int_a^b f(t)\,\mathrm{d}t \quad \text{and} \int_a^b g(t)\,\mathrm{d}t \quad \text{have the same nature}$$

Remarks :

1. The conditions $f(t) \leqslant g(t)$, $f(t) \geqslant 0$ and $g(t) \geqslant 0$ may be satisfied only in an interval $[c,b[\subset [a,b[$ (if b is the improper bound), or in an interval $]a,c] \subset]a,b]$ (if a is the improper bound).
2. If the function f and g are negative instead of being positive, these rules can easily be adapted. Just multiply the functions by -1.

1.3 Examples

Example 1 : determine the nature of $\displaystyle\int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t$.

The function $f(t) = \dfrac{e^{-t}}{\sqrt{t}}$ is continuous on $]0,+\infty[$. There are two improper bounds.

1. As t tends to 0, $f(t) \sim \dfrac{1}{\sqrt{t}} = \dfrac{1}{t^{\frac{1}{2}}}$. Since the functions are positive and $\displaystyle\int_0^1 \frac{1}{t^{\frac{1}{2}}}$ converges (Riemann, $\alpha = \frac{1}{2} < 1$), then $\displaystyle\int_0^1 f(t)\,\mathrm{d}t$ converges.
2. As t tends to $+\infty$, $f(t) \ll \dfrac{1}{t^2}$ because $t^2 f(t) = t^{\frac{3}{2}} e^{-t} \xrightarrow[t\to+\infty]{} 0$. Since $\displaystyle\int_1^{+\infty} \frac{1}{t^2}\,\mathrm{d}t$ converges (Riemann, $\alpha = 2 > 1$, then $\displaystyle\int_1^{+\infty} f(t)\,\mathrm{d}t$ converges.
3. Finally, $\displaystyle\int_0^{+\infty} f(t)\,\mathrm{d}t$ converges.

Example 2 : determine the nature of $\displaystyle\int_0^1 \frac{1}{\sin(t)}\,\mathrm{d}t$.

The function $f(t) = \dfrac{1}{\sin(t)}$ is continuous on $]0,1]$. 0 is an improper bound.

As t approaches 0, $\sin(t) \sim t \Longrightarrow \dfrac{1}{\sin(t)} \sim \dfrac{1}{t} > 0$.

Since $\displaystyle\int_0^1 \frac{1}{t}\,\mathrm{d}t$ diverges (Riemann, $\alpha = 1$), then $\displaystyle\int_0^1 \frac{1}{\sin(t)}\,\mathrm{d}t$ diverges.

2 Case when f has no constant sign

In this case, no theorem can prove the divergence of the integral. There is one theorem :

Theorem : let $\displaystyle\int_a^b f(t)\,\mathrm{d}t$ be an integral with one or two improper bounds. Then :

$$\int_a^b \left|f(t)\right| \mathrm{d}t \text{ converges} \Longrightarrow \int_a^b f(t)\,\mathrm{d}t \text{ converges}$$

In this case, we say that $\displaystyle\int_a^b f(t)\,\mathrm{d}t$ «converges absolutely».

Example : determine the nature of the integral $\displaystyle\int_0^{+\infty} \frac{\sin(t)}{t^{\frac{3}{2}}}\,\mathrm{d}t$.

The function $f(t) = \dfrac{\sin(t)}{t^{\frac{3}{2}}}$ is continuous on $]0,+\infty[$. The integral has two improper bounds.

1. In 0, $\sin(t) \sim t \Longrightarrow f(t) \sim \dfrac{t}{t^{\frac{3}{2}}} = \dfrac{1}{t^{\frac{1}{2}}}$.

   Since $\displaystyle\int_0^1 \frac{1}{t^{\frac{1}{2}}}\,\mathrm{d}t$ converges (Riemann, $\alpha = \frac{1}{2} < 1$), we can deduce that $\displaystyle\int_0^1 f(t)\,\mathrm{d}t$ converges.

2. In $+\infty$, $\left|f(t)\right| \leqslant \dfrac{1}{t^{\frac{3}{2}}}$. Since the functions are positive and since $\displaystyle\int_1^{+\infty} \frac{1}{t^{\frac{3}{2}}}\,\mathrm{d}t$ converges (Riemann, $\alpha = \frac{3}{2} > 1$), we can deduce that $\displaystyle\int_1^{+\infty} \left|f(t)\right| \mathrm{d}t$ converges.

   Hence, $\displaystyle\int_1^{+\infty} f(t)\,\mathrm{d}t$ converges absolutely, so it converges.

3. Finally, $\displaystyle\int_0^{+\infty} \frac{\sin(t)}{t^{\frac{3}{2}}}\,\mathrm{d}t$ converges.

3 Other methods of integration

When an integral has no improper bounds, two methods may be used to evaluate its value :

1. Integration by parts (IBP).
2. Integration by substitution.

In this section, we specify how these methods may be extended to improper integrals.

3.1 Integration by parts

Consider an integral having the form $\displaystyle\int_a^b u(t)v'(t)\,\mathrm{d}t$. If u and v are of class C^1 on $[a,b]$, then there are no improper bounds and the IBP formula is :

$$\int_a^b u(t)v'(t)\,\mathrm{d}t = \Big[u(t)v(t)\Big]_a^b - \int_a^b u'(t)v(t)\,\mathrm{d}t$$

The question is : how to extend this formula when there are one or two improper bounds ?

3.1.1 The IBP rule

Defining $\Big[u(t)v(t)\Big]_a^b$: we define it as

$$\Big[u(t)v(t)\Big]_a^b = \lim_{t\to b} u(t)v(t) - \lim_{t\to b} u(t)v(t)$$

Remarks :

1. If one of these limit does not exist or is infinite, then the expression above does not have sense and cannot be used.

   In practical situations, when you use it, *you have to justify later on your worksheet that both limits exist.*

   For example, you can write : $\displaystyle\int_1^{+\infty} \frac{1}{t^2}\,\mathrm{d}t = \left[-\frac{1}{t}\right]_1^{+\infty} = 1$ because $\displaystyle\lim_{t\to+\infty} -\frac{1}{t} = 0$.

   But you cannot write : $\displaystyle\int_1^{+\infty} \frac{1}{t^{\frac{1}{2}}}\,\mathrm{d}t = \left[2t^{\frac{1}{2}}\right]_1^{+\infty} = +\infty$ because $\displaystyle\lim_{t\to+\infty} 2t^{\frac{1}{2}} = +\infty$.

2. In the particular case where there are no improper bounds, then

$$\lim_{t\to b} u(t)v(t) = u(b)v(b) \quad \text{and} \quad \lim_{t\to a} u(t)v(t) = u(a)v(a)$$

   Thus, the definition of $\Big[u(t)v(t)\Big]_a^b$ is not altered in that case.

Theorem

Consider an integral having the form $\displaystyle\int_a^b u(t)v'(t)\,\mathrm{d}t$. Assume that :

1. u and v are of class C^1 on $]a,b[$
2. The expression $\Big[u(t)v(t)\Big]_a^b$ has a sense : $u(t)v(t)$ has finite limits as t approaches a and b.

Then :

1. The integrals $\displaystyle\int_a^b u(t)v'(t)\,\mathrm{d}t$ and $\displaystyle\int_a^b u'(t)v(t)\,\mathrm{d}t$ have the same nature.
2. If they converge, then their values satisfy the relation

$$\int_a^b u(t)v'(t)\,\mathrm{d}t = \Big[u(t)v(t)\Big]_a^b - \int_a^b u'(t)v(t)\,\mathrm{d}t$$

3.1.2 Examples

Example 1 : determine the nature of $\displaystyle\int_1^{+\infty} \frac{\ln(t)}{t^2}\,\mathrm{d}t$. If it converges, determine its value.
We use an integration by parts, setting :

$$\left|\begin{array}{lcl} u(t) & = & \ln(t) \\ v'(t) & = & \dfrac{1}{t^2} \end{array}\right. \quad \Longrightarrow \quad \left|\begin{array}{lcl} u'(t) & = & \dfrac{1}{t} \\ v(t) & = & -\dfrac{1}{t} \end{array}\right.$$

We get : $\displaystyle\int_1^{+\infty} \frac{\ln(t)}{t^2}\,\mathrm{d}t = \left[-\frac{\ln(t)}{t}\right]_1^{+\infty} + \int_1^{+\infty} \frac{1}{t^2}\,\mathrm{d}t$

The expression $\displaystyle\left[-\frac{\ln(t)}{t}\right]_1^{+\infty}$ is allowed and equal to

$$\left[-\frac{\ln(t)}{t}\right]_1^{+\infty} = \lim_{t\to+\infty} -\frac{\ln(t)}{t} + \frac{\ln(1)}{1} = 0$$

Hence, the integral has the nature of $\displaystyle\int_1^{+\infty} \frac{1}{t^2}\,\mathrm{d}t$, which is a convergent Riemann integral.
Furthermore,

$$\int_1^{+\infty} \frac{\ln(t)}{t^2}\,\mathrm{d}t = \underbrace{\left[-\frac{\ln(t)}{t}\right]_1^{+\infty}}_{0} + \int_1^{+\infty} \frac{1}{t^2}\,\mathrm{d}t = \left[-\frac{1}{t}\right]_1^{+\infty} = 1$$

since $\displaystyle\lim_{t\to+\infty} -\frac{1}{t} = 0$.

Example 2 : determine the nature of $\displaystyle\int_1^{+\infty} \frac{\cos(t)}{t^{\frac{1}{2}}}\,\mathrm{d}t$.
We do an IBP, setting :

$$\left|\begin{array}{lcl} u(t) & = & \dfrac{1}{t^{\frac{1}{2}}} \\ v'(t) & = & \cos(t) \end{array}\right. \implies \left|\begin{array}{lcl} u'(t) & = & -\dfrac{1}{2t^{\frac{3}{2}}} \\ v(t) & = & \sin(t) \end{array}\right.$$

Then we get :

$$\int_1^{+\infty} \frac{\cos(t)}{t^{\frac{1}{2}}}\,\mathrm{d}t = \left[\frac{\sin(t)}{t^{\frac{1}{2}}}\right]_1^{+\infty} + \int_1^{+\infty} \frac{\sin(t)}{2t^{\frac{3}{2}}}\,\mathrm{d}t$$

The term $\displaystyle\left[\frac{\sin(t)}{t^{\frac{1}{2}}}\right]_1^{+\infty}$ makes sense because $\displaystyle\frac{\sin(t)}{t^{\frac{1}{2}}} \xrightarrow[t\to+\infty]{} 0$.
Hence, $\displaystyle\int_1^{+\infty} \frac{\cos(t)}{t^{\frac{1}{2}}}\,\mathrm{d}t$ has the same nature than $\displaystyle\int_1^{+\infty} \frac{\sin(t)}{2t^{\frac{3}{2}}}\,\mathrm{d}t$. But the latter converges absolutely : for any $t \in [1, +\infty[$,

$$\left|\frac{\sin(t)}{2t^2}\right| \leqslant \frac{1}{2t^2}$$

and $\displaystyle\int_1^{+\infty} \frac{1}{2t^2}\,\mathrm{d}t = \frac{1}{2}\int_1^{+\infty} \frac{1}{t^2}\,\mathrm{d}t$ is a convergent Riemann integral.
Finally, $\displaystyle\int_1^{+\infty} \frac{\cos(t)}{t^{\frac{1}{2}}}\,\mathrm{d}t$ converges.

3.2 Integration by substitution

Consider an integral $\displaystyle\int_a^b f(t)\,\mathrm{d}t$. If there are no improper bounds, the following theorem holds :

Theorem
Let φ be defined on an interval $[\alpha, \beta]$ (or $[\beta, \alpha]$ if $\beta < \alpha$) such that :

1. φ is of class C^1 on $[\alpha, \beta]$.
2. $\varphi(\alpha) = a$ and $\varphi(\beta) = b$.
3. φ is bijective from $[\alpha, \beta]$ to $[a, b]$.

Then $\displaystyle\int_a^b f(t)\,\mathrm{d}t = \int_\alpha^\beta f\big(\varphi(x)\big)\,\mathrm{d}x$

In the right hand side integral, the variable t has been substituted by a new variable, x.
Note that the condition that φ should be bijective may be relaxed. Just take care that $\varphi(x)$ does not take values for which $f\big(\varphi(x)\big)$ may be undefined.

If the initial integral has improper bounds, the theorem is altered as follows :

Theorem for improper integrals
Let φ be defined on an interval $]\alpha, \beta[$ (or $]\beta, \alpha[$ if $\beta < \alpha$) such that :

1. φ is of class C^1 on $]\alpha, \beta[$.
2. $\lim\limits_{x\to\alpha} \varphi(x) = a$ and $\lim\limits_{x\to\beta} \varphi(x) = b$.
3. φ is bijective from $]\alpha, \beta[$ to $]a, b[$.

Then the integrals $\displaystyle\int_a^b f(t)\,\mathrm{d}t$ and $\displaystyle\int_\alpha^\beta f\big(\varphi(x)\big)\,\mathrm{d}x$ have the same nature.
If they converge, their values are equal.

Example : using the substitution $x = \sqrt{t}$, determine the nature of $I = \displaystyle\int_0^4 \frac{1}{(1+t)\sqrt{t}}\,\mathrm{d}t$.

1. Bounds for x : $\sqrt{0} = 0$ and $\sqrt{4} = 2$
2. $\mathrm{d}x$: we have $t = x^2 \implies \mathrm{d}t = 2x\,\mathrm{d}x$

3. Calculation : we get

$$I = \int_0^2 \frac{1}{(1+t)\sqrt{t}} \underbrace{2x \, \mathrm{d}x}_{\mathrm{d}t} = \int_0^2 \frac{2}{1+x^2} \, \mathrm{d}x = \Big[2\arctan(x) \Big]_0^2 = 2\arctan(2)$$