

Chapter 6



ELECTROSTATICS

GAUSS' THEOREM

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1 - Introduction

Gauss's theorem allows to calculate the electric field \vec{E} from its flux $\Phi(\vec{E})$.

The flux is easier to use for systems with geometrical symmetries (spherical, cylindrical,...).

To calculate the flux one needs to know the direction of the field \vec{E} which can be deduced from the symmetries.

2 – Gauss' theorem

The flux of the electric field \vec{E} through a closed surface, called Gaussian surface S_G , is equal to the sum of the charges enclosed in the surface S_G divided by ϵ

$$\oiint_{S_g} \vec{E} \cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon}$$

where ϵ is called the permittivity of the medium and ϵ_0 the permittivity of the air and/or of free space. The relation $\epsilon = \epsilon_r \epsilon_0$ links both ϵ and ϵ_0 where ϵ_r is the relative permittivity $\epsilon_r = 1 \Rightarrow \epsilon = \epsilon_0$

- \oiint_S stems for an integration on a closed surface
- S_G is the Gaussian surface which is a fictitious surface with no material reality which has the following properties:
 - S_G is closed
 - S_G must be chosen such that the normal component of the electric field is constant thanks to geometrical symmetries
 - S_G passes through the point M where you want to calculate the field $\vec{E}(M)$
 - S_G is chosen such that its geometry is related (coherent) with the physical system's geometry.
- Q_{int} is the total charge enclosed in the surface S_G

3 – Meaning of the notions used in the theorem

1 – Definition of a flux through a surface

Mathematically the flux of any vector field through a surface is defined by:

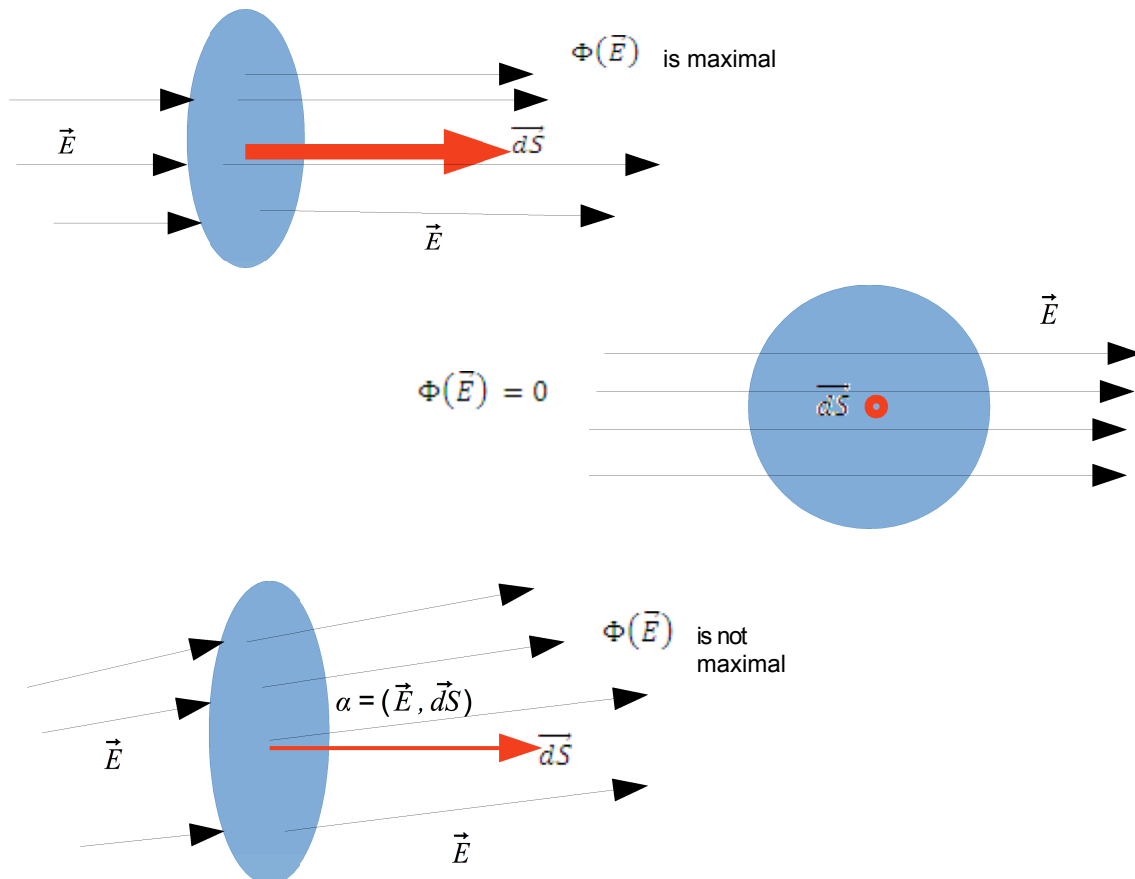
$$\Phi(\vec{E}) = \oiint \vec{E} \cdot \vec{dS} = \oiint \vec{E} \cdot dS \cos(\alpha) \quad \text{where} \quad \alpha = (\vec{E}, \vec{dS})$$

The flux represents the intensity of « the field lines' flow of the vector field » through a given surface.

It's rather intuitive to understand what is a flux of something (electric field, water, etc.) passing through a surface and what it is depending of: the magnitude of the field, the size of the surface and the orientation of the field with respect to the orientation of the surface.

Example :

a- Field lines of \vec{E} crossing a disk of surface S



1 - If \vec{E} is parallel to \vec{dS} (the normal to the surface) then the flux $\Phi(\vec{E})$ is maximal

2 - If \vec{E} skims (is parallel to) the surface S then the flux $\Phi(\vec{E})$ is zero.

3 - If \vec{E} is not parallel to \vec{dS} then the flux $\Phi(\vec{E}) = E \cdot S \cdot \cos(\alpha)$ where $\alpha = (\vec{E}, \vec{dS})$

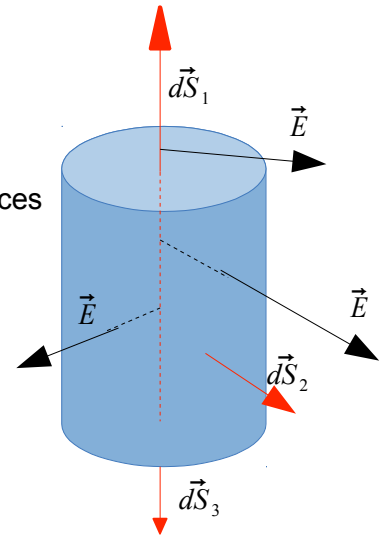
b- Radial electric field

\vec{dS}_1 and \vec{dS}_3 are the normal of the circular basis of the cylinder, they have opposite orientation.

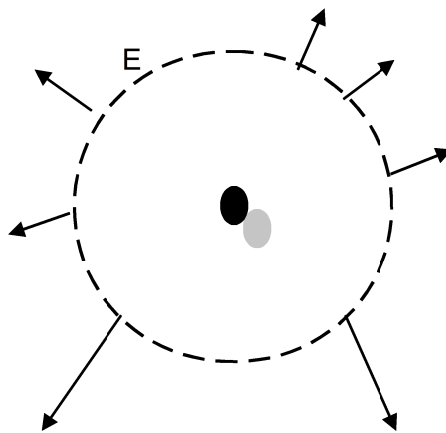
\vec{dS}_2 is the normal to the lateral surface.

- $\Phi_{S_1}(\vec{E}) = 0 = \Phi_{S_3}(\vec{E})$ because the field lines skim the surfaces S_1 and S_3 without crossing them.

- The flux is non zero and even maximum through the lateral surface S_2



Upper view



$$\left\{ \begin{array}{l} \vec{E} \cdot \vec{dS}_2 = E dS_2 \cos(0) \Rightarrow \Phi(S_2) \text{ max} \\ \vec{E} \cdot \vec{dS}_3 = E dS_3 \cos\left(\frac{\pi}{2}\right) \Rightarrow \Phi(S_3) = 0 \end{array} \right.$$

2 – Properties of the surface element \vec{dS}

◇ \vec{dS} must verify:

$$\left\{ \begin{array}{l} \vec{dS} \text{ is perpendicular to the gaussian surface} \\ \vec{dS} \text{ is oriented wrt to the rule of the right hand} \\ \text{Its magnitude is } dS \end{array} \right.$$

$$dS = \begin{cases} dx dy & \text{: for a plane surface} \\ r dr d\theta & \text{: for a circular base surface} \\ r d\theta dz & \text{: for a cylindrical lateral surface} \end{cases}$$

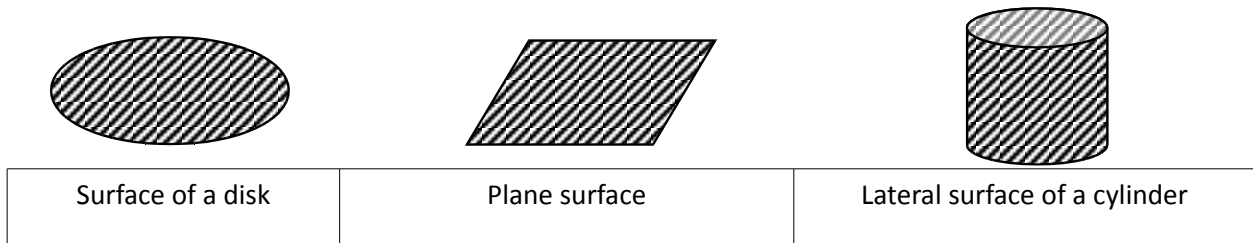
3 - Notions of closed surface

A closed surface separates or isolates the external medium from the internal one.

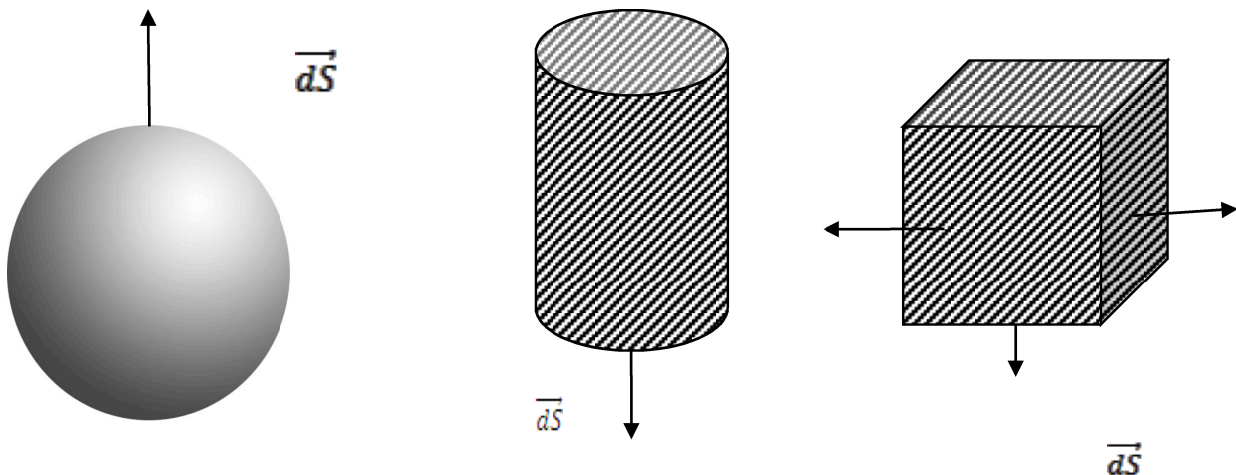
For a closed surface, the \vec{dS} vector is oriented outward (the surface).

Examples :

a) Non – closed surfaces:



b) Closed surfaces:



4 - Expression of the internal charge

The internal charge represents the total charge enclosed inside the gaussian surface S_g . Its expressions depending of the type of distribution are:

For a lineic distribution: $Q = \int_L \lambda \cdot dl$; where λ is the lineic charge density.

For a surfacic distribution: $Q = \iint_S \sigma \cdot dS$; where σ is the surfacic charge density.

For a volumic distribution: $Q = \iiint_\tau \rho \cdot d\tau$; where ρ is the volumic charge density.

4 - Applications

- Infinite wire
- Hollow cylinder charged in surface
- Volumic charged sphere