Assignment 3

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https://github.com/KBVijayVarma/AI1103−Assignment−3

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Problem GATE 2015 (MA), Q. 8

Let $X \sim B(5, \frac{1}{2})$ and $Y \sim U(0,1)$. Then $\frac{P(X+Y\leq 2)}{P(X+Y\geq 5)}$ is equal to
(where
$B(n,p)$: Binomial distribution with n trials and success probability p; $n \in \{1,2,\dots\}$ and $p \in (0,1)$
U(a,b) : Uniform distribution on the interval $(a,b), -\infty < a < b < \infty$)

Solution

Given X is a Binomial Random Variable with 5 trails and success probability $p = 0.5$ and Y is a Continuous Random Variable over the interval $(0,1)$.
So, $X \in \{0,1,2,3,4,5\}$ and $Y = U(0,1)$
Since X and Y are Independent Random Variables,

$$\Pr(X+Y\leq 2) = \Pr(X=a, Y\leq 2-a) \tag{0.0.1}$$

$$= \sum_{a=0}^{a=2} \Pr(X=a)\Pr(Y\leq 2-a) \tag{0.0.2}$$

$$\Pr(X+Y\leq 2) = \Pr(X=0)\Pr(Y\leq 2) + \Pr(X=1)\Pr(Y\leq 1) + \Pr(X=2)\Pr(Y\leq 0) \tag{0.0.3}$$

Since X is a Binomial Random Variable,

$$\Pr(X=k) = \begin{cases} {}^{n}C_k p^{n-k}(1-p)^k & 0\leq k\leq 5 \\ 0 & otherwise \end{cases} \tag{0.0.4}$$

Substituting the values of n = 5 and p = $\frac{1}{2}$ in (0.0.4), we get

$$\Pr(X=k) = {}^{5}C_k \left(\frac{1}{2}\right)^{5-k}\left(\frac{1}{2}\right)^{k} = {}^{5}C_k \left(\frac{1}{2}\right)^{5}$$

Also, the Cumulative Distribution Function of Y is defined as

$$CDF(Y) = F_Y(a) = \Pr(Y\leq a) = \begin{cases} 0 & a\leq 0 \\ a & 0<a<1 \\ 1 & a\geq 1 \end{cases} \tag{0.0.5}$$

By substituting the probability values from (0.0.4) and (0.0.5) in (0.0.3), we get

$$\Pr(X+Y\leq 2) = {}^{5}C_0\left(\frac{1}{2}\right)^5(1) + {}^{5}C_1\left(\frac{1}{2}\right)^5(1) + {}^{5}C_2\left(\frac{1}{2}\right)^5(0) \tag{0.0.6}$$

$$= (1)\left(\frac{1}{32}\right) + (5)\left(\frac{1}{32}\right) + 0 \tag{0.0.7}$$

$$= \left(\frac{1}{32}\right) + \left(\frac{5}{32}\right) \tag{0.0.8}$$

$$= \frac{6}{32} \tag{0.0.9}$$

$$\Pr(X+Y\leq 2) = \frac{3}{16} \tag{0.0.10}$$

Now,

$$\Pr(X+Y\geq 5) = 1 - \Pr(X+Y<5) \tag{0.0.11}$$

$$= 1 - [\Pr(X+Y\leq 5) - \Pr(X+Y=5)] \tag{0.0.12}$$

But, as Y is a Continuous Random Variable over $(0,1)$, so $\Pr(Y=k) = 0 \ \forall \ k \in [0,1]$. Therefore considering all possible cases,

$$\Pr(X+Y=5) = \Pr(X=4)\Pr(Y=1) + \Pr(X=5)\Pr(Y=0) \quad (0.0.13)$$

$$= \Pr(X=4)(0) + \Pr(X=5)(0) \quad (0.0.14)$$
$$= 0 + 0 \quad (0.0.15)$$

$$\Pr(X+Y=5) = 0 \quad (0.0.16)$$

Hence, by substituting (0.0.16) in (0.0.12), we get

$$\Pr(X+Y\geq 5) = 1 - [\Pr(X+Y\leq 5) - 0] \quad (0.0.17)$$
$$\Pr(X+Y\geq 5) = 1 - \Pr(X+Y\leq 5) \quad (0.0.18)$$
$$\Pr(X+Y\geq 5) = 1 - \Pr(X=a, Y\leq 5-a) \quad (0.0.19)$$

$$= 1 - \left[\sum_{a=0}^{a=5} \Pr(X=a)\Pr(Y\leq 5-a)\right] \quad (0.0.20)$$

$$= 1 - [\Pr(X=0)\Pr(Y\leq 5) + \Pr(X=1)\Pr(Y\leq 4) + \Pr(X=2)\Pr(Y\leq 3) + \Pr(X=3)\Pr(Y\leq 2) + \Pr(X=4)\Pr(Y\leq 1) + \Pr(X=5)\Pr(Y\leq 0)] \quad (0.0.21)$$

By substituting the probability values from (0.0.4) and (0.0.5) in (0.0.21), we get

$$\Pr(X+Y\geq 5) = 1 - [{}^5C_0\left(\frac{1}{2}\right)^5(1) + {}^5C_1\left(\frac{1}{2}\right)^5(1) + {}^5C_2\left(\frac{1}{2}\right)^5(1) + {}^5C_3\left(\frac{1}{2}\right)^5(1) + {}^5C_4\left(\frac{1}{2}\right)^5(1) + {}^5C_5\left(\frac{1}{2}\right)^5(0)] \quad (0.0.22)$$

$$\Pr(X+Y\geq 5) = 1 - \left(\frac{1}{2}\right)^5 [{}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4] \quad (0.0.23)$$

$$= 1 - \left(\frac{1}{32}\right)[1+5+10+10+5] \quad (0.0.24)$$
$$= 1 - \left(\frac{1}{32}\right)[31] = \frac{1}{32} \quad (0.0.25)$$

Hence, $\Pr(X+Y\leq 2) = \frac{3}{16}$ and $\Pr(X+Y\geq 5) = \frac{1}{32}$.

$$\therefore \frac{\Pr(X+Y\leq 2)}{\Pr(X+Y\geq 5)} = \frac{\frac{3}{16}}{\frac{1}{32}} = 6.$$
$$\therefore \frac{\Pr(X+Y\leq 2)}{\Pr(X+Y\geq 5)} = 6$$

Hence, the required ratio is 6 .

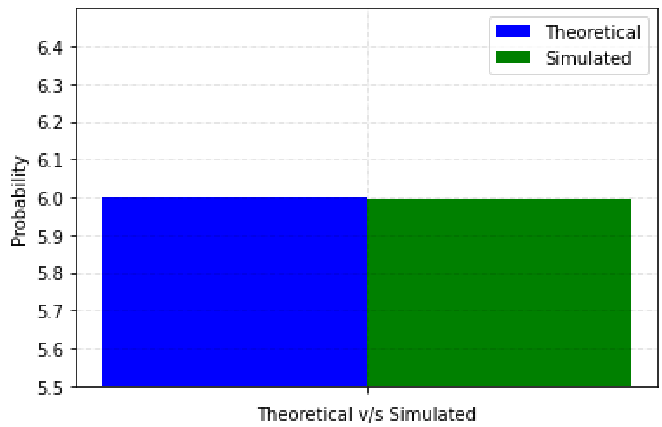

Fig. 0