

# Projects 1 (Not Ready Yet)

## Chapter 1 projects

### Project 1

(Shores 62)

**Problem Description:** You are given a long tube of still dry air in which there are 7 sampling/insertion points equally spaced  $1/6$  meters apart from each other. The position of each point is measured by setting the leftmost point at 0.0 meters and rightmost at 1.0 meters. Initially, a small amount of a certain gas is inserted in the central insertion point. Subsequently, measurements of the concentration of the gas at each sampling/insertion point are taken at later times in seconds. The results of these measurements, which you may assume are accurate to about 2-3 digits, are specified in Table 1.1. Based on this information, your task is to determine the best estimate you can find for the true value of the diffusion coefficient  $D$  of this gas in a motionless air medium. Use this estimate and a marching method to calculate values of the material density function on the interval  $[0, 1]$  at times  $t = 210$  and  $t = 300$  and at the given spatial nodes.

**Procedure:** You should use equation (1.10) or some variant to move backward and forward in time. These will result in linear systems, which ALAMA calculator or another technology tool can solve. One way to proceed is simply to use trial and error until you think you've hit on a reasonable value of  $D$ , that is, the one that gives the best approximation to  $t = 180$  from the  $t = 360$

Sec \ Meter	0	1/6	1/3	1/2	2/3	5/6	1
$t = 240$	0.0	0.032	1.23	3.69	1.23	0.032	0.0
$t = 270$	0.0	0.051	1.21	3.48	1.21	0.051	0.0

Table 1.1: Concentration data measurements of a gaseous material.

values. Do not expect perfect matches - the data is relatively sparse. Then march backwards in time once more to get the initial values at  $t = 0$ . Finally, march forward in time to compute and plot the resulting approximate density function.

Output: Discuss your results and provide a graph of profiles of the material density function at times in the data table along with your computed profiles.

Comments: This project introduces you to a very interesting area of mathematics called “inverse theory.” The idea is, rather than proceeding from problem (the governing equations for concentration values) to solution (concentration profiles), you are given the “solution,” namely the measured solution values at various points, and are to determine from this information the “problem,” i.e., the diffusion coefficient needed to define the governing equations.

## **Chapter 2 Projects**

### **Project 1: LU Factorization**

(Shores p. 177) Write a program module that implements Theorem 2.14 using partial pivoting and implicit row exchanges. This means that space is allocated for the  $n \times n$  matrix  $A = [a[i, j]]$  and an array of row indices, say  $\text{indx}[i]$ . Initially,  $\text{indx}$  should consist of the integers  $1, 2, \dots, n$ . Whenever two rows need to be exchanged, say the first and third, then the indices  $\text{indx}[1]$  and  $\text{indx}[3]$  are exchanged. References to array elements throughout the Gaussian elimination process should be indirect: Refer to the  $(1, 4)$  th entry of  $A$  as the element  $a[\text{indx}[1], 4]$ . This method of reference has the same effect as physically exchanging rows, but without the work. It also has the appealing feature that we can design the algorithm as though no row exchanges have taken place provided we replace the direct reference  $a[i, j]$  by the indirect reference  $a[\text{indx}[i], j]$ . The module should return the lower/upper matrix in the format of Example 2.70 as well as the permuted array  $\text{indx}[i]$ . Effectively, this index array tells the user what the permutation matrix  $P$  is.

Use this module to implement an LU system solver module that uses the LU factorization to solve a general linear system. Also write a module that finds the inverse of an  $n \times n$  matrix  $A$  by first using the LU factorization module, then making repeated use of the LU system solver to solve  $A\mathbf{x}^{(i)} = \mathbf{e}_i$ , where  $\mathbf{e}_i$  is the  $i$  th column of the identity. Then we will have

$$A^{-1} = [\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}]$$

Be sure to document and test your code and report on the results.

### **Project 2: Markov Chains**

(Shores p. 177)

Refer to Example 2.19 and Section 2.3 for background. Three automobile insurance firms compete for a fixed market of customers. Annual premiums are sold to these customers. Label the companies A, B, and C. You work for Company A, and your team of market analysts

has done a survey that draws the following conclusions: In each of the past three years, the number of A customers switching to B is 20%, and to C is 30%. The number of B customers switching to A is 20%, and to C is 20%. The number of C customers switching to A is 30%, and to B is 10%. Those who do not switch continue to use their current company's insurance for the next year. Model this market as a Markov chain. Display the transition matrix for the model. Illustrate the workings of the model by showing what it would predict as the market shares three years from now if currently A, B, and C owned equal shares of the market.

The next part of your problem is as follows: Your team has tested two advertising campaigns in some smaller test markets and are confident that the first campaign will convince 20% of the B customers who would otherwise stay with B in a given year to switch to A. The second advertising campaign would convince 20% of the C customers who would otherwise stay with C in a given year to switch to A. Both campaigns have about equal costs and would not change other customers' habits. Make a recommendation, based on your experiments with various possible initial state vectors for the market. Will these campaigns actually improve your company's market share? If so, which one do you recommend? Write up your recommendation in the form of a report, with supporting evidence. It's a good idea to hedge on your bets a little by pointing out limitations to your model and claims, so devote a few sentences to those points.

It would be a plus to carry the analysis further (your manager might appreciate that). For instance, you could turn the additional market share from, say B customers, into a variable and plot the long-term gain for your company against this variable. A manager could use this data to decide whether it was worthwhile to attempt gaining more customers from B.

### ***Project 3 Sports Ranking***

(Shores p. 180)

Refer to Example 2.24 and Section 2.3 for background. As a sports analyst you are given the following data about a league of seven teams numbered 1-7, where the pair  $(j, k)$  represents a game in which team  $j$  defeated team  $k$  :

$$E = \{(1, 2), (7, 3), (2, 4), (4, 5), (3, 2), (5, 1), (6, 1), (3, 1), (7, 2), (2, 6), \\ (3, 4), (7, 4), (5, 7), (6, 4), (3, 5), (5, 6), (7, 1), (5, 2), (7, 6), (1, 4), (6, 3)\}$$

Based on these data you are to rank the teams. To this end, begin with the simplest method, ranking by win/loss record. Next, treat the data as defining a digraph. Begin this analysis by constructing the adjacency matrix of this digraph and drawing a picture of the digraph either by hand or using some software. Then rank the teams by using the following methods: First use the method of Example 2.26 to find a power ranking of each team. Then use the reverse PageRank idea of Example 2.47 to rank the the teams.

Next, suppose you are given additional information, namely, the game margins (winning score minus losing score) for each game. Following is a list of these margins matching the order of matches in the definition of  $E$  :

$$M = \{4, 8, 7, 3, 7, 7, 23, 15, 6, 18, 13, 14, 7, 13, 7, 18, 45, 10, 19, 14, 13\}$$

In order to utilize these data examine your picture of the digraph and label each edge with the margin that matches it in  $M$ . You are now dealing with a weighted graph and one can construct a different sort of “adjacency matrix” by entering this margin in the  $(i, j)$  th entry according as team  $i$  defeated team  $j$  by that margin. Use this approach to calculate “power ranking”.

## Chapter 3 Projects

### Project 4

### Projects: Modeling with Directed Graphs

Project Descriptions: These projects introduce more applications of digraphs as mathematical modeling tools. You are given that the digraph  $G$  has vertex set  $V = \{1, 2, 3, 4, 5, 6\}$  and edge set

$$E = \{(1, 2), (2, 3), (3, 4), (4, 2), (1, 4), (3, 1), (3, 6), (6, 3), (4, 5), (5, 6)\}$$

Address the following points regarding  $G$ .

1. (a) Draw a picture of this digraph. You may leave space in your report and draw this by hand, or if you prefer, you may use the computer drawing applications available to you on your system.
- (b) Exhibit the incidence matrix  $A$  of this digraph and find a basis for  $\mathcal{N}(A)$  using its reduced row echelon form. Some of the basis elements may be algebraic but not directed loops. Use this basis to find a basis of directed loops (e.g., non-directed basis element  $\mathbf{c}_1$  might be replaced by directed  $\mathbf{c}_1 + \mathbf{c}_2$  ).
2. Think of the digraph as representing an electrical circuit where an edge represents some electrical object like a resistor or capacitor. Each node represents the circuit space between these objects. and we can attach a potential value to each node, say the potentials are  $x_1, \dots, x_6$ . The potential difference across an edge is the potential value of head minus tail. Kirchhoff's second law of electrical circuits says that the sum of potential differences around a circuit loop must be zero. Assume and use the fact (p. 422) that

$A\mathbf{x} = \mathbf{b}$  implies that for all  $\mathbf{y} \in \mathcal{N}(A^T)$ ,  $\mathbf{y}^T \mathbf{b} = 0$  to find conditions that a vector  $\mathbf{b}$  must satisfy in order for it to be a vector of potential differences for some potential distribution on the vertices.

### **Project 5**

3. Assume that across each edge of a circuit a current flows. Thus, we can assign to each edge a “weight,” namely the current flow along the edge. This is an example of a weighted digraph. However, not just any set of current weights will do, since Kirchhoff’s first law of circuits says that the total flow of current in and out of any node should be 0. Use this law to find a matrix condition that must be satisfied by the currents and solve it to exhibit some current flows.
4. Think of the digraph as representing a directed communications network. Here loops determine which nodes have bidirectional communication since any two nodes of a loop can only communicate with each other by way of a loop. By examining only a basis of directed loops how could you determine which nodes in the network can communicate with each other?

### **Project 6**

5. Think of vertices of the digraph as representing airports and edges representing flight connections between airports for Gamma Airlines. Suppose further that for each connection there is a maximum number of daily flights that will be allowed by the destination airport from an origin airport and that, in the order that the edges in  $E$  are listed above, these limits are

$$M = \{4, 3, 8, 7, 2, 6, 7, 10, 5, 8\}.$$

Now suppose that Gamma wants to maximize the flow of flights into airport 1 and out of airport 6. Count inflows into an airport as positive and outflows as negative. Assume that the net in/outflow of Gamma flights at each airport 1 to 5 is zero, while the net inflow of such flights into airport 1 matches the net outflow from 6.

- (a) Describe the problem of maximizing this inflow to airport 1 as a linear programming problem and express it in a standard form (block matrices are helpful.) Note that the appropriate variables are all outflows from one airport to another, i.e., along edges, together with the net inflow into airport 1.
- (b) Solve the problem of part (a). Also solve the reverse problem: Maximize inflow into airport 6 and matching outflow from 1. Explain and justify your answers.

6. With the same limits on allowable flights into airports as in item 5, suppose that Gamma Airlines wants to determine an allocation of planes that will maximize their profits, given the following constraints: (1) Airports 1 and 6 have repair facilities for their planes, so no limit is placed on the inflow or outflow of their planes other than the airport limits. (2) Flights through airports 2-5 of Gamma planes are pass through, i.e., inflow and outflow must match. (3) Gamma has 32 planes available for this network of airports. (4) The profits per flight in thousands are, in the order that the edges in  $E$  are listed above,

$$P = \{5, 6, 7, 9, 10, 8, 9, 5, 6, 10\}$$

- (a) Set this problem up as a linear programming problem in standard form. Clearly identify the variables and explain how the constraints follow.
- (b) Solve this problem explicitly and specify the operations taken to do so. Example 3.56 is instructive for this problem, so be aware of it. Use a technology tool that allows you to use elementary operations (ALAMA calculator has this capability).