

Homework 2

1

Given that a linear system in the unknowns x_1, x_2, x_3, x_4 has general solution $(x_2 + 3x_4 + 4, x_2, 2 - x_4, x_4)$ for free variables x_2, x_4 , find a minimal reduced row echelon for this system.

Exercise 2.1.13

Solution:

We know that there are at least two equations in the system (since we have two constraints in the general solution.) We can write these two constraints as:

$$R = \begin{bmatrix} 1 & -1 & 0 & -3 & 4 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

We could also have written

$$R = \begin{bmatrix} 1 & -1 & 0 & -3 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ which has the same solutions.}$$

Just for kicks, checking in Sympy:

```
from sympy import Matrix
R=Matrix([[1, -1, 0, -3], [0, 0, 1, 1]])
rhs = Matrix([4,2])
R.gauss_jordan_solve(rhs)
```

```
(Matrix([
    [tau0 + 3*tau1 + 4],
```

```
[
    tau0],
[
    2 - tau1],
[
    tau1]]),
Matrix([
    tau0],
    tau1]))
```

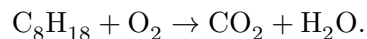
```
from sympy import Matrix, print_latex
R=Matrix([[1, -1, 0, -3], [0, 0, 1, 1], [0, 0, 0, 0], [0, 0, 0, 0]])
rhs = Matrix([4,2,0,0])
print_latex(R.gauss_jordan_solve(rhs), itex=False,mode='equation')
```

$$\left(\begin{bmatrix} \tau_0 + 3\tau_1 + 4 \\ \tau_0 \\ 2 - \tau_1 \\ \tau_1 \end{bmatrix}, \begin{bmatrix} \tau_0 \\ \tau_1 \end{bmatrix} \right) \quad (1)$$

2

Use the technique of Example 2.10 in your textbook to balance the following chemical equation:

Exercise 2.2.23



3

Express the following functions, if linear, as matrix operators. (If not linear, explain why.)

Exercise 2.3.3

- (a) $T((x_1, x_2)) = (x_1 + x_2, 2x_1, 4x_2 - x_1)$ (b) $T((x_1, x_2)) = (x_1 + x_2, 2x_1x_2)$
- (b) $T((x_1, x_2, x_3)) = (2x_3, -x_1)$
- (c) $T((x_1, x_2, x_3)) = (x_2 - x_1, x_3, x_2 + x_3)$

4

A *fixed-point* of a linear operator T_A is a vector \mathbf{x} such that $T_A(\mathbf{x}) = \mathbf{x}$. Find all fixed points, if any, of the linear operators in the previous exercise.

Exercise 2.3.9

5

A linear operator on \mathbb{R}^2 is defined by first applying a scaling operator with scale factors of 2 in the x -direction and 4 in the y -direction, followed by a counterclockwise rotation about the origin of $\pi/6$ radians. Express this operator and the operator that results from reversing the order of the scaling and rotation as matrix operators.

Exercise 2.3.5

6

Find a scaling operator S and shearing operator H such that the concatenation $S \circ H$ maps the points $(1, 0)$ to $(2, 0)$ and $(0, 1)$ to $(4, 3)$.

Exercise 2.3.7

7

Given transition matrices for discrete dynamical systems

Exercise 2.3.11

$$(a) \begin{bmatrix} .1 & .3 & 0 \\ 0 & .4 & 1 \\ .9 & .3 & 0 \end{bmatrix} \quad (b) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (c) \begin{bmatrix} .5 & .3 & 0 \\ 0 & .4 & 0 \\ .5 & .3 & 1 \end{bmatrix}$$

$$(d) \begin{bmatrix} 0 & 0 & 0.9 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0.1 \end{bmatrix} \text{ and initial state vector } \mathbf{x}^{(0)} =$$

$\frac{1}{2}(1, 1, 0)$, calculate the first and second state vector for each system and determine whether it is a Markov chain.

8

For each of the dynamical systems of the previous exercise, determine by calculation whether the system tends to a limiting steady-state vector. If so, what is it?

Exercise 2.3.12

9

A population is modeled with two states, immature and mature, and the resulting structured population model transition matrix is $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$.

Exercise 2.3.13

- (a) Explain what this matrix says about the two states.
- (b) Starting with a population of $(30, 100)$, does the population stabilize, increase or decrease over time? If it stabilizes, to what distribution?

10

A digraph G has vertex set $V = \{1, 2, 3, 4, 5\}$ and edge set $E = \{(2, 1), (1, 5), (2, 5), (5, 4), (4, 2), (4, 3), (3, 2)\}$. Sketch a picture of the graph G and find its adjacency matrix. Use this to find the power of each vertex of the graph and determine whether this graph is dominance-directed.

Exercise 2.3.15

11

Consider the linear difference $y_{k+2} - y_{k+1} - y_k = 0$.

Exercise 2.3.19

- (a) Express this difference in matrix form.
- (b) Find the first ten terms of the solution to this difference given the initial conditions $y_0 = 0, y_1 = 1$. (This is the well-known Fibonacci sequence.)

12

Suppose that in Example 2.27 you invest \$1,000 initially (the zeroth year) and no further amounts. Make a table of the value of your investment for years 0 to 12. Also include a column that calculates the annual interest rate that your investment is earning each year, based on the current and previous year's values. What conclusions do you draw? You will need a technology tool for this exercise.

Problem 2.3.23