# Homework 2

## 1

Given that a linear system in the unknowns $x_1, x_2, x_3, x_4$ has general solution $(x_2 + 3x_4 + 4, x_2, 2 - x_4, x_4)$ for free variables $x_2, x_4$, find a minimal reduced row echelon for this system.

Exercise 2.1.13

## 2

Use the technique of Example 2.10 in your textbook to balance the following chemical equation:

Exercise 2.2.23

$$C_8H_{18} + O_2 \to CO_2 + H_2O.$$

## 3

Express the following functions, if linear, as matrix operators. (If not linear, explain why.)

Exercise 2.3.3

(a) $T((x_1, x_2)) = (x_1 + x_2, 2x_1, 4x_2 - x_1)$ (b) $T((x_1, x_2)) = (x_1 + x_2, 2x_1 x_2)$

(b) $T((x_1, x_2, x_3)) = (2x_3, -x_1)$

(c) $T((x_1, x_2, x_3)) = (x_2 - x_1, x_3, x_2 + x_3)$

## 4

A *fixed-point* of a linear operator $T_A$ is a vector $\mathbf{x}$ such that $T_A(\mathbf{x}) = \mathbf{x}$. Find all fixed points, if any, of the linear operators in the previous exercise.

Exercise 2.3.9

## 5

A linear operator on $\mathbb{R}^2$ is defined by first applying a scaling operator with scale factors of 2 in the $x$-direction and 4 in the $y$-direction, followed by a counterclockwise rotation about the origin of $\pi/6$ radians. Express this operator and the operator that results from reversing the order of the scaling and rotation as matrix operators.

Exercise 2.3.5

## 6

Find a scaling operator $S$ and shearing operator $H$ such that the concatenation $S \circ H$ maps the points $(1, 0)$ to $(2, 0)$ and $(0, 1)$ to $(4, 3)$.

Exercise 2.3.7

## 7

Given transition matrices for discrete dynamical systems

Exercise 2.3.11

(a) $\begin{bmatrix} .1 & .3 & 0 \\ 0 & .4 & 1 \\ .9 & .3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} .5 & .3 & 0 \\ 0 & .4 & 0 \\ .5 & .3 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0.9 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0.1 \end{bmatrix}$ and initial state vector $\mathbf{x}^{(0)} = \frac{1}{2}(1, 1, 0)$, calculate the first and second state vector for each system and determine whether it is a Markov chain.

## 8

For each of the dynamical systems of the previous exercise, determine by calculation whether the system tends to a limiting steady-state vector. If so, what is it?

Exercise 2.3.12

## 9

A population is modeled with two states, immature and mature, and the resulting structured population model transition matrix is $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$.

Exercise 2.3.13

(a) Explain what this matrix says about the two states.
(b) Starting with a population of $(30, 100)$, does the population stabilize, increase or decrease over time? If it stabilizes, to what distribution?

## 10

A digraph $G$ has vertex set $V = \{1, 2, 3, 4, 5\}$ and edge set $E = \{(2,1), (1,5), (2,5), (5,4), (4,2), (4,3), (3,2)\}$. Sketch a picture of the graph $G$ and find its adjacency matrix. Use this to find the power of each vertex of the graph and determine whether this graph is dominance-directed.

Exercise 2.3.15

## 11

Consider the linear difference $y_{k+2} - y_{k+1} - y_k = 0$.

Exercise 2.3.19

(a) Express this difference in matrix form.

(b) Find the first ten terms of the solution to this difference given the initial conditions $y_0 = 0, y_1 = 1$. (This is the well-known Fibonacci sequence.)

## 12

Suppose that in Example 2.27 you invest $\$1,000$ initially (the zeroth year) and no further amounts. Make a table of the value of your investment for years 0 to 12. Also include a column that calculates the annual interest rate that your investment is earning each year, based on the current and previous year's values. What conclusions do you draw? You will need a technology tool for this exercise.

Problem 2.3.23