Homework 2

```
from sympy import print_latex
def my_print(x, *args, **kwargs):
    print_latex(x, itex=False, mode='equation', *args, **kwargs)
```

1

Given that a linear system in the unknowns x_1, x_2, x_3, x_4 has general solution $(x_2 + 3x_4 + 4, x_2, 2 - x_4, x_4)$ for free variables x_2, x_4, find a minimal reduced row echelon for this system.

Solution:

Exercise 2.1.13

We know that there are at least two equations in the system (since we have two constraints in the general solution.) We can write these two constraints as:

$$R = \begin{bmatrix} 1 & -1 & 0 & -3 & 4 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

We could also have written

$$R = \begin{bmatrix} 1 & -1 & 0 & -3 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ which has the same solutions.}$$

Just for kicks, checking in Sympy:

```
from sympy import Matrix
R=Matrix([[1, -1, 0, -3], [0, 0, 1, 1]])
rhs = Matrix([4,2])
R.gauss_jordan_solve(rhs)
my_print(R.gauss_jordan_solve(rhs))
```

```
\begin{equation}\left( \left[\begin{matrix}\tau_{0} + 3 \tau_{1} + 4\\\tau_{0}\\2 - \tau_{1}\
```

```
from sympy import Matrix, print_latex
R=Matrix([[1, -1, 0, -3], [0, 0, 1, 1], [0, 0, 0, 0], [0, 0, 0, 0]])
rhs = Matrix([4,2,0,0])
my_print(R.gauss_jordan_solve(rhs))
```

$$\left(\left[\begin{matrix}\tau_{0} + 3 \tau_{1} + 4\\\tau_{0}\\2 - \tau_{1}\\\tau_{1}\end{matrix}\right], \ \left[\begin{matrix}\tau_{0}\\\tau_{1}\end{matrix}\right]\right) \tag{1}$$

2

Use the technique of Example 2.10 in your textbook to balance the following chemical equation:

$$C_8H_{18} + O_2 \to CO_2 + H_2O.$$

Solution:

Exercise 2.2.23

With vectors indicating amount of C, H and O, variables the number of molecules of each compound occurring, system represented is

$$x_1 \begin{bmatrix} 8 \\ 18 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

resulting in coefficient matrix

$$\begin{bmatrix} 8 & 0 & -1 & 0 \\ 18 & 0 & 0 & -2 \\ 2 & 2 & -2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{9} \\ 0 & 1 & 0 & \frac{25}{18} \\ 0 & 0 & 1 & \frac{8}{9} \end{bmatrix}$$

```
from sympy import Matrix
R=Matrix([[8, 0, -1, 0], [18, 0, 0, -2], [2, 2, -2, -1]])
rhs = Matrix([0,0,0])
soln=R.gauss_jordan_solve(rhs)[0]
my_print(soln.subs('tau0',18))
my_print(R.rref()[0])
```

$$\begin{bmatrix} 2 \\ 23 \\ 16 \\ 18 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{9} \\ 0 & 1 & 0 & -\frac{23}{18} \\ 0 & 0 & 1 & -\frac{8}{9} \end{bmatrix} \tag{3}$$

3

Express the following functions, if linear, as matrix operators. (If not linear, explain why.)

Exercise 2.3.3

(a) $T((x_1, x_2)) = (x_1 + x_2, 2x_1, 4x_2 - x_1)$ (b) $T((x_1, x_2)) = (x_1 + x_2, 2x_1x_2)$

(b) $T((x_1, x_2, x_3)) = (2x_3, -x_1)$

(c) $T((x_1, x_2, x_3)) = (x_2 - x_1, x_3, x_2 + x_3)$

4

A *fixed-point* of a linear operator T_A is a vector $\mathbf{x}$ such that $T_A(\mathbf{x}) = \mathbf{x}$. Find all fixed points, if any, of the linear operators in the previous exercise.

Exercise 2.3.9

5

A linear operator on $\mathbb{R}^2$ is defined by first applying a scaling operator with scale factors of 2 in the x-direction and 4 in the y-direction, followed by a counterclockwise rotation about the origin of $\pi/6$ radians. Express this operator and the operator that results from reversing the order of the scaling and rotation as matrix operators.

Exercise 2.3.5

6

Find a scaling operator S and shearing operator H such that the concatenation $S \circ H$ maps the points $(1,0)$ to $(2,0)$ and $(0,1)$ to $(4,3)$.

Exercise 2.3.7

7

Given transition matrices for discrete dynamical systems

Exercise 2.3.11

(a) $\begin{bmatrix} .1 & .3 & 0 \\ 0 & .4 & 1 \\ .9 & .3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} .5 & .3 & 0 \\ 0 & .4 & 0 \\ .5 & .3 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0.9 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0.1 \end{bmatrix}$ and initial state vector $\mathbf{x}^{(0)} = \frac{1}{2}(1,1,0)$, calculate the first and second state vector for each system and determine whether it is a Markov chain.

8

For each of the dynamical systems of the previous exercise, determine by calculation whether the system tends to a limiting steady-state vector. If so, what is it?

Exercise 2.3.12

9

A population is modeled with two states, immature and mature, and the resulting structured population model transition matrix is $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$.

Exercise 2.3.13

(a) Explain what this matrix says about the two states.
(b) Starting with a population of $(30, 100)$, does the population stabilize, increase or decrease over time? If it stabilizes, to what distribution?

10

A digraph G has vertex set $V = \{1, 2, 3, 4, 5\}$ and edge set $E = \{(2, 1), (1, 5), (2, 5), (5, 4), (4, 2), (4, 3), (3, 2)\}$. Sketch a picture of the graph G and find its adjacency matrix. Use this to find the power of each vertex of the graph and determine whether this graph is dominance-directed.

Exercise 2.3.15

11

Consider the linear difference $y_{k+2} - y_{k+1} - y_k = 0$.

Exercise 2.3.19

(a) Express this difference in matrix form.

(b) Find the first ten terms of the solution to this difference given the initial conditions $y_0 = 0, y_1 = 1$. (This is the well-known Fibonacci sequence.)

12

Suppose that in Example 2.27 you invest $\$1,000$ initially (the zeroth year) and no further amounts. Make a table of the value of your investment for years 0 to 12. Also include a column that calculates the annual interest rate that your investment is earning each year, based on the current and previous year's values.

Problem 2.3.23

What conclusions do you draw? You will need a technology tool for this exercise.