Homework 2

1

Given that a linear system in the unknowns x_1, x_2, x_3, x_4 has general solution $(x_2 + 3x_4 + 4, x_2, 2 - x_4, x_4)$ for free variables x_2, x_4, find a minimal reduced row echelon for this system.

> Solution:
>
> We know that there are at least two equations in the system (since we have two constraints in the general solution.) We can write these two constraints as:
>
> $$R = \begin{bmatrix} 1 & -1 & 0 & -3 & 4 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$
>
> We could also have written
>
> $$R = \begin{bmatrix} 1 & -1 & 0 & -3 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ which has the same solutions.}$$
>
> Just for kicks, checking in Sympy:
>
> ```
> from sympy import Matrix
> R=Matrix([[1, -1, 0, -3], [0, 0, 1, 1]])
> rhs = Matrix([4,2])
> R.gauss_jordan_solve(rhs)
> ```
>
> ```
> (Matrix([
>   [tau0 + 3*tau1 + 4],
> ```

```
     [              tau0],
     [        2 - tau1],
     [            tau1]]),
  Matrix([
   [tau0],
   [tau1]]))
```
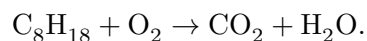
```
from sympy import Matrix
R=Matrix([[1, -1, 0, -3], [0, 0, 1, 1], [0, 0, 0, 0], [0, 0, 0, 0]])
rhs = Matrix([4,2,0,0])
R.gauss_jordan_solve(rhs)
```

```
 (Matrix([
  [tau0 + 3*tau1 + 4],
  [              tau0],
  [        2 - tau1],
  [            tau1]]),
  Matrix([
   [tau0],
   [tau1]]))
```

2

Use the technique of Example 2.10 in your textbook to balance
the following chemical equation:

$$C_8H_{18} + O_2 \rightarrow CO_2 + H_2O.$$

3

Express the following functions, if linear, as matrix operators.
(If not linear, explain why.)

(a) $T((x_1, x_2)) = (x_1 + x_2, 2x_1, 4x_2 - x_1)$ (b) $T((x_1, x_2)) = (x_1 + x_2, 2x_1x_2)$

(b) $T((x_1, x_2, x_3)) = (2x_3, -x_1)$

(c) $T\left((x_1, x_2, x_3)\right) = (x_2 - x_1, x_3, x_2 + x_3)$

4

A *fixed-point* of a linear operator T_A is a vector $\mathbf{x}$ such that $T_A(\mathbf{x}) = \mathbf{x}$. Find all fixed points, if any, of the linear operators in the previous exercise.

5

A linear operator on $\mathbb{R}^2$ is defined by first applying a scaling operator with scale factors of 2 in the x-direction and 4 in the y-direction, followed by a counterclockwise rotation about the origin of $\pi/6$ radians. Express this operator and the operator that results from reversing the order of the scaling and rotation as matrix operators.

6

Find a scaling operator S and shearing operator H such that the concatenation $S \circ H$ maps the points $(1, 0)$ to $(2, 0)$ and $(0, 1)$ to $(4, 3)$.

7

Given transition matrices for discrete dynamical systems

(a) $\begin{bmatrix} .1 & .3 & 0 \\ 0 & .4 & 1 \\ .9 & .3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} .5 & .3 & 0 \\ 0 & .4 & 0 \\ .5 & .3 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0.9 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0.1 \end{bmatrix}$ and initial state vector $\mathbf{x}^{(0)} = \frac{1}{2}(1, 1, 0)$, calculate the first and second state vector for each system and determine whether it is a Markov chain.

8

For each of the dynamical systems of the previous exercise, determine by calculation whether the system tends to a limiting steady-state vector. If so, what is it?

9

A population is modeled with two states, immature and mature, and the resulting structured population model transition matrix is $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$.

(a) Explain what this matrix says about the two states.
(b) Starting with a population of $(30, 100)$, does the population stabilize, increase or decrease over time? If it stabilizes, to what distribution?

10

A digraph G has vertex set $V = \{1, 2, 3, 4, 5\}$ and edge set $E = \{(2, 1), (1, 5), (2, 5), (5, 4), (4, 2), (4, 3), (3, 2)\}$. Sketch a picture of the graph G and find its adjacency matrix. Use this to find the power of each vertex of the graph and determine whether this graph is dominance-directed.

11

Consider the linear difference $y_{k+2} - y_{k+1} - y_k = 0$.

(a) Express this difference in matrix form.

(b) Find the first ten terms of the solution to this difference given the initial conditions $y_0 = 0, y_1 = 1$. (This is the well-known Fibonacci sequence.)

12

Suppose that in Example 2.27 you invest $1,000$ initially (the zeroth year) and no further amounts. Make a table of the value of your investment for years 0 to 12. Also include a column that calculates the annual interest rate that your investment is earning each year, based on the current and previous year's values. What conclusions do you draw? You will need a technology tool for this exercise.

Problem 2.3.23