Homework 1

1

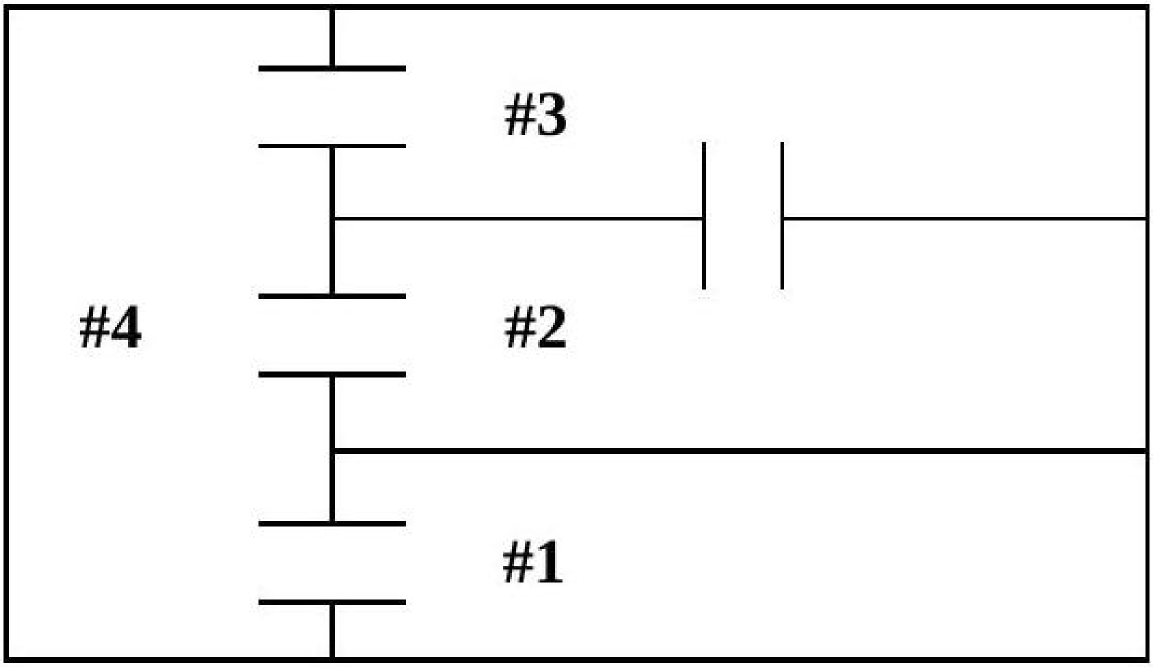
testing

By solving a 3×3 system, find the coefficients in the equation of the parabola $y = \alpha + \beta x + \gamma x^2$ that passes through the points $(1, 1), (2, 2)$, and $(3, 0)$.

(Meyer 1.2.10) $\alpha = -3, \beta = \frac{11}{2}$, and $\gamma = -\frac{3}{2}$

2

Suppose that 100 insects are distributed in an enclosure consisting of four chambers with passageways between them as shown below.



At the end of one minute, the insects have redistributed themselves. Assume that a minute is not enough time for an insect to visit more than one chamber and that at the end of a minute 40% of the insects in each chamber have not left the chamber they occupied at the beginning of the minute. The insects that leave a chamber disperse uniformly among the chambers that are directly accessible from the one they initially occupied-e.g., from #3, half move to #2 and half move to #4.

(a) If at the end of one minute there are $12, 25, 26$, and 37 insects in chambers $\#1, \#2, \#3$, and $\#4$, respectively, determine what the initial distribution had to be.

(b) If the initial distribution is $20, 20, 20, 40$, what is the distribution at the end of one minute?

(Meyer 1.2.11) (a) If x_i = the number initially in chamber $\#i$, then

$$\begin{aligned} .4x_1 + 0x_2 + 0x_3 + .2x_4 &= 12 \\ 0x_1 + .4x_2 + .3x_3 + .2x_4 &= 25 \\ 0x_1 + .3x_2 + .4x_3 + .2x_4 &= 26 \\ .6x_1 + .3x_2 + .3x_3 + .4x_4 &= 37 \end{aligned}$$

and the solution is $x_1 = 10, x_2 = 20, x_3 = 30$, and $x_4 = 40$.

(b) $16, 22, 22, 40$

3

Suppose that $[\mathbf{A} \mid \mathbf{b}]$ is the augmented matrix associated with a linear system. You know that performing row operations on $[\mathbf{A} \mid \mathbf{b}]$ does not change the solution of the system. However, no mention of column operations was ever made because column operations can alter the solution.

(a) Describe the effect on the solution of a linear system when columns $\mathbf{A}_{*j}$ and $\mathbf{A}_{*k}$ are interchanged.

(b) Describe the effect when column $\mathbf{A}_{*j}$ is replaced by $\alpha\mathbf{A}_{*j}$ for $\alpha \neq 0$.

(c) Describe the effect when $\mathbf{A}_{*j}$ is replaced by $\mathbf{A}_{*j} + \alpha\mathbf{A}_{*k}$. Hint: Experiment with a 2×2 or 3×3 system.

(Meyer 1.2.13) a. This has the effect of interchanging the order of the unknowns $-x_j$ and x_k are p b. The solution to the new system is the same as the solution to the old system except that the solution for the j^{th} unknown of the new system is $\hat{x}_j = \frac{1}{\alpha}x_j$. This has the effect of "changing the units" of the j^{th} unknown. c. The solution to the new system is the same as the solution for the old system except that the solution for the k^{th} unknown in the new system is $\hat{x}_k = x_k - \alpha x_j$.

4

Use the Gauss-Jordan method to solve the following three systems at the same time.

$$\begin{array}{rl|c|c} 2x_1 - x_2 & = 1 & 0 & 0 \\ -x_1 + 2x_2 - x_3 & = 0 & 1 & 0 \\ -x_2 + x_3 & = 0 & 0 & 1 \end{array}$$

(Meyer 1.3.3)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

5

Consider the following system:

$$\begin{aligned} 10^{-3}x - y &= 1, \\ x + y &= 0. \end{aligned}$$

(a) Use 3-digit arithmetic with no pivoting to solve this system.

(b) Find a system that is exactly satisfied by your solution from part (a), and note how close this system is to the original system.

(c) Now use partial pivoting and 3-digit arithmetic to solve the original system.

(d) Find a system that is exactly satisfied by your solution from part (c), and note how close this system is to the original system.

(e) Use exact arithmetic to obtain the solution to the original system, and compare the exact solution with the results of parts (a) and (c).

(f) Round the exact solution to three significant digits, and compare the result with those of parts (a) and (c).

(Meyer 1.5.1) (a)

$\left[\begin{smallmatrix} 0.001 & -1 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right] \overrightarrow{E_{2,1}(1000)} \left[\begin{smallmatrix} 0.001 & -1.0 & 1.0 \\ 0 & 1.0{\cdot}10^3 & -1.0{\cdot}10^3 \end{smallmatrix}\right] \overrightarrow{E_1(1000)} \left[\begin{smallmatrix} 1.00 & -1.00{\cdot}10^3 & 1.00{\cdot}10^3 \\ 0 & 1.00{\cdot}10^3 & -1.00{\cdot}10^3 \end{smallmatrix}\right]$

The solution is $(0, -1)$.

(b) One example of a system whose exact solution is $0, -1$ would be

$$\begin{aligned} 2x - y &= 1, \\ x + y &= -1. \end{aligned}$$

This is not close at all to the original system!

(c)

$\left[\begin{smallmatrix} 0.001 & -1 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right] \overrightarrow{E_{2,1}} \left[\begin{smallmatrix} 1.0 & 1.0 & 0 \\ 0.001 & -1.0 & 1.0 \end{smallmatrix}\right] \overrightarrow{E_{2,1}(1/1000)} \left[\begin{smallmatrix} 1.00 & 1.00 & 0 \\ -7.25{\cdot}10^{-8} & -1.00 & 1.00 \end{smallmatrix}\right]$

Solution is $(1, -1)$.

(d) One example of a system whose exact solution is $1, -1$ would be

$$\begin{aligned} 0x - y &= 1, \\ x + y &= 0. \end{aligned}$$

This is very close to the original system!

(e) $\left(\frac{1}{1.001}, \frac{-1}{1.001}\right)$

This is close to our solution in (c).

6

Consider the following well-scaled matrix:

$$\mathbf{W}_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 1 \\ -1 & -1 & 1 & \ddots & 0 & 0 & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \ddots & 1 & 0 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 1 \end{pmatrix}$$

(a) Reduce $\mathbf{W}_n$ to an upper-triangular form using Gaussian elimination with partial pivoting, and determine the element of maximal magnitude that emerges during the elimination procedure.

(Meyer 1.5.7) 2^{n-1}

7

Answer True/False and explain your answers: 1. If a linear system is inconsistent, then the rank of the augmented matrix exceeds the number of unknowns. 2. Any homogeneous linear system is consistent. 3. A system of 3 linear equations in 4 unknowns has infinitely many solutions. 4. Every matrix can be reduced to only one matrix in row echelon form. 5. Any homogeneous linear system with more equations than unknowns has a nontrivial solution.

(Shores 1.4.19)

(a) False. One very simple counterexample: $0x = 1$.

Another: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The rank of the augmented matrix is 2, but the system is inconsistent.

We know that b must not be a linear combination of the columns of A for the system to be inconsistent; however, there is no requirement that the columns of A be linearly independent. Therefore the rank of the augmented matrix could be less than the number of unknowns.

(b) True, a homogeneous system must have the trivial solution.
(c) False, it could be inconsistent (and therefore have zero solutions).

(d) False. Here is a counterexample. The matrix $\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$ can be reduced to either $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (Only the latter is in *reduced* row echelon form, which must be unique.)

(e) False, a counterexample is $\begin{bmatrix} 1 \\ 2 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.