Multiplying Eqn (3.6) by m_1 and Eqn (3.5) by m_2 and subtracting, the equation for the relative motion of the bodies can be cast in the form

$$\mu_r \frac{d^2\mathbf{r}_{12}}{dt^2} = \mu_r \frac{d^2(\mathbf{r}_1 - \mathbf{r}_2)}{dt^2} = -\frac{G\mu_r M}{r_{12}^3}\mathbf{r}_{12}, \tag{3.8}$$

where $\mu_r \equiv m_1 m_2/(m_1 + m_2)$ is called the reduced mass and $M \equiv m_1 + m_2$ is the total mass. Thus, the relative motion is completely equivalent to that of a particle of reduced mass μ_r orbiting *a fixed* central mass M. For known masses, specifying the elements of the relative orbit and the positions and velocities of the center of mass is completely equivalent to specifying the positions and velocities of both bodies. A detailed solution of the equation of motion (Eqn 3.8) is discussed in any elementary text on orbital mechanics and in most general classical mechanics books. In the remainder of Section 2, a few key results are given.

2.3. Energy, Circular Velocity, and Escape Velocity

The centripetal force necessary to keep an object of mass μ_r in a circular orbit of radius r with speed υ_c is $\mu_r \upsilon_c^2/r$. Equating this to the gravitational force exerted by the central body of mass M, the circular velocity is

$$\upsilon_c = \sqrt{\frac{GM}{r}}. \tag{3.9}$$

Thus the orbital period (the time to move once around the circle) is

$$P = 2\pi r/\upsilon_c = 2\pi\sqrt{\frac{r^3}{GM}}. \tag{3.10}$$

The total (kinetic plus potential) energy E of the system is a conserved quantity:

$$E = T + V = \frac{1}{2}\mu_r \upsilon^2 - \frac{GM\mu_r}{r}, \tag{3.11}$$

where the first term on the right is the kinetic energy of the system, T, and the second term is the potential energy of the system, V. If $E < 0$, the absolute value of the potential energy of the system is larger than its kinetic energy, and the system is bound. The body will orbit the central mass on an elliptical path. If $E > 0$, the kinetic energy is larger than the absolute value of the potential energy, and the system is unbound. The relative orbit is then described mathematically as a hyperbola. If $E = 0$, the kinetic and potential energies are equal in magnitude, and the relative orbit is a parabola. By setting the total energy equal to zero, the escape velocity at any separation can be calculated:

$$\upsilon_e = \sqrt{\frac{2GM}{r}} = \sqrt{2}\upsilon_c. \tag{3.12}$$

For circular orbits it is easy to show (using Eqns (3.9) and (3.11)) that both the kinetic energy and the total energy of the system are equal in magnitude to half the potential energy:

$$T = -\frac{1}{2}V, \tag{3.13}$$

$$E = -\frac{GM\mu_r}{2r}. \tag{3.14}$$

For an elliptical orbit, Eqn (3.14) holds if the radius r is replaced by the semimajor axis a:

$$E = -\frac{GM\mu_r}{2a}. \tag{3.15}$$

Similarly, for an elliptical orbit, Eqn (3.10) becomes Newton's generalization of Kepler's third law:

$$P^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}. \tag{3.16}$$

It can be shown that Kepler's second law follows immediately from the conservation of angular momentum, $\mathbf{L}$:

$$\frac{d\mathbf{L}}{dt} = \frac{d(\mu_r \mathbf{r} \times \mathbf{v})}{dt} = 0. \tag{3.17}$$

2.4. Orbital Elements: Elliptical, Parabolic, and Hyperbolic Orbits

As noted earlier, the relative orbit in the two-body problem is either an ellipse, a parabola, or a hyperbola depending on whether the energy is negative, zero, or positive, respectively. These curves are known collectively as conic sections and the generalization of Eqn (3.1) is

$$r = \frac{p}{1 + e\cos f}, \tag{3.18}$$

where r and f have the same meaning as in Eqn (3.1), e is the generalized **eccentricity**, and p, the semilatus rectum, is a conserved quantity that depends on the initial conditions. For an ellipse, $p = a(1 - e^2)$, as in Eqn (3.1). For a parabola, $e = 1$ and $p = 2q$, where q is the pericentric separation (distance of closest approach). For a hyperbola, $e > 1$ and $p = q(1 + e)$, where q is again the pericentric separation. For all orbits, the three orientation angles i, Ω, and ω are defined as in the elliptical case.

3. PLANETARY PERTURBATIONS AND THE ORBITS OF SMALL BODIES

Gravity is not restricted to interactions between the Sun and the planets or individual planets and their satellites, but rather all bodies feel the gravitational force of one another.

Within the solar system, one body typically produces the dominant force on any given body, and the resultant motion can be thought of as a Keplerian orbit about a primary, subject to small perturbations by other bodies. In this section, some important examples of the effects of these perturbations on the orbital motion are considered.

Classically, much of the discussion of the evolution of orbits in the solar system used perturbation theory as its foundation. Essentially, the method involves writing the equations of motion as the sum of a part that describes the independent Keplerian motion of the bodies about the Sun plus a part (called the disturbing function) that contains terms due to the pairwise interactions among the planets and minor bodies and the indirect terms associated with the backreaction of the planets on the Sun. In general, one can then expand the disturbing function in terms of the small parameters of the problem (such as the ratio of the planetary masses to the solar mass, the eccentricities and inclinations, etc.), as well as the other orbital elements of the bodies, including the mean longitudes (i.e. the location of the bodies in their orbits), and attempt to solve the resulting equations for the time dependence of the orbital elements.

3.1. Perturbed Keplerian Motion and Resonances

Although perturbations on a body's orbit are often small, they cannot always be ignored. They must be included in short-term calculations if high accuracy is required, for example, for predicting when an object passes in front of a star (stellar occultation) or targeting spacecraft. Most long-term perturbations are periodic in nature, their directions oscillating with the relative longitudes of the bodies or with some more complicated function of the bodies' orbital elements.

Small perturbations can produce large effects if the forcing frequency is commensurate or nearly commensurate with the natural frequency of oscillation of the responding elements. Under such circumstances, perturbations add coherently, and the effects of many small tugs can build up over time to create a large-amplitude, long-period response. This is an example of resonance forcing, which occurs in a wide range of physical systems.

An elementary example of resonance forcing is given by the simple one-dimensional harmonic oscillator, for which the equation of motion is

$$m\frac{d^2x}{dt^2} + m\Gamma^2 x = F_o \cos \varphi t. \qquad (3.19)$$

In Eqn (3.19), m is the mass of the oscillating particle, F_o is the amplitude of the driving force, Γ is the natural frequency of the oscillator, and φ is the forcing or resonance frequency. The solution to Eqn (3.19) is

$$x = x_o \cos \varphi t + A \cos \Gamma t + B \sin \Gamma t, \qquad (3.20a)$$

where

$$x_o \equiv \frac{F_o}{m\left(\Gamma^2 - \varphi^2\right)}, \qquad (3.20b)$$

and A and B are constants determined by the initial conditions. Note that if $\varphi \approx \Gamma$, a large-amplitude, long-period response can occur even if F_o is small. Moreover, if $\varphi = \Gamma$, this solution to Eqn (3.19) is invalid. In this case, the solution is given by

$$x = \frac{F_o}{2m\Gamma} t \sin \Gamma t + A \cos \Gamma t + B \sin \Gamma t. \qquad (3.21)$$

The t in front of the first term on the right-hand side of Eqn (3.21) leads to **secular** growth. Often this linear growth is moderated by the effects of nonlinear terms that are not included in the simple example provided here. However, some perturbations have a secular component.

Nearly exact orbital commensurabilities exist at many places in the solar system. Io orbits Jupiter twice as frequently as Europa does, which in turn orbits Jupiter twice as frequently as Ganymede does. Conjunctions (at which the bodies have the same longitude) always occur at the same position of Io's orbit (its perijove). How can such commensurabilities exist? After all, the probability of randomly picking a rational from the real number line is 0, and the number of small integer ratios is infinitely smaller still! The answer lies in the fact that orbital resonances may be held in place as stable locks, which result from nonlinear effects not represented in the foregoing simple mathematical example. For example, differential tidal recession (see Section 7.5) brings moons into resonance, and nonlinear interactions among the moons can keep them there. Other examples of resonance locks include the Hilda asteroids, the Trojan asteroids, Neptune–Pluto, and the pairs of moons about Saturn, Mimas–Tethys and Enceladus–Dione.

Resonant perturbation can also force material into highly eccentric orbits that may lead to collisions with other bodies; this is thought to be the dominant mechanism for clearing the Kirkwood gaps in the asteroid belt (see Section 5.1). Spiral density waves can propagate away from resonant locations in a self-gravitating particle disk perturbed by an orbiting satellite. Density waves are seen at many resonances in Saturn's rings; they explain most of the structure seen in Saturn's A ring. The vertical analogs of density waves, bending waves, are caused by resonant perturbations perpendicular to the ring plane due to a satellite in an orbit that is inclined to the ring. Spiral bending waves excited by the moons Mimas and Titan have been seen in Saturn's rings. In the next few subsections, these manifestations of resonance effects that do not explicitly involve chaos are discussed. Chaotic motion produced by resonant forcing is discussed later in the chapter.

3.2. Examples of Resonances: Lagrangian Points and Tadpole and Horseshoe Orbits

Many features of the orbits considered in this section can be understood by examining an idealized system in which two massive (but typically of unequal mass) bodies move in circular orbits about their common center of mass. If a third body is introduced that is much less massive than either of the first two, its motion can be followed by assuming that its gravitational force has no effect on the orbits of the other bodies. By considering the motion in a frame corotating with the massive pair (so that the pair remain fixed on a line that can be taken to be the *x*-axis), Lagrange found that there are five points where particles placed at rest would feel no net force in the rotating frame. Three of the so-called **Lagrange points** (L_1, L_2, and L_3) lie along a line joining the two masses m_1 and m_2. The other two Lagrange points (L_4 and L_5) form equilateral triangles with the two massive bodies.

Particles displaced slightly from the first three Lagrangian points will continue to move away and hence these locations are unstable. The triangular Lagrangian points are potential energy maxima, which are stable for sufficiently large primary to secondary mass ratio due to the Coriolis force. Provided that the most massive body has at least 25 times the mass of the secondary (which is the case for all known examples in the solar system larger than the Pluto—Charon system), the Lagrangian points L_4 and L_5 are stable points. Thus, a particle at L_4 or L_5 that is perturbed slightly will start to "orbit" these points in the rotating coordinate system. Lagrangian points L_4 and L_5 are important in the solar system. For example, the Trojan asteroids in Jupiter's Lagrangian points and both Neptune and Mars confine their own Trojans. There are also small moons in the triangular Lagrangian points of Tethys and Dione, in the Saturnian system. The L_4 and L_5 points in the Earth—Moon system have been suggested as possible locations for space stations.

3.2.1. Horseshoe and Tadpole Orbits

Consider a moon on a circular orbit about a planet. Figure 3.3 shows some important dynamical features in the frame corotating with the moon. All five Lagrangian points are indicated in the picture. A particle just interior to the moon's orbit has a higher angular velocity than the moon in the stationary frame and thus moves with respect to the moon in the direction of corotation. A particle just outside the moon's orbit has a smaller angular velocity and moves away from the moon in the opposite direction. When the outer particle approaches the moon, the particle is slowed down (loses angular momentum) and, provided the initial difference in semimajor axis is not too large, the particle drops to an orbit lower than that of the moon. The particle then recedes in the forward direction. Similarly, the particle at the lower orbit is

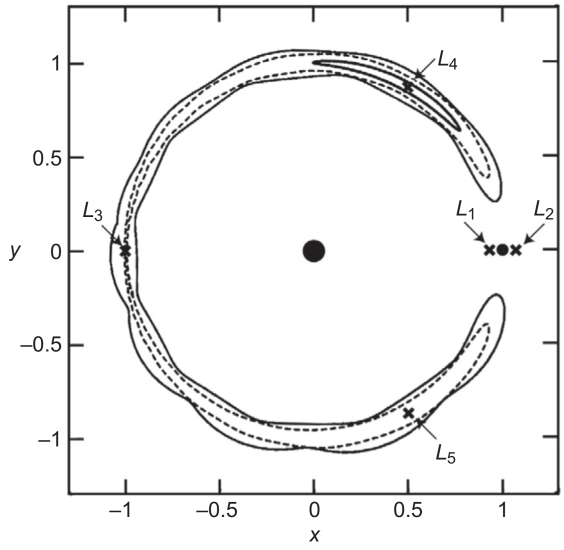

FIGURE 3.3 Diagram showing the five Lagrangian equilibrium points (denoted by crosses) and three representative orbits near these points for the circular restricted three-body problem. In this example, the secondary's mass is 0.001 times the total mass. The coordinate frame has its origin at the barycenter and corotates with the pair of bodies, thereby keeping the primary (large solid circle) and secondary (small solid circle) fixed on the *x*-axis. Tadpole orbits remain near one or the other of the L_4 and L_5 points. An example is shown near the L_4 point on the diagram. Horseshoe orbits enclose all three of L_3, L_4, and L_5 but do not reach L_1 or L_2. The outermost orbit on the diagram illustrates this behavior. There is a critical curve dividing tadpole and horseshoe orbits that encloses L_4 and L_5 and passes through L_3. A horseshoe orbit near this dividing line is shown as the dashed curve in the diagram.

accelerated as it catches up with the moon, resulting in an outward motion toward the higher, slower orbit. Orbits like these encircle the L_3, L_4, and L_5 points and are called **horseshoe orbits**. Saturn's small moons Janus and Epimetheus execute just such a dance, changing orbits every 4 years.

Since the Lagrangian points L_4 and L_5 are stable, material can librate about these points individually: such orbits are called **tadpole orbits**. The tadpole **libration** width at L_4 and L_5 is roughly equal to $(m/M)^{1/2}r$, and the horseshoe width is $(m/M)^{1/3}r$, where M is the mass of the planet, m the mass of the satellite, and r the distance between the two objects. For a planet of Saturn's mass, $M = 5.7 \times 10^{29}$ g, and a typical small moon of mass $m = 10^{20}$ g (e.g. an object with a 30-km radius, with density of ~ 1 g/cm^3), at a distance of 2.5 Saturnian radii, the tadpole libration half-width is about 3 km and the horseshoe half-width is about 60 km.

3.2.2. Hill Sphere

The approximate limit to a planet's gravitational dominance is given by the extent of its **Hill sphere**,

$$R_H = \left[\frac{m}{3(M+m)}\right]^{1/3} a, \qquad (3.22)$$

where m is the mass of the planet and M is the Sun's mass. A test body located at the boundary of a planet's Hill sphere is subjected to a gravitational force from the planet that is comparable in magnitude to the tidal difference between the force of the Sun on the planet and that on the test body. The Hill sphere essentially stretches out to the L_1 point and is roughly the limit of the Roche lobe (maximum extent of an object held together by gravity alone) of a body with $m \ll M$. Planetocentric orbits that are stable over long periods are those well within the boundary of a planet's Hill sphere; the overwhelming majority of natural satellites lie in this region. The trajectories of the outermost planetary satellites, which lie closest to the boundary of the Hill sphere, show large variations in planetocentric orbital paths (Figure 3.4). Stable heliocentric orbits are those that are always well outside the Hill sphere of any planet.

3.3. Examples of Resonances: Ring Particles and Shepherding

In the discussions in Section 2, the gravitational force produced by a spherically symmetric body was described. In this section, the effects of deviations from spherical symmetry must be included when computing the force. This is most conveniently done by introducing the gravitational potential $\Phi(r)$, which is defined such that the acceleration $\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$ of a particle in the gravitational field is

$$\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2 = \nabla\Phi. \tag{3.23}$$

In empty space, the Newtonian gravitational potential $\Phi(r)$ always satisfies Laplace's equation

$$\nabla^2\Phi = 0. \tag{3.24}$$

Most planets are very nearly axisymmetric, with the major departure from sphericity being due to a rotationally induced equatorial bulge. Thus, the gravitational potential can be expanded in terms of Legendre polynomials instead of the complete spherical harmonic expansion, which would be required for the potential of a body of arbitrary shape:

$$\Phi(r,\phi,\theta) = -\frac{Gm}{r}\left[1 - \sum_{n=2}^{\infty} J_n P_n(\cos\theta)(R/r)^n\right]. \tag{3.25}$$

This equation uses standard spherical coordinates, so that θ is the angle between the planet's symmetry axis and the vector to the particle. The terms $P_n(\cos\theta)$ are the Legendre polynomials and J_n are the gravitational moments determined by the planet's mass distribution. If the planet's mass distribution is symmetrical about the planet's equator,

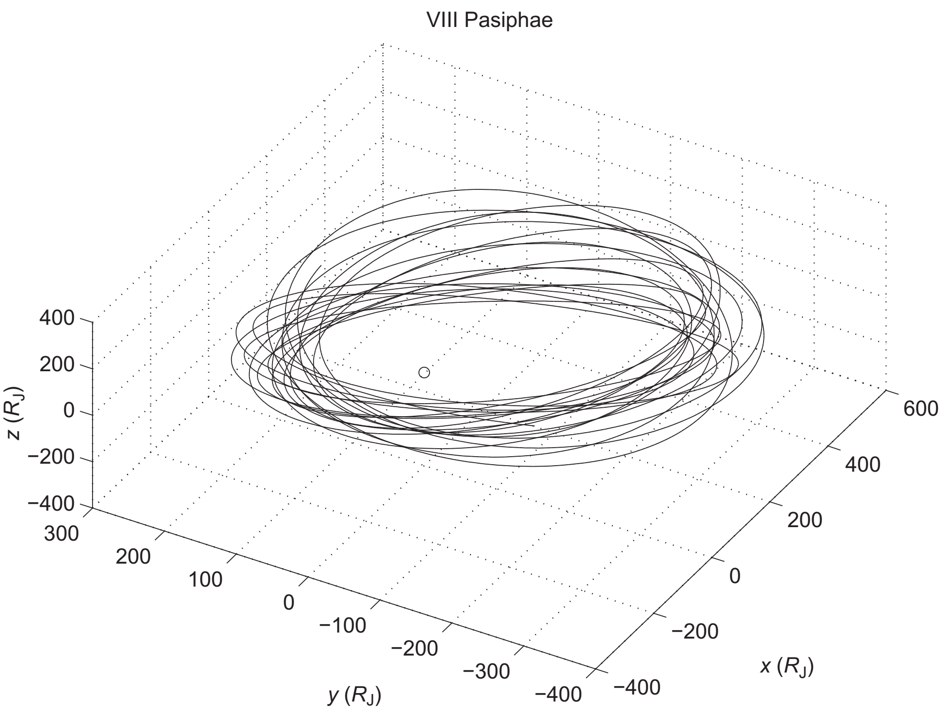

FIGURE 3.4 The orbit of J VIII Pasiphae, a distant retrograde satellite of Jupiter, is shown in a nonrotating coordinate system with Jupiter at the origin (open circle). The satellite was integrated as a massless test particle in the context of the circular restricted three-body problem for approximately 38 years. The unit of distance is Jupiter's radius, R_J. During the course of this integration, the distance to Jupiter varied from 122 to 548 R_J. Note how the large solar perturbations produce significant deviations from a Keplerian orbit. *Figure reprinted with permission from Jose Alvarellos (1996). "Orbital Stability of Distant Satellites of Jovian planets," M.Sc. thesis, San Jose State University.*

J_n are zero for odd n. For large bodies, J_2 is generally substantially larger than the other gravitational moments.

Consider a particle in Saturn's rings, which revolves around the planet in a circular orbit in the equatorial plane ($\theta = 90°$) at a distance r from the center of the planet. The centripetal force must be provided by the radial component of the planet's gravitational force (see Eqn (3.9)), so the particle's angular velocity n satisfies

$$rn^2(r) = \left[\frac{\partial \Phi}{\partial r}\right]_{\theta = 90°}. \tag{3.26}$$

If the particle suffers an infinitesimal displacement from its circular orbit, it will oscillate freely in the horizontal and vertical directions about the reference circular orbit with radial (epicyclic) frequency $\kappa(r)$ and vertical frequency $\mu(r)$, respectively, given by

$$\kappa^2(r) = r^{-3}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(r^2 n\right)^2\right], \tag{3.27}$$

$$\mu^2(r) = \left[\frac{\partial^2 \Phi}{\partial z^2}\right]_{z = 0}. \tag{3.28}$$

From Eqns (3.24)–(3.28), the following relation is found between the three frequencies for a particle in the equatorial plane:

$$\mu^2 = 2n^2 - \kappa^2. \tag{3.29}$$

For a perfectly spherically symmetric planet, $\mu = k = n$. Since Saturn and the other ringed planets are oblate, μ is slightly higher and k is slightly lower than the orbital frequency n.

Using Eqns (3.24)–(3.29), one can show that the orbital and epicyclic frequencies can be written as

$$n^2 = \frac{GM}{r^3}\left[1 + \frac{3}{2}J_2\left(\frac{R}{r}\right)^2 - \frac{15}{8}J_4\left(\frac{R}{r}\right)^4 + \frac{35}{16}J_6\left(\frac{R}{r}\right)^6 + \ldots\right], \tag{3.30}$$

$$\kappa^2 = \frac{GM}{r^3}\left[1 - \frac{3}{2}J_2\left(\frac{R}{r}\right)^2 + \frac{45}{8}J_4\left(\frac{R}{r}\right)^4 - \frac{175}{16}J_6\left(\frac{R}{r}\right)^6 + \ldots\right], \tag{3.31}$$

$$\mu^2 = \frac{GM}{r^3}\left[1 + \frac{9}{2}J_2\left(\frac{R}{r}\right)^2 - \frac{75}{8}J_4\left(\frac{R}{r}\right)^4 + \frac{245}{16}J_6\left(\frac{R}{r}\right)^6 + \ldots\right]. \tag{3.32}$$

Thus, for a particle orbit that is nearly equatorial, the oblateness of a planet causes the line of periapse to precess and the line of nodes to regress.

Resonances occur where the radial (or vertical) frequency of the ring particles is equal to the frequency of a component of a satellite's horizontal (or vertical) forcing, as experienced in the rotating frame of the particle. In this case, the resonating particle is always near the same phase in its radial (or vertical) oscillation when it experiences a particular phase of the satellite's forcing. This situation enables continued coherent "kicks" from the satellite to build up the particle's radial (or vertical) motion, and significant forced oscillations may thus result. The location and strengths of resonances with any given satellite can be determined by decomposing the gravitational potential of the satellite's effect on the ring particle into its Fourier components. The disturbance frequency, ϖ, can be written as the sum of integer multiples of the satellite's angular, vertical, and radial frequencies:

$$\varpi = jn_s + k\mu_s + \ell\kappa_s, \tag{3.33}$$

where the azimuthal symmetry number, j, is a nonnegative integer, and k and ℓ are integers, with k being even for horizontal forcing and odd for vertical forcing. The subscript s refers to the satellite. A particle placed at distance $r = r_L$ will undergo horizontal (Lindblad) resonance if r_L satisfies

$$\varpi - jn(r_L) = \pm\kappa(r_L). \tag{3.34}$$

It will undergo vertical resonance if its radial position, r_v, satisfies

$$\varpi - jn(r_L) = \pm\mu(r_v). \tag{3.35}$$

When Eqn (3.34) is valid for the lower (upper) sign, r_L is referred to as the inner (outer) Lindblad or horizontal resonance. The distance r_v is called an inner (outer) vertical resonance if Eqn (3.35) is valid for the lower (upper) sign. Since all of Saturn's large satellites orbit the planet well outside the main ring system, the satellite's angular frequency n_s is less than the angular frequency of the particle and inner resonances are more important than the outer ones. When $j \neq 1$, the approximation $\mu \approx n \approx k$ may be used to obtain the ratio

$$\frac{n\left(r_{L,v}\right)}{n_s} = \frac{j + k + \ell}{j - 1}. \tag{3.36}$$

The notation $(j + k + \ell)/(j - 1)$ or $(j + k + \ell):(j - 1)$ is commonly used to identify a given resonance.

The strength of the forcing by the satellite depends, to lowest order, on the satellite's eccentricity, e, and inclination, i, as $e^{|\ell|}(\sin i)^{|k|}$. The strongest horizontal resonances have $k = \ell = 0$, and are of the form $j:(j - 1)$. The strongest vertical resonances have $k = 1$, $\ell = 0$, and are of the form