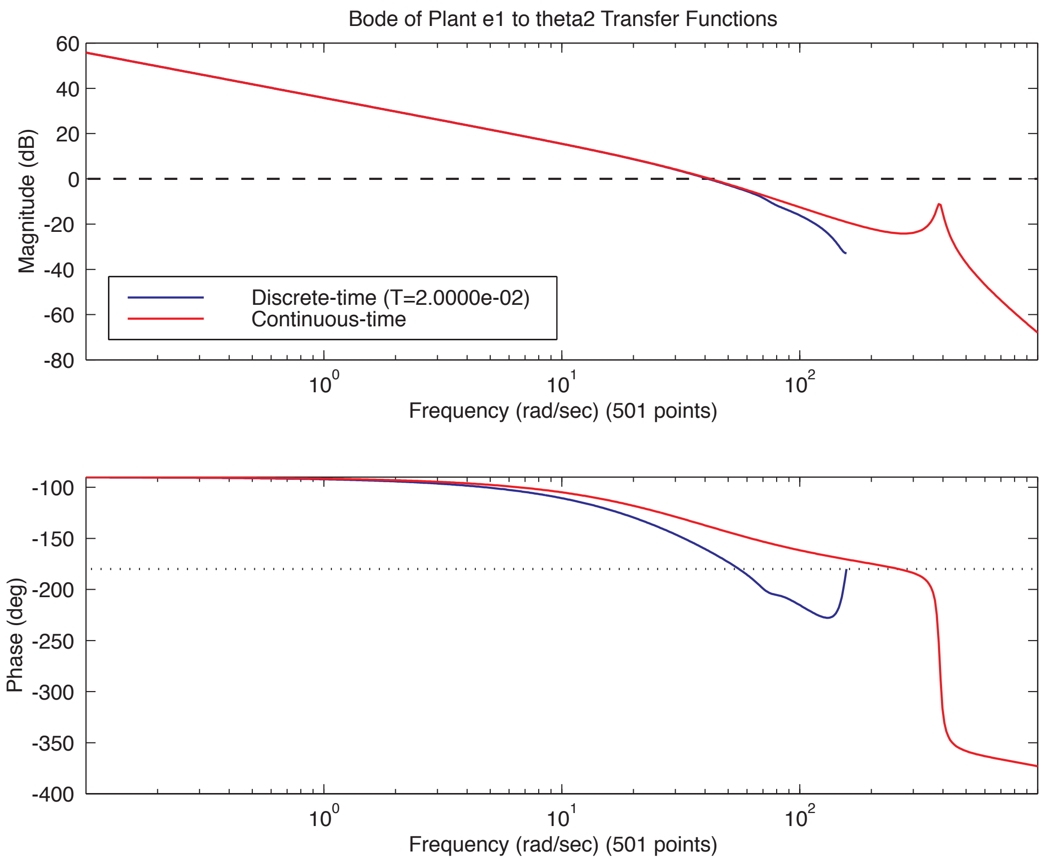
AA/EE/ME 581
Digital Control

HW 8 Solution

Special Problem



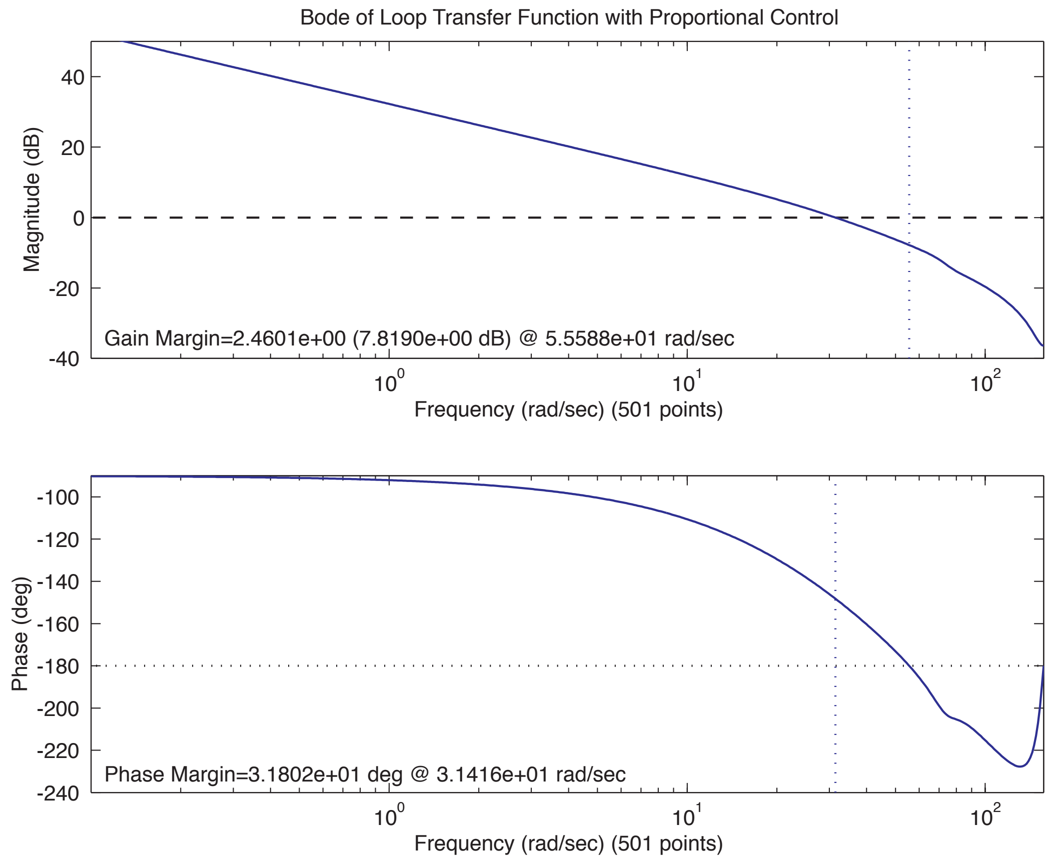
Using the discrete-time open-loop plant e_1 to θ_2 frequency response shown above and the rule-of-thumb that the closed-loop bandwidth will be about twice the loop transfer function crossover frequency, I started with the proportional controller

$$C(z) = K \quad \text{V/rad}$$

and selected

$$K = 0.66703$$

to get the loop transfer function crossover frequency to be 10π rad/sec. The loop transfer function frequency response with this proportional controller is shown below.



With this controller, the loop transfer function has 28.2 deg too much phase lag at the loop transfer function crossover frequency to meet the phase margin objective. So I used the loop shaping lead compensator design formulas from the Classical Control Notes to determine the continuous-time lead compensator dynamics

$$\sqrt{\frac{\omega_p}{\omega_z}}\;\frac{s+\omega_z}{s+\omega_p}$$

that provide unity gain and 29.2 deg of phase lead at the loop transfer function crossover frequency $10\,\pi$ rad/sec. The one degree extra of phase lead here turns out to be necessary to correct for the approximately one degree of phase lag that the lag compensator to be added later will contribute at the loop transfer function crossover frequency $10\,\pi$ rad/sec. This yields

$$\omega_z \;=\; 18.432 \text{ rad/sec} \qquad \text{and} \qquad \omega_p \;=\; 53.547 \text{ rad/sec}$$

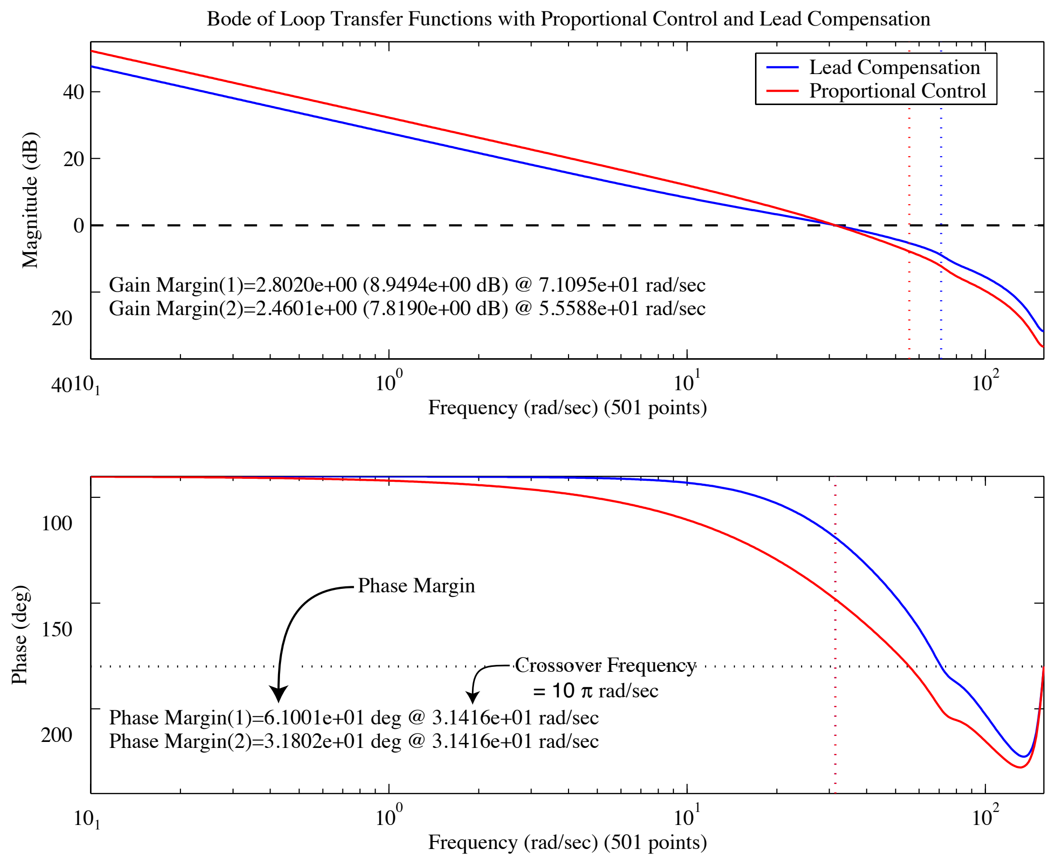
Next I discretized the complete continuous-time lead compensator

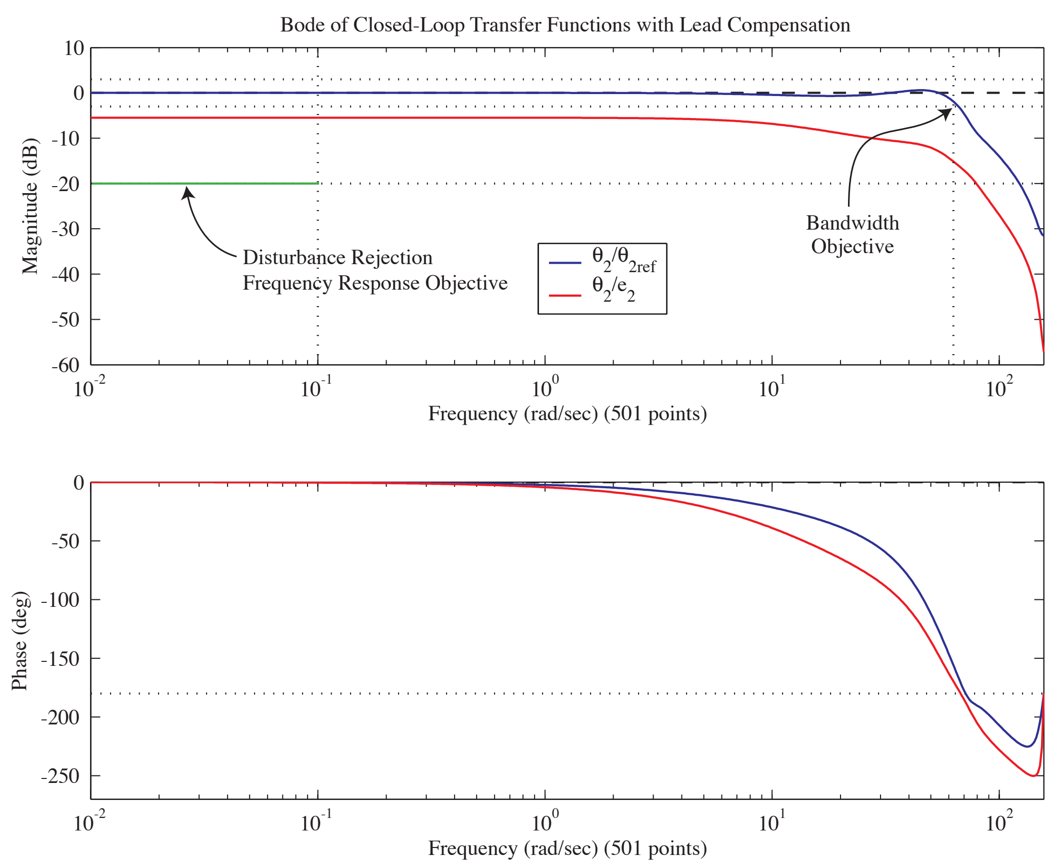
$$C(s) \;=\; K\sqrt{\frac{\omega_p}{\omega_z}}\;\frac{s+\omega_z}{s+\omega_p}$$

using Tustin's with prewarping, and using the loop transfer function crossover frequency $10\,\pi$ rad/sec as the prewarping frequency, to obtain the discrete-time lead compensator

$$C_{lead}(z) \;=\; 0.87118\;\frac{z-0.67978}{z-0.28716}$$

The resulting loop transfer function and closed-loop transfer function frequency responses are shown below.


Bode of Loop Transfer Functions with Proportional Control and Lead Compensation
Lead Compensation
Proportional Control
Magnitude (dB)
Gain Margin(1)=2.8020e+00 (8.9494e+00 dB) @ 7.1095e+01 rad/sec
Gain Margin(2)=2.4601e+00 (7.8190e+00 dB) @ 5.5588e+01 rad/sec
Frequency (rad/sec) (501 points)
Phase (deg)
Phase Margin
Crossover Frequency
= 10 π rad/sec
Phase Margin(1)=6.1001e+01 deg @ 3.1416e+01 rad/sec
Phase Margin(2)=3.1802e+01 deg @ 3.1416e+01 rad/sec
Frequency (rad/sec) (501 points)


Bode of Closed-Loop Transfer Functions with Lead Compensation
Magnitude (dB)
Disturbance Rejection
Frequency Response Objective
θ2/θ2ref
θ2/e2
Bandwidth
Objective
Frequency (rad/sec) (501 points)
Phase (deg)
Frequency (rad/sec) (501 points)

The bandwidth and phase margin objectives are now satisfied, but the disturbance rejection objectives are not. To meet the disturbance rejection objectives: (1) the magnitude of the $e_2(z)$ to $\theta_2(z)$ frequency response needs to be zero at zero frequency and (2) the magnitude of the $e_2(z)$ to $\theta_2(z)$ frequency response needs to be less than 0.1 rad/V at all frequencies up to 0.1 rad/sec. Knowing that the closed-loop $e_2(z)$ to $\theta_2(z)$ transfer function has $1 + L(z)$ in it's denominator, where $L(z)$ is the loop transfer function, I cascaded with my lead compensator a lag compensator. I found this lag compensator by discretizing, using Tustin's with prewarping, and using the loop transfer function crossover frequency 10 π rad/sec as the prewarping frequency, a continuous-time lag compensator of the form

$$C(s) = \frac{s + \omega_z}{s + \omega_p}$$

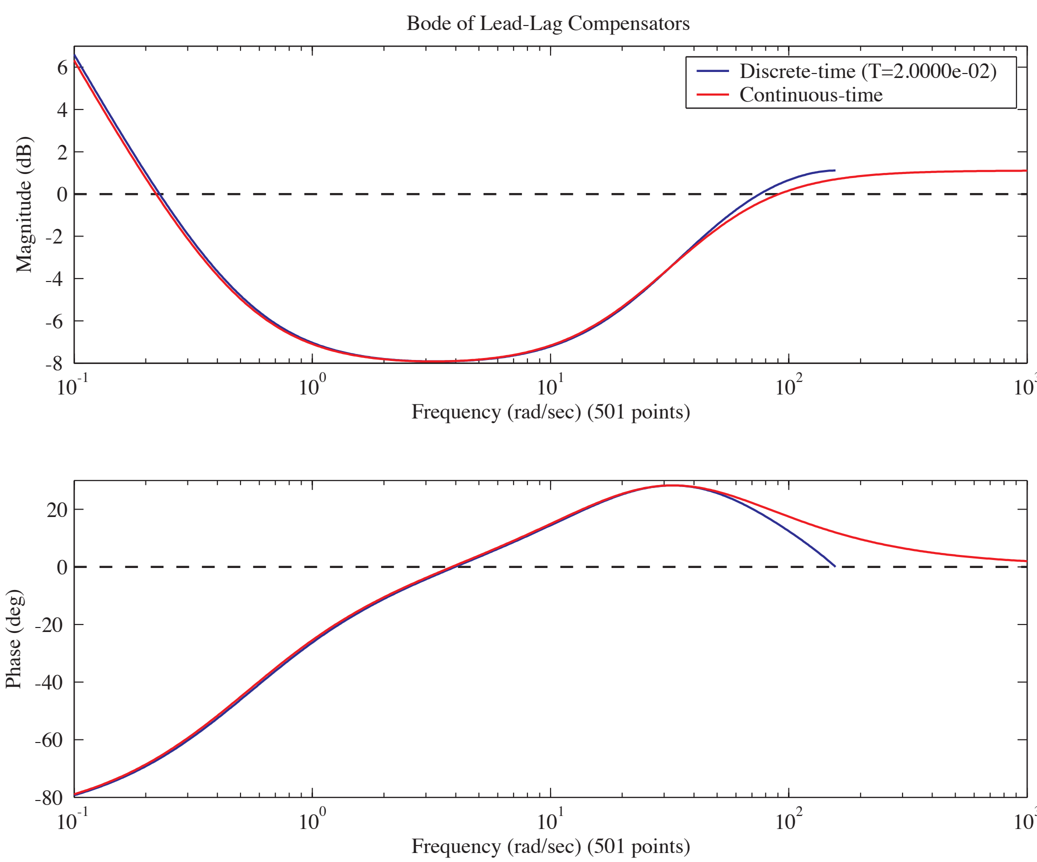
To satisfy Objective 2 ($\theta_2 = 0$ in the steady state in response to an e_2 step input when $\theta_{2ref} = 0$), $\omega_p = 0$ was required. To satisfy Objective 3 (steady-state θ_2 response to $e_2(t) = \sin(\omega t)$ V less than 0.1 rad for all $\omega \le 0.1$ rad/sec), ω_z needed to be sufficiently large. To keep the lag compensator's phase lag sufficiently small at the crossover frequency, ω_z needed to be small compared to the crossover frequency. Trial and error with a few different values of ω_z quickly lead me to choose $\omega_z = 0.52$ rad/sec. Tustin-with-prewarping discretization of this continuous-time lag compensator using the crossover frequency as the prewarping frequency then resulted in

$$C_{lag}(z) = 1.0052\,\frac{z - 0.98930}{z - 1}$$

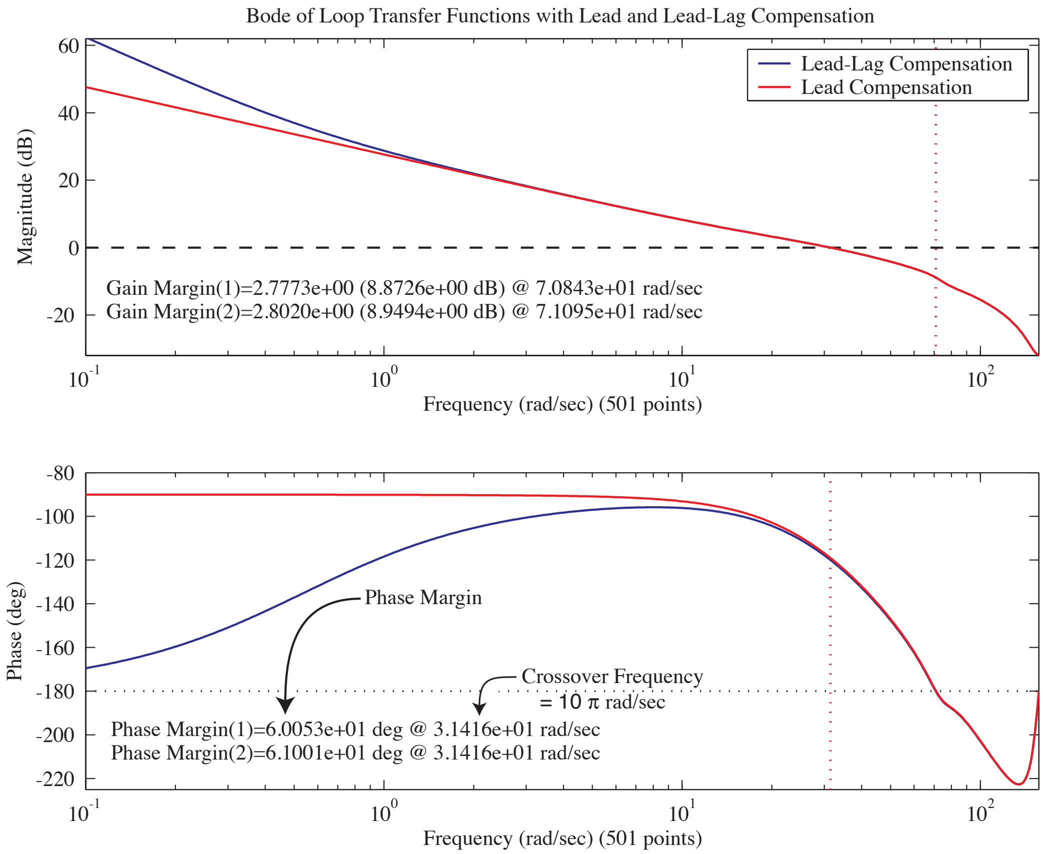
My final discrete-time lead-lag compensator was then

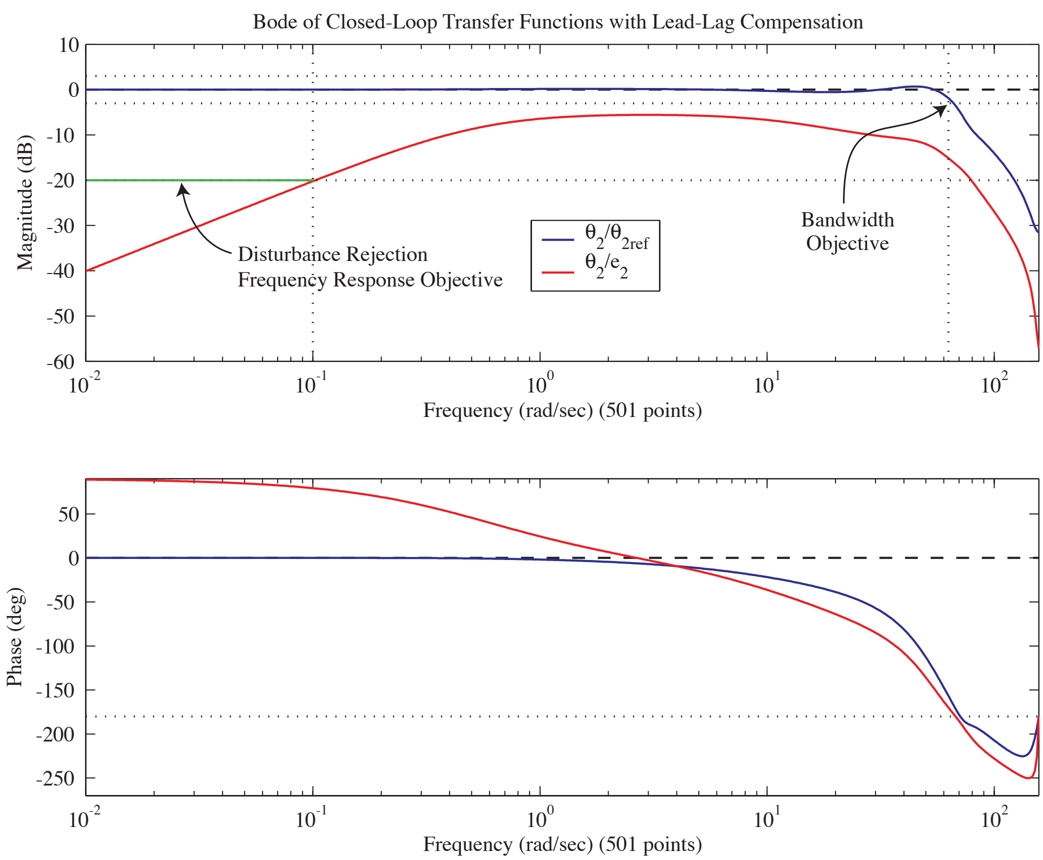
$$C(z) = C_{lead}(z)C_{lag}(z) = 0.87574\,\frac{z - 0.67978}{z - 0.28716}\,\frac{z - 0.98930}{z - 1}$$

It's frequency response is shown below.



The loop transfer function and closed-loop transfer function frequency responses with this lead-lag compensator are shown below.





That the steady-state θ_2 response of the closed-loop system to step e_2 input is zero is indicated by the θ_2 response of the closed-loop system to unit step e_2 input shown here: