Investigating the constructability of regular polygons through Origami.

Subject: Mathematics

Word Count: 3657

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1 Introduction

The art of paper folding, Origami, has been a traditional Japanese practice that utilizes a flat square piece of paper, often with colored designs, to construct elegant figures dating back as far as the 15th century. Despite the varying styles of Origami, I took a particular interest in the original square-papered, non-cutting, techniques of paper folding; the concept of creating something detailed out of a flat sheet of paper seemed magical. Every constructed model has a crease pattern that acts as a blueprint unique to that figure, and its constructability depends heavily on the constructability of its creases. In pursuit of the mathematics behind paper folding, I came across an elegantly constructed pattered coaster that was made by my aunt in Japan (Figure 1 to Figure 5). This 14-sided polygon, tetradecagon, was purely made out of a square paper without being cut. The coaster intrigued me into the simple research question: “to what extent is a regular polygon theoretically constructable?” Further research led me into the field of polygon constructions, and claims how Origami is more powerful as it able to produce ratios that Euclidean Geometry Methodology (aka Straight-edge and Compass constructions) cannot. Being interested in geometry and origami as a child, I was enlightened by the fact that my traditional custom was proven to be superior than ancient Greek geometry in a theoretical field. This Extended Essay is an opportunity for me to combine my love for geometry and origami into a single research paper.

Figure 1 – Tetradecagon Origami Coaster 1

Figure 2 – Tetradecagon Origami Coaster 2

Figure 3 – Tetradecagon Origami Coaster 3

In this paper, I will be exploring the constructable numbers using the Origami Axioms to ultimately deduce how many regular polygons are buildable within its restrictions, and demonstrate the constructions of tetradecagon, heptagon, and tridecagon.

Figure 4 – Octagon Ninja Star 1

Figure 5 – Octagon Ninja 2

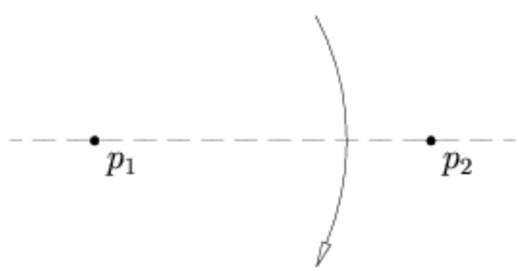
2 Elementary Origami Constructions

In order to deduce how many regular polygons can be folded on origami, it is important to note to what kind of rules we can use. Starting from a given unit length, the elementary operation allows us to extrapolate the constructable numbers and provide us insight to what we can create. In this field, instead of a one-by-one unit paper, we imagine the paper to be an infinitely extended geometrical plane with negligible thinness.

2.1. Huzita-Hatori Axioms

Origami follows a set of rules in order to create a number; it follows a total of 7 Axioms. Axioms 1 through 6 were first discovered by Jacques Justin in 1986, and Humiaki Huzita, a Japanese-Italian mathematician, rediscovered and popularized the first 6 Axioms, hence the name "Huzita's Axioms." The 7th Axioms was found by Koshiro Hatori in 2001, and Robert J. Lang, an American physicist, proved it to be the last of the Axioms (Lang). The Huzita-Hatori Axioms are given as follows:

1. Given two distinct points p_1 and p_2 there is a unique fold that passes through both of them.



(Source: Wikipedia)

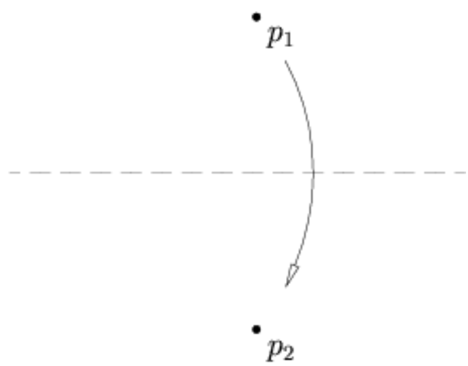
To present this in a Cartesian formula, we must first define $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. If a point on the folded crease line is $P_l = (x_l, y_l)$, it must satisfy the equation:

$$y_l - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x_l - x_1) \qquad (1)$$

In the $y = mx + b$ form:

$$y_l = \frac{y_2 - y_1}{x_2 - x_1} x_l + \left(y_1 - x_1 \frac{y_2 - y_1}{x_2 - x_1} \right) \tag{2}$$

2. Given two distinct points p_1 and p_2 there is a unique fold that places p_1 onto p_2.



(Source: Wikipedia)

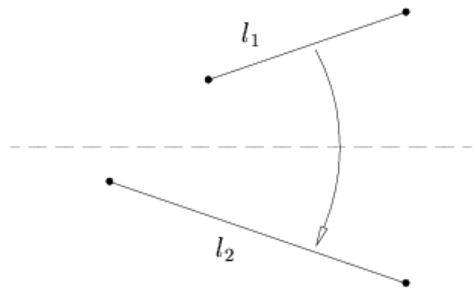
The folded crease line must be perpendicular to the slope derived from Axiom 1. The folded line also must go through the middle point $P_m = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Therefore, a point on the creased line must satisfy:

$$y_l - \frac{y_1 + y_2}{2} = -\frac{x_2 - x_1}{y_2 - y_1}\left(x_l - \frac{x_1 + x_2}{2}\right) \tag{3}$$

In the $y = mx + b$ form:

$$y_l = -\frac{x_2 - x_1}{y_2 - y_1} x_l + \left(\frac{y_1 + y_2}{2} + \frac{x_2^2 - x_1^2}{2(y_2 - y_1)} \right) \tag{4}$$

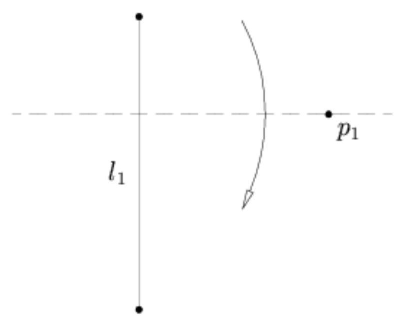
3. Given two lines l_1 and l_2, there is a fold that places l_1 onto l_2.



(Source: Wikipedia)

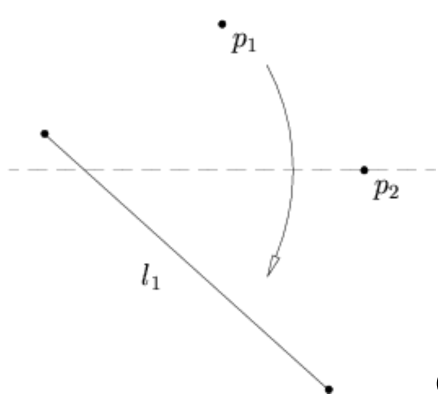
This Axiom allows bisection of an arbitrary angle made by 2 distinct lines.

4. Given a point p_1 and a line l_1, there is a unique fold perpendicular to l_1 that passes through point p_1.



(Source: Wikipedia)

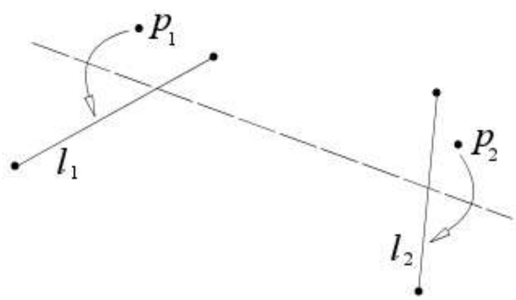
5. Given two points p_1 and p_2 and a line l_1, there is a fold that places p_1 onto l_1 and passes through point p_2.



(Source: Wikipedia)

Because the distance between p_1 and p_2 are preserved, the point on l_1 that is p_1 when folded is also the same distance from p_2. This allows a compass like property that draws a circle centered at p_1.

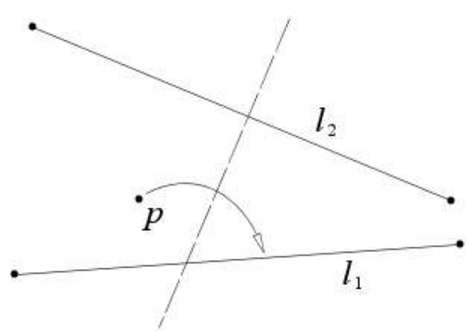
6. Given two points p_1 and p_2 and two lines l_1 and l_2 , there is a fold that places p_1 onto l_1 and p_2 onto l_2.



(Source: Wikipedia)

Note that Axiom 6 involves four given different definitions: p_1, p_2, l_1, and l_2. This makes the Axiom unique in comparison to the others as it is able to solve any cubic equation with real solutions.

7. Given one point p and two lines l_1 and l_2, there is a fold that places p onto l_1 and is perpendicular to l_2.



(Source: Wikipedia)

3 Constructing a regular 16-gon

Making a regular tetradecagon may sound difficult at first; however, once one notices 16 is merely a square of 4, it is possible to formulate the angle of a sector just by bisecting a 90º angle twice.

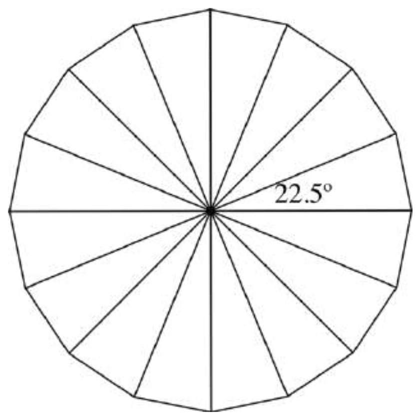


Figure 6 – Regular Tetradecagon

As seen in Figure 6, to create a polygon, one must be able produce the angle respective of that polygon. In this case, the tetradecagon has $\frac{360}{16} = 22.5^{\circ}$ for each sector, which is just bisecting a 45º angle.

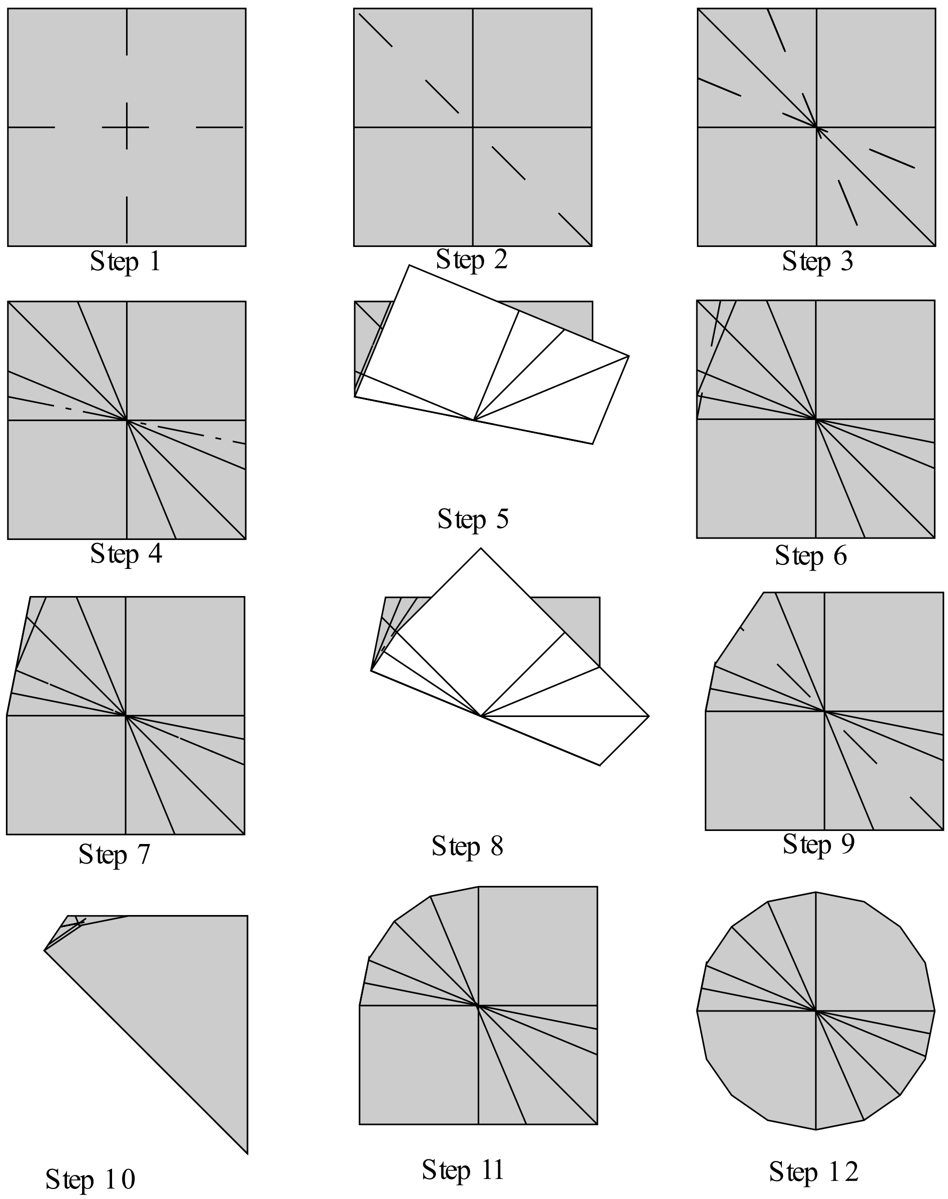


Figure 7 – Instructions to fold a tetradecagon (Created on Inkscape)

Steps 1 to 3 in Figure 7 demonstrates the construction of 22.5º only using Axiom 3, whereas steps 4-11 show the building of sides mainly utilizing Axiom 5 because of its compass like property.

Figure 8 – Origami Tetradecagon

One can conclude that because of Axiom 3, any constructable regular polygon with their sides multiplied by a power of two is also constructable.

4 Rational Numbers

In the article "Origami and Geometric Constructions," by Robert J. Lang, he claims to have constructed all possible rational numbers given a unit squared paper. The article suggests multiple methods in producing all rational numbers, however, it is mainly focused on comparing the "ranks" (the number of folds necessary) to create a certain rational. One of the methods presented in this article is "Noma's Construction." The following proves the constructability of a fraction, $\frac{a}{b} \leq 1$, where $a, b \in \mathbb{Z}^{+}$, using the mentioned method and the earlier cartesian derived equation of Axiom 2.

Note that by creating a fraction on a unit square, it is equivalent to producing all rational numbers. We can see this as a ratio between the side length of a paper and the constructed number as $1 : \frac{a}{b}, where\ a \leq b$. By using a square paper with side-length k, we are essentially proving the constructability of $k\frac{a}{b}$.

4.1.Noma's Construction

Lang presents Noma's construction by first understanding the Binary Divisions. Binary Division is simple as using Axiom 2 to bisect a segment, in this case the edge of the paper.

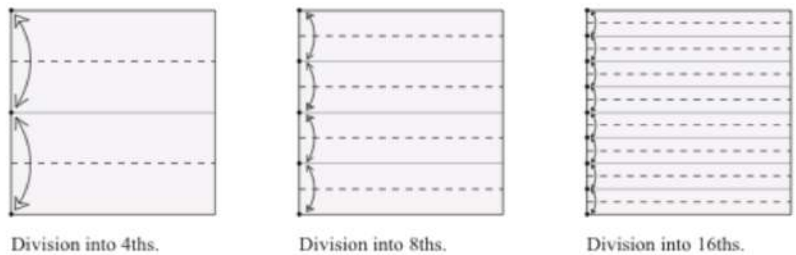


Figure 9 – Binary Division (Lang)

This means a segment can be divided into in proportions of $\frac{1}{2^n}$, where n is a natural number, which also indicates the buildability of $\frac{m}{2^n}$, where the positive integer $m \leq 2^n$. The method does require considerable number of folds; however, Lang devises a Binary Folding Algorithm that folds the fraction $\frac{m}{2^n}$ in the most efficient way possible, which we will not get into.

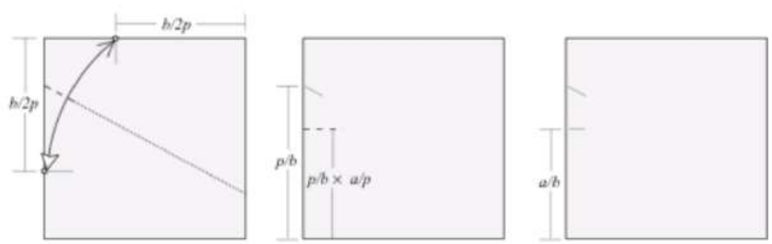


Figure 10 – Noma's Construction (Lang)

Noma's method creates the denominator initially, and then uses the binary division method to achieve the objective fraction $\frac{a}{b}$. We may consider the origami to be a graph with an x and y axis as shown in Figure 11.

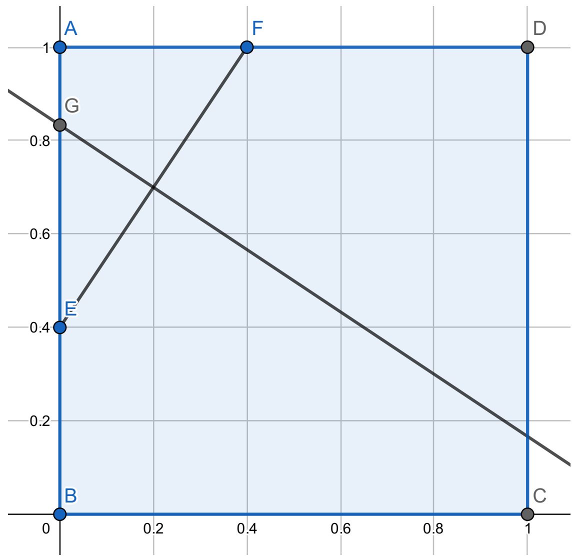


Figure 11 – Noma's Construction Example

We define the points $E = P_1 = (0, w)$ and $F = P_2 = (w, 1)$ and substitute them into Axiom-2-equation (equation 4) to obtain the line in slope intercept form.

$$y_l = -\frac{1-w}{1-w}x_l + \left(\frac{1-w}{2} + \frac{w^2}{2(1-w)}\right) \tag{5}$$

Because the line's y-intercept is the value we want ($x_l = 0$),

$$y_l = \frac{1}{2(1-w)} \tag{6}$$

Lang claims that y_l will give us the desired denominator by defining $w = 1 - \frac{b}{2 \cdot 2^n}$:

$$y_l = \frac{1}{2\left(1 - \left(1 - \frac{b}{2 \cdot 2^n}\right)\right)}$$

$$y_l = \frac{2^n}{b} \tag{7}$$

Now that we have the denominator, we may again use the binary division (Figure 10) to obtain the desired proportion.

$$\frac{2^n}{b} \times \frac{a}{2^n} = \frac{a}{b} \tag{8}$$

5 Solving Polynomials

Lill's Method, discovered by an Austrian engineer, Eduard Lill, is a graphical representation of finding a polynomial's real roots. Margherita Piazzola Beloch, an Italian Mathematician, discovered an connection between Lill's method and the Origami's Axiom 6 and presented the Beloch's Fold to solve any arbitrary cubic equation just by folding paper.

5.1. Lill's Method

Lill's method, as described by Thomas C. Hull in the article "Solving Cubics With Creases: The Work of Beloch and Lill", starts by a turtle traveling a_n distance away from the starting point O. The turtle then turns 90º anti-clockwise to travel a_{n-1}. This process repeats until the turtle travels a_0 towards their respective angle and reaches a terminal point T shown in Figure 11.1. The method claims that by taking a "bullet" through an alternative route to reach the terminal point, primarily by traveling from an angle θ and having 90º reflections along every turtle-traveled line, the negative value of the slope of the initial "bullet" path is the solution of the polynomial; in other words, $-\tan\theta$ is the solution.

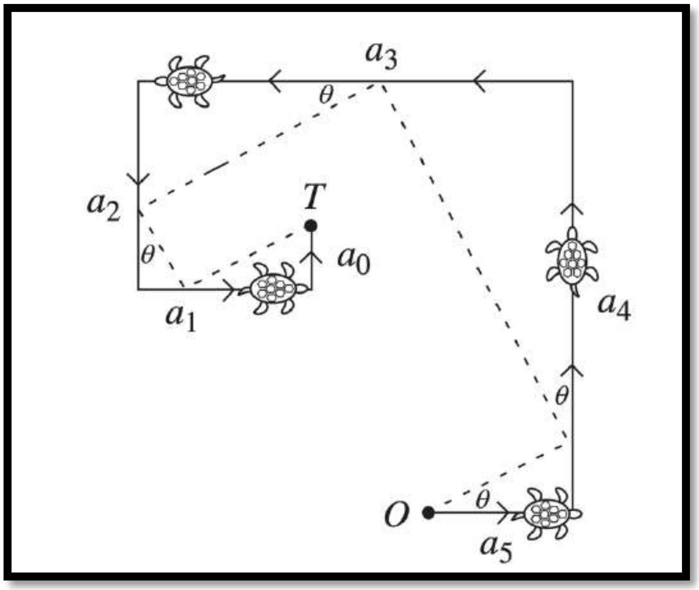


Figure 11.1 – Lill's Method Visual Representation (Hull, "Solving Cubics with creases")

This can be proved by viewing a polynomial as (in this case a cubic function):

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \tag{9}$$
$$f(x) = x(a_3x^2 + a_2x + a_1) + a_0$$

$$f(x) = a_0 + x\big(a_1 + x(a_2 + a_3x)\big)$$

$$0 = a_0 + x\big(a_1 + x(a_2 + a_3x)\big) \tag{10}$$

The process is done just from factoring x out repeatedly until the all x are in the first degree.

Thomas Hull further proves Lill's Method by deriving the formula as follows:

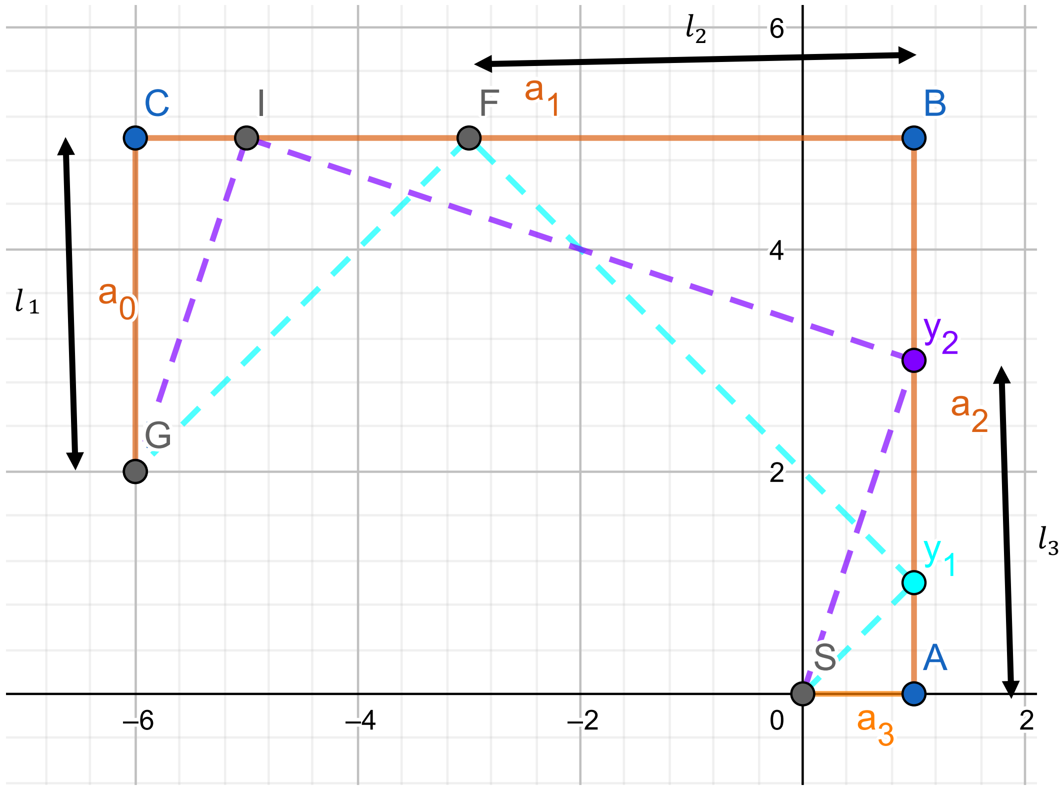


Figure 12 – Lill's Method Proof Diagram

$$l_3 = a_3 \tan\theta$$

$$l_2 = \tan\theta\,(a_2 - l_3)$$

$$l_1 = \tan\theta\,(a_1 - l_2)$$

$$l_0 = 0 = a_0 - l_1$$

By substituting the equations, we get:

$$l_0 = 0 = a_0 - \tan\theta\left(a_1 - \tan\theta\,(a_2 - a_3\tan\theta)\right) \tag{11}$$

$$0 = a_0 + x\left(a_1 + x(a_2 + a_3 x)\right) \tag{12}$$

If we compare equation 11 and 12, it is clear that $x = -\tan\theta$ and is therefore a valid solution to the equation.

5.2. Solving Quadratics

Lill's method helps us translate the Axiom 5 into solving arbitrary quadratic equation.

Let's take for example the equation:

$$2x^2 + 3x + 1 = 0 \tag{13}$$

$$a_2 = 2, a_1 = 3, a_0 = 1$$

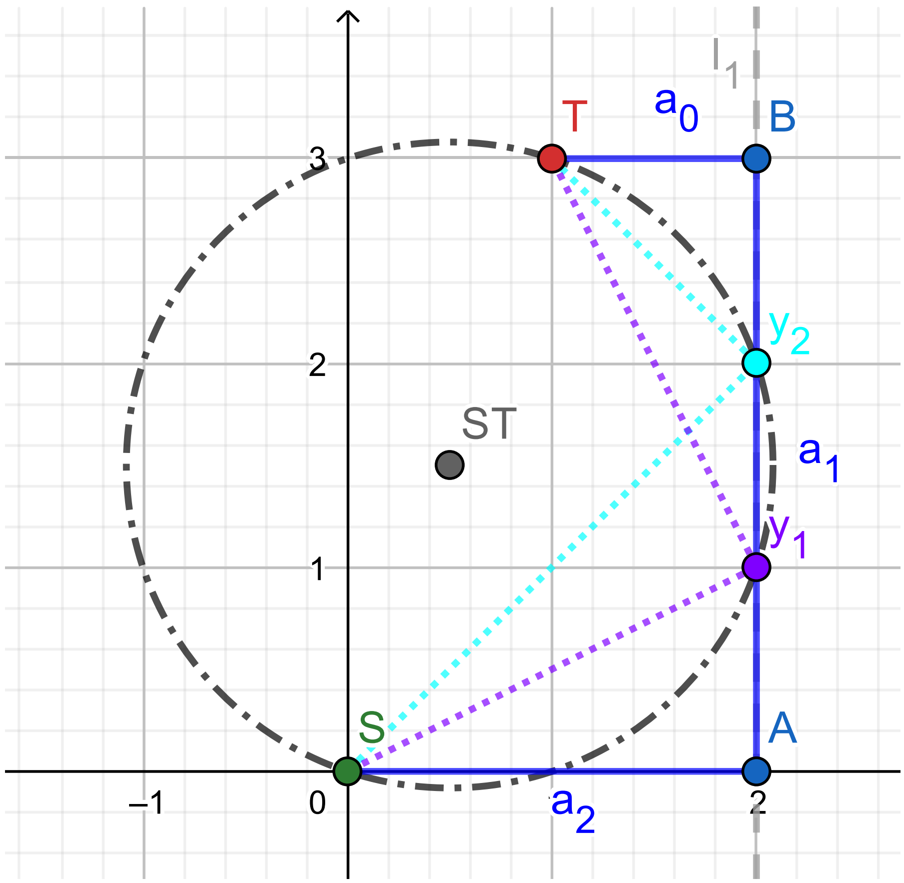


Figure 13 – Lill's Method: Quadratic roots

Solving a quadratic equation via Lill's Method is fairly simple. We just need a right-angle triangle where its hypotenuse is the segment S to T. The perpendicular vertex of the right-angle triangle with a fixed hypotenuse creates a circle. The solution must be a point on $l_1$where the perpendicular vertex of the said triangle must meet. Because we are looking at an intersection of a circle whose center is the midpoint of the segment S to T (Point ST) and a line, (Segment AB), we must always get 2 or less solutions, which stays consistent with the roots of a quadratic.

Lill's method states that the solution is:

$$x = -\tan\theta = -\frac{Distance\ from\ A\ to\ y_1\ or\ y_2}{a_2} \tag{14}$$

$$x = -\frac{1}{2}, -1$$

Note that y_1 and y_2 both can be found via Axiom 5 by placing point T on l_1 where the fold line passes through point ST. The position where point T was placed on is either y_1 or y_2.

If a coefficient is negative, the turtle must travel equal in magnitude but in the opposite direction than it would with a positive coefficient. The following example equation is used for convenience as it will be used later on:

$$x^2 - 4x - 9 = 0 \tag{15}$$

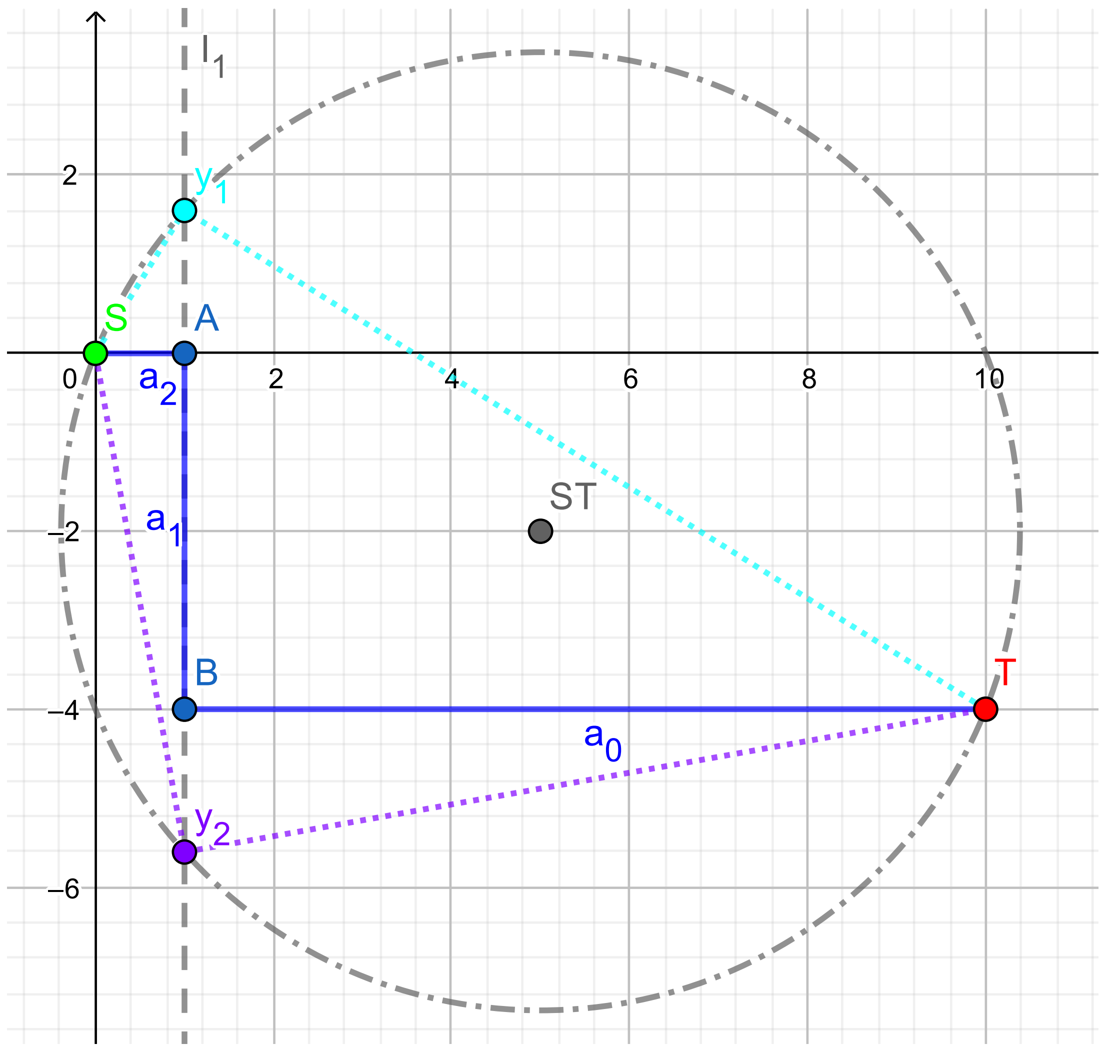


Figure 14 – Lill's Method: Quadratic roots from negative coefficient

Although there is no intersection between the circle ST and segment AB, there is two solutions for the extension of the segment, l_1. Note that this circle method is valid even with negative coefficients.

$$x = -\frac{Distance\ from\ A\ to\ y_1\ or\ y_2}{a_2} \approx -\frac{1.6056}{1}, -\frac{-5.6056}{1}$$

$$x = 2 - \sqrt{13}, 2 + \sqrt{13}$$

5.3. Beloch's Fold

The circle method does not work on trinomials as it requires 2 right angled corners instead of one. Beloch's Fold integrates the Lill's method for trinomials to origami. It suggests that by creating a vertical line, l_1, a_3 distance away from point A, and a horizontal line, l_2, a_0 distance away from point C as seen in Figure 15, we are able to fold it so that the points S and T to l_1 and l_2 lines up respectively (this is the application of Axiom 6). The negative distance from point A to y_1 divided by $a_3$line is the solution to the cubic function.

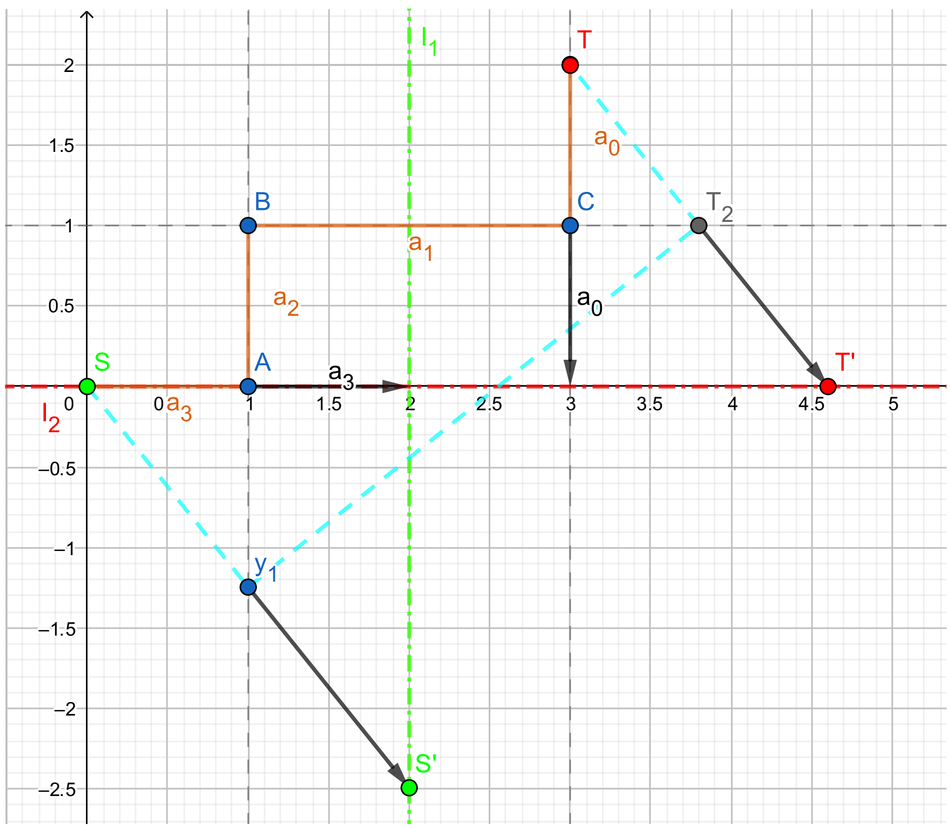


Figure 15 – Lill's Method: Cubic roots

Figure 15 displays one of the solutions of the example equation:

$$x^3 + x^2 - 2x - 1 = 0 \quad (16)$$

By computationally measuring the distance from $A\ to\ y_1$ to the 4th decimal, we get the solution.

$$x = -\frac{Distance\ from\ A\ to\ y_1}{a_3} \approx -\frac{-2.4940}{2}$$

$$x \approx 1.2470$$

Which is one of the solution to the trinomial 16.

6 *Trisection*

Axiom 3 evidently shows the Origami's ability to bisect a given angle; however, is trisecting an angle possible, where the Euclidean method could not? It turns out that Origami with the special Axiom 6 that allows the construction of cubic solutions, which also allows trisection of an arbitrary angle. This is due to the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ considering the initial angle $\cos\alpha = \cos 3\theta$, where $\cos\alpha$ is constructable, and we are able to solve for $\cos\theta$ in the equation.

$$\cos\alpha = 4\cos^3\theta - 3\cos\theta$$

$$0 = 4\cos^3\theta - 3\cos\theta - \cos\alpha \tag{17}$$

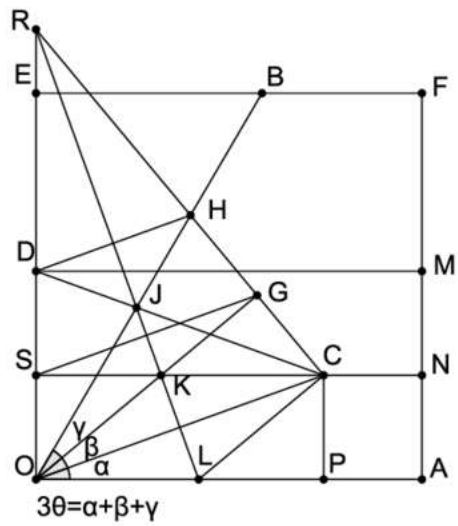


Figure 16 – Trisection Proof (Bhat)

Bha proves this argument geometrically as seen in Figure 16. If trisection is possible of an arbitrary angle then, with a similar reasoning with the bisection of an angle, every constructable regular polygon with their sides multiplied by a power of 3 is also constructable. Considering the use of bisection and trisection to a constructable k-gon, one can deduce that:

$$n = 3^p 2^q k, \ \ p, q \in \mathbb{Z}^+ \tag{18}$$

If k-gon is constructable, then n-gon is also constructable for all positive integers p and q.

7 *Polygon in the complex plane*

What does it mean to construct a polygon? How do we know if a regular polygon is constructable? The previous deduction suggests that "any constructable polygon with their sides multiplied by either the power of 2 or 3 is also constructable," then how can we determine the constructability of polygons with sides that are not a product of the powers of 2 or 3? A useful tool in mathematics to understand regular polygons is the complex plane geometry. Roots of

Unity is a mathematical phenomenon that when solving an equation $z^n - 1 = 0$, there are evenly spaced complex solutions on a unit circle of the complex plane.

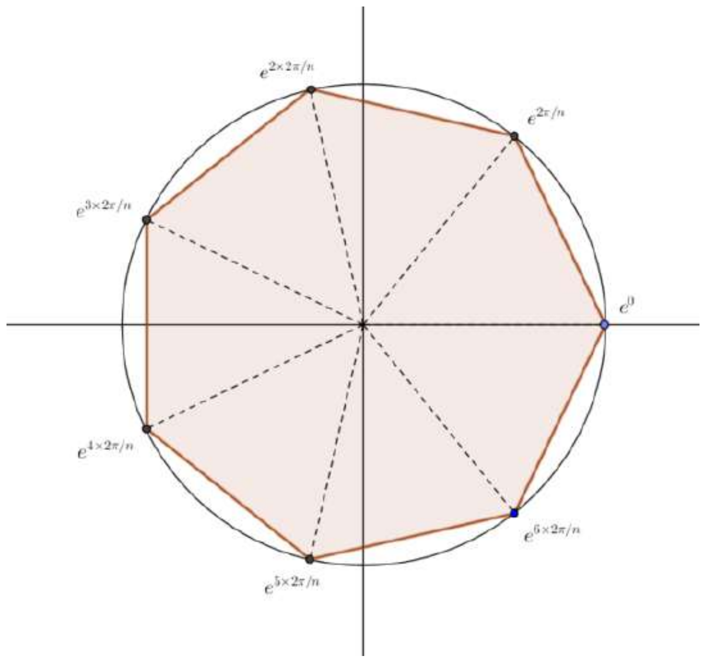


Figure 17 – Roots of Unity (Socratic)

If we let $z^n = e^{2k\pi i}$, where $k \in [1, n]$ and ζ is the primitive root, then we obtain n solutions:

$$\zeta = e^{\frac{2\pi i}{n}}, \qquad \zeta^2 = e^{2\frac{2\pi i}{n}} \ldots \zeta^{n-1} = e^{(n-1)\frac{2\pi i}{n}}, \qquad \zeta^n = e^{n\frac{2\pi i}{n}} = 1$$

Then we must prove an equation by factoring out 1 out of the solution, and because $z^n - 1 = 0$:

$$\frac{z^n - 1}{z - 1} = 1 + \omega + \omega^2 \ldots \omega^{n-1} = 0 \qquad (19)$$

These points on the complex plane can be thought of vertices on polygon. Note that the bottom half of the vertices are reflected top half along the imaginary axis; therefore, the imaginary values are just negatives of those on the top half. With this in mind, De Moivre's Theorem shows that

$$\omega = e^{\frac{2\pi i}{n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} \qquad (20)$$

Whereas the bottom half is its conjugate as shown below:

$$\omega^{n-1} = e^{(n-1)\frac{2\pi i}{n}} = e^{-\frac{2\pi i}{n}} = \omega^{-1} \tag{21}$$

$$\omega^{-1} = \cos -\frac{2\pi}{n} + i \sin -\frac{2\pi}{n}$$

$$\omega^{-1} = \cos \frac{2\pi}{n} - i \sin \frac{2\pi}{n} \tag{22}$$

Therefore:

$$\omega + \omega^{-1} = 2\cos \frac{2\pi i}{n} \tag{23}$$

If we can construct $2\cos\frac{2\pi}{n}$ where its solutions are within the constructable range – a solution to any combinations of linear, quadratic, and cubic functions – it would signify that the nth polygon is constructable.

8 Constructing a Regular Hendecagon – 7-gon

Utilizing the appropriate construction methods mentioned, Thomas C. Hull demonstrated a folding of a Heptagon on a one-by-one sheet of paper. He begins by corresponding the vertices to the 7th roots of unity:

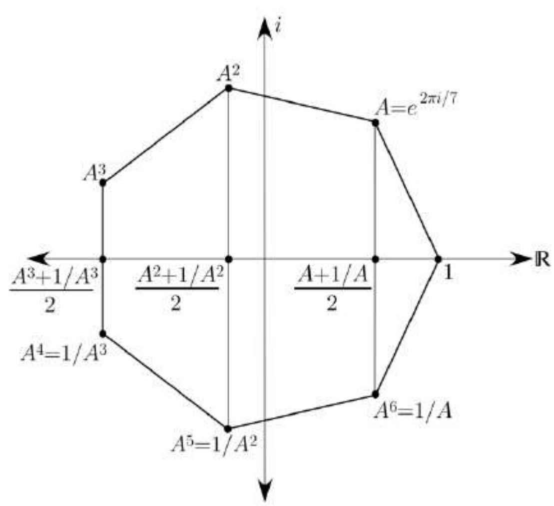


Figure 18 – 7th Roots of Unity (Hull, "Folding Regular Heptagons")

$$\zeta = e^{\frac{2\pi i}{7}}, \quad \zeta^2 = e^{2\frac{2\pi i}{7}}, \quad \zeta^3 = e^{3\frac{2\pi i}{7}}, \quad \zeta^4 = e^{4\frac{2\pi i}{7}}, \quad \zeta^5 = e^{5\frac{2\pi i}{7}}, \quad \zeta^6 = e^{6\frac{2\pi i}{7}}$$

Eventually reaching $\zeta^7 = e^{2\pi i} = 1$. From equation 21, we can derive:

$$1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6 = 0 \tag{24}$$
$$1 + \zeta + \zeta^2 + \zeta^3 + \zeta^{-1} + \zeta^{-2} + \zeta^{-3} = 0 \tag{25}$$

We can expand and simplify to get the sets in terms of $\zeta + \zeta^{-1} = 2\cos\frac{2\pi}{n}$.:

$$\zeta^2 + \zeta^{-2} = (\zeta + \zeta^{-1})^2 - 2 \tag{26}$$
$$\zeta^3 + \zeta^{-3} = (\zeta + \zeta^{-1})^3 - 3(\zeta + \zeta^{-1}) \tag{27}$$

Substituting these into the derived equation gives us:

$$(\zeta + \zeta^{-1})^3 + (\zeta + \zeta^{-1})^2 - 2(\zeta + \zeta^{-1}) - 1 = 0 \tag{28}$$
$$t^3 + t^2 - 2t - 1 = 0 \tag{29}$$

Using the Beloch's Fold, we are able to solve the cubic equation giving us a solution for $t = 2\cos\frac{2\pi}{n}$. Equation 29 is demonstrated via Lill's Method in Figure 15.

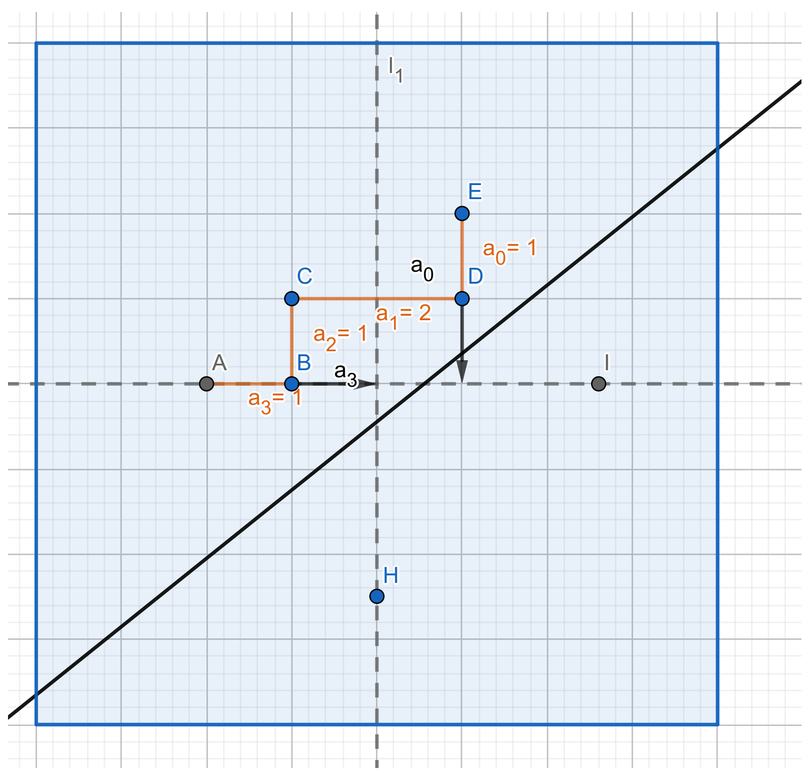


Figure 19 – Lill's Method: Constructing Heptagon

If we mirror and rotate the Beloch's fold 90º clockwise on Figure 19, then we get the following.

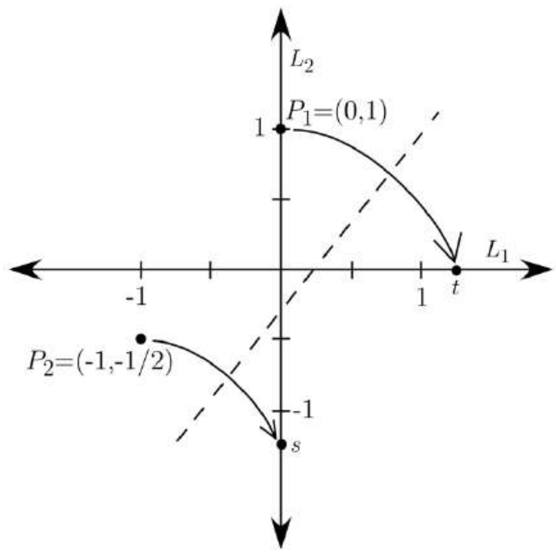


Figure 20 – Solving Heptagon (Hull, "Folding Regular Heptagons")

Robert Geretschläger has demonstrated how to fold a heptagon as follows:

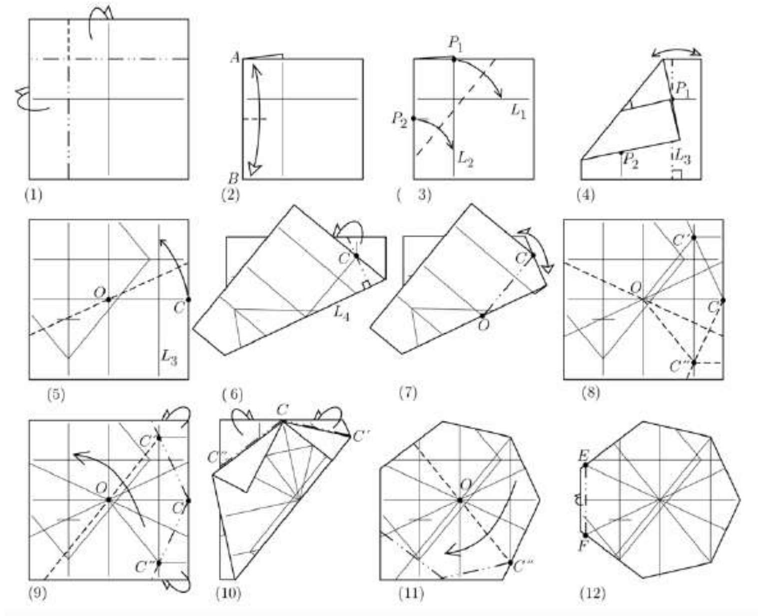


Figure 21 – Heptagon construction instructions (Hull, "Folding Regular Heptagons")

Geretschläger first created a four-by-four grid making its center the coordinates $(0, 0)$. He identified the points P_1 and P_2 in Figure 21 on the origami, and applied Axiom 6 as demonstrated in step 3. At step 4, the value $2\cos\frac{2\pi}{7}$ has been formed; the point P_1 on L_2 is $2\cos\frac{2\pi}{7}$ units away from the center. Step 5 produces the angle $\frac{2\pi}{7}$; Step 6-8 produces the first sector; and the rest of the sectors are produced as shown in steps 8-12, which are self-explanatory. (Similar process is demonstrated and explained in the subsection 9.1.1.)

9 Constructing a Regular Tridecagon 13-gon

Unlike the Heptagon, the equation for solving $2\cos\frac{2\pi}{13}$ is a non-factorable sextic polynomial, and seemingly cannot be solved through roots of cubic and quadratic functions.

$$1+\zeta+\zeta^2+\zeta^3+\zeta^4+\zeta^5+\zeta^6+\frac{1}{\zeta^6}+\frac{1}{\zeta^5}+\frac{1}{\zeta^4}+\frac{1}{\zeta^3}+\frac{1}{\zeta^2}+\frac{1}{\zeta}=0 \qquad (30)$$

$$t^6+t^5-5t^4-4t^3+6t^2+3t-1=0 \qquad (31)$$

However, using a similar method Gauss did in 1796, at the age of 19 (Andrews University), it is possible to find $2\cos\frac{2\pi}{13}$. Gauss has found the exact value of $2\cos\frac{2\pi}{17}$ in terms of quadratic solutions, thereby implying the constructability of a regular 17-gon through straight-edge & compass and Origami.

The book, "Tafeln complexer Primzahlen," provides definition on solving roots of unity via polynomials functions, similar to Gauss' method. It only suggests to use the equations 33 and 34.

$$\alpha_n = \zeta^n + \frac{1}{\zeta^n}, n \in [1, 2, 3, 4, 5, 6] \tag{32}$$

Define β_1 and β_2, so that $\beta_1 + \beta_2 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ which is equivalent to -1. According to the book, "Tafeln complexer Primzahlen,"

$$\beta_1 = \alpha_1 + \alpha_3 + \alpha_4 \tag{33}$$
$$\beta_2 = \alpha_2 + \alpha_5 + \alpha_6 \tag{34}$$

If β_1 and β_2 are a solution of a quadratic where its coefficients are constructable, then β_1 and β_2 are constructable.

$$(x - \beta_1)(x - \beta_2) = 0$$

$$x^2 - (\beta_1 + \beta_2)x + \beta_1\beta_2 = 0 \tag{35}$$

$\beta_1 + \beta_2 = -1$ as they are the sum of all roots except 1.

Finding $\beta_1\beta_2$ is a little tricky, as:

$$\beta_1\beta_2 = (\alpha_1 + \alpha_3 + \alpha_4)(\alpha_2 + \alpha_5 + \alpha_6) \tag{36}$$
$$\beta_1\beta_2 = \left(\zeta + \zeta^3 + \zeta^4 + \frac{1}{\zeta^4} + \frac{1}{\zeta^3} + \frac{1}{\zeta}\right)\left(\zeta^2 + \zeta^5 + \zeta^6 + \frac{1}{\zeta^6} + \frac{1}{\zeta^5} + \frac{1}{\zeta^2}\right)$$

13th Roots of Unity

$$\zeta^7 = \zeta^{-6}$$
$$\zeta^8 = \zeta^{-5}$$
$$\zeta^9 = \zeta^{-4}$$
$$\zeta^{10} = \zeta^{-3}$$
$$\zeta^{11} = \zeta^{-2}$$
$$\zeta^{12} = \zeta^{-1}$$
$$\zeta^{n-13} = \zeta^n$$

$$\beta_1\beta_2 = \zeta^3 + \zeta^6 + \frac{1}{\zeta^6} + \frac{1}{\zeta^5} + \frac{1}{\zeta^4} + \frac{1}{\zeta}$$
$$+\zeta^5 + \frac{1}{\zeta^5} + \frac{1}{\zeta^4} + \frac{1}{\zeta^3} + \frac{1}{\zeta^2} + \zeta$$
$$+\zeta^6 + \frac{1}{\zeta^4} + \frac{1}{\zeta^3} + \frac{1}{\zeta^2} + \frac{1}{\zeta} + \zeta^2$$
$$+\frac{1}{\zeta^2} + \frac{1}{\zeta} + \zeta^2 + \zeta^3 + \zeta^4 + \frac{1}{\zeta^6}$$
$$+\frac{1}{\zeta} + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \frac{1}{\zeta^5}$$
$$+\zeta + \zeta^4 + \zeta^5 + \zeta^6 + \frac{1}{\zeta^6} + \frac{1}{\zeta^3}$$

Different conjugates are coloured differently.

$$\zeta + \frac{1}{\zeta}$$
$$\zeta^2 + \frac{1}{\zeta^2}$$
$$\zeta^3 + \frac{1}{\zeta^3}$$
$$\zeta^4 + \frac{1}{\zeta^4}$$
$$\zeta^5 + \frac{1}{\zeta^5}$$
$$\zeta^6 + \frac{1}{\zeta^6}$$

By grouping each conjugate pairs, the equation is reduced to:

$$\beta_1\beta_2 = 3(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) \tag{37}$$
$$\beta_1\beta_2 = 3(-1) = -3 \tag{38}$$

Therefore, $x^2 - (\beta_1 + \beta_2)x + \beta_1\beta_2 = 0$ is the same as

$$x^2 - (-1)x + (-3) = 0$$

$$x^2 + x - 3 = 0 \tag{39}$$

$$x = \frac{3 \pm \sqrt{1 - 4(-3)(1)}}{2}$$

$$x = \frac{-1 \pm \sqrt{13}}{2} = \beta_1, \beta_2 \tag{40}$$

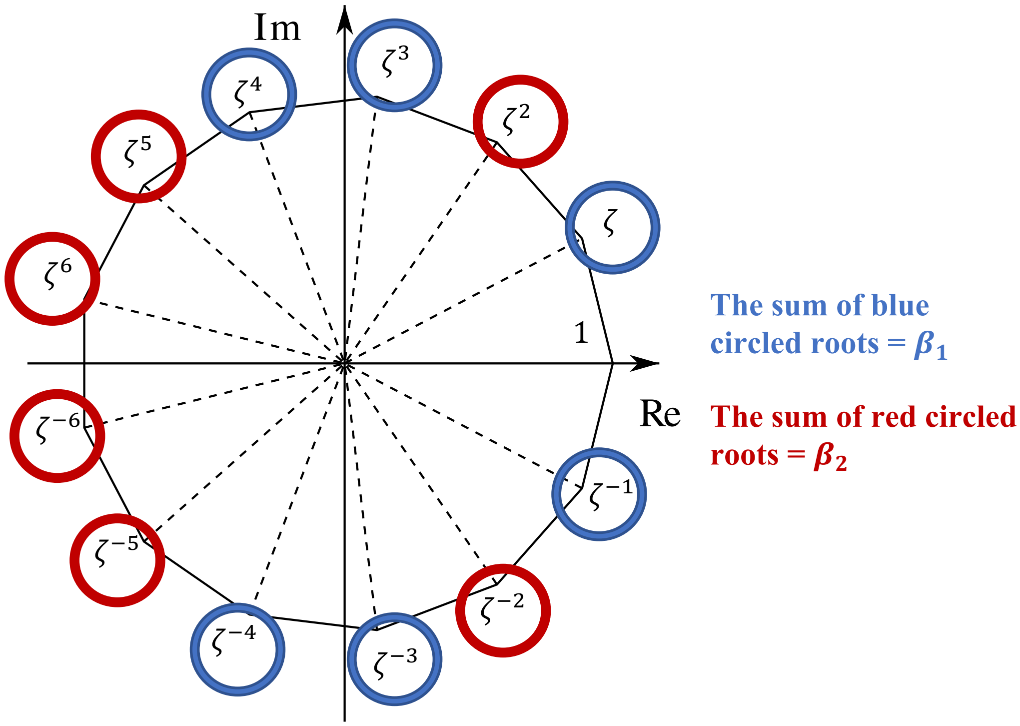


Figure 22 – Comparing $\boldsymbol{\beta_1}$ *and* $\boldsymbol{\beta_1}$

Figure 22 visually shows that components of β_1 is leaning towards the positive side more than those of β_1.

$$\therefore \beta_1 > \beta_1$$

Therefore,

$$\beta_1 = \frac{-1+\sqrt{13}}{2} \tag{41}$$

$$\beta_2 = \frac{-1-\sqrt{13}}{2} \tag{42}$$

Similar to the idea with β_1 and β_2, if α_1, α_3, and α_4 are the solutions to a trinomial with the constructable coefficients, then $\alpha_1 = 2\cos\frac{2\pi}{13}$ is constructable.

$$(x-\alpha_1)(x-\alpha_3)(x-\alpha_4) = 0$$

$$x^3 - (\alpha_1 + \alpha_3 + \alpha_4)x^2 + (\alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_3\alpha_4)x - \alpha_1\alpha_3\alpha_4 \tag{43}$$

Where,

$$\alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_3\alpha_4 = \left(\zeta + \frac{1}{\zeta}\right)\left(\zeta^3 + \frac{1}{\zeta^3}\right) + \left(\zeta + \frac{1}{\zeta}\right)\left(\zeta^4 + \frac{1}{\zeta^4}\right) + \left(\zeta^3 + \frac{1}{\zeta^3}\right)\left(\zeta^4 + \frac{1}{\zeta^4}\right)$$

$$= \zeta^6 + \frac{1}{\zeta^6} + \zeta^5 + \frac{1}{\zeta^5} + \zeta^4 + \frac{1}{\zeta^4} + \zeta^3 + \frac{1}{\zeta^3} + \zeta^2 + \frac{1}{\zeta^2} + \zeta + \frac{1}{\zeta}$$

$$= \alpha_6 + \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1$$

$$= -1$$

Thus,

$$\alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_3\alpha_4 = -1 \tag{44}$$

On the other hand,

$$\alpha_1\alpha_3\alpha_4 = \left(\zeta + \frac{1}{\zeta}\right)\left(\zeta^3 + \frac{1}{\zeta^3}\right)\left(\zeta^4 + \frac{1}{\zeta^4}\right) \tag{45}$$

$$= \zeta^5 + \frac{1}{\zeta^5} + \zeta^6 + \frac{1}{\zeta^6} + \zeta^2 + \frac{1}{\zeta^2} + 2$$

$$\alpha_1\alpha_3\alpha_4 = \alpha_6 + \alpha_5 + \alpha_2 + 2 \tag{46}$$

And since, $\beta_2 = \alpha_2 + \alpha_5 + \alpha_6$

$$\alpha_1\alpha_3\alpha_4 = \beta_2 + 2 \tag{47}$$

And lastly because $\beta_1 = \alpha_1 + \alpha_3 + \alpha_4$, the trinomial with the solution can be represented by

$$x^3 - (\beta_1)x^2 + (-1)x - (\beta_2 + 2) \tag{48}$$

Which is a constructable trinomial.

In other words, we must solve the equation 49 in origami.

$$x^3 - \frac{\sqrt{13} - 1}{2}x^2 - x + \frac{\sqrt{13} - 3}{2} = 0 \tag{49}$$

Beloch's Fold must be applied by visualizing the trinomial through Lill's Method like so in Figure 23.

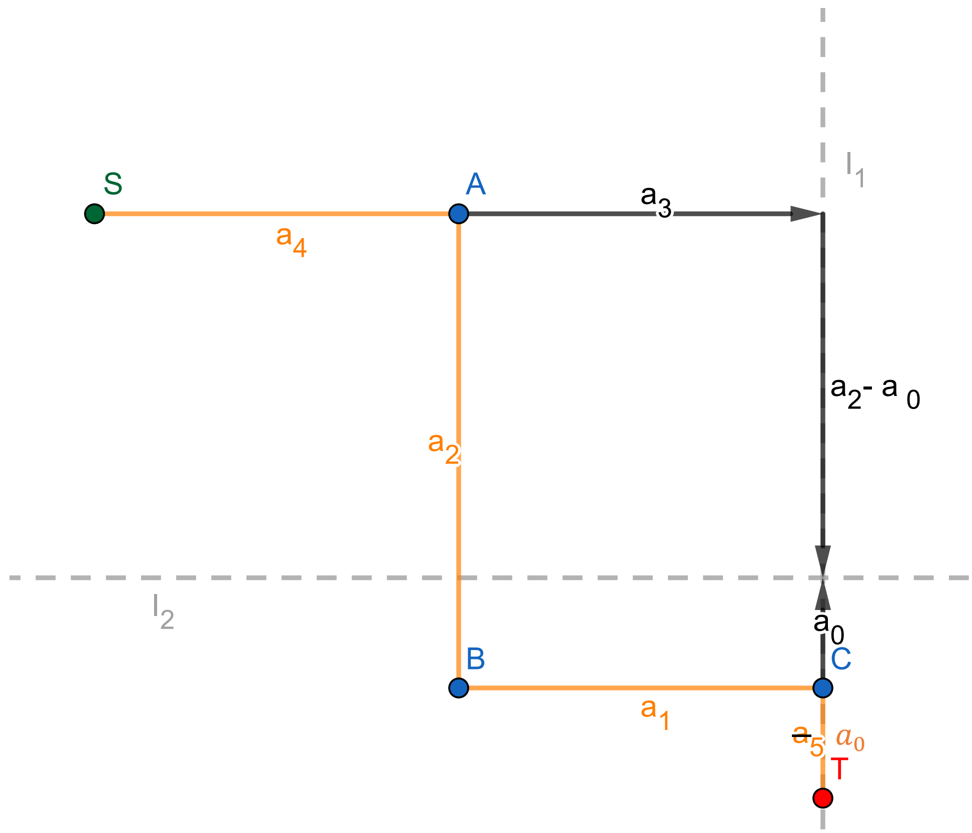


Figure 23 – Tridecagon solving trinomial represented via Lill's Method

$$a_3 = 1$$
$$a_2 = \frac{\sqrt{13}-1}{2}$$
$$a_1 = 1$$
$$a_0 = \frac{\sqrt{13}-3}{2}$$

$$a_2 - a_0 = \frac{\sqrt{13}-1}{2} - \frac{\sqrt{13}-3}{2}$$
$$a_2 - a_0 = 1$$

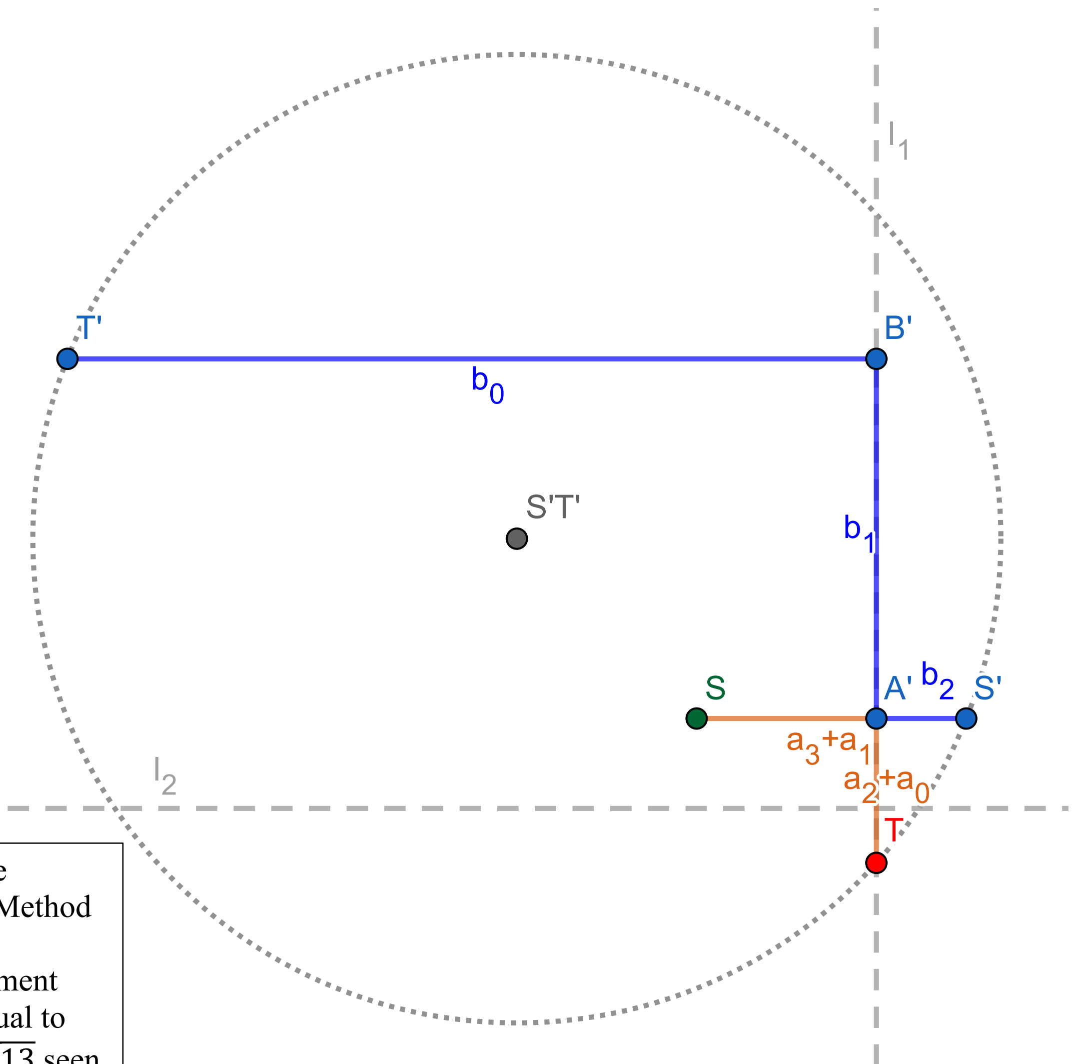
$$a_3 + a_3 = 2$$

$$a_2 + a_0 = \frac{\sqrt{13}-1}{2} - \frac{\sqrt{13}-3}{2}$$
$$a_2 + a_0 = 2 + \sqrt{13}$$

To construct $2+\sqrt{13}$, Lill's method can be used again for its minimal polynomial

$$x^2 - 4x - 9 = 0$$

And placed in a way so that it looks like in Figure 24. The Lill's method visualization of the equation above is seen in Figure 14 – Lill's Method: Quadratic roots from negative coefficient.



The blue lines are indicating Lill's Method of equation 15 to construct the segment A'T, which is equal to $a_2 + a_0 = 2 + \sqrt{13}$ seen in Figure 23.

Figure 24 – Lill's method used to solve a trinomial coefficient

The point S and T must lie on top of line 1 and 2 respectively.

The following are the three solutions:

(Note that point T', B', is removed as it is unnecessary, and not used when constructing the polygon.)

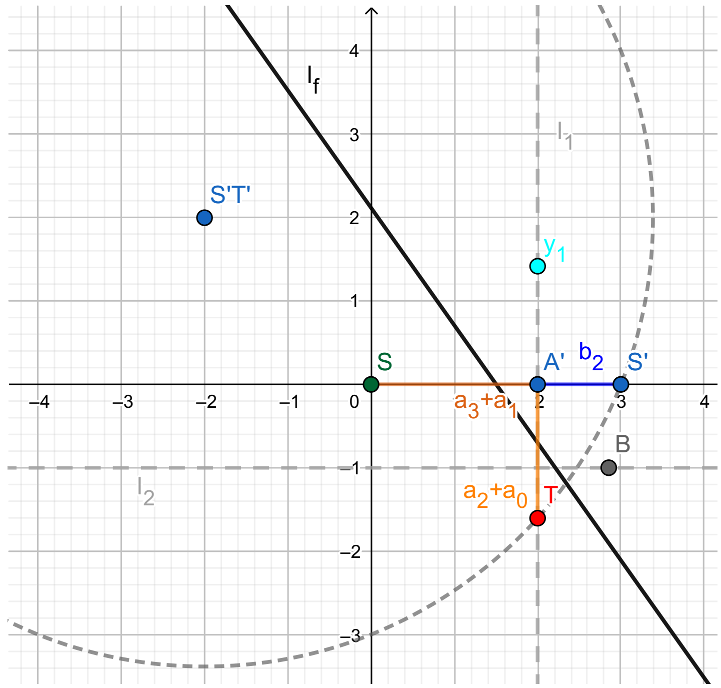


Figure 25 – Solution to the trinomial: α_4

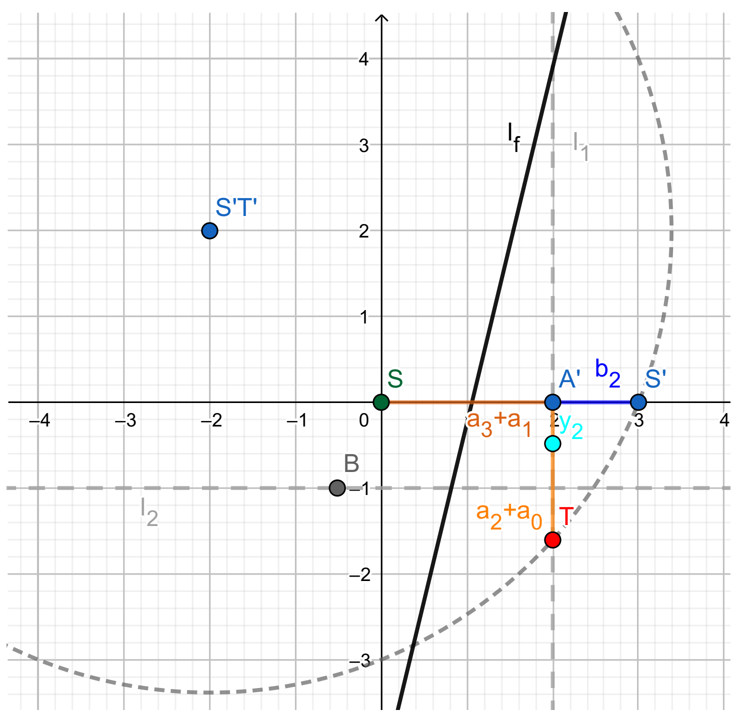


Figure 26 – Solution to the trinomial: α_3

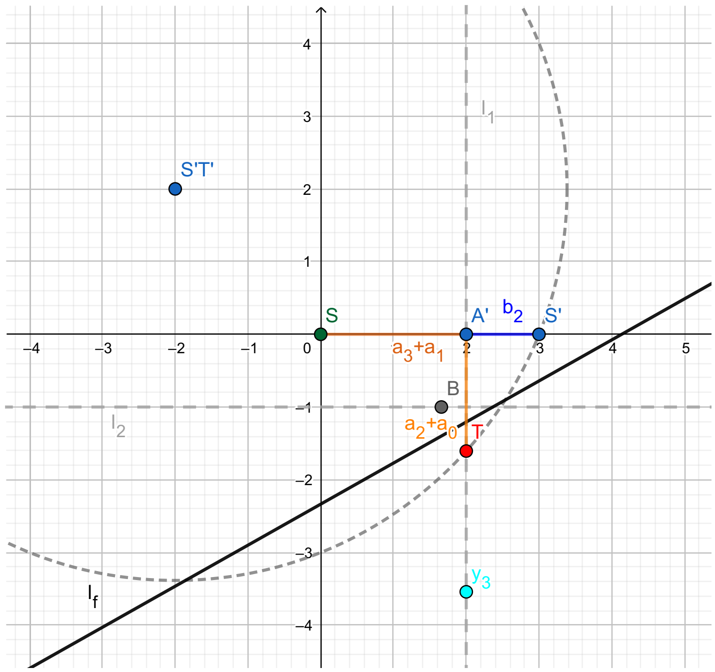


Figure 27 – Solution to the trinomial: α_1

For convenience, Figure 27 will be used on the origami as the negative of the slope of line S to y_3 is equivalent to $\alpha_1 = 2\cos\frac{2\pi}{13}$.

What should the dimensions on the square origami be? The following explains the reason to use an 8x8 grid.

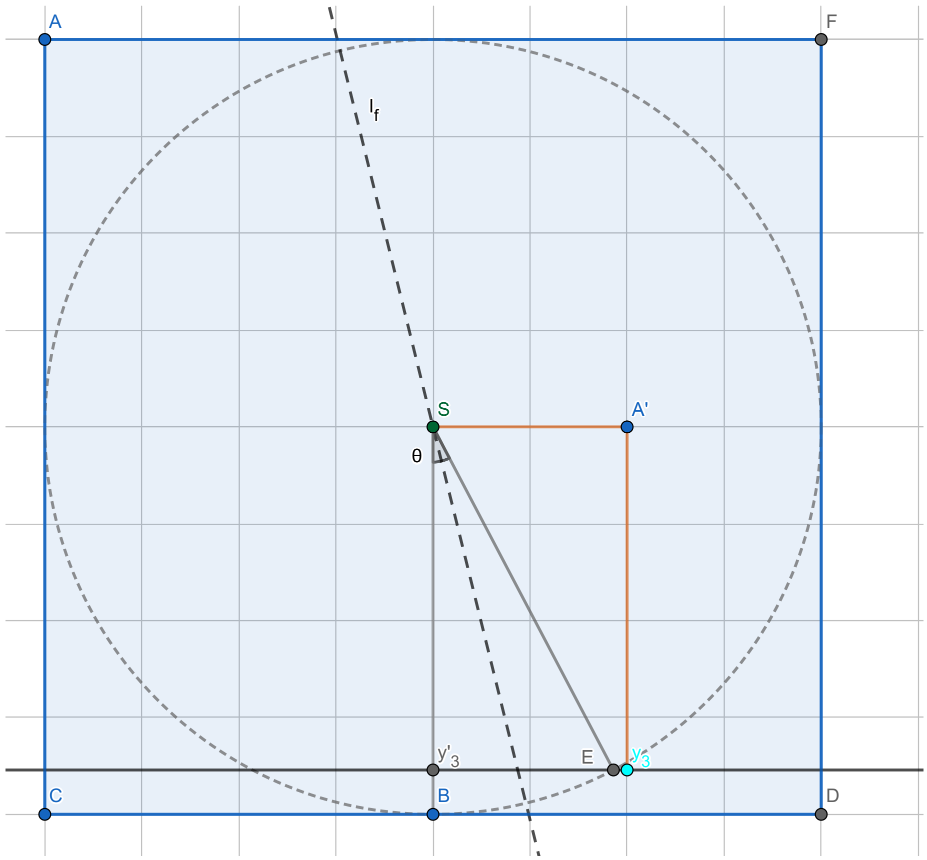


Figure 28 – 8x8 gird paper blueprint

$$x = -\frac{Distance\ from\ A'\ to\ y_3}{a_3}$$

$$2\cos\frac{2\pi}{13} = -\frac{-|\overline{A'y_3}|}{2}$$

$$|\overline{A'y_3}| = 4\cos\frac{2\pi}{13} \tag{50}$$

In Figure 28, the bottom line is folded on point y_3 so that it is perpendicular to line DF (using Axiom 4). If the paper is 8x8 with point S in the center, the segment SB must be 4 units in length. We may utilize Axiom 5 to fold point B on the bottom line such that the folding line, l_f, crosses point S. This means:

$$|\overline{Sy_3'}| = |\overline{A'y_3}| = 4\cos\frac{2\pi}{13} \tag{51}$$

$$|\overline{SE}| = |\overline{SB}| = 4 \tag{52}$$

Therefore, the angle $\theta = \angle y_3' SE$ is defined as:

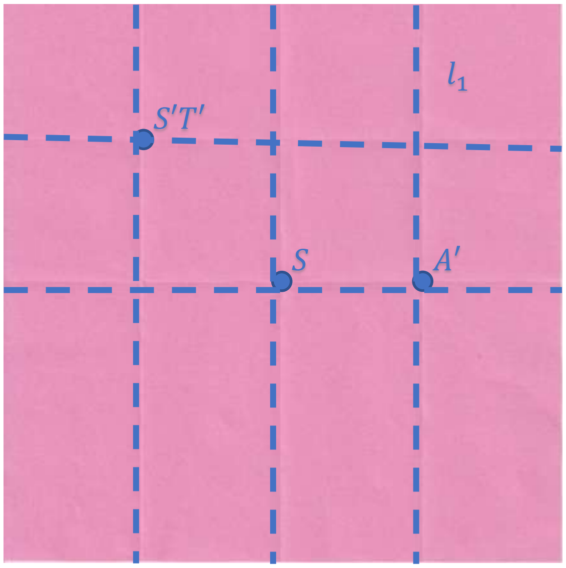
$$\cos\theta = \frac{\left|\overline{Sy_3'}\right|}{|\overline{SE}|} \tag{53}$$

$$\cos\theta = \frac{4\cos\frac{2\pi}{13}}{4}$$

$$\theta = \frac{2\pi}{13} \tag{54}$$

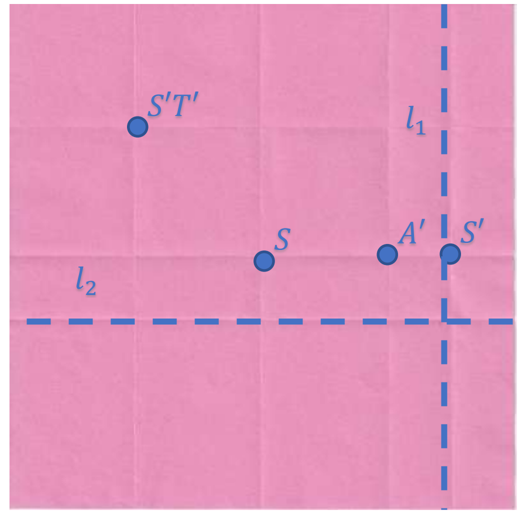
The 8x8 grid must be used so that $|\overline{SE}| = |\overline{SB}| = 4$, and thus $\cos\theta$ can be cleanly cancelled out. A similar idea is followed when constructing a heptagon as well.

9.1.1. Instructions for the tridecagon



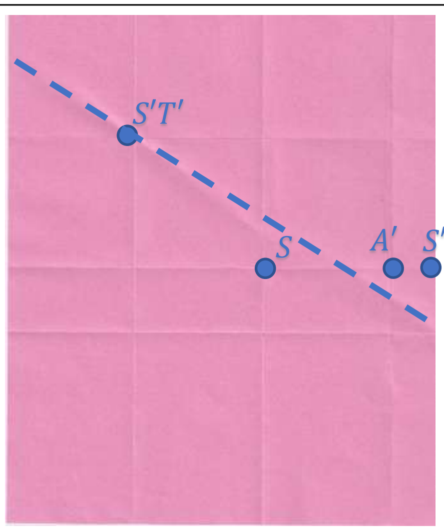
Step 1

The above creases are created to locate the points $S'T'$, S, and A'. l_1 is also determined.



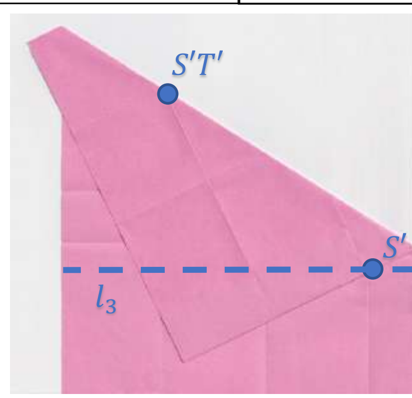
Step 2

An additional bisection on the right side is folded to determine point S'. The fold is kept. l_2 is folded as well.

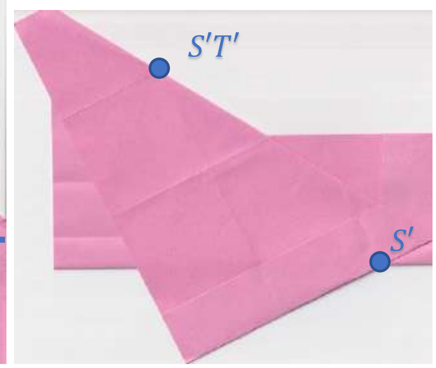


Step 3

Now that the right side is folded, it is clearer to see where point S' is when doing Axiom 5. Point S' is folded onto l_1 so that the crease line passes through point $S'T'$.

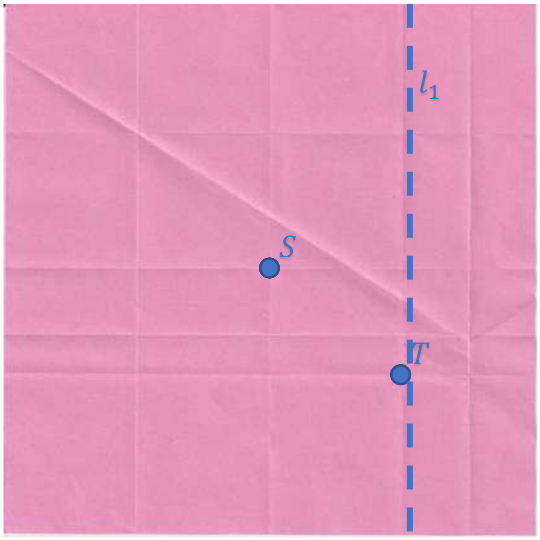


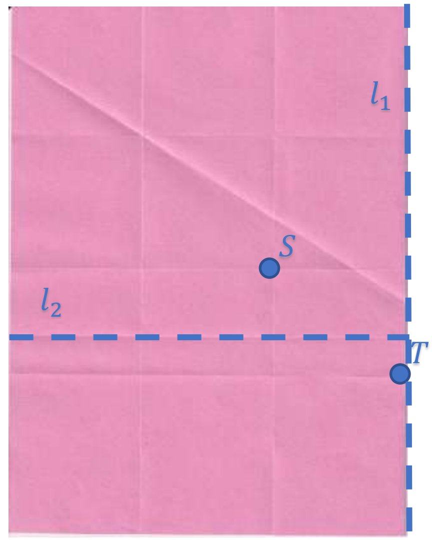
Step 4



Step 5

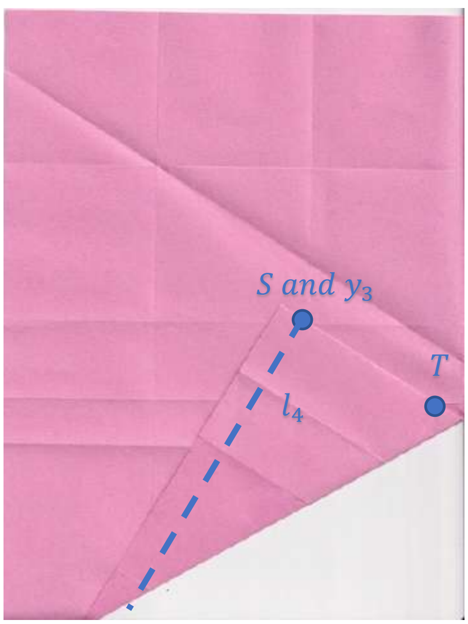
To place a mark on where point S' is on l_1, the line, l_3, is folded so that it intersects the point of interest, T.

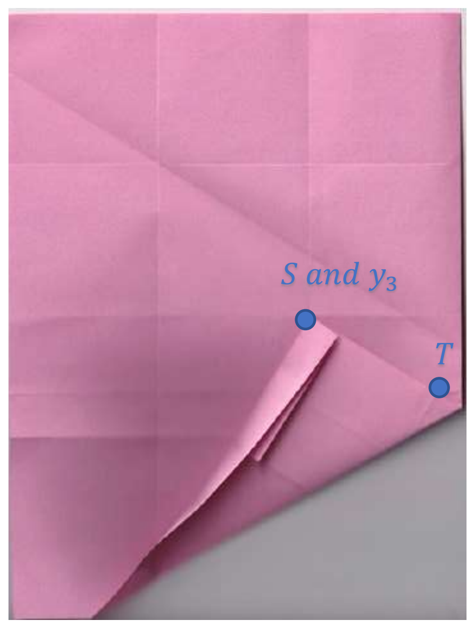




Step 6 Step 7

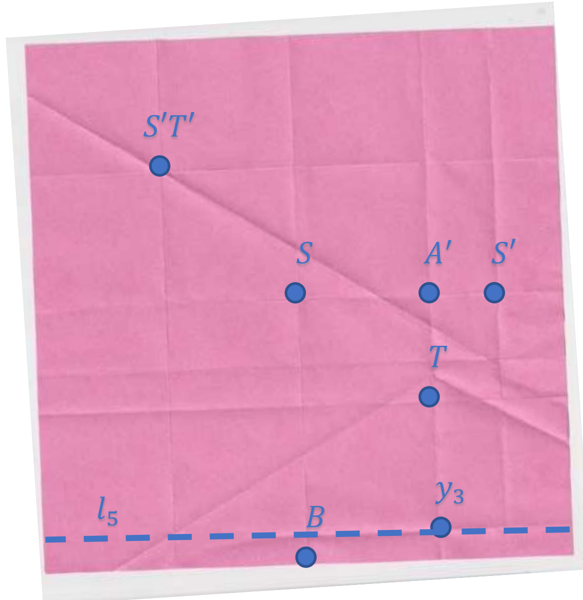
Crease line significance overview. We must now fold point S on l_1 and point T on l_2. l_1 can be folded as it is clearer to see point T.





Step 8 Step 9

The point S on l_1 represents point y_3. The point is marked as a line is perpendicularly folded to l_1 whilst hitting point y_3. “and” implies the overlapping of points.

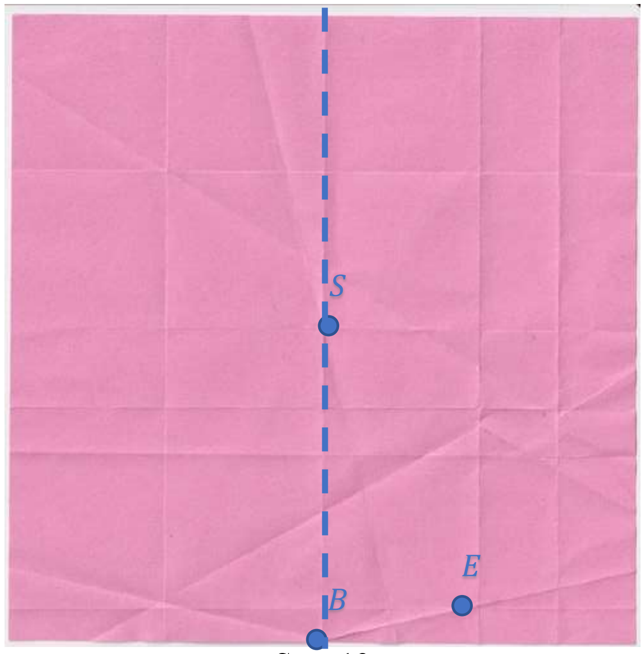


Step 10

The bottom line may be refolded to clearly visualize the y_3 intersection. As previously explained in Figure 28, point B is folded on to the bottom line whilst its crease line is passing through point S.

Step 11 Step 12

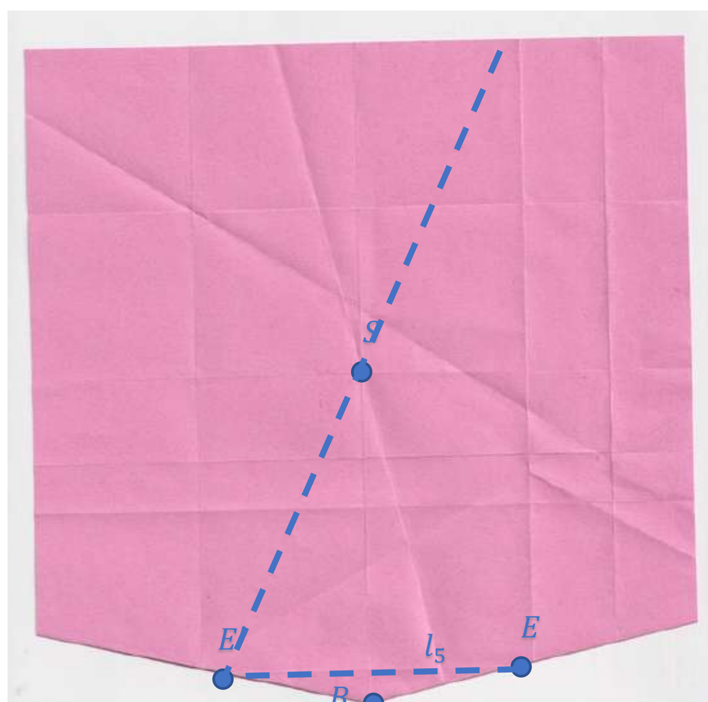
Point B on the bottom line, l_5, is point E. Axiom 4 is used to perpendicularly fold across point E to mark it.



Step 13

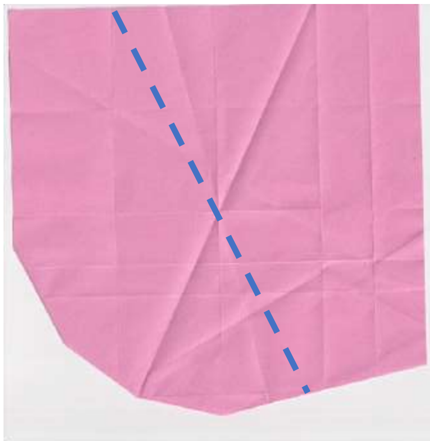
Once everything is unfolded, we can see the tridecagon sector with points B, S, and E. Now, the next few steps are to create the sides, which is a very similar process Geretschläger took when constructing the heptagon in Figure 21.

Step 14



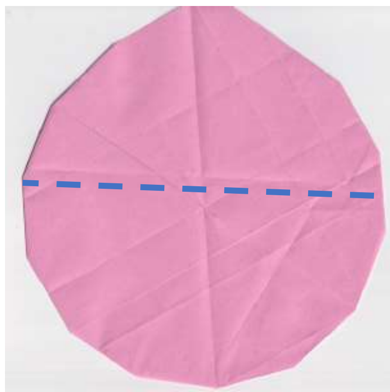
Step 15

The side is reciprocated by folding along the middle axis. The intersection of the folded edge and l_5 must be the mirrored point E, E'.



Step 16

Step 17



Step 18

Steps 16 to 17 presents a similar process as steps 13 and 14. The origami is folded along the dotted lines and the sides folded so it matches the other side. This is repeated until all sides are folded.

Figure 29 – Step 19 of Tridecagon Construction

Figure 29 – Step 19 of Tridecagon Construction and Figure 30 – Ideal Tridecagon Crease Patternhows a complete tridecagon. Although very disproportionate, if carefully done with precision and accuracy, an exact regular tridecagon should be formed.

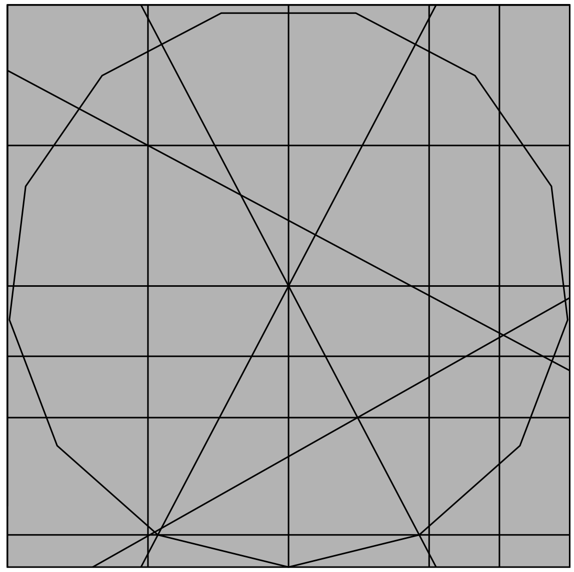


Figure 30 – Ideal Tridecagon Crease Pattern

Conclusion

There are, in terms of theoretical constructions, infinitely many possible regular polygons that can be constructed by following the rules of Origami Axioms. We first investigated the possible values that can be made just from Origami Axioms. Axiom 1-5 could simulate Euclidean constructions and solve arbitrary quadratic equations, whereas Axiom 6, as elaborated by Eduard Lill and Margherita Piazzola Beloch, has the unique ability to solve cubic equations.

The Axiom equations were then used to prove the constructability of any rational fraction. This showed that the problem when creating polygons was the construction of irrational numbers. In order to find what values we must construct when forming a regular polygon, the roots of unity was used to determine an equation to solve. Utilizing the roots of unity and its properties, one must conclude that any nth sided regular polygon can be created if $2\cos\frac{2\pi}{n}$ is constructable. Furthermore, because of origami's ability to bisect and trisect arbitrary angles, one can conclude that:

$$n = 3^p 2^q k$$

$$p, q \in \mathbb{Z}^+$$

If k-gon is constructable, then n-gon is also constructable.

To create a new strand of constructable n-gon, k must be a number that is not divisible by 2^q nor 3^p, where $p, q \in \mathbb{Z}^+$. Using Gauss' method, if there is an equation on the kth roots of unity in terms of only cubic and quadratic equation that solves for $2\cos\frac{2\pi}{k}$ (the sum of conjugate pairs) then k-gon must be constructable.

James Pierpoint, an American mathematician, concluded through the similar methods presented in this paper, that the k-value is constructable if and only when k is a distinct Pierpont prime (Lee):

$$k = 3^r 2^s + 1, \quad r, s \in \mathbb{Z}^+$$

The Extended Essay supported this argument as 7-gon and 13-gon constructions are demonstrated where they are both Pierpont primes.

Nonetheless, higher the sides of a regular polygon there is, the less accurate we can fold in practice. The uncertainties add up to a disproportionate outcome as seen in Figure 29 – Step 19 of Tridecagon Construction. Infinitely many regular polygons can be folded (with exclusions) on an origami, however, one must have extreme precision and accuracy in order to fold those of higher sided regular polygons.

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