Graphs

CPE348

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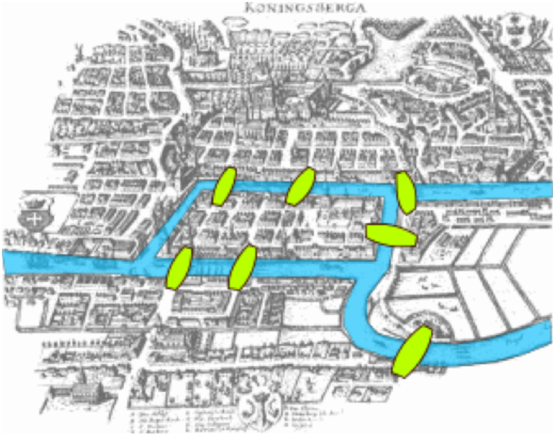
Graph Theory

- Father of graph theory: Leonhard Euler

- Swiss mathematician
- *Seven Bridges of Königsberg* 1736.

Seven Bridges of Königsberg



Is there a walk that traverses each bridge exactly once?

Seven Bridges of Königsberg

Theorem

There is a walk through a graph that traverses each edge exactly once if and only if the graph is connected and there are exactly two or zero vertices of odd degree.

What is a graph?

- Vertices and edges.
- Nodes and links.
- People and relationships.

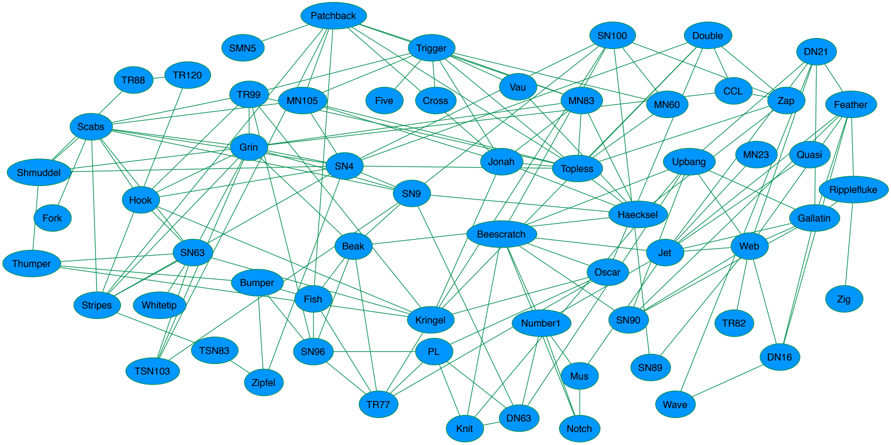
10.1 Graphs

Graphs

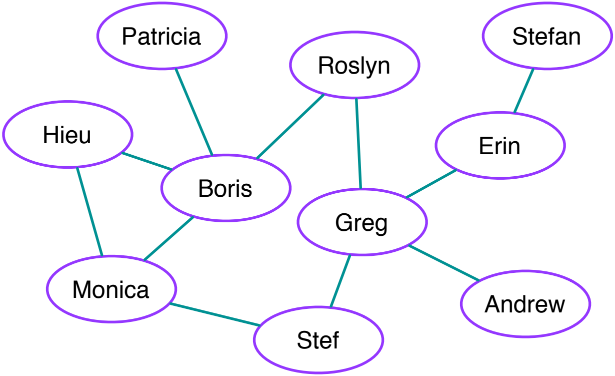
Definition

A **graph**, $G = (V, E)$, consists of a set of vertices, V, and a set of edges, E. Each edge has two vertices associated with it, call its **endpoints**. An edge is said to **connect** its endpoints.

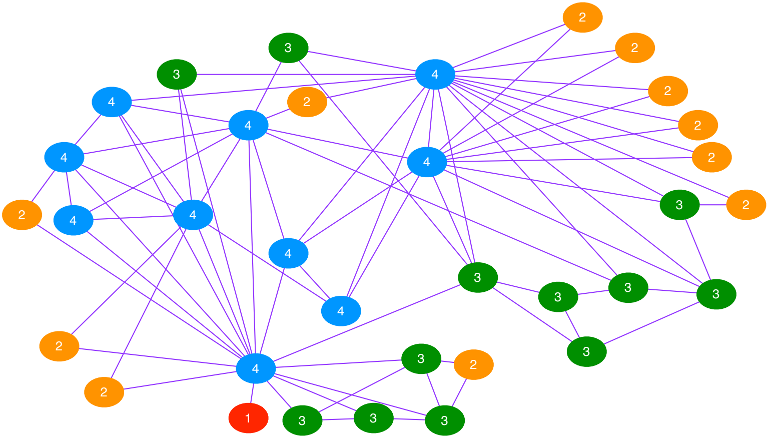
What is a graph? - Dolphin network



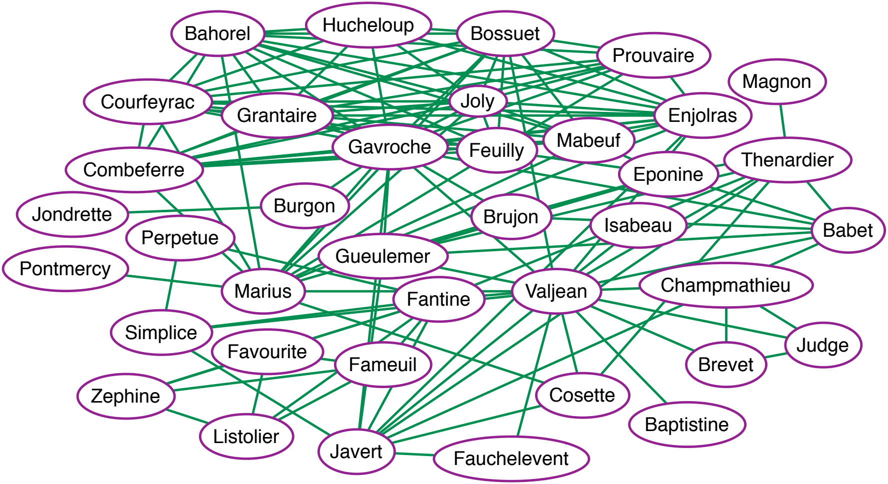
What is a graph? - Friends network



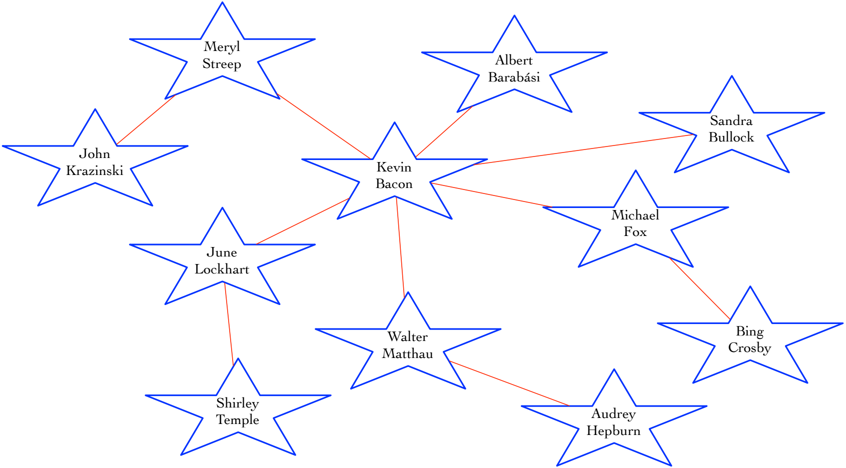
What is a graph? - Karate network



What is a graph? - Les Misérables network



What is a graph? - Actors network



Types of Graphs

- Collaboration networks
- Who-talks-to-whom graphs
- Information linkage graphs
- Technological networks
- Biological networks

Directed Graphs

What if we want to imply one directional relationships?

- Family trees
- Sewage networks
- Food webs
- Webpage network
- Epidemiological networks...

Directed Graphs

We can place a direction on the edges of our graphs:

Definition

A **directed graph**, $G = (V, E)$, consists of a set of vertices, V, and a set of *directed* edges, E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

Simple Graphs

Definition

A **loop** in a graph is an edge that connects a vertex with itself.

Definition

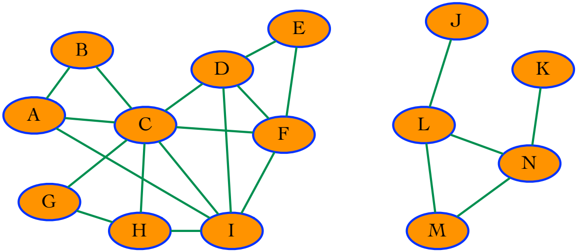
A **multigraph** is a graph that allows for multiple edges connecting the same vertices.

Definition

A graph with no multiple edges and no loops is called a **simple graph**.

10.2 Graph Terminology and Special Types of Graphs

Connectivity

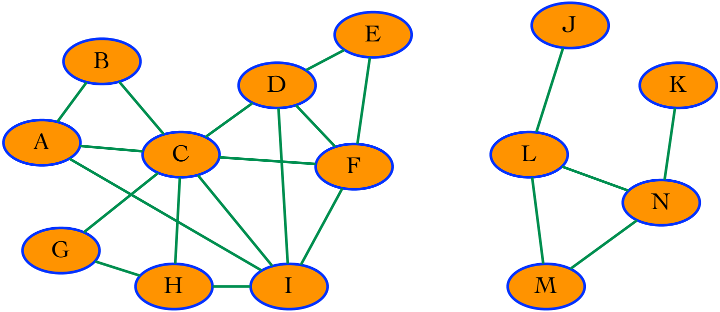


Definition

A vertex A and a vertex B are **neighbors** if there is an edge, (A, B), between A and B. We say that A and B are **adjacent** if they are neighbors. We say that edge (A, B) **connects** A and B.

D is neighbors with E, F, and C, but not B.

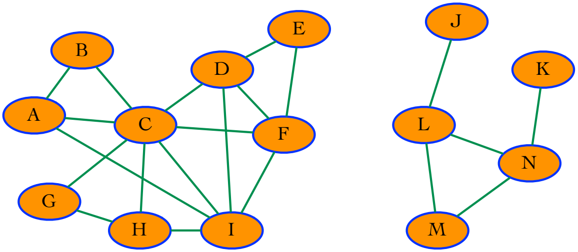
Neighborhoods



Definition

The set of all neighbors of a vertex A is called the **neighborhood** of A, denoted $N(A)$.
If $U \subseteq V$ then the neighborhood of U, $N(U)$, is the set of all vertices that are adjacent to at least one vertex in U.

Degree



Definition

The **degree** of a vertex is the number of edges adjacent to it (or the number of neighbors).

C has degree 7. J has degree 1.

Degree

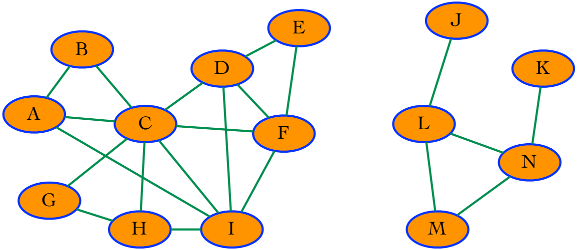
Theorem

Let $G = (V, E)$ be an undirected graph with m edges. Then $2m = \sum_{v \in V} deg(v)$.
Note: this even applies if multiple edges or self-loops are present.

Corollary

An undirected graph has an even number of vertices of odd degree.

Degree Distribution

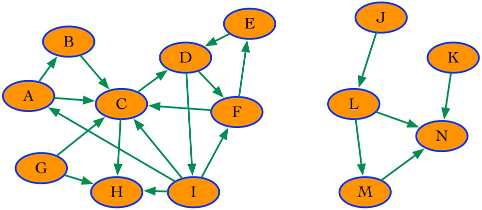


Definition

The **degree distribution** of a graph is the number of vertices of each degree.

$\{0, 2, 4, 4, 2, 1, 0, 1\}$ or $\{0, 1/7, 2/7, 2/7, 1/7, 1/14, 0, 1/14\}$

Directed Degree



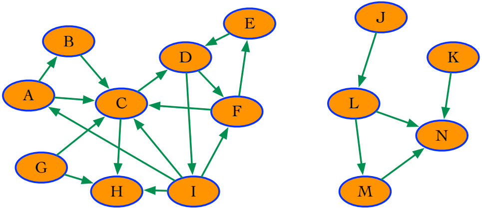
Here $(F, C) \in E$ but $(C, F) \notin E$.

Definition

The **indegree** of a vertex, v, $deg^{-}(v)$, in a directed graph is the number of edges directed into v.
The **outdegree** of a vertex, v, $deg^{+}(v)$ in a directed graph is the number of edges directed out of v.

The indegree of I is 1. The outdegree of I is 4.

Directed Degree



Theorem

Let $G = (V, E)$ be a directed graph. Then
$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$

Special Types of Graphs

- Complete graphs
- Cycle graphs
- Wheel graphs
- Star graphs

Bipartite Graphs

Definition

A simple graph G is called **bipartite** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.

Theorem

A simple graph is bipartite if and only if it is possbile to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Matchings

Definition

A **matching**, M, in a simple graph, $G = (V, E)$, is a subset of E such that no two edges of M are incident with the same vertex.

- A **maximal matching** is a matching with the property that if any edge was added to the matching, it would no longer be a matching.
- A **maximum matching** is a matching with the largest number of edges.
- A matching M in a bipartite graph with bipartition (V_1, V_2) is a **complete matching** from V_1 to V_2 if every vertex in V_1 is the endpoint of an edge in the matching ($|M| = |V_1|$).

Are maximal and maximum matchings equivalent?

Matchings

Theorem

The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

Subgraphs

Definition

A **subgraph** of a graph $G = (V, E)$ is a graph $H = (W.F)$, where $W \subseteq V$ and $F \subseteq E$.
A subgraph H of G is a **proper subgraph** if $H \neq G$.

Definition

Let $G = (V, E)$ be a simple graph. The **subgraph induced** by a subset W of V is the graph (W, F), where the edge set of F contains an edge in E if and only if both endpoints are in W.

Definition

The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

10.3 Representing Graphs and Graph Isomorphism

Basic Graph Representations

There are two basic ways to represent a graph, $G = (V, E)$, $V = \{v_1, v_2, \ldots v_n\}$:

1. An **adjacency matrix** is an $n \times n$ array where the (i, j) entry is:
   - $a_{ij} = 1$ if there is an edge from v_i to v_j.
   - $a_{ij} = 0$ otherwise.
2. An **adjacency list** is a set of n linked lists, one for each vertex.
   - The linked list for vertex v holds the names of all vertices, u, such that there is an edge from v to u.

- What is the size of each data structure?
- How long does it take to find a particular edge for each data structure?

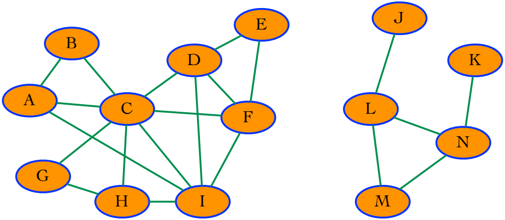
Isomorphism

Definition

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2 for all $a, b \in V_1$.
Such a function f is called an **isomorphism**.

10.4 Connectivity

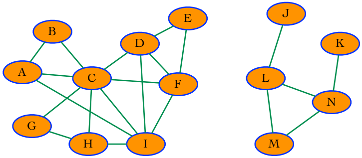
Paths



Definition

A **path** between two vertices is a sequence of vertices with the property that each consecutive pair in the sequence is connected by an edge.
The **length** of a path is the number of edges in the path.
A path or cycle is **simple** if it doesn't contain the same edge more than once.

Connectivity

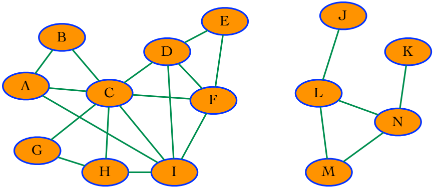


Definition

We say that a graph is **connected** if for each pair of vertices, there is a path between them.

The above graph is not connected.

Components

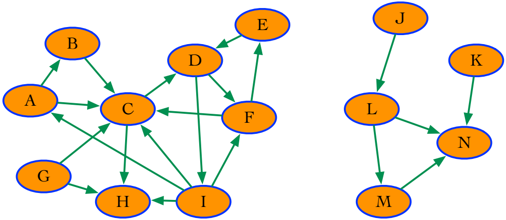


Definition

A **connected component** (or just **component**) of a graph is a subset of vertices such that every vertex in the subset has a path to every other vertex in the subset and the subset is not a part of some larger subset with the property that there is a path between every pair of vertices.

There are two components in the graph $A, B, C, D, E, F, G, H, I$ and J, K, L, M, N. Note that L, M, N is not a component.

Directed Paths

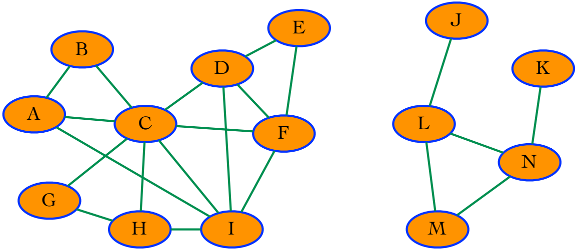


Definition

A **path** in a directed graph from a vertex x to a vertex y is a sequence of vertices with the property that each consecutive pair in the sequence is connected with an edge and all edges are directed in the same direction (out of x).

There is a path from G to E (G, C, D, F, E). There is not a path from H to D.

Cycles

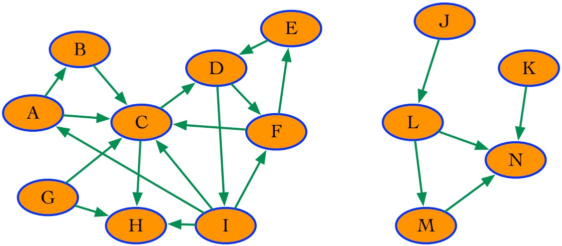


Definition

A **cycle** (in an undirected graph) is a path with at least 3 edges in which the first and last vertices are the same, but otherwise all vertices are distinct.

L, M, N is a cycle, so is A, C, F, I, and many more...

Directed Cycles

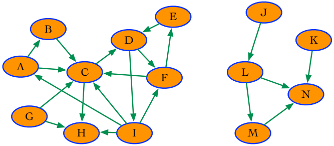


Definition

A **cycle** in a directed graph is a (directed) path with at least 2 edges in which the first and last vertices are the same, but otherwise all vertices are distinct.

A, B, C, D, I is a cycle. C, D, E, F, I is not a cycle.

Directed Connectivity

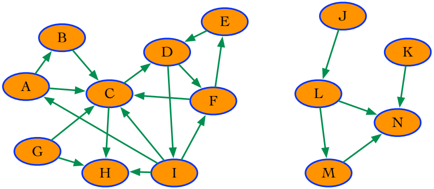


Definition

Two vertices, x and y, are **connected** in a directed graph if there is a path from x to y and y to x.

A and D are connected. L and M are not.

Directed Connectivity



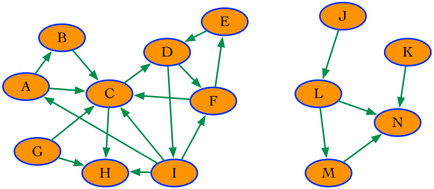
Definition

A directed graph, $G = (V, E)$ is **strongly connected** if for all pairs of vertices $u, v \in V$, u and v are connected.

Definition

The **strongly connected components** of a directed graph partition the graph into strongly connected subgraphs.

DAGs



Definition

A **directed acyclic graph** or **DAG** is a directed graph with no cycles.

Theorem

Every directed graph is a DAG of its strongly connected components.

10.7 Planar Graphs

Planar Graphs

Definition

A graph is called **planar** if it can be drawn in the plane without any edges crossing. Such a drawing (embedding) is called a planar representation (embedding) of the graph.

Examples:

- Is K_3 planar?
- Is K_4 planar?
- Is K_5 planar?
  - No - We will be able to prove this in a bit.

Planar Graphs

Theorem (Euler's Formula)

Let G be a planar simple connected graph with e edges and v vertices. Let r be the number of regions in a planar representation of G.
Then $r = e - v + 2$.

Example:

- Suppose that there is a planar simple connected graph with 16 vertices, each with degree 5.
  - Into how many regions does a planar embedding of this graph split the plane?

Corollaries for Euler's Formula

Corollary

Let G be a planar simple connected graph with e edges and $v \geq 3$ vertices.
Then $e \leq 3v - 6$.

Show that K_5 is nonplanar.

Corollary

Let G be a planar simple connected graph.
Then G has a vertex of degree less than or equal to 5.

Corollary

Let G be a planar simple connected graph with e edges and $v \geq 3$ vertices and no cycles of length 3.
Then $e \leq 2v - 4$.

Kuratowski's Theorem

If a graph is planar, so will any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$.
Such an operation is called an **elementary subdivision**.

Definition

Two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called **homeomorphic** if they can be obtained from the same graph by a sequence of elementary subdivisions.

Theorem

A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

10.8 Graph Coloring

Maps

Consider a simple map of any region in the world. Mapmakers prefer to color adjacent regions different colors for contrast. How can we relate this to graph theory?

- A map in the plane can be represented by a graph.
- Each region in the map can be represented by a vertex.
- Edges connect two vertices if the regions represented by these vertices have a common border.
- The resulting graph is called the **dual** graph of the map.

Graph Colorings

Definition

A **coloring** of a simple graph is an assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Definition

The **chromatic number** of a graph is the least number of colors needed for a coloring of this graph.
The chromatic number of a graph, G, is denoted by $\chi(G)$

Theorem

The chromatic number of a planar graph is less than or equal to 4.

Graph Colorings - Application

How can final exams be scheduled (in the fewest time slots) such that no student has two exams at the same time?

- Model this problem with a graph where vertices represent classes and two vertices are connected if there is a student common to the two classes they represent.
- Each time slot for a final exam is represented by a different color.
- A scheduling of exams corresponds to a coloring of the graph.