Counting

CPE348

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6.1 The Basics of Counting

Basic Counting Principles

Theorem (The Product Rule)

Suppose that a procedure can be broken down into a sequence of two tasks.
If there are n_1 ways to do the first task and for each of these ways of doing the first task there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Examples:

- There are 2 roosters and 10 coops. How many ways can the roosters be assigned to coops?
  - 10×9

We can generalize the Product Rule:

- How many bit strings of length 7 are there?
  - $2^7 = 128$

The Product Rule

More Examples:

- How many functions are there from a set with m elements to a set with n elements?
  - n^m
- How many one-to-one functions are there from a set with m elements to a set with n elements?
  - $n(n-1)(n-2)\cdots(n-m+1)$

The Sum Rule

Theorem (The Sum Rule)

If a task can be done either in one of n_1 ways or one of n_2 ways, where none of the n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to complete the task.

Examples:

- Suppose for a recipe, you can either add one of 12 types of chocolate chips or one or 7 types of dried fruits. How many different choices for the recipe are there?
  - 19
- A child can choose a sticker from one of four boxes. The four boxes contain 34, 7, 2, and 100 stickers respectively. How many possible stickers are there to choose from?
  - 143

The Subtraction Rule

Theorem (The Subtraction Rule - Principle of Inclusion-Exclusion)

If a task can be done in either n_1 ways or n_2 ways, then the number of ways to complete the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Example:

- How many bit strings of length 5 are there that start with a 0 bit or end with the two bits 10?
  - There are $2^4 = 16$ bit strings of length 5 that start with 0.
  - There are $2^3 = 8$ bit strings of length 5 that end with 10.
  - There are $2^2 = 4$ bit strings that start with a 0 and end with 10.
  - Thus, there are $16 + 8 - 4 = 20$ bit strings that start with a 0 bit or end with the two bits 10.

The Division Rule

Theorem (The Division Rule)

There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

Example:

- How many different ways are there to seat five people around a circular table, where two seatings are considered the same when each person has the same left neighbor and right neighbor?
  - $5!/5 = 24$

6.2 The Pigeonhole Principle

The Pigeonhole Principle

Theorem (The Pigeonhole Principle)

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more objects.

Examples:

- Among any group of 367 people, there must be at least two with the same birthday.
  - Why?
- In any group of 27 English words, there must be at least two that begin with the same letter.
  - Why?
- A function from a set with $k+1$ or more elements to a set with k elements is not one-to-one.
  - Why?

The Generalized Pigeonhole Principle

Theorem (The Generalized Pigeonhole Principle)

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Examples:

- Among 50 people there are at least 5 people born in the same month.
  - Why?
- How many cards must be selected from a deck of 52 cards to guarantee that 5 cards of the same suit are chosen?
  - 17
  - Why?

6.3 Permutations and Combinations

Permutations

Definition

A **permutation** of a set of distinct objects is an ordered arrangement of the objects.
An ordered arrangement of r elements of a set is called an **r-permutation**.

Example:

- How many ways are there to select 4 students from a group of 20 students to stand in line?
  - $20 \times 19 \times 18 \times 17$ ways
- How many ways are there to line up all 20 students?
  - $20 \times 19 \times 18 \times \cdots \times 2 \times 1 = 20!$ ways

Permutations

Theorem

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are
$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$
r-permutations of a set with n distinct elements.

Proof: Use the product rule.

Corollary

If n and r are positive integers with $0 \leq r \leq n$, then there are
$P(n,r) = \frac{n!}{(n-r)!}$

Permutations

Examples:

- How many permutations of $ABCDEFGHIJ$ contain the string BIG?
  - Find the permutation of the 8 objects, namely, A, C, D, E, F, H, J, BIG.
  - $8!$
- Suppose that a shuttle driver must start at the airport, pick up 9 passengers (in any order she wishes), and drop them all off at the airport (at the same time). How many possible orders of pick ups are there?
  - $9! = 362880$

Combinations

Now we consider problems where order doesn't matter.

Definition

An **r-combination** of elements of a set is an unordered selection of r elements from the set.
Alternatively, an r-combination is a subset of some set, with r elements.
The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$ or $\binom{n}{r}$.

Examples:

- What is $C(4, 3)$?
  - 4
- What is $C(5, 2)$?
  - 10

Combinations

Theorem

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$ is:
$C(n, r) = \frac{n!}{r!(n-r)!}$

Proof: By the product rule.
Examples:

- How many possible poker hands of 5 cards can be dealt from a deck of 52 cards?
- How many ways are there to select 47 cards from a deck of 52 cards?

Corollary

Let n and r be nonnegative integers with $r \leq n$.
Then $C(n, r) = C(n, n - r)$.

6.4 Binomial Coefficients and Identities

The Binomial Theorem

Theorem

Let x and y be variables and n a nonnegative integer. Then $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$.

Examples:

- What is the expansion of $(x+y)^4$?
- What is the coefficient of $x^{10}y^{13}$ in the expansion of $(x+y)^{23}$?
- What is the coefficient of $x^{10}y^{13}$ in the expansion of $(2x-y)^{23}$?

Corollaries

Corollary

Let n be a nonnegative integer. Then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Corollary

Let n be a positive integer. Then $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Theorem

Let n and k be positive integers with $n \geq k$. Then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$