Sets, Functions, Sequences, and Sums

CPE348

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2.1 Sets

Sets

Definition

A **set** is an unordered collection of objects called **elements** or **members** of the set. A set is said to **contain** its elements.

- We write $a \in A$ to denote that a is an element of set A.
- We write $a \notin A$ to denote that a is not an element of A.

Examples:

- The set of animals in a barn:
  {cow, horse, chicken, sheep}
- The set of even natural numbers less than 20:
  $\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$
- A mixed set:
  {pink, green, 5, CPE348, 3.4}

Set Builder Notation

We use **set builder notation** to characterize all elements in a set by stating the property or properties that they must have to be members.

Examples:

- The set of all negative even integers can be written as:
  $\{x|x \text{ is an even negative integer}\}$
- The set of natural numbers can be written as:
  $\mathbb{N} = \{x|x \text{ is a natural number}\} = \{0, 1, 2, 3, \dots\}$
- The set of rational numbers can be written as:
  $\mathbb{Q} = \{\frac{p}{q}|p \text{ is an integer and } q \text{ is a nonzero integer}\}$
  $= \{\frac{p}{q}|p \in \mathbb{Z}, q \in \mathbb{Z}, \text{ and } q \neq 0\}$
- The real interval $[a, b)$ can be written as:
  $[a, b) = \{x|a \leq x < b\}$

Set Equality

Definition

Two sets are **equal** if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.We write $A = B$ if A and B are equal sets.

Examples:

Are the following sets equal?

- $\{2, 4, 6, 8\}$ and $\{8, 4, 6, 2\}$
- $\{1, 2, 3, 4, 5\}$ and $\{2, 3, 4, 5, 6\}$
- $\{\text{red, blue, red, red}\}$ and $\{\text{blue, red}\}$

The Empty Set

Definition

The set with no elements is called the **empty set** (denoted $\emptyset$).

Example:

- $\{x | x > 4.7 \text{ and } x > x^2\} = \emptyset$

The Singleton Set

Definition

A set with exactly one element is called a **singleton set**.

Examples:

- $\{4\}$
- $\{\text{blue}\}$
- $\{\emptyset\}$

Venn Diagrams

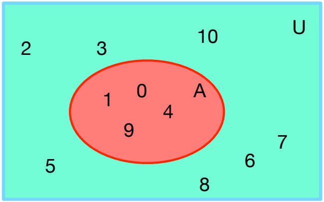
Sets can be represented graphically using **Venn diagrams**.

- In Venn diagrams there is a **universal set** that contains all objects under consideration.
  - The universal set is represented using a rectangle, usually denoted U.
- Inside the universal set, sets are represented using circles.

Example:

$U = \{$The set of all natural numbers less than 11$\}$.

$A = \{x|x$ is a perfect square$\}$.



Subsets

Definition

A set A is a subset of B ($A \subseteq B$) if and only if every element of A is also an element of B.

Examples:

- $\{2, 8\} \subseteq \{1, 2, 4, 7, 8\}$
- {red} $\subseteq$ {blue, red, red, green}
- $\{1, 2, 3\} \nsubseteq \{1, 3, 5, 7\}$

To show that $A \subseteq B$:

- Take an arbitrary element, x, of A and show that $x \in B$.

To show that $A \nsubseteq B$:

- Find a single $x \in A$ such that $x \notin B$.

Subsets

Theorem

For every set S:

1. $\emptyset \subseteq S$
2. $S \subseteq S$

Proof (on board).

Definition

Let A and B be sets. We say that A is a **proper subset** of B ($A \subset B$) if $A \subseteq B$ and $A \neq B$.

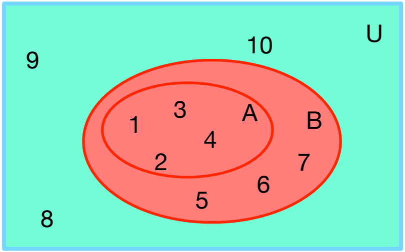
Example:

- $\{1, 3\} \subset \{1, 2, 3\}$

Subsets

Example:

- $A = \{1, 2, 3, 4\}$
- $B = \{1, 2, 3, 4, 5, 6, 7\}$
- $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- Draw the Venn diagram representing this situation.



Subsets

To show that two sets, A and B are equal, show $A \subseteq B$ and $B \subseteq A$.
Example:
Show that the following sets are equal.

- $A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
- $B = \{x | x \text{ is a subset of the set } \{a, b\}\}$

The Size of a Set

Definition

Let S be a set. If there are exactly n distinct elements in S, we say that S is **finite** with **cardinality** n. We denote the cardinality of S by $|S|$.

Definition

A set is said to be **infinite** if it is not finite.

Examples:

- $|\{1, 5, 8, 9\}| = 4$
- $|\{\text{red, red, blue, blue, blue}\}| = 2$
- $|\mathbb{Z}| = \infty$

Power Sets

Definition

Given a set, S, the **power set** of S is the set of all subsets of the set S.
The power set of S is denoted by $\mathcal{P}(S)$

Examples:

- $\mathcal{P}(\{2,4,6\}) = \{\emptyset, \{2\}, \{4\}, \{6\}, \{2,4\}, \{2,6\}, \{4,6\}, \{2,4,6\}\}$
- $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$
- $\mathcal{P}(\emptyset) = \{\emptyset\}$

Can you come up with a conjecture about the cardinality of a power set of a set with n elements?

Cartesian Product

Definition

The **ordered n-tuple** $(a_1, a_2, \ldots a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element,..., and a_n as its nth element.

Examples:

- $(5, 2, 9)$
  - Note $(5, 2, 9) \neq (2, 5, 9)$
- $(3, 4)$
  - 2-tuples are called **ordered pairs**.

Cartesian Product

Definition

Let A and B be sets. The **Cartesian product** of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \land b \in B\}$$

Examples:

- Let $A = \{1, 2, 3\}$ and $B = \{\text{red, blue}\}$
- What is $A \times B$?
- What is $B \times A$

Cartesian Product

Definition

The **Cartesian product** of the sets $A_1, A_2, \dots A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \dots a_n)$, where $a_i \in A_i$ for $i = 1, 2, \dots n$.

$$A_1 \times A_2 \times \cdots A_n = \{(a_1, a_2, \dots a_n) | a_i \in A_i \text{ for } i = 1, 2, \dots n\}$$

Example:
Find $A \times B \times C$ where:

- $A = \{1, 2, 3\}$
- $B = \{2, 4\}$
- $C = \{\text{SLO, SF}\}$

Truth Sets

Definition

Given a predicate P, and a domain D, we define the **truth set** of P to be the set of elements $x \in D$ for which $P(x)$ is true.

Examples:
What are the truth sets for the following predicates if the domain is the set of integers?

- $P(x)$ is "$|x| < 3$"
- $Q(x)$ is "$-|x| > 0$"

2.2 Set Operations

Combining Sets

Definition

Let A and B be sets. The **union** of the sets A and B (denoted by $A \cup B$) is the set that contains those elements that are either in A, or in B, or in both.

$$A \cup B = \{x | x \in A \vee x \in B\}$$

Examples:

- $\{2, 4, 6, 8\} \cup \{1, 4, 9\} = \{1, 2, 4, 6, 8, 9\}$
- $\{1, 2, 3\} \cup \{\text{water, tea, coffee}\} = \{1, 2, 3, \text{ water, tea, coffee}\}$

Combining Sets

Definition

Let A and B be sets. The **intersection** of the sets A and B (denoted by $A \cap B$) is the set that contains those elements that are both in A and B.

$$A \cap B = \{x | x \in A \land x \in B\}$$

Examples:

- $\{2, 4, 6, 8\} \cap \{1, 4, 9\} = \{4\}$
- $\{1, 2, 3\} \cap \{\text{water, tea, coffee}\} = \emptyset$
  - Two sets are called **disjoint** if their intersection is the empty set.

$|A \cup B| = |A| + |B| - |A \cap B|$

Combining Sets

Definition

Let A and B be sets. The **difference** of the sets A and B (denoted by $A - B$ or $A \setminus B$) is the set that contains those elements that are in A and not in B.

$$A - B = \{x | x \in A \wedge x \notin B\}$$

Examples:

- $\{2, 4, 6, 8\} - \{1, 4, 9\} = \{2, 6, 8\}$
- $\{1, 2, 3\} - \{\text{water, tea, coffee}\} = \{1, 2, 3\}$

Combining Sets

Definition

Let U be the universal set. The **complement** of the set A with respect to U (denoted by $\overline{A}$) is the set $U - A$.

$$\overline{A} = \{x \in U | x \notin A\}$$

Examples:

- Suppose $A = \{2, 4, 6, 8\}$ and $U = \{x | x \text{ is a natural number less than or equal to } 10\}$.
  - $\overline{A} = \{0, 1, 3, 5, 7, 9, 10\}$
- Suppose $A = \{\text{water, tea, coffee}\}$ and $U = \{\text{lemonade, water, tea, coffee, soda}\}$.
  - $\overline{A} = \{\text{lemonade, soda}\}$

Venn Diagrams

Draw the Venn diagrams for:

- $A \cup B$
- $A \cap B$
- $A - B$
- $\overline{A}$

Theorem

$A - B = A \cap \overline{B}$

Proof on board.

Set Identities

Examples:
Prove the following set identities:

- $A \cap U = A$
- $A \cap B = B \cap A$
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Generalized Unions and Intersections

Definition

The **union** of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We denote the union of sets $A_1, A_2, \ldots A_n$ by

$$A_1 \cup A_2 \cup, \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

Definition

The **intersection** of a collection of sets is the set that contains those elements that are members all of the sets in the collection.

We denote the intersetion of sets $A_1, A_2, \ldots A_n$ by

$$A_1 \cap A_2 \cap, \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

2.3 Functions

Functions

Definition

Let A and B be nonempty sets. A **function** f from A to B is an assignment of exactly one element of B to each element of A.
We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
If f is a function from A to B, we write $f : A \to B$.

Examples:

Functions

Definition

If f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f.
If $f(a) = b$, we say that b is the **image** of a and a is the **preimage** of b.
The **range** or **image** of f is the set of all images of elements of A.
We say "f maps A to B".

Example:
Let f be the function that assigns the last two bits of a bit string (with length greater than or equal to two) to that string.

- $f(10011) = 11$
- $f(1001) = 01$

What are the domain, codomain, and range for this function?

Functions

Examples:

- Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function that assigns the cube of an integer to that integer.
  - $f(2) = 8$
  - $f(-3) = -27$

  What are the domain, codomain, and range for this function?
- Let $f : \mathbb{R} \to \mathbb{R}$ be the floor function.
  - $f(2.75) = 2$
  - $f(-5.67) = -6$

  What are the domain, codomain, and range for this function?

Definition

A function is called **real-valued** if its codomain is the set of real numbers and is called **integer-valued** if its codomain is the set of integers.

Functions

Definition

Let f_1 and f_2 be functions from A to $\mathbb{R}$.Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined for all $x \in A$ by

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$,
- $f_1 f_2(x) = f_1(x) f_2(x)$

Examples:
Let $f_1(x) = 2x^2$ and $f_2(x) = 4x - 17$.
Determine the following functions:

- $f_1 + f_2$
- $f_1 + f_1$
- $f_1 f_2$

Functions

Definition

Let f be a function from A to B and let S be a subset of A. The **image** of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$:
$f(S) = \{t | \exists s \in S \text{ such that } t = f(s)\}$

Example:
Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with
$f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1, f(e) = 1.$
What is the image of the subset $S = \{b, c, d\}$?

One-to-One Functions

Definition

A function f is said to be **one-to-one**, or an **injunction** if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be **injective** if it is one-to-one.

Examples:

- Is $f(x) = x^2$ one-to-one when the domain is all integers?
  - What if the domain is all natural numbers?
- Is $f(x) = x - 17$ one-to-one when the domain is all real numbers?

Increasing and Decreasing Functions

Definition

A function f whose domain and codomain are subsets of the set of real numbers is called **increasing** if $f(x) \leq f(y)$, and **strictly increasing** if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f.
Similarly, f is called **decreasing** if $f(x) \geq f(y)$, and **strictly decreasing** if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f.

Examples?
Can you create a conjecture involving the relationship between a function being one-to-one and a function being strictly increasing or decreasing?

Onto Functions

Definition

A function $f : A \to B$ is called **onto** or a **surjection** if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
A function f is called **surjective** if it is onto.

Examples:

- Is the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ onto?
  - What about when the codomain is positive real numbers?
- Is the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 4$ onto?

Bijections

Definition

A function f is a **one-to-one correspondence** or a **bijection** if it is both one-to-one and onto.
We say such a function is **bijective**.

Examples?

Functions

Suppose that $f : A \to B$.

- **To show that f is injective:** Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.
- **To show that f is not injective:** Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.
- **To show that f is surjective:** Consider an arbitrary element $y \in B$ and find an element $a \in A$ such that $f(a) = y$.
- **To show that f is not surjective:** Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Inverse Functions

Definition

Let f be a bijection from the set A to the set B. The **inverse** function of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.
The inverse function of f is denoted f^{-1}.
Thus, $f^{-1}(b) = a$ when $f(a) = b$.

Example:
Find the inverse of the function $f(x) = x + 4$

Composition of Functions

Definition

Let $g : A \to B$ and $f : B \to C$ be functions. The **composition** of the functions f and g (denoted by $f \circ g$) is defined by:
$(f \circ g)(a) = f(g(a))$

Examples:

- Suppose $f(x) = 2x + 4$ and $g(x) = x^2$.
  - What is $f \circ g$?
  - What is $g \circ f$?

The Graph of a Function

Definition

Suppose $f : A \to B$. The **graph** of the function f is the set of ordered pairs $\{(a, b) | a \in A \text{ and } f(a) = b\}$.

Examples:

- Display the graph of $f : \mathbb{R} \to \mathbb{R}$. Where $f(x) = x^2$.
- Display the graph of $f : \mathbb{Z} \to \mathbb{Z}$. Where $f(x) = x^2$.

Sequences

Definition

A **sequence** is a function from a subset of the set of integers (usually either $\{0, 1, 2, 3, \dots\}$ or $\{1, 2, 3, 4 \dots\}$) to a set S.
We use the notation a_n to denote the image of the integer n. We call a_n a **term** of the sequence.

Examples:

- Consider the sequence $\{a_n\}$ where $a_n = 2n$.
  - List the first 7 terms of the sequence.
- Consider the sequence $\{a_n\}$ where $a_n = 1/2n$.
  - List the first 5 terms of the sequence.

The Geometric Progression

Definition

A **geometric progression** is a sequence of the form:
$a, ar, ar^2, ar^3, \ldots$
where the **initial term** a and the **common ratio** r are real numbers.

Examples:

- Consider the geometric sequence $\{a_n\}$ where $a_n = -2 \cdot 2^n$.
  - List the first 4 terms of the sequence.
- Consider the geometric sequence $\{b_n\}$ where $b_n = 5 \cdot (1/4)^n$.
  - List the first 5 terms of the sequence.

The Arithmetic Progression

Definition

An **arithmetic progression** is a sequence of the form:
$a, a+d, a+2d, a+3d, \ldots$
where the **initial term** a and the **common difference** d are real numbers.

Examples:

- Consider the arithmetic sequence $\{a_n\}$ where $a_n = -2 + 2n$.
  - List the first 4 terms of the sequence.
- Consider the arithmetic sequence $\{b_n\}$ where $b_n = 5 + (1/3)n$.
  - List the first 5 terms of the sequence.

Recurrence Relations

So far we have only defined sequences with explicit formulas for their terms.

- There are many other ways to specify a sequence.
- One such way is to provide one or more initial terms together with a rule for determining subsequent terms from those that proceed them.

Definition

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms in the sequence $(\{a_0, a_1, a_2, \ldots, a_{n-1}\})$.
A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

Recurrence Relations

Examples:

- Let $\{a_n\}$ be the sequence that satisfies the recurrence relation $a_n = a_{n-1} - 4$ where $a_0 = 10$.
  - What are the first 4 terms of the sequence?
- Let $\{b_n\}$ be the sequence that satisfies the recurrence relation $b_n = b_{n-2} * 2$ where $b_0 = 10$ and $b_1 = 2$.
  - What are the first 4 terms of the sequence?
- Let $\{f_n\}$ be the sequence that satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ where $f_0 = 0$ and $f_1 = 1$.
  - This is called the **Fibonacci sequence**.
  - What are the first 4 terms of the sequence?
- Let $\{c_n\}$ be the sequence that satisfies the recurrence relation $c_n = nc_{n-1}$ where $c_1 = 1$.
  - What are the first 4 terms of the sequence?

Solving Recurrence Relations

We say that we have **solved** a recurrence relation together with its initial conditions when we find an explicit formula, called a **closed formula** for the terms of the sequence.

Example:

Solve the recurrence relation: $a_n = a_{n-1} - 4$ where $a_0 = 10$.

$$\begin{aligned}
a_n &= a_{n-1} - 4 \\
&= (a_{n-2} - 4) - 4 = a_{n-2} - 2 \times 4 \\
&= (a_{n-3} - 4) - 2 \times 4 = a_{n-3} - 3 \times 4 \\
&= \dots \\
&= a_0 - n \times 4 \\
&= 10 - n \times 4
\end{aligned}$$

This technique is called **iteration**.

Summations

We begin with notation for the addition of terms of a sequence.

- Suppose we are given a sequence $\{a_n\}$ and we wish to sum terms m through n $(a_m + a_{m+1} + \cdots + a_{n-1} + a_n)$. We could write this sum as:
  - $\sum_{i=m}^{n} a_i$ or
  - $\sum_{m \leq i \leq n} a_i$
- Here m is called the **lower limit** and n is called the **upper limit**.

Example:

- Use summation notation to express the sum of the first 50 terms of the sequence $\{a_j\}$ where $a_j = 7 \times (1/j)^2$

Summations

Examples:

- What is the value of $\sum_{i=4}^{8} 12 \times i$?
- What is the value of $\sum_{i=2}^{8} 12 \times i$?
- What is the value of $\sum_{i=1}^{5} (-1)^i i^2$?
- What is the value of $\sum_{i=2}^{10} i$?

The sum of the terms of geometric progressions (**geometric series**) often arise in computer science:

Theorem

If a and r are real numbers ($r \neq 0$), then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1}, & if\ r \neq 1 \\ (n+1)a & if r = 1 \end{cases}$$

Summations

For double sums, first expand the inner sum, then evaluate the outer sum:
Example:

$$\begin{aligned}\sum_{i=1}^{4}\sum_{j=1}^{3} ij &= \sum_{i=1}^{4}(i + 2i + 3i) \\ &= \sum_{i=1}^{4} 6i \\ &= 60\end{aligned}$$

2.5 Cardinality of Sets

Cardinality of Sets

Definition

Sets A and B have the same **cardinality** if and only if there is a bijection (a one-to-one and onto function) from A to B.
When A and B have the same cardinality, we write $|A| = |B|$.

Definition

If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$

Countable Sets

Definition

A set that is either finite or has the same cardinality as the set of positive integers is called **countable**.

Examples:

- Show that the set of even positive integers is a countable set.
  - Show that there is a one-to-one, onto function from the positive integers, $\mathbb{Z}^+$, to the set of even positive integers.
- Show that the set of all integers is a countable set.
  - Show that there is a one-to-one, onto function from the positive integers, $\mathbb{Z}^+$, to the set of all integers.
- Show that the set of positive rational numbers is countable.
  - Show that there is a one-to-one, onto function from the positive integers, $\mathbb{Z}^+$, to the set of positive rational numbers.

An Uncountable Set

Example:
Show that the set of real numbers is an uncountable set.

- We will use the *Cantor Diagonalization method.*

Countability

Theorem

If A and B are countable sets, then $A \cup B$ is also countable.

Proof.

By cases (on board). $\square$

Theorem

If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

The statement of the above theorem is straightforward. The proof, however, is very difficult.