Outline : 1. What is algebraic K-thy. why we care ?

2. Universal characterization of alg K.

3. Idea of proof.

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1. Concise background in algebraic K.

- Why do we care ?

1) Vandiver conj :

$\forall$ max subfield $F$ of $\mathbb{Q}(\zeta_p)$. $p \nmid h(F)$ = class number.

Kurihara 92' : $\Leftrightarrow$ $K_{4n}\mathbb{Z} = 0$

2) s-cobordism :

$X \hookrightarrow W$ h-cobordism. Then the obstruction of $W$ to be cylinder lies in $K_1(\mathbb{Z}[\pi_1 X])$.

3) Quillen-Lichtenbaum conj :

K does not satisfy the étale descent. but for sufficiently large $i$. $K_i(S, \mathbb{Z}/n) \cong H^{-i}_{ét}(S, F^{ét}/n)$, for $S$ noetherian scheme of finite Krull dim. $n$ invertible in $S$. $F^{ét}$ = sheafification of $(F : X \mapsto K(X))$ on the small étale site of $S$. As consequence of Bloch-Kato. proved by Voevodsky and Rost.

- Motivic SS ( Thomason 85' ) in étale (hyper) descent :

$$H^*_{ét}(X, \pi^{ét}_* K/\ell^\nu[\beta^{-1}]) \Rightarrow \pi_* K/\ell^\nu(X)[\beta^{-1}]$$

for $\ell$ invertible in $X$ separated regular noetherian scheme,

$\ell^\nu$ prime power. $\beta$ = Bott elet. $\pi_*^{ét}$ étale htpy sheaf.

- What is algebraic $K$?

Generally speaking. $\forall$ $R$ assoc. unital ring. $1_R \neq 0_R$.

$$K_n R := \pi_n(BGL(R)^+) . \quad \forall n \geq 1.$$

where $(-)^+$ = Quillen "+" - construction. Write

$$K(R) = K_0 R \times BGL(R)^+$$

Then
$$K_n R = \pi_n K(R) . \quad \forall n \geq 0.$$

e.g. $K_0(R) = \mathbb{Z}\{[P] : [P] = \text{iso class of proj modules}\}/\sim$

$$[P] + [Q] \sim [P \oplus Q]$$

$K_0(\text{fields}) = \mathbb{Z}$

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/(q^i - 1) & , n = 2i-1 \\ 0 & . \text{ else} \end{cases}$$

- Different constructions of alg. $K$.

1) "+" construction : as above.

2) "Q" construction : $\mathcal{C}$ $\infty$-cat. $K_n\mathcal{C} = \pi_n K\mathcal{C} = \pi_n \Omega BQ\mathcal{C}$

$$= \pi_{n+1}(BQ\mathcal{C}).$$

where $BQ\mathcal{C} = |NQ\mathcal{C}|$. and

$Q : \mathrm{Cat}_\infty^{ex} \longrightarrow \mathrm{Cat}_\infty$ . $Q\mathcal{C}$ has n-simplices given by

$$\begin{tikzcd}[column sep=tiny]
 & & & X_{0,n} \arrow[dl] \arrow[dr] & & & \\
 & & X_{0,n-1} \arrow[dl] \arrow[dr] & & X_{1,n} \arrow[dl] \arrow[dr] & & \\
 & \cdots & & \cdots & & \cdots & \\
 & X_{0,1} \arrow[dl] \arrow[dr] & & & & X_{n-1,n} \arrow[dl] \arrow[dr] & \\
X_{0,0} & & X_{1,1} & \cdots & X_{n-1,n-1}, & & X_{n,n}
\end{tikzcd}$$

3) Waldhansen $S_\bullet$ construction : $\forall\ \mathcal{C} \in \mathrm{Cat}_\infty$.

$S_0\mathcal{C} \simeq *$

$S_1\mathcal{C} \simeq \mathcal{C}$

$$\begin{tikzcd}
* \arrow[r] & X \in \mathcal{C} \arrow[d] \\
 & *
\end{tikzcd}$$

$S_2\mathcal{C} \simeq \mathrm{Fun}(\Delta^1, \mathcal{C})$

$$\begin{tikzcd}
* \arrow[r] & X(0,0) \arrow[r] \arrow[d] & X(0,1) \arrow[d] \\
 & * \arrow[r] & X(1,1) \arrow[d] \\
 & & *
\end{tikzcd}$$

cofib seq $X(0,0) \to X(0,1) \to X(1,1)$

$\vdots$

$S_n\mathcal{C} \simeq \mathrm{Fun}(\Delta^{n-1}, \mathcal{C})$

$$\begin{tikzcd}
* \arrow[r] & X(0,0) \arrow[r] \arrow[d] & X(0,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & X(0,n) \arrow[d] \\
 & * \arrow[r] & X(1,1) \arrow[r] & \cdots \arrow[r] & X(1,n) \arrow[d] \\
 & & & \ddots & \vdots
\end{tikzcd}$$

s.t. each square is cocartesian

( or pushout )

**Thm** (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) . via $\mathcal{C} = N\mathcal{P}(R)$.

nerve of cat of f.g. projective R-mod.

**Rk** Use Waldhausen $S_\bullet$ - construction from now on.

- Some consequences : suppose $\mathcal{C}$ is small stable $\infty$-cat.

  1° $S_n\mathcal{C}$ is an stable $\infty$-cat. $\forall n$.

  2° Can construct an alg K spectrum $K\mathcal{C}$ w/

  $K\mathcal{C}_n = |(S_\bullet^{(n)}\mathcal{C})^{\simeq}|$ . where

  $(-)^{\simeq}$ = taking max subgroupoid.

  $S_\bullet^{(n)}\mathcal{C} = \underbrace{S_\bullet \circ \cdots \circ S_\bullet}_{n \text{ times}} \mathcal{C}$

  the structure map is induced by

  $\Sigma(-)^{\simeq} \longrightarrow |(S_\bullet\mathcal{C})^{\simeq}|$

  obtained by restriction to 1-skeleton. Thus

  $\Omega^\infty K\mathcal{C} \simeq \Omega|(S_\bullet\mathcal{C})^{\simeq}|$

  3° $K(\mathcal{C}) = K(Sp\,\mathcal{C})$

  Here $Sp\,\mathcal{C}$ = $\infty$-cat of spectrum objects in $\mathcal{C}$. which

  is an $\infty$-functor $X : \mathbb{Z}\times\mathbb{Z} \longrightarrow \mathcal{C}$ s.t. $\forall i \neq j$.

$X(i,j) = 0 \in \mathcal{C}$. Better to write as $X[n] := X(n,n)$.

4° $K: \mathrm{Cat}_\infty^{st} \longrightarrow Sp$ is lax symmetric monoidal.

Here $\mathrm{Cat}_\infty^{st}$ = $\infty$-cat of stable $\infty$-cats (small) whose morphisms are exact functors, i.e. preserves finite limits / colimits.

Aside: lax sym mon: $F: (\mathcal{C}, \otimes_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}})$ s.t.

$$\begin{tikzcd}
FA \otimes_{\mathcal{D}} FB \arrow[r] \arrow[d] & FB \otimes_{\mathcal{D}} FA \arrow[d] \\
F(A \otimes_{\mathcal{C}} B) \arrow[r] & F(B \otimes_{\mathcal{C}} A)
\end{tikzcd}$$

commute.

2. Universal characterization of alg $K$.

Write $\mathrm{Cat}_\infty^{perf} \subseteq \mathrm{Cat}_\infty^{st}$ be the idempotent complete small stable $\infty$-cats. We have

$$\mathrm{Idem}: \mathrm{Cat}_\infty^{st} \rightleftarrows \mathrm{Cat}_\infty^{perf} : \mathrm{Forget}$$

which is an adjunction pair. Some notations and definitions:

Def $\mathcal{C}$ cat. $X, Y \in \mathcal{C}$. $Y$ retract of $X$ iff

$$\begin{tikzcd}
 & X \arrow[dr, "r"] & \\
Y \arrow[ur, "\iota"] \arrow[rr, "id"] & & Y
\end{tikzcd} \qquad (*)$$

where $\iota$ is a monomorphism. $r$ is a retraction.

$\iota \circ r$ idempotent.

In $\infty$-cat. we need to take care of higher coherence.

Def $\mathcal{C}$ $\infty$-cat. $X, Y \in \mathcal{C}$. $Y$ retract of $X$ iff $Y$ retract of $X$ in $h\mathcal{C}$

$\Leftrightarrow$ $\exists$ 2-simplex $\Delta^2 \to \mathcal{C}$ corresponds to (*) as above.

Write $\mathrm{Idem}^+$ = simplicial set s.t.

$\forall J \neq \phi$, $|J| < \infty$ totally ordered finite index set.

$\mathrm{Hom}_{\mathrm{Set}}(\Delta^J, \mathrm{Idem}^+) = \{ (J_0, \sim) : J_0 \subseteq J,$

$\sim$ satisfies $(*) \}$.

$(*)$: $i \le j \le k$ in $J$. $i, k \in J_0$, $i \sim k \Rightarrow j \in J_0$ and $i \sim j \sim k$.

$\mathrm{Idem} \subseteq \mathrm{Idem}^+$ : subset of $\mathrm{Idem}^+$ s.t. $J_0 = J$ in pairs $(J_0, \sim)$.

Def $\mathcal{C}$ $\infty$-cat is idempotent complete if any functor $F \in \mathrm{Fun}(\mathrm{Idem}, \mathcal{C})$ is effective, i.e. $F$ can be extended to $\mathrm{Fun}(\mathrm{Idem}^+, \mathcal{C})$.

Classically. $\mathcal{C}$ 1-cat is idempotent complete if $\forall$ idempotent $f: X \to X$ $\overset{1-1}{\leftrightsquigarrow}$ retracts $X \xrightarrow{r} Y$. Write $\mathcal{C}^{perf}$ to be idempotent completion of $\mathcal{C}$. or $\mathrm{Idem}\ \mathcal{C}$.

Def. 1) $f: \mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}_\infty^{st}$ is Morita equiv if

$$\mathrm{Idem}\ f : \mathrm{Idem}\ \mathcal{C} \xrightarrow{\sim} \mathrm{Idem}\ \mathcal{D}$$

2) $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{E}$ is exact (or Karoubi) in $\mathrm{Cat}_\infty^{perf}$ if

① $f$ is fully faithful

② $\mathcal{D}/\mathcal{C} \simeq \mathcal{E}$

③ $g \circ f \simeq 0$

Def $F: \mathrm{Cat}_\infty^{st} \longrightarrow \mathrm{Sp}$.

1) $F$ is additive invariant if

① $F$ inverts Morita equiv.

(i.e. $f: \mathcal{C} \to \mathcal{D}$ Morita equiv. $F(f)$ equiv.)

② $F$ preserves filtered colims

③ $F$ takes split exact sequences to split cofib sequences.

2) $F$ is localizing invariant if ① ② hold and

③' $F$ takes exact sequences to cofib seqs.

Write $PSh^{add}_{Sp}(Cat^{st}_\infty)$ to be the cat of "presheaves" $F: ((Cat^{st}_\infty)^{w})^{op} \longrightarrow Sp$ s.t. ③ is satisfied.

Here $\mathcal{C}^{w}$ = full subcat of $\mathcal{C}$ consisting of cpt objs in $\mathcal{C}$.

Recall $\forall X \in \mathcal{C}$ is compact if $Map_{\mathcal{C}}(X, -)$ preserves filtered colim.

We have a pair of adjunction:

$$\text{Forget}: PSh^{add}_{Sp}(Cat^{st}_\infty) \rightleftarrows PSh_{Sp}((Cat^{st}_\infty)^{w}) : L^{add}$$

forget the condition ③

Consider

$$\mathcal{U}_{add}: Cat^{st}_\infty \xrightarrow{\text{Yoneda}} PSh_{Sp}((Cat^{st}_\infty)^{w}) \xrightarrow{L^{add}} PSh^{add}_{Sp}(Cat^{st}_\infty)$$

$$\mathcal{C} \longmapsto \mathcal{U}_{add}(\mathcal{C})$$

Here $\mathcal{U}_{add}(\mathcal{C})$ is called the additive non-commutative motive.

Same story happens if we replace ③ by ③'. Then we get the localizing non-comm. motive:

$$\mathcal{U}_{loc}: Cat^{st}_\infty \longrightarrow PSh^{loc}_{Sp}(Cat^{st}_\infty)$$

$$\mathcal{C} \longmapsto \mathcal{U}_{loc}(\mathcal{C}).$$

<u>Thm</u> (Blumberg - Gepner - Tabuada, 2013)

$\forall \mathcal{C} \in Cat^{perf}_\infty$. there are nat. equiv

$$Map(\mathcal{U}_{add}(Sp^{w}), \mathcal{U}_{add}(\mathcal{C})) \simeq K(\mathcal{C})$$

$$Map(\mathcal{U}_{loc}(Sp^{w}), \mathcal{U}_{loc}(\mathcal{C})) \simeq \mathbb{K}(\mathcal{C})$$

where $\mathbb{K}$ = non-connective alg. K-thy spectrum.

Equivalently, the functor

$$\Psi : \mathrm{PSh}_{\mathrm{Sp}}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{st}}) \longrightarrow \mathrm{Sp}$$

is corepresented by $K \in \mathrm{PSh}_{\mathrm{Sp}}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{st}})$. Similar story happens in localizing invariants.

<u>Cor</u> $\forall\, n \in \mathbb{Z}$, iso of abelian gps

$$K_n \mathcal{C} \simeq \mathrm{Hom}(\mathcal{M}_{\mathrm{add}}(\mathrm{Sp}^{\omega}), \Sigma^{-n}\mathcal{M}_{\mathrm{add}}(\mathcal{C}))$$

$$\mathbb{K}_n \mathcal{C} \simeq \mathrm{Hom}(\mathcal{M}_{\mathrm{loc}}(\mathrm{Sp}^{\omega}), \Sigma^{-n}\mathcal{M}_{\mathrm{loc}}(\mathcal{C})).$$

in $h(\mathcal{M}_{\mathrm{add}})$ and $h(\mathcal{M}_{\mathrm{loc}})$, respectively.

<u>Cor</u> $\forall\, F \in \mathrm{PSh}_{\mathrm{Sp}}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{st}})$ (resp. $\mathrm{PSh}_{\mathrm{Sp}}^{\mathrm{loc}}$),

$$\mathrm{Map}_{\mathrm{add}}(K, F) \simeq F(\mathrm{Sp}^{\omega})$$

$$\mathrm{Map}_{\mathrm{loc}}(\mathbb{K}, F) \simeq F(\mathrm{Sp}^{\omega})$$

note $\mathrm{Sp}^{\omega}$ = $\infty$-cat of finite spectra, so

$$\pi_0 \mathrm{Map}_{\mathrm{add}}(K, F) \simeq \pi_0 F(\mathrm{Sp}^{\omega})$$

For $F$ "good functor" e.g. $F = \mathrm{THH}$, $F(\mathrm{Sp}^{\omega})$ can be defined by some good nerve (e.g. cyclic nerve for $F = \mathrm{THH}$). In these cases, after geometric realization, $\pi_0 F(\mathrm{Sp}^{\omega}) = \pi_0 F(\mathbb{S})$

3. Idea of proof (c.f. Thm 7.13 in BGT)

For the connective case (additive invariant), $\forall\, \mathcal{A} \in \mathrm{Cat}_\infty^{\mathrm{st}}$,

$\mathcal{B} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$. $\mathcal{B}$ compact in $\mathrm{Cat}_\infty^{\mathrm{perf}}$, then one can show

$\mathcal{M}_{add}(\mathcal{A}) \simeq K_{\mathcal{A}}$, where $K_{\mathcal{A}} \in PSh_{Sp}((\mathrm{Cat}_\infty^{st})^{\omega})$

$$K_{\mathcal{A}} : \mathcal{C} \longmapsto K(\mathrm{Fun}^{ex}(\mathcal{C}, \mathrm{Idem}\,\mathcal{A}))$$

$\mathrm{Fun}^{ex}$ = exact functors.

Note $K_{\mathcal{A}}(Sp^{\omega}) = K(\mathcal{A})$. One can also show that $K_{\mathcal{A}}$ is local, i.e. $\forall$ split exact seqs $\mathcal{B} \to \mathcal{C} \to \mathcal{D}$ in $(\mathrm{Cat}_\infty^{\mathrm{perf}})^{\omega}$,

$$\mathrm{Map}(\psi(\mathcal{D}), K_{\mathcal{A}}) \simeq \mathrm{Map}(\psi(\mathcal{C})/\psi(\mathcal{B}), K_{\mathcal{A}}) \quad (*)$$

where $\psi : \mathrm{Cat}_\infty^{\mathrm{perf}} \longrightarrow PSh_{Sp}((\mathrm{Cat}_\infty^{\mathrm{perf}})^{\omega})$ is the Yoneda embedding. Therefore,

$$\mathrm{Map}(\mathcal{M}_{add}(\mathcal{B}), \mathcal{M}_{add}(\mathcal{A})) \simeq \mathrm{Map}(\psi(\mathcal{B}), K_{\mathcal{A}}).$$

$\simeq K_{\mathcal{A}}(\mathcal{B})$ by spectral Yoneda Lemma applied to $\psi(\mathcal{B})$.

Here we implicitly use the fact that

$$\mathcal{M}_{add} \simeq PSh_{Sp}((\mathrm{Cat}_\infty^{\mathrm{perf}})^{\omega})[\text{"map"}^{-1}],$$

where "map" is given by $\psi(\mathcal{C})/\psi(\mathcal{B}) \longrightarrow \psi(\mathcal{D})$ for the split exact $\mathcal{B} \to \mathcal{C} \to \mathcal{D}$.

Thus

$$\mathrm{Map}(\mathcal{M}_{add}(\mathcal{B}), \mathcal{M}_{add}(\mathcal{A})) \simeq \mathrm{Map}(\psi(\mathcal{B})/\text{"map"}, \mathcal{M}_{add}(\mathcal{A}))$$

$$\simeq \mathrm{Map}(\psi(\mathcal{C})/\psi(\mathcal{D}), \mathcal{M}_{add}(\mathcal{A}))$$

$$\simeq \mathrm{Map}(\psi(\mathcal{B}), \mathcal{M}_{add}(\mathcal{A})). \quad \text{by } (*)$$

$$\simeq \mathrm{Map}(\psi(\mathcal{B}), K_{\mathcal{A}}).$$

Several minor issues:

1° $\mathcal{M}_{add}(\mathcal{A}) = K_{\mathcal{A}}$:

IDEA: consider simplicial fiber seq

$$\mathcal{A}_\bullet \to PS_\bullet\mathcal{A} \to S_\bullet\mathcal{A}$$

where $\mathcal{A}_\bullet$ = const simplicial obj

$PS_\bullet\mathcal{A}$ = simplicial path obj of $S_\bullet\mathcal{A}$

$S_\bullet\mathcal{A}$ = Waldhausen $S_\bullet$-construction.

We have a split exact sequence

$$\mathcal{A} \longrightarrow PS_n\mathcal{A} = S_{n+1}\mathcal{A} \longrightarrow S_n\mathcal{A}.$$

Apply $\mathcal{M}_{add}(-)$ to $\mathcal{A}_\bullet \to PS_\bullet^{(\infty)}\mathcal{A} \to S_\bullet^{(\infty)}\mathcal{A}$

At each $n$.

$$\mathcal{M}_{add}(PS_n^{(\infty)}\mathcal{A})/\mathcal{M}_{add}(\mathcal{A}) \xrightarrow{\simeq} \mathcal{M}_{add}(S_n^{(\infty)}\mathcal{A})$$

$$\Rightarrow \quad \Sigma(\mathcal{M}_{add}(\mathcal{A})) \simeq |\mathcal{M}_{add}(S_\bullet^{(\infty)}\mathcal{A})|$$

But $|\mathcal{M}_{add}(S_\bullet^{(\infty)}\mathcal{A})| \simeq \Sigma K_{\mathcal{A}}$

2° $K_{\mathcal{A}}$ is local:

By spectral Yoneda lemma. equiv to show

$$K(\mathrm{Fun}^{ex}(\mathcal{D}.\, \mathrm{Idem}\,\mathcal{A})) \to K(\mathrm{Fun}^{ex}(\mathcal{C}.\, \mathrm{Idem}\,\mathcal{A})) \to K(\mathrm{Fun}^{ex}(\mathcal{B}.\, \mathrm{Idem}\,\mathcal{A}))$$

is a cofib seq. Suffice to show

$$K(\mathrm{rep}(\mathcal{D}.\mathcal{A})) \to K(\mathrm{rep}(\mathcal{C}.\mathcal{A})) \to K(\mathrm{rep}(\mathcal{B}.\mathcal{A}))$$

cofib seq for split exact seq of "spectral cats"

$$\mathrm{rep}(\mathcal{D}.\mathcal{A}) \longrightarrow \mathrm{rep}(\mathcal{C}.\mathcal{A}) \longrightarrow \mathrm{rep}(\mathcal{B}.\mathcal{A}).$$

The apply the Waldhausen fibration theorem to conclude the proof

3° Spectral categories and Yoneda Lemma:

Def $\mathcal{A}$ spectral category if it is enriched in the cat of symmetric spectra. Specifically. it is given by

1) a class of obj

2) $\forall (x.y) \in \mathcal{A}$. a sym. spectrum $\mathcal{A}(x.y)$

3) $\forall (x.y.z) \in \mathcal{A}$. a composition in sym. mon. simplicial model cat of sym. spectra (denoted by SymSp)

$$\mathcal{A}(y.z) \wedge \mathcal{A}(x.y) \longrightarrow \mathcal{A}(x.z)$$

satisfies assoc.

4) $\forall x \in \mathcal{A}$. a morphism in SymSp

$$\mathbb{S} \longrightarrow \mathcal{A}(x.x)$$

unit in SymSp.

Let $\hat{\mathcal{A}} = \mathrm{Fun}^{\mathrm{spec}}(\mathcal{A}^{op}, \mathrm{SymSp})$. $\mathrm{Fun}^{\mathrm{spec}}$ = spectral functor

$\hat{\mathcal{A}}$ called the spectral cat of $\mathcal{A}$-mod.

Yoneda embedding:

$$\mathcal{A} \longrightarrow \hat{\mathcal{A}}$$

$$z \longmapsto \mathcal{A}(-.z)$$

Spectral Yoneda embedding: $\mathcal{C} \in \mathrm{Cat}_\infty^{st}$

$$\mathcal{C} \simeq \mathrm{Sp}(\mathcal{C}_*) \longrightarrow \mathrm{Sp}(\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}_\infty)_*) \simeq \mathrm{Fun}(\mathcal{C}^{op}. \mathrm{Sp}).$$

Non-connective case is much harder. I will skip.