The algebraic K-theory of finite fields

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November, 2022

Abstract

This note mainly comes from the article [1] of D. Quillen, in which he uses the Adams operation to calculate the algebraic K-theory groups of finite fields.

Contents

1 Introduction

Let's first recall the definition of D. Quillen's higher algebraic K-theory groups.

Definition 1.1. (Algebraic K-theory groups) Given a unital ring R, not necessarily commutative, the commutator of the general linear group $GL(R)$ (abbr. GLR) is the elementary matrix group $E(R)$ (abbr. ER), let BG denotes the classifying space of a group G. There exists a universal object $BGL(R)^+_{E(R)}$ (abbr. $BGLR^+$) (Quillen's plus construction) satisfying that

$$\pi_1(BGLR^+) = GLR/ER$$
$$H_*(BGLR^+;A) = H_*(BGLR;A) \text{ for any coefficient } A$$

Then the i-th algebraic K-theory group of R is defined as: $K_i(R) := \pi_i(BLGR^+)$.

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This definition was given by Daniel Quillen and he gave the first non-trivial result of algebraic K-theory groups, namely those of finite fields.

For notation, we denote the finite field of q elements as $k = \mathbb{F}_q$, where p is the character of the field, $q = p^d$. Given a prime l such that $l \neq p$, we have a minimal number r satisfying that $q^r \equiv 1 \pmod{l}$.

In [1], Quillen constructed a homotopy equivalent space $F\Psi^q$ for computation. The final result is :

Theorem 5.2. *For any $i > 0$,*

$$K_{2i}(\mathbb{F}_q) = 0;$$
$$K_{2i-1}(\mathbb{F}_q) = \mathbb{Z}/(q^i - 1)$$

We need the following important theorems to get the final result.

Theorem 3.4. *We have an algebra isomorphism*

$$H_*(F\Psi^q; \mathbb{Z}/l) \cong P[\xi_1, \xi_2, ...] \bigotimes \Lambda[\eta_1, \eta_2, ...]$$

where deg $\xi_j = 2jr$, deg $\eta_j = 2jr - 1$ for $j \geq 1$, and P, Λ denote respectively the polynomial ring and the exterior algebra.

Proof. By the Eilenberg-Moore spectral sequence and techniques of homological algebra. □

Theorem 1.2. *We have an algebra isomorphism*

$$H_*(GLk; \mathbb{Z}/l) \cong P[\hat{\xi_1}, \hat{\xi_2}, ...] \otimes \Lambda[\hat{\eta_1}, \hat{\eta_2}, ...]$$

where deg $\hat{\xi_j} = 2jr$, deg $\hat{\eta_j} = 2jr - 1$.

Proof. Mostly, by detection of cyclic groups. □

Theorem 5.1. *We have a homotopy equivalence $\tau^+ : BLGk^+ \longrightarrow F\Psi^q$.*

Proof. By Theorem 1.3 and 1.4, the mod l homology algebras of $BGLk^+$ and $F\Psi^q$ are isomorphic. Besides, we can prove that their rational and mod p homology algebra are trivial. Thus by universal coefficient theorem, their integral homology algebras are also isomorphic. Since $BGLk^+$ and $F\Psi^q$ are simple, according to the generalized Whitehead theorem ([3]), they are homotopy equivalent. □

Theorem 2.4. *For any $i > 0$,*

$$K_{2i}(F\Psi^q) = 0;$$
$$K_{2i-1}(F\Psi^q) = \mathbb{Z}/(q^i - 1)$$

Proof. By definition and the homotopy exact sequence. □

Thus we have the Theorem 5.2.

Quillen left an unsolved question in this paper, namely the determination of $H_*(GL_n k; \mathbb{Z}/p)$, which is yet unknown today. We shall talk about it in the last section.

2 The space $F\Psi^q$

In [2], J.F. Adams constructed a series of operations $\Psi^q : \tilde{K}(X) \longrightarrow \tilde{K}(X)$, known as the q-th Adams operation, where $\tilde{K}(X) := [X, BU]$ is the topological K-theory group. The following lemma could be found in [2].

Lemma 2.1. *The Adams oprations*

$$\Psi^q : \tilde{K}(S^{2i}) \to \tilde{K}(S^{2i})$$

are given by

$$\Psi^q(\iota) = q^i \iota.$$

We have that $\tilde{K}(S^{2i}) \cong \pi_{2i}(BU)$, which will be important in computing $F\Psi^q$'s homotopy groups (Theorem 2.4).

By Yoneda lemma, we have a bijection $Nat(Hom(-, A), G) \to G(A)$. Take G as $\tilde{K}$ and A as $(BU)^n$, we have a bijection $Nat((\tilde{K})^n, \tilde{K}) \to [(BU)^n, BU]$. Thus we have the following maps: $\sigma : BU \to BU$ representing Ψ^q and $d : BU \times BU \to BU$ representing subtraction on $\tilde{K}$.

Definition 2.2. The space $F\Psi^q$ is defined as the fibre product of (id, σ) and the diagonal mapping Δ, namely

$$\begin{tikzcd} F\Psi^q \arrow[r, "\gamma"] \arrow[d, "\phi"] & BU^I \arrow[d, "\Delta"] \\ BU \arrow[r, "{(id,\sigma)}"] & BU \times BU \end{tikzcd}$$

Here $\Delta : p \in BU^I \mapsto (p(0), p(1))$, so explicitly $F\Psi^q = \{(x, p) | x \in BU, p \in BU^I, p(0) = x, p(1) = \sigma(x)\}$, $\phi : (x, p) \in F\Psi^q \mapsto x$.

Intuitiveyly speaking, $F\Psi^q$ is the homotopy-theoretical fixpoint set of Ψ^q, which explains the choice of its notation F.

Lemma 2.3. *The space $F\Psi^q$ is homotopy equivalent to the fibre of the map $d \circ (id, \sigma)$: $BU \to BU$ representing $1 - \Psi^q$ on $\tilde{K}$.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd} F\Psi^q \arrow[r, "\gamma"] \arrow[d, "\phi"] & BU^I \arrow[r, "m"] \arrow[d, "\Delta"] & BU^I \times \{b\} \arrow[d, "n"] \\ BU \arrow[r, "{(id,\sigma)}"] & BU \times BU \arrow[r, "d"] & BU \end{tikzcd}$$

where $m(p)(t) = d(p(t), p(1))$, $n(p) = p(0)$.

The vertical mappings are fibrations and have the same fibre ΩBU. We define the fibre product of n and $d \circ (id, \sigma)$ as F, and construct a mapping $f : F\Psi^q \to F, (x, p) \mapsto (x, m(p))$. By the commutativity of the above diagram, f is well-defined and naturally continuous. Since Δ and n have the same fibre, f is bijective and thus a homotopy equivalence. Therefore $F\Psi^q$ is homotopy equivalent to the fibre of $d \circ (id, \sigma)$.

□

Theorem 2.4. *For any $i > 0$,*

$$\pi_{2i}(F\Psi^q) = 0;$$
$$\pi_{2i-1}(F\Psi^q) = \mathbb{Z}/(q^i - 1)$$

Proof. Using lemma 2.3, we have a long exact sequence:

$$\dots \longrightarrow \pi_j(BU) \xrightarrow{1-\Psi^q} \pi_j(BU) \xrightarrow{\partial} \pi_{j-1}(F\Psi^q) \longrightarrow \dots$$

By Bott periodicity, $\pi_{2i-1}(BU) = 0$ and $\pi_{2i}(BU) = \tilde{K}(S^{2i}) = \mathbb{Z}$ with $1 - \Psi^q$ acting by multiplying by $1 - q^i$, whence the formulae for $\pi_*(F\Psi^q)$. □

Besides, since $\pi_1(BU)$ acts trivially on $\pi_*(F\Psi^q)$, so is $\pi_1(F\Psi^q)$. Thus $F\Psi^q$ is simple.

Lemma 2.5. *If $[X, U] = 0$, then*

$$\phi_* : [X, F\Psi^q] \longrightarrow [X, BU]^{\Psi^q}$$

is an isomorphism.

Proof. Using lemma 2.3, we have a long exact sequence:

$$\dots \longrightarrow [X, BU] \xrightarrow{1-\Psi^q} [X, BU] \xrightarrow{\partial} [X, F\Psi^q] \longrightarrow \dots$$

□

Lemma 2.6. *Given a finite group G, $[BG, U] = 0$.*

Proof. In [4], when G is finite, we have a homeomorphism

$$\widehat{R(G)} \cong K^*(BG),$$

where the former has the $I(G)$-adic topology and the latter has the filtration topology.

Since the image is totally in $K^0(BG)$, $K^1(BG) = [BG, U] = 0$. □

Given this lemma and the natural ntations of G to complex vector bundles, we have a mapping $\# : R(G)^{\Psi^q} \longrightarrow [BG, F\Psi^q], \alpha \mapsto \alpha^{\#}$.

3 The algebras $H^*(F\Psi^q)$ and $H_*(F\Psi^q)$

Lemma 3.1. *The rational and mod p homology groups of $F\Psi^q$ are trivial, i.e.*

$$H_*(F\Psi^q;\mathbb{Q}) = 0;$$
$$H_*(F\Psi^q;\mathbb{Z}/p) = 0.$$

Proof. We recall that a class $\mathcal{C}$ is called Serre class if for every exact sequence $0 \to A' \to A \to A'' \to 0$, $A \in \mathcal{C}$ if and only if $A', A'' \in \mathcal{C}$.

The generalized Hurewicz theorem (or Hurewicz theorem mod $\mathcal{C}$) says that if X is simple and for $i < n$, $\pi_i(X) \in \mathcal{C}$, then $H_i(X;\mathbb{Z}) \in \mathcal{C}$.([5]). The class $\mathcal{C}_f$ of abelian finite groups and the class $\mathcal{C}_p$ of abelian groups whose orders are prime to p are Serre classes ([5]).

Since the homotopy groups of $F\Psi^q$ are finite and of orders prime to p, so are its integral homology groups. By the universal coefficient theorem, the rational and mod p homology groups of $F\Psi^q$ are trivial.

□

Thus we are interested in the mod l cohomology and homology of $F\Psi^q$, where $l \neq p$.

Lemma 3.2. *We have a ring isomorphism*

$$H^*(F\Psi^q;\mathbb{Z}/l) \cong P[c_r, c_{2r}, ...] \bigotimes \Lambda[e_r, e_{2r}, ...]$$

where deg $c_{jr} = 2jr$, deg $e_{jr} = 2jr-1$ for $j \geq 1$, and P, Λ denote respectively the polynomial ring and the exterior algebra.

Proof. References to [1].

□

Theorem 3.3. *We have an algebra isomorphism*

$$H^*(F\Psi^q;\mathbb{Z}/l) \cong P[c_r, c_{2r}, ...] \bigotimes \Lambda[e_r, e_{2r}, ...]$$

where deg $c_{jr} = 2jr$, deg $e_{jr} = 2jr - 1$ for $j \geq 1$.

Proof. Denote C as $k(\mu_l)$, where μ_l is the group of l-th roots of unity in $\bar{k}$. $[k(\mu_l) : k] = r$,thus as a group, $C \cong \mathbb{Z}/(q^r - 1)$.

Consider a reprensentation $\zeta : C \to GL_1(\mathbb{C}) = \mathbb{C}^*, 1 \mapsto e^{(2\pi i/(q^r-1))}$ and $W = \zeta \oplus \zeta^q \oplus ... \oplus \zeta^{q^{r-1}}$, then $\Psi^q W = W$, which gives a mapping $W^\# : BC \longrightarrow F\Psi^q$.

Let $W_i (1 \leq i \leq m)$ be the copies of W and $T_m = \bigoplus W_i$. Then T_m is a representation of C^m and we have a mapping $T_m^\# : BC^m \longrightarrow F\Psi^q$.

The group C being cyclic, one knows that (by Lens space,[6])

$$H^*(BC) = \begin{cases} P[u] \otimes \Lambda[v], if\ l \neq 2\ or\ l = 2\ and\ q \equiv 1\ (mod 4) \\ P[v]\ with\ u = v^2, if\ l = 2\ and\ q \equiv 3\ (mod 4) \end{cases} \tag{3.1}$$

$T_m^\#$ induces a homomorphism $(T_m^\#)^* : H^*(F\Psi^q) \longrightarrow H^*(BC^m)$, which maps c_{jr}, e_{jr} to $\bar{c_{jr}}, \bar{e_{jr}}$ respectively. We can prove that ([1],Proposition 1)

$$(W^\#)^*(c_i) = \begin{cases} (-1)^{r-1}u^r = x_i, & if\ r|i \\ 0 & else \end{cases} \tag{3.2}$$

$$(W^{\#})^*(e_{jr}) = \begin{cases} (-1)^{r-1}u^{r-1}v = y_j, & if\ j = 1 \\ 0 & else \end{cases} \tag{3.3}$$

Moreove $\bar{c_{jr}} = \Sigma x_{i_1}...x_{i_j}$, $\bar{e_{jr}} = \Sigma x_{i_1}...\hat{x_{i_k}}...x_{i_j}y_{i_k}$([1], Lemma 8). Denote $y_{i_k} = dx_{i_k}$, then $\bar{c_{jr}} = \sigma_j$, $\bar{e_{jr}} = d\sigma_j$. We have an injection (By de Rham complexes, references to [1], Lemma 9)

$$P[\bar{c_{jr}}, ...] \bigotimes \Lambda[\bar{e_{jr}}, ...] \longrightarrow P[u_1, ...] \bigotimes \Lambda[v_1, ...]$$

which deduces that $\bar{c_r}^{\alpha_1}...\bar{c_{mr}}^{\alpha_m}\bar{e_r}^{\beta_1}...\bar{e_{mr}}^{\beta_m}$, $0 \leq \alpha_i, 0 \leq \beta_j \leq 1$, are linearly independent. With m tending to infinity, we have that $\bar{c_r}^{\alpha_1}\bar{c_{2r}}^{\alpha_2}...\bar{e_r}^{\beta_1}\bar{e_{2r}}^{\beta_2}...$, $0 \leq \alpha_i, 0 \leq \beta_j \leq 1, \Sigma\alpha_i + \beta_j = m$, constitute a basis of degree m of $H^*(F\Psi^q)$, whence the algebra isomorphism. □

Theorem 3.4. *We have an algebra isomorphism*

$$H_*(F\Psi^q; \mathbb{Z}/l) \cong P[\xi_r, \xi_{2r}, ...] \bigotimes \Lambda[\eta_r, \eta_{2r}, ...]$$

where deg $\xi_{jr} = 2jr$, *deg* $\eta_{jr} = 2jr - 1$ *for* $j \geq 1$.

Proof. By duality (references to [1], Theorem 2). The notations come from that

$$\begin{gathered} \xi'_j \in H_{2jr}(BC), \eta'_j \in H_{2jr-1}(BC) \text{ for } j \geq 1, \\ < \xi'_j, x^j >=< \eta'_j, x^{j-1}y >= 1. \\ \xi_j = (T^{\#})_*(\xi'_j) \in H_{2jr}(F\Psi^q), \\ \eta_j = (T^{\#})_*(\eta'_j) \in H_{2jr-1}(F\Psi^q); \end{gathered}$$

□

4 $H_*(GL_n\mathbb{F}_q; \mathbb{Z}/l)$

Our goal is to establish a chain of compatible mappings:

$$H_*(C^m)_{\Sigma_m \rtimes \pi^m} \xrightarrow{surjection} H_*(GL_n\mathbb{F}_q) \xrightarrow{\tau_{n*}} H_*(F\Psi^q)$$

We need the following important lemma.

Lemma 4.1. *Let* H *be a Sylow* p*-subgroup of* G*, where* $p|\ |G|$ *is a prime. Then the inclusion of* H *induces an injection:*

$$i^* : H^*(G; \mathbb{Z}/p) \longrightarrow H^*(H; \mathbb{Z}/p)$$

Proof. Suppose that $[G : H] = k$, $|G| = p^\alpha k, (p, k) = 1$ and the representors of cosets are $\{g_1, ...g_k\}$.

Define $\tau : H^*(H) \longrightarrow H^*(G)$, i.e. $Hom_H(F, \mathbb{Z}/p) \longrightarrow Hom_G(F, \mathbb{Z}/p)$, where F is a term of the injective resolution of $\mathbb{Z}$ (The definition of group cohomology), satisfying that $(\tau f)(c) = \Sigma g_i f(g_i^{-1}c)$. One can verify that τ is a group homomorphism and τf is a $\mathbb{Z}[G]$-morphism.

Moreover, $i^*\tau f(c) = \Sigma g_i(i^*f)(g_i^{-1}c) = \Sigma g_i f(g_i^{-1}c) = kf(c)$, thus $i^*\tau = k\ id$. Since $k = [G : H]$ is prime to p, $i^*\tau$ is the identity map when mod p. Thus i^* is injective.

□

By duality, we have a surjection $i_* : H_*(G; \mathbb{Z}/p) \longrightarrow H_*(H; \mathbb{Z}/p)$.

Corollary 4.2. *If $H < G$ satisfies that $[G : H]$ is prime to p, then we have an injection*

$$i^* : H^*(G; \mathbb{Z}/p) \longrightarrow H^*(H; \mathbb{Z}/p)$$

An n-dimensional vector space over $k(\mu_l)$ could be thought as an nr-dimensional vector space over k, thus we have a mapping $GL(n, k(\mu_l)) \longrightarrow GL(nr, k)$. When $n = 1$, this becomes $C = GL(1, k(\mu_l)) \longrightarrow GL(r, k)$. Suppose that $n = mr + e, 0 \leq e < r$, we have a mapping $C^m \longrightarrow GL(mr, k) \longrightarrow GL(n, k)$, where the latter is the direct sum of the inclusion and trivial representations. Thus C^m can be thought as a subgroup of $GL(n, k)$. Further, the symmetric group Σ_m acts naturally on it, which gives a mapping $\Sigma_m \rtimes C^m \longrightarrow GL_n(k)$.

Lemma 4.3. *$[GL_n(k) : \Sigma_m \rtimes C^m]$ is prime to l.*

Proof. $|GL_n(k)| = q^{n(n-1)/2} \prod_{i=1}^{n} (q^i - 1)$, $|\Sigma_m \rtimes C^m| = \prod_{j=1}^{m} j(q^j - 1)$. Since $v_l(q^{jr} - 1) = v_l(q^r - 1) + v_l(j)$, we have that $v_l([GL_n(k) : \Sigma_m \rtimes C^m]) = 0$. □

By Lemma 4.3 and Corollary 4.2, we have a surjection $H_*(C^m)_{\Sigma_m \rtimes \pi^m} \longrightarrow H_*(GL_n\mathbb{F}_q)$. By Künneth formula, $H_*(C^m)_{\Sigma_m \rtimes \pi^m} \cong ((H_*(C)_\pi)^{\otimes m})_{\Sigma_m}$, whose basis is the set $(\xi_0')^{\otimes \alpha_0} \otimes ...(\eta_1')^{\otimes \beta_1} \otimes ...$, where $0 \leq \alpha_i, \beta_i, \Sigma(\alpha_i + \beta_i) = m$ and deg $\xi_j' = 2jr$, deg $\eta_j' = 2jr - 1$. Thus we have elements in $H_*(GL_n\mathbb{F}_q)$: $\varepsilon^\alpha \xi_1^{\alpha_1} ... \eta_1^{\beta_1} ...$, where ε is the generator of $H_0(GL_1 k)$, $\xi_0 = \varepsilon^r$, $\alpha = e + r\alpha_0$, $\alpha + \Sigma(\alpha_i + \beta_i) = n$.

To get a mapping from $GL_n k$ to $F\Psi^q$, we need Brauer lifting $\hat{\rho} : R_{\bar{k}}(G) \longrightarrow R_{\mathbb{C}}(G)$ that the character of $\hat{\rho}(E)$ satisfies $\chi_E(g) = \Sigma \rho(\lambda_i)$, where λ_i are the eigenvalues of $E(g)$ and $\rho : \bar{k}^* \longrightarrow \mathbb{C}^*$ is an embedding ([7]).

Thus for $id \in R_{\bar{k}}(GL_n k)$, we have $\hat{\rho}(id) \in R_{\mathbb{C}}(GL_n k)$. Set $(\hat{\rho}(id))^\# = \tau_n$, we can prove that W is the Brauer lift of L(References to [1]), which states a chain of compatible morphisms:

$$H_*(C^m)_{\Sigma_m \rtimes \pi^m} \xrightarrow{surjection} H_*(GL_n\mathbb{F}_q) \xrightarrow{\tau_{n*}} H_*(F\Psi^q)$$

τ takes $\hat{\xi_j}, \hat{\eta_j}$ to ξ_j, η_j respectively, which proves that $\varepsilon^\alpha \xi_1^{\alpha_1} ... \eta_1^{\beta_1} ...$ of $H_*(GL_n\mathbb{F}_q)$ are linear independent. Moreover, by its linearly independence and the first mapping being surjective, they constitute a basis. Thus we have the following theorem.

Theorem 4.4. *We have an algebra isomorphism*

$$\bigoplus_n H_*(GL_n k; \mathbb{Z}/l) \cong P[\varepsilon, \hat{\xi_1}, \hat{\xi_2}, ...] \otimes \Lambda[\hat{\eta_1}, \hat{\eta_2}, ...]$$

where deg $\hat{\xi_j} = 2jr$, deg $\hat{\eta_j} = 2jr - 1$.

5 The Algebraic K-Theory

Since $GLk = \varinjlim GL_n k$, there exist a mapping $\tau : Glk \longrightarrow F\Psi^q$ which is compatible to τ_n. Thus we have a chain of compatible mappings:

$$\bigoplus_n H_*(GL_n k; \mathbb{Z}/l) \xrightarrow{i_*} H_*(GLk; \mathbb{Z}/l) \xrightarrow{\tau_*} H_*(F\Psi^q; \mathbb{Z}/l)$$

and $ker\ i_* \circ \tau_*$ is the ideal generated by $\varepsilon - 1$. Further, since $H_*(GLk) = \varinjlim H_*(GL_n k)$, $H_*(GL_n k) \cong P[\hat{\xi_1}, ...\hat{\xi_n}] \otimes \Lambda[\hat{\eta_1}, ...\hat{\eta_n}]$, so is $ker\ i_*$. Thus j is an isomorphism.

For mod p situation, we can prove that $\forall 0 < i < d(p-1)$, $H^i(GL_n k) = 0$ (By the Sylow -p subgroup, i.e. upper triangular matrix group U of $GL_n k$, [1]), which could further prove that $H_*(GL_n k; \mathbb{Z}/p) = 0$.

Since $GL_n k$ are finite groups and GLk is the limit of them, by the Tor definition of group cohomology , $H_*(GLk; \mathbb{Q})$ are trivial.

The rational and mod p cohomology of $F\Psi^q$ are also trivial, thus by universal coefficient theorem, the integral cohomology and homology of $F\Psi^q$ and GLk are isomorphic.

$\pi_1(F\Psi^q)$ being abelian, by the universal property of Quillen's plus construction, there exists a mapping $\tau^+ : BGLk^+ \longrightarrow F\Psi^q$ which is compatible with τ. $BGLk^+$ and $F\Psi^q$ being simple ([8]), and theire integral homology groups being isomorphic, by the generalized Whitehead theorem ([3]), they are homotopy equivalent.

Theorem 5.1. *We have a homotopy equivalence $\tau^+ : BLGk^+ \longrightarrow F\Psi^q$.*

Thus by Theorem 5.1 and Theorem 2.4, we have the algebraic theory groups of finite fields:

Theorem 5.2. *For any $i > 0$,*

$$K_{2i}(\mathbb{F}_q) = 0;$$
$$K_{2i-1}(\mathbb{F}_q) = \mathbb{Z}/(q^i - 1)$$

Moreover, Quillen left the unstable situation unsolved, i.e. the determination of mod p cohomology of $GL_n k$. The problem is yet unknown today but has some progress. In [9], Friedlander and Parshall proved that $\forall 0 < i < d(2p-3)$, $H^i(GL_n k) = 0$. In [10], Maazen proved that $\forall 0 < i < n/2$, $H^i(GL_n k) = 0$. In [11], Milgram and Priddy proved the existence of a non-trivial term in high dimensions. In [12], Sprehn proved that when $i = d(2p-3)$ and the Coxeter number $< p$, $dim\ H^i(GL_n k; \mathbb{Z}/p) = 1$.

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