## Lecture 6: Interlude on Model Category Theory

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So far, in our lecture series, we have seen an  $\infty$ -categorical treatment of the motivic world. In this lecture, we will go a bit classical to see a model theoretic perspective that builds the stable motivic homotopy category classically.

## 1 Model Category

To do this, we will talk the concept of model categories. A **model category** is some sort of an abstraction of homotopy theory in the classical sense, that allows us to talk about fibrations, cofibrations, and weak equivalences in a very general sense.

**Definition 1.1.** Let  $\mathcal{C}$  be a 1-category. Given  $f:A\to B$  and  $g:X\to Y$  in  $\mathcal{C}$ , we say that f has the **left lifting property** w.r.t to g (or equivalently g has the **right lifting property** w.r.t to f) if for every commutative diagram below

$$\begin{array}{ccc}
A & \longrightarrow & X \\
f \downarrow & & \downarrow g \\
B & \longrightarrow & Y
\end{array}$$

there exists  $h: B \to X$  making the diagram commute.

**Remark 1.2.** This choice of h is generally speaking not unique.

**Definition 1.3.** Given  $f: X \to X'$  and  $g: Y \to Y'$  in  $\mathcal{C}$ , we say that f is a **retract of** g if there is a commutative diagram

$$X \xrightarrow{id} X$$

$$f \downarrow \qquad g \downarrow \qquad \downarrow f$$

$$X' \xrightarrow{id} X'$$

**Definition 1.4.** A **model category** is a category C together with classes of morphisms W, C, F (called weak equivalences, cofibrations, and fibrations respectively), satisfying the following axioms.

- C is complete and co-complete.
- All of W, C, F are closed under retracts.

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- If  $f \in \mathcal{C}$  and  $g \in F \cap W$  (these are called **acyclic fibrations**), then f has the LLP with respect to g.
- If  $f \in \mathcal{C} \cap W$  (there are called **acyclic cofibrations**) and  $g \in F$ , then f has the LLP with respect to g.
- Any morphism can be factorized as  $g \circ f$  for  $f \in C$  and  $g \in F \cap W$ .
- Any morphism can also be factorized as  $g \circ f$  where  $f \in C \cap W$  and  $g \in F$ .
- W satisfies the 2-out-of-3 property that is if 2 out of the 3 morphisms  $f, g, f \circ g$  are in W, then the third one is in W.

We denote elements of W by  $\xrightarrow{\sim}$ , elements of C by  $\hookrightarrow$ , and elements of F by  $\rightarrow$ .

Exercise 1.5. The cofibrations are precisely the maps having the LLP with respect to all acyclic fibrations.

Now we discuss in some sense the fundamental example of a model category.

**Example 1.6.** The category Top is a model category with:

- W being the weak homotopy equivalences.
- F being the Serre fibrations.
- C being the retracts of inclusion maps of the form  $X \hookrightarrow X \cup \{\text{cells}\}$ .

It turns out this is not the only model category structure on Top - Top also admits one with:

- W being the homotopy equivalences.
- F being the Hurewicz fibrations.
- C being the closed Hurewicz fibrations.

Here are some other examples.

**Example 1.7.** The category of simplicial sets sSet is a model category with:

- W being maps whose induced map between geometric realizations are weak equivalences.
- F are Kan fibrations.
- C being categorical monomorphisms (ie. levelwise injections).

This model structure has a name and is called the **Quillen model structure**.

**Definition 1.8.** Let  $\mathcal{C}$  be a model category, an object  $X \in \mathrm{Obj}(\mathcal{C})$  is **fibrant** if the unique map  $X \to *$  (the terminal object) is a fibration. Dually, we say X is **cofibrant** if the unique map  $0 \to X$  (0 being the initial object) is a cofibration.

Given  $Y \in \mathrm{Obj}(\mathcal{C})$ , a **fibrant replacement** is a fibrant object X together with a weak equivalence  $Y \xrightarrow{\sim} X$ . Similarly, we can define a **cofibrant replacements**.

**Proposition 1.9.** A fibrant replacement (resp. cofibrant replacement) always exists.

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*Proof.* We can axiomatically factorize a map  $Y \to *$  as  $Y \xrightarrow{\sim} X \to *$  where  $X \to *$  is fibrant. We can do this similarly with cofibrations.

We write RY for the fibrant replacement of Y and QY for the cofibrant replacement of Y. We also note that given a morphism f taking  $X \to Y$ , we get morphisms  $Qf : QX \to QY$  and  $Rf : RX \to RY$ .

We do this, for fibrant replacement R for example, by applying the lifting property

That being said, we also know the lifting exists, this is in general not functorial because we <u>lack uniqueness</u>. Similarly, we can also get one for cofibrant replacement Q.

**Definition 1.10.** Let  $X \in \text{obj}(\mathcal{C})$ , a **cylinder object** for X is an object, usually denoted cyl(X), such that there is a factorization

$$\begin{array}{c} X \sqcup X \xrightarrow{id \sqcup id} X \\ \downarrow & \\ \text{Cyl}(X) \end{array}$$

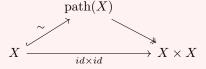
where the map  $X \sqcup X \to \operatorname{Cyl}(X)$  is a cofibration and  $\operatorname{Cyl}(X) \to X$  is an acyclic fibration.

**Remark 1.11.** Some sources do not require conditions on maps into and out of Cyl(X) - this is called a good cylinder instead.

**Definition 1.12.** If  $f, g: X \to Y$ , a **left homotopy** between f and g is a factorization of  $f \sqcup g: X \sqcup X \to Y$  through  $X \sqcup X \to \operatorname{Cyl}(X)$  (for some cylinder object).

Dual to the notion of **cylinder object**, there is also the concept of **path object**.

**Definition 1.13.** A path object is an object path(X) such that there is a factorization



where the map  $X \to \text{path}(X)$  is an acyclic cofibration and the map  $\text{path}(X) \to X \times X$  is a fibration.

**Definition 1.14.** A **right homotopy** is a factorization of  $(f,g):X\to Y\times Y$  through  $\mathrm{Path}(Y)\to Y\times Y$ .

**Remark 1.15.** The factorization axioms imply cylinder and path objects exist.

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**Proposition 1.16.** Suppose  $f, g: X \to Y$  are **left homotopic**. Given  $h: X' \to X$  and  $k: Y \to Y'$ , then  $h \triangleright f \triangleright k$  and  $h \triangleright g \triangleright k$  are left-homotopic. Here whenever we write  $a \triangleright b$ , we mean  $a \circ b$ .

*Proof.* Pick a left homotopy  $H: \mathrm{Cyl}(X) \to Y$  between f and g. Pick a cylinder object  $\mathrm{Cyk}(X')$  of X', and apply lifting to

$$X' \sqcup X' \xrightarrow{h \sqcup h} X \sqcup X \xrightarrow{} \operatorname{Cyl}(X)$$

$$\downarrow \qquad \qquad \downarrow \sim$$

$$\operatorname{Cyl}(X') \xrightarrow{} X' \xrightarrow{h} X$$

to get a map  $h': \mathrm{Cyl}(X') \to \mathrm{Cyl}(X)$ . After working out long enough, we get a commutative diagram

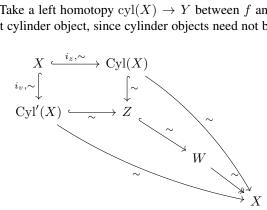
$$\begin{array}{ccc} \operatorname{Cyl}(X') & \xrightarrow{h'} & \operatorname{Cyl}(X) \\ \uparrow & & \uparrow & & \downarrow \\ X' \sqcup X' & \xrightarrow{h \sqcup h} & X \sqcup X & \xrightarrow{f \sqcup g} & Y & \xrightarrow{k} & Y' \end{array}$$

The composition  $k \circ H \circ h'$  is the desired homotopy.

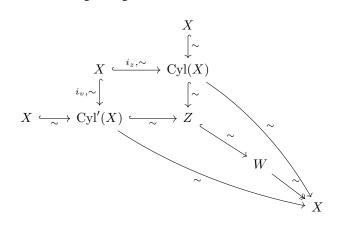
The more general notion of homotopy theory for us is not transitive, but we do have the following result.

**Proposition 1.17.** Suppose X is cofibrant and Y is fibrant, then **left homotopy** is a transitive relation on  $\mathcal{C}(X,Y)$ .

*Proof.* Take  $f, g, h: X \to Y$ . Take a left homotopy  $\operatorname{cyl}(X) \to Y$  between f and g, and consider a left homotopy  $\operatorname{cyl}'(X) \to Y$  (possibly different cylinder object, since cylinder objects need not be unique) between g and h. Let Z be the pushout



with a map induced from  $Z \to X$  factoring through W. Now consider the additional



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Some reasoning around this tells us that W is the cylinder for X. From here, solving the lift

shows us that  $W \to Y$  is a left homotopy between f and h.

Corollary 1.18. Suppose we have maps  $f_1, f_2 : X \to Y$  and  $g_1, g_2 : Y \to Z$ , such that Z is fibrant, and  $f_1, f_2$  and  $g_1, g_2$  are left-homotopic. Then  $f_1 \triangleright g_1$  is left homotopic to  $f_2 \triangleright g_2$ .

**Proposition 1.19.** Suppose that X is cofibrant. If  $f, g: X \to Y$  that are left homotopic, then they are also right homotopic.

*Proof.* Take a left homotopy  $H: \mathrm{Cyl}(X) \to X$  between f and g. Write  $q: \mathrm{Cyl}(X) \to X$  for the map factorizing  $\mathrm{id} \sqcup \mathrm{id}: X \sqcup X \to X$ . Let  $\mathrm{path}(Y)$  be a path object for Y - since X is **cofibrant**, the endpoint maps  $X \xrightarrow{\sim} \mathrm{Cyl}(X)$  are cofibrations. Thus, we can find lift to the following square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y & \xrightarrow{\sim} & \mathrm{path}(Y) \\ \sim & & & \downarrow & \\ \mathrm{Cyl}(X) & \xrightarrow{(g \times f, H)} & Y \times Y \end{array}$$

to get a map  $k : \mathrm{Cyl}(X) \to \mathrm{Path}(Y)$ . If we unwrap the constructions we have done, the map  $i_1 \triangleright k : X \to Y \times Y$  is a right homotopy between f and g.

The takeaway is that the general notion of homotopy is not that well-behaved, unless we introduce some fibrancy and cofibrancy to make them more well-behaved. If we summarize what we have done so far, we have proven the following.

**Theorem 1.20.** Let  $C_{fc}$  be the full subcategory of C composing of fibrant and cofibrant objects. In  $C_{fc}$ , left and right homotopy coincides, respect compositions, and define an equivalence relation on hom-sets.