Intro to Motivic Homotopy Theory A'-htpy
Plan: O Zariski & Nisnevich descent
2 Motivic spaces.
3 Motivic Eilenberg-MacLane spaces.
S 1 D . 7
§ 1. Descent Theory.
Assume the base scheme S is 909s & Noetherian.
$Sm_S := cot of smooth S-schemes of finite type.$
guarantee Sms is ess. Small.
grangitee sms is ess. small.
Recall A (Grothendieck) site is a cat equipped w/a
top. which is a choice of collection of families of maps
$f_i: U_i \rightarrow X_{i\in I}$ a.k.a. coverings s.t. it satisfies
1. Base change $g: X \to Y \rightsquigarrow \{U_i \times_X Y \to Y\}$
2. Local character $\{g_j: V_j \to X\}$ s.t. $\{V_j \times_X U_i \to U_i\}$
Sj } covering
3. Identity V iso φ. (φ) covering.
(Have to ask that C has pullbacks)
$\sim$ For $C = Sm_S$ , have different top:
For $C = Sm_S$ have different top:  - Zariski top $X = U_i$ fi(U <sub>i</sub> )
$X = \bigcup_{i \in \mathcal{V}} f_i(U_i)$
$\{U_i \xrightarrow{f_i} X\}$ open immersion. Jointly surjective.

-	étale	top
		ı

·· étale morphism . · · ·

- Nisnevich top

" Étale, s.t.  $\forall x \in X$ .  $\exists i$  and  $y \in Ui$  g.t.  $y \mapsto x$  induces iso on residue fields. Ox.x/mx fi Ovi.y/

FACT Zar = Nis = ét.

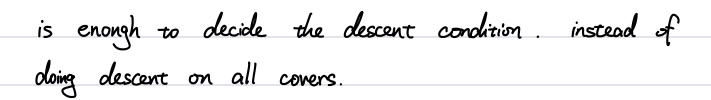
Def Consider the presheaf  $F: \operatorname{Sch}_{S}^{\operatorname{op}} \longrightarrow \mathcal{C}$ . Schs has a anothendieck top T. Then F is a T-sheaf if  $\forall$  covering  $\{f_i: U_i \longrightarrow X\} =: U$ 

 $F(X) \xrightarrow{\sim} \lim_{\Delta} F(\mathcal{U}).$ applying F to Cech nerve  $N(\mathcal{U})$ .

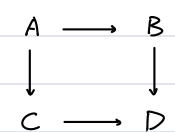
Rk If C is 1-cost, then all higher info are forgotten. So replace  $\triangle$  by  $[0] \rightarrow [1]$  . get (equalizer)  $F(X) \xrightarrow{\sim} \lim \left( \prod_{i \in B} F(U_i) \xrightarrow{\alpha} \prod_{i \in B} F(U_{ij}) \right)$ 

If C is abelian, then recover the sheaf condition  $0 \to F(X) \to \overline{\prod_i} F(U_i) \xrightarrow{\alpha \cdot \beta} \overline{\prod_{i \in j}} F(U_{ij})$ left exact.

Rk Previous def is known as the descent condition. Actually for specific top, a small collection of covers

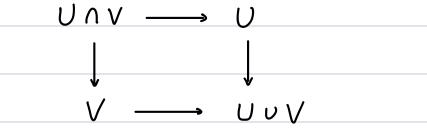


Def A col-str (completely decomposable) is a collection of comm. square in C closed under iso:



Given this, can define its top being the coarset top s.t.  $\{B \to D: C \to D\}$  is a covering for every such square.

e.g. Zariski top is gen. by distinguished Zar. square.



i open immersion. p restricts to iso  $p^{-1}(X-U) \rightarrow X-U$ .

 $\frac{\text{Def}/\text{Thm}}{\text{ID}} \text{ I)} \quad F \in PSh^{e}(Sch_{s}^{\circ p}) \quad \text{is a Zaviski sheaf}$   $\iff F(\phi) = * \text{ and sends distinguished Zar. square to}$ 

(htpy) pnllback.

2) F & PSh (Smg) is a Nis. sheaf (=>)

F( $\phi$ ) = \* and sends distinguished Nis. Sq. to (htpy) pullback.

§ 2. Motivic Spaces

Working in  $(Sm_s, Nis.)$ . Let  $F \in PSh^{e}(Sm_s)$ 

Def 1. F is A'-inv if  $\forall X \in Sms$ .  $X \times A' \to X$  includes  $F(X) \xrightarrow{\sim} F(X \times A')$ 

2. F is Strongly htpy inv if  $\forall Y \to X$  Zariski loc. trivial affine morphism w/ fibers iso to affine space, then  $F(X) \xrightarrow{\sim} F(Y)$ 

Write  $PSh_{A'}(Sm_S) = full subcoit of <math>PSh(Sm_S)$  gen. by 1.  $PSh_{he}(Sm_S) = \cdots$  2.

Strict!

Then PShhe (Sms) < PShai (Sms) < PSh(Sms)

Def The cat of motivic spaces is  $Spc(k) = Shv_{Nis}(Sm_{k}) \cap PSh_{A^{1}}(Sm_{k})$ i.e. the full subcat of  $PSh(Sm_{k})$  s.t. objs are Nis. sheaves and are  $A^{1}$ -inv.

FACT Shv<sub>Nis</sub> (Sm<sub>k</sub>) \(\int \text{PSh}\_{A'}\) (Sm<sub>k</sub>)  $= \text{Shv}_{Nis} (\text{Sm}_k) \(\int \text{PSh}_{ht} (\text{Sm}_k).$ 

In general, hard to write down objs in Spc(k). Instead, there 's a universal way to turn any presheaf on Smk into a motivic space:

 $PSh(Sm_k)$  LNis LNis LAI LA

where  $L_{Nis} = sheafification \ w.r.t. \ Nis. \ top.$   $L_{A'} = (ocalization, \ a left adjoint to inclusion$   $(L_{A'}: Shv(Sm_k) \longrightarrow Shv_{A'}(Sm_k))$   $L_{mot} \simeq L_{A'} \circ L_{Nis}.$ 

Explicitly, Lmot = colin (Lnis -> La' O Lnis -> Lnis La' Lnis

computed in presheaf cat.

Rk  $X \in Smk$  or Sms, y(X) = assoc. presheof of set by Yoneda. Then

Liniot (y(X)) = motivic space of X.

Def Pted motivic spaces  $Spc(S)_*$  is given by  $(-)_+: Spc(S) \Longrightarrow Spc(S)_*: Forget$  where  $X_+:=X \coprod S$ 

Notation ① cofiber of  $Y \rightarrow X$ :  $Y \rightarrow X \quad \text{pushout}$   $\downarrow \quad \downarrow \quad \downarrow$   $* \rightarrow X/Y$   $2 \quad X \land Y := \quad \begin{array}{c} \times \times Y / \times Y \\ \text{for } X \lor Y \quad \text{coproduct in } \text{Spc}(S)_{*} \end{array}$   $3 \quad \Sigma X := \quad X \longrightarrow * \quad \text{pushout}$   $\downarrow \quad \downarrow \quad \downarrow$   $* \longrightarrow \Sigma X$   $\text{or } \Sigma X \cong S' \land X$ 

Rk "=" in motivic spaces means  $Y \xrightarrow{f} X$ , then Lnot f equiv. e.g.  $\forall F \in PSh(Sm_S)$ .

 $F \times A^n \xrightarrow{\simeq} F$ 

e.g. 
$$P_{k}^{l} = A_{k}^{l} \cup A_{k}^{l}$$
 intersection =  $E_{m}$ .

$$E_{m} \stackrel{Z \mapsto Z}{\Longrightarrow} A_{k}^{l} \simeq *$$

$$\downarrow^{1/2} A_{k}^{l} \longrightarrow P_{k}^{l} \simeq \sum E_{m}$$

$$E_{m} = S_{pec} \ k[z^{\pm 1}] = A^{l} \setminus S_{o}S_{o}.$$

Def motivic spheres are

$$S^{1\cdot 1} := Gm . S^{1\cdot 0} := S^1$$
 $S^{p\cdot 0} := S^{p\cdot 0} \wedge (Gm)^{n} \wedge .$ 

FACT  $O$   $A^n - 503 \simeq \sum^{n-1} ((Gm)^{n})$ 
 $A^n \times Gm \longrightarrow Gm \times A^1$ 
 $A^1 \times Gm \longrightarrow A^2 - 503$ 
 $Collapse A^1 \text{ and observe}$ 
 $Gm \longrightarrow X$ 
 $Cm \longrightarrow X$ 

Def 
$$A'$$
 - htpy gps

 $X \in PSh(Sm_S)$ . Then

 $To X := Nis. sheef assoc w/$ 
 $U \in Sm_S \longmapsto To(X(U))$ 

If  $(X, x) \in PSh(Sm)_*$ . then

 $Ti(X, x) := \cdots$ 
 $U \in Sm_S \longmapsto Ti(X(U), x)$ 

We write  $A'$  - htpy gps to be:

 $Ti_o^{A'} X := Ti(Lmot X)$ 
 $Ti_o^{A'}(X, x) := Ti(Lmot X, x)$ .

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8 3. Eilenberg - MacLane Spaces
   One important example of motivic htpy thy is the Eilenberg
  MacLane space.
   \overline{Def} \forall A \in Ab_{Nis}(k) := cat of Nis. sheaves of abelian
                                    gps on X \in Sm_k.
          denote
                   K(A.n) := DK(A[n]) \in PSh(Smk)
           where DK(A[n]) = Dold - Kan construction of chain cpx
                                      w/ A concentrated at deg n.
  Rk Dold-Kan correspondence:
                Cheo (ANis(k)) = Fun( Dor. Abnis(k))
                 chain cpx -> simplicial sheaf.
       Consider (forget leveluise sheaf str)
         \operatorname{Fun}(\Delta^{\operatorname{op}}, \operatorname{Ab}_{\operatorname{Nis}}(k)) \subseteq \operatorname{Fun}(\Delta^{\operatorname{op}}, \operatorname{Fun}(\operatorname{Sm}_{k}^{\operatorname{op}}, \operatorname{Ab}))
                                    Fun ( \Delta^{\circ p} . Fun ( Sm_k^{\circ p} . Set ))
                                          Forget
                                      PSh(Smk)
       Write DK: Chro (AbNis (k)) -> PSh (Smk) be the
       composition.
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Prop 1) K(A.n) & Shunis (Smk) 2) Ti  $K(A, n) = \begin{cases} A & i=n \\ o & else \end{cases}$ 3) I not identification TTO  $Map_{ShV_{Nis}}(-, K(A, n)) \cong H_{Nis}^{n}(-, A).$ Rk K(A, n) not necess. a motivic space. Examples ?

Note  $K(A,n)(X) = Hom_{PSh(Shk)}(X, K(A,n))$  $= H_{Nis}^{n}(X.A).$ 

if A is A'-local.

X pretty worse like BGL