Special Seminar: Modern Techniques in Homotopy Theory
Summer 2025

Seminar Information

Website: To appear
Time: TBD
Classroom: Zoom
Organizers: Albert Jinghui Yang & Mattie Ji

Seminar Description

This is a reading seminar on modern techniques in homotopy theory. The goal of this seminar is to explore the connections between homotopy theory and other fields of math.

Homotopy theory, as a branch of algebraic topology, has become increasingly important and influential in modern mathematics. Recent developments in homotopy theory have demonstrated its power by solving major problems across various fields – for example, in classical topology with the solution to the Kervaire invariant problem, and in number theory through the proof approach of the Weil conjecture via $\mathbb{A}^1$-homotopy theory. These advances are paralleled by significant progress in (∞-)category theory.

Conversely, modern developments in other fields have also had a profound impact on homotopy theory. For example, Bhatt–Morrow–Scholze [1] constructed some cohomology theory on certain nice schemes valued in Breuil–Kisin modules, which are powerful tools in abstract integral p-adic Hodge theory. By comparison theorems, this construction recovers many well-behaved cohomology theories, such as crystalline cohomology. Building on this, they later provided an alternative construction via TC (topological cyclic homology) and its variants ($\mathrm{THH}, \mathrm{TP}, \mathrm{TC}^{-}$), resulting in some even better-behaved cohomology theory with desirable descent properties. These constructions also laid the groundwork for motivic filtrations on TC and its variants, analogous to the motivic filtration on algebraic K-theory via motivic cohomology. These filtrations were further refined into the *even filtrations* introduced in [2], which are more suitable for computational purposes. With the aid of computers, Antieau–Krause–Nicolaus [3] were able to compute $\mathrm{K}_i(\mathbb{Z}/p^n)$ using the remarkable Dundas–Goodwillie–McCarthy theorem, and this was one of the major breakthroughs of 2024.

The main topics of the seminar will be <u>a certain subset of</u> the following (mainly one of the three), in an expository sense:

1. **<u>Algebraic Geometry Viewpoint</u>: Motivic Homotopy Theory**
    - Basics in algebraic geometry: schemes, varieties, divisors, cohomology, categories. Functor of points view.
    - Grothendieck topology and sites. Zariski topology, Nisnevich topology, étale topology. Sheaf over sites. Nisnevich sheaves.
    - Motivic spaces. Sheafification. Milnor-Witt K-theory. Motivic spheres. Eilenberg-MacLane spaces.

- Fundamental groups of $\mathbb{P}^1$. Motivic Freudenthal suspension theorem. More motivic homotopy groups.
- Introduction to stable motivic homotopy theory. Motivic (ring) spectra. Cohomology theories. Chern classes.
- Motivic cohomology via the Voevodsky motivic complexes. A roadmap to normal residue theorem, and Lichtenbaum-Quillen conjecture.
- Algebraic cobordisms. BGL, KGL and MGL. Formal group laws. Computations.

2. **Arithmetic Geometry & Number Theory Viewpoint: Topological Cyclic Homology**

- Dundas-Goodwillie-McCarthy theorem, relations to the algebraic K-theory.
- $\mathbb{E}_\infty$-rings. Viewpoint of higher algebra. Definition(s) of topological Hochschild homology (THH), change-of-basis formula and universal property. Easy calculations and examples.
- Hopkins-Mahowald theorem, Bökstedt periodicity. Some discussions in Dyer-Lashof algebra. Baby case of the descent spectral sequence.
- S^1-action on THH, cyclotomic spectra. Definitions of TP, TC^-. Canonical and Frobenius. Definition of TC. Some spectral sequences (homotopy fixed point spectral sequences and Tate spectral sequences).
- Fontaine rings $\mathbb{A}_{\text{inf}}$. Witt vectors. Tilting. Perfectoid rings. Computations of TC of perfectoid rings with coefficients in p-adic integers.
- Quasisyntomic sites. Syntonmic cohomology. Prisms and prismatic cohomology. Nygaard filtration. Motivic filtrations for TC and its variants.
- (OPTIONAL) More descent spectral sequences and the computation of algebraic K-theory groups.

3. **Chromatic Viewpoint: Chromatic Homotopy Theory**

- Formal group laws, complex orientations, complex cobordism, Quillen's theorem, Landweber exactness.
- Construction of Morava K-theory and Morava E-theory, the stabilizer group, the chromatic tower.
- Nilpotence, periodicity, and thick-subcategory theorem.
- Hopkins-Kuhn-Ravenel character theory.
- Elliptic cohomologies and topological modular forms, connections to the higher real K-theories $\mathrm{EO}_n(G)$. Geometric applications of tmf (e.g. [4])
- The stack perspective on chromatic homotopy theory, Lubin-Tate theory.
- $\mathrm{K}(2)$-local computations, the topological and algebraic duality resolutions.

We will discuss some potential research problems for each of the viewpoints. If time permits, some of the following topics will be considered:

1. **Basics in synthetic spectra**

- *Synthetic spectra*, briefly speaking, provide a framework that packages classical spectra together with their associated Adams spectral sequences – tools directly related to computations. This approach offers a clearer and more accessible method for computing homotopy groups and understanding their elements, without getting lost in the complexities of spectral sequences. It has proven to be a highly effective way to study the homotopy groups of spheres. Recently, Lin-Wang-Xu (2024) announced a complete proof of the last Kervaire invariant problem, relying primarily on computations of stable stems via synthetic methods.
- Main references: Pstrągowski, P. (2023) *Synthetic spectra and the cellular motivic category*. Invent. math. 232, 553–681. arXiv version.

2. **Applications of Higher Algebra**

- Higher algebra, as the name suggests, is a "generalization" of classical algebra that incorporates higher notions of associativity and commutativity. The main objects of interest are spectra, which serve as the higher analogues of classical algebraic structures. For example, the sphere spectrum $\mathbb{S}$ plays a role analogous to $\mathbb{Z}$ in the new setting. The "higher" morphisms are encoded within spectra, leading to concepts such as A_∞-spectra and $\mathbb{E}_\infty$-spectra. Many famous algebraic notions can be lifted to the realm of higher algebra. For example, classical Witt vectors have higher analogues known as spherical Witt vectors. We will explore this phenomenon in detail and examine how it is used to tackle deep problems in modern mathematics. A review of background material in ∞-category theory will also be included.
- Main references: Lurie's *Higher Topos Theory* and *Higher Algebra*.

3. **Topological modular forms (tmf)**

- For many affine elliptic curves C, it is possible to associated an elliptic (co)homology theory associated to C that recovers the underlying formal group law of C. Topological modular forms (tmf) represent the vague notion of an universal elliptic cohomology over these elliptic curves. tmf can be thought of as a "higher real K-theory" has admitted many geometric applications, including detecting exotic smooth structures and classifying complex vector bundles on projective spaces.
- Main references: Topological modular forms by Douglas, Francis, Henriques, and Hill.

Prerequisites

The following knowledge are somewhat important to know:

1. **Required**: graduate-level algebraic topology, spectra, basics in algebraic K-theory.
2. **Not necessary but helpful**: ∞-categories, graduate-level algebraic geometry and algebraic number theory.

Suggested Reading

The main books or notes that is used, or at least helpful:

1. **For Motivic Homotopy Theory**

- Very nice notes by Thomas Brazelton Unstable motivic homotopy theory, at Harvard, Fall 2024.
- Fabien Morel. (2004) *An Introduction to $\mathbb{A}^1$-homotopy Theory*. In: Contemporary Developments in Algebraic K-theory. Vol. XV. ICTP Lect. Notes. Trieste: Abdus Salam Int. Cent. Theoret. Phys., pp. 357–441. Link.
- Marc Hoyois. (2015) *From algebraic cobordism to motivic cohomology*. In: J. Reine Angew.Math. 702, pp. 173–226. arXiv version.
- Notes from PCMI 2024. See here.
- Very friendly notes on Milnor-Witt K-theory, by Frédéric Déglise.
- Another good note of motivic homotopy theory, by Frédéric Déglise, PCMI 2021.

2. **For Topological Cyclic Homology**

- For a detailed story: [1].
- MIT Talbot Workshop, TX 2024.
- A. Krause, T. Nikolaus. (2019) *Bökstedt periodicity and quotients of DVRs*. preprint.
- Arbeitsgemeinschaft: Topological Cyclic Homology (2018). Notes here.
- B. Bhatt, M. Morrow, P. Scholze. (2016) *Integral p-adic Hodge Theory*. arXiv version.
- A short but fun survey paper on perfectoid space by Bhargav Bhatt. Also see this Overflow post.
- Deeper things in prismatic F-gauges: here, by Bhargav Bhatt. Only for people interested in this area.
- *The Local Structure of Algebraic K-theory* (see Springer book) by Dundas, Goodwillie, and McCarthy.

3. **For Chromatic Homotopy Theory**

- D.C. Ravenel. *Complex Cobordism and the Stable Homotopy Groups of Spheres* and *Nilpotence and Periodicity in Stable Homotopy Theory*.
- Rognes' friendly course notes on the topic.
- Hopkins' COCTALOS and Lurie's course on chromatic homotopy theory for the stack perspective.
- Strickland's expository account of algebraic geometry for chromatic homotopy theory.
- Morgan Opie's thesis on application of tmf to classify rank 3 vector bundles on $\mathbb{CP}^5$ [4].

References

[1] Bhatt, B., Morrow, M. & Scholze, P. (2019) *Topological Hochschild homology and integral p-adic Hodge theory*. Publ.math.IHES 129, 199–310. https://doi.org/10.1007/s10240-019-00106-9

[2] Hahn, J., Raksit, A. & Wilson, D. (2022) *A motivic filtration on the topological cyclic homology of commutative ring spectra*. Annals of Math. *To appear*.

[3] Antieau, B., Krause, A. & Nicolaus, T. (2024) *On the K-theory of $\mathbb{Z}/p^n$*. *preprint*. arXiv: 2405.04329.

[4] Opie, M. *A classification of complex rank 3 vector bundles on complex projective 5-space*. *preprint*. arXiv: 2301.04313.