Lecture 6: Interlude on Model Category Theory

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So far, in our lecture series, we have seen an ∞-categorical treatment of the motivic world. In this lecture, we will go a bit classical to see a model theoretic perspective that builds the stable motivic homotopy category classically.

1 Model Category

To do this, we will talk the concept of model categories. A **model category** is some sort of an abstraction of homotopy theory in the classical sense, that allows us to talk about fibrations, cofibrations, and weak equivalences in a very general sense.

Definition 1.1. Let $\mathcal{C}$ be a 1-category. Given $f : A \to B$ and $g : X \to Y$ in $\mathcal{C}$, we say that f has the **left lifting property** w.r.t to g (or equivalently g has the **right lifting property** w.r.t to f) if for every commutative diagram below

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

there exists $h : B \to X$ making the diagram commute.

Remark 1.2. This choice of h is generally speaking not unique.

Definition 1.3. Given $f : X \to X'$ and $g : Y \to Y'$ in $\mathcal{C}$, we say that f is a **retract of** g if there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "id"] & Y \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ X' \arrow[r] \arrow[rr, bend right, "id"'] & Y' \arrow[r] & X' \end{tikzcd}$$

Definition 1.4. A **model category** is a category $\mathcal{C}$ together with classes of morphisms W, C, F (called weak equivalences, cofibrations, and fibrations respectively), satisfying the following axioms.

- $\mathcal{C}$ is complete and co-complete.
- All of W, C, F are closed under retracts.

- If $f \in \mathcal{C}$ and $g \in F \cap W$ (these are called **acyclic fibrations**), then f has the LLP with respect to g.
- If $f \in \mathcal{C} \cap W$ (there are called **acyclic cofibrations**) and $g \in F$, then f has the LLP with respect to g.
- Any morphism can be factorized as $g \circ f$ for $f \in C$ and $g \in F \cap W$.
- Any morphism can also be factorized as $g \circ f$ where $f \in C \cap W$ and $g \in F$.
- W satisfies the 2-out-of-3 property - that is if 2 out of the 3 morphisms $f, g, f \circ g$ are in W, then the third one is in W.

We denote elements of W by $\xrightarrow{\sim}$, elements of C by $\hookrightarrow$, and elements of F by $\twoheadrightarrow$.

Exercise 1.5. The cofibrations are precisely the maps having the LLP with respect to all acyclic fibrations.

Now we discuss in some sense the fundamental example of a model category.

Example 1.6. The category Top is a model category with:

- W being the weak homotopy equivalences.
- F being the Serre fibrations.
- C being the retracts of inclusion maps of the form $X \hookrightarrow X \cup \{\text{cells}\}$.

It turns out this is not the only model category structure on Top - Top also admits one with:

- W being the homotopy equivalences.
- F being the Hurewicz fibrations.
- C being the closed Hurewicz fibrations.

Here are some other examples.

Example 1.7. The category of simplicial sets sSet is a model category with:

- W being maps whose induced map between geometric realizations are weak equivalences.
- F are Kan fibrations.
- C being categorical monomorphisms (ie. levelwise injections).

This model structure has a name and is called the **Quillen model structure**.

Definition 1.8. Let $\mathcal{C}$ be a model category, an object $X \in \mathrm{Obj}(\mathcal{C})$ is **fibrant** if the unique map $X \to *$ (the terminal object) is a fibration. Dually, we say X is **cofibrant** if the unique map $0 \to X$ (0 being the initial object) is a cofibration.

Given $Y \in \mathrm{Obj}(\mathcal{C})$, a **fibrant replacement** is a fibrant object X together with a weak equivalence $Y \xrightarrow{\sim} X$. Similarly, we can define a **cofibrant replacements**.

Proposition 1.9. A fibrant replacement (resp. cofibrant replacement) always exists.

Proof. We can axiomatically factorize a map $Y \to *$ as $Y \xrightarrow{\sim} X \to *$ where $X \to *$ is fibrant. We can do this similarly with cofibrations. ■

We wrtie RY for the fibrant replacement of Y and QY for the cofibrant replacement of Y. We also note that given a morphism f taking $X \to Y$, we get morphisms $Qf : QX \to QY$ and $Rf : RX \to RY$.

We do this, for fibrant replacement R for example, by applying the lifting property

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[r, "\sim"] & RY \arrow[d, two heads] \\ RX \arrow[urr, dashed, "Rf"] \arrow[rr, two heads] & & * \end{tikzcd}$$

That being said, we also know the lifting exists, this is in general not functorial because we lack uniqueness. Similarly, we can also get one for cofibrant replacement Q.

Definition 1.10. Let $X \in \text{obj}(\mathcal{C})$, a **cylinder object** for X is an object, usually denoted $\text{cyl}(X)$, such that there is a factorization

$$\begin{tikzcd} X \sqcup X \arrow[r, "id \sqcup id"] \arrow[d, hook] & X \\ \text{Cyl}(X) \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

where the map $X \sqcup X \to \text{Cyl}(X)$ is a cofibration and $\text{Cyl}(X) \to X$ is an acyclic fibration.

Remark 1.11. Some sources do not require conditions on maps into and out of $\text{Cyl}(X)$ - this is called a good cylinder instead.

Definition 1.12. If $f, g : X \to Y$, a **left homotopy** between f and g is a factorization of $f \sqcup g : X \sqcup X \to Y$ through $X \sqcup X \to \text{Cyl}(X)$ (for some cylinder object).

Dual to the notion of **cylinder object**, there is also the concept of **path object**.

Definition 1.13. A path object is an object $\text{path}(X)$ such that there is a factorization

$$\begin{tikzcd} & \text{path}(X) \arrow[dr, two heads] & \\ X \arrow[ur, hook, "\sim"] \arrow[rr, "id \times id"'] & & X \times X \end{tikzcd}$$

where the map $X \to \text{path}(X)$ is an acyclic cofibration and the map $\text{path}(X) \to X \times X$ is a fibration.

Definition 1.14. A **right homotopy** is a factorization of $(f, g) : X \to Y \times Y$ through $\text{Path}(Y) \to Y \times Y$.

Remark 1.15. The factorization axioms imply cylinder and path objects exist.

Proposition 1.16. Suppose $f, g : X \to Y$ are **left homotopic**. Given $h : X' \to X$ and $k : Y \to Y'$, then $h \triangleright f \triangleright k$ and $h \triangleright g \triangleright k$ are left-homotopic. Here whenever we write $a \triangleright b$, we mean $a \circ b$.

Proof. Pick a left homotopy $H : \mathrm{Cyl}(X) \to Y$ between f and g. Pick a cylinder object $\mathrm{Cyk}(X')$ of X', and apply lifting to

$$\begin{tikzcd}
X' \sqcup X' \arrow[r, "h \sqcup h"] \arrow[d, hook] & X \sqcup X \arrow[r] & \mathrm{Cyl}(X) \arrow[d, two heads, "\sim"] \\
\mathrm{Cyl}(X') \arrow[r] \arrow[urr, dashed, "h'"] & X' \arrow[r, "h"'] & X
\end{tikzcd}$$

to get a map $h' : \mathrm{Cyl}(X') \to \mathrm{Cyl}(X)$. After working out long enough, we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Cyl}(X') \arrow[r, "h'"] & \mathrm{Cyl}(X) \arrow[dr, "H"] & & \\
X' \sqcup X' \arrow[u] \arrow[r, "h \sqcup h"'] & X \sqcup X \arrow[u] \arrow[r, "f \sqcup g"'] & Y \arrow[r, "k"] & Y'
\end{tikzcd}$$

The composition $k \circ H \circ h'$ is the desired homotopy. ■

The more general notion of homotopy theory for us is not transitive, but we do have the following result.

Proposition 1.17. Suppose X is cofibrant and Y is fibrant, then **left homotopy** is a transitive relation on $\mathcal{C}(X, Y)$.

Proof. Take $f, g, h : X \to Y$. Take a left homotopy $\mathrm{cyl}(X) \to Y$ between f and g, and consider a left homotopy $\mathrm{cyl}'(X) \to Y$ (possibly different cylinder object, since cylinder objects need not be unique) between g and h. Let Z be the pushout

$$\begin{tikzcd}
X \arrow[r, hook, "i_z, \sim"] \arrow[d, hook, "i_v, \sim"'] & \mathrm{Cyl}(X) \arrow[d, hook, "\sim"] \arrow[ddrr, bend left, "\sim"] & & \\
\mathrm{Cyl}'(X) \arrow[r, hook, "\sim"'] \arrow[drrr, bend right, "\sim"'] & Z \arrow[dr, "\sim"] & & \\
& & W \arrow[r, two heads, "\sim"] & X
\end{tikzcd}$$

with a map induced from $Z \to X$ factoring through W. Now consider the additional

$$\begin{tikzcd}
& & X \arrow[d, hook, "\sim"] & & \\
& X \arrow[r, hook, "i_z, \sim"] \arrow[d, hook, "i_v, \sim"'] & \mathrm{Cyl}(X) \arrow[d, hook, "\sim"] \arrow[ddrr, bend left, "\sim"] & & \\
X \arrow[r, hook, "\sim"'] & \mathrm{Cyl}'(X) \arrow[r, hook, "\sim"'] \arrow[drrr, bend right, "\sim"'] & Z \arrow[dr, "\sim"] & & \\
& & & W \arrow[r, two heads, "\sim"] & X
\end{tikzcd}$$

Some reasoning around this tells us that W is the cylinder for X. From here, solving the lift

$$\begin{tikzcd} Z \arrow[r] \arrow[d, hook, "i"'] & Y \arrow[d] \\ W \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

shows us that $W \to Y$ is a left homotopy between f and h. ■

Corollary 1.18. Suppose we have maps $f_1, f_2 : X \to Y$ and $g_1, g_2 : Y \to Z$, such that Z is fibrant, and f_1, f_2 and g_1, g_2 are left-homotopic. Then $f_1 \triangleright g_1$ is left homotopic to $f_2 \triangleright g_2$.

Proposition 1.19. Suppose that X is cofibrant. If $f, g : X \to Y$ that are left homotopic, then they are also right homotopic.

Proof. Take a left homotopy $H : \mathrm{Cyl}(X) \to X$ between f and g. Write $q : \mathrm{Cyl}(X) \to X$ for the map factorizing $\mathrm{id} \sqcup \mathrm{id} : X \sqcup X \to X$. Let $\mathrm{path}(Y)$ be a path object for Y - since X is **cofibrant**, the endpoint maps $X \xrightarrow{\sim} \mathrm{Cyl}(X)$ are cofibrations. Thus, we can find lift to the following square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "\sim"'] & Y \arrow[r, hook, "\sim"] & \mathrm{path}(Y) \arrow[d, two heads] \\ \mathrm{Cyl}(X) \arrow[rr, "(g \times f, H)"'] & & Y \times Y \end{tikzcd}$$

to get a map $k : \mathrm{Cyl}(X) \to \mathrm{Path}(Y)$. If we unwrap the constructions we have done, the map $i_1 \triangleright k : X \to Y \times Y$ is a right homotopy between f and g. ■

The takeaway is that the general notion of homotopy is not that well-behaved, unless we introduce some fibrancy and cofibrancy to make them more well-behaved. If we summarize what we have done so far, we have proven the following.

Theorem 1.20. Let $\mathcal{C}_{fc}$ be the full subcategory of $\mathcal{C}$ composing of fibrant and cofibrant objects. In $\mathcal{C}_{fc}$, **left and right homotopy coincides, respect compositions, and define an equivalence relation on** $\hom$**-sets**.