Database Systems, CSCI 4380-01
Homework # 1 Practice Problem for Question 3 Version 1.02

Question 3 [30 points]. For the following relations, find and list all the keys.

1. $R1(A, B, C, D, E, F, G)$, $\mathcal{F} = \{AB \rightarrow CD, DEF \rightarrow ACG, G \rightarrow BD, D \rightarrow E\}$

Solution.

1. Relation R has seven attributes. The total number of subsets of attributes would be $2^7 - 1 = 127$ (excluding the empty set). In principle, we have to consider each from these subsets but hopefully we can take some shortcuts, so that we don't have to test every single one of them. Let's start with the full list of subsets though:

A	AB	ABC	ABCD	ABCDE	ABCDEF	ABCDEFG
B	AC	ABD	ABCE	ABCDF	ABCDEG	
C	AD	ABE	ABCF	ABCDG	ABCDFG	
D	AE	ABF	ABCG	ABCEF	ABCEFG	
E	AF	ABG	ABDE	ABCEG	ABDEFG	
F	AG	ACD	ABDF	ABCFG	ACDEFG	
G	BC	ACE	ABDG	ABDEF	BCDEFG	
	BD	ACF	ABEF	ABDEG		
	BE	ACG	ABEG	ABDFG		
	BF	ADE	ABFG	ABEFG		
	BG	ADF	ACDE	ACDEF		
	CD	ADG	ACDF	ACDEG		
	CE	AEF	ACDG	ACDFG		
	CF	AEG	ACEF	ACEFG		
	CG	AFG	ACEG	ADEFG		
	DE	BCD	ACFG	BCDEF		
	DF	BCE	ADEF	BCDEG		
	DG	BCF	ADEG	BCDFG		
	EF	BCG	ADFG	BCEFG		
	EG	BDE	AEFG	BDEFG		
	FG	BDF	BCDE	CDEFG		
		BDG	BCDF			
		BEF	BCDG			
		BEG	BCEF			
		BFG	BCEG			
		CDE	BCFG			
		CDF	BDEF			
		CDG	BDEG			
		CEF	BDFG			
		CEG	BEFG			
		CFG	CDEF			
		DEF	CDEG			
		DEG	CDFG			
		DFG	CEFG			
		EFG	DEFG			

Now, looking at the functional dependencies, we see that none of them have attribute F on the right-hand side. It means that this attribute will be included in every key. Therefore, attribute sets without this attribute cannot be keys and can be crossed out:

~~A~~	~~AB~~	~~ABC~~	~~ABCD~~	~~ABCDE~~	ABCDEF	ABCDEFG
~~B~~	~~AC~~	~~ABD~~	~~ABCE~~	ABCDF	~~ABCDEG~~	
~~C~~	~~AD~~	~~ABE~~	ABCF	~~ABCDG~~	ABCDFG	
~~D~~	~~AE~~	ABF	~~ABCG~~	ABCEF	ABCEFG	
~~E~~	AF	~~ABG~~	~~ABDE~~	~~ABCEG~~	ABDEFG	
F	~~AG~~	~~ACD~~	ABDF	ABCFG	ACDEFG	
~~G~~	~~BC~~	~~ACE~~	~~ABDG~~	ABDEF	BCDEFG	
	~~BD~~	ACF	ABEF	~~ABDEG~~		
	~~BE~~	~~ACG~~	~~ABEG~~	ABDFG		
	BF	~~ADE~~	ABFG	ABEFG		
	~~BG~~	ADF	~~ACDE~~	ACDEF		
	~~CD~~	~~ADG~~	ACDF	~~ACDEG~~		
	~~CE~~	AEF	~~ACDG~~	ACDFG		
	CF	~~AEG~~	ACEF	ACEFG		
	~~CG~~	AFG	~~ACEG~~	ADEFG		
	~~DE~~	~~BCD~~	ACFG	BCDEF		
	DF	~~BCE~~	ADEF	~~BCDEG~~		
	~~DG~~	BCF	~~ADEG~~	BCDFG		
	EF	~~BCG~~	ADFG	BCEFG		
	~~EG~~	~~BDE~~	AEFG	BDEFG		
	FG	BDF	~~BCDE~~	CDEFG		
		~~BDG~~	BCDF			
		BEF	~~BCDG~~			
		~~BEG~~	BCEF			
		BFG	~~BCEG~~			
		~~CDE~~	BCFG			
		CDF	BDEF			
		~~CDG~~	~~BDEG~~			
		CEF	BDFG			
		~~CEG~~	BEFG			
		CFG	CDEF			
		DEF	~~CDEG~~			
		~~DEG~~	CDFG			
		DFG	CEFG			
		EFG	DEFG			

Now, we only have 64 sets to test. Much easier than 127! Then let's pick a set and check if it is a key. For example, we consider $ACEF$: $\{A, C, E, F\}^+ = \{A, C, E, F\}\}$

It is not a key, so we can cross it out. More importantly, we can make an observation that any set of attributes that does not contain all the attributes from at least one of the following sets $\{A, B\}$, $\{D, E, F\}$, $\{G\}$, and $\{D\}$ would not be a key. It is due to the fact that we would not be able to apply any of the functional dependencies to add more attributes to the closure of such a set of attributes.

~~A~~	~~AB~~	~~ABC~~	~~ABCD~~	~~ABCDE~~	ABCDEF	ABCDEFG
~~B~~	~~AC~~	~~ABD~~	~~ABCE~~	ABCDF	~~ABCDEG~~	
~~C~~	~~AD~~	~~ABE~~	ABCF	~~ABCDG~~	ABCDFG	
~~D~~	~~AE~~	ABF	~~ABCG~~	ABCEF	ABCEFG	
~~E~~	~~AF~~	~~ABG~~	~~ABDE~~	~~ABCEG~~	ABDEFG	
~~F~~	~~AG~~	~~ACD~~	ABDF	ABCFG	ACDEFG	
~~G~~	~~BC~~	~~ACE~~	~~ABDG~~	ABDEF	BCDEFG	
	~~BD~~	~~ACF~~	ABEF	~~ABDEG~~		
	~~BE~~	~~ACG~~	~~ABEG~~	ABDFG		
	~~BF~~	~~ADE~~	ABFG	ABEFG		
	~~BG~~	ADF	~~ACDE~~	ACDEF		
	~~CD~~	~~ADG~~	ACDF	~~ACDEG~~		
	~~CE~~	~~AEF~~	~~ACDG~~	ACDFG		
	~~CF~~	~~AEG~~	~~ACEF~~	ACEFG		
	~~CG~~	AFG	~~ACEG~~	ADEFG		
	~~DE~~	~~BCD~~	ACFG	BCDEF		
	DF	~~BCE~~	ADEF	~~BCDEG~~		
	~~DG~~	~~BCF~~	~~ADEG~~	BCDFG		
	~~EF~~	~~BCG~~	ADFG	BCEFG		
	~~EG~~	~~BDE~~	AEFG	BDEFG		
	FG	BDF	~~BCDE~~	CDEFG		
		~~BDG~~	BCDF			
		~~BEF~~	~~BCDG~~			
		~~BEG~~	~~BCEF~~			
		BFG	~~BCEG~~			
		~~CDE~~	BCFG			
		CDF	BDEF			
		~~CDG~~	~~BDEG~~			
		~~CEF~~	BDFG			
		~~CEG~~	BEFG			
		CFG	CDEF			
		DEF	~~CDEG~~			
		~~DEG~~	CDFG			
		DFG	CEFG			
		EFG	DEFG			

We are down to just 52 sets to test. Let's pick a set $ABCDF$: $\{A,B,C,D,F\}^+ = \{A,B,C,D,F\} \cup \{E\}$ (after using $D \to E$) $\cup$ $\{A,C,G\}$ (after using $DEF \to ACG$) $= \{A,B,C,D,E,F,G\}$. So, we found at least a superkey. But we don't know whether it is minimal or not. Let's list all of the subsets of $\{A,B,C,D,F\}$ that have not been crossed out:

- $\{D,F\}$
- $\{A,B,F\}$
- $\{A,D,F\}$
- $\{B,D,F\}$
- $\{C,D,F\}$
- $\{A,B,C,F\}$
- $\{A,B,D,F\}$
- $\{A,C,D,F\}$
- $\{B,C,D,F\}$

- $\{A, B, C, D, F\}$

Let's compute the closure of $\{A, B, F\}$:
$\{A, B, F\}^+ = \{A, B, F\} \cup \{C, D\}$ (after using $AB \to CD$) $\cup$
$\{E\}$ (after using $D \to E$) $\cup$ $\{A, C, G\}$ (after using $DEF \to ACG$) $= \{A, B, C, D, E, F, G\}$.
We conclude that $\{A, B, C, D, F\}$ was not a key since it was not minimal. $\{A, B, F\}$ is a superkey. Is it a key? Yes, because all proper subsets of $\{A, B, F\}$ would contain at most two elements and would also be subsets of $\{A, B, C, D, F\}$. However, there are no single-element subsets of $\{A, B, C, D, F\}$ listed above. The only two-element subset of $\{A, B, C, D, F\}$ listed above is $\{D, F\}$ which is not a subset of $\{A, B, F\}$. It means that $\{A, B, F\}$ is a key. Since we found a key, let's mark it and cross out all of its superkeys:

~~A~~	~~AB~~	~~ABC~~	~~ABCD~~	~~ABCDE~~	~~ABCDEF~~	~~ABCDEFG~~
~~B~~	~~AC~~	~~ABD~~	~~ABCE~~	~~ABCDF~~	~~ABCDEG~~	
~~C~~	~~AD~~	~~ABE~~	~~ABCF~~	~~ABCDG~~	~~ABCDFG~~	
~~D~~	~~AE~~	ABF	~~ABCG~~	~~ABCEF~~	~~ABCEFG~~	
~~E~~	~~AF~~	~~ABG~~	~~ABDE~~	~~ABCEG~~	~~ABDEFG~~	
~~F~~	~~AG~~	~~ACD~~	~~ABDF~~	~~ABCFG~~	ACDEFG	
~~G~~	~~BC~~	~~ACE~~	~~ABDG~~	~~ABDEF~~	BCDEFG	
	~~BD~~	~~ACF~~	~~ABEF~~	~~ABDEG~~		
	~~BE~~	~~ACG~~	~~ABEG~~	~~ABDFG~~		
	~~BF~~	~~ADE~~	~~ABFG~~	~~ABEFG~~		
	~~BG~~	ADF	~~ACDE~~	ACDEF		
	~~CD~~	~~ADG~~	ACDF	~~ACDEG~~		
	~~CE~~	~~AEF~~	~~ACDG~~	ACDFG		
	~~CF~~	~~AEG~~	~~ACEF~~	ACEFG		
	~~CG~~	AFG	~~ACEG~~	ADEFG		
	~~DE~~	~~BCD~~	ACFG	BCDEF		
	DF	~~BCE~~	ADEF	~~BCDEG~~		
	~~DG~~	~~BCF~~	~~ADEG~~	BCDFG		
	~~EF~~	~~BCG~~	ADFG	BCEFG		
	~~EG~~	~~BDE~~	AEFG	BDEFG		
	FG	BDF	~~BCDE~~	CDEFG		
		~~BDG~~	BCDF			
		~~BEF~~	~~BCDG~~			
		~~BEG~~	~~BCEF~~			
		BFG	~~BCEG~~			
		~~CDE~~	BCFG			
		CDF	BDEF			
		~~CDG~~	~~BDEG~~			
		~~CEF~~	BDFG			
		~~CEG~~	BEFG			
		CFG	CDEF			
		DEF	~~CDEG~~			
		~~DEG~~	CDFG			
		DFG	CEFG			
		EFG	DEFG			

We have 36 sets left to test. Let's pick a set AFG:
$\{A, F, G\}^+ = \{A, F, G\} \cup \{B, D\}$ (after using $G \to BD$) $\cup$
$\{E\}$ (after using $D \to E$) $\cup$ $\{A, C, G\}$ (after using $DEF \to ACG$) $= \{A, B, C, D, E, F, G\}$.
So, we found at least a superkey. But we don't know whether it is minimal or not. Let's list all of the subsets of $\{A, F, G\}$ that have not been crossed out:

- $\{F, G\}$
- $\{A, F, G\}$

Let's compute the closure of $\{F, G\}$:
$\{F, G\}^+ = \{F, G\} \cup \{B, D\}$ (after using $G \to BD$) $\cup$
$\{E\}$ (after using $D \to E$) $\cup$ $\{A, C, G\}$ (after using $DEF \to ACG$) $= \{A, B, C, D, E, F, G\}$.
We conclude that $\{A, F, G\}$ was not a key since it was not minimal. $\{F, G\}$ is a key since its closure contains all attributes, and none of the proper subsets of $\{F, G\}$ remains uncrossed in our list. Since we found a key, let's mark it and cross out all of its superkeys:

~~A~~	~~AB~~	~~ABC~~	~~ABCD~~	~~ABCDE~~	~~ABCDEF~~	~~ABCDEFG~~
~~B~~	~~AC~~	~~ABD~~	~~ABCE~~	~~ABCDF~~	~~ABCDEG~~	
~~C~~	~~AD~~	~~ABE~~	~~ABCF~~	~~ABCDG~~	~~ABCDFG~~	
~~D~~	~~AE~~	ABF	~~ABCG~~	~~ABCEF~~	~~ABCEFG~~	
~~E~~	~~AF~~	~~ABG~~	~~ABDE~~	~~ABCEG~~	~~ABDEFG~~	
~~F~~	~~AG~~	~~ACD~~	~~ABDF~~	~~ABCFG~~	~~ACDEFG~~	
~~G~~	~~BC~~	~~ACE~~	~~ABDG~~	~~ABDEF~~	~~BCDEFG~~	
	~~BD~~	~~ACF~~	~~ABEF~~	~~ABDEG~~		
	~~BE~~	~~ACG~~	~~ABEG~~	~~ABDFG~~		
	~~BF~~	~~ADE~~	~~ABFG~~	~~ABEFG~~		
	~~BG~~	ADF	~~ACDE~~	ACDEF		
	~~CD~~	~~ADG~~	ACDF	~~ACDEG~~		
	~~CE~~	~~AEF~~	~~ACDG~~	~~ACDFG~~		
	~~CF~~	~~AEG~~	~~ACEF~~	~~ACEFG~~		
	~~CG~~	~~AFG~~	~~ACEG~~	~~ADEFG~~		
	~~DE~~	~~BCD~~	~~ACFG~~	BCDEF		
	DF	~~BCE~~	ADEF	~~BCDEG~~		
	~~DG~~	~~BCF~~	~~ADEG~~	~~BCDFG~~		
	~~EF~~	~~BCG~~	~~ADFG~~	~~BCEFG~~		
	~~EG~~	~~BDE~~	~~AEFG~~	~~BDEFG~~		
	FG	BDF	~~BCDE~~	~~CDEFG~~		
		~~BDG~~	BCDF			
		~~BEF~~	~~BCDG~~			
		~~BEG~~	~~BCEF~~			
		~~BFG~~	~~BCEG~~			
		~~CDE~~	~~BCFG~~			
		CDF	BDEF			
		~~CDG~~	~~BDEG~~			
		~~CEF~~	~~BDFG~~			
		~~CEG~~	~~BEFG~~			
		~~CFG~~	CDEF			
		DEF	~~CDEG~~			
		~~DEG~~	~~CDFG~~			
		~~DFG~~	~~CEFG~~			
		~~EFG~~	~~DEFG~~			

With that, we are down to only 12 sets to check. Note that all remaining sets are supersets of $\{D, F\}$. If we feel lucky, we can try to test $\{D, F\}$ and see if it is a key.

Let's compute the closure of $\{D, F\}$:
$\{D, F\}^+ = \{D, F\} \cup \{E\}$ (after using $D \to E$) $\cup$
$\{A, C, G\}$ (after using $DEF \to ACG$) $\cup$ $\{B, D\}$ (after using $G \to BD$) $= \{A, B, C, D, E, F, G\}$.
So, $\{D, F\}$ is a superkey. Since none of the proper subsets of $\{D, F\}$ remains uncrossed in our list, it is a key. Since we found a key, let's mark it and cross out all of its superkeys:

~~A~~	~~AB~~	~~ABC~~	~~ABCD~~	~~ABCDE~~	~~ABCDEF~~	~~ABCDEFG~~
~~B~~	~~AC~~	~~ABD~~	~~ABCE~~	~~ABCDF~~	~~ABCDEG~~	
~~C~~	~~AD~~	~~ABE~~	~~ABCF~~	~~ABCDG~~	~~ABCDFG~~	
~~D~~	~~AE~~	ABF	~~ABCG~~	~~ABCEF~~	~~ABCEFG~~	
~~E~~	~~AF~~	~~ABG~~	~~ABDE~~	~~ABCEG~~	~~ABDEFG~~	
~~F~~	~~AG~~	~~ACD~~	~~ABDF~~	~~ABCFG~~	~~ACDEFG~~	
~~G~~	~~BC~~	~~ACE~~	~~ABDG~~	~~ABDEF~~	~~BCDEFG~~	
	~~BD~~	~~ACF~~	~~ABEF~~	~~ABDEG~~		
	~~BE~~	~~ACG~~	~~ABEG~~	~~ABDFG~~		
	~~BF~~	~~ADE~~	~~ABFG~~	~~ABEFG~~		
	~~BG~~	~~ADF~~	~~ACDE~~	~~ACDEF~~		
	~~CD~~	~~ADG~~	~~ACDF~~	~~ACDEG~~		
	~~CE~~	~~AEF~~	~~ACDG~~	~~ACDFG~~		
	~~CF~~	~~AEG~~	~~ACEF~~	~~ACEFG~~		
	~~CG~~	~~AFG~~	~~ACEG~~	~~ADEFG~~		
	~~DE~~	~~BCD~~	~~ACFG~~	~~BCDEF~~		
	DF	~~BCE~~	~~ADEF~~	~~BCDEG~~		
	~~DG~~	~~BCF~~	~~ADEG~~	~~BCDFG~~		
	~~EF~~	~~BCG~~	~~ADFG~~	~~BCEFG~~		
	~~EG~~	~~BDE~~	~~AEFG~~	~~BDEFG~~		
	FG	~~BDF~~	~~BCDE~~	~~CDEFG~~		
		~~BDG~~	~~BCDF~~			
		~~BEF~~	~~BCDG~~			
		~~BEG~~	~~BCEF~~			
		~~BFG~~	~~BCEG~~			
		~~CDE~~	~~BCFG~~			
		~~CDF~~	~~BDEF~~			
		~~CDG~~	~~BDEG~~			
		~~CEF~~	~~BDFG~~			
		~~CEG~~	~~BEFG~~			
		~~CFG~~	~~CDEF~~			
		~~DEF~~	~~CDEG~~			
		~~DEG~~	~~CDFG~~			
		~~DFG~~	~~CEFG~~			
		~~EFG~~	~~DEFG~~			

There are no subsets of arguments that are not keys and that remain uncrossed in our list. It means that we found all keys of $R1$.

Answer: Keys of $R1$ are DF, FG, and ABF.