The time of the maximum of fractional Brownian motion

Fractional Brownian motion(fBm) is used quite commonly as a stochastic, or random, model in finance and many fields of science such as solid state physics,hydrology and geophysics,to name a few. This report documents the methodology and some of the results of an undergraduate research project, in which various aspects of the trajectories of fractional Brownian motion were analysed. In section 1 we shall define the fractional Brownian motion. Here the distribution of the time at which fractional Brownian motion achieves it maximum is explored with an exploratory data analysis approach. This report is based on more a more extensive investigation done with Patrick He under the supervision of Konstantin Borovkov.

1 Introduction

Strictly speaking, fractional Brownian motion is a family of stochastic processes, indexed by the so-called *Hurst parameter*, $H \in (0, 1]$.

Definition 1. *fractional Brownian motion of Hurst parameter H is a zero mean Gaussian process with the covariance structure*

$$E\left(B_t^H B_S^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

Some properties of fractional Brownian motion are:

1. self-similarity: for any $a \geq 0$, $B_{at}^H =^d a^H B_t^H$. Hence the term *fractional.*
2. stationary increments: for any $s \geq 0$, $(B_{t+s}^H - B_t^H) =^d B_t^H$.

However fBm, unlike many commonly used stochastic models, is neither a semi-martingale nor a Markovian process. In general the increments of fBm are dependent on each other, with non-overlapping increments being negatively correlated when $H \in (0, 1/2)$ and positively correlated when $H \in (1/2, 1)$; only when $H = 1/2$, and fBm collapses to the Wiener process, does fBm have independent increments as well as the Markov and martingale properties. It is for these reasons that analysis of fBm for a general Hurst parameter is an often impenetrable task, and exploratory simulations should be considered the first step in exploring the properties of fractional Brownian motion.

Suppose that we have an fBm process on the time interval $[0, T]$. For each realisation, the process will attain its maximum M at some time $\tau = \arg\max\{B_t^H\} \in [0, T]$. The distribution of the latter random variables(RV), a functional of fBm, is investigated. First, fBm is simulated using an 'exact method', so-called as the procedure produces a sampled realisation of fBm. Our investigation utilised the Davies-Harte method,built in-house, due to the procedure's relatively fast computational speed of $M\log_2(M)$ to produce a sample of size $M = 2^k$. The τ can then be calculated from the simulated process. Finally, we note that simulations need only be performed on the unit interval $[0, 1]$, as any results obtained here can be generalised using the self-similarity of fBm to $[0, T]$.

2 The time of attainment of the maximum

We turn our attention first to the distribution of the time of attainment of the maximum of fractional Brownian motion τ. When $H = 1/2$, the distribution of τ is the arcsine distribution, i.e a symmetric beta distribution $B(\alpha, \beta)$ with $\alpha = \beta = 1/2$. A question at the outset of our investigation is whether τ has a similar distribution for fractional Brownian motion. In figure 1, simulations show that τ has a bi-modal distribution with a similar structure to the arcsine distribution, with the probability density clustering and increasing towards both the ends points $t = 0$ and $t = 1$. Whilst τ no longer has an arcsine distribution for general H, the distribution could still be a symmetric beta distribution with $\alpha(H) = \beta(H)$. Using method of moments estimates $\hat{\alpha}$ and $\hat{\beta}$, one sees that the corresponding beta distribution visually seems very close to the empirical distribution of τ for each Hurst parameter tested, a striking feature of figure 1 being the large overlap between the histograms of the simulated τ and the distribution $B(\hat{\alpha}, \hat{\beta})$.

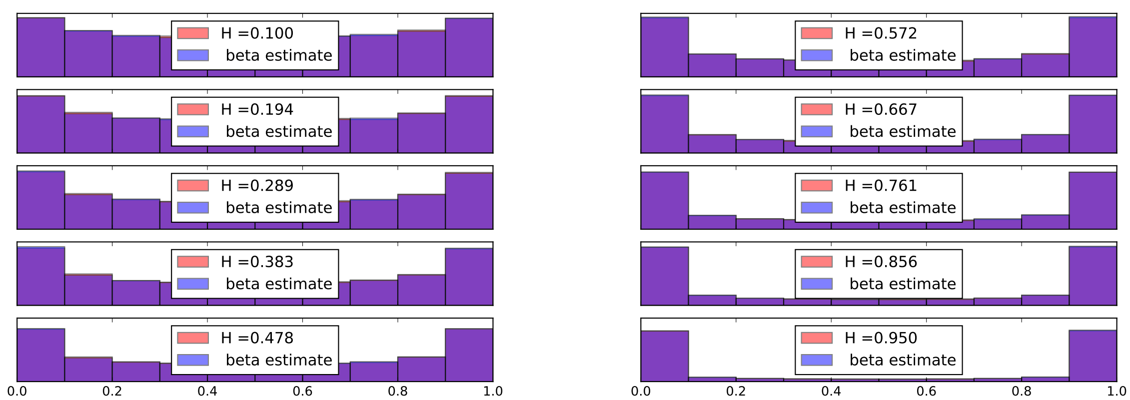

Figure 1: Simulated τ (red) and the corresponding estimated beta distribution(blue). A sample of $N = 50,000$ observations of fBm trajectories was used to produce these histograms. The number of sampled points in the fBm trajectories was $M = 2^8$.

Moreover, the estimated α and β are roughly equal, giving a symmetric beta distribution estimate. Both parameters estimates monotonically decrease as H varies. This seems sensible. As $H \to 1$, fractional Brownian motion becomes more positively correlated until it becomes a straight line with a Gaussian distributed slope when $H = 1$, in which case fBm achieves its maximum value either at $\tau = 0$ or $\tau = 1$ with equal probability. One would then expect that $\alpha \to 0^+$ and $\beta \to 0^+$ as $H \to 1^-$. Similarly, fBm approaches a white noise process (in finite dimensional distribution) as $H \to 0^+$, in which case one would expect τ to be uniformly distributed on $[0, 1]$. It should be then that $\alpha \to 1^-$ and $\beta \to 1^-$.

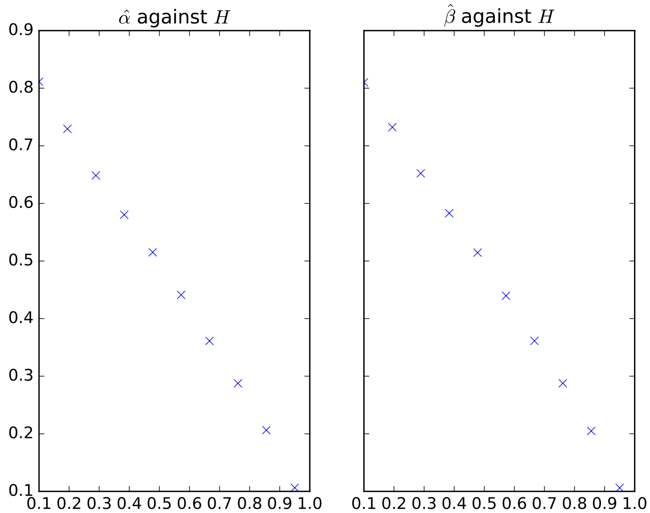

Figure 2: Estimated beta distribution parameters for various Hurst parameters, using a sample of $N = 50000$ simulated τ values.

Table 1 shows the D test statistic and p-value of a two sample Kolmogorov-Smirnov test is used to test the null-hypothesis that τ has the Beta distribution $B(\hat{\alpha}, \hat{\beta})$. The lowest p-value obtained is $p = 0.128$, and so the null hypothesis that the distribution of τ is symmetric beta cannot be rejected for any of the Hurst parameters tested here.

The results presented for τ need to be reaffirmed with a larger sample size. Due to a lack of computational resources, this result can only be reported with an fBm sampled on $M = 2^{10}$ grid points. To perform the two sample KS test, we have taken a sample size of only $N = 1000$ to avoid using significantly more samples than grid points in the KS test statistic,

$$D_{n,m} = \sup_x \left| F^1_n(x) - F^2_m(x) \right|,$$

where F^1_n and F^2_m are the empirical distribution functions of the corresponding samples each of size n and m, than points are sampled from the fBm in the Davies-Harte method. A better resourced investigation will include more points in the mesh of the fBm sample and use a correspondingly larger sample size to perform this statistical analysis. Nonetheless, this exploratory data analysis lends weight to an argument that τ could follow a symmetric beta distribution, with parameters $\alpha(H) = \beta(H)$, with the Wiener process $W_t = B^{1/2}_t$ having $\tau \sim B(1/2, 1/2)$.

H	D	p
0.95	0.141	0.982
0.856	0.272	0.395
0.761	0.294	0.303
0.667	0.281	0.356
0.572	0.15	0.967
0.478	0.355	0.128
0.383	0.229	0.617
0.289	0.295	0.230
0.194	0.16	0.943
0.1	0.244	0.535

Table 1: Two sided Kolmogorov-Smirnoff test. Results are for a sample of $N = 1000$ simulated τ values using a fBm trajectories sampling $M = 2^{10}$ mesh points.