

Mathematics of the Earth Atmosphere and Oceans

1 The Stommel problem and boundary current intensification

The first section gives an overview of the western boundary intensification model of Stommel, as presented in Vallis' *Atmospheric and Oceanic fluid dynamics*. In the large scale flows of the atmosphere and oceans, regions close to boundaries of the domain, for example air-land or ocean-air interfaces, host a transition away from the typical behaviour in the interior of the domain to the different behaviour at the boundary. As the Coriolis force is a dominant force in geophysical fluid dynamics, the behaviour within this *Ekman layer* is determined by a balance between the Coriolis force and turbulent Eddy dissipation of momentum. Under the Boussinesq approximation, the governing equation for the horizontal fluid velocity \mathbf{u} is

$$\mathbf{f} \times \mathbf{u} = -\nabla_z \phi + K \frac{\partial^2 \mathbf{u}}{\partial z^2} \quad (1)$$

where \mathbf{f} is the Coriolis parameter, ϕ is the pressure and K the turbulent diffusivity. We take the beta-plane approximation, writing

$$\mathbf{f} = (f_0 + \beta y) \hat{\mathbf{k}}$$

and supposing that it is wind stress $\boldsymbol{\tau}$ which drives Eddy diffusion at an air ocean interface and bottom frictional drag that provides an Ekman layer at the ocean bottom, (1) can be rewritten as

$$\mathbf{f} \times \mathbf{u} = -\nabla_z \phi + \frac{\partial \boldsymbol{\tau}}{\partial z}.$$

Note the stress in the above equation is a kinematic stress. We can write equation (2) under multiple assumptions. The first is to approximation flow by the depth integrated quantities and the second is to apply a rigid lid and flat bottom approximation (the rigid lid approximation being the more reasonable of the latter two approximations over a long length scale).

$$\beta v = \text{curl}_z (\boldsymbol{\tau}_t - \boldsymbol{\tau}_b). \quad (2)$$

In dealing with depth integrated variables, there is allowed to exist a scalar function $\psi(x, y)$ such that

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

The simplest hypothesis for the bottom drag in (2) is a dissipative force proportional flow velocity

$$\boldsymbol{\tau}_b = r \mathbf{u}$$

We suppose a square domain $D = [0, a] \times [0, a]$ with basin length scale a . On that square domain the model wind stress is supposed to be

$$\boldsymbol{\tau}_t = -\tau_0 \cos(\pi y) \hat{\mathbf{x}}.$$

Cast in terms of the stream function ψ , the Stommel problem is equation (3):

$$\beta \frac{\partial \psi}{\partial x} = -\pi \tau_0 \sin(\pi y) - r \nabla^2 \psi. \quad (3)$$

To write this equation, the vorticity $\zeta(x, y) = \text{curl}_z(\mathbf{u})$ was substituted for the Laplacian of the streamfunction. Non-dimensionalization in terms of the characteristic length scale a (in both the horizontal and vertical), force scale τ_0 , and velocity scale $\tau_0/\beta a$ gives equation (4),

$$\epsilon \nabla^2 \psi + \frac{\partial \psi}{\partial x} = -\pi \sin(\pi y) \quad (4)$$

where $\epsilon = r/a\beta$ is typically a small parameter. The Stommel problem (4) can be solved numerically to see the behaviour of the flow in this Ocean basin. Boundary conditions must first be selected and applied: a no-flow condition is imposed so that ψ and ψ_n must both be zero on the basin boundary. A simple Chebychev spectral collocation is used to solve (4) with no-flow conditions numerically.

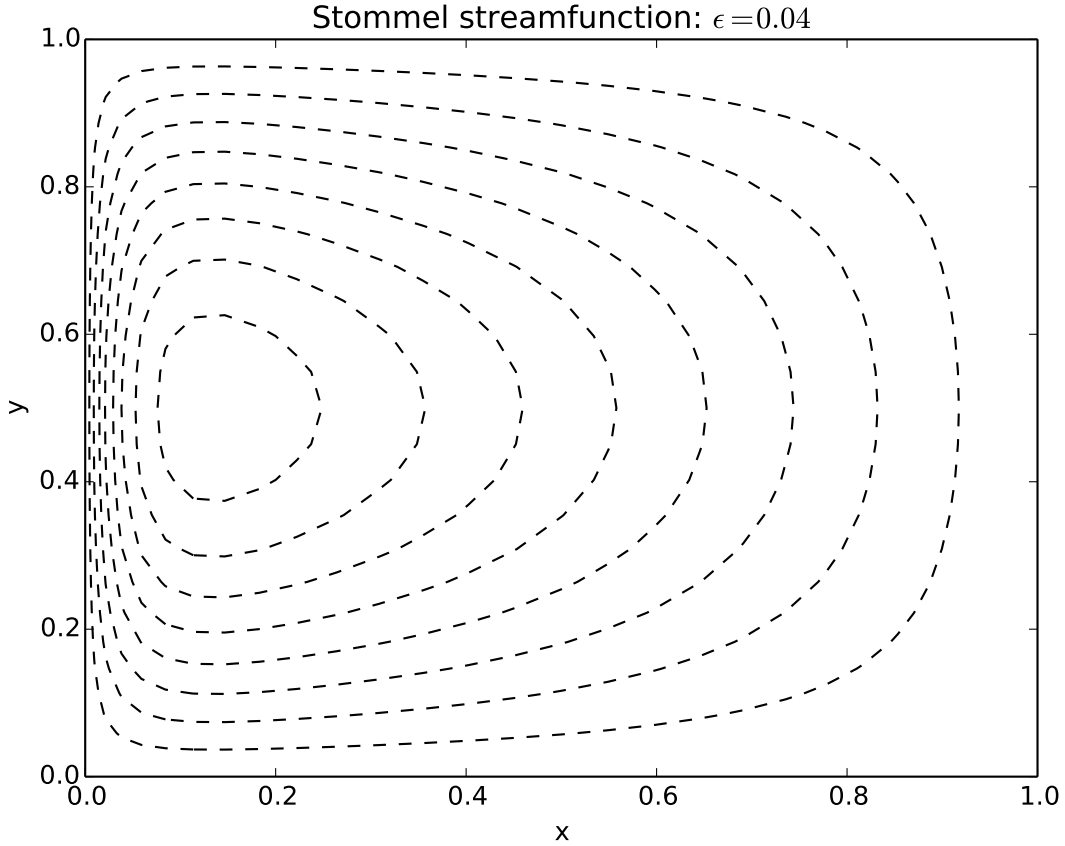


Figure 1: Stommel streamfunction in a no-flow square basin. The streamfunction is produced using $N = 2^{10}$ grid points in a Chebychev collocation.

The numerical solution displays a vortex motion that is slower towards the right boundary and faster towards the left. In the oceanographic context the Stommel model qualitatively predicts an intensification of currents towards the western boundary of an ocean boundary, a phenomenon that is indeed seen in, for example, the Agulhas current and the Kuroshiro.

2 Cloud droplets

This section gives insight into nucleation of water droplets, with cloud formation mechanisms in mind. It is based entirely off of my solution to problem 5.46 in Schroeder's *An Introduction to Thermal Physics*. We assume familiarity with the fundamental quantities considered in thermodynamics such as internal energy, pressure, chemical potentials energies. Recall that the Gibbs Free energy of a thermodynamic system is

$$G \equiv U + PV - TS \quad (5)$$

$$G = \mu N \quad (6)$$

In the initial formation of very small water droplets, the incredibly small size of the droplets means a considerable proportion of the particles will be at the boundary so that the contribution to the Gibbs free energy of a boundary term to account for the energy required to hold the water droplets within the particular surface of the water-air interface cannot be neglected. The problem is as follows: a small spherical droplet of water with N_l molecules is supposed to be surrounded $N - N_l$ water vapour molecules, with the vapour phase assumed to behave as an ideal gas. The Gibbs free energy of the system these two phases form is

$$G = \mu_l N_l + \mu_g N_g + \sigma A,$$

where the l subscript denotes quantities of the liquid phase and the g subscript denotes quantities of the gaseous phase and σ is the energy required per unit area to hold the interface area. In terms of the radius r of the droplet and the total number of particles N , the Gibbs free energy becomes:

$$G(r) = -\frac{4\pi(\mu_g - \mu_l)r^3}{3v_l} + 4\pi\sigma r^2 + \mu_g N.$$

Notice the substitution of $N_l = V/v_l$ for v_l the volume per molecule of water in the liquid. Figure 2 shows two cases of importance: when $(\mu_g - \mu_l) > 0$ and $(\mu_g - \mu_l) < 0$. [MAKE FIGURE 2] We see when in the former case there is a positive equilibrium radius while the latter case only presents a zero equilibrium radius. This makes sense as one shouldn't expect a droplet equilibrium if the potential energy per unit molecule for water in the liquid phase is greater than in the vapour phase. On the other hand, the equilibrium at the radius

$$r_c = \frac{2v_l\sigma}{\mu_g - \mu_l}$$

gives an unstable equilibrium as it occurs at a local maximum in the Gibbs free energy. A flaw in the model can be seen by expressing $\mu_g - \mu_l$ in terms of the partial pressure of water. Reconsider (5) and (6), considering a differential change in the Gibbs free energy under constant temperature in (5):

$$dG = dU + PdV + VdP - SdT - TdS = VdP - SdT = VdP$$

Holding the number of particles constant, a differential change in (6) gives the equality

$$Nd\mu = VdP$$

Therefore for the vapour-liquid system

$$d(\mu_g - \mu_l) = (v_g - v_l) dP \approx v_g dP.$$

In the last approximation we have made use of the much greater molecular volume of vapour water. Using the ideal gas law $PV = NkT$, where k is Boltzmann's constant, we have the simple differential equation

$$\frac{d}{dP}(\mu_g - \mu_l) = \frac{kT}{P}$$

whose solution is

$$\mu_g - \mu_l = kT \ln \left(\frac{P}{P_v} \right). \quad (7)$$

In (7), P_v is the vapour pressure of water in diffusive equilibrium. Thus P/P_v is the relative humidity. Strikingly the result predicts that for droplets to form the relative humidity is required to be greater than 100 %, and large enough to ensure the critical radius is small. The former requirement is rare, and the last is also difficult because of the slow rate of increase of the logarithm. The conclusion is that cloud formation by spontaneous aggregation of water molecules is not likely. Instead particles acting as 'nuclei' for droplet formation are usually required.

3 The dry and wet atmosphere temperature gradients

The discussion here concerns simple models of how thermal gradients drive the convection of both dry and moist air. It is based on my solution to problem 5.45 in Schroeder's *An Introduction to Thermal Physics*. We begin by determining how pressure varies vertically within the atmosphere. Before deriving an approximate law for this variation, we should note that vertical variations in pressure, called the pressure gradient, are balanced by buoyancy forces, an equilibrium in the vertical called hydrostatic balance. The hydrostatic balance is given by

$$\frac{dP}{dz} = -\rho g \quad (8)$$

where ρ is the fluid density and g is the gravitational acceleration. The ideal gas law is

$$PV = NkT.$$

If we transpose the above equation to

$$\frac{MP}{NkT} = \frac{M}{V} = \rho$$

Substitution into (8) yields

$$\frac{dP}{dz} = -\frac{mg}{kT}P. \quad (9)$$

Here m is the molecular mass, the mass per molecule. With this expression, we ask how temperature varies in the troposphere. We suppose that parcels of air expand adiabatically. By the first law of the thermodynamics, the change in internal energy of that parcel will be

$$dU = -PdV$$

By the equipartition of energy, the internal energy of a thermodynamic system at temperature T with N particles that have f degrees of freedom in their motion is $U = fNkT/2$ so that $dU = fNkdT$. Thus

$$\frac{f}{2}NkdT = -PdV.$$

A well-known result relating the volume and pressure of an ideal gas undergoing an adiabatic process is

$$V^\gamma P = C$$

where $\gamma = f/(f + 2)$ is the so-called *adiabatic exponent* and C is a constant. By the chain rule dV/dP is

$$\frac{dV}{dP} = -\frac{V}{\gamma P}$$

Substitution of dV/dP into the first law of thermodynamics gives

$$\begin{aligned} fNkdT &= 2\frac{V}{\gamma}dP \\ \implies \frac{dT}{dP} &= \frac{2}{f+2} \frac{V}{Nk} = \frac{2}{f+2} \frac{T}{P} \end{aligned} \tag{10}$$

An expression for the derivative of atmosphere temperature with respect to the pressure allows one, using the chain rule, to determine the vertical variation of the temperature:

$$\frac{dT}{dz} = \frac{dP}{dz} \frac{dT}{dP} = -\frac{2mg}{k(f+2)} = -\Gamma. \tag{11}$$

The model presented therefore predicts a temperature that decreases linearly with height. The gradient is referred to as the dry adiabatic lapse rate, so-called because it is usually interpreted as the critical temperature gradient at which convection occurs. Notice that we have specified dry air, indeed there has been no consideration of the partial pressure of water. However moisture in the air should be an important consideration in, for example, the dynamics of clouds. After all, clouds are in essence droplets of water condensed from moist air. We shall still suppose that a parcel of air expands adiabatically, but as it does so liquid water condenses from this parcel. Suppose that there are N particles in total and n_w water molecules. The total change in energy of the system is

$$dU = dU_{\text{exp}} + dU_{\text{cond}}$$

with the first term representing the loss energy to adiabatic expansion and the second term the loss arising from condensation. Therefore

$$dU = -PdV - Ldn_w. \tag{12}$$

The second term is the *latent heat* released as gaseous water transitions to the aqueous state. If by n_d we denote the moles of the dry component of the air, so that $n = n_w + n_d$, the left hand is

$$dU = \frac{f}{2}d((n_d + n_w)RT) = \frac{f}{2}nRdT + \frac{f}{2}dn_wRT.$$

The right hand side can be rewritten as

$$VdP - d(PV) - Ldn_w = VdP - nRdT - dn_wRT$$

Rearrangement gives

$$\begin{aligned} \frac{f+2}{2}dT &= \frac{VdP}{nR} - dn_w \left(\frac{L}{nR} + \frac{(f+2)T}{2n} \right) \\ \Rightarrow \frac{f+2}{2}dT &= \frac{T}{P}dP - dn_w \frac{L}{nR} \left(1 + \frac{(f+2)PV}{2nL} \right) \end{aligned}$$

Since L is the latent heat per mole of water released, the right most term will have an order of magnitude

$$\frac{PV}{nL} \sim \frac{(f+2)PV}{2Q} \cdot \frac{n_w}{n}$$

where Q is the heat released from the fusion of water. Assuming that PV/Q is not too large, this term will be negligible since $n_w/n \ll 1$. Substituing $f = 5$ for the degrees of freedom of gaseous air, the differential change in temperature is therefore

$$dT = \frac{2}{7} \frac{T}{P} dP - \frac{2}{7} \frac{L}{nR} dn_w \quad (13)$$

Using this relation, the temperature gradient will be

$$\frac{dT}{dz} = \frac{2}{7} \frac{T}{P} \frac{dP}{dz} - \frac{2}{7} \frac{L}{nR} \frac{dn_w}{dz}. \quad (14)$$

The vapour pressure of water satisfies the relation

$$\frac{n_w}{n} = \frac{P_v}{P}. \quad (15)$$

The right term is a function of temperature and pressure so by the chain rule

$$\frac{dn_w}{dz} = \frac{\partial n_w}{\partial T} \frac{dT}{dz} + \frac{\partial n_w}{\partial P} \frac{dP}{dz}$$

The partial derivative with respect to pressure P is

$$\frac{\partial n_w}{\partial P} = -\frac{nP_v}{P^2}$$

whilst the temperature derivative is

$$\frac{\partial n_w}{\partial T} = \frac{n}{P} \frac{\partial n_w}{\partial T} = \frac{nL.n_w}{PV_wT} = \frac{nLP_v}{PRT^2}.$$

In the second last equality the Clausius-Clayperon(CC) relation has been used. This is an expression relating the change of temperature with respect pressure at a phase boundary to. Notice the

presence of the total heat of condensation Ln_w , as opposed to the heat per mole, in the CC relation. In the last line the ideal gas law has been applied to the water component only of the air parcel. Substitution of dn_w/dz into (14) gives

$$\left(1 + \frac{2}{7} \frac{L}{nR} \frac{nLP_v}{PRT^2}\right) \frac{dT}{dz} = \frac{2}{7} \left(\frac{T}{P} + \frac{L}{nR} \frac{nP_v}{P^2}\right) \frac{dP}{dz}$$

The pressure gradient, assuming the 'exponential atmosphere', is given by (9). We then have

$$\left(1 + \frac{2}{7} \left(\frac{L}{nR}\right)^2 \frac{P_v}{P}\right) \frac{dT}{dz} = -\frac{2}{7} \left(\frac{Mg}{R}\right) \left(1 + \left(\frac{L}{RT}\right) \frac{P_v}{P}\right)$$

where M is the molecular weight of water. One therefore arrives a simple model expression for the temperature gradient in a saturated atmosphere:

$$\frac{dT}{dz} = -\frac{2}{7} \left(\frac{Mg}{R}\right) \frac{1 + \frac{P_v}{P} \frac{L}{RT}}{1 + \frac{2}{7} \left(\frac{L}{nR}\right)^2 \frac{P_v}{P}} \quad (16)$$

This gradient is the *wet adiabatic* lapse rate, and gives the temperature gradient at which water saturated air will begin to convect.