Norm and Distance

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Outline

Norm

- the *Euclidean norm* (or just *norm*) of an n-vector x is

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^T x}$$

- used to measure the size of a vector
- reduces to absolute value for $n = 1$

Properties

for any n-vectors x and y, and any scalar β

- *Homogeneity.* $\|\beta x\| = |\beta|\|x\|$
- *Triangle inequality.* $\|x + y\| \leq \|x\| + \|y\|$
- *Nonnegativity.* $\|x\| \geq 0$
- *Definiteness.* $\|x\| = 0$ only if $x = 0$

easy to show except triangle inequality, which we show later

RMS value

- *mean-square value* of n-vector x is

$$\frac{x_1^2 + \cdots + x_n^2}{n} = \frac{\|x\|^2}{n}$$

- *root-mean-square value* (RMS value) is

$$\mathbf{rms}(x) = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} = \frac{\|x\|}{\sqrt{n}}$$

- $\mathbf{rms}(x)$ gives 'typical' value of $|x_i|$
- *e.g.*, $\mathbf{rms}(\mathbf{1}) = 1$ (independent of n)
- RMS value useful for comparing sizes of vectors of different lengths

Norm of block vectors

- suppose a, b, c are vectors
- $\|(a,b,c)\|^2 = a^T a + b^T b + c^T c = \|a\|^2 + \|b\|^2 + \|c\|^2$
- so we have

$$\|(a,b,c)\| = \sqrt{\|a\|^2 + \|b\|^2 + \|c\|^2} = \|(\|a\|, \|b\|, \|c\|)\|$$

  (parse RHS very carefully!)
- we'll use these ideas later

Chebyshev inequality

- suppose that k of the numbers $|x_1|, \ldots, |x_n|$ are $\geq a$
- then k of the numbers $x_1^2, \ldots, x_n^2$ are $\geq a^2$
- so $\|x\|^2 = x_1^2 + \cdots + x_n^2 \geq ka^2$
- so we have $k \leq \|x\|^2/a^2$
- number of x_i with $|x_i| \geq a$ is no more than $\|x\|^2/a^2$
- this is the *Chebyshev inequality*
- in terms of RMS value:

$$\text{fraction of entries with } |x_i| \geq a \text{ is no more than } \left(\frac{\mathbf{rms}(x)}{a}\right)^2$$

- example: no more than 4% of entries can satisfy $|x_i| \geq 5\,\mathbf{rms}(x)$

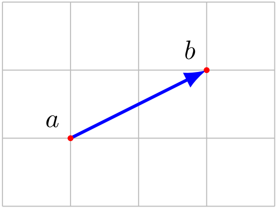
Outline

Distance

- (Euclidean) *distance* between n-vectors a and b is

$$\mathbf{dist}(a, b) = \|a - b\|$$

- agrees with ordinary distance for $n = 1, 2, 3$



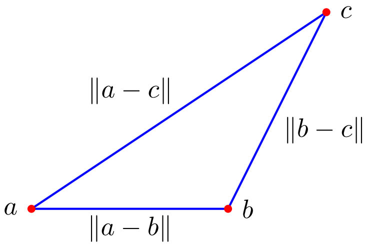
- $\mathbf{rms}(a - b)$ is the *RMS deviation* between a and b

Triangle inequality

- triangle with vertices at positions a, b, c
- edges lengths are $\|a-b\|$, $\|b-c\|$, $\|a-c\|$
- by triangle inequality

$$\|a-c\| = \|(a-b)+(b-c)\| \leq \|a-b\| + \|b-c\|$$

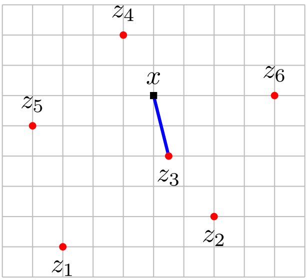
i.e., third edge length is no longer than sum of other two



Feature distance and nearest neighbors

- when x and y are feature vectors for two entities, $\|x - y\|$ is their *feature distance*
- if $z_1, \ldots, z_m$ is a list of vectors, z_j is the *nearest neighbor* of x if

$$\|x - z_j\| \leq \|x - z_i\|, \quad i = 1, \ldots, m$$



- these simple ideas are very widely used

Document dissimilarity

- 5 Wikipedia articles: ‘Veterans Day’, ‘Memorial Day’, ‘Academy Awards’, ‘Golden Globe Awards’, ‘Super Bowl’
- word count histograms, dictionary of 4423 words
- pairwise distances shown below

	Veterans Day	Memorial Day	Academy Awards	Golden Globe Awards	Super Bowl
Veterans Day	0	0.095	0.130	0.153	0.170
Memorial Day	0.095	0	0.122	0.147	0.164
Academy A.	0.130	0.122	0	0.108	0.164
Golden Globe A.	0.153	0.147	0.108	0	0.181
Super Bowl	0.170	0.164	0.164	0.181	0

Outline

Standard deviation

- for n-vector x, $\mathbf{avg}(x) = \mathbf{1}^T x/n$
- *de-meaned vector* is $\tilde{x} = x - \mathbf{avg}(x)\mathbf{1}$ (so $\mathbf{avg}(\tilde{x}) = 0$)
- *standard deviation* of x is

$$\mathbf{std}(x) = \mathbf{rms}(\tilde{x}) = \frac{\|x - (\mathbf{1}^T x/n)\mathbf{1}\|}{\sqrt{n}}$$

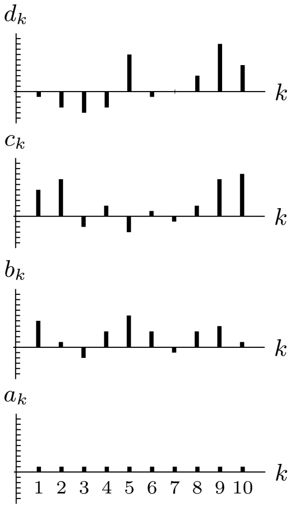
- $\mathbf{std}(x)$ gives 'typical' amount x_i vary from $\mathbf{avg}(x)$
- $\mathbf{std}(x) = 0$ only if $x = \alpha\mathbf{1}$ for some α
- greek letters μ, σ commonly used for mean, standard deviation
- a basic formula:

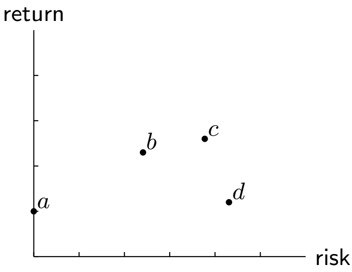
$$\mathbf{rms}(x)^2 = \mathbf{avg}(x)^2 + \mathbf{std}(x)^2$$

Mean return and risk

- x is time series of returns (say, in %) on some investment or asset over some period
- $\mathbf{avg}(x)$ is the mean return over the period, usually just called *return*
- $\mathbf{std}(x)$ measures how variable the return is over the period, and is called the *risk*
- multiple investments (with different return time series) are often compared in terms of return and risk
- often plotted on a *risk-return plot*

Risk-return example





Chebyshev inequality for standard deviation

- x is an n-vector with mean $\mathbf{avg}(x)$, standard deviation $\mathbf{std}(x)$
- rough idea: most entries of x are not too far from the mean
- by Chebyshev inequality, fraction of entries of x with $|x_i - \mathbf{avg}(x)| \geq \alpha\,\mathbf{std}(x)$ is no more than $1/\alpha^2$ (for $\alpha > 1$)
- for return time series with mean 8% and standard deviation 3%, loss ($x_i \leq 0$) can occur in no more than $(3/8)^2 = 14.1\%$ of periods

Outline

Cauchy-Schwarz inequality

- for two n-vectors a and b, $|a^Tb| \leq \|a\|\|b\|$
- written out,

$$|a_1b_1 + \cdots + a_nb_n| \leq \left(a_1^2 + \cdots + a_n^2\right)^{1/2} \left(b_1^2 + \cdots + b_n^2\right)^{1/2}$$

- now we can show triangle inequality:

$$\|a+b\|^2 = \|a\|^2 + 2a^Tb + \|b\|^2 \leq \|a\|^2 + 2\|a\|\|b\| + \|b\|^2 = (\|a\| + \|b\|)^2$$

Derivation of Cauchy-Schwarz inequality

- it's clearly true if either a or b is 0
- so assume $\alpha = \|a\|$ and $\beta = \|b\|$ are nonzero
- we have

$$\begin{array}{rcl} 0 & \leq & \|\beta a - \alpha b\|^2 \\ & = & \|\beta a\|^2 - 2(\beta a)^T(\alpha b) + \|\alpha b\|^2 \\ & = & \beta^2 \|a\|^2 - 2\beta\alpha(a^T b) + \alpha^2 \|b\|^2 \\ & = & 2\|a\|^2 \|b\|^2 - 2\|a\|\,\|b\|(a^T b) \end{array}$$

- divide by $2\|a\|\,\|b\|$ to get $a^T b \leq \|a\|\,\|b\|$
- apply to $-a$, b to get other half of Cauchy-Schwarz inequality

Angle

- *angle* between two nonzero vectors a, b defined as

$$\angle(a,b) = \arccos\left(\frac{a^T b}{\|a\|\,\|b\|}\right)$$

- $\angle(a,b)$ is the number in $[0,\pi]$ that satisfies

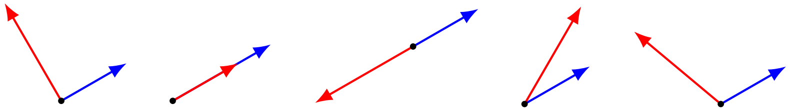
$$a^T b = \|a\|\,\|b\| \cos\left(\angle(a,b)\right)$$

- coincides with ordinary angle between vectors in 2-D and 3-D

Classification of angles

$\theta = \angle(a, b)$

- $\theta = \pi/2 = 90^\circ$: a and b are *orthogonal*, written $a \perp b$ ($a^T b = 0$)
- $\theta = 0$: a and b are *aligned* ($a^T b = \|a\|\|b\|$)
- $\theta = \pi = 180^\circ$: a and b are *anti-aligned* ($a^T b = -\|a\| \, \|b\|$)
- $\theta \le \pi/2 = 90^\circ$: a and b make an *acute angle* ($a^T b \ge 0$)
- $\theta \ge \pi/2 = 90^\circ$: a and b make an *obtuse angle* ($a^T b \le 0$)



Document dissimilarity by angles

- measure dissimilarity by angle of word count histogram vectors
- pairwise angles for 5 Wikipedia pages shown below

	Veterans Day	Memorial Day	Academy Awards	Golden Globe Awards	Super Bowl
Veterans Day	0	60.6	85.7	87.0	87.7
Memorial Day	60.6	0	85.6	87.5	87.5
Academy A.	85.7	85.6	0	58.7	85.7
Golden Globe A.	87.0	87.5	58.7	0	86.0
Super Bowl	87.7	87.5	86.1	86.0	0

Correlation coefficient

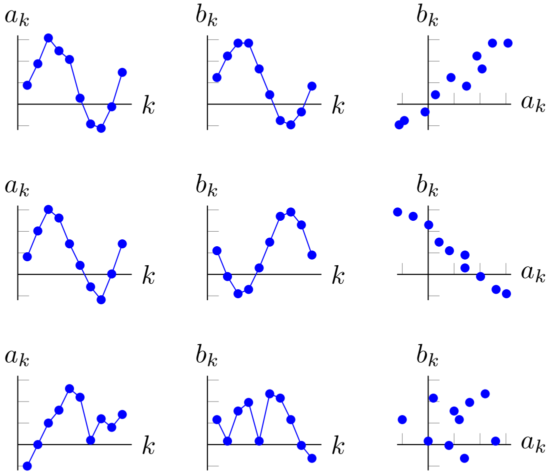
- vectors a and b, and de-meaned vectors $\tilde{a} = a - \mathbf{avg}(a)\mathbf{1}$, $\tilde{b} = b - \mathbf{avg}(b)\mathbf{1}$
- *correlation coefficient* (between a and b, with $\tilde{a} \neq 0$, $\tilde{b} \neq 0$)

$$\rho = \frac{\tilde{a}^T \tilde{b}}{\|\tilde{a}\| \, \|\tilde{b}\|}$$

- $\rho = \cos \angle(\tilde{a}, \tilde{b})$
  - $\rho = 0$: a and b are *uncorrelated*
  - $\rho > 0.8$ (or so): a and b are *highly correlated*
  - $\rho < -0.8$ (or so): a and b are *highly anti-correlated*
- very roughly: highly correlated means a_i and b_i are typically both above (below) their means together

Examples

$\rho = 97\%$ (top row); $\rho = -99\%$ (middle row); $\rho = -0.4\%$ (bottom row)



Examples

- highly correlated vectors:
  - rainfall time series at nearby locations
  - daily returns of similar companies in same industry
  - word count vectors of closely related documents (*e.g.*, same author, topic, . . .)
  - sales of shoes and socks (at different locations or periods)
- approximately uncorrelated vectors
  - unrelated vectors
  - audio signals (even different tracks in multi-track recording)
- (somewhat) negatively correlated vectors
  - daily temperatures in Palo Alto and Melbourne