Linear Dynamical Systems

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Outline

Linear dynamical systems

Population dynamics

Epidemic dynamics

State sequence

- sequence of n-vectors $x_1, x_2, \ldots$,
- t denotes time or period
- x_t is called *state* at time t; sequence is called *state trajectory*
- (assuming t is current time)
  - x_t is current state
  - x_{t-1} is previous state
  - x_{t+1} is next state
- examples: x_t represents
  - age distribution in a population
  - economic output in n sectors
  - mechanical variables

Linear dynamics

- linear dynamical system: $x_{t+1} = A_t x_t$
- A_t are $n \times n$ dynamics matrices
- $(A_t)_{ij}(x_t)_j$ is contribution to $(x_{t+1})_i$ from $(x_t)_j$
- called *time-invariant* if $A_t = A$ doesn't depend on time
- can simulate evolution of x_t using recursion $x_{t+1} = A_t x_t$

Variations

- $x_{t+1} = A_t x_t + B_t u_t + c_t$
  - u_t is an *input* m-vector
  - B_t is $n \times m$ *input matrix*
  - c_t is *offset*
- used in many application areas

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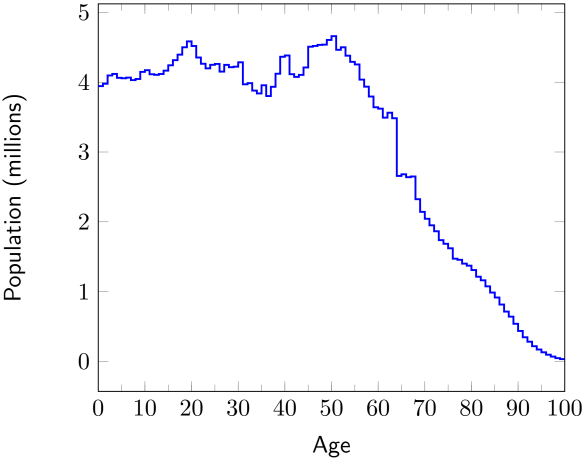
Population distribution

- $x_t \in \mathbf{R}^{100}$ gives population distribution in year $t = 1, \ldots, T$
- $(x_t)_i$ is the number of people with age $i - 1$ in year t (say, on January 1)
- total population in year t: $\mathbf{1}^T x_t$
- number of people age 70 or older in year t: $(0_{70}, \mathbf{1}_{30})^T x_t$

Population distribution of the U.S.

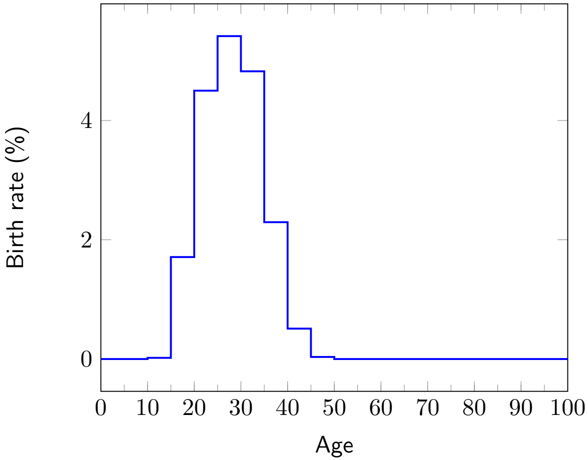
(from 2010 census)



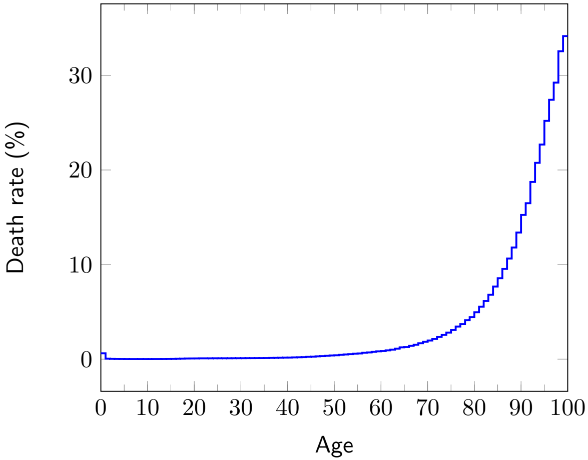
Birth and death rates

- birth rate $b \in \mathbf{R}^{100}$, death (or mortality) rate $d \in \mathbf{R}^{100}$
- b_i is the number of births per person with age $i - 1$
- d_i is the portion of those aged $i - 1$ who will die this year
  (we'll take $d_{100} = 1$)
- b and d can vary with time, but we'll assume they are constant

Birth rate in the U.S.



Death rate in the U.S.



Dynamics

- let's find next year's population distribution x_{t+1} (ignoring immigration; we'll add that later)
- number of 0-year-olds next year is total births this year:

$$(x_{t+1})_1 = b^T x_t$$

- number of i-year-olds next year is number of $(i-1)$-year-olds this year, minus those who die:

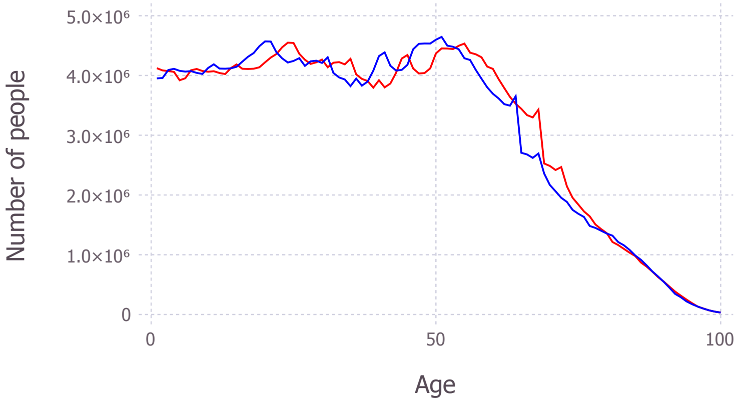
$$(x_{t+1})_{i+1} = (1 - d_i)(x_t)_i, \quad i = 1, \ldots, 99$$

- $x_{t+1} = Ax_t$, where

$$A = \begin{bmatrix} b_1 & b_2 & \cdots & b_{99} & b_{100} \\ 1 - d_1 & & & \cdots & 0 \\ 0 & 1 - d_2 & & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & 1 - d_{99} & 0 \end{bmatrix}$$

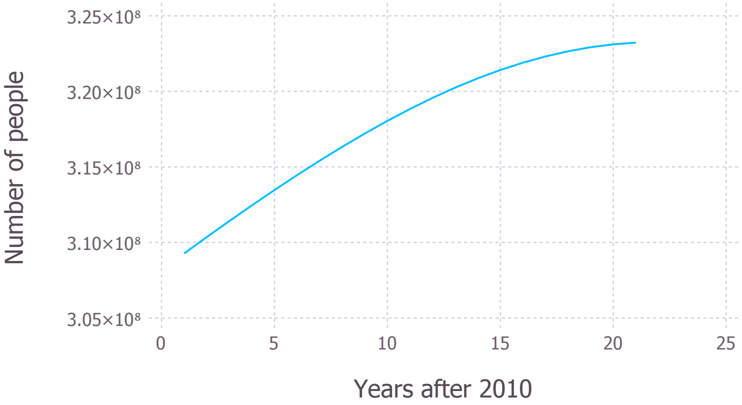
Predicting future population distributions

predicting U.S. 2015 distribution from 2010 (ignoring immigration)



Predicting population growth

predicted population growth (ignoring immigration)



Initial population distributions

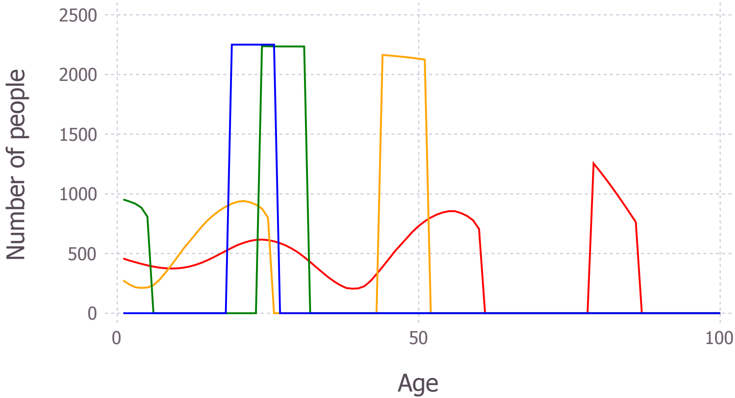
- what if we changed x_0?
- instead of U.S. Census data, let's use a "college nation"

$$(x_0)_i = \begin{cases} 2200 & i = 19, 20, \ldots, 27 \\ 0 & \text{otherwise} \end{cases}$$

(approximate population distribution of Stanford students)

Predicting future population distributions

predict s years into the future



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SIR model

- 4-vector x_t gives proportion of population in 4 infection states:
  - *Susceptible:* can acquire the disease the next day
  - *Infected:* have the disease
  - *Recovered:* had the disease, recovered, now immune
  - *Deceased:* had the disease, and unfortunately died

  (sometimes called *SIR model*)
- *e.g.*, $x_t = (0.75, 0.10, 0.10, 0.05)$

Epidemic dynamics

over each day,

- among susceptible population,
  - 5% acquires the disease
  - 95% remain susceptible
- among infected population,
  - 1% dies
  - 10% recovers with immunity
  - 4% recover without immunity (*i.e.*, become susceptible)
  - 85% remain infected
- (100% of immune and dead people remain in their state)

Epidemic dynamics as linear dynamical system

$$x_{t+1} = \begin{bmatrix} 0.95 & 0.04 & 0 & 0 \\ 0.05 & 0.85 & 0 & 0 \\ 0 & 0.10 & 1 & 0 \\ 0 & 0.01 & 0 & 1 \end{bmatrix} x_t$$

Simulation from $x_1 = (1, 0, 0, 0)$