In statistics, linear regression is a linear approach for modelling the relationship between a scalar response and one or more explanatory variables (also known as dependent and independent variables). The case of one explanatory variable is called simple linear regression; for more than one, the process is called multiple linear regression.[1] This term is distinct from multivariate linear regression, where multiple correlated dependent variables are predicted, rather than a single scalar variable.[2]

In linear regression, the relationships are modeled using linear predictor functions whose unknown model parameters are estimated from the data. Such models are called linear models.[3] Most commonly, the conditional mean of the response given the values of the explanatory variables (or predictors) is assumed to be an affine function of those values; less commonly, the conditional median or some other quantile is used. Like all forms of regression analysis, linear regression focuses on the conditional probability distribution of the response given the values of the predictors, rather than on the joint probability distribution of all of these variables, which is the domain of multivariate analysis.

Linear regression was the first type of regression analysis to be studied rigorously, and to be used extensively in practical applications.[4] This is because models which depend linearly on their unknown parameters are easier to fit than models which are non-linearly related to their parameters and because the statistical properties of the resulting estimators are easier to determine.

Linear regression has many practical uses. Most applications fall into one of the following two broad categories:

If the goal is error reduction in prediction or forecasting, linear regression can be used to fit a predictive model to an observed data set of values of the response and explanatory variables. After developing such a model, if additional values of the explanatory variables are collected without an accompanying response value, the fitted model can be used to make a prediction of the response.

If the goal is to explain variation in the response variable that can be attributed to variation in the explanatory variables, linear regression analysis can be applied to quantify the strength of the relationship between the response and the explanatory variables, and in particular to determine whether some explanatory variables may have no linear relationship with the response at all, or to identify which subsets of explanatory variables may contain redundant information about the response.

Linear regression models are often fitted using the least squares approach, but they may also be fitted in other ways, such as by minimizing the "lack of fit" in some other norm (as with least absolute deviations regression), or by minimizing a penalized version of the least squares cost function as in ridge regression (L2-norm penalty) and lasso (L1-norm penalty). Conversely, the least squares approach can be used to fit models that are not linear models. Thus, although the terms "least squares" and "linear model" are closely linked, they are not synonymous.

Standard linear regression models with standard estimation techniques make a number of assumptions about the predictor variables, the response variables and their relationship. Numerous extensions have been developed that allow each of these assumptions to be relaxed (i.e. reduced to a weaker form), and in some cases eliminated entirely. Generally these extensions make the estimation procedure more complex and time-consuming, and may also require more data in order to produce an equally precise model.

Example of a cubic polynomial regression, which is a type of linear regression. Although polynomial regression fits a nonlinear model to the data, as a statistical estimation problem it is linear, in the sense that the regression function E(y | x) is linear in the unknown parameters that are estimated from the data. For this reason, polynomial regression is considered to be a special case of multiple linear regression.

The following are the major assumptions made by standard linear regression models with standard estimation techniques (e.g. ordinary least squares):

Weak exogeneity. This essentially means that the predictor variables x can be treated as fixed values, rather than random variables. This means, for example, that the predictor variables are assumed to be error-free—that is, not contaminated with measurement errors. Although this assumption is not realistic in many settings, dropping it leads to significantly more difficult errors-in-variables models.

Linearity. This means that the mean of the response variable is a linear combination of the parameters (regression coefficients) and the predictor variables. Note that this assumption is much less restrictive than it may at first seem. Because the predictor variables are treated as fixed values (see above), linearity is really only a restriction on the parameters. The predictor variables themselves can be arbitrarily transformed, and in fact multiple copies of the same underlying predictor variable can be added, each one transformed differently. This technique is used, for example, in polynomial regression, which uses linear regression to fit the response variable as an arbitrary polynomial function (up to a given degree) of a predictor variable. With this much flexibility, models such as polynomial regression often have "too much power", in that they tend to overfit the data. As a result, some kind of regularization must typically be used to prevent unreasonable solutions coming out of the estimation process. Common examples are ridge regression and lasso regression. Bayesian linear regression can also be used, which by its nature is more or less immune to the problem of overfitting. (In fact, ridge regression and lasso regression can both be viewed as special cases of Bayesian linear regression, with particular types of prior distributions placed on the regression coefficients.)

Constant variance (a.k.a. homoscedasticity). This means that the variance of the errors does not depend on the values of the predictor variables. Thus the variability of the responses for given fixed values of the predictors is the same regardless of how large or small the responses are. This is often not the case, as a variable whose mean is large will typically have a greater variance than one whose mean is small. For example, a person whose income is predicted to be $100,000 may easily have an actual income of $80,000 or $120,000—i.e., a standard deviation of around $20,000—while another person with a predicted income of $10,000 is unlikely to have the same $20,000 standard deviation, since that would imply their actual income could vary anywhere between −$10,000 and $30,000. (In

fact, as this shows, in many cases—often the same cases where the assumption of normally distributed errors fails—the variance or standard deviation should be predicted to be proportional to the mean, rather than constant.) The absence of homoscedasticity is called heteroscedasticity. In order to check this assumption, a plot of residuals versus predicted values (or the values of each individual predictor) can be examined for a "fanning effect" (i.e., increasing or decreasing vertical spread as one moves left to right on the plot). A plot of the absolute or squared residuals versus the predicted values (or each predictor) can also be examined for a trend or curvature. Formal tests can also be used; see Heteroscedasticity. The presence of heteroscedasticity will result in an overall "average" estimate of variance being used instead of one that takes into account the true variance structure. This leads to less precise (but in the case of ordinary least squares, not biased) parameter estimates and biased standard errors, resulting in misleading tests and interval estimates. The mean squared error for the model will also be wrong. Various estimation techniques including weighted least squares and the use of heteroscedasticity-consistent standard errors can handle heteroscedasticity in a quite general way. Bayesian linear regression techniques can also be used when the variance is assumed to be a function of the mean. It is also possible in some cases to fix the problem by applying a transformation to the response variable (e.g., fitting the logarithm of the response variable using a linear regression model, which implies that the response variable itself has a log-normal distribution rather than a normal distribution).

To check for violations of the assumptions of linearity, constant variance, and independence of errors within a linear regression model, the residuals are typically plotted against the predicted values (or each of the individual predictors). An apparently random scatter of points about the horizontal midline at 0 is ideal, but cannot rule out certain kinds of violations such as autocorrelation in the errors or their correlation with one or more covariates.

Independence of errors. This assumes that the errors of the response variables are uncorrelated with each other. (Actual statistical independence is a stronger condition than mere lack of correlation and is often not needed, although it can be exploited if it is known to hold.) Some methods such as generalized least squares are capable of handling correlated errors, although they typically require significantly more data unless some sort of regularization is used to bias the model towards assuming uncorrelated errors. Bayesian linear regression is a general way of handling this issue.

Lack of perfect multicollinearity in the predictors. For standard least squares estimation methods, the design matrix X must have full column rank p; otherwise perfect multicollinearity exists in the predictor variables, meaning a linear relationship exists between two or more predictor variables. This can be caused by accidentally duplicating a variable in the data, using a linear transformation of a variable along with the original (e.g., the same temperature measurements expressed in Fahrenheit and Celsius), or including a linear combination of multiple variables in the model, such as their mean. It can also happen if there is too little data available compared to the number of parameters to be estimated (e.g., fewer data points than regression coefficients). Near violations of this assumption, where predictors are highly but not perfectly correlated, can reduce the precision of parameter estimates (see Variance inflation factor). In the case of perfect multicollinearity, the parameter vector β will be non-identifiable—it has no unique solution. In such a case, only some of the parameters can be identified (i.e., their values can only be estimated within some linear subspace of the full parameter space Rp). See partial least squares regression. Methods for fitting linear models with multicollinearity have been developed,[5][6][7][8] some of which require additional assumptions such as "effect sparsity"—that a large fraction of the effects are exactly zero.

Note that the more computationally expensive iterated algorithms for parameter estimation, such as those used in generalized linear models, do not suffer from this problem.

Beyond these assumptions, several other statistical properties of the data strongly influence the performance of different estimation methods:

The statistical relationship between the error terms and the regressors plays an important role in determining whether an estimation procedure has desirable sampling properties such as being unbiased and consistent.

The arrangement, or probability distribution of the predictor variables x has a major influence on the precision of estimates of β. Sampling and design of experiments are highly developed subfields of statistics that provide guidance for collecting data in such a way to achieve a precise estimate of β.