Computer Exercice:
Classical Loop-Shaping

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Abstract − **This paper resume our work for the first homework in the course *Control, Theory & Practice*. In the first part we deal with controlling a simple single-input/single-output (SISO) system. In the second part, we still have a SISO system but we have to take care of the disturbances, rising the difficulty to design the controller.**

4 Exercises

4.1 Basics

We consider a system which can be modeled by the transfer function:

$$G(s) = \frac{3(-s+1)}{(5s+1)(10s+1)}$$

. . . .   *Exercise 4.1.1*   . . . .

We want to design a lead-lag controller which eliminates the static control error for a step response in the reference signal. The controller transfer function is the following:

$$F(s) = K\frac{\tau_D s + 1}{\beta\tau_D s + 1}\frac{\tau_I s + 1}{\tau_I s + \gamma}$$

We want to fulfill the following criteria:

- Phase margin of 30° at the cross-over frequency $\omega_c = 0.4$ rad/s.
- No static control error for a step response

Phase-lag action

Setting the parameters $\tau_I = 1$s and $\gamma = 0$ leads to an integral part in the system. This guarantee no static control error. $(\lim_{t\to\infty} e(t) = 0)$.

Phase-lead action

The lead-action of the controller leads to the following values in the frequency domain:

- Phase margin: $\phi_0 = -50°$
- Gain: $|G(\omega = 0.4 \text{ rad/s}| = 0.9436$

Thus:

- $\beta = \frac{1-\sin(30°-\phi_0)}{1+\sin(30°-\phi_0)} = 0.0086$
- $\tau_D = \frac{1}{\omega_c\sqrt{a}} = 26.9459$
- $K = \frac{\sqrt{a}}{m} = 0.0983$

Controller

The two previous paragraph lead to the design of the controller as:

$$F(s) = K\frac{\tau_D s + 1}{\beta\tau_D s + 1}\frac{\tau_I s + 1}{\tau_I s + \gamma} = 0.0983\frac{26.9459s + 1}{0.2319s + 1}\frac{s + 1}{s}$$

Figure 1 shows the bode diagrams of the different function of the system and the step response.
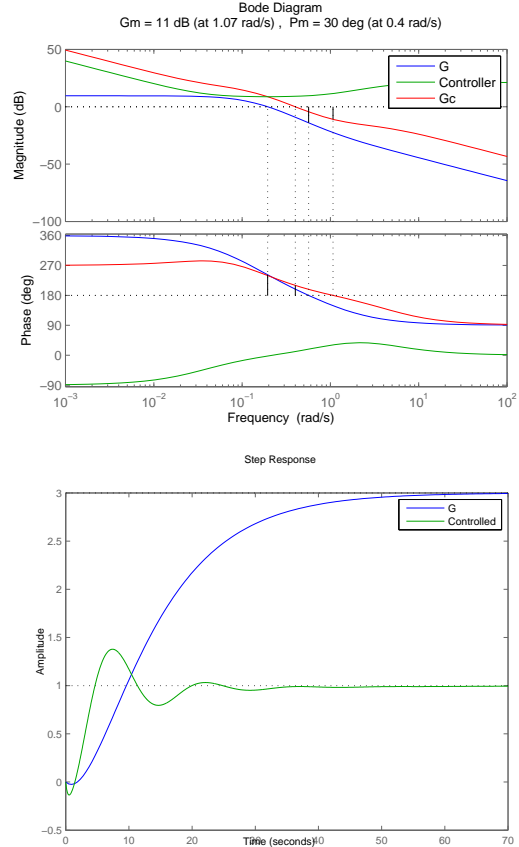The two criteria (phase-margin and static error null) are fulfilled.

Figure 1: Bode diagram of the system's functions
Step response of the system with and without the controller
Phase-margin: 30°

. . . .   *Exercise 4.1.2*   . . . .

Adding the lead lag controller to the system leads to the characteristics reported in the following table

Bandwith $(-3dB)$	$[0, 0.75](\text{rad/s})$
Resonance peak M_T	5.8dB at 0.4(rad/s)
Rise time t_r	2.46 s
Overshoot $D\%$	37.8%

. . . .   *Exercise 4.1.3*   . . . .

Using the method depicted in Exercise 4.1.1 with $\tau_I = 1$s places the integral action too close to the cross-over frequency: The lag-action and the lead-action of the controller are overlapping.
Therefore, we increase the value of τ_I to $\tau_I = 10$s and apply the same method as in Exercise 4.1.1 to get the values of τ_D and β.
The resulting controller fulfill all the criteria. (see Figure 2)
The lead lag controller with a phase-margin of 50° has the following characteristics:

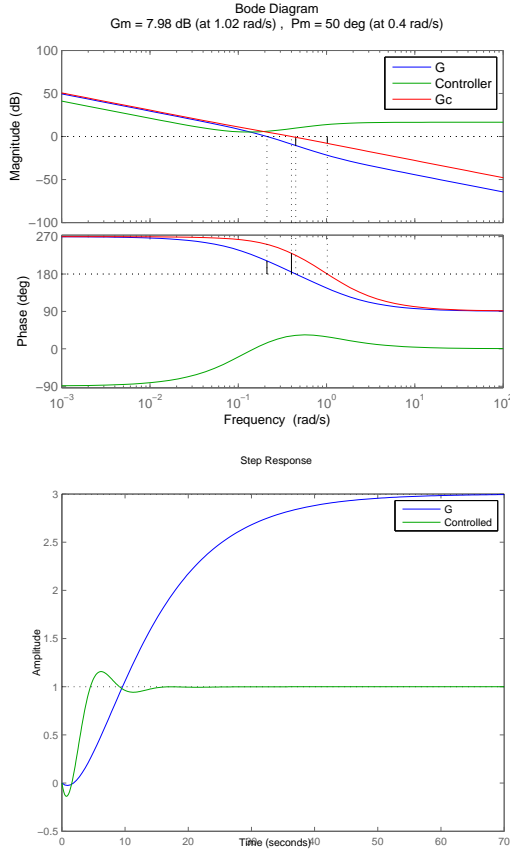|	
Bandwith $(-3dB)$	$[0, 0.99]$(rad/s)
Resonance peak M_T	2.06dB at 0.56(rad/s)
Rise time t_r	2.22 s
Overshoot $D\%$	15.7%

Figure 2: Bode diagram of the system's functions
Step response of the system with and without the controller
Phase-margin: $50°$

4.2 Disturbance attenuation

In this subsection, we need to design a controller which both tracks the reference signal and attenuates the disturbances. The transfer function of the system is:

$$G(s) = \frac{20}{(s+1)((\frac{s}{20})^2 + \frac{s}{20} + 1)}$$

The disturbance transfer function is:

$$G_d(s) = \frac{10}{s+1}$$

The purpose of this subsection is to designed the prefilter function and the feedback function (F_r and F_y) with the following specifications:

- Rise time:
$$t_r \leq 0.20\text{s}$$

- Overshoot:
$$D(\%) \leq 10\%$$

- Step in the disturbance:
$$|y(t)| \leq 1, \forall t \text{ and } |y(t)| \leq 0.1, \forall t \geq 0.5 \text{ s}$$

- Control signal obeys:
$$|u(t)| \leq 1, \forall t$$

Since $G_d(s) = \frac{10}{s+1}$, the cross-over frequency is $\omega_c = 10$ rad/s.
We have the following result:

$$\forall \omega < \omega_c, |Gd(j\omega)| > 1$$

Control action is needed at least for $\omega \in [0, \omega_c]$.
Here, we will try to design F_y in such a way that:

$$L(s) = F_y G \approx \frac{\omega_c}{s}$$

Unproper feedback

Let:

$$F_y = G^{-1}\frac{\omega_c}{s} = \frac{200s^3 + 4200s^2 + 84000s + 80000}{160000s}$$

This controller has 3 zero and 1 pole. Thus, it is not proper. However – using `Matlab` – we can plot the step response of the system and the step response to a step perturbation (see Figure 4).
Figure 4 shows that even if the performance are poor (With a step disturbance, $|y(t)| \leq 0.1$ for $t \geq t_d = 2.5$ s) we still have an attenuation of the perturbation and a step response of good quality (No overshoot and $t_r = 0.22$s).

Proper feedback

Assuming that this controller is the one we want to implement, we need to design it in a proper way. For now, it has 3 zeros and 1 pole. Therefore it is needed to add two more poles.
Let:

$$F_y(s) = G^{-1}\frac{\omega_c}{s} \frac{\omega_1 \omega_2}{(s+\omega_1)(s+\omega_2)}$$

We need to choose $\omega_{1,2}$ in order to not change the system performance two much.
We use the following criteria:

- $\omega_1 = \omega_2$
- ω_1 and ω_2 take action after ω_c

Let's pick $\omega_1 = \omega_2 = \omega_0 = 100\omega_c$. Figure 3 shows the bode diagram of $L(s) = F_y(s)G(s)$ (we can see that $\forall \omega < \omega_c, |L_{proper}|_{dB} \approx |L_{unproper}|_{dB}$). Figure 4 shows that the step response to a perturbation with L_{proper} is the same than with $L_{unproper}$.
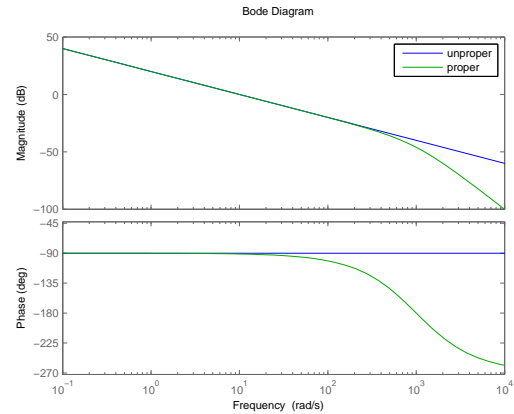

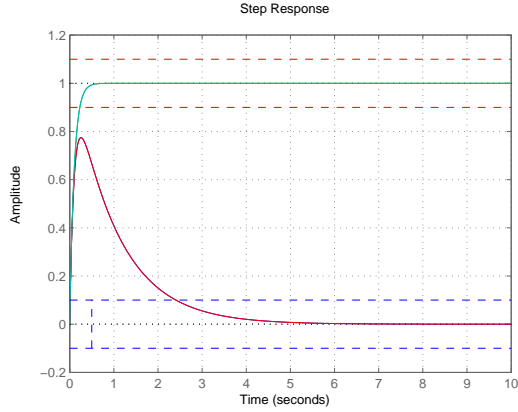Figure 3: Bode diagram of $L(s)$ with the proper & unproper F_y

Figure 4: Step response of the system with the proper & unproper F_y
Both responses are merged

Figure 5: Step response of the system and response to a step in the disturbance with the proper (red, magenta) and unproper (blue, green) feedback

.... *Exercise 4.2.2* ....

The previous controller did not offer good control performance. Indeed, a loop gain of slope -1 at all frequencies gives poor disturbance attenuation. It is needed to design a better controller.

Unproper feedback

As a starting point, we can choose:

$$F_{y,unproper} = \frac{s + \omega_I}{s} G^{-1} G_d$$

Even if this controller is not proper, we can design ω_I – using `Matlab`.
Tuning this parameter leads to the following result: The bigger ω_I is, the fastest perturbation d is attenuates. However, the bigger ω_I is, the bigger the disturbance becomes. Therefore we need to find a value which balance this two behaviors.
$\omega_I = 5$ rad/s was the value that gives us the best results:

$$|y(t)| \leq 1, \forall t \text{ and } |y(t)| \leq 0.1, \forall t \geq 0.5 \text{ s}$$

Proper feedback

As in Exercise 4.2.1, the previous controller is not proper. In order to make it proper, we need to add to poles in ω_0 and ω_1.
Let:

$$F_{y,proper} = \frac{s + \omega_I}{s} \frac{\omega_0 \omega_1}{(s + \omega_0)(s + \omega_1)} G^{-1} G_d$$

This controller is indeed proper. As mentioned in the subject, if $G_d \approx 1$, the controller should contain the inverse of the system. Since for all ω in $[0, \omega_c]$, $G_d \approx 1$, the two poles should be placed *after* ω_c.
We pick:

$$\omega_0 = \omega_1 = 10\omega_I$$

This poles guarantee that the disturbance criteria on $|y(t)|$ is still met.
Figure 5 display the step response of the system and the response to a step in the disturbance.

.... *Exercise 4.2.3* ....

Now that our controller fulfill the criteria about the disturbance attenuation we need to improve it by accelerating the system and controlling the overshoot.
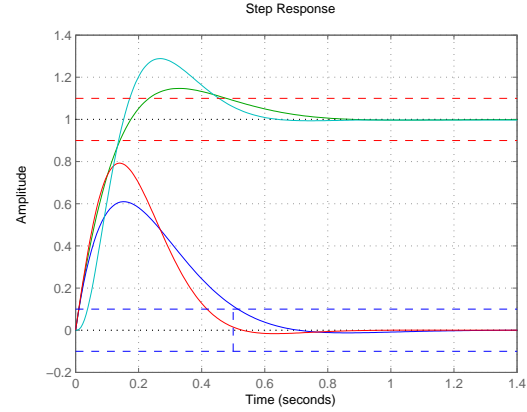
Lead-controller design

To do so we add a lead-controller to F_y: $F_{lead} = K \frac{\tau_D s + 1}{\beta \tau_D s + 1}$.
Using the same method as the one depicted in Exercise 4.1.1, we have the following values for the parameters:

- $K = 1.35$
- $\beta = 0.76$
- $\tau_D = 0.078$

We meet the following specifications:

Phase margin	$15°$ at $\omega_C = 15$ rad/s
Rise time t_r	0.07s
Overshoot $D\%$	24%

The overshoot is still bigger than the 10% required.
Figure 6 shows the step response of the system and the response to a step in the perturbation.
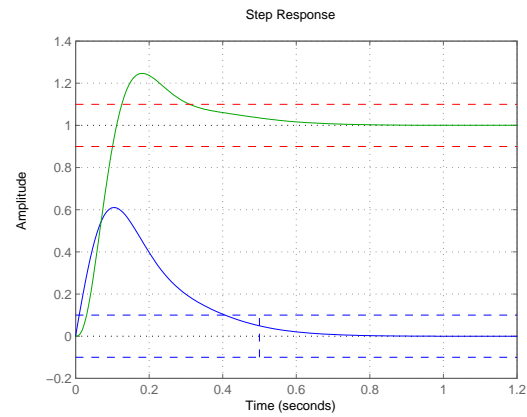

Figure 6: Step response of the system (green) and response to a step in the disturbance (blue) with the addition of the lead-controller

Now that we have added the lead-controller, the expression of F_y is the following:

$$F_y = K \frac{\tau_D s + 1}{\beta \tau_D s + 1} \frac{s + \omega_I}{s} \frac{\omega_0 \omega_1}{(s + \omega_0)(s + \omega_1)} G^{-1} G_d$$

Prefilter design

Once this lead-controller is added to the feedback controller we want to create a prefilter F_r to fulfill all specifications.

Let:

$$F_r = \frac{1}{1 + \tau s}$$

The value of τ is tuned in order to get $|u(t)| < 1$ for all t.
With $\tau = 0.14$, **all the criteria are met** and $\max_t |u(t)| = 0.98$.
Figure 7 shows the step response of the system and the response to a step in the perturbation.

.... *Exercise 4.2.4* ....

The two previous exercises lead to the design of the following controller:

$$F_r(s) = \frac{1}{1 + \tau s}$$

$$F_y(s) = K \frac{\tau_D s + 1}{\beta \tau_D s + 1} \frac{s + \omega_I}{s} \frac{\omega_0^2}{(s + \omega_0)^2} G(s)^{-1} G_d(s)$$

With:

$$
\begin{array}{rcl}
\tau & = & 0.14 \text{ s} \\
K & = & 1.35 \\
\tau_D & = & 0.078 \text{ s} \\
\beta & = & 0.75 \\
\omega_I & = & 5 \text{ rad.s}^{-1} \\
\omega_0 & = & 50 \text{ rad.s}^{-1}
\end{array}
$$

All the criteria are met:

- Rise time:
$$t_r = 0.19 \text{s}$$

- Overshoot:
$$D(\%) = 2.9\%$$

- Step in the disturbance:
$$|y(t)| \leq 1, \forall t \text{ and } |y(t)| \leq 0.1, \forall t \geq 0.4 \text{ s}$$

- Control signal obeys:
$$|u(t)| \leq \max_t |u(t)| = 0.98 \leq 1, \forall t$$

Figure 7 shows the step response of the system, the response to a step in the disturbance and the bode diagram of the sensitivity and complementary sensitivity functions.

Conclusion − **Controlling a single-input/single-output system can be very complicated, since all of the transfer function has to be considered and checked for stability. This exercise was the opportunity for us to design a complex controller taking care of the disturbances.**
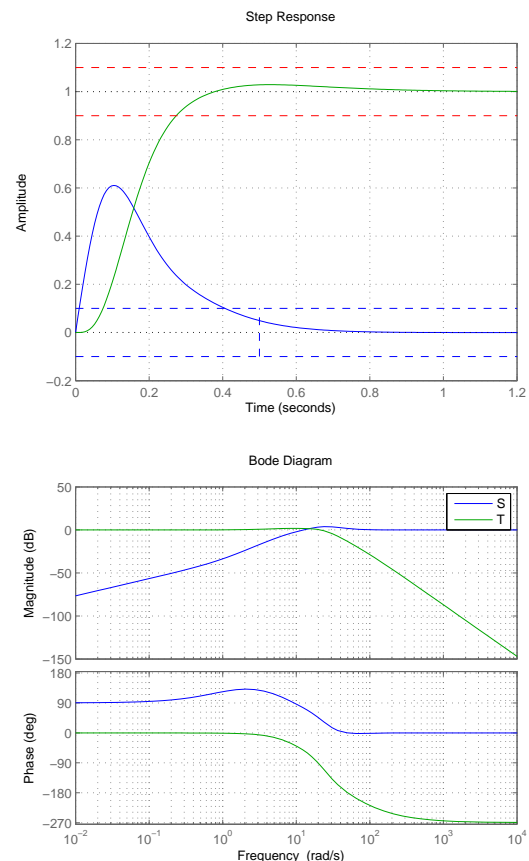

Figure 7: Step response of the system (green) and response to a step in the disturbance (blue) with the addition of the lead-controller
Bode diagram of the sensitivity and complementary sensitivity functions