Computer Exercice:
MULTIVARIABLE SYSTEMS

Jean-Alix DAVID
jadavid@kth.se

Kilian DEMEULEMEESTER
kiliande@kth.se

April 15, 2014

Abstract – **This paper summarizes our work for the second homework in the course *Control, Theory & Practice*. In this exercise linear models of a four-tank process is investigated for two different settings of the valves (see Figure 1). The first part deals with the analysis of the poles and zeros of the system. The second part deals with the decentralized control of the system.**

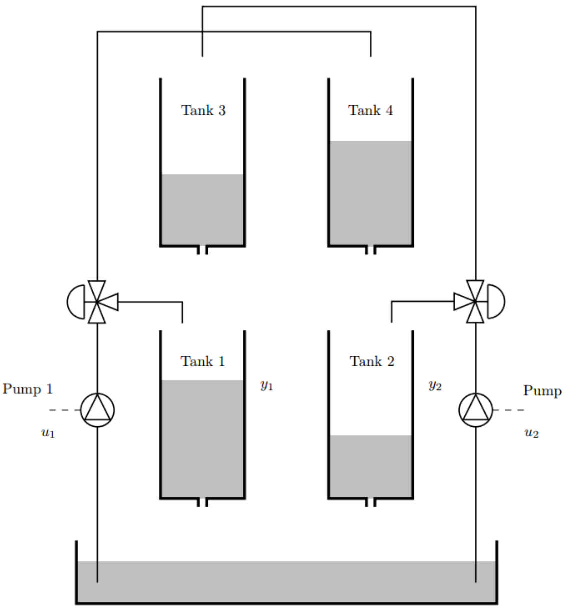

Figure 1: The four tank process

3 Exercises

3.1 Poles, zeros and RGA

The system is modelized as a MIMO system with two inputs (u_1&u_2) and two ouputs (y_1&y_2). The multivariable model with 2 inputs and 2 outputs is given by $Y(s) = G(s)U(s)$. Depending on the settings of the valves – minimum/non-minimum phase model – there are different $G(s)$. Each exercise in the following take care of the two settings.
In this repport the following notation is used:

- $G_{mp}(s)$: Transfer matrix for the minimum phase model;
- $G_{nmp}(s)$: Transfer matrix for the non-minimum phase model.

.... ***Exercise 3.1.1*** ....

Minimum phase model

The transfer matrix $G_{mp}(s)$ is:

$$G_{mp}(s) = \begin{pmatrix} \frac{.035}{s+.056} & \frac{.00046}{s^2+.082s+.0015} \\ \frac{.00044}{s^2+.073s+.0011} & \frac{.030}{s+.052} \end{pmatrix}$$

Poles and zeros of the elements are given in the following table:

$$\text{Poles}(G_{mp,i,j}(s)) = \begin{pmatrix} -.056 & \begin{pmatrix} -.056 \\ -.026 \end{pmatrix} \\ \begin{pmatrix} -.052 \\ -.021 \end{pmatrix} & -.052 \end{pmatrix}$$

$$\text{Zeros}(G_{mp,i,j}(s)) = \emptyset$$

Non-minimum phase model

The transfer matrix $G_{nmp}(s)$ is:

$$G_{nmp}(s) = \begin{pmatrix} \frac{.021}{s+.051} & \frac{.0026}{s^2+.14s+.0044} \\ \frac{.0032}{s^2+.14s+.0043} & \frac{.018}{s+.047} \end{pmatrix}$$

Poles and zeros of the elements are given in the following table:

$$\text{Poles}(G_{nmp,i,j}(s)) = \begin{pmatrix} -.051 & \begin{pmatrix} -.086 \\ -.051 \end{pmatrix} \\ \begin{pmatrix} -.091 \\ -.047 \end{pmatrix} & -.047 \end{pmatrix}$$

$$\text{Zeros}(G_{nmp,i,j}(s)) = \emptyset$$

.... ***Exercise 3.1.2*** ....

Minimum phase model

Poles and zeros of the multivariable system in the minimum phase model case are:

$$\text{Poles}(G_{mp}(s)) = \begin{pmatrix} -.056 \text{ (double)} \\ -.052 \text{ (double)} \\ -.021 \\ -.026 \end{pmatrix}$$

$$\text{Zeros}(G_{mp}(s)) = \begin{pmatrix} -.0093 \\ -.038 \\ -.056 \\ -.052 \end{pmatrix}$$

Non-minimum phase model

Poles and zeros of the multivariable system in the non-minimum phase model case are:

$$\text{Poles}(G_{nmp}(s)) = \begin{pmatrix} -.051 \text{ (double)} \\ -.047 \text{ (double)} \\ -.091 \\ -.086 \end{pmatrix}$$

$$\text{Zeros}(G_{nmp}(s)) = \begin{pmatrix} -.24 \\ .059 \\ -.051 \\ -.047 \end{pmatrix}$$

We see that there is one zero of $G_{nmp}(s)$ is in the right-hand side of the complex plane. Therefore the system is *(indeed)* a non-minimum phase systems. This system offers a gain in the wrong direction for rapid inputs: the cross over frequency should be small.
Using the rule of thumb defined in the course the bandwith limitation is :

$$\omega_{BS} \leq \frac{z}{2} = .029$$

.... ***Exercise 3.1.3*** ....

The largest and smallest singular values for the system at different frequencies for the minimum phase model are visible on Figure 2.

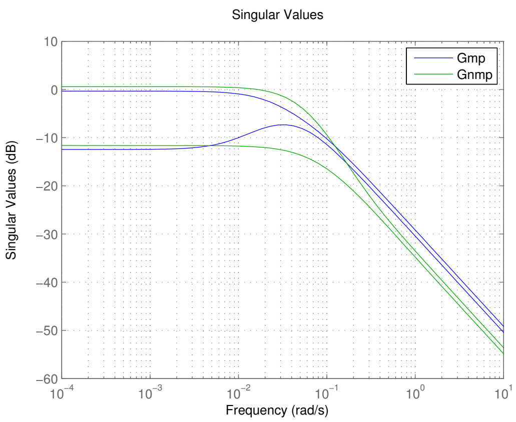

Figure 2: $\underline{\sigma}(G(i\omega))$ and $\bar{\sigma}(G(i\omega))$ for the minimum phase model (blue) and the non-minimum phase model (green)

We have the following value for the H_∞ norm of the systems:

$$||G||_{\infty,mp} = .96 \quad \textbf{and} \quad ||G||_{\infty,nmp} = 1.1$$

.... ***Exercise 3.1.4*** ....

Minimum phase model

The RGA of the minimum phase system at frequency 0 is:

$$\text{RGA}(G_{mp}(0)) = G_{mp}(0).*[G_{mp}^{-1}(0)]^T = \begin{pmatrix} 1.6 & -0.56 \\ -0.56 & 1.6 \end{pmatrix}$$

Using the second rule of thumb expressed in the subject gives the following result:

	u_1	u_2
y_1	1	0
y_2	0	1

(0 – avoid pairing, 1 – pairing possible).

Non-minimum phase model

The RGA of the non-minimum phase system at frequency 0 is:

$$\text{RGA}(G_{nmp}(0)) = G_{nmp}(0).*[G_{nmp}^{-1}(0)]^T = \begin{pmatrix} -0.56 & 1.6 \\ 1.6 & -0.56 \end{pmatrix}$$

Using the second rule of thumb expressed in the subject gives the following result:

	u_1	u_2
y_1	0	1
y_2	1	0

(0 – avoid pairing, 1 – pairing possible).

Remark: The two RGA matrix at frequency 0 are symetrical. This seems legit since the alimentation of tanks 1 & 4 is made by pump 1 and the alimentation of tanks 2 & 3 is made by pump 2.

.... ***Exercise 3.1.5*** ....

Figure 3 shows the step response of the systems (minimum phase & non-minimum phase). As we can see the system is indeed coupled, with the same correlation as the one depicted by the properties of RGA.

.... ***Exercise 3.1.6*** ....

The impact of each input can be measured as:

$$\frac{|y_{\text{paired to } u_i}|}{|y_{\text{not-paired}}|}$$

We have the following relation:

- Input (1):

$$\frac{|y_{1,m}|_\infty}{|y_{2,m}|_\infty} = \frac{.6}{.4} = 1.5 \leq \frac{|y_{2,nm}|_\infty}{|y_{1,nm}|_\infty} = \frac{.7}{.4} = 1.75$$

- Input (2):

$$\frac{|y_{2,m}|_\infty}{|y_{1,m}|_\infty} = \frac{.6}{.3} = 2 \geq \frac{|y_{1,nm}|_\infty}{|y_{2,nm}|_\infty} = \frac{.6}{.4} = 1.5$$

This result can be sum up as:

Desired control	y_1	y_2
$G_m(s)$	Act on u_1 *Stronger coupling*	Act on u_2 *Weaker coupling*
$G_{nm}(s)$	Act on u_2 *Weaker coupling*	Act on u_1 *Stronger coupling*

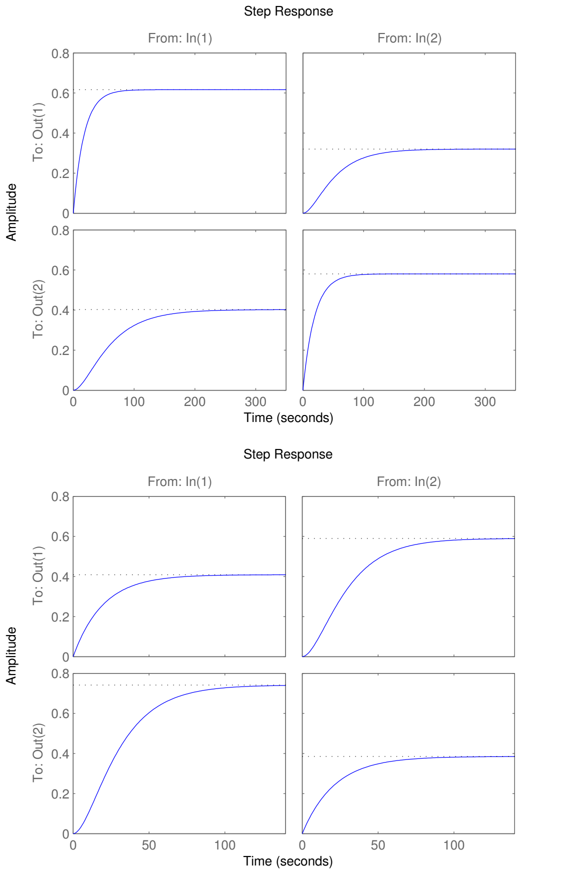

Figure 3: Step responses of the systems
Minimum phase system (top) and non-minimum phase system (bottom)

3.2 Decentralized control

In this section we will investigate the control of the four-tank process with a decentralized control.

Minimum phase system

In the minimum phase system the input u_1 is coupled with y_1 and u_2 is coupled with y_2.
We will therefore use:

$$F_m(s) = \begin{bmatrix} f_1(s) & 0 \\ 0 & f_2(s) \end{bmatrix}$$

Non-minimum phase system

In the non-minimum phase system the input u_1 is coupled with y_2 and u_2 is coupled with y_1.
We will therefore use:

$$F_{nm}(s) = \begin{bmatrix} 0 & f_1(s) \\ f_2(s) & 0 \end{bmatrix}$$

.... ***Exercise 3.2.1*** ....

The specification of the controlled system are the following:

- Phase margin: $\varphi_m = \pi/3$
- Crossover frequency:
  - Minimum phase system: $\omega_{c,m} = .1$rad/s
  - Non-minimum phase system: $\omega_{c,nm} = .02$rad/s

As we can see $\omega_{c,nm} \leq .029$ (see Exercise 3.1.2). Therefore the RHP zero will not be a problem.

The controller are designed using the method depicted in the subject. We have the following values:

	$f_1(s)$	$f_2(s)$
$F_m(s)$	$1.68(1+\frac{1}{5.90s})$	$2.01(1+\frac{1}{6.39s})$
$F_{nm}(s)$	$0.15(1+\frac{1}{3.94s})$	$0.14(1+\frac{1}{4.81s})$

Figure 4 shows the bode diagrams of the transfer function L for the coupled input–output. The margin and cross-over criteria are fullfilled. Moreover the phase margin for the non-minimum phase system is bigger ($\approx 80°$) than the one expected, which is even better.

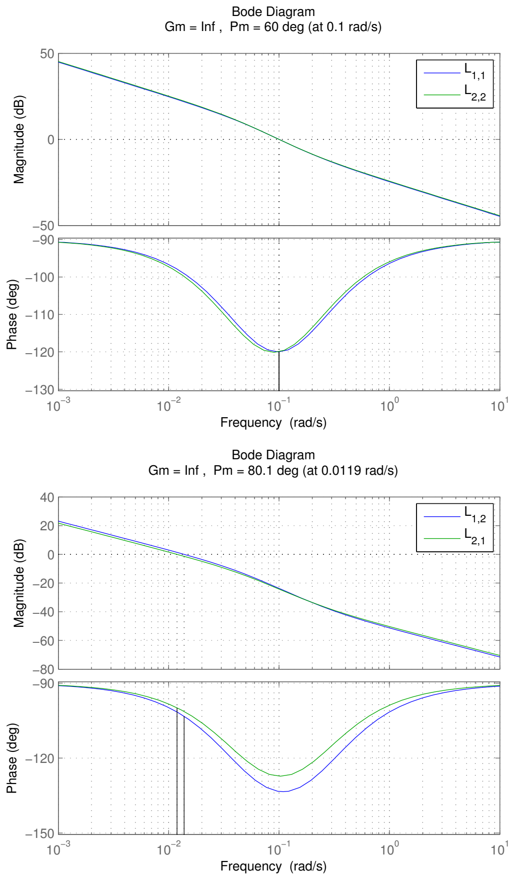

Figure 4: Bode diagram of L for the minimum phase system (top) and the non-minimum phase system (bottom)

.... ***Exercise 3.2.2*** ....

Minimum phase model

Figure 5 shows the maximum and minimum singular value of S and T as a function of the frequency. In this minimum-phase case, no restricting criteria was formulated regarding to S and T.

Non-minimum phase model

Figure 6 shows the maximum and minimum singular value of S and T as a function of the frequency. In this non-minimum-phase case, since there is a RHP zero in the transfer matrix we had the following criteria regarding the bandwith limitation : $\omega_{BS} \leq 0.029$. This criteria is fulfilled.

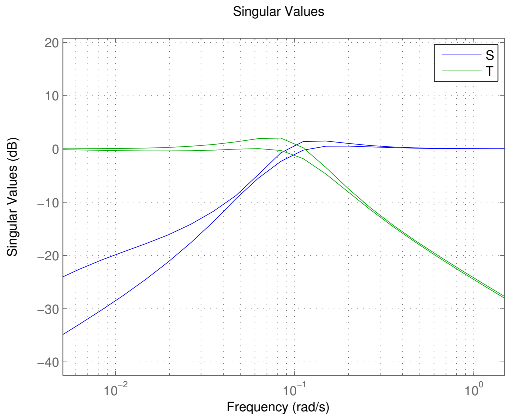

Figure 5: Singular values of S and T for the minimum phase system

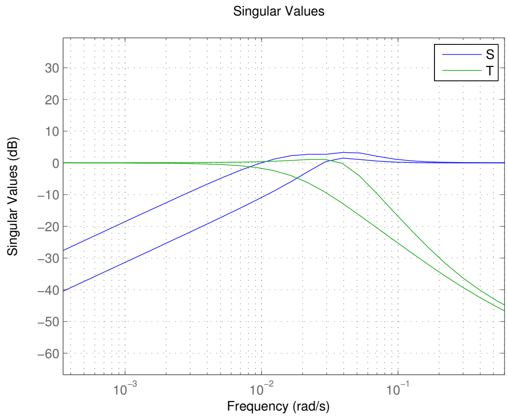

Figure 6: Singular values of S and T for the non-minimum phase system

Exercise 3.2.3

Using simulink we simulate the closed-loop system for an input:

$$r(t) = \begin{cases} r_1(t) & = & u_{t\geq 100} \\ r_2(t) & = & u_{t\geq 500} \end{cases}$$

Minimum phase model

Figure 7 shows the output y_1 and y_2 of the closed-loop system in the minimum phase case. The control quality is the following:

	Overshoot coupled	Overshoot non-coupled	Rise time
y_1	12%	0.1	30s
y_2	13%	0.1	30s

Non-minimum phase model

Figure 8 shows the output y_1 and y_2 of the closed-loop system in the non-minimum phase case. The control quality is the following:

	Overshoot coupled	Overshoot non-coupled	Rise time
y_1	0%	0.5	350s
y_2	0%	0.35	200s

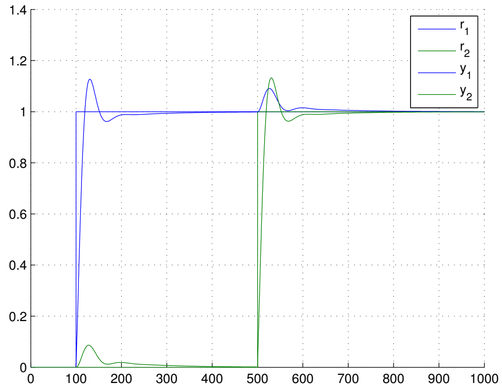

Figure 7: Response of the minimum phase system to $r(t)$

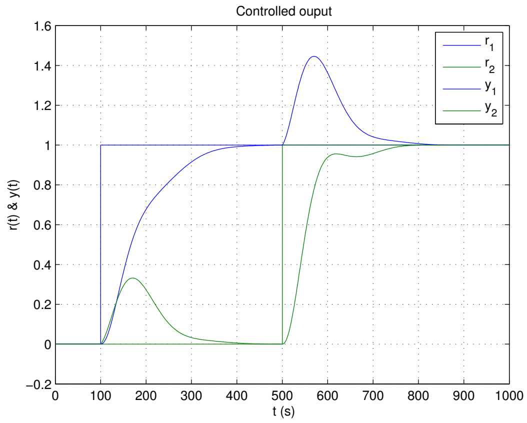

Figure 8: Response of the non-minimum phase system to $r(t)$

Results

Figure 7 & 8 shows that the outputs are indeed coupled. However the control performance are more satisfying in the minimum-phase case. This is legit since the minimum-phase system do not present a RHP zero.

Exercise 3.2.4

As we have seen in the previous exercise, the control performance are better in the minimum phase case. (Even if its a bad idea in most cases to implement a decentralized controller). Moreover since the non-minimum phase system presents a zero in the RHP it is more complicated to design this kind of controller

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Conclusion – **This exercise was the opportunity to implement a decentralized controller and understand the impact of a ZHP zero. As seen in the lectures, controlling a MIMO system as if it was two SISO system is not the best that can be done.**