Computer Exercice:
Classical Loop-Shaping

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Abstract **– This paper resume our work for the first homework in the course *Control, Theory & Practice*. In the first part we deal with controlling a simple single-input/single-output (SISO) system. In the second part, we still have a SISO system but we have to take care of the disturbances, rising the difficulty to design the controller.**

4 Exercises

4.1 Basics

We consider a system which can be modeled by the transfer function:

$$G(s) = \frac{3(-s+1)}{(5s+1)(10s+1)}$$

.... ***Exercise 4.1.1*** ....

We want to design a lead-lag controller which eliminates the static control error for a step response in the reference signal. The controller transfer function is the following:

$$F(s) = K\frac{\tau_D s+1}{\beta\tau_D s+1}\frac{\tau_I s+1}{\tau_I s+\gamma}$$

We want to fulfill the following criteria:

- Phase margin of 30° at the cross-over frequency $\omega_c = 0.4$ rad/s.
- No static control error for a step response

Phase-lag action

Setting the parameters $\tau_I = 1$s and $\gamma = 0$ leads to an integral part in the system. This guarantee no static control error. ($\lim_{t\to\infty} e(t) = 0$).

Phase-lead action

The lead-action of the controller leads to the following values in the frequency domain:

- Phase margin: $\phi_0 = -50°$
- Gain: $|G(\omega = 0.4 \text{ rad/s}| = 0.9436$

Thus:

- $\beta = \frac{1-\sin(30°-\phi_0)}{1+\sin(30°-\phi_0)} = 0.0086$
- $\tau_D = \frac{1}{\omega_c\sqrt{a}} = 26.9459$
- $K = \frac{\sqrt{a}}{m} = 0.0983$

Controller

The two previous paragraph lead to the design of the controller as:

$$F(s) = K\frac{\tau_D s+1}{\beta\tau_D s+1}\frac{\tau_I s+1}{\tau_I s+\gamma} = 0.0983\frac{26.9459s+1}{0.2319s+1}\frac{s+1}{s}$$

Figure 1 shows the bode diagrams of the different function of the system and the step response.
The two criteria (phase-margin and static error null) are fulfilled.

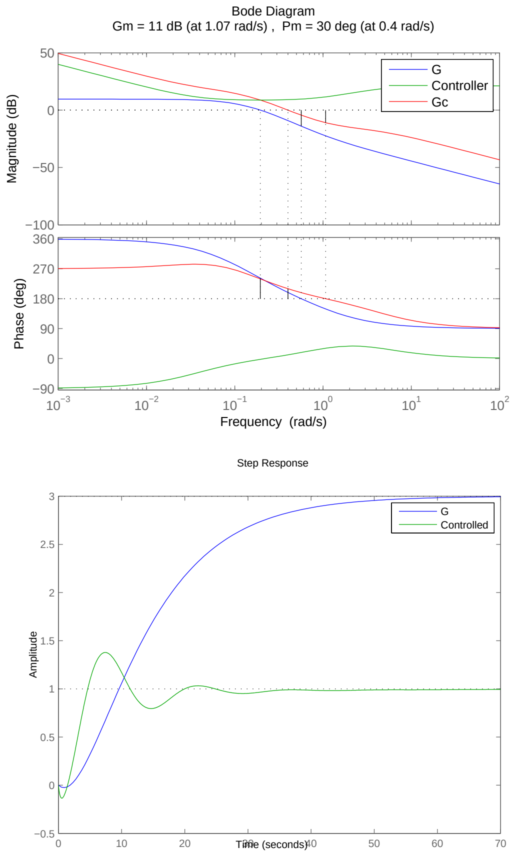

Figure 1: Bode diagram of the system's functions
Step response of the system with and without the controller
Phase-margin: 30°

.... ***Exercise 4.1.2*** ....

Adding the lead lag controller to the system leads to the characteristics reported in the following table

Bandwith $(-3dB)$	$[0, 0.75]$(rad/s)
Resonance peak M_T	5.8dB at 0.4(rad/s)
Rise time t_r	2.46 s
Overshoot $D\%$	37.8%

.... ***Exercise 4.1.3*** ....

Using the method depicted in Exercise 4.1.1 with $\tau_I = 1$s places the integral action too close to the cross-over frequency: The lag-action and the lead-action of the controller are overlapping.
Therefore, we increase the value of τ_I to $\tau_I = 10$s and apply the same method as in Exercise 4.1.1 to get the values of τ_D and β.
The resulting controller fulfill all the criteria. (see Figure 2)
The lead lag controller with a phase-margin of 50° has the following characteristics:

Bandwith ($-3dB$)	$[0, 0.99]$(rad/s)
Resonance peak M_T	2.06dB at 0.56(rad/s)
Rise time t_r	2.22 s
Overshoot $D\%$	15.7%

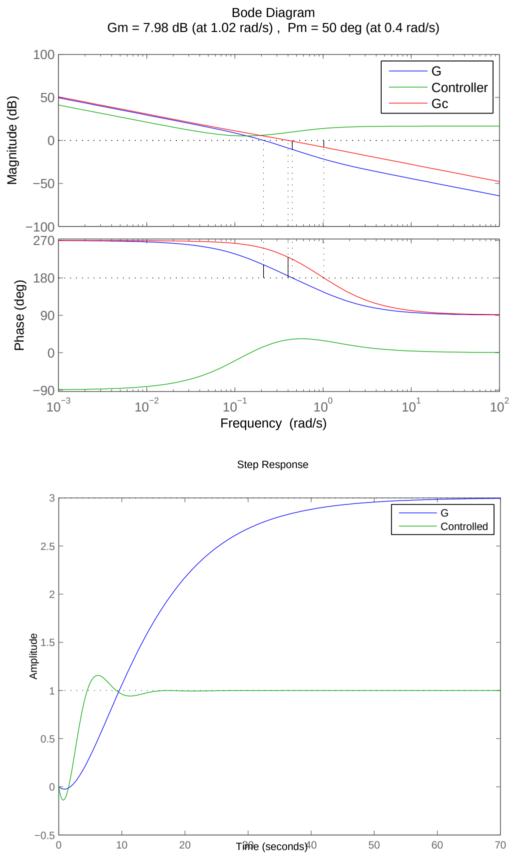

Figure 2: Bode diagram of the system's functions
Step response of the system with and without the controller
Phase-margin: 50°

4.2 Disturbance attenuation

In this subsection, we need to design a controller which both tracks the reference signal and attenuates the disturbances. The transfer function of the system is:

$$G(s) = \frac{20}{(s+1)((\frac{s}{20})^2 + \frac{s}{20} + 1)}$$

The disturbance transfer function is:

$$G_d(s) = \frac{10}{s+1}$$

The purpose of this subsection is to designed the prefilter function and the feedback function (F_r and F_y) with the following specifications:

- Rise time:
$$t_r \leq 0.20\text{s}$$
- Overshoot:
$$D(\%) \leq 10\%$$
- Step in the disturbance:
$$|y(t)| \leq 1, \forall t \text{ and } |y(t)| \leq 0.1, \forall t \geq 0.5 \text{ s}$$
- Control signal obeys:
$$|u(t)| \leq 1, \forall t$$

Exercise 4.2.1

Since $G_d(s) = \frac{10}{s+1}$, the cross-over frequency is $\omega_c = 10$ rad/s.
We have the following result:

$$\forall \omega < \omega_c, |Gd(j\omega)| > 1$$

Control action is needed at least for $\omega \in [0, \omega_c]$.
Here, we will try to design F_y in such a way that:

$$L(s) = F_y G \approx \frac{\omega_c}{s}$$

Unproper feedback

Let:

$$F_y = G^{-1}\frac{\omega_c}{s} = \frac{200s^3 + 4200s^2 + 84000s + 80000}{160000s}$$

This controller has 3 zero and 1 pole. Thus, it is not proper. However – using `Matlab` – we can plot the step response of the system and the step response to a step perturbation (see Figure 4).
Figure 4 shows that even if the performance are poor (With a step disturbance, $|y(t)| \leq 0.1$ for $t \geq t_d = 2.5$ s) we still have an attenuation of the perturbation and a step response of good quality.

Proper feedback

Assuming that this controller is the one we want to implement, we need to design it in a proper way. For now, it has 3 zeros and 1 pole. Therefore it is needed to add two more poles.
Let:

$$F_y(s) = G^{-1}\frac{\omega_c}{s}\frac{\omega_1\omega_2}{(s+\omega_1)(s+\omega_2)}$$

We need to choose $\omega_{1,2}$ in order to not change the system performance two much.
We use the following criteria:

- $\omega_1 = \omega_2$
- ω_1 and ω_2 take action after ω_c

Let's pick $\omega_1 = \omega_2 = \omega_0 = 100\omega_c$. Figure 3 shows the bode diagram of $L(s) = F_y(s)G(s)$ (we can see that $\forall\omega < \omega_c, |L_{proper}|_{dB} \approx |L_{unproper}|_{dB}$). Figure 4 shows that the step response to a perturbation with L_{proper} is the same than with $L_{unproper}$.

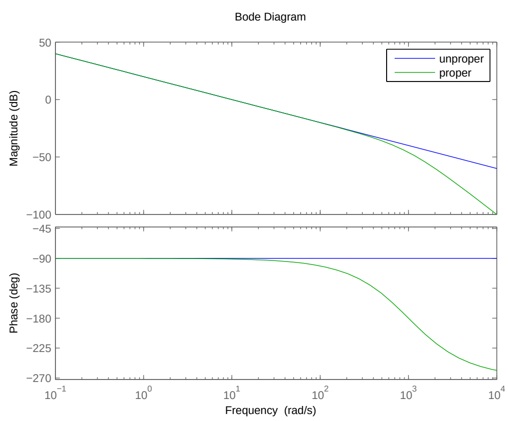

Figure 3: Bode diagram of $L(s)$ with the proper & unproper F_y

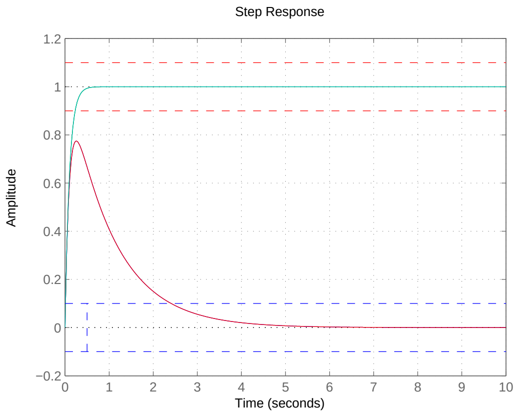

Figure 4: Step response of the system with the proper & unproper F_y
Both responses are merged

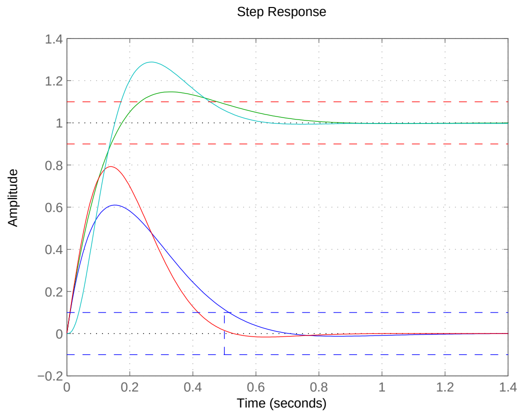

Figure 5: Step response of the system and response to a step in the disturbance with the proper (red, magenta) and unproper (blue, green) feedback

Exercise 4.2.2

The previous controller did not offer good control control performance. Indeed, a loop gain of slope -1 at all frequencies gives poor disturbance attenuation. It is needed to design a better controller.

Unproper feedback

As a starting point, we can choose:

$$F_{y,unproper} = \frac{s+\omega_I}{s} G^{-1} G_d$$

Even if this controller is not proper, we can design ω_I – using `Matlab`.
Tuning this parameter leads to the following result: The bigger ω_I is, the fastest perturbation d is attenuates. However, the bigger ω_I is, the bigger the disturbance becomes. Therefore we need to find a value which balance this two behaviors.
$\omega_I = 5$ rad/s was the value that gives us the best results:

$$|y(t)| \leq 1, \forall t \text{ and } |y(t)| \leq 0.1, \forall t \geq 0.5 \text{ s}$$

Proper feedback

As in Exercise 4.2.1, the previous controller is not proper. In order to make it proper, we need to add to poles in ω_0 and ω_1.
Let:

$$F_{y,proper} = \frac{s+\omega_I}{s} \frac{\omega_0\omega_1}{(s+\omega_0)(s+\omega_1)} G^{-1} G_d$$

This controller is indeed proper. As mentioned in the subject, if $G_d \approx 1$, the controller should contain the inverse of the system. Since for all ω in $[0, \omega_c]$, $G_d \approx 1$, the two poles should be placed *after* ω_c.
We pick:

$$\omega_0 = \omega_1 = 10\omega_I$$

This poles guarantee that the disturbance criteria on $|y(t)|$ is still met.
Figure 5 display the step response of the system and the response to a step in the disturbance.

Exercise 4.2.3

Now that our controller fulfill the criteria about the disturbance attenuation we need to improve it by accelerating the system and controlling the overshoot.

Lead-controller design

To do so we add a lead-controller to F_y: $F_{lead} = K\frac{\tau_D s+1}{\beta\tau_D s+1}$.
Using the same method as the one depicted in Exercise 4.1.1, we have the following values for the parameters:

- $K = 1.35$
- $\beta = 0.76$
- $\tau_D = 0.078$

We meet the following specifications:

Phase margin	$15°$ at $\omega_C = 15$ rad/s
Rise time t_r	0.07s
Overshoot $D\%$	24%

The overshoot is still bigger than the 10% required.
Figure 6 shows the step response of the system and the response to a step in the perturbation.

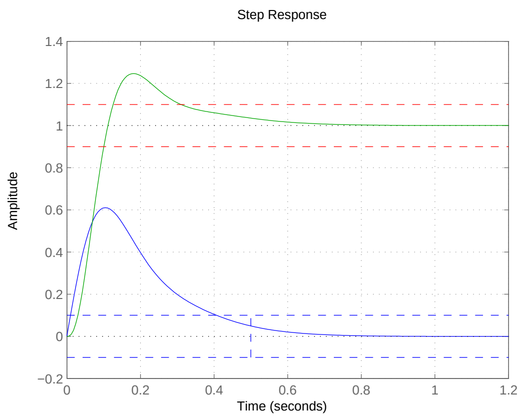

Figure 6: Step response of the system (green) and response to a step in the disturbance (blue) with the addition of the lead-controller

Now that we have added the lead-controller, the expression of F_y is the following:

$$F_y = K\frac{\tau_D s+1}{\beta\tau_D s+1} \frac{s+\omega_I}{s} \frac{\omega_0\omega_1}{(s+\omega_0)(s+\omega_1)} G^{-1} G_d$$

Prefilter design

Once this lead-controller is added to the feedback controller we want to create a prefilter F_r to fulfill all specifications.

Let:

$$F_r = \frac{1}{1+\tau s}$$

The value of τ is tuned in order to get $|u(t) < 1$ for all t.
With $\tau = 0.14$, **all the criteria are met** and $\max_t |u(t)| = 0.98$.
Figure 7 shows the step response of the system and the response to a step in the perturbation.

Exercise 4.2.4

The two previous exercises lead to the design of the following controller:

$$F_r(s) = \frac{1}{1+\tau s}$$

$$F_y(s) = K\frac{\tau_D s+1}{\beta\tau_D s+1}\frac{s+\omega_I}{s}\frac{\omega_0^2}{(s+\omega_0)^2}G(s)^{-1}G_d(s)$$

With:

$$\begin{aligned} \tau &= 0.14 \text{ s} \\ K &= 1.35 \\ \tau_D &= 0.078 \text{ s} \\ \beta &= 0.75 \\ \omega_I &= 5 \text{ rad.s}^{-1} \\ \omega_0 &= 50 \text{ rad.s}^{-1} \end{aligned}$$

All the criteria are met:

- Rise time:
$$t_r \leq 0.20\text{s}$$
- Overshoot:
$$D(\%) \leq 10\%$$
- Step in the disturbance:
$$|y(t)| \leq 1, \forall t \text{ and } |y(t)| \leq 0.1, \forall t \geq 0.5 \text{ s}$$
- Control signal obeys:
$$|u(t)| \leq 1, \forall t$$

Figure 7 shows the step response of the system, the response to a step in the disturbance and the bode diagram of the sensitivity and complementary sensitivity functions.

Conclusion **– Controlling a single-input/single-output system can be very complicated, since all of the transfer function has to be considered and checked for stability. This exercise was the opportunity for us to design a complex controller taking care of the disturbances.**

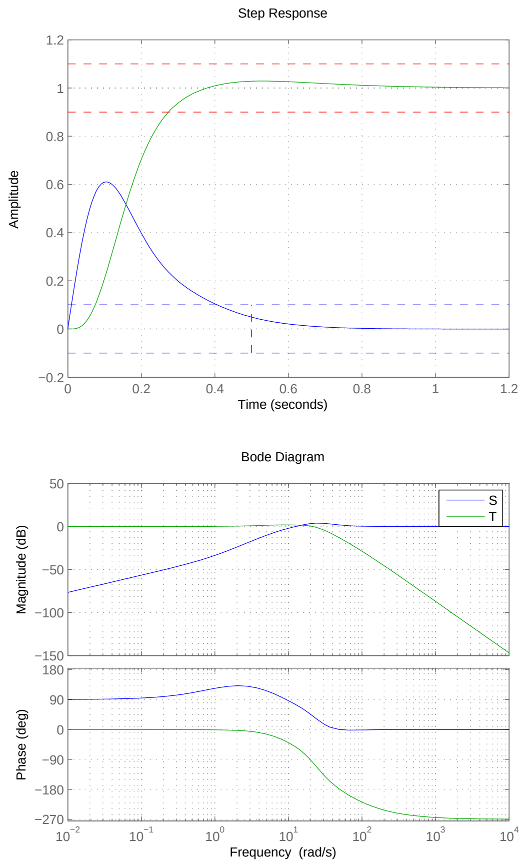

Figure 7: Step response of the system (green) and response to a step in the disturbance (blue) with the addition of the lead-controller
Bode diagram of the sensitivity and complementary sensitivity functions