Gaussian Profile Solutions to the 1-D wave and diffusion equations

restart;

 # Gaussian profile leaving sigma(t) as yet unspecified

$$f := (t, x) \rightarrow \frac{1}{\mathrm{sqrt}(2 \cdot \mathrm{Pi}) \cdot \mathrm{sigma}(t)} \cdot \exp\left(-\frac{(x - c \cdot t)^2}{2 \cdot \mathrm{sigma}(t)^2}\right);$$

$$(t, x) \rightarrow \frac{e^{-\frac{1}{2} \frac{(x-c t)^2}{\sigma(t)^2}}}{\sqrt{2 \pi}\, \sigma(t)} \tag{1}$$

 # Area under the curve and energy

$int(f(t, x), x = -\mathrm{infinity}..\mathrm{infinity})$ assuming $\mathrm{sigma}(t) > 0$;

$int\left(\frac{f(t, x)^2}{2}, x = -\mathrm{infinity}..\mathrm{infinity}\right)$ assuming $\mathrm{sigma}(t) > 0$;

$$1$$

$$\frac{1}{4\, \sigma(t) \sqrt{\pi}} \tag{2}$$

 # Evaluate the diffusion equation, set c=0

$de1 := simplify(diff(f(t, x), t) - \mathrm{nu} \cdot diff(f(t, x), x, x));$

$de2 := subs(c = 0, de1);$

$$-\frac{1}{2} \frac{1}{\sqrt{\pi}\, \sigma(t)^5} \left(\sqrt{2}\, e^{-\frac{1}{2} \frac{(c t - x)^2}{\sigma(t)^2}} \left(-\left(\frac{d}{dt} \sigma(t)\right) \sigma(t)\, c^2 t^2 + 2 \left(\frac{d}{dt} \sigma(t)\right) \sigma(t)\, c\, t\, x + \sigma(t)^2 c^2 t + c^2 \nu t^2 + \left(\frac{d}{dt} \sigma(t)\right) \sigma(t)^3 - \left(\frac{d}{dt} \sigma(t)\right) \sigma(t)\, x^2 - \sigma(t)^2 c\, x - 2\, c\, \nu\, t\, x - \sigma(t)^2 \nu + \nu x^2\right)\right)$$

$$-\frac{1}{2} \frac{\sqrt{2}\, e^{-\frac{1}{2} \frac{x^2}{\sigma(t)^2}} \left(\left(\frac{d}{dt} \sigma(t)\right) \sigma(t)^3 - \left(\frac{d}{dt} \sigma(t)\right) \sigma(t)\, x^2 - \sigma(t)^2 \nu + \nu x^2\right)}{\sqrt{\pi}\, \sigma(t)^5} \tag{3}$$

 # Factor the expression

$factor(de2)$

$$-\frac{1}{2} \frac{\sqrt{2}\, e^{-\frac{1}{2} \frac{x^2}{\sigma(t)^2}} (\sigma(t) - x)\, (\sigma(t) + x) \left(\left(\frac{d}{dt} \sigma(t)\right) \sigma(t) - \nu\right)}{\sqrt{\pi}\, \sigma(t)^5} \tag{4}$$

 # Solve for sigma(t) - diffusion

$s1 := dsolve(diff(\mathrm{sigma}(t), t) \cdot \mathrm{sigma}(t) = \mathrm{nu});$

$subs(\_C1 = sigma\_0, s1[1]);$

$$\sigma(t) = \sqrt{2 \nu t + \_C1}, \sigma(t) = -\sqrt{2 \nu t + \_C1}$$

$$\sigma(t) = \sqrt{2 \nu t + sigma\_0} \tag{5}$$

 # Peak value

$f(t, 0);$

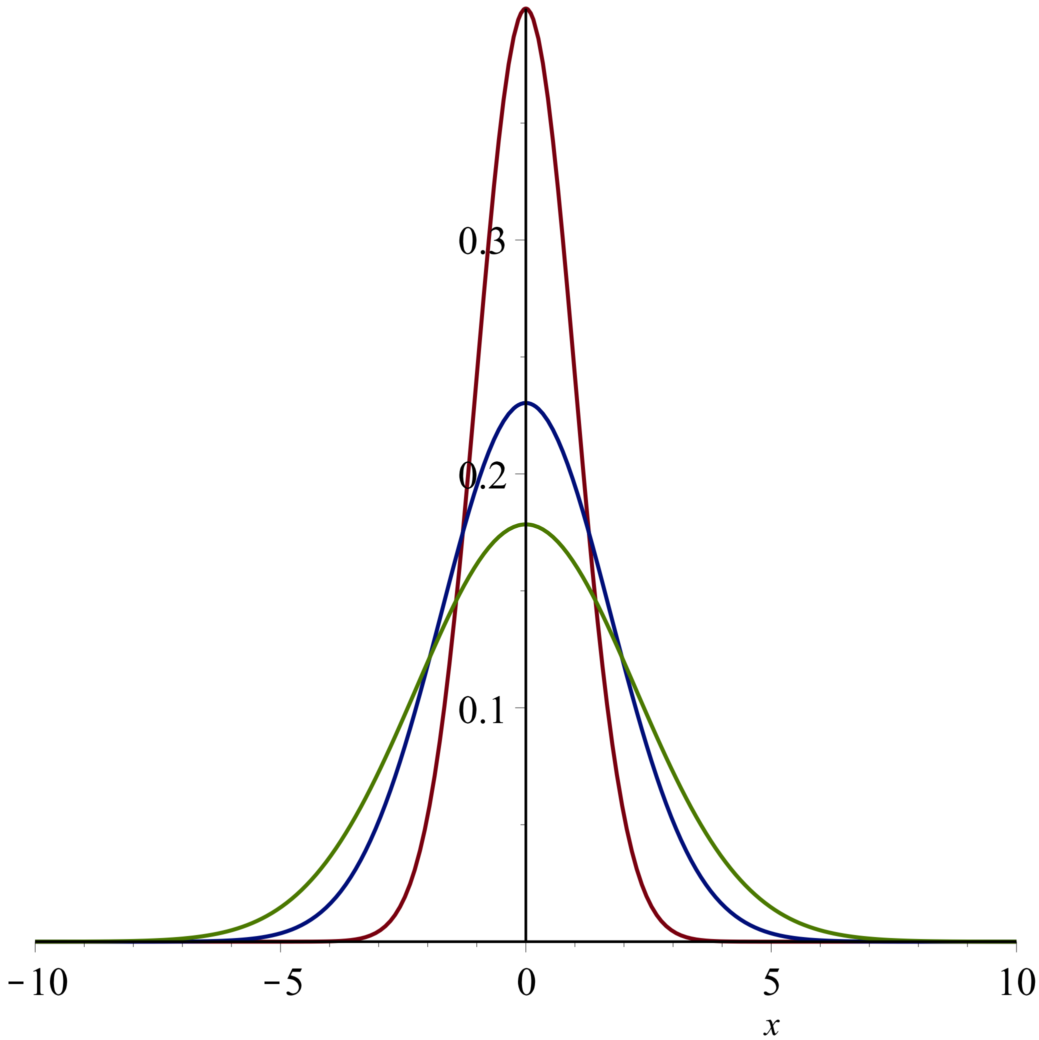
$$\frac{1}{2}\,\frac{\sqrt{2}\,\mathrm{e}^{-\frac{1}{2}\,\frac{c^2 t^2}{1+2t}}}{\sqrt{\pi}\,\sqrt{1+2t}} \tag{6}$$

 # Plot profiles - diffusion with c=0, nu=1, profile spreads out

sigma := $t \rightarrow \mathrm{sqrt}(1 + 2\cdot 1\cdot t)$;

$plot([subs(\{c=0\}, f(0,x)), subs(\{c=0\}, f(1,x)), subs(\{c=0\}, f(2,x))], x=-10..10)$;

$$t \rightarrow \sqrt{1+2t}$$



 # Evaluate wave equation

$we1 := simplify(diff(f(t,x),t) + c\cdot diff(f(t,x),x))$;

$$\frac{1}{2}\,\frac{\sqrt{2}\,\mathrm{e}^{-\frac{1}{2}\,\frac{(ct-x)^2}{1+2t}}\left(c^2 t^2 - 2\,c\,t\,x + x^2 - 2\,t - 1\right)}{\sqrt{\pi}\,(1+2t)^{5/2}} \tag{7}$$

 # Factor the expression

$factor(we1)$

$$-\frac{1}{2}\,\frac{\sqrt{2}\,\mathrm{e}^{-\frac{1}{2}\,\frac{(c\,t-x)^2}{\sigma(t)^2}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\sigma(t)\right)\left(c\,t+\sigma(t)-x\right)\left(-c\,t+\sigma(t)+x\right)}{\sqrt{\pi}\,\sigma(t)^4} \tag{8}$$

 # Solve for sigma(t) - wave

$s2 := dsolve(diff(\text{sigma}(t), t) = 0);$

$$\sigma(t) = \_C1 \tag{9}$$

 # Plot profiles - wave, semi-width remains constant and profile translates with speed c=1

sigma $:= t \to 1;$

$plot([subs(\{c = 1\}, f(0, x)), subs(\{c = 1\}, f(1, x)), subs(\{c = 1\}, f(2, x))], x = -10..10);$

$$t \to 1$$