Sub-Grid Models

1. Smagorinski

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = 2\nu_{SG}\hat{S}_{ij}$$

$$\hat{S}_{ij} = \frac{1}{2}\left(\frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i}\right)$$

$$\nu_{SG} = (C_0\Delta)^2 \mathcal{S}$$

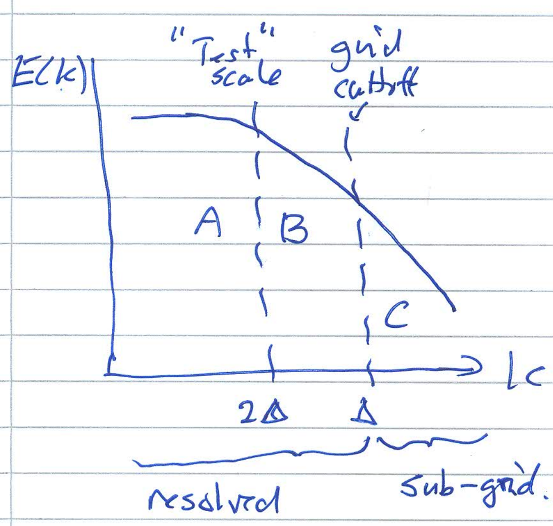
$$\mathcal{S} = (2\hat{S}_{ij}\hat{S}_{ij})^{\frac{1}{2}}$$

$$\Delta = (\Delta x\,\Delta y\,\Delta z)^{1/3}$$ or equivalent grid scale.

$C_0 \simeq 0.095 - 0.15$
Smagorinsky model constant.

Need to ensure that $\nu_{SG} \to 0$ at walls, $\Rightarrow$ various forms of wall damping employed, + Wall treatment to ensure that $\hat{u}$ limits to law-of-wall close to boundaries.

2. Dynamic SGS (Germano 1990)



Identity relates statistics of scales at A|B boundary to scales at B|C boundary, $\Rightarrow$ Test scale of 2Δ, then use this to determine set of least squares equations for C_0 (at B|C boundary) in Smagorinsky model.

$\Rightarrow$ is a Dynamic Tuning procedure applied to Smagorinsky.

3. Approximate Deconvolution (Stolz, others 1999)

General idea is that the filter can be considered a convolution of a filter kernel G against the flow $u(\underline{x}, t)$.

$$\Rightarrow \qquad \hat{u} = G * u(\underline{x}, t).$$

where $*$ denotes a convolution.

Then try to find an inverse G^{-1} so that we can recover the sg-scales.

$$\underline{u} = G^{-1} * \hat{u}.$$

but if (as required for LES) the filter removes completely the small scales then the inverse is likely to be singular.

Nevertheless, Singular Value Decomposition (SVD) techniques are able to recover a 'plausable' inverse.

Reynolds Average Navier Stokes (RANS or uRANS)

u = "unsteady"

Extend idea of filter to include time + ensemble average so that all (or nearly all, uRANS) of turbulence motion is represented in the model and only the quasi-steady filtered flow is resolved on the grid.

We define the (kinematic) Reynolds stress $\tau_{ij} \equiv -\overline{u_i' u_j'}$

and TKE $k = -\frac{1}{2}\tau_{kk} \geq 0$

Eddy viscosity concept, we model the deviatoric part of τ_{ij} ;

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = 2\nu_T \bar{S}_{ij}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

we've included $-\frac{1}{3}\delta_{ij}\tau_{kk}$ on the left so that we allow $k = -\frac{1}{2}\tau_{kk}$ TKE to be non-zero given that $\bar{S}_{kk} = 0$ by continuity.

Here ν_T is the eddy viscosity

$$\nu_T \sim v \cdot l$$

where v is a characteristic velocity scale and l a characteristic turbulent eddy scale. These are related to $k \sim v^2$ and dissipation $\varepsilon \sim v^3/l$.

k-ε model

Here, we derive ν_T directly from knowledge of TKE k and dissipation ε,

$$v \sim k^{1/2}, \quad l \sim k^{3/2}/\varepsilon$$

$$\Rightarrow \quad \nu_T \sim k^2/\varepsilon$$

model constant

$$\nu_T = C_\mu \cdot k^2/\varepsilon, \quad C_\mu = 0.09$$

and we provide model transport equations for k and ε,

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_j k\right) = \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right) + P_k - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u}_j \varepsilon\right) = \frac{\partial}{\partial x_j}\left(\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right) + C_{\varepsilon 1}\frac{\varepsilon}{k}P_k - C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

where $P_k = 2\nu_T \bar{S}_{ij}\bar{S}_{ij}$ is the production, and

$$C_{\varepsilon 1} \simeq 1.44, \quad C_{\varepsilon 2} \simeq 1.92, \quad \sigma_k = 1, \quad \sigma_\varepsilon \simeq 1.3$$

are model constants.

Wall behaviour – need to supply boundary conditions on k and ε as approach wall, however ε varies rapidly here so care + resolution needed. Also out in free-stream, $k \to 0$ and $\varepsilon \to 0$, $\Rightarrow$ care needed to avoid $0/0$ issue.

<u>k–ω Model</u> Define $\omega \equiv \varepsilon / k$ specific dissipation rate (idea here is to avoid problems with modelling ε as $\to$ free stream, and $\to$ wall).

$$v \sim k^{1/2}, \quad l \sim v/\omega, \quad \nu_T \sim k^{1/2} \cdot k^{1/2}/\omega$$

$$\Rightarrow \quad \nu_T \simeq k/\omega.$$

Provide transport equations for k and ω. The ratio $\omega = \varepsilon/k$ is better behaved in free-stream and at walls.

<u>Spalart – Allmaras</u> This is a one-equation model, using just $\tilde{\nu}_T$ as the transported quantity, BUT the ~~rapid~~ variation in ν_T near a wall at $y^+ \sim O(1) - O(30)$ is captured using a pre-computed wall profile function f_ν, so that

$$\nu_T = \tilde{\nu} \cdot f_\nu$$

$$f_\nu = \frac{(\tilde{\nu}/\nu)^3}{(\tilde{\nu}/\nu)^3 + C_\nu^3}$$

Variation in $\tilde{\nu}$ is much smoother / linear as $y^+ \to 1$ at a wall.

Equation for $\tilde{\nu}$ is optimized for aerofoil applications.