Equations for average velocities

It is convenient to analyse turbulent flows by decomposing it in to the mean and the fluctuation

Thus, the instantaneous velocities and stresses can be written as,

$$\begin{aligned} \tilde{u}_i &= U_i + u_i \\ \tilde{p} &= P + p \\ \tilde{T}_{ij}^{(v)} &= T_{ij}^{(v)} + \tau_{ij}^{(v)} \end{aligned} \tag{3.11}$$

where U_i, p, and $T_{ij}^{(v)}$ represent the mean motion, and u_i, p, and τ_{ij} the fluctuating motions. This technique for decomposing the instantaneous motion is referred to as the *Reynolds decomposition.* Note that if the averages are defined as ensemble means, they are, in general, *time-dependent*.

source: Turbulence for the 21st century, W. K. George

Equations for average velocities

Substitution of equations 3.11 into equations 3.10

$$\rho \left[\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} \right] = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial \tilde{T}_{ij}^{(v)}}{\partial x_j} \tag{3.10}$$

$$\rho \left[\frac{\partial (U_i + u_i)}{\partial t} + (U_j + u_j) \frac{\partial (U_i + u_i)}{\partial x_j} \right] = -\frac{\partial (P + p)}{\partial x_i} + \frac{\partial (T_{ij}^{(v)} + \tau_{ij}^{(v)})}{\partial x_j} \tag{3.12}$$

This equation can now be averaged to yield an equation expressing momentum conservation for the averaged motion. Note that the operations of averaging and differentiation commute; i.e., the average of a derivative is the same as the derivative of the average. Also, the average of a fluctuating quantity is zero.[2] Thus the equation for the averaged motion reduces to:

$$\rho \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right] = -\frac{\partial P}{\partial x_i} + \frac{\partial T_{ij}^{(v)}}{\partial x_j} - \rho \langle u_j \frac{\partial u_i}{\partial x_j} \rangle \tag{3.13}$$

source: Turbulence for the 21st century, W. K. George

Equations for average velocities

$$\rho\left[\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}\right]=-\frac{\partial P}{\partial x_i}+\frac{\partial T_{ij}^{(v)}}{\partial x_j}-\rho\langle u_j\frac{\partial u_i}{\partial x_j}\rangle \tag{3.13}$$

where the remaining fluctuating product term has been moved to the right-hand side of the equation. Whether or not this last term is zero like the other fluctuating terms depends on the correlation of terms in the product. In general, these correlations are *not* zero.

The mass conservation equation can be similarly decomposed. In incompressible form, substitution of equations 3.11 into equation 3.4 yields:

$$\frac{\partial(U_j+u_j)}{\partial x_j}=0 \tag{3.14}$$

of which the average is:

$$\frac{\partial U_j}{\partial x_j}=0 \tag{3.15}$$

source: Turbulence for the 21st century, W. K. George

Equations for average velocities

Equation 3.15 can be subtracted from equation 3.14 to yield an equation for the instantaneous motion alone; i.e.,

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{3.16}$$

Again, like the mean, the form of the original instantaneous equation is seen to be preserved. The reason, of course, is obvious: the continuity equation is linear. The momentum equation, on the other hand, is not; hence the difference.

The entire problem of turbulence would be so much easier if the same thing were true for momentum equation

source: Turbulence for the 21st century, W. K. George

Equations for average velocities

Equation 3.16 can be used to rewrite the last term in equation 3.13 for the mean momentum. Multiplying equation 3.16 by u_i and averaging yields:

$$\langle u_i \frac{\partial u_j}{\partial x_j} \rangle = 0 \tag{3.17}$$

This can be added to $\langle u_j \partial u_i / \partial x_j \rangle$ to obtain:

$$\langle u_j \frac{\partial u_i}{\partial x_j} \rangle + 0 = \langle u_j \frac{\partial u_i}{\partial x_j} \rangle + \langle u_i \frac{\partial u_j}{\partial x_j} \rangle = \frac{\partial}{\partial x_j} \langle u_i u_j \rangle \tag{3.18}$$

where again the fact that arithmetic and averaging operations commute has been used.

The equation for the averaged momentum, equation 3.13 can now be rewritten as:

$$\rho \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right] = -\frac{\partial P}{\partial x_i} + \frac{\partial T_{ij}^{(v)}}{\partial x_j} - \frac{\partial}{\partial x_j} \rho \langle u_i u_j \rangle \tag{3.19}$$

source: Turbulence for the 21st century, W. K. George

Equations for average velocities

The last two terms on the right-hand side are both divergence terms and can be combined; the result is:

$$\rho\left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}\right] = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left[T_{ij}^{(v)} - \rho\langle u_i u_j\rangle\right] \tag{3.20}$$

Now the terms in square brackets on the right have the dimensions of stress. The first term is, in fact, the *viscous stress*. The second term, on the other hand, is not a stress at all, but simply a re-worked version of the fluctuating contribution to the non-linear acceleration terms. The fact that it can be written this way, however, indicates that *at least as far as the mean motion is concerned*, it *acts* as though it were a *stress* — hence its name, the **Reynolds stress**.

source: Turbulence for the 21st century, W. K. George

The problem with turbulence

$$\rho\left[\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}\right]=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left[T_{ij}^{(v)}-\rho\langle u_i u_j\rangle\right] \tag{3.20}$$

Appearance of Reynolds stress is the problem

They are created by the flow that we want to study/understand/predict in the first place!

- The equations are not closed
- Simple models do not work (more later)
- Compiling tables/handbooks carry substantial risk

source: Turbulence for the 21st century, W. K. George

Closure and eddy viscosity

$$\rho \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right] = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[T_{ij}^{(v)} - \rho \langle u_i u_j \rangle \right] \qquad (3.20)$$

There are 6 unknowns for the 3 equations

$$\langle u_1^2 \rangle, \langle u_2^2 \rangle, \langle u_3^2 \rangle, \langle u_1 u_2 \rangle, \langle u_1 u_3 \rangle \& \langle u_2 u_3 \rangle$$

These have to be related to the mean motion in order to be able to solve for the mean motions

The fact that we do not have these relations is referred to as the closure problem

source: Turbulence for the 21st century, W. K. George

Turbulence: lecture

Closure and eddy viscosity

Analogy with viscous stress

$$\left[\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}\right] = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial \tilde{T}_{ij}^{(v)}}{\partial x_j}$$

We wrote down a constitutive relationship,

$$\tilde{T}_{ij}^{(v)} = 2\mu \left[\tilde{s}_{ij} - \frac{1}{3}\tilde{s}_{kk}\delta_{ij}\right] \tag{3.5}$$

The viscosity, μ, is a *property of the fluid* that can be measured in an independent experiment. $\tilde{s}_{ij}$ is the instantaneous strain rate tensor defined by

$$\tilde{s}_{ij} \equiv \frac{1}{2}\left[\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right] \tag{3.6}$$

The viscosity only depends on the fluid and not on its motions

source: Turbulence for the 21st century, W. K. George

Closure and eddy viscosity

We can write a similar equations for the Reynolds stress

$$-\rho\langle u_i u_j\rangle + \frac{1}{3}\langle u_i u_i\rangle = \mu_t\left[S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right] \tag{3.23}$$

where μ_t is the turbulence "viscosity" (also called the eddy viscosity), and S_{ij} is the *mean* strain rate defined by:

$$S_{ij} = \frac{1}{2}\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right] \tag{3.24}$$

The second term on the right hand side is zero (continuity)

source: Turbulence for the 21st century, W. K. George

Turbulence: lecture

Closure and eddy viscosity

For a simple case, 2D shear flow

$$-\rho\langle u_1 u_2\rangle = \mu_t \frac{\partial U_1}{\partial x_2} \tag{3.25}$$

Note this "model" is the direct analogy to the Newtonian model for viscous stress in a fluid. The Reynolds stresses, $\langle -u_i u_j\rangle$ replaces the viscous stress, $\tau_{ij}^{(v)}$.

Direct analogy to the Newtonian model for viscous stress

The counterpart to the mechanical pressure is the the normal stress

Reynolds stress depends on mean strain at a single location /instant in the flow and has no history or non-local dependence

This is fatal!!!

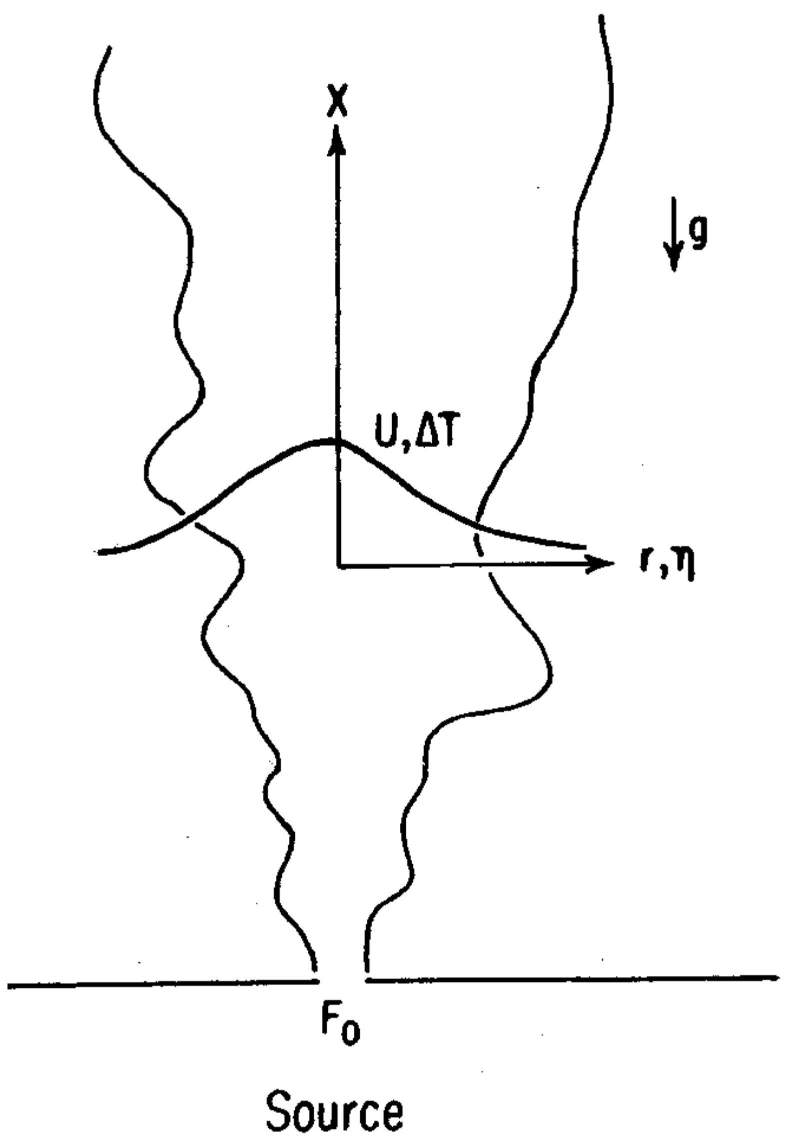
source: Turbulence for the 21st century, W. K. George

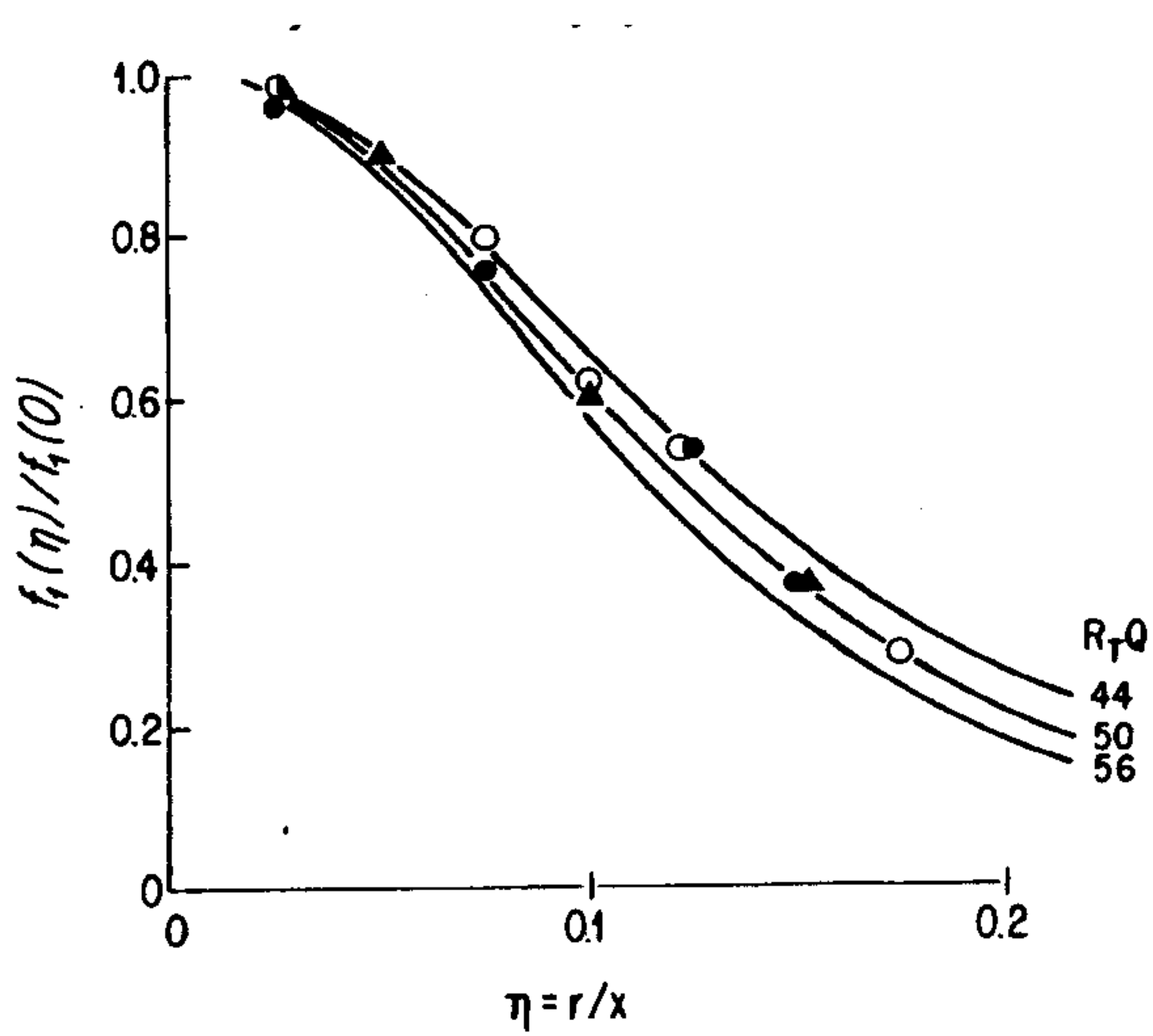
Turbulence: lecture

Closure and eddy viscosity

Example, buoyant plume

Want to "predict" the mean flow at different downstream locations





source: Turbulence for the 21st century, W. K. George

Turbulence: lecture

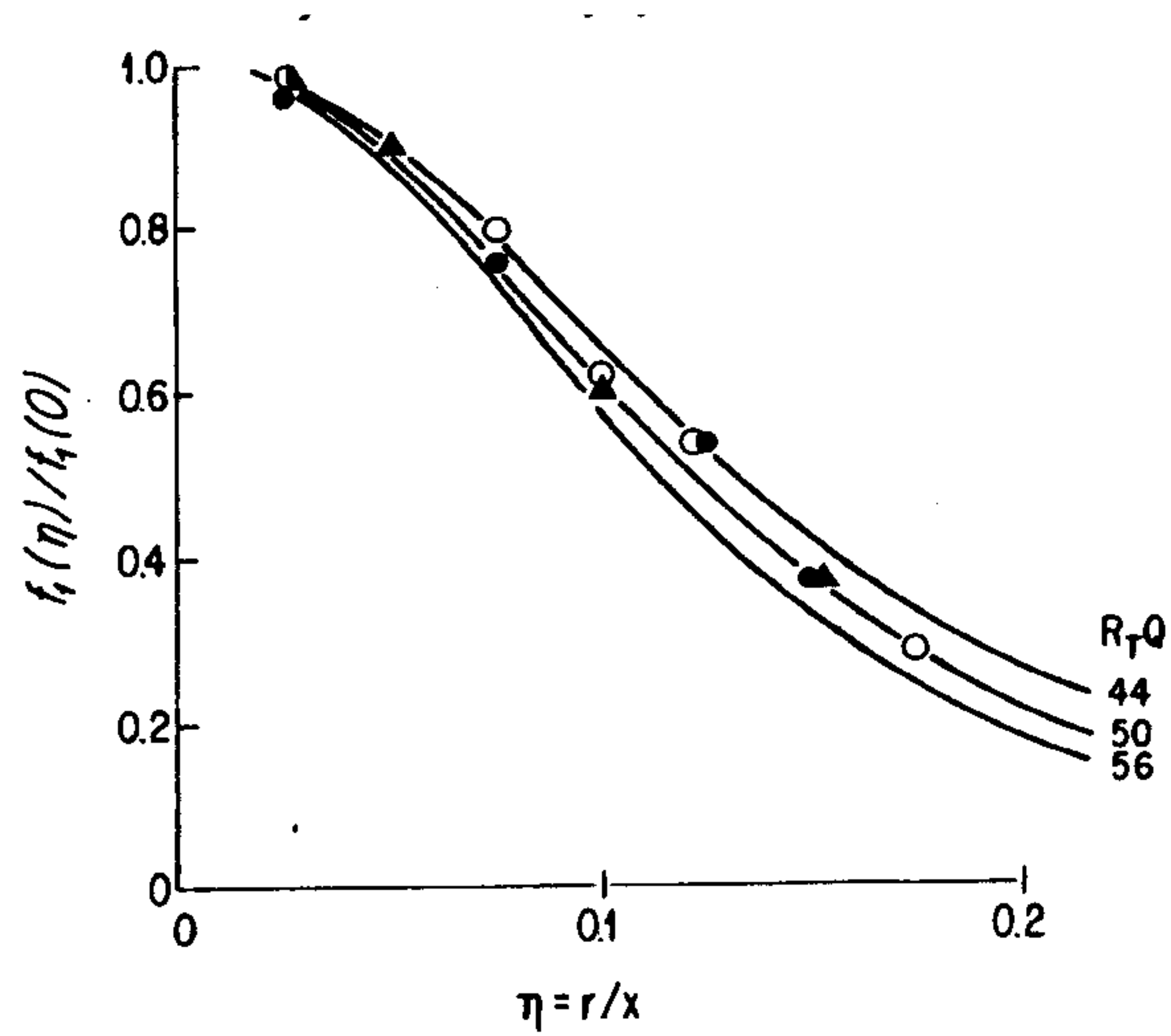
Closure and eddy viscosity

Example, buoyant plume

Model generated from measurements of Reynolds stresses

Only good for this particular set of conditions

As good as predicting yesterday's weather!



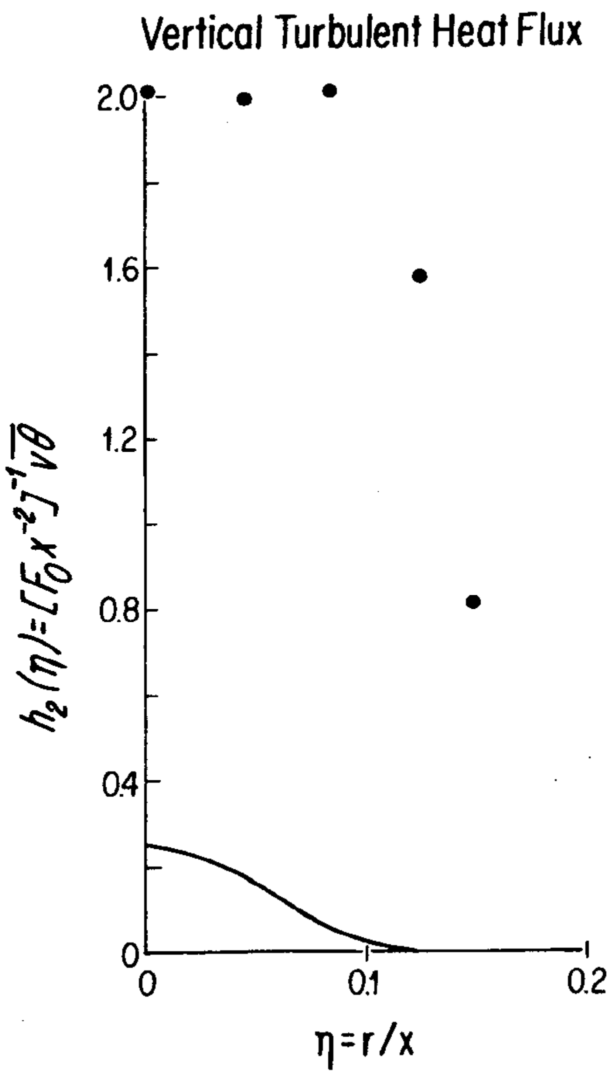
source: Turbulence for the 21st century, W. K. George

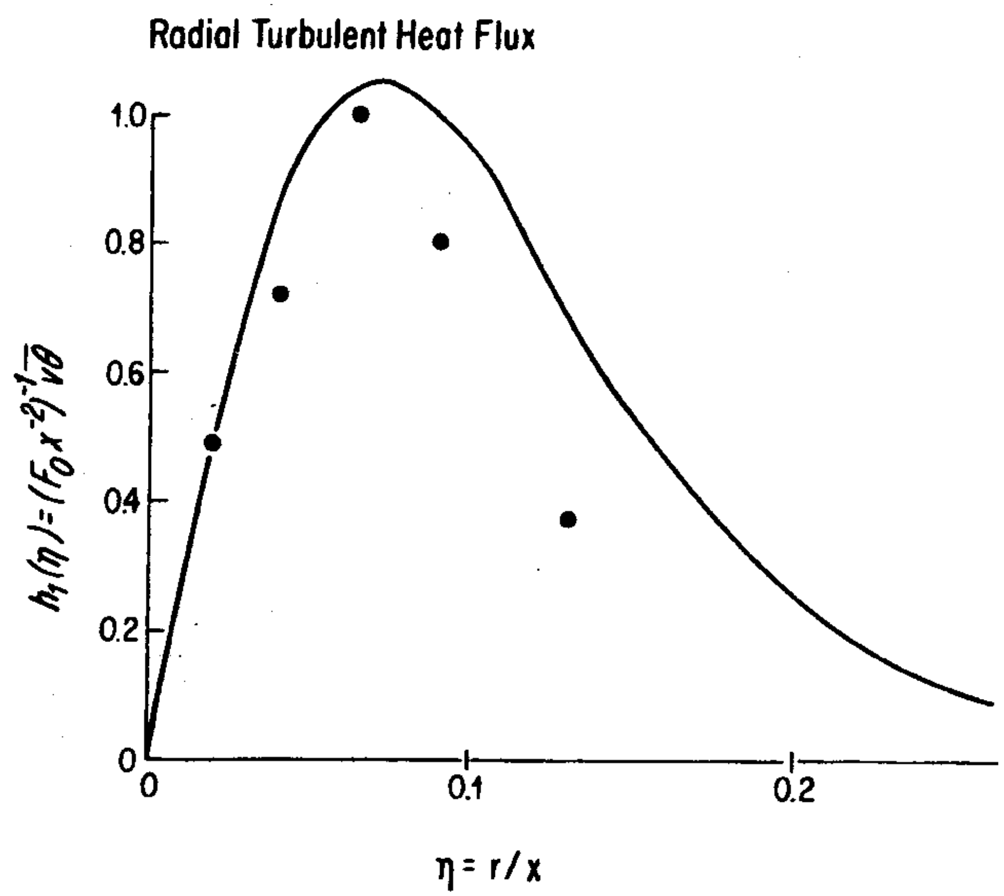
Turbulence: lecture

Closure and eddy viscosity

Example, buoyant plume

Does not really work for other quantities of interest such as vertical components of Reynolds stress





source: Turbulence for the 21st century, W. K. George

Closure and eddy viscosity

Example, buoyant plume

This means that turbulence is not an isotropic medium

A more general form of constitutive equation which would allow for the non-isotropic nature of the "medium" (in this case the turbulence itself) would be

$$-\rho\langle u_i u_j\rangle + \frac{1}{3}\langle u_k u_k\rangle\delta_{ij} = \mu_{ijkl}\left[S_{kl} - \frac{1}{3}S_{mm}\delta_{kl}\right] \tag{3.26}$$

We have traded 6 unknowns for 81 unknowns!!!

Closure problem remains!

There are no general solutions to this problem
Flow is turbulent and the turbulence depends on circumstances

source: Turbulence for the 21st century, W. K. George

Turbulence: lecture

Reynolds stress equations

It is clear that eddy viscosity approach is flawed.

So, we need dynamical equations for the Reynolds stresses that recognises that these stresses depend on velocities everywhere now and in past times

Equation for instantaneous fluctuating velocity,

$$\rho\left[\frac{\partial u_i}{\partial t}+U_j\frac{\partial u_i}{\partial x_j}\right]=-\frac{\partial p}{\partial x_i}+\frac{\partial \tau_{ij}^{(v)}}{\partial x_j}-\rho\left[u_j\frac{\partial U_i}{\partial x_j}\right]-\rho\left\{u_j\frac{\partial u_i}{\partial x_j}-\langle u_j\frac{\partial u_i}{\partial x_j}\rangle\right\} \quad (3.27)$$

Note that the free index in this equation is i. Also, since we are now talking about turbulence again, the capital letters represent mean or averaged quantities.

Take NS equation and subtract mean momentum equation

source: Turbulence for the 21st century, W. K. George

Reynolds stress equations

Multiplying equation 3.27 by u_k and averaging yields:

$$\rho\left[\langle u_k\frac{\partial u_i}{\partial t}\rangle + U_j\langle u_k\frac{\partial u_i}{\partial x_j}\rangle\right] = - \langle u_k\frac{\partial p}{\partial x_i}\rangle + \langle u_k\frac{\partial \tau_{ij}^{(v)}}{\partial x_j}\rangle - \rho\left[\langle u_k u_j\rangle\frac{\partial U_i}{\partial x_j}\right] - \rho\left\{\langle u_k u_j\frac{\partial u_i}{\partial x_j}\rangle\right\} \quad (3.28)$$

Now since both i and k are free indices they can be interchanged to yield a second equation given by[7]:

$$\rho\left[\langle u_i\frac{\partial u_k}{\partial t}\rangle + U_j\langle u_i\frac{\partial u_k}{\partial x_j}\rangle\right] = - \langle u_i\frac{\partial p}{\partial x_k}\rangle + \langle u_i\frac{\partial \tau_{kj}^{(v)}}{\partial x_j}\rangle - \rho\left[\langle u_i u_j\rangle\frac{\partial U_k}{\partial x_j}\right] - \rho\left\{\langle u_i u_j\frac{\partial u_k}{\partial x_j}\rangle\right\} \quad (3.29)$$

source: Turbulence for the 21st century, W. K. George

Reynolds stress equations

Equations 3.28 and 3.29 can be added together to yield an equation for the Reynolds stress,

$$
\begin{aligned}
\frac{\partial \langle u_i u_k \rangle}{\partial t} + U_j \frac{\partial \langle u_i u_k \rangle}{\partial x_j} &= -\frac{1}{\rho}\left[\langle u_i \frac{\partial p}{\partial x_k} \rangle + \langle u_k \frac{\partial p}{\partial x_i} \rangle \right] \\
&- \left[\langle u_i u_j \frac{\partial u_k}{\partial x_j} \rangle + \langle u_k u_j \frac{\partial u_i}{\partial x_j} \rangle \right] \\
&+ \frac{1}{\rho}\left[\langle u_i \frac{\partial \tau^{(v)}_{kj}}{\partial x_j} \rangle + \langle u_k \frac{\partial \tau^{(v)}_{ij}}{\partial x_j} \rangle \right] \\
&- \left[\langle u_i u_j \rangle \frac{\partial U_k}{\partial x_j} + \langle u_k u_j \rangle \frac{\partial U_i}{\partial x_j} \right]
\end{aligned}
\tag{3.30}
$$

Let's look at this equation term-by-term

source: Turbulence for the 21st century, W. K. George

Reynolds stress equations

Equations 3.28 and 3.29 can be added together to yield an equation for the Reynolds stress,

$$\begin{aligned}\frac{\partial\langle u_i u_k\rangle}{\partial t} + U_j \frac{\partial\langle u_i u_k\rangle}{\partial x_j} &= \boxed{-\frac{1}{\rho}\left[\langle u_i \frac{\partial p}{\partial x_k}\rangle + \langle u_k \frac{\partial p}{\partial x_i}\rangle\right]} \\ &- \left[\langle u_i u_j \frac{\partial u_k}{\partial x_j}\rangle + \langle u_k u_j \frac{\partial u_i}{\partial x_j}\rangle\right] \\ &+ \frac{1}{\rho}\left[\langle u_i \frac{\partial \tau_{kj}^{(v)}}{\partial x_j}\rangle + \langle u_k \frac{\partial \tau_{ij}^{(v)}}{\partial x_j}\rangle\right] \\ &- \left[\langle u_i u_j\rangle \frac{\partial U_k}{\partial x_j} + \langle u_k u_j\rangle \frac{\partial U_i}{\partial x_j}\right]\end{aligned} \qquad (3.30)$$

Let's look at this equation term-by-term

source: Turbulence for the 21st century, W. K. George

Reynolds stress equations

It is customary to rearrange the first term on the right hand side in the following way:

$$\left[\langle u_i \frac{\partial p}{\partial x_k}\rangle + \langle u_k \frac{\partial p}{\partial x_i}\rangle\right] = \langle p\left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right]\rangle + \frac{\partial}{\partial x_j}\left[\langle p u_i\rangle \delta_{kj} + \langle p u_k\rangle \delta_{ij}\right] \tag{3.31}$$

The first term on the right is pressure-strain term

The second term, written in divergence form, is pressure-diffusion term

source: Turbulence for the 21st century, W. K. George

Turbulence: lecture

Reynolds stress equations

Equations 3.28 and 3.29 can be added together to yield an equation for the Reynolds stress,

$$
\begin{aligned}
\frac{\partial\langle u_i u_k\rangle}{\partial t} + U_j \frac{\partial\langle u_i u_k\rangle}{\partial x_j} \quad &= \boxed{-\frac{1}{\rho}\left[\langle u_i \frac{\partial p}{\partial x_k}\rangle + \langle u_k \frac{\partial p}{\partial x_i}\rangle\right]} \\
&- \quad \left[\langle u_i u_j \frac{\partial u_k}{\partial x_j}\rangle + \langle u_k u_j \frac{\partial u_i}{\partial x_j}\rangle\right] \\
&+ \boxed{\frac{1}{\rho}\left[\langle u_i \frac{\partial \tau^{(v)}_{kj}}{\partial x_j}\rangle + \langle u_k \frac{\partial \tau^{(v)}_{ij}}{\partial x_j}\rangle\right]} \\
&- \quad \left[\langle u_i u_j\rangle \frac{\partial U_k}{\partial x_j} + \langle u_k u_j\rangle \frac{\partial U_i}{\partial x_j}\right]
\end{aligned}
\tag{3.30}
$$

Let's look at this equation term-by-term

source: Turbulence for the 21st century, W. K. George

Reynolds stress equations

The third term on the right-hand side of equation 3.30 can similarly be re-written as:

$$\left[\langle u_i \frac{\partial \tau_{kj}^{(v)}}{\partial x_j}\rangle + \langle u_k \frac{\partial \tau_{ij}^{(v)}}{\partial x_j}\rangle\right] = - \left[\langle \tau_{ij}^{(v)} \frac{\partial u_k}{\partial x_j}\rangle + \langle \tau_{kj}^{(v)} \frac{\partial u_i}{\partial x_j}\rangle\right] + \frac{\partial}{\partial x_j}[\langle u_i \tau_{kj}^{(v)}\rangle + \langle u_k \tau_{ij}^{(v)}\rangle] \tag{3.32}$$

The second term is a divergence term

The first term on the right hand side is,
"Dissipation of Reynolds stress" by turbulent viscous stress

If we substitute the Newtonian constitutive relation,

$$\frac{1}{\rho}\left[\langle \tau_{ij}^{(v)} \frac{\partial u_k}{\partial x_j}\rangle + \langle \tau_{kj}^{(v)} \frac{\partial u_i}{\partial x_j}\rangle\right] = 2\nu \left[\langle s_{ij} \frac{\partial u_k}{\partial x_j}\rangle + \langle s_{kj} \frac{\partial u_i}{\partial x_j}\rangle\right] \tag{3.33}$$

source: Turbulence for the 21st century, W. K. George

Reynolds stress equations

Equations 3.28 and 3.29 can be added together to yield an equation for the Reynolds stress,

$$\begin{aligned}\frac{\partial\langle u_i u_k\rangle}{\partial t} + U_j\frac{\partial\langle u_i u_k\rangle}{\partial x_j} \quad = \quad & \boxed{-\frac{1}{\rho}\left[\langle u_i\frac{\partial p}{\partial x_k}\rangle + \langle u_k\frac{\partial p}{\partial x_i}\rangle\right]} \\ - & \boxed{\left[\langle u_i u_j\frac{\partial u_k}{\partial x_j}\rangle + \langle u_k u_j\frac{\partial u_i}{\partial x_j}\rangle\right]} \\ + & \boxed{\frac{1}{\rho}\left[\langle u_i\frac{\partial \tau^{(v)}_{kj}}{\partial x_j}\rangle + \langle u_k\frac{\partial \tau^{(v)}_{ij}}{\partial x_j}\rangle\right]} \\ - & \left[\langle u_i u_j\rangle\frac{\partial U_k}{\partial x_j} + \langle u_k u_j\rangle\frac{\partial U_i}{\partial x_j}\right]\end{aligned} \tag{3.30}$$

Let's look at this equation term-by-term

source: Turbulence for the 21st century, W. K. George

Reynolds stress equations

Now if we use the same trick from before using the continuity equation, we can rewrite the second term on the right-hand side of equation 3.30 to obtain:

$$\left[\langle u_i u_j \frac{\partial u_k}{\partial x_j}\rangle + \langle u_k u_j \frac{\partial u_i}{\partial x_j}\rangle\right] = \frac{\partial}{\partial x_j}\langle u_i u_k u_j\rangle \tag{3.34}$$

Note that in each of the three terms on the RHS of equation 3.30, we have now identified a divergence term

Now, lets group terms that has this divergence form and rewrite equation 3.30.

source: Turbulence for the 21st century, W. K. George

Turbulence: lecture

Reynolds stress equations

Rate of change of Reynolds stress following mean motion

Pressure-strain term

$$\boxed{\frac{\partial}{\partial t}\langle u_i u_k\rangle + U_j \frac{\partial}{\partial x_j}\langle u_i u_k\rangle} = \boxed{\langle \frac{p}{\rho}\left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right]\rangle}$$

$$\boxed{+ \frac{\partial}{\partial x_j}\left\{-\frac{1}{\rho}[\langle p u_k\rangle \delta_{ij} + \langle p u_i\rangle \delta_{kj}] - \langle u_i u_k u_j\rangle + 2\nu[\langle s_{ij} u_k\rangle + \langle s_{kj} u_i\rangle]\right\}}$$

Transport

$$\boxed{- \left[\langle u_i u_j\rangle \frac{\partial U_k}{\partial x_j} + \langle u_k u_j\rangle \frac{\partial U_i}{\partial x_j}\right]}$$

production

$$\boxed{- 2\nu \left[\langle s_{ij} \frac{\partial u_k}{\partial x_j}\rangle + \langle s_{kj} \frac{\partial u_i}{\partial x_j}\rangle\right]} \qquad (3.35)$$

dissipation

source: Turbulence for the 21st century, W. K. George

Reynolds stress equations

Obviously these equations do not involve only U_i and $\langle u_i u_j \rangle$, but depend on many more new unknowns.

It is clear that, contrary to our hopes, we have not derived a single equation relating the Reynolds stress to the mean motion. Instead, our Reynolds stress transport equation is exceedingly complex. Whereas the process of averaging the equation for the mean motion introduced only six new independent unknowns, the Reynolds stress, $\langle u_i u_j \rangle$, the search for a transport equation which will relate these to the mean motion has produced many more unknowns. They are:

$$\langle p u_i \rangle \quad - \quad 3 \quad unknowns \tag{3.36}$$

$$\langle u_i s_{jk} \rangle \quad - \quad 27 \tag{3.37}$$

$$\langle s_{ij} s_{jk} \rangle \quad - \quad 9 \tag{3.38}$$

$$\langle u_i u_k u_j \rangle \quad - \quad 27 \tag{3.39}$$

$$\langle p \frac{\partial u_i}{\partial x_j} \rangle \quad - \quad 9 \tag{3.40}$$

$$TOTAL \quad - \quad 75 \tag{3.41}$$

source: Turbulence for the 21st century, W. K. George

Turbulence: lecture

Reynolds stress equations

Closure problem cannot be solved

By deriving new dynamical equations,

or by using “eddy” viscosity models

We need to understand how the turbulence behaves

This **may** allow us to develop new constitutive models for turbulence and also understands its limitations

Turbulence is a topic that we are still studying!
So, this module by definition is incomplete

source: Turbulence for the 21st century, W. K. George