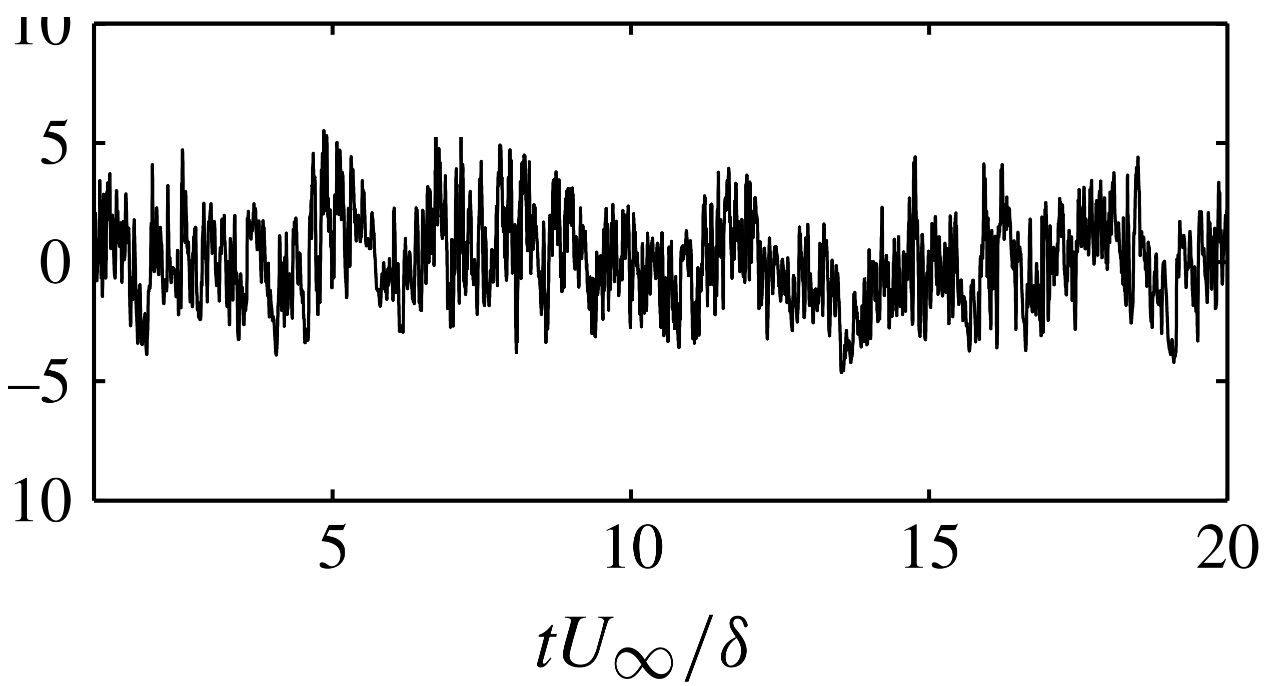
Turbulence: lecture

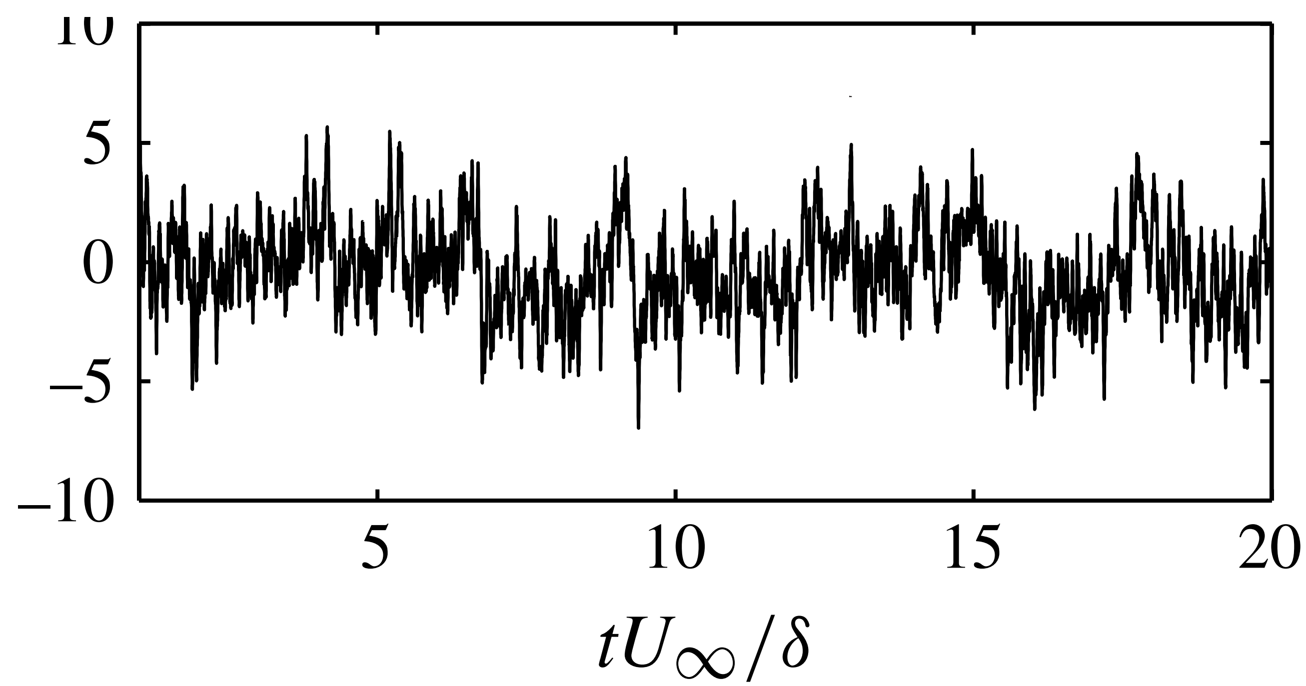
Time/space domain analysis

So far, we have looked at amplitude statistics. These do not tell you anything about temporal/spatial structures

Turbulent flow



Synthetic signal



We need additional tools to tell these apart

Time/space domain analysis

Autocorrelation

One of the most useful statistical moments in the study of stationary random processes (and turbulence, in particular) is the **autocorrelation** defined as the average of the product of the random variable evaluated at two times, i.e. $\langle u(t)u(t')\rangle$. Since the process is assumed stationary, this product can depend only on the time difference $\tau = t' - t$. Therefore the autocorrelation can be written as:

$$C(\tau) \equiv \langle u(t)u(t+\tau)\rangle \tag{8.1}$$

Measure of memory of the process

You can replace time with space and it will tell you about longevity of the process

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Autocorrelation

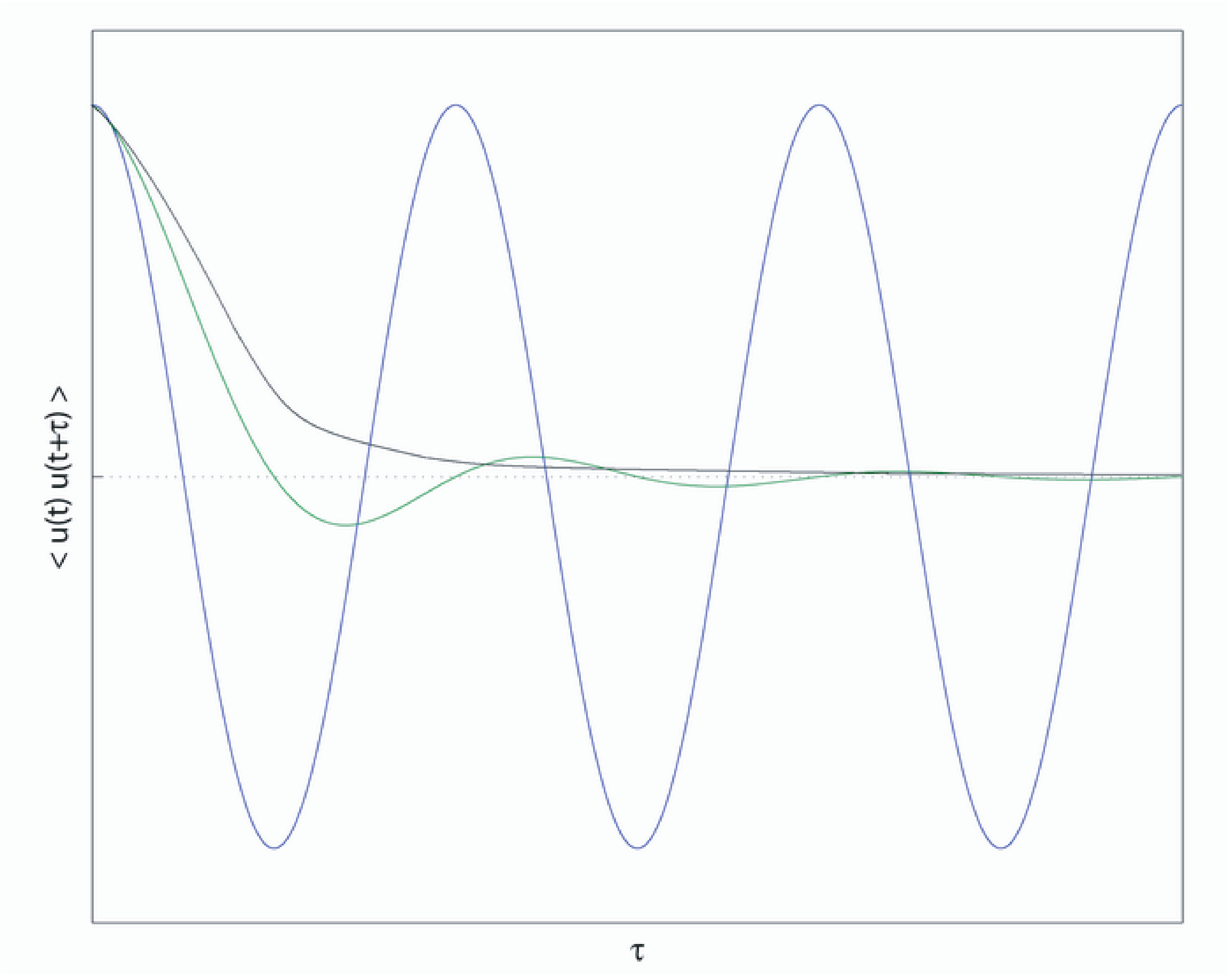


Figure 8.1: Autocorrelations for two random processes and a periodic one.

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Autocorrelation

Since a random process can never be more than perfectly correlated, it can never achieve a correlation greater than is value at the origin. Thus

$$|C(\tau)| \leq C(0) \tag{8.2}$$

An important consequence of stationarity is that the autocorrelation is symmetric in the time difference, $\tau = t' - t$. To see this simply shift the origin in time backwards by an amount τ and note that independence of origin implies:

$$\langle u(t)u(t+\tau)\rangle = \langle u(t-\tau)u(t)\rangle = \langle u(t)u(t-\tau)\rangle \tag{8.3}$$

Since the right hand side is simply $C(-\tau)$, it follows immediately that:

$$C(\tau) = C(-\tau) \tag{8.4}$$

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Autocorrelation

It is convenient to define the *autocorrelation coefficient* as:

Sometimes, R = $$\rho(\tau) \equiv \frac{C(\tau)}{C(0)} = \frac{\langle u(t)u(t+\tau)\rangle}{\langle u^2\rangle} \tag{8.5}$$

where

$$\langle u^2\rangle = \langle u(t)u(t)\rangle = C(0) = var[u] \tag{8.6}$$

Since the autocorrelation is symmetric, so is its coefficient, i.e.,

$$\rho(\tau) = \rho(-\tau) \tag{8.7}$$

It is also obvious from the fact that the autocorrelation is maximal at the origin that the autocorrelation coefficient must also be maximal there. In fact from the definition it follows that

$$\rho(0) = 1 \tag{8.8}$$

and

$$\rho(\tau) \leq 1 \tag{8.9}$$

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

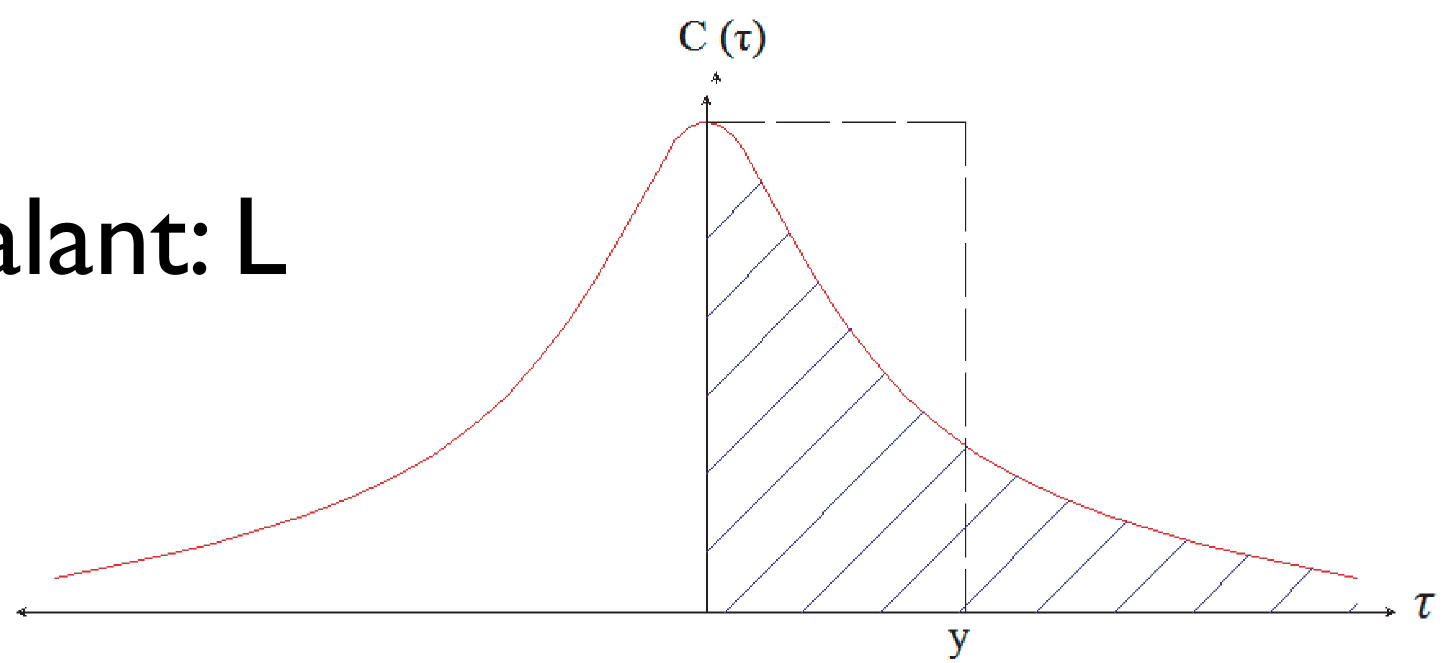
Integral length scale

One of the most useful measures of the length of time a process is correlated with itself is the integral scale defined by

$$T_{int} \equiv \int_0^\infty \rho(\tau) d\tau \tag{8.10}$$

It is easy to see why this works by looking at Figure 8.2. In effect we have replaced the area under the correlation coefficient by a rectangle of height unity and width T_{int}.

Spatial equivalant: L



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Turbulence: lecture

Time/space domain analysis

Taylor microscale

The autocorrelation can be expanded about the origin in a MacClaurin series; i.e.,

$$C(\tau) = C(0) + \tau \left. \frac{dC}{d\tau} \right|_{\tau=0} + \frac{1}{2}\tau^2 \left. \frac{d^2C}{d\tau^2} \right|_{\tau=0} + \frac{1}{3!}\tau^3 \left. \frac{d^3C}{dt} \right|_{\tau=0} \tag{8.11}$$

But we know the autocorrelation is symmetric in τ, hence the odd terms in τ must be identically zero (i.e., $dC/d\tau|_{\tau=0} = 0$, $d^3/d\tau^3|_{\tau=0}$, etc.). Therefore the expansion of the autocorrelation near the origin reduces to:

$$C(\tau) = C(0) + \frac{1}{2}\tau^2 \left. \frac{d^2C}{d\tau^2} \right|_{\tau=0} + \cdots \tag{8.12}$$

Similarly, the autocorrelation coefficient near the origin can be expanded as:

$$\rho(\tau) = 1 + \frac{1}{2} \left. \frac{d^2\rho}{d\tau^2} \right|_{\tau=0} \tau^2 + \cdots \tag{8.13}$$

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Taylor microscale

$$\rho(\tau) = 1 + \frac{1}{2}\rho''(0)\tau^2 + \cdots \tag{8.14}$$

Since $\rho(\tau)$ has its maximum at the origin, obviously $\rho''(0)$ must be negative.

We can use the correlation and its second derivative at the origin to *define* a special time scale, λ_τ (called the Taylor microscale [1]) by:

$$\lambda_\tau^2 \equiv -\frac{2}{\rho''(0)} \tag{8.15}$$

Using this in equation 8.14 yields the expansion for the correlation coefficient near the origin as:

$$\rho(\tau) = 1 - \frac{\tau^2}{\lambda_\tau^2} + \cdots \tag{8.16}$$

source: Turbulence for the 21st century, W. K. George

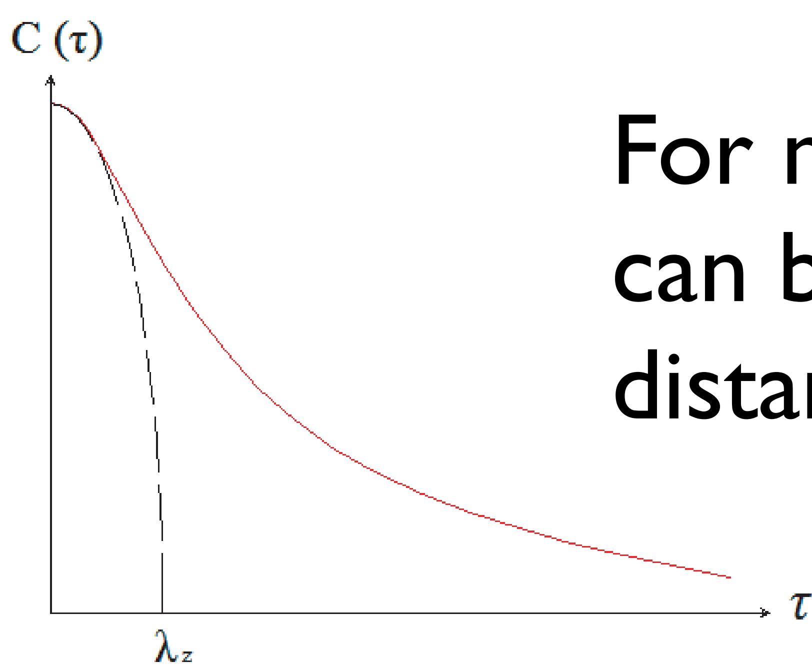
Time/space domain analysis

Taylor microscale

Thus very near the origin the correlation coefficient (and the autocorrelation as well) simply rolls off parabolically; i.e.,

$$\rho(\tau) \approx 1 - \frac{\tau^2}{\lambda_\tau^2} \tag{8.17}$$

This parabolic curve is shown in Figure 8.3 as the osculating (or 'kissing') parabola which approaches zero exactly as the autocorrelation coefficient does. The intercept of this osculating parabola with the τ-axis is the Taylor microscale, λ_τ.



For many random processes, Taylor microscale can be shown to be equal to the average distance between zero-crossings

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Taylor microscale

is related to the mean squared of the derivative of the signal

stationary random signals, say u and u', we obtain by evaluating the same signal at two different times, say $u = u(t)$ and $u' = u(t')$. The first is only a function of t, and the second is only a function of t'. The derivative of the first signal is du/dt and the second du'/dt'. Now lets multiply these together and rewrite them as:

$$\frac{du'}{dt'}\frac{du}{dt} = \frac{d^2}{dtdt'}u(t)u'(t') \tag{8.18}$$

where the right-hand side follows from our assumption that u is not a function of t' nor u' a function of t.

Using standard calculus and some manipulation,

$$\langle\left(\frac{du}{dt}\right)^2\rangle = 2\frac{\langle u^2\rangle}{\lambda_\tau^2}$$

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Cross correlation

It is convenient to define the *autocorrelation coefficient* as:

$$\rho(\tau) \equiv \frac{C(\tau)}{C(0)} = \frac{\langle u(t)u(t+\tau)\rangle}{\langle u^2\rangle} \tag{8.5}$$

What if the second signal is not "u", but another variable?

$$\rho_{AB}(\tau) = \frac{\langle A(t)B(t+\tau)\rangle}{\sqrt{\langle A^2\rangle}\sqrt{\langle B^2\rangle}}$$

Cross correlation between A & B

Gives you information about how one signal is related to another - We will come back to this in Reynolds shear stress

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Matlab implementation

```
>>[R,lags] = xcorr(A,B,'unbiased');
>> R = R./(std(A).*std(B));
>> plot(lags.*dt,R);
```

```
If A = B, then we have autocorrelation
```

`Unbiased' is crucial since we need to account for the limited number of samples as we go farther towards the edge of the signals

Ideally, we want to compute correlations within the central part of the signal

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Fourier analysis

This provides a frequency analysis of a signal - amplitude or energy content in different frequencies

If we have a periodic signal (i.e., a signal that repeats itself every time interval T) it can be developed in a Fourier series as:

$$u(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(2\pi n \frac{t}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(2\pi n \frac{t}{T}\right) \tag{C.1}$$

The frequencies present in this decomposition, $f_n = n/T$, are *harmonics* (or integer multiples) of the fundamental frequency $1/T$. The Fourier coefficients, A_n and B_n, are given by:

$$A_n = \frac{1}{T}\int_{-T/2}^{T/2} u(t) \cos\left(2\pi n \frac{t}{T}\right) dt \tag{C.2}$$

$$B_n = \frac{1}{T}\int_{-T/2}^{T/2} u(t) \sin\left(2\pi n \frac{t}{T}\right) dt \tag{C.3}$$

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Fourier analysis

It is sometimes convenient to use a complex notation in the formulation of Fourier series. We can define a complex coefficient as $C_n = A_n - iB_n$ and rewrite equations (C.2–C.3) as:

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} u(t)e^{-i2\pi nt/T}dt \tag{C.5}$$

The ratio between the real and imaginary values can provide phase information about the signal.

It is also convenient to introduce negative values of n. This corresponds to negative frequencies (which can be thought of as waves going backwards in time). Then the values are symmetric in the sense that $A(-n) = A(n)$ and $B(-n) = B(n)$. The use of negative values of n means that the reconstruction formula now can be written very compactly as

$$u(t) = \sum_{n=-\infty}^{\infty} C_n e^{+i2\pi nt/T} \tag{C.6}$$

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Fourier analysis

Fourier series is only valid for time-periodic signals.

We need to deal with signals that result from random processes

We define then, the *Fourier transform* of the function $u(t)$ as:

$$\hat{u}(f) = \int_{-\infty}^{\infty} e^{-i2\pi f t} u(t) dt \tag{C.7}$$

These are really the continuous counterpart to the Fourier series coefficients of a periodic signal, and can similarly be used to reconstruct the original signal. We call this reconstruction the *inverse Fourier transform* and define it as:

$$u(t) = \int_{-\infty}^{\infty} e^{+i2\pi f t} \hat{u}(f) df \tag{C.8}$$

We assume that these integrals converge!

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Fourier analysis

But, we cannot collect data for infinite time

In any application of Fourier analysis we are always limited by the length of the time record, T. This means that the most we can expect to be able to transform is the finite time transform given by:

$$\hat{u}_{iT}(f) = \int_{-T/2}^{T/2} e^{-i2\pi f t} u(t) dt \tag{C.14}$$

where for convenience we have written it over the symmetric interval in time $(-T/2, T/2)$.

The length of the time-record becomes really important

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Fourier analysis

Now with a little thought, it is clear that we are actually taking the Fourier transform of the product of two functions, the correlation (the part we want) plus the window function; i.e.,

$$\hat{u}_{iT}(f) = \mathcal{F}[u(t)w_T(t)] = \int_{-\infty}^{\infty} e^{-i2\pi ft} u(t) w_T(t) dt \tag{C.15}$$

where $w_T(\tau)$ is defined by:

$$w_T(\tau) = \begin{cases} 1, & -T/2 \leq \tau \leq T/2 \\ 0, & |\tau| > T/2 \end{cases} \tag{C.16}$$

What does this window function do?

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Fourier analysis

The window contaminates the amplitude at a given frequency by leaking information from other frequencies

Most people use a top-hat window without actually realising that they are using a top-hat window

Need to ensure the record length is much much larger than the largest scale to be considered

Conservatively, keep the record length at least one order of magnitude larger than the integral scale

source: Turbulence for the 21st century, W. K. George

Turbulence: lecture

Time/space domain analysis

Fourier analysis

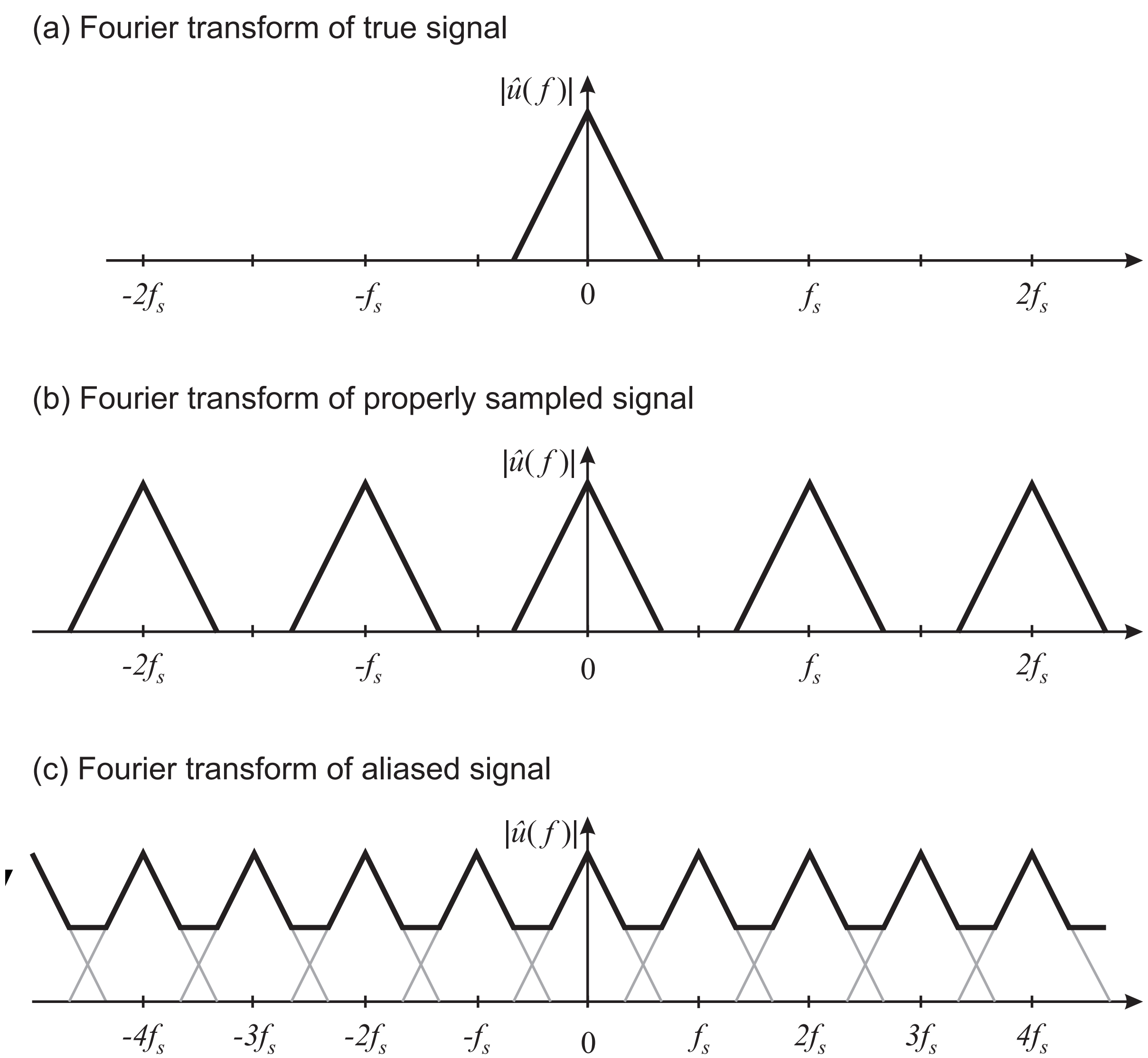
Usually, we sample signals digitally for a length of time at a given frequency of acquisition

The frequency of acquisition is very important

We will incur ALIASING if sampling frequency is too low

Satisfy “Nyquist criterion”

Sample at least twice the frequency as the maximum frequency



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Turbulence: lecture

Time/space domain analysis

Fourier analysis

Continuous fourier transform

$$\hat{u_T}(f) = \int_0^T e^{-i2\pi f t} u(t) dt \tag{D.14}$$

What we have is discrete data: Discrete fourier transform

$$u_n = u(n\Delta t) \qquad\qquad n = 0, 1, 2, \ldots, N-1 \tag{D.15}$$

Using the basic definitions of integral calculus, we can discretize the integral of equation (D.14) as:

$$\hat{u_T}(f) = \sum_{n=0}^{N-1} e^{-i2\pi f n\Delta t} u_n \Delta t \tag{D.16}$$

N is the total number of samples and $T = N\Delta t$ is the total sample time. The time between samples, Δt, is given by the sampling frequency, $\Delta t = 1/f_s = T/N$. This is, of course, an approximation which becomes exact in the limit as the number of points, N, becomes infinite and as the interval between them, Δt, goes to zero.

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Fourier analysis

Now since we only have N data points, we can only calculate N independent Fourier coefficients. In fact, since we are in the complex domain, we can only calculate $N/2$, since the real and imaginary parts are independent of each other. So we might as well pick the frequencies for which we will evaluate the sum of equation D.16 for maximum convenience. For almost all applications this turns out to be integer multiples of the inverse record length; i.e.,

$$f_m = \frac{m}{T} = \frac{m}{N\Delta t} \qquad m = 0, 1, 2, \ldots, N-1 \tag{D.17}$$

Substituting this into equation D.16 yields our discretized Fourier transform as:

$$\hat{u}_T(f_m) = T\left\{\frac{1}{N}\sum_{n=0}^{N-1} e^{-i2\pi mn/N} u_n\right\} \qquad m = 0, 1, 2, \ldots, N-1 \tag{D.18}$$

This equation can be evaluated numerically for each of the frequencies f_m defined in eq. (D.17). The Fourier coefficients, $\hat{u}_T(f_m)$, found from eq. (D.18) are complex numbers. For future reference note they negative frequencies are mapped into $f_m = N-1, N-2, N-3, N-m$ instead of at negative values of m, as illustrated in Figure D.1.

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Fourier analysis

It is easy to show the original time series data points can be recovered by using the inverse discrete Fourier transform:

$$u_n = \frac{1}{T}\left\{\sum_{m=0}^{N-1} e^{+i2\pi mn/N}\hat{u_T}(f_m)\right\} \tag{D.19}$$

Evaluating these discrete coefficients is numerically taxing

Fast Fourier Transform (FFT) allows inexpensive computation

Matlab implementation

```
>> uh = fft(A);
```

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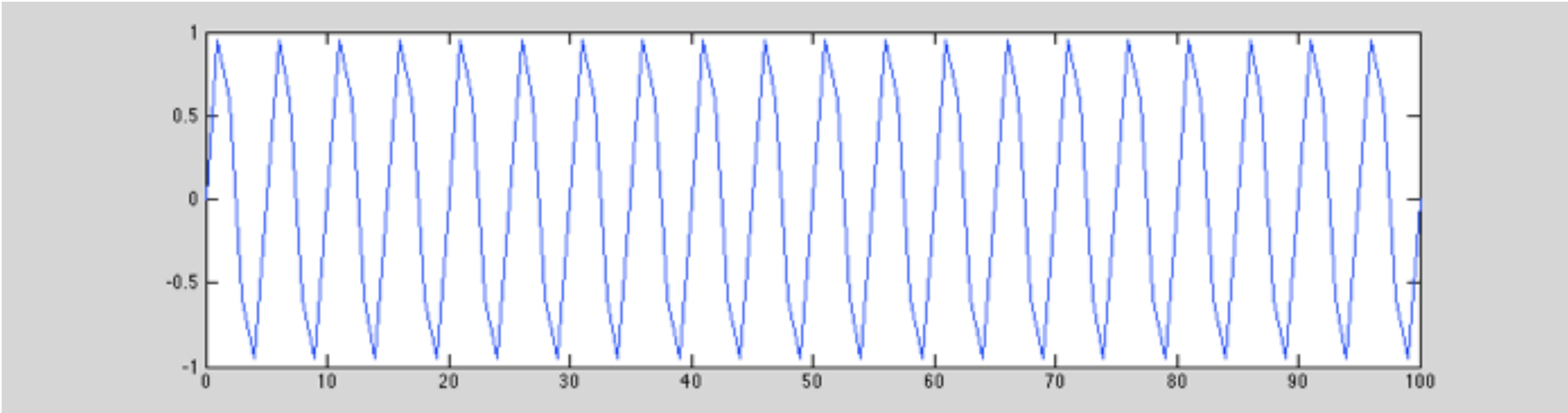
Time/space domain analysis

Fourier analysis

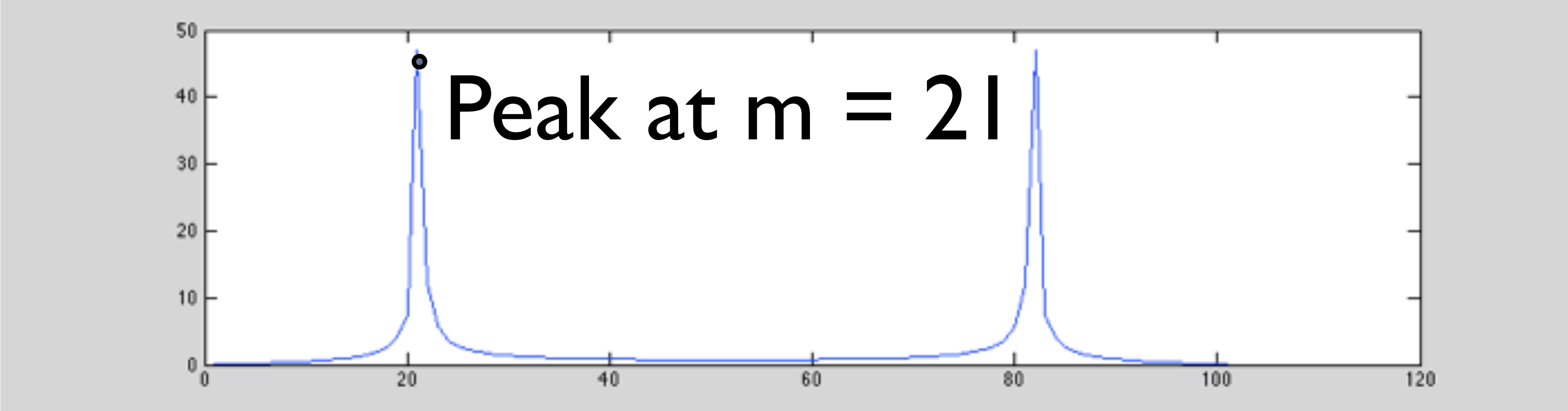
$a = \sin(2*pi*t/5);$

T = 101

dt = 1




```
uh = fft(a);
ah = abs(uh);
plot(ah);
```



You only need to consider till N/2

In matlab, zero index does not exist

So, m=1 is in fact zero frequency

$f = (m-1)/T$

f = 0.2 (= 1/5 that we put in)

Turbulence: lecture

Time/space domain analysis

Power spectral density function

Now that we know how to compute fourier transforms, we can use this to see the energy/power content within signals

$$\text{Define: } S_{AB}(f) = \hat{A}^*(f) \ \ \hat{B}(f)$$

R (in dimensional form) and S form perfect fourier transform pairs

Different people use different symbols of S

Obviously, the symbol or letter does not matter

$$S_{AB}(f) = \int_{-\infty}^{\infty} e^{-i2\pi f\tau} R_{AB}(f) d\tau$$

$$R_{AB}(\tau) = \int_{-\infty}^{\infty} e^{i2\pi f\tau} S_{AB}(f) df$$

Time/space domain analysis

Power spectral density function

$$R_{AB}(\tau) = \int_{-\infty}^{\infty} e^{i2\pi f\tau} S_{AB}(f)df$$

For $\tau = 0$, if we evaluate this integral,

$$R_{AB}(0) = \int_{-\infty}^{\infty} S_{AB}(f)df$$

This tells us how the cross correlation at zero time lag comes from the entire distribution of frequencies

source: Turbulence for the 21st century, W. K. George

Turbulence: lecture

Time/space domain analysis

Power spectral density function

$$\text{If A} = \text{B} = \text{u}$$

$$R_{uu}(0) = \int_{-\infty}^{\infty} S_{uu}(f)df$$

Experimentally, negative frequencies do not make sense

S_{uu} is an even function and hence we can write this as,

$$R_{uu}(0) = 2\int_{0}^{\infty} S_{uu}(f)df$$

$$R_{uu}(0) = \int_{0}^{\infty} E_{uu}(f)df$$

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Power spectral density function

$$\langle u^2 \rangle = \quad R_{uu}(0) = \int_0^\infty E_{uu}(f)df$$

$R_{uu}(0)$ has units of energy per unit mass (u^2)

df has units of frequency

$E_{uu}(f)$ has units of energy per unit mass per frequency

source: Turbulence for the 21st century, W. K. George

Time/space domain analysis

Matlab implementation: Power spectrum

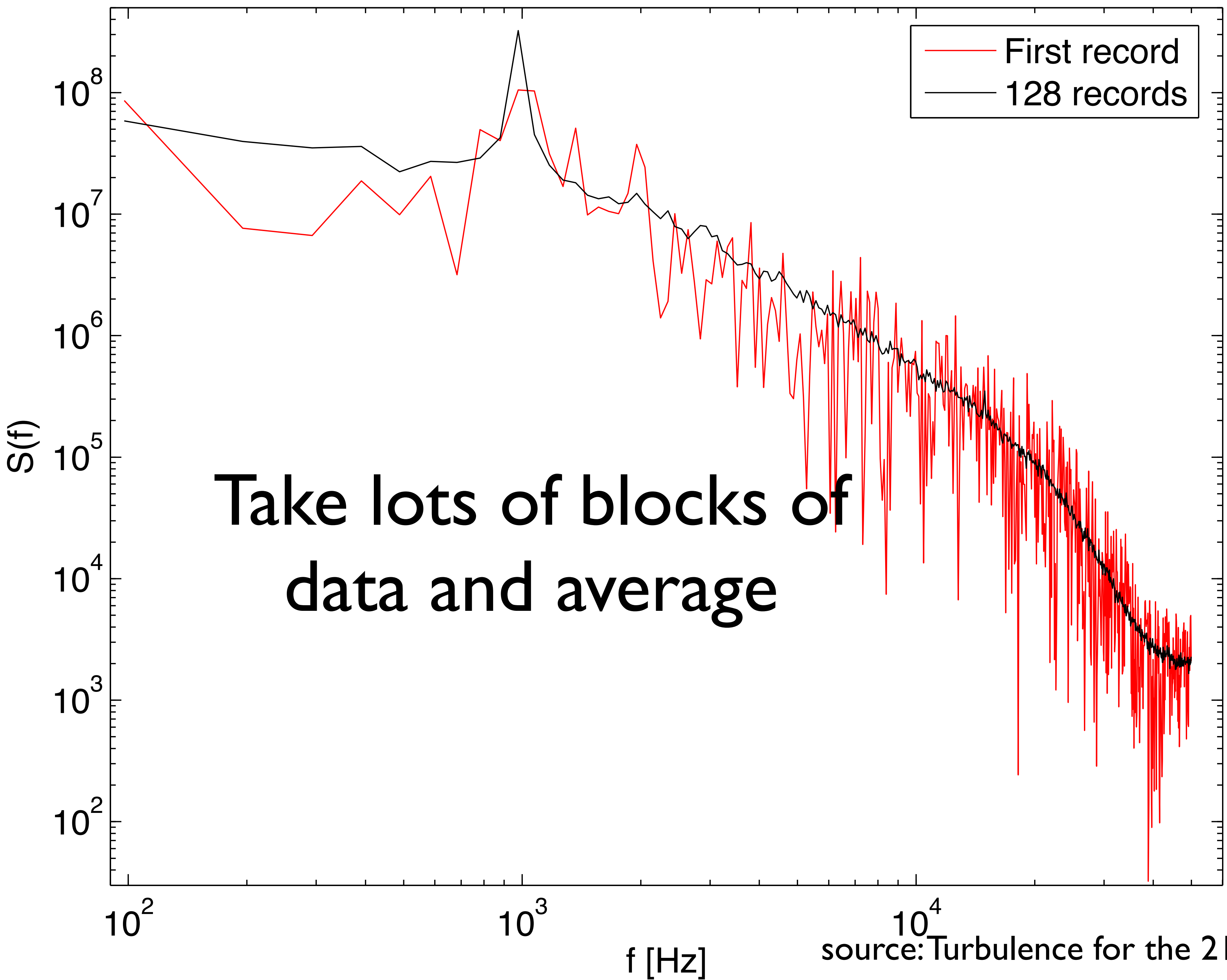
```
>> u = load('hwdata.dat'); % load data
>> N = length(u) % number of samples
>> uh = fft(u); %Do FFT
>> f = [0:N-1]./(N*dt)
"dt is the time separation between successive time
records"

>> eh = (conj(uh).*uh)/(N*dt);
% calculate spectrum. Dividing by N*dt ensure %that
the units are correct

>>plot(f(2:N/2),eh(2:N/2));
% The area under the curve here should be equal to
the variance of the signal
```

source: Turbulence for the 21st century, W. K. George

Turbulence: lecture

Time/space domain analysis

Matlab implementation: Power spectrum



Take lots of blocks of data and average

source: Turbulence for the 21st century, W. K. George

Turbulence: lecture

Time/space domain analysis

Taylor's hypothesis

Most of the time, we obtained temporal data

Almost always, the theories in turbulence are related to space

Need a way to convert time to space

If the velocity of the air stream which carries the eddies is very much greater than the turbulent velocity, one may assume that the sequence of changes in u at the fixed point are simply due to the passage of an unchanging pattern of turbulent motion over the point, i.e. one may assume that

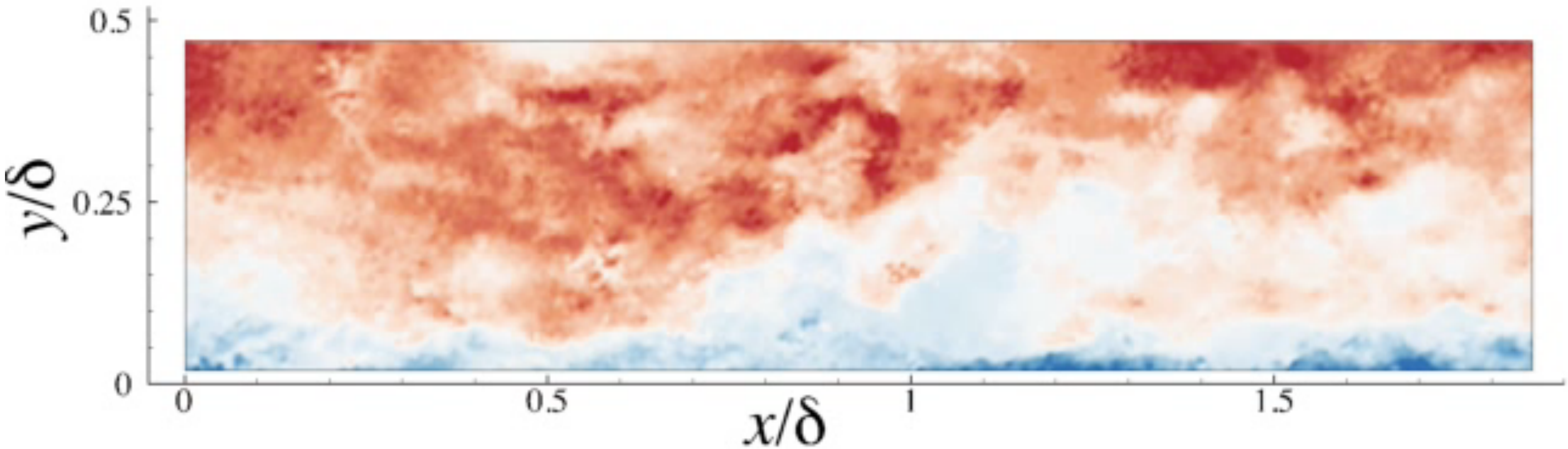
$$u = \phi(t) = \phi\left(\frac{x}{U}\right), \qquad (7)$$

Proc. R. Soc. Lond. A February 18, **1938**, Vol 164 No. 919 pp 476-490; doi:10.1098/rspa.1938.0032

$$x = -Ut \text{ or } l = f/U$$

Time/space domain analysis

Taylor's hypothesis



At a first glance, it looks pretty frozen

But, if you start peeling the layers, then not really frozen

There is an entire range of convection velocities

It is not really correct, but, we are limited at this time