Coursework Project I

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Question 1

Investigate the stability and accuracy of the four schemes in terms of the Courant-Friedrichs-Lewy (CFL) numbers $\lambda = c\frac{dt}{dx}$ *and* $\mu = \nu\frac{dt}{dx^2}$*. In particular use numerical experimentation and/or von Neumann stability analysis to determine stability boundaries.*

Listing 1: Code used to find the maximum time step and the resulting CFL parameters

```
function [lamda, mu] = probe2()
N=30;       % No of iterations
nu_range = NaN(2, N);
nu = linspace(1e-5, 0.01, N);
i=1;
cr = linspace(1e-5, 4, N);
% Compute for each value of c
for c = cr
    flags = compNu(c, nu);
    pos1 = find(flags, 1, 'first');
    pos2 = find(flags, 1, 'last');
    if ~isempty(pos1) && ~isempty(pos2)
        nu_range(1, i) = nu(pos1);    % min
        nu_range(2, i) = nu(pos2);     % max
    end
    i=i+1;
end
% Obtain CFL values
dt = 0.003;
dx = 0.0073;
lamda = nu_range * dt/dx^2;
mu = cr * dt/dx;
plot(cr, lamda(1, :), cr, lamda(2, :))
end
```

```
function [flags] = compNu(c, nu)
% See if the function is stable for the given c and nu values
% c      float        a wavespeed value
% nu     vector       spaced viscosity values
% flags  vector       1 - stable, 0 - unstable
N = size(nu, 2);
flags = ones(1,N);
for i = 1:N
    flags(i) = wave(c, nu(i));
end
end
```

The stability boundary is probed numerically through the script in Listing 1 by iterating the provided `wave.m`, which was changed to accept values of c and ν and return a boolean flag to whether the function is stable or not. Since the energy is supposed to decay with each iteration due to losses from viscosity, a final energy value higher than the initial should indicate that the problem has become unstable. The ranges of c and ν are created using a `linspace` command and starting from values close to 0 to avoid instabilities especially for Euler based evolutions. Figures 1 and 2 show the stability for the four combinations of space and time differencing formulas.

The main differences between using 2^{nd} and 4^{th} Central Differences is that the upper stability boundary is lowered from $\lambda \approx 0.5$ in the CD2 case to $\lambda \approx 0.37$ in the CD4 case. The lower value for CD4 is expected due to the fact that more nearby spatial mesh points are required to propagate the solution to the next time step accurately. The lower stability boundary (in orange) for Euler time-stepping is more linear than for Runge-Kutta 2 due to the fact that in RK2 the time step is 'divided' into a predicted value and a corrected one which can be compared to taking a smaller step size.

The coarseness of the two ranges means that the exact moment when the instabilities start to grow are not captured, so the displayed graphs should be taken as an indication of general behaviour of the methods and not precise quantitative truth. It should also be noted that in some of the methods the values of μ become larger than 1, which even though it is not displayed at that moment as an energy increase, it is an instability due to the fact that the lower frequencies are separated from the main wave and move slower. This behaviour is not captured by the comparison charts in the figures below and they should be seen, again, as an indication of general behaviour and not quantitative accuracy.

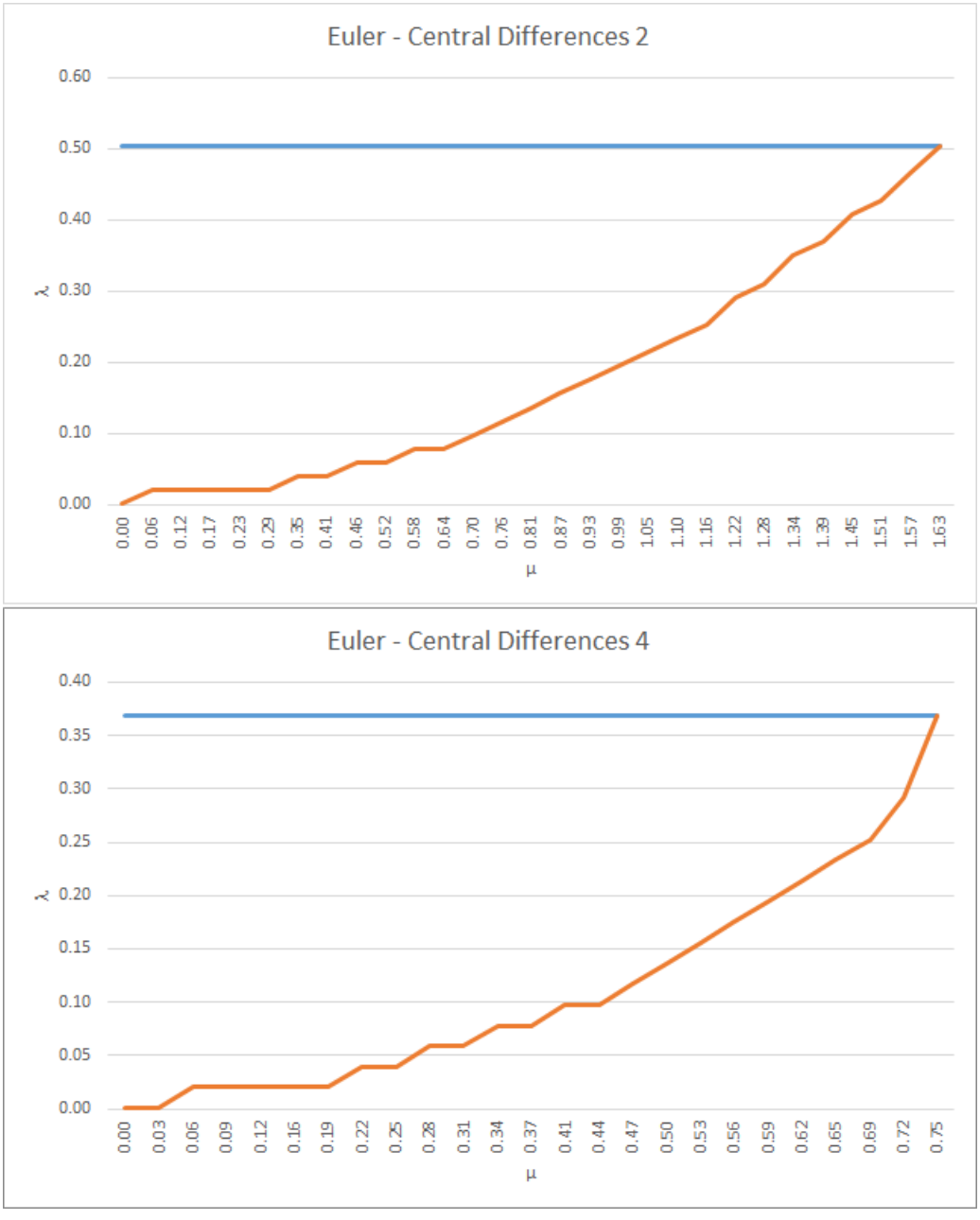

Figure 1: Euler time stepping with 2^{nd} and 4^{th} order Central Differences

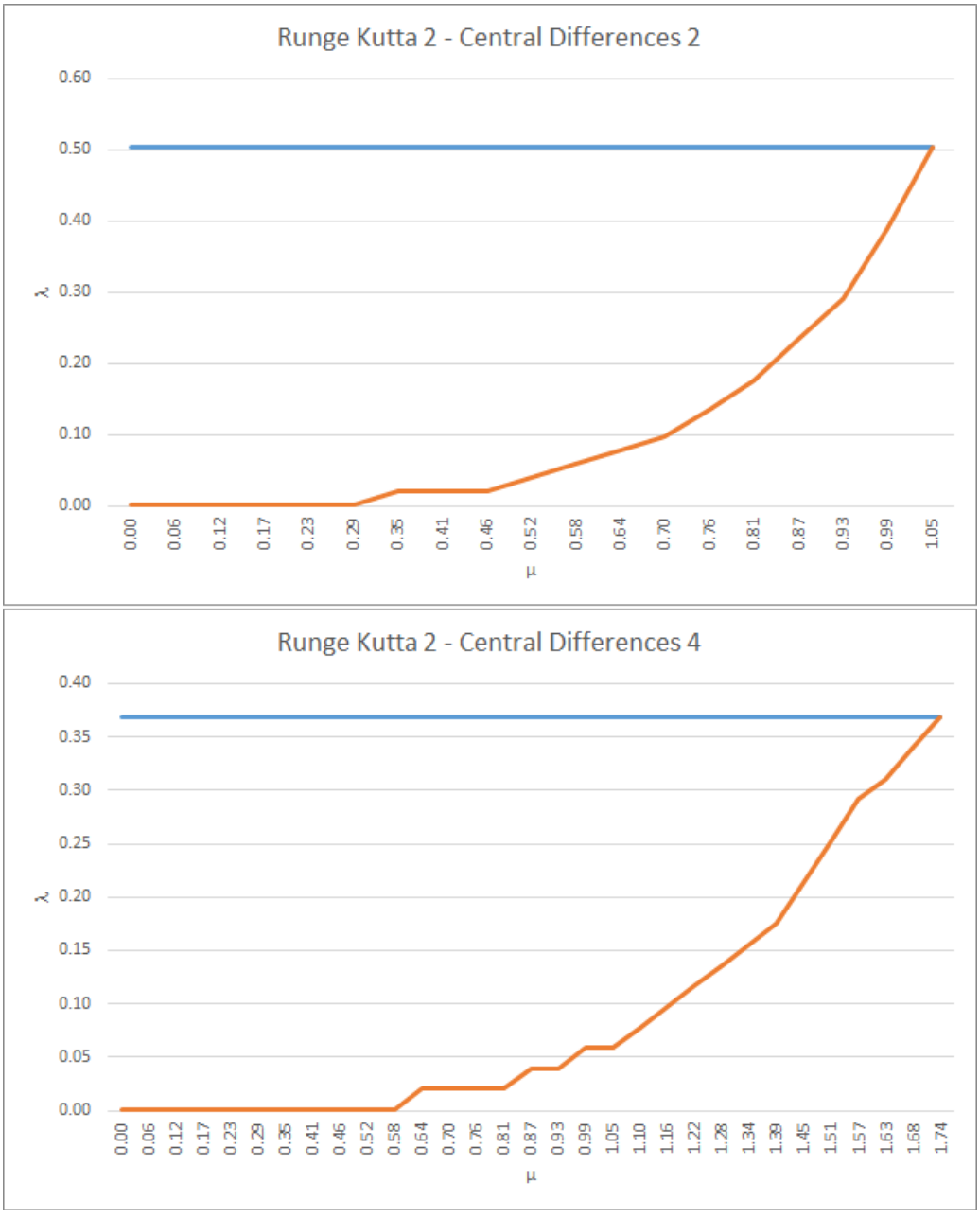

Figure 2: Runge-Kutta 2 time stepping with 2^{nd} and 4^{th} order Central Differences

Question 2

Use the exact solution as an initial condition for your simulations and compare the numerical results against the exact ones at a later time. Consider whether the integral 'constants' $A = \int_{-\infty}^{\infty} f dx$ and $E = \int_{-\infty}^{\infty} \frac{1}{2} f^2 dx$ are conserved (or not) as appropriate.

The *Area* (or volume) is conserved in both the diffusion and the convection parts of the equation, unless the numerical scheme becomes unstable. In convection this is evident due to the fact that it is a Gaussian translating from left to right. The diffusion part of the equation has the effect of increasing the Gaussian's variance (σ value) and decreasing its amplitude proportionally and thus the underlying area remains unchanged. The four graphs shown in Figure 3 show that this is the case.

The *Energy* is again conserved in the convection part due to the translation of the Gaussian. Diffusion has the role of reducing larger eddies into smaller ones until all the energy is dissipated from a flow. In the present case, the larger wavelengths are reduced to smaller ones and the energy of the Gaussian decays. If the default values are used in `wave.m` it can be seen that the RK2 based schemes decay as expected, while in the Euler schemes, after a certain number of time steps the solution 'explodes' (to note that the value is by 10^{100} higher in CD4). Energy can be thus be used as an indicator that the system is unstable and to test if decays are too large (i.e. shorter wavelengths get dissipated to quickly) compared to previous iterations or if the energy increases from one iteration to the next.

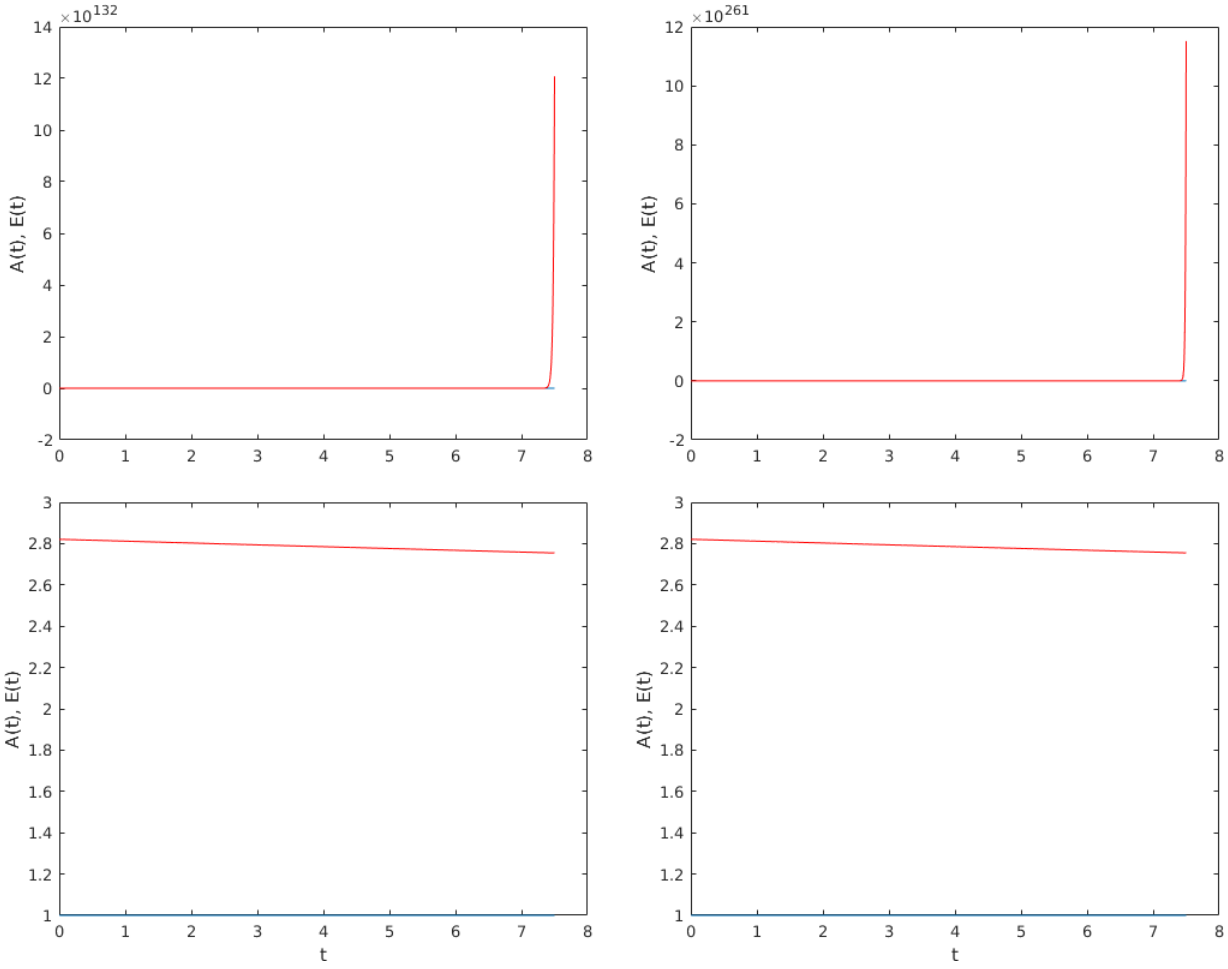

Figure 3: Illustration of the conservation for the default values

Question 3

Investigate the solutions when the viscosity is very small (in effect, zero) so that we reduce to the one-dimensional wave equation. Look the amplitude (growth/decay, stability) and dispersion (phase speed of wave components. One way to retain stability is to add an optimal small amount of artificial viscosity (e.g. the Lax-Wendroff method, Euler+CDiff2, using $\nu_{Euler} = 0.5c^2dt, \nu_{RK2} = 0.03c^4dt^3/dx^2$*).*

When viscosity is ≈ 0 (figure 4) the solution tends to explode as the energy and area go to infinity due to numerical error. The reason for this is that some gain exists due to numerical approximation that is accumulated with each iteration. The effect is strongest in Euler based methods of time propagation since any previous errors will be added to the new time step. RK2 is much better behaved and the errors accumulate more slowly (from 4 it seems to be linear), thus requiring many more time steps to reach infinity values.

The Lax-Wendroff artificial viscosity stabilises all discretisation methods but it introduces inaccuracies in the simulation. Figure 5 shows that the four methods are stable (i.e. energy does not increase), but for all cases except Euler with 4^{th} order central differences some energy is dissipated, thus it is not in agreement with the physics requirement and the solution cannot be deemed accurate. A solution to this may be to adaptively change the 0.5 or 0.03 constants to avoid this behaviour and keep energy constant.

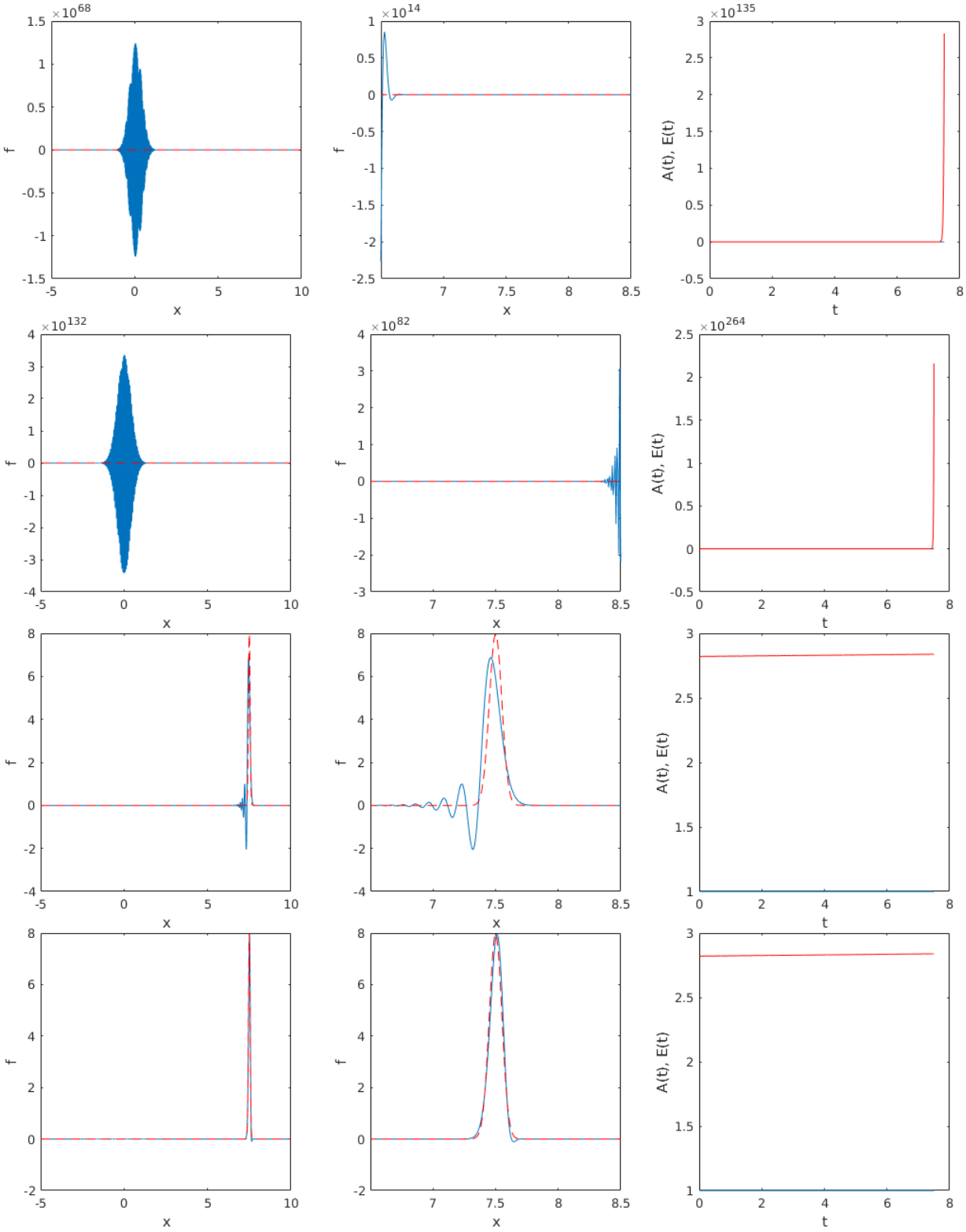

Figure 4: Behaviour of the numerical schemes for a viscosity $\nu = 0$: 1st Euler-CD2, 2nd Euler - CD4, 3rd RK2 - CD2 and 4th RK2 - CD4

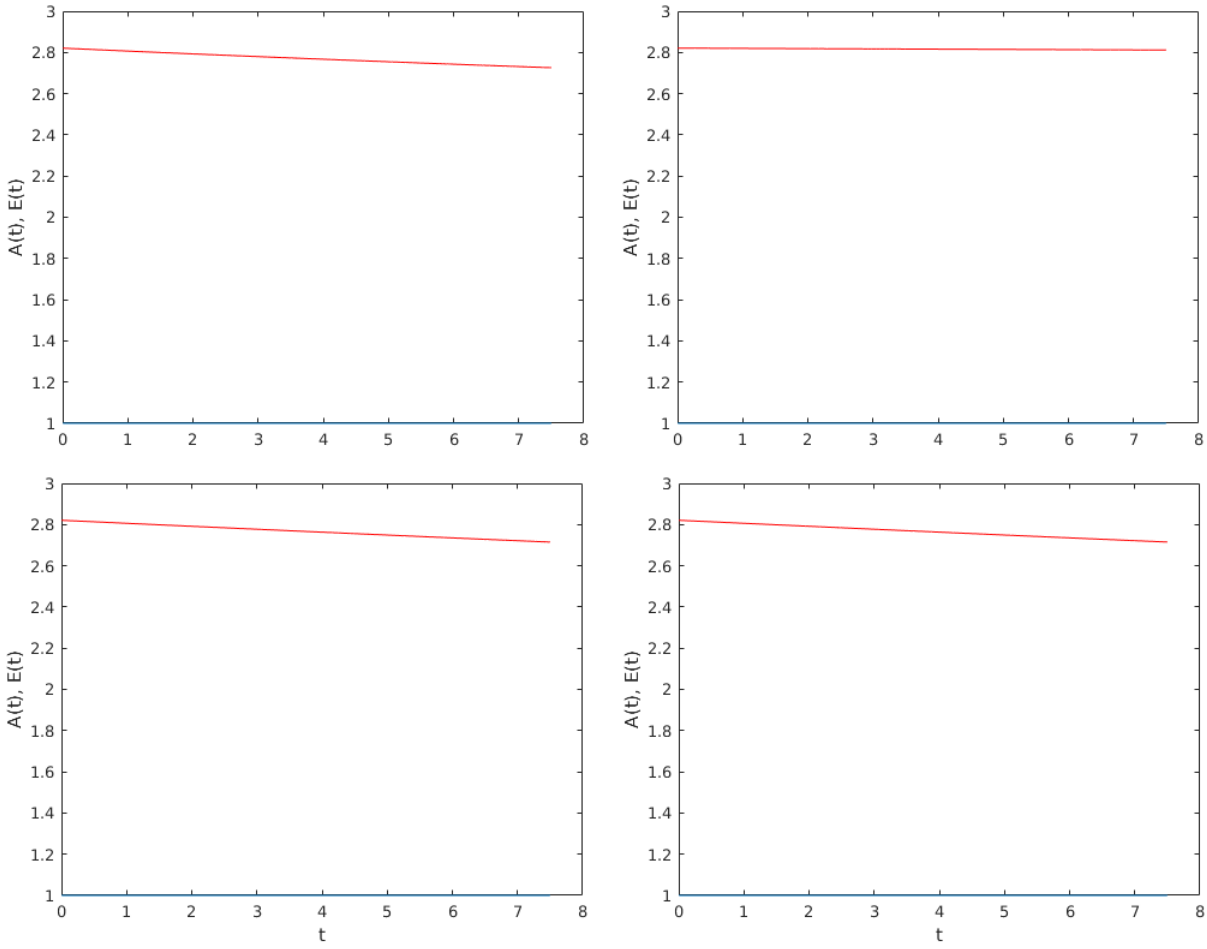

Figure 5: Energy and Area variation when the Lax-Wendroff artificial viscosity is used.

Question 4

Investigate the solutions when convection is negligible ($c = 0$). Look at the accuracy and stability limit in terms of $\mu = \nu \frac{dt}{dx^2}$ for the different schemes.

Table 1 shows that the lower limit is 0 as expected for all methods and the upper limit depends on the order of the central differencing scheme, with a maximum value $\mu = 5.592e-02$ for the second order scheme and $\mu = 3.355e-01$ for the fourth order scheme. The higher order scheme is more stable for larger values of ν (or larger step sizes) due to the larger number of values used to predict the next step and thus the smaller proportion of error that is propagated in time.

Table 1: Diffusion CFL $\mu = \nu \frac{dt}{dx^2}$ limit comparison.

	Lower limit	Upper limit
Euler - CD 2	0	$5.592e-02$
Euler - CD 4	0	$3.355e-01$
Runge-Kutta 2 - CD 2	0	$5.592e-02$
Runge-Kutta 2 - CD 4	0	$3.355e-01$

Question 5

Finally, discuss your results in terms of which schemes are best suited to be used for simulations of turbulent flows.

The best schemes are the ones based on Runge-Kutta time stepping because they are more accurate but they have the disadvantage of performing twice the number of steps when compared to Euler based schemes. RK2 allows for larger time step sizes (i.e. it accepts a larger CFL range), however, which may reduce the total time of the simulation. Euler based schemes are also much more unstable, as it was seen in Q2, but this fast growing instabilities could be useful for triggering adaptive schemes to improve the CFL coefficients. RK2 schemes are most useful when pure convection cases are simulated since they are partly stable in the sense that energy does not increase to a high degree if a small amount of time is considered and from figure 4 it is visible that when coupled with a CD4 scheme, the gaussian wave is not changed to a high degree.

4^{th} order central differences produce more accurate results but they also need more time to compute for two reasons: firstly, they need to access and operate on more values in a vector and secondly, they place a lower upper limit on $\lambda = 0.35$ compared to $\lambda = 0.5$ for CD2. For pure diffusion the CD2 method seems to be slower due to the one order of magnitude higher diffusion CFL.

In conclusion, as a general choice, the Runge-Kutta 2 coupled with 4^{th} order central differences seems to be the best default setting for most cases that produces even in extreme cases some reasonably accurate results.