SESA6061 Turbulence: Physics and Modelling
2015/16 Coursework Project II (15% of module mark)
Due Date: January

In this project you will investigate several numerical schemes for the discretisation of the advection-diffusion equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \nu\frac{\partial^2 f}{\partial x^2} \tag{1}$$

where $c = 1$ denotes the convection speed and $\nu \geq 0$ the diffusion coefficient. Your task is to discretization the equation in the form

$$\frac{\partial u}{\partial t} = \mathrm{RHS}(f) \quad \text{where} \quad \mathrm{RHS}(f) = -c\frac{\partial f}{\partial x} + \nu\frac{\partial^2 f}{\partial x^2} \tag{2}$$

with time derivatives approximated by (i) first-order Euler and (ii) a second-order Runge-Kutta schemes and the RHS approximated by (a) a second-order central-difference and (b) fourth order central-difference schemes, as below

The temporal schemes are:

- A first-order accurate explicit Euler time integration:

$$f_i^{n+1} = f_i^n + \Delta t \cdot \mathrm{RHS}(f_i^n) \tag{3}$$

- A second-order accurate Runge–Kutta time integration:

$$\begin{aligned} f_i^1 &= f_i^n + \Delta t \cdot \mathrm{RHS}(f_i^n) \\ f_i^{n+1} &= f_i^n + \frac{\Delta t}{2} \cdot \left[\mathrm{RHS}(f_i^n) + \mathrm{RHS}(f_i^1)\right] \end{aligned} \tag{4}$$

The finite-difference approximations for the spatial derivative are:

- A second-order accurate central scheme:

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \tag{5}$$

- A fourth-order accurate central scheme:

$$\frac{\partial u}{\partial x} \approx \frac{-u_{i+2}^n + 8u_{i+1}^n - 8u_{i-1}^n + u_{i-2}^n}{12\Delta x^2} \tag{6}$$

Using the numerical schemes given above, conduct the following analysis.

1. Investigate the stability and accuracy of the four schemes in terms of the Courant-Friedrichs-Lewy (CFL) numbers $\lambda = cdx/dx$ and $\mu = \nu dt/dx^2$. In particular use numerical experimentation and/or von Neumann stability analysis to determine stability boundaries.

2. Use the exact solution as an initial condition for your simulations and compare the numerical results against the exact ones at a later time. Consider whether the integral 'constants' $A = \int_{-\infty}^{\infty} f\, dx$ and $E = \int_{-\infty}^{\infty} \frac{1}{2} f^2\, dx$ are conserved (or not) as appropriate.

$$f(t,x) = \frac{1}{\sqrt{2\pi}\sigma(t)} \exp\left(-\frac{(x-ct)^2}{2\sigma(t)^2}\right) \tag{7}$$

3. Investigate the solutions when the viscosity is very small (in effect, zero) so that we reduce to the one-dimensional wave equation. Look the amplitude (growth/decay, stability) and dispersion (phase speed of wave components. One way to retain stability is to add an optimal small amount of artificial viscosity (e.g. the Lax-Wendroff method, Euler+CDiff2, using $\nu = 0.5c^2\, dt$).

4. Investigate the solutions when convection is negligible ($c = 0$). Look at the accuracy and stability limit in terms of $\mu = \nu\, dt/dx^2$ for the different schemes.

5. Finally, discuss your results in terms of which schemes are best suited to be used for simulations of turbulent flows.

Your project writeup need not be a full technical report but should state all approximations made and use figures of results to back up any conclusions. Handwritten workings of the wavenumber characteristics and von Neumann analysis are permitted but the final results should be included in the typed report. Be sure to include enough detail (using appendices as necessary) so that your results could be reproduced by another researcher (or yourself at a future date!) wishing to check or extend your findings. Your work will be primarily assessed on (1) the completeness of the results, and the visual/logical effectiveness of the manner in which they are presented, and (2) recommendations regarding cost/accuracy trade-offs, and practical guidelines for CFD users, implied by your findings.