Von Neumann Stability Analysis

Idea is to calculate the growth G of a wave-number over a single time step, then ask whether $|G| > 1$ (unstable) or $|G| \leq 1$ (stable). The system is stable if $|G| \leq 1$ for all wavenumbers $0 < k\Delta x < \pi$

Example: Euler time stepping and 2nd order central differences, applied to

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = \nu \frac{\partial^2 f}{\partial x^2}$$

discretized on a uniform grid of spacing Δ, so that $x_j = x_0 + \Delta \cdot j$ at successive grid points.

Let $f^n = e^{ikx}$ and $f^{n+1} = G \cdot e^{ikx}$, n = time step. and G is the gain / amplification over the timestep.

$$f^{n+1} = G f^n = G e^{ikx} =$$

$$= e^{ikx} + \frac{c\Delta t}{2\Delta x} \cdot e^{ikx}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) + \frac{\nu \Delta t}{\Delta x^2}\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)$$

using Euler t-step and 2nd order central differences for $\partial/\partial x$ and $\partial^2/\partial x^2$.

We can remove the common factor e^{ikx} and re-write the exponentials in terms of trig functions sin/cos

$$G = 1 + \left(\frac{c\Delta t}{\Delta x}\right) i \sin(k\Delta x) - \left(\frac{\nu \Delta t}{\Delta x^2}\right) 2\left(1 - \cos(k\Delta x)\right)$$

and then (easier to test $|G|^2 > 1$ than $|G| > 1$)

$$|G|^2 = 1 + \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x) + 4\left(\frac{\nu\Delta t}{\Delta x^2}\right)^2 4\left(1 - \cos(k\Delta x)\right)^2$$
$$- 4\left(\frac{\nu\Delta t}{\Delta x}\right)\left(1 - \cos(k\Delta x)\right)$$

The worst cases are (a) convection term at $k\Delta x = \pi/2$ and (b) diffusion at $k\Delta x = \pi$, i.e. when $\sin^2 = 1$ and $1 - \cos() = 2$. We can check how large we can make the CFL numbers (Courant, Friedrichs, Lewy)

convection CFL $\boxed{\frac{c\Delta t}{\Delta x}}$ and $\boxed{\frac{\nu \Delta t}{\Delta x^2}}$ viscous CFL while still satisfying stability, i.e. such that $|G|^2 \leq 1$ for all $0 \leq k\Delta x \leq \pi$.

(a) Pure diffusion ($c=0$, $\nu>0$) and $k\Delta x = \pi$

$$|G|^2 = 1 + 4\left(\frac{\nu\Delta t}{\Delta x^2}\right)^2 (2)^2 - 4\left(\frac{\nu\Delta t}{\Delta x^2}\right).2$$

and we get the stability boundary by solving for $|G|^2 = 1$, therefore

$$16\left(\frac{\nu\Delta t}{\Delta x^2}\right)^2 - 8\left(\frac{\nu\Delta t}{\Delta x^2}\right) = 0$$

and

$$\left(\frac{\nu\Delta t}{\Delta x^2}\right)\left[2\frac{\nu\Delta t}{\Delta x^2} - 1\right] = 0$$

solving gives

$$\frac{\nu\Delta t}{\Delta x^2} = \left(0 \text{ or } \frac{1}{2}\right)$$

and $|G|^2 \leq 1$ requires $0 \leq \frac{\nu\Delta t}{\Delta x^2} \leq \frac{1}{2}$ for stability.

(b) Pure convection ($c \neq 0$, $\nu = 0$), $k\Delta x = \pi/2$

$$|G|^2 = 1 + \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x)$$

this is $|G|^2 > 1$ for any finite $\left|\frac{c\Delta t}{\Delta x}\right| > 0$ and $0 < k\Delta x \leq \pi$ $\Rightarrow$ unconditionally unstable.

But, can be stabilised by adding a very small amount of 'artificial' viscosity, this is the basis of the Lax-Wendroff approach.

(c) Small but finite viscosity ($c \neq 0$, $\nu \neq 0$ but small)

This approximates convection and is stabilised by small-but-finite viscosity, here

$$|G(k)|^2 = 1 + \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x) + 4\left(\frac{\nu\Delta t}{\Delta x^2}\right)(1-\cos(k\Delta x))^2$$
$$+ 4\left(\frac{\nu\Delta t}{\Delta x^2}\right)(1-\cos(k\Delta x))$$

We want $G(k)$ to be close to unity (and NOT exceed it) in as wide a range of k around $k \sim 0$.

For small $k\Delta$, we expand as a series

$$|G|^2 = 1 + \left[\left(\frac{c\Delta t}{\Delta x}\right)^2 - 2\left(\frac{\nu\Delta t}{\Delta x^2}\right)\right](k\Delta x)^2$$
$$+ \left[-\frac{1}{3}\left(\frac{c\Delta t}{\Delta x}\right)^2 + \left(\frac{\nu\Delta t}{\Delta x^2}\right)^2 + \frac{1}{6}\left(\frac{\nu\Delta t}{\Delta x^2}\right)\right](k\Delta x)^4 + \ldots$$
$$+ O(k\Delta x)^6$$

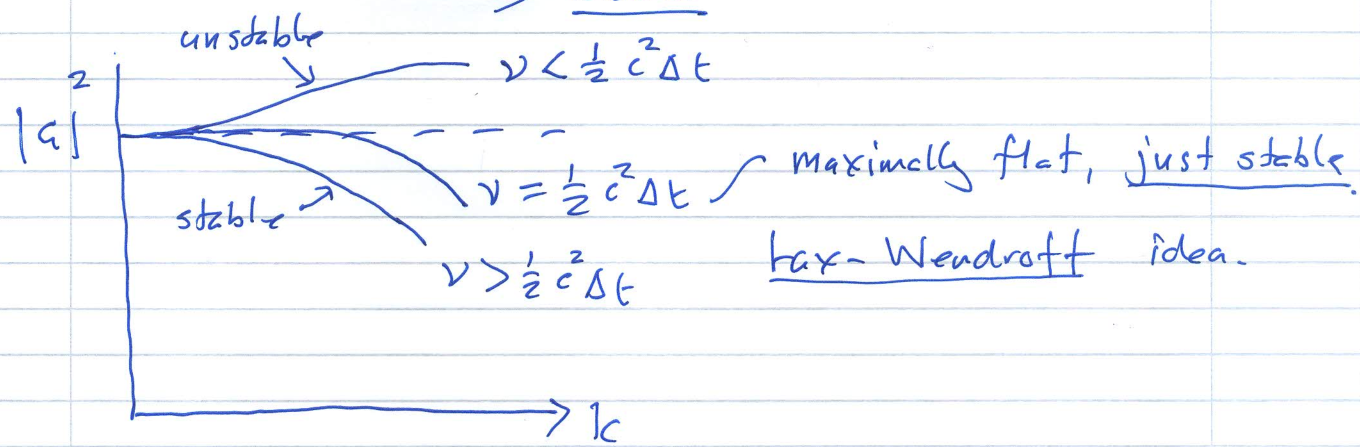
We are closest to $|G|^2 = 1$ if we zero the $k^2\Delta x^2$ term by putting

$$\frac{\nu\Delta t}{\Delta x} = \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2$$

ie by setting the 'artificial' viscosity to $\nu = \frac{1}{2}c^2\Delta t$,
In this case

$$|G|^2 = 1 \underbrace{-\frac{1}{4}\left(\frac{c\Delta t}{\Delta x}\right)^2\left(1 - \left(\frac{c\Delta t}{\Delta x}\right)^2\right)}_{\text{always } <0 \text{ for any } \left|\frac{c\Delta t}{\Delta x}\right| < 1}(k\Delta x)^4 + O(k\Delta x)^6 + \ldots$$

$\Rightarrow$ stable



Stability Boundary: Euler + central-diff 2nd order