1.7 Rigorously prove a limit or a limit law.

As close as I want to — by making r close enough to

1000 cm^2 — $\sqrt{\frac{1000}{9}}$ cm

ε	δ
5 cm^2	.0446 cm
1 cm^2	.008751 cm

$$\lim_{x \to a} f(x) = L$$ ← This is true if:

For every number $\varepsilon > 0$, (near 0, approaching 0.)
there exists a number $\delta > 0$, such that
whenever $0 < |x-a| < \delta$,
we have $|f(x) - L| < \varepsilon$.

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#19

$$\lim_{x \to 1} \frac{2+4x}{3} = 2$$

DSP depends on limit def!

I want to show: For every $\varepsilon > 0$,
there exists a $\delta > 0$, s.t.
whenever $0 < |x-1| < \delta$,
we have $\left|\frac{2+4x}{3} - 2\right| < \varepsilon$.

what is equivalent?

$$\Leftrightarrow \left|\frac{2+4x}{3} - \frac{6}{3}\right| < \varepsilon$$

$$\Leftrightarrow \left|\frac{4x-4}{3}\right| < \varepsilon$$

$$\Leftrightarrow \left|\frac{4}{3}(x-1)\right| < \varepsilon$$

$\boxed{|cx| = c|x| \quad c > 0}$

$\boxed{|ab| = |a||b|}$

$\iff \frac{4}{3}|x-1| < \varepsilon$

$\iff |x-1| < \frac{3}{4}\varepsilon$

"whenever $|x-1|<\delta$, we have ~~$\frac{4}{3}|x-1|<\varepsilon$~~." $|x-1|<\frac{3}{4}\varepsilon$.

Let $\delta = \frac{3}{4}\varepsilon$.

Then for every ε,
there is a δ, $(\frac{3}{4}\varepsilon)$ such that
when $0<|x-1|<\delta$, we have
$\left|\frac{2+4x}{3} - 2\right| < \varepsilon$, since all the steps are reversible.

$\therefore \lim_{x\to 1} \frac{2+4x}{3} = 2$. $\square$

Limit law: $\lim_{x\to a} x = a$?

I want to show: For every $\varepsilon>0$,
there is a $\delta>0$, s.t.
whenever $0<|x-a|<\delta$,
we have $|x-a|<\varepsilon$.

Let $\delta = \varepsilon$. Then $\lim_{x\to a} x = a$. $\square$

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\# 29 $\lim_{x\to 2} (x^2-4x+5) = 1$

I want to show: For every $\varepsilon>0$,
There is a $\delta>0$, s.t
whenever $0<|x-2|<\delta$,
we have $|x^2-4x+5-1| < \varepsilon$.

$\iff |x^2-4x+4| < \varepsilon$

$\iff |(x-2)^2| < \varepsilon$

$(x-2)^2 < \varepsilon$ $\quad$ $(x-2)^2$ is positive, so $|(x-2)^2| = (x-2)^2$.

$$-\sqrt{\varepsilon} < x-2 < \sqrt{\varepsilon}$$ Square root principle, inequalities

or $|x-2| < \sqrt{\varepsilon}$

For "whenever $|x-2| < \delta$, we have $|x-2| < \sqrt{\varepsilon}$" to work,

Let $\delta = \sqrt{\varepsilon}$.

Then $\lim_{x \to 2} (x^2 - 4x + 5) = 1$. $\square$

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$$\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}.$$

I want to show: For every $\varepsilon > 0$,
There is a $\delta > 0$, s.t.
whenever $0 < |x-2| < \delta$,
we have $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$.

$$\left|\frac{2}{2x} - \frac{x}{2x}\right| < \varepsilon$$

$$\left|\frac{2-x}{2x}\right| < \varepsilon$$

$$\frac{|2-x|}{|2x|} < \varepsilon$$

$$\frac{1}{|2x|}|x-2| < \varepsilon$$

$|x-2| = |2-x|$

since $(x-2) = -(2-x)$

~~$|x-2| < \varepsilon|2x|$~~

~~Let $\delta = \varepsilon|2x|$.~~

δ must be in terms of ε, not x.

$x \to 2$

$\boxed{x \text{ is near } 2.}$

Restrict
$1 < x < 3$.
Then
$$\frac{1}{3} < \frac{1}{x} < 1,$$
$$\frac{1}{6} < \frac{1}{|2x|} < \frac{1}{2}.$$

$$\frac{1}{6}|x-2| < \frac{1}{|2x|}|x-2| < \frac{1}{2}|x-2|.$$

but also $\frac{1}{|2x|}|x-2| < \varepsilon$.

"When $|x-2|<\delta$, $\frac{1}{|2x|}|x-2|<\varepsilon$" — I want a δ in terms of ε that makes this true.

$\frac{1}{6}|x-2|<\varepsilon$ $\qquad$ $\frac{1}{2}|x-2|<\varepsilon$

$|x-2|<6\varepsilon$ $\qquad$ $|x-2|<2\varepsilon$

If $\delta = 6\varepsilon$, then: whenever $|x-2|<\delta$ $(|x-2|<6\varepsilon)$, $\overset{?}{\Rightarrow}$ $\frac{1}{|2x|}|x-2|<\varepsilon$

$\Rightarrow \frac{1}{6}|x-2|<\varepsilon$

$|x-2|<6\varepsilon$ $\qquad$ $\frac{1}{6}|x-2| < \frac{1}{|2x|}|x-2| < \varepsilon$

Don't assume what you want to prove!

If $\delta = 2\varepsilon$, then:

$$|x-2|<\delta \Rightarrow |x-2|<2\varepsilon \Rightarrow \frac{1}{2}|x-2|<\varepsilon$$

but also, $\frac{1}{|2x|} < \frac{1}{2}$.

$$\frac{1}{|2x|}|x-2| < \frac{1}{2}|x-2| < \varepsilon$$

$$\therefore \frac{1}{|2x|}|x-2| < \varepsilon.$$

$$\Rightarrow \left|\frac{1}{x}-\frac{1}{2}\right| < \varepsilon.$$

$$\therefore \lim_{x\to 2} \frac{1}{x} = \frac{1}{2} \quad \square.$$

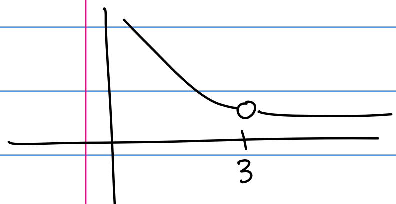
1.8 Determine whether a function is continuous (at a point, in an interval, everywhere).

Continuity: A function f is continuous at a (number / point) if:

$$\underbrace{\lim_{x\to a} f(x)}_{\text{exists}} = \underbrace{f(a)}_{\text{is defined}}.$$

$f(x) = \dfrac{x-3}{x^2-9}$

$\lim_{x \to 3} f(x) = \lim_{x \to 3} \dfrac{1}{x+3} = \dfrac{1}{6}$



$f(3) =$ undefined.

Cont at 3? no.

$f(x) = \dfrac{x^2-1}{x^2+1}$

$\lim_{x \to 1} f(x) = 0$

$=$

$f(1) = 0$

DSP works on rational functions.

f cont at $x = 1$.

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#39 modified

$f(x) = \begin{cases} 1-x^2 & \text{if } x \leq 1 \\ x-1 & \text{if } x > 1 \end{cases}$

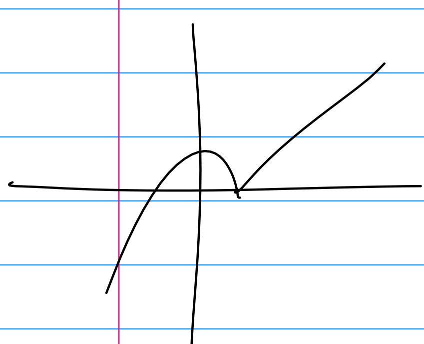
f is continuous on an interval if f is cont. at every point in the interval.

Continuous on $(1, \infty)$? yes.

Cont. on $(-\infty, 1)$? yes.

$\boxed{\text{Thm Polynomials and rational functions are continuous on their domains.}}$

$\left.\begin{array}{l} \left.\begin{array}{l} \lim_{x \to 1^+} f(x) = 0 \\ \lim_{x \to 1^-} f(x) = 0 \end{array}\right] \Rightarrow \lim_{x \to 1} f(x) = 0 \\ \qquad\qquad\qquad\qquad\qquad\quad = \\ \qquad\qquad\qquad\qquad\qquad f(1) = 0 \end{array}\right\} \Rightarrow f$ is continuous at 1.



f cont everywhere.