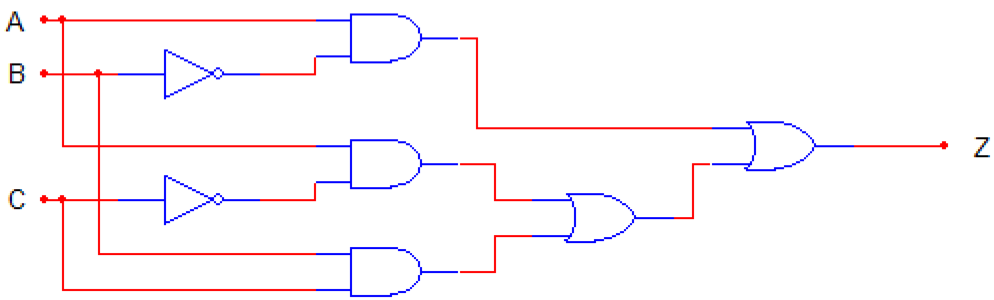
Circuit Simplification: Boolean Algebra

What is Boolean Algebra ?

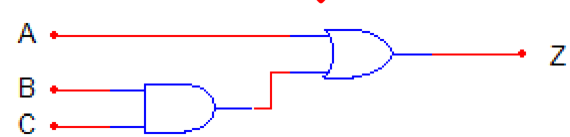
Boolean Algebra is a mathematical technique that provides the ability to algebraically simplify logic expressions. These simplified expressions will result in a logic circuit that is equivalent to the original circuit, yet requires fewer gates.



$$Z = BC + A\overline{B} + A\overline{C}$$

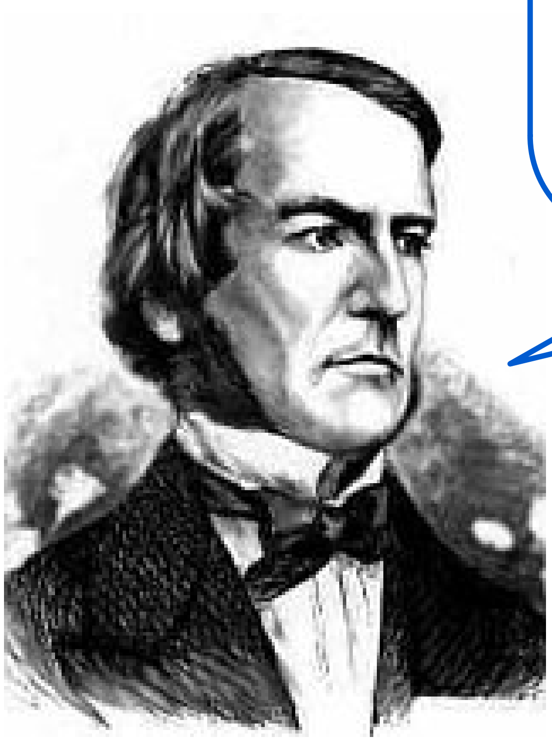
Simplification With Boolean Algebra

$$Z = A + BC$$



George Boole

My name is George Boole and I lived in England in the 19th century. My work on mathematical logic, algebra, and the binary number system has had a unique influence upon the development of computers. Boolean Algebra is named after me.

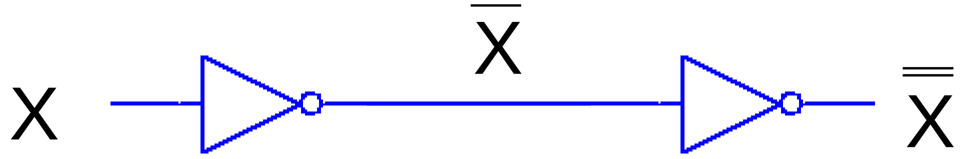


Boolean Theorems

Single Variable - Invert Function

Theorem #9

$$\overline{\overline{X}} = X$$



X	$\overline{X}$	$\overline{\overline{X}}$
0	1	0
1	0	1

Boolean Theorems

Commutative Law

<u>Theorem #11</u> – AND Function

$$X \cdot Y = Y \cdot X$$

<u>Theorem #12</u> – OR Function

$$X + Y = Y + X$$

Boolean Theorems

Associative Law

Theorem #13– AND Function

$$X\,(Y\,Z) = (X\,Y)\,Z$$

Theorem #14 – OR Function

$$X + (Y + Z) = (X + Y) + Z$$

Boolean Theorems

Distributive Law

<u>Theorem #15</u> – AND Function

$$X(Y+Z)=XY+XZ$$

<u>Theorem #16</u> – OR Function

$$(X+Y)(W+Z)=XW+XZ+YW+YZ$$

Boolean Theorems

Consensus Theorem

Theorem #16

$$X + \overline{X}\,Y = X + Y$$

Summary

Boolean Theorems

1) $X \cdot 0 = 0$

2) $X \cdot 1 = X$

3) $X \cdot X = X$

4) $X \cdot \overline{X} = 0$

5) $X + 0 = X$

6) $X + 1 = 1$

7) $X + X = X$

8) $X + \overline{X} = 1$

9) $\overline{\overline{X}} = X$

10) $X \cdot Y = Y \cdot X$

11) $X + Y = Y + X$

Commutative Law

12) $X(YZ) = (XY)Z$

13) $X + (Y + Z) = (X + Y) + Z$

Associative Law

14) $X(Y + Z) = XY + XZ$

15) $(X + Y)(W + Z) = XW + XZ + YW + YZ$

Distributive Law

16) $X + \overline{X}Y = X + Y$

Consensus Theorem

Duality Principle

- A Boolean equation remains valid if we take the dual of the expressions on both sides of the equals sign
- Dual of expression it means,
  - Interchange 1's with 0's (and Vice-versa)
  - Interchange AND (.) with OR (+) (and Vice-versa)

$X+0=X \xrightarrow{\text{Duality}} X\cdot 1=X$

$X+X\cdot Y=X \xrightarrow{\text{Duality}} X\cdot(X+Y)=X$

$X+X=X \xrightarrow{\text{Duality}} X\cdot X=X$

$X+X\cdot Y=X \xrightarrow{\text{Duality}} X\cdot(X+Y)=X$

Find the complement of the functions

$$F_1 = x'yz' + x'y'z$$

$$F_2 = x(y'z' + yz).$$

1. $F_1 = x'yz' + x'y'z.$
   The dual of F_1 is $(x' + y + z')(x' + y' + z).$
   Complement each literal: $(x + y' + z)(x + y + z') = F_1'.$

2. $F_2 = x(y'z' + yz).$
   The dual of F_2 is $x + (y' + z')(y + z).$
   Complement each literal: $x' + (y + z)(y' + z') = F_2'.$

DeMorgan’s Law

Theorem 1: NAND = Bubbled OR Complement of product is equal to addition of the compliments.

$$\overline{A \cdot B} = \overline{A} + \overline{B}$$

Theorem 2: NOR = Bubbled AND Complement of sum is equal to product of the compliments.

$$\overline{A + B} = \overline{A} \cdot \overline{B}$$

Simplify the following Boolean functions to a minimum number of literals.

1. $x(x' + y) = xx' + xy = 0 + xy = xy.$

2. $x + x'y = (x + x')(x + y) = 1(x + y) = x + y.$

3. $(x + y)(x + y') = x + xy + xy' + yy' = x(1 + y + y') = x.$

4.
$$\begin{aligned} xy + x'z + yz &= xy + x'z + yz(x + x') \\ &= xy + x'z + xyz + x'yz \\ &= xy(1 + z) + x'z(1 + y) \\ &= xy + x'z. \end{aligned}$$

Sum-of-products

- ANDed product–input combination for which output is true
- Each variable appears exactly once, in true or inverted form (but not both)

From the pervious example:

A	B	C		minterms	
0	0	0	0	A'B'C'	m0
0	0	1	1	A'B'C	m1
0	1	0	0	A'BC'	m2
0	1	1	1	A'BC	m3
1	0	0	0	AB'C'	m4
1	0	1	1	AB'C	m5
1	1	0	1	ABC'	m6
1	1	1	1	ABC	m7

short-hand notation for minterms of 3 variables

F in canonical form:

$$
\begin{aligned}
F(A, B, C) &= \Sigma m(1,3,5,6,7) \\
&= m1 + m3 + m5 + m6 + m7 \\
&= A'B'C + A'BC + AB'C + ABC' + ABC
\end{aligned}
$$

canonical form $\neq$ minimal form

$$
\begin{aligned}
F(A, B, C) &= A'B'C + A'BC + AB'C + ABC + ABC' \\
&= (A'B' + A'B + AB' + AB)C + ABC' \\
&= ((A' + A)(B' + B))C + ABC' \\
&= C + ABC' \\
&= ABC' + C \\
&= AB + C
\end{aligned}
$$

Product-of-sums

- Sum term (or maxterm)
- ORed sum of literals –input combination for which output is false
- Each variable appears exactly once, in true or inverted form (but not both)

A	B	C	maxterms	
0	0	0	A+B+C	M0
0	0	1	A+B+C'	M1
0	1	0	A+B'+C	M2
0	1	1	A+B'+C'	M3
1	0	0	A'+B+C	M4
1	0	1	A'+B+C'	M5
1	1	0	A'+B'+C	M6
1	1	1	A'+B'+C'	M7

short-hand notation for Maxterms of 3 variables

F in canonical form:

$$\begin{aligned} F(A, B, C) &= \Pi M(0,2,4) \\ &= M0 \cdot M2 \cdot M4 \\ &= (A + B + C)(A + B' + C)(A' + B + C) \end{aligned}$$

canonical form $\neq$ minimal form

$$\begin{aligned} F(A, B, C) &= (A + B + C)(A + B' + C)(A' + B + C) \\ &= (A + B + C)(A + B' + C) \\ &\quad (A + B + C)(A' + B + C) \\ &= (A + C)(B + C) \end{aligned}$$

SOP and POS Represent the same function

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F = \Sigma m(1,3,5,6,7) \Rightarrow F = \Pi M(0,2,4)$$

Conversion Procedure for Standard SOP

Example: Convert the expression into the standard SOP form

$$Y = AB + AC' + BC$$

Solution: Given expression $Y = AB + AC' + BC$

Step 1: Find the missing literals

$$Y = AB + AC' + BC$$

C B A

Step 2: AND each term with (Missing literal + its compliment)

$$Y = AB(C + C') + AC'(B + B') + BC(A + A')$$

Simplify the expression to get the standard SOP form

$$Y = ABC + ABC' + ABC' + AB'C' + ABC + A'BC$$

A + A = A

$$Y = ABC + ABC' + AB'C' + A'BC$$

(Each term consists of all literal)

Conversion Procedure for Standard POS

- Example: Convert the expression into the standard SOP form

$$Y = (A+B)(A+C)(B+C')$$

Step 1: Find the missing literals

Step 2: OR each term with (Missing literal + its compliment)

$$Y = (A + B + CC')(A + BB' + C)(AA' + B + C')$$

$$Y = (A + B + C)(A + B + C')(A + B + C)(A + B' + C)(A + B + C')(A' + B + C')$$

$$Y = (A + B + C)(A + B + C')(A + B' + C)(A' + B + C')$$