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APPROXIMATIONS TO FINITE SAMPLE MOMENTS OF ESTIMATORS WHOSE EXACT SAMPLING DISTRIBUTIONS ARE UNKNOWN

BY T. N. SRINIVASAN[1]

The exact sampling distributions of estimators of structural parameters of econometric models are unknown except for a few simple cases. In this situation two alternative approaches towards evaluating finite sample properties of various estimators have been adopted in the literature: (i) Monte Carlo experiments, and (ii) the approach pioneered by Nagar and his students in which the sampling error of an estimator is expressed as the sum of an infinite series of random variables, successive terms of which are of decreasing order of sample size in probability. It is claimed that the small sample properties of the estimator under consideration can be approximated by those of the first few terms of such an infinite series. This paper shows through examples that the Nagar approach can be misleading in the sense that it can yield an estimate for finite sample bias that differs from the true finite sample bias to the same order of sample size. And it can yield estimates of bias which are finite (infinite) while the true bias is infinite (finite). The paper also draws attention to some of the pitfalls to be avoided in studying the properties of an infinite sequence of random variables.

1. INTRODUCTION

A FREQUENTLY ENCOUNTERED situation in the estimation of parameters of an econometric relation is one where the finite sample distributions of the estimators are unknown. The known results in such a situation usually relate only to the asymptotic properties such as consistency or the convergence to the standard normal distribution of the estimating sequence (after suitable normalisation). To choose one among many examples, the finite sample distribution of the estimators of structural parameters of a system of simultaneous stochastic equations, such as two stage least squares or k-class estimators, is not known except for some special cases [**1**]. In such a situation two alternative approaches towards evaluating finite sample properties of various estimators have been adopted in the literature: (i) Monte Carlo experiments, [**8**, **17**] and others, and (ii) the approach pioneered by Nagar and his students [**7**–**15**], in which the sampling error of an estimator is expressed as the sum of an infinite series of random variables, successive terms of which are of decreasing order of sample size (in probability). It is claimed that the small sample properties (in particular, the moments) of the estimator under consideration, can then be approximated (to the desired order of sample size) by the properties of the first few terms of the infinite series.

Basmann [**1**] has suggested that the Monte Carlo investigations of the bias, variance, mean squared error, etc., of estimators can be misleading particularly in situations where the population counterparts of these are not even defined. The present paper shows that the second approach, which for convenience we shall call the Nagar approach, can also be misleading in that: (a) it can yield an estimate for finite sample bias (to the specified order of sample size) that differs from the

[1] I wish to thank Professor A. L. Nagar and Mr. S. C. Srivastava for their helpful comments on an earlier version. The comments of the referees were extremely helpful. Errors remaining are mine.

true finite sample bias; (b) it may suggest that the bias is infinite while the true bias is finite; (c) it may result in finite valued expressions for bias while the moments of the exact sampling distribution are infinite.[2] *This is not to suggest that the results obtained by Nagar* [**7**–**15**] *are necessarily invalid; only that further investigation is necessary to establish their validity. This certainly has not been done by these authors.*

Some of the difficulties with the Nagar approach can in part be attributed to the inadequate attention given to the distinction between probability limit of a sequence of random variables and the limit of the sequence of their expected values. To make this and other such distinctions clear is another purpose of this note.

Finally, an alternative interpretation of the Nagar approach due to Basmann [**1**] is examined. According to Basmann, the Nagar approach leads to an approximation of the *exact finite sample distribution* of the estimator considered, rather than of the *moments* of such distribution. We shall argue that this interpretation is useful only if the purpose of such approximation and the criterion by which closeness of approximation is to be judged are clearly defined.

2. CONCEPTS AND DEFINITIONS

Suppose we are considering a sequence of estimators $\{X_T\}$, T being the size of the sample, estimating a parameter θ belonging to some set. The following definitions are useful. Let E denote the expectation operator, $\text{plim}_{T\to\infty}$ denote the probability limit of a sequence of random variables.

DEFINITION 1: The sequence $\{X_T\}$ of estimators of θ is said to be *asymptotically unbiased*[3] if

$$\lim_{T\to\infty} EX_T = \theta.$$

In particular an unbiased sequence of estimators is also asymptotically unbiased.

DEFINITION 2: The sequence $\{X_T\}$ is said to be *asymptotically bounded*[4] *in probability* if for each $\varepsilon > 0$ there exists an M such that probability $\{|X_T| > M\} < \varepsilon$ for T sufficiently large. Clearly if $\text{plim}_{T\to\infty}\{X_T\}$ exists, then $\{X_T\}$ is asymptotically bounded in probability.

DEFINITION 3: Let $\{X_T\}$ be a sequence of random variables and let H be the set of real numbers h such that $\{T^h X_T\}$ is asymptotically bounded in probability. If H is nonempty and bounded above, then the sequence $\{X_T\}$ is said to be of order T^{-k} in probability where $k = \sup H$.

[2] Professor Phoebus Dhrymes, on receiving the present note, informed me that he has made a similar criticism of the Nagar approach in his forthcoming book. He was kind enough to send me photostat copies of the relevant pages.

[3] This definition follows Rao [**16**, p. 152f]. It is different from that of Hood and Koopmans [**4**, p. 129], who define asymptotic bias as probability limit of $\{X_T - \theta\}$ if such a limit exists. Such a definition is not a natural extension of the usual definition of bias for finite sample sizes.

[4] This term is used at the suggestion of one of the referees. Feller [**2**, p. 247] uses the term "stochastically bounded" instead.

This definition is more restrictive than that of Mann and Wald [**6**, Definition 3]. It can be easily shown, following their definition, that X_T is of probability order $O_p(T^{-h})$ for each h belonging to the set H of our Definition 3. By taking the supremum of H, we avoid this nonuniqueness.

It is clear that if $\operatorname{plim}_{T\to\infty} \{T^k X_T\}$ exists and is not zero then $\{X_T\}$ is of order T^{-k} in probability. Situations can arise of course where the supremum k of the set H of Definition 3 does not belong to the set H. In such cases, $\{T^k X_T\}$ will not be asymptotically bounded in probability but $\{T^{k-\varepsilon} X_T\}$ will be for any $\varepsilon > 0$ and sufficiently small and the term "$\{X_T\}$ is at most of order T^{-k} in probability" may be preferable.

DEFINITION 4: Suppose $\{X_T\}$ is a sequence of estimators of θ. Suppose further that $EX_T - \theta = \Sigma \alpha_m T^{-m}$. Then the bias of order T^{-k} of the estimator sequence $\{X_T\}$ is obtained by retaining terms up to and including T^{-k} in the sum $\Sigma \alpha_m T^{-m}$.

It is clear that if the expansion is valid, then the coefficients α_m will be unique. The reason is that we can view $EX_T - \theta$ as a function of $1/T$ and the expansion as a power series expansion in terms of powers of $1/T$. Then by a well known theorem it follows that if the expansion is valid for a neighbourhood of a value of $1/T$, then it is unique for that neighbourhood.[5]

3. THE VALIDITY OF THE NAGAR PROCEDURE

The Nagar approach amounts to the following. Suppose we have a consistent estimator sequence $\{X_T\}$ for a parameter θ. Let $Z_T \equiv X_T - \theta$. Suppose further, that we can express Z_T formally as an infinite sum (say) $\Sigma_{k=0}^{\infty} \alpha_k U_T^k$ where $\{U_T\}$ is of order $1/T$ in probability. Then it is claimed that the "bias" to the order T^{-m} of the sequence $\{X_T\}$ is $E\,\Sigma_{k=0}^{m} \alpha_k U_T^k$.

The following examples establish that this procedure can yield values for the "bias" to the order T^{-m}, which are different from the true "bias" to the same order when the latter is well defined or even in situations when the true "bias" does not exist. It is also shown that examples can be found where the true bias is finite while this approach will suggest that it is infinite. These examples, it must be emphasized, are chosen not for their realism or their arising out of an actual estimation problem, but for the ease with which they demonstrate the invalidity of this procedure in general, though not necessarily for the cases considered by Nagar. This is not a restrictive feature since a single counterexample, however far fetched, is enough to show the invalidity of an alleged theorem.

Before we proceed to the examples a few comments on this approach are in order. First, it is possible that the sum $\Sigma_{k=0}^{\infty} \alpha_k U_T^k$ may not be convergent for some set of permissible values of U_T. Presumably, the proponents of this approach will consider only cases in which the probability measure of the set of values of U_T at which the series does not converge tends to zero as $T \to \infty$. Second, even if

[5] See Knopf [**5**, p. 104, Corollary 2]. The statement of result is "If it is at all possible to expand a function $f(Z)$ in a power series for a neighbourhood of a point z_0, then this can be done in only one way."

$\Sigma_{k=0}^{\infty} \alpha_k U_T^k$ is convergent with probability one, it does not follow that $\Sigma_{k=0}^{\infty} \alpha_k E U_T^k$ is convergent. Thirdly, even if $\Sigma_0^{\infty} \alpha_k U_T^k$ is convergent and $\Sigma_0^{\infty} \alpha_k E U_T^k$ is convergent, it is not necessarily true that $E(\Sigma_0^{\infty} \alpha_k U_T^k) = \Sigma_0^{\infty} \alpha_k E U_T^k$. Fourthly, and this is the basis of some of the difficulties of this procedure, even if both $\Sigma_{k=0}^{\infty} \alpha_k U_T^k$ and $\Sigma_0^{\infty} \alpha_k E U_T^k$ are convergent and $E(\Sigma_0^{\infty} \alpha_k U_T^k) = \Sigma_{k=0}^{\infty} \alpha_k E U_T^k$, the fact that U_T is of order T^{-1} in probability does not imply that EU_T^k is of order T^{-k}.

Consider now the following example:

EXAMPLE 1: Let $Z_T = X_T - \theta \equiv U_T/(1 - U_T)$ where, for all $T > 1$,

$$\text{probability}\left\{U_T = \frac{\beta}{T}\right\} = 1 - \frac{1}{T};$$

$$\text{probability}\left\{U_T = T\right\} = \frac{1}{T} \quad \text{where } 0 < \beta < 1.$$

Hence,

$$\text{probability}\left\{Z_T = \frac{\beta}{T - \beta}\right\} = 1 - \frac{1}{T},$$

and

$$\text{probability}\left\{Z_T = \frac{T}{1 - T}\right\} = \frac{1}{T}.$$

Then

$$\begin{aligned} EZ_T &= \frac{\beta}{T - \beta}\left(1 - \frac{1}{T}\right) + \frac{T}{1 - T}\cdot\frac{1}{T} \\ &= \frac{\beta}{T - \beta} + \frac{1}{1 - T} - \frac{\beta}{T(T - \beta)} = \frac{\beta - 1}{T} + \frac{\beta^2 - \beta - 1}{T^2} + \dots. \end{aligned}$$

Thus the true bias to the order $1/T$ of $\{X_T\}$ is $(\beta - 1)/T$. Note that $\{X_T\}$ is consistent and asymptotically unbiased.

Let us now follow the Nagar procedure. Express Z_T formally as $\Sigma_{k=1}^{\infty} U_T^k$. Now $\{U_T\}$ is of order $1/T$ in probability since $\text{plim}_{T\to\infty} TU_T = \beta$. Thus U_T^2, U_T^3, etc., are of order $1/T^2$, $1/T^3$, etc., in probability. Hence the Nagar bias to the order $1/T$ is obtained by omitting U_T^2, U_T^3, etc., from the sum and taking the expectation of U_T.

Thus Nagar bias equals

$$\begin{aligned} EU_T &= \frac{\beta}{T}\left(1 - \frac{1}{T}\right) + T\cdot\frac{1}{T} = 1 + \frac{\beta}{T} - \frac{\beta}{T^2} \\ &= 1 + \frac{\beta}{T} \qquad \text{to the order } \frac{1}{T}. \end{aligned}$$

It can be easily verified that the true bias and the Nagar bias to the same order

$1/T$ differ. In particular the true bias to order $1/T$ vanishes if $\beta = 1$ while the Nagar bias does not.

EXAMPLE 2: Let Z_T be defined as in Example 1. Let probability $\{U_T = -\beta/T\} = 1 - (1/T)$ and let U_T have the density function

$$f_T(u) = \frac{1}{T}\frac{\beta}{T}u^{-2} \quad \text{if} \quad u < -\frac{\beta}{T},$$
$$= 0 \quad \text{if} \quad u > -\frac{\beta}{T} \quad \text{where} \quad \beta > 0.$$

Clearly $\{U_T\}$ is of order $1/T$ in probability since $\text{plim}_{T\to\infty}\{TU_T\} = -\beta$. But EU_T does not exist and hence Nagar bias is not even defined. However it is easy to check that probability $\{Z_T = -\beta/(T+\beta)\} = 1 - (1/T)$ and Z_T has the density function

$$g_T(z) = \frac{1}{T}\frac{\beta}{T}z^{-2} \quad \text{for} \quad -1 \leqslant z < -\frac{\beta}{T+\beta}$$
$$= 0 \quad \text{for} \quad z < -1 \quad \text{or} \quad z > -\frac{\beta}{T+\beta}$$

Evaluating EZ_T we get:

$$EZ_T = -\frac{\beta}{T+\beta}\left(1 - \frac{1}{T}\right) + \frac{\beta}{T^2}\int_{-1}^{-\beta/(T+\beta)} z \cdot z^{-2}\,dz$$
$$= -\frac{\beta}{T+\beta}\left(1 - \frac{1}{T}\right) + \frac{\beta}{T^2}\log\left(\frac{\beta}{T+\beta}\right)$$
$$= -\frac{\beta}{T} \quad \text{to the order} \quad \frac{1}{T}.$$

Thus the true bias to the order $1/T$ is well defined while Nagar bias is not.

EXAMPLE 3: This is in a sense the obverse of Example 2. Here the true bias does not exist while the Nagar bias does. Let $\{Z_T\}$ be defined as in Example 1.

Let probability $\{U_T = \beta/T\} = 1 - (1/T)$ and let U_T have the density function

$$f_T(u) = \frac{1}{T}\frac{\beta}{T-\beta}u^{-2} \quad \text{for} \quad 1 \geqslant u > \frac{\beta}{T},$$
$$= 0 \quad \text{for} \quad u \geqslant 1 \quad \text{or} \quad \frac{\beta}{T} > u.$$

One can verify that probability $\{Z_T = \beta/(T-\beta)\} = 1 - (1/T)$ and Z_T has the density function

$$g_T(z) = \frac{1}{T}\frac{\beta}{T-\beta}z^{-2} \quad \text{for} \quad z > \frac{\beta}{T-\beta},$$
$$= 0 \quad \text{for} \quad z < \frac{\beta}{T-\beta}.$$

Now EZ_T does not exist and hence the true bias of any order T^{-k} is not defined. However $\{U_T\}$ is of order $1/T$ in probability and following Nagar we obtain the Nagar bias as

$$\begin{aligned} EU_T &= \left(1 - \frac{1}{T}\right)\frac{\beta}{T} + \frac{1}{T}\frac{\beta}{T-\beta}\int_{\beta/T}^{1} u \cdot u^{-2}\,du \\ &= \left(1 - \frac{1}{T}\right)\frac{\beta}{T} - \frac{1}{T}\frac{\beta}{T-\beta}\log\frac{\beta}{T} \\ &= \frac{\beta}{T} \quad \text{to the order} \quad \frac{1}{T}. \end{aligned}$$

Thus Nagar bias to order $1/T$ is finite while the true bias is not.

In the above three examples, the purely formal expansion of Z_T as an infinite series in powers of U_T had the property that the set of values of U_T for which the infinite sum did not converge had a probability measure approaching zero as $T \to \infty$. In the following example, the set of values for which the infinite series does not converge has a probability measure zero for *each value* of T, and not merely asymptotically as $T \to \infty$. Still the Nagar procedure does not go through.

EXAMPLE 4: Let Z_T be defined as in Example 1. Let U_T have the density function $g_T(u)$ given by

$$g_T(u) = \frac{1}{T} + \left(1 - \frac{1}{T}\right)\frac{e^{-\frac{1}{2}(u/\beta_T)^2}}{\sqrt{2\pi}\beta_T\left[F\left(\frac{1}{2\beta_T}\right) - F\left(\frac{-1}{2\beta_T}\right)\right]} \quad \text{for} \quad -\tfrac{1}{2} \leqslant u \leqslant \tfrac{1}{2},$$

and $g_T(u) = 0$ otherwise, where β_T is positive and $F(x)$ is the cumulative probability function of a standard normal variable. Since the range of the random variable U_T is $[-\frac{1}{2}, \frac{1}{2}]$ for all T, the infinite series expansion of Z_T as $\Sigma_{k=1}^{\infty} U_T^k$ is valid for all relevant values of U_T for each T. Choose $\beta_T = 1/T \log(T+1)$. It is easy to verify that $\operatorname{plim}_{T\to\infty}\{T^{1-\alpha}U_T\} = 0$, while $\operatorname{plim}_{T\to\infty}\{T^{1+\alpha}U_T\}$ does not exist for any $\alpha > 0$. Hence $\{U_T\}$ is of order $1/T$ in probability. It is also easily checked that all odd moments of U_T are zero while *all* even moments of $\{U_T\}$ are of order $1/T$. Hence by taking the expectation of U_T, or even the expectation of any finite number of terms of the expansion $\Sigma_{k=1}^{\infty} U_T^k$, one cannot obtain the true bias to order $1/T$ of the estimating sequence $\{X_T\}$.

4. SOME USEFUL DISTINCTIONS

It is quite common in the econometric literature to use the terms "consistent estimate" and asymptotically unbiased estimate interchangeably. These are distinct concepts, however, if we define an asymptotically unbiased estimate by Definition 1 of Section 2. It can be easily shown that a consistent estimate need not be asymptotically unbiased and vice versa.[6] For instance, consider the

[6] Goldberger wrongly asserts [3, p. 128] that "a consistent estimator is asymptotically unbiased but the converse is not true."

sequence of random variables $\{X_T\}$ where

$$X_T = \begin{cases} \theta + \alpha_1 T & \text{with probability } \frac{1}{3}, \\ \theta - \alpha_2 T & \text{with probability } \frac{2}{3}, \end{cases}$$

where α_1, α_2 are both different from zero. Clearly $\{X_T\}$ is not a consistent sequence for estimating θ. However, $EX_T = \theta + T(\alpha_1 - 2\alpha_2)/3$. If $\alpha_1 = 2\alpha_2$ then $\{X_T\}$ is unbiased for all T. Consider the following example.

$$X_T = \begin{cases} \theta + \dfrac{\alpha_1}{T} & \text{with probability } 1 - \dfrac{1}{T}, \\ \theta - \alpha_2 T & \text{with probability } \dfrac{1}{T}. \end{cases}$$

Here $\{X_T\}$ is clearly consistent. But $EX_T = \theta + (\alpha_1/T)(1 - 1/T) - \alpha_2$. As long as $\alpha_2 \neq 0$, $\{X_T\}$ is neither unbiased nor asymptotically unbiased.

The example that follows shows that a sequence of random variables can have a limiting distribution with well defined moments while each member of the sequence has no moments.

Let X_T have the following density function:

$$f_T(x) = \alpha^T(\delta - 1)x^{-\delta} + (1 - \alpha^T)\frac{1}{\sqrt{2\pi}\theta_T} e^{-\frac{1}{2}[(x-\beta_T)/\theta_T]^2} \quad \text{for} \quad x \geqslant 1,$$

$$= (1 - \alpha^T)\frac{1}{\sqrt{2\pi}\theta_T} e^{-\frac{1}{2}[(x-\beta_T)/\theta_T]^2} \quad \text{for} \quad x < 1,$$

where $0 < \alpha < 1$ and $1 < \delta < 2$. It is easy to verify that all the raw moments of X_T are infinite. However the sequence of random variables $\{(X_T - \beta_T)/\theta_T\}$ has a limiting distribution, namely, the normal distribution with mean zero and variance one. If we choose the $\beta_T = 1/T$ and $\theta_T = 1/\sqrt{T}$, we get $\{X_T\}$ as a consistent estimating sequence for β. However, note that the result that $\{(X_T - \beta_T)/\theta_T\}$ tends to a normal distribution with zero mean and unit variance does not depend upon the existence of the moments of $\{X_T\}$ or for that matter β_T or θ_T being bounded or having any limits.

The above example showed that the normalised sequence $\{(X_T - \beta_T)/\theta_T\}$ had no moments for each T while its asymptotic distribution had an expectation of zero and all other moments were finite. The following final example shows that while the normalised sequence $\{(X_T - \beta_T)/\theta_T\}$ has a finite expectation for each T, the limit as $T \to \infty$ of the sequence of expectations is *not* the same as the expectation of the asymptotic distribution. Let $Z_T \equiv (X_T - \beta_T)/\theta_T$. Let Z_T have the following density function:

$$f(Z) = \frac{1}{\sqrt{2\pi}}[\alpha^T e^{-\frac{1}{2}(Z-\alpha^{-T})^2} + (1 - \alpha^T) e^{-\frac{1}{2}Z^2}],$$

where $0 < \alpha < 1$. It is clear that $EZ_T = \alpha^T \cdot \alpha^{-T} + (1 - \alpha^T) \cdot 0 = 1$ for all T and hence $\lim_{T\to\infty} EZ_T = 1$. However, the limiting distribution is normal with mean zero and variance one.

5. BASMANN'S INTERPRETATION OF THE NAGAR PROCEDURE

Basmann has suggested that the distribution of the truncated sum used in the Nagar procedure provides an approximation to the true finite sample distribution of the estimator considered. Thus, according to this interpretation, in the examples of Section 3, the distribution of $U_T, U_T + U_T^2, U_T + U_T^2 + U_T^3$ provide closer and closer approximations to the distribution of Z_T or equivalently of X_T. Basmann's interpretation will also mean that these approximations can have finite moments, even if the distribution approximated, namely Z_T, has no moments. There are a few difficulties, however, with this interpretation.

First, a careful reading of the articles [**7**–**15**] suggests their authors certainly did not have this interpretation in mind and believed that they were approximating the moments of Z_T. Secondly, and more importantly, it is not at all clear why the distribution of Z_T is being approximated by $U_T, U_T + U_T^2, \ldots$, etc., and what criterion is used to judge the closeness of approximation. All that can be said is that $\operatorname{plim}_{T\to\infty} T(Z_T - U_T) = 0$ and $\operatorname{plim}_{T\to\infty} T^2(Z_T - U_T - U_T^2) = 0$, etc. This is not saying very much, since the distribution of any sequence of random variables which converges in probability to a constant can be used to generate an approximation to the distribution of any other sequence of random variables which also converges to a constant. In other words, suppose $\operatorname{plim}_{T\to\infty} \{X_T\} = \theta$ and $\operatorname{plim}_{T\to\infty} \{Y_T\} = \delta$. Then $\operatorname{plim}_{T\to\infty} [(X_T - \theta) - (Y_T - \delta)] = 0$. Thus the distribution of $Y_T - \delta$ provides an approximation to that of $X_T - \theta$ in the Basmann sense.

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