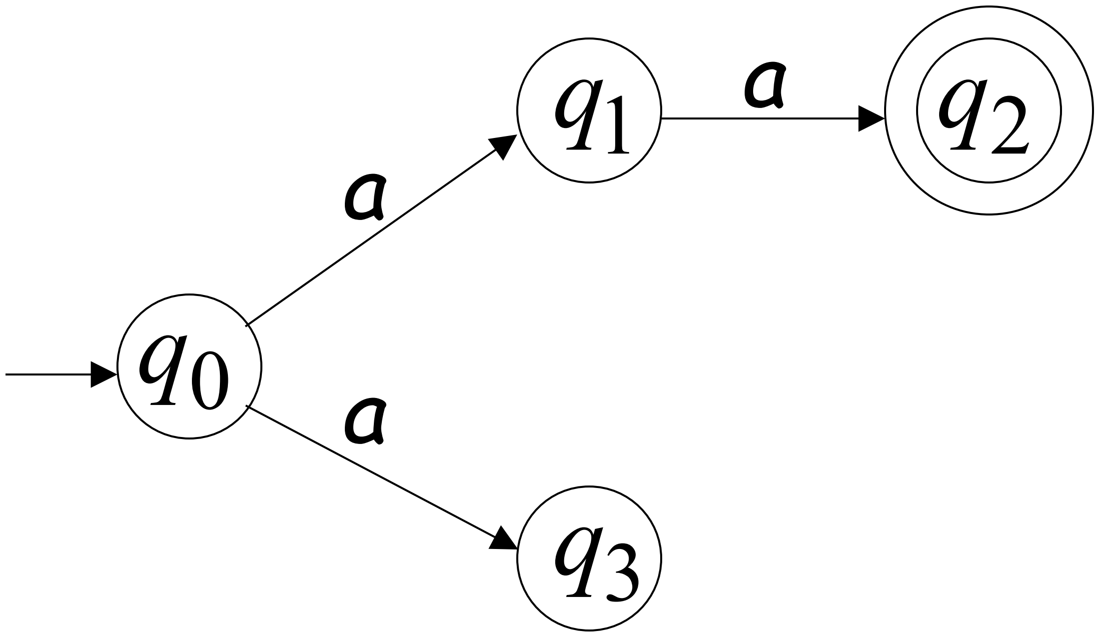
Non Deterministic Automata

Nondeterministic Finite Accepter (NFA)

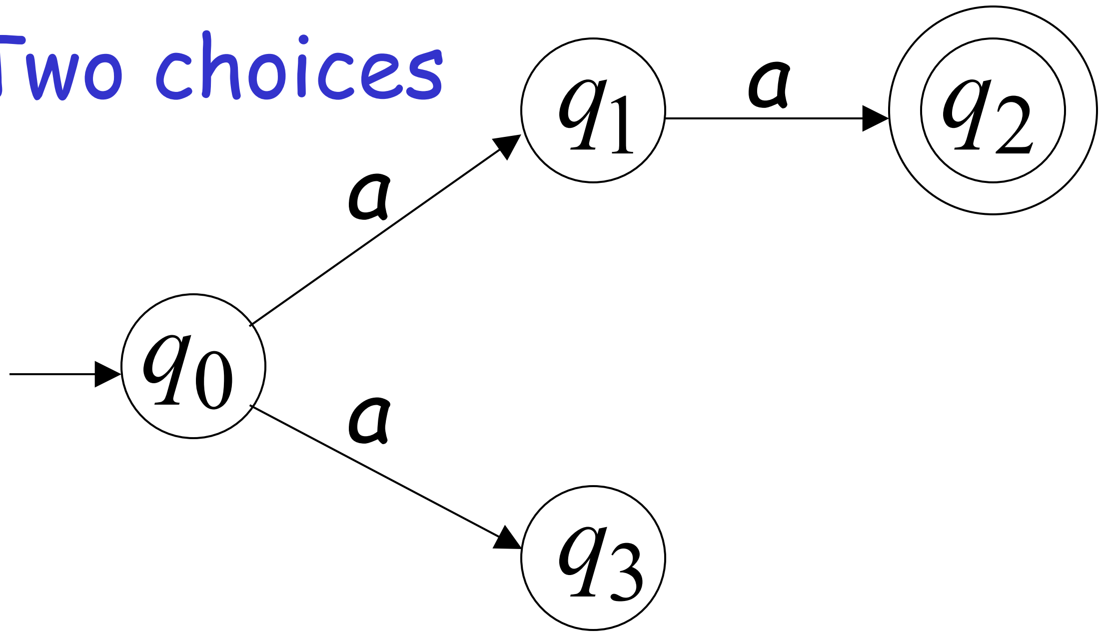
Alphabet = $\{a\}$



Nondeterministic Finite Accepter (NFA)

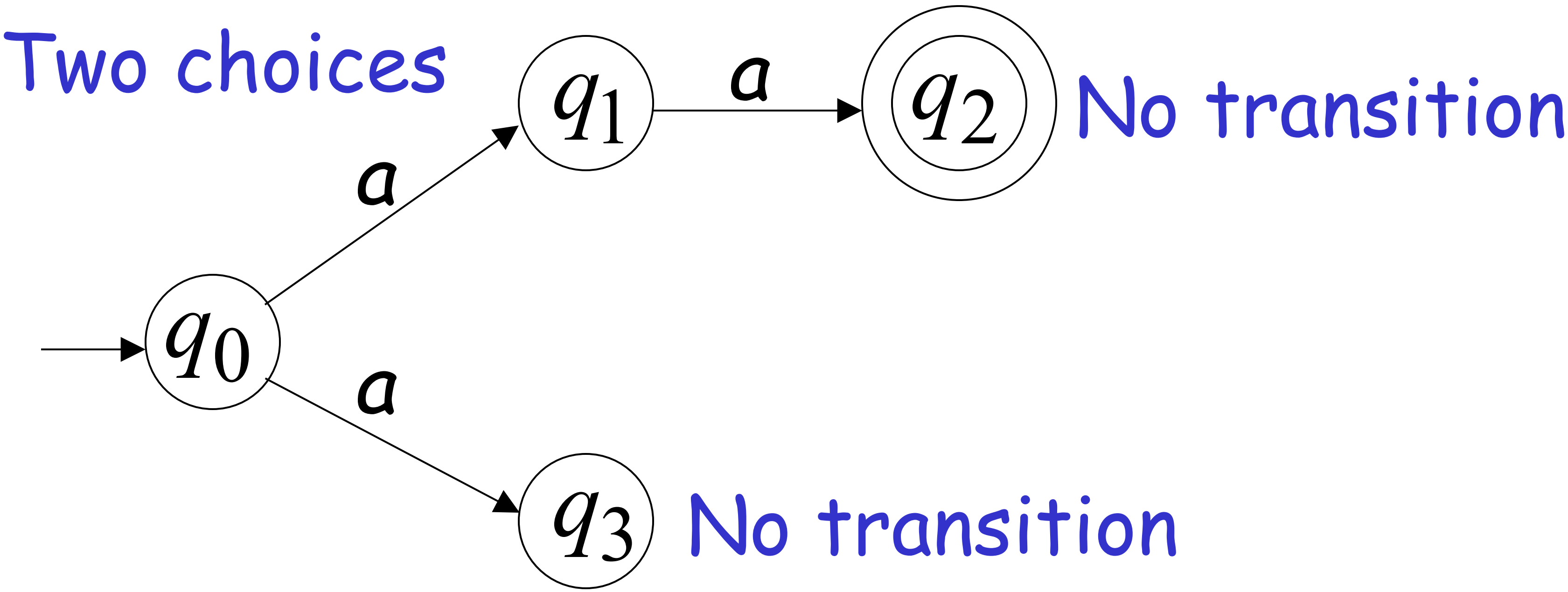
Alphabet = $\{a\}$

Two choices

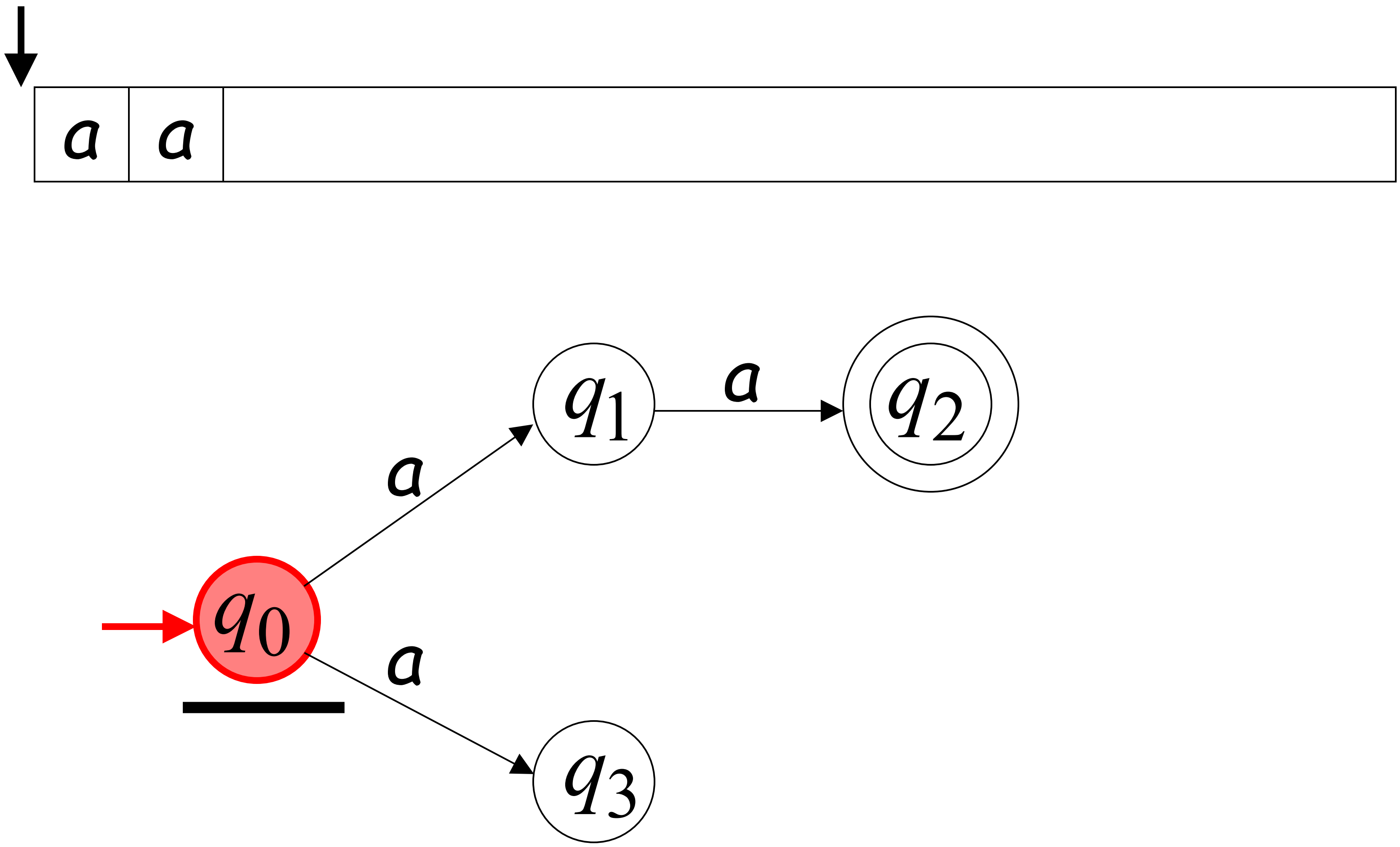


Nondeterministic Finite Accepter (NFA)

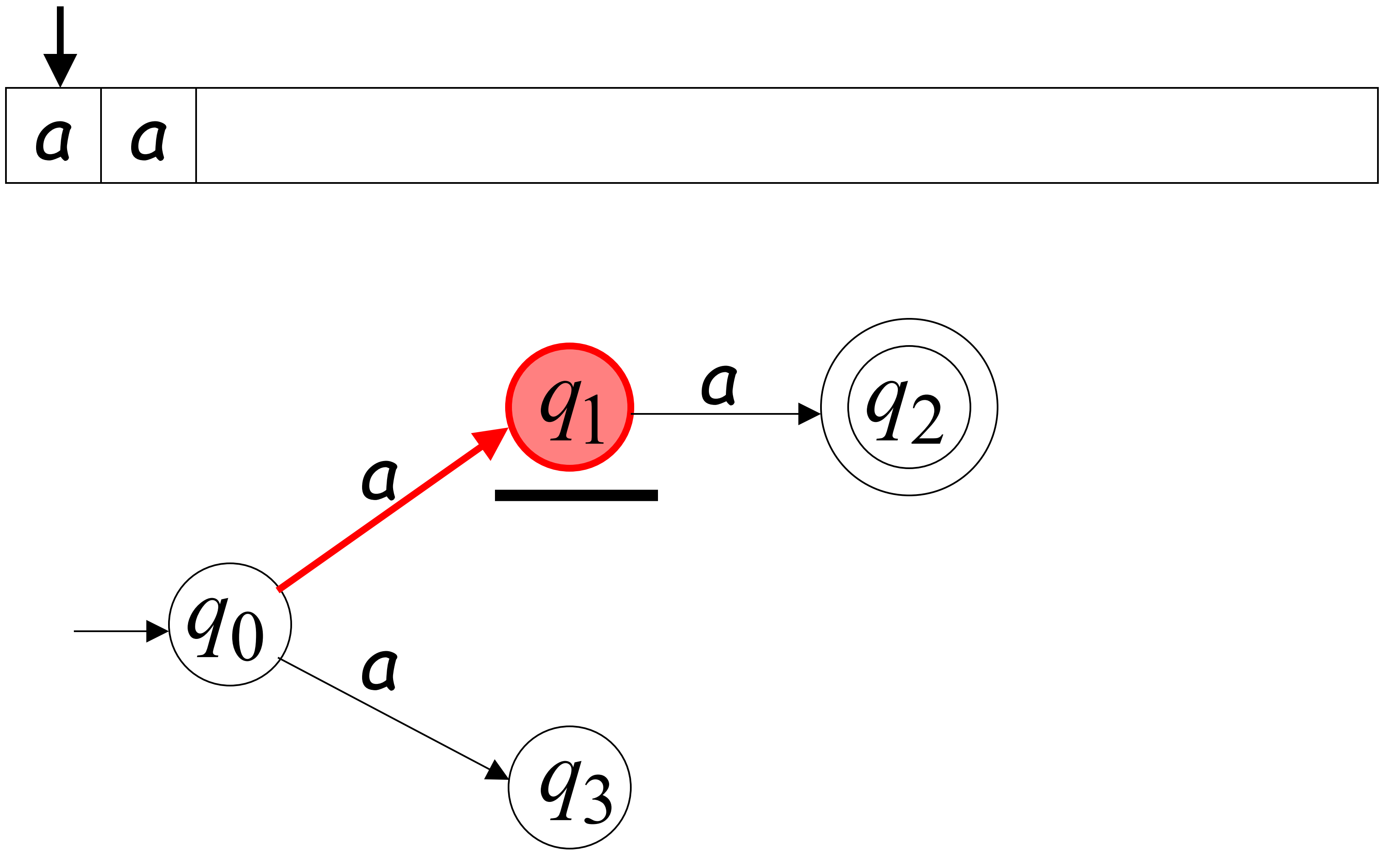
Alphabet = $\{a\}$



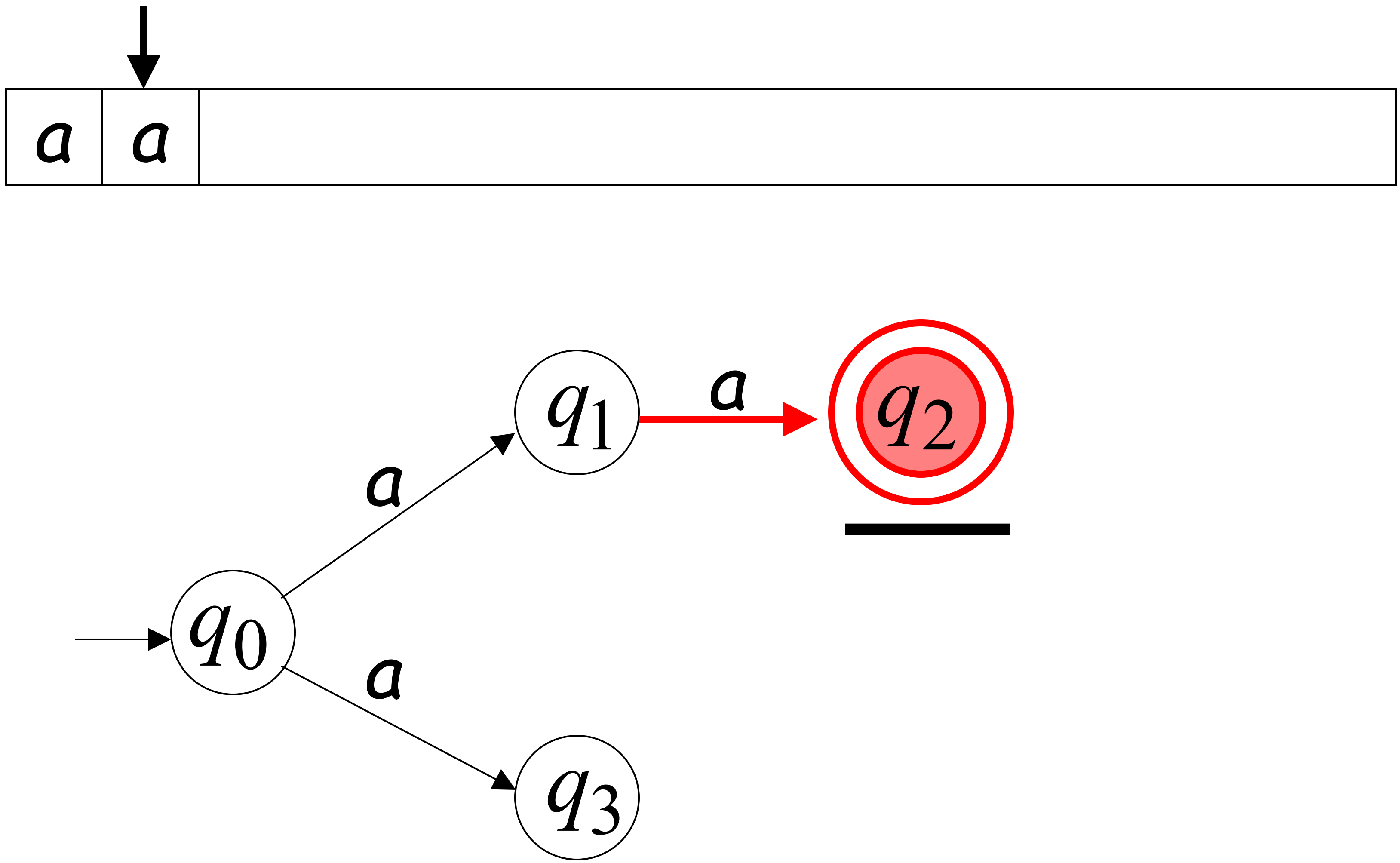
First Choice



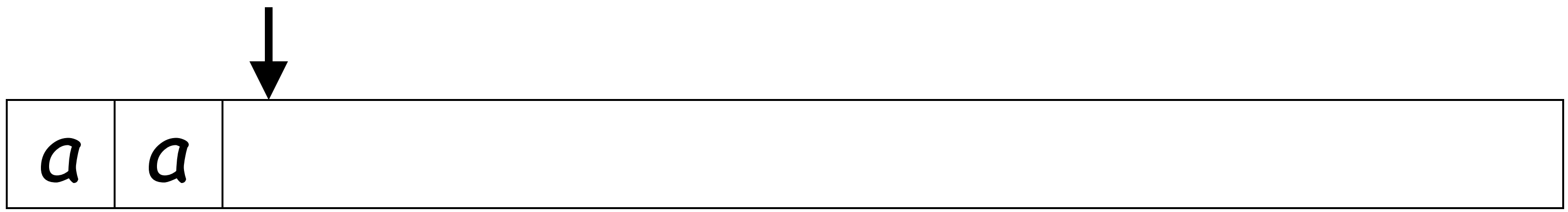
First Choice



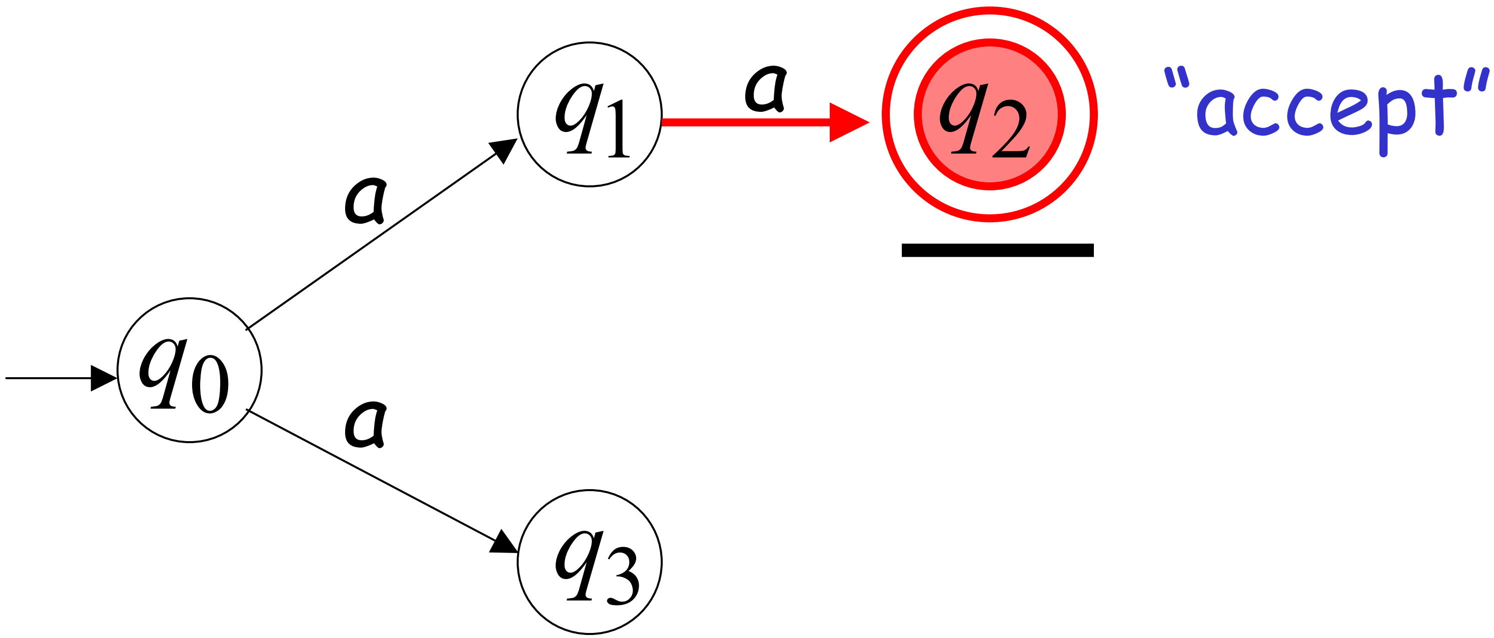
First Choice



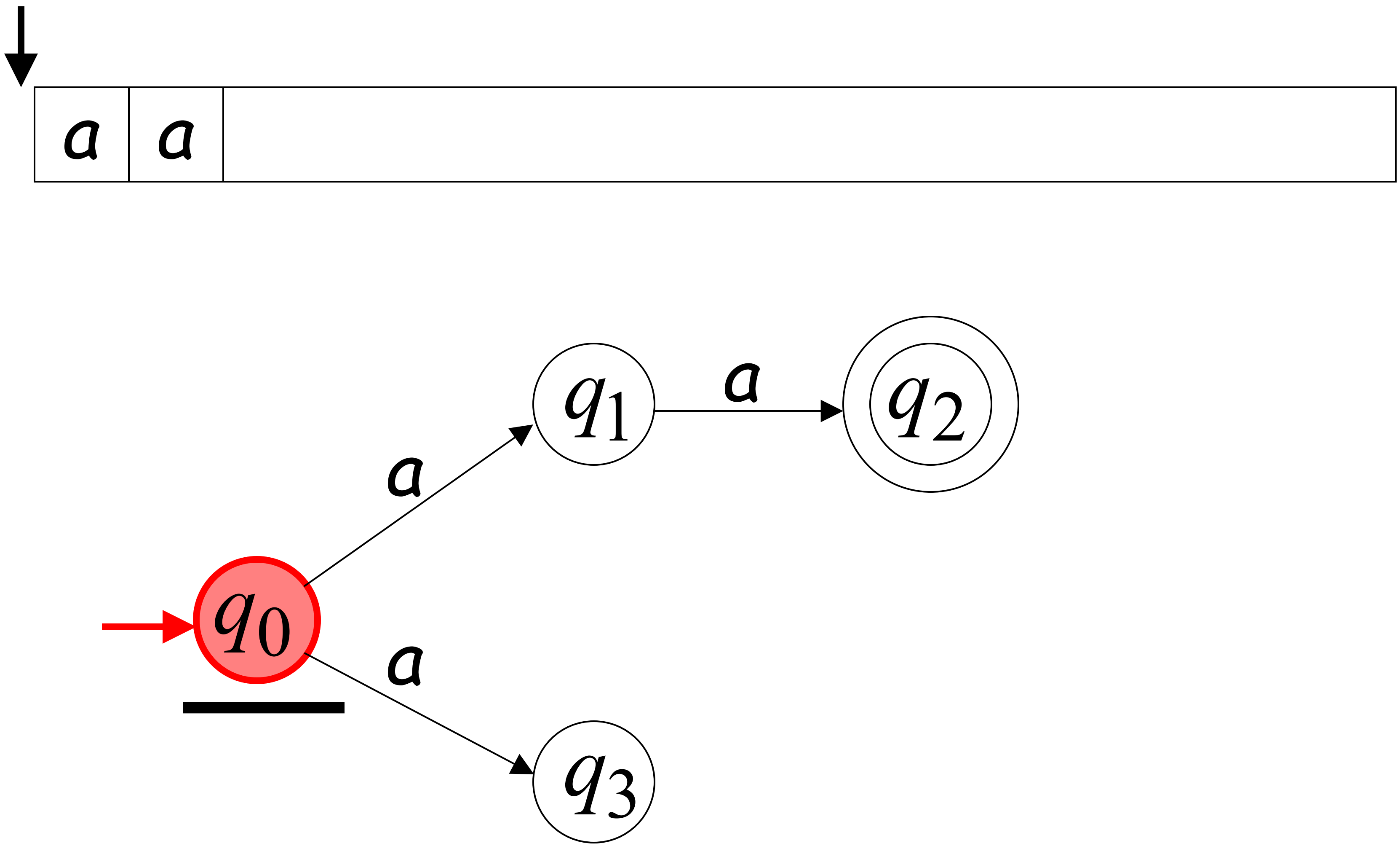
First Choice



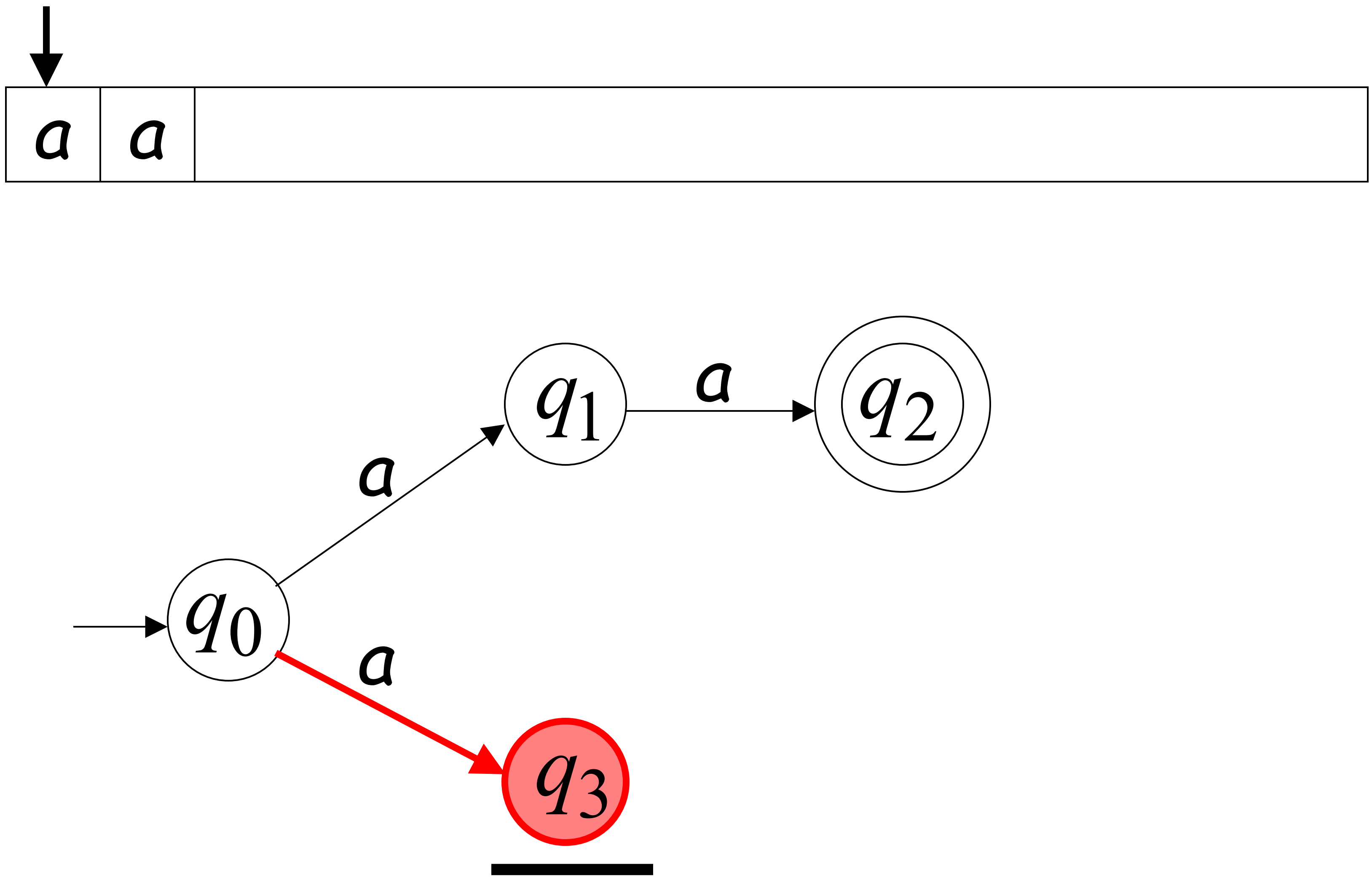
All input is consumed



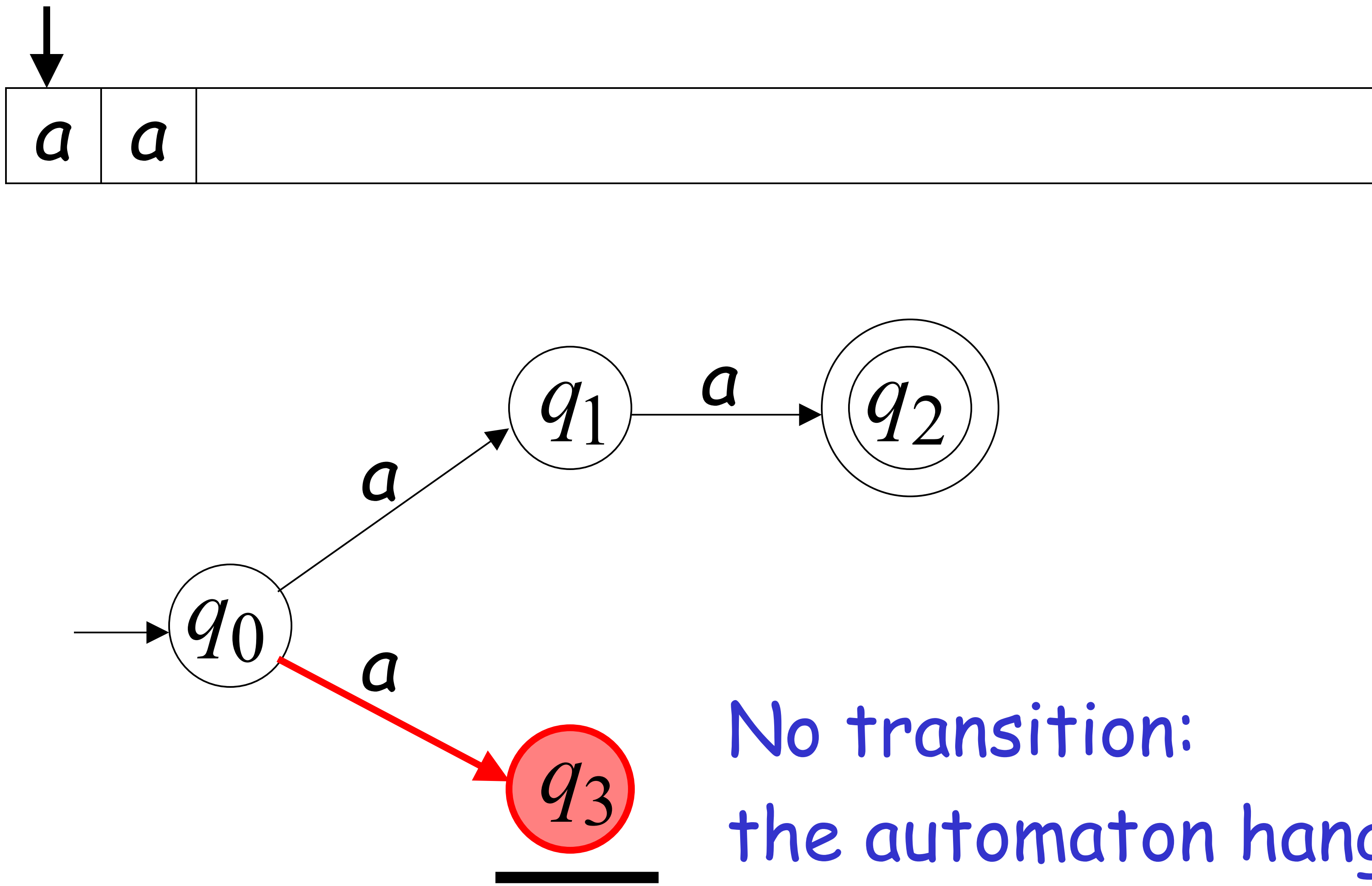
Second Choice



Second Choice

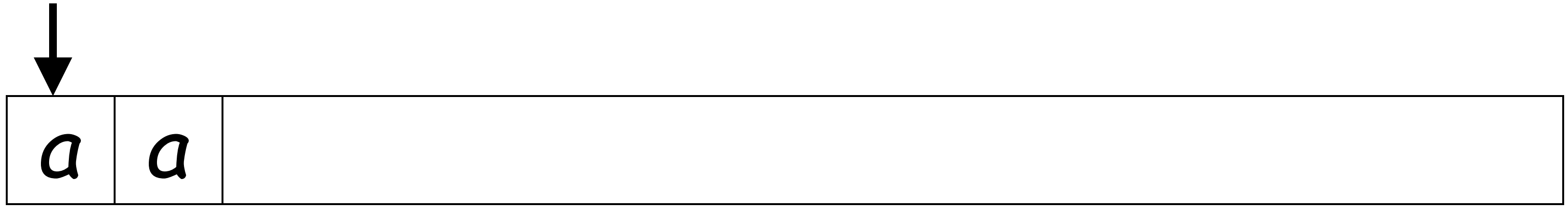


Second Choice

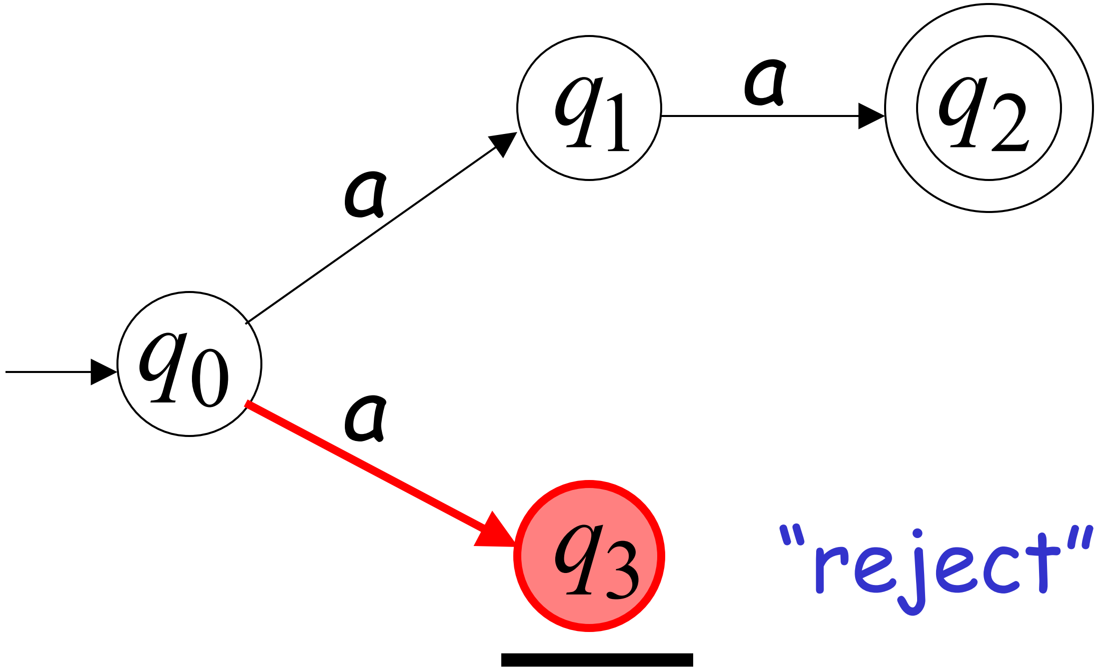


No transition:
the automaton hangs

Second Choice



Input cannot be consumed



"reject"

An NFA accepts a string:
when there is a computation of the NFA
that accepts the string

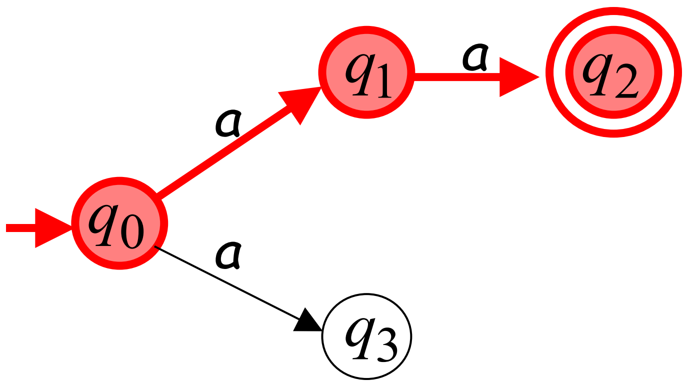
AND

all the input is consumed and the automaton
is in a final state

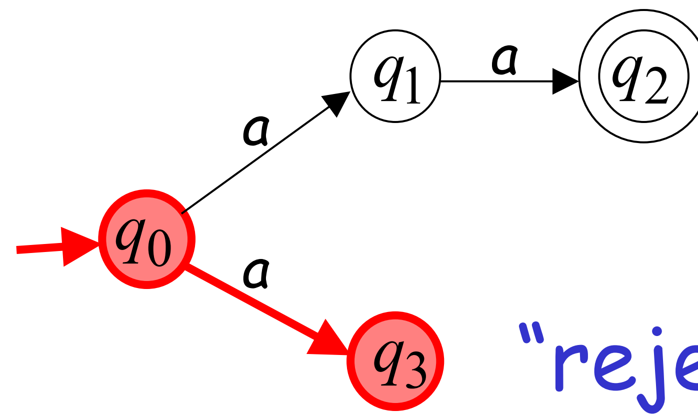
Example

aa is accepted by the NFA:

"accept"

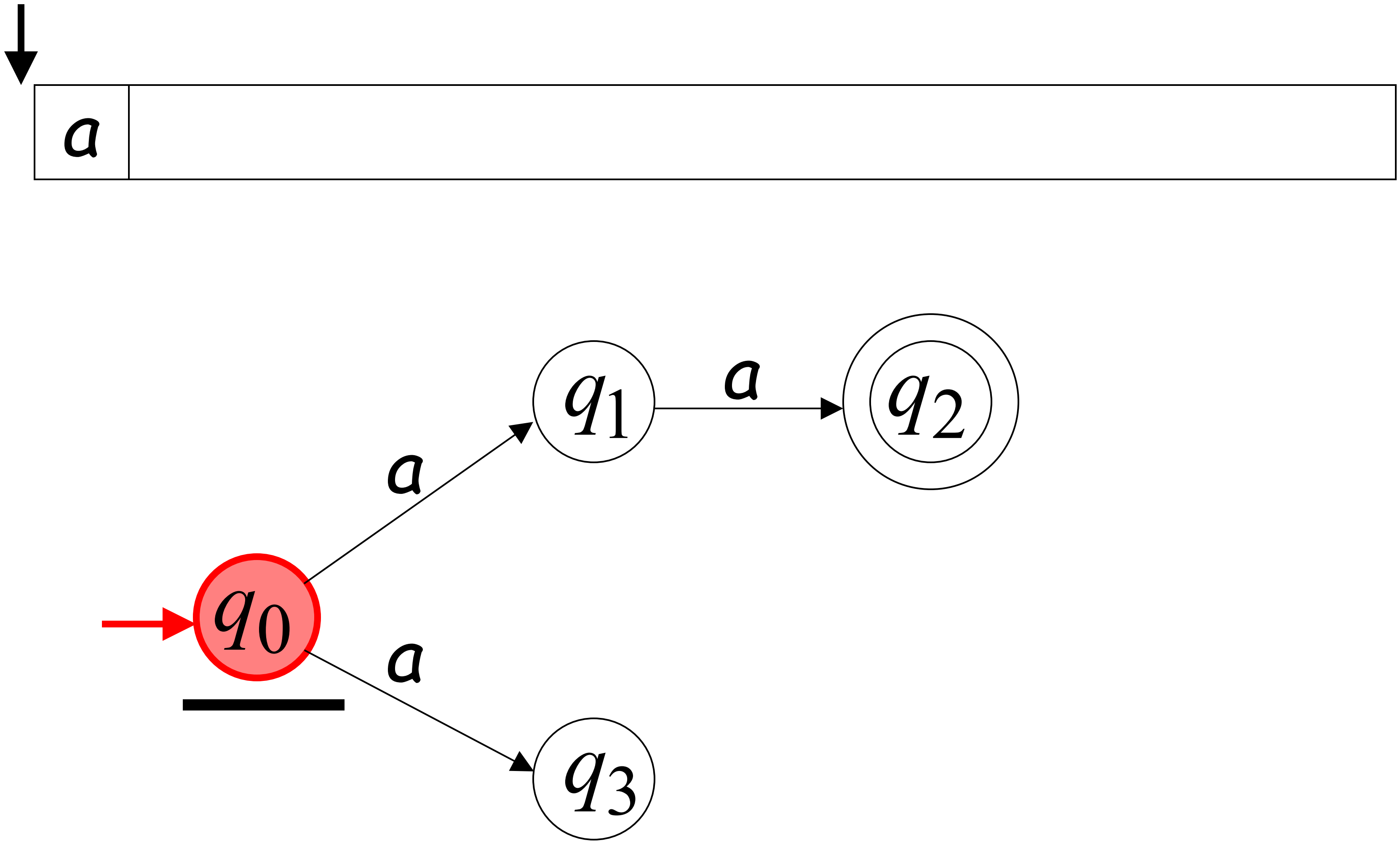


because this computation accepts aa

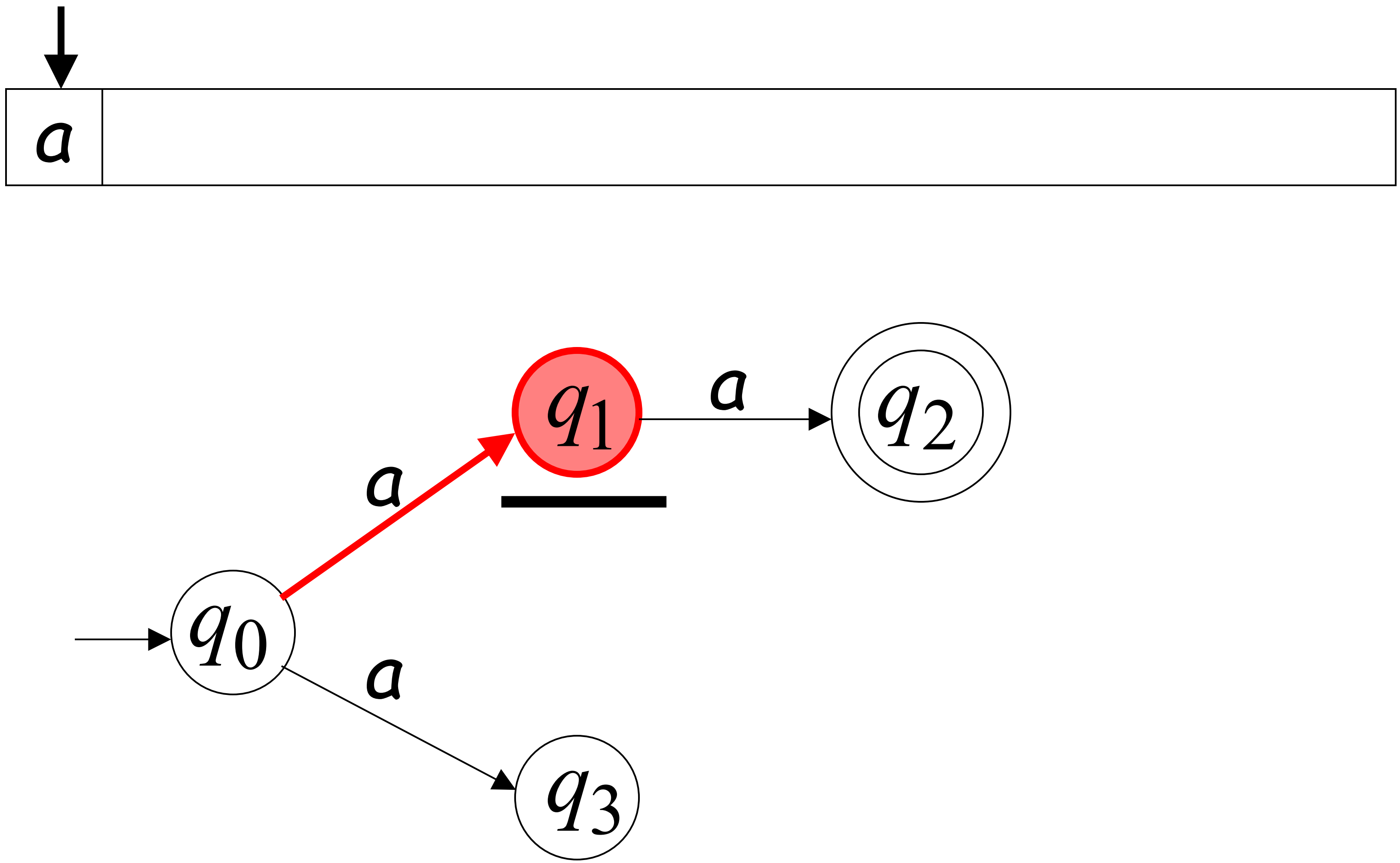


"reject"

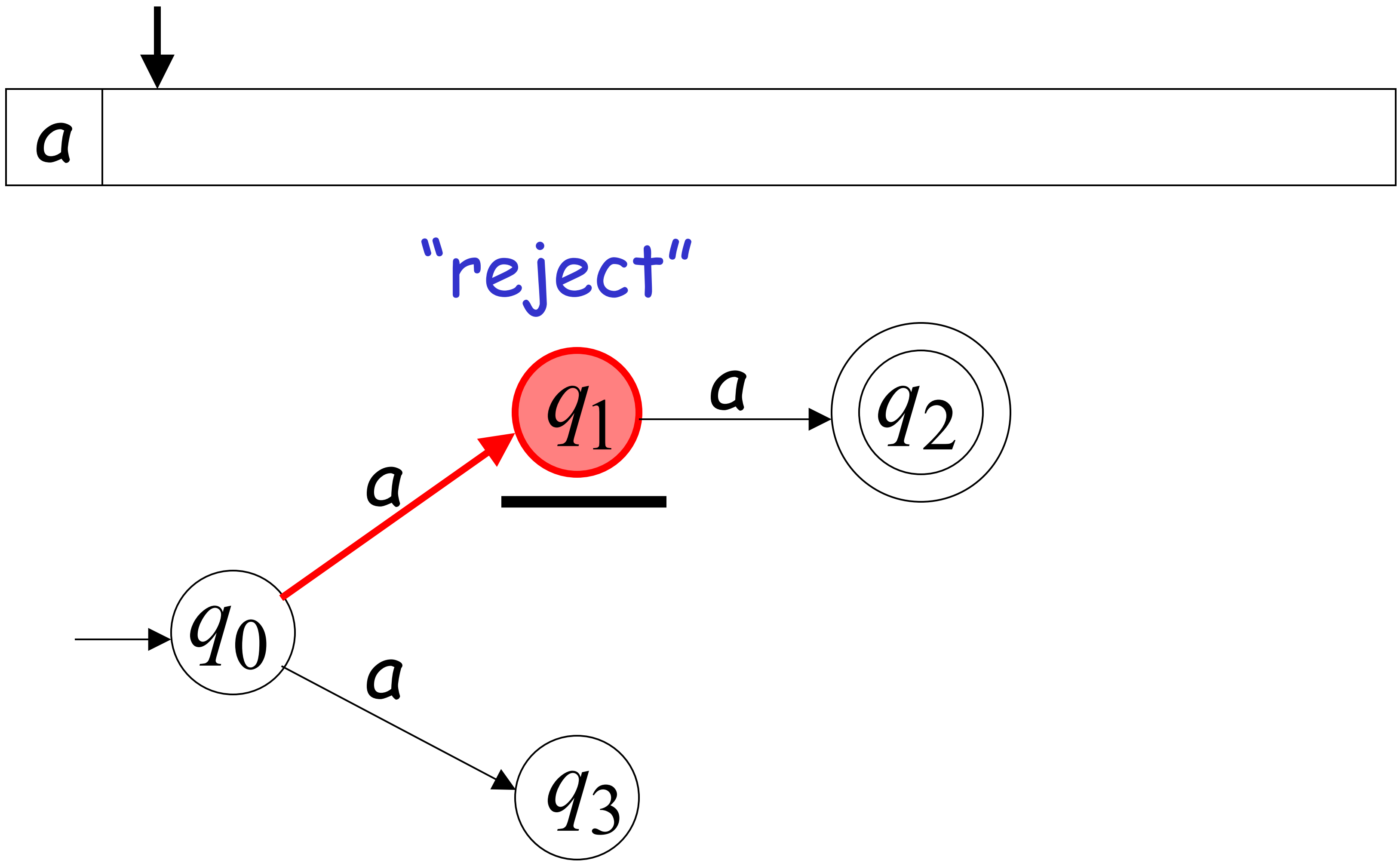
Rejection example



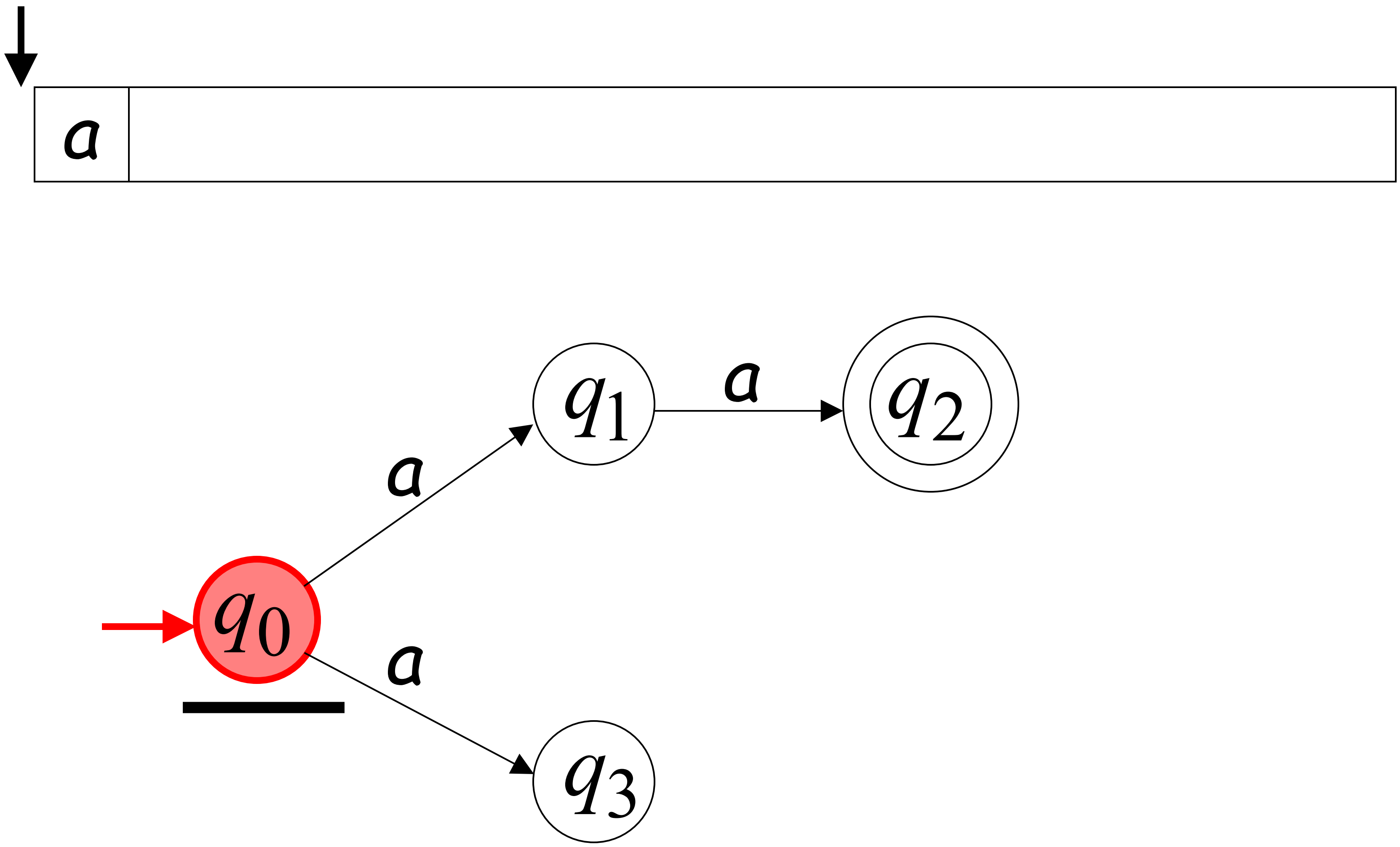
First Choice



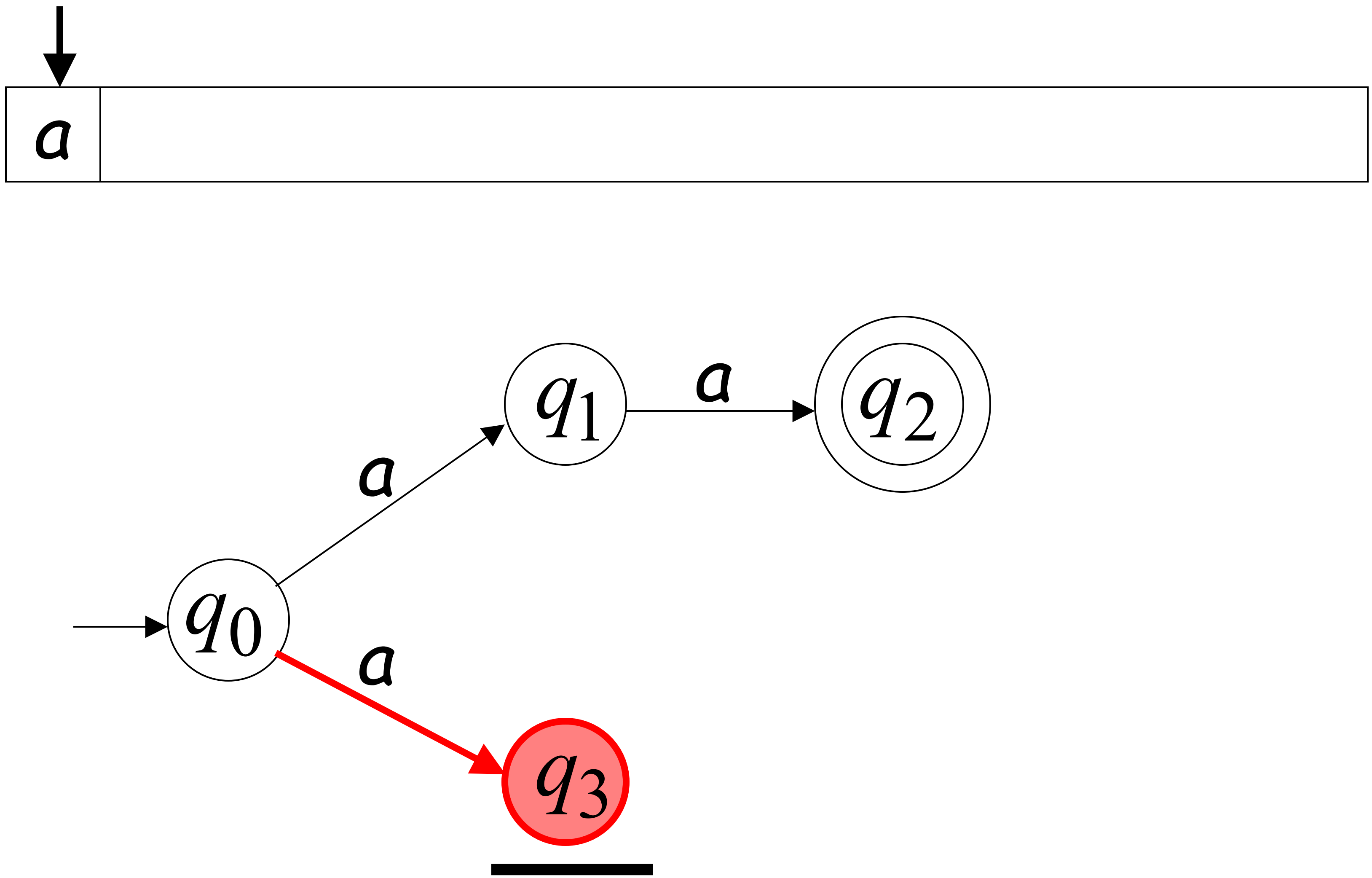
First Choice



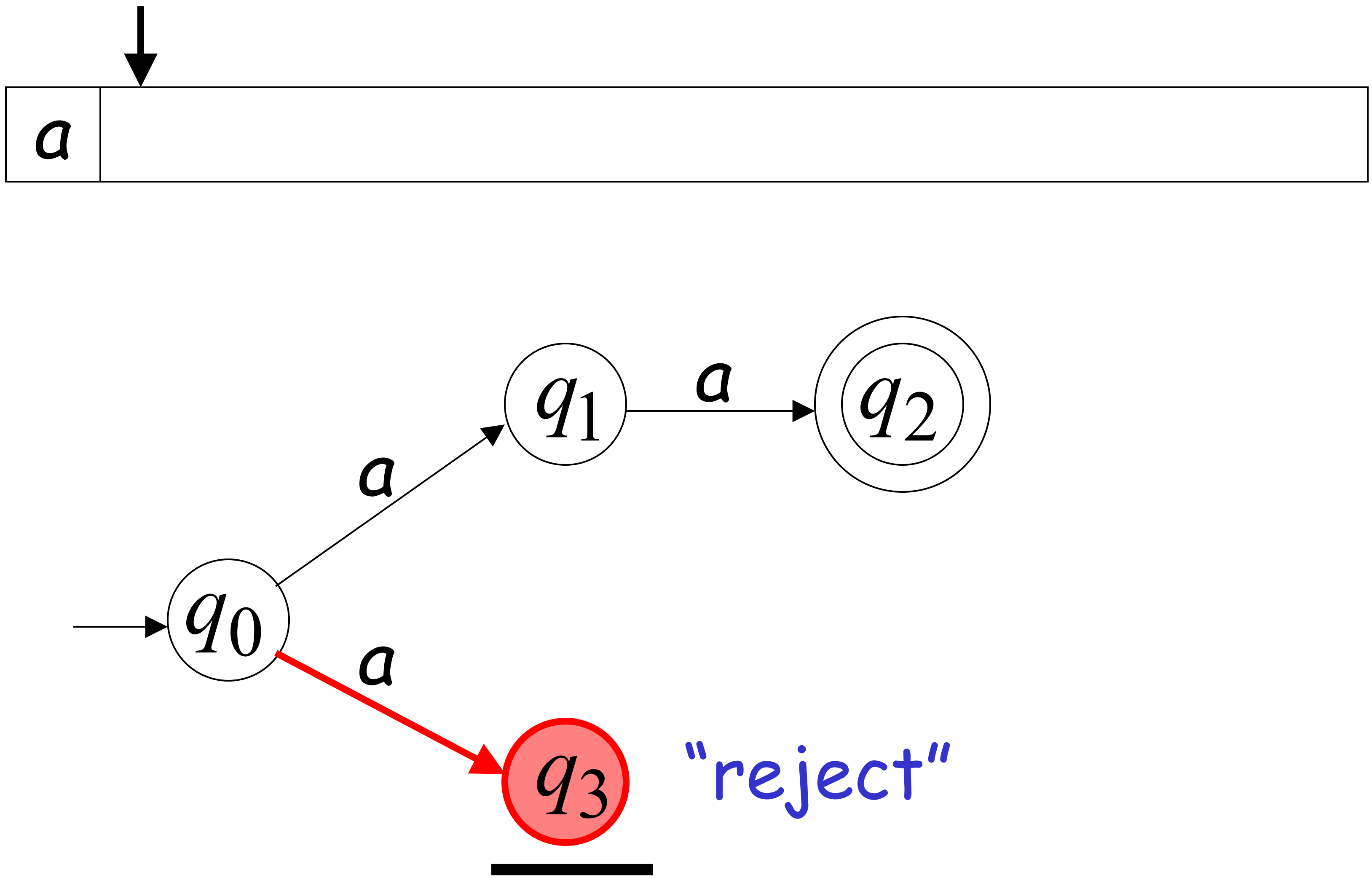
Second Choice



Second Choice



Second Choice



An NFA rejects a string:

when there is no computation of the NFA that accepts the string:

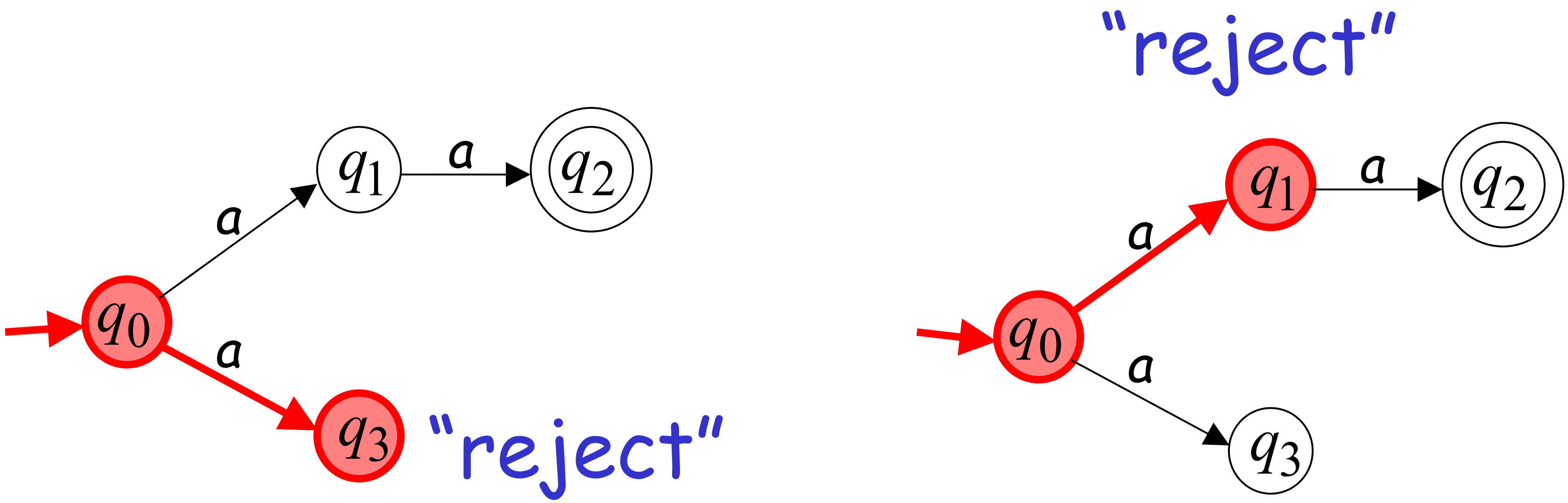
- All the input is consumed and the automaton is in a non final state

OR

- The input cannot be consumed

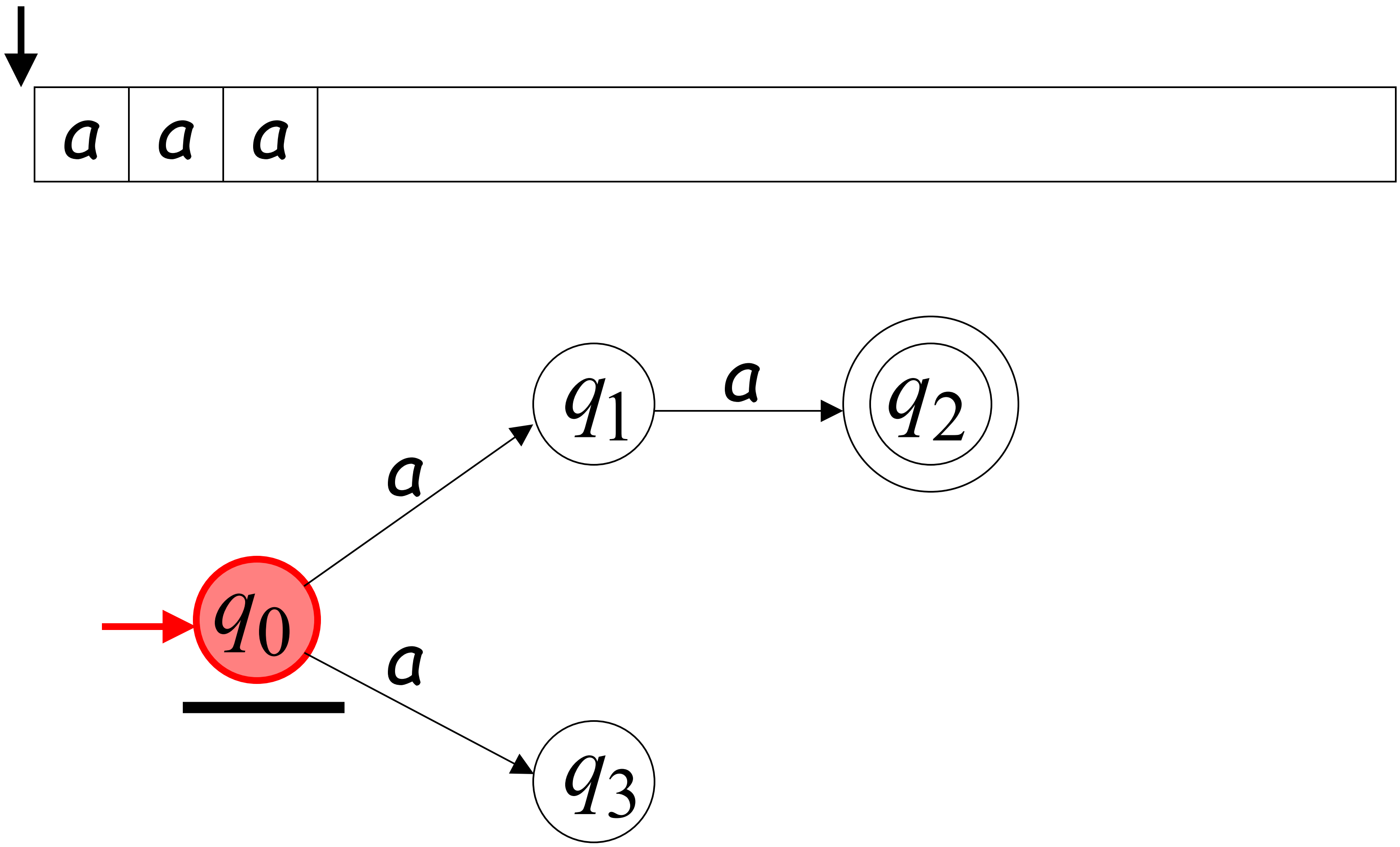
Example

a is rejected by the NFA:

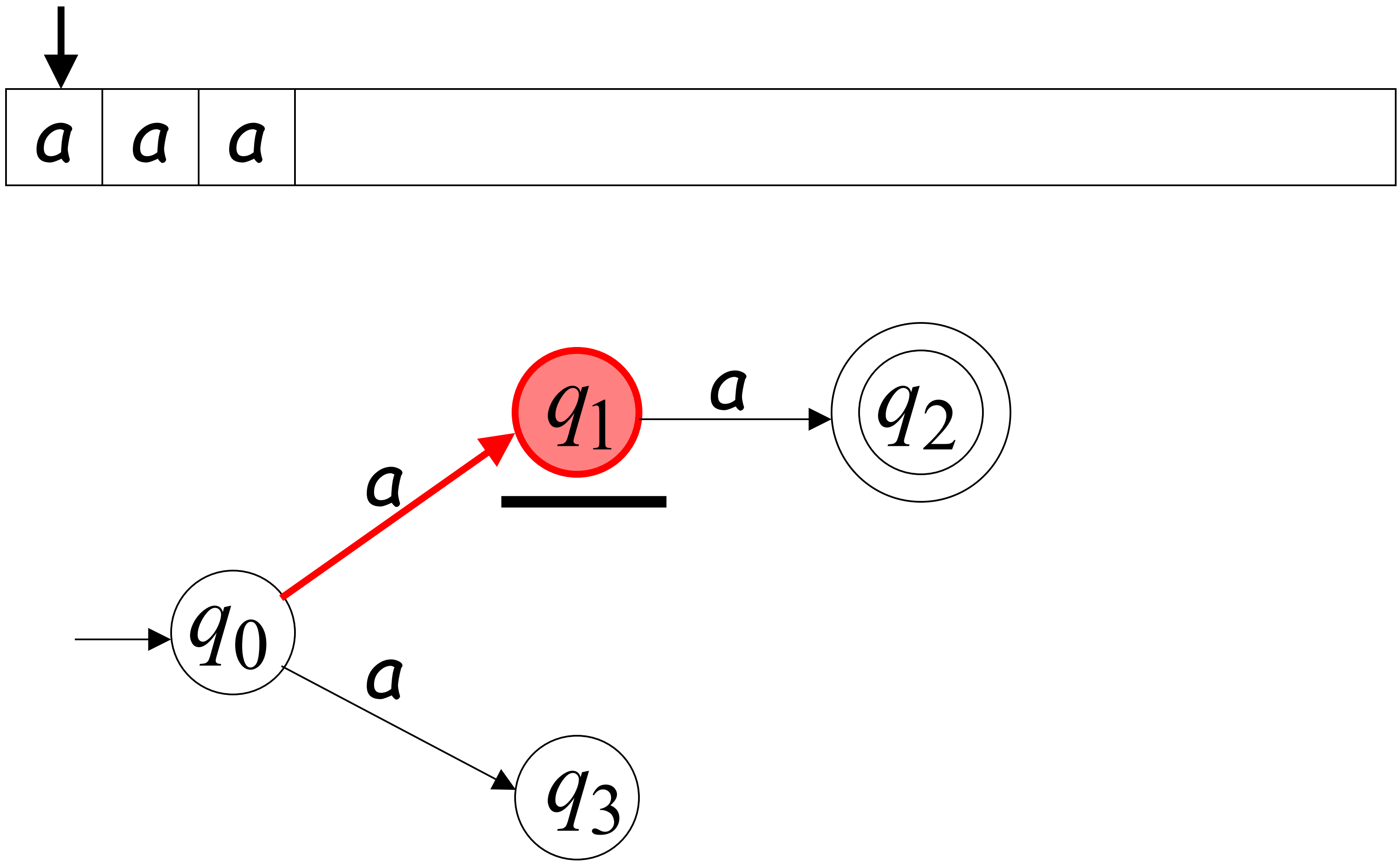


All possible computations lead to rejection

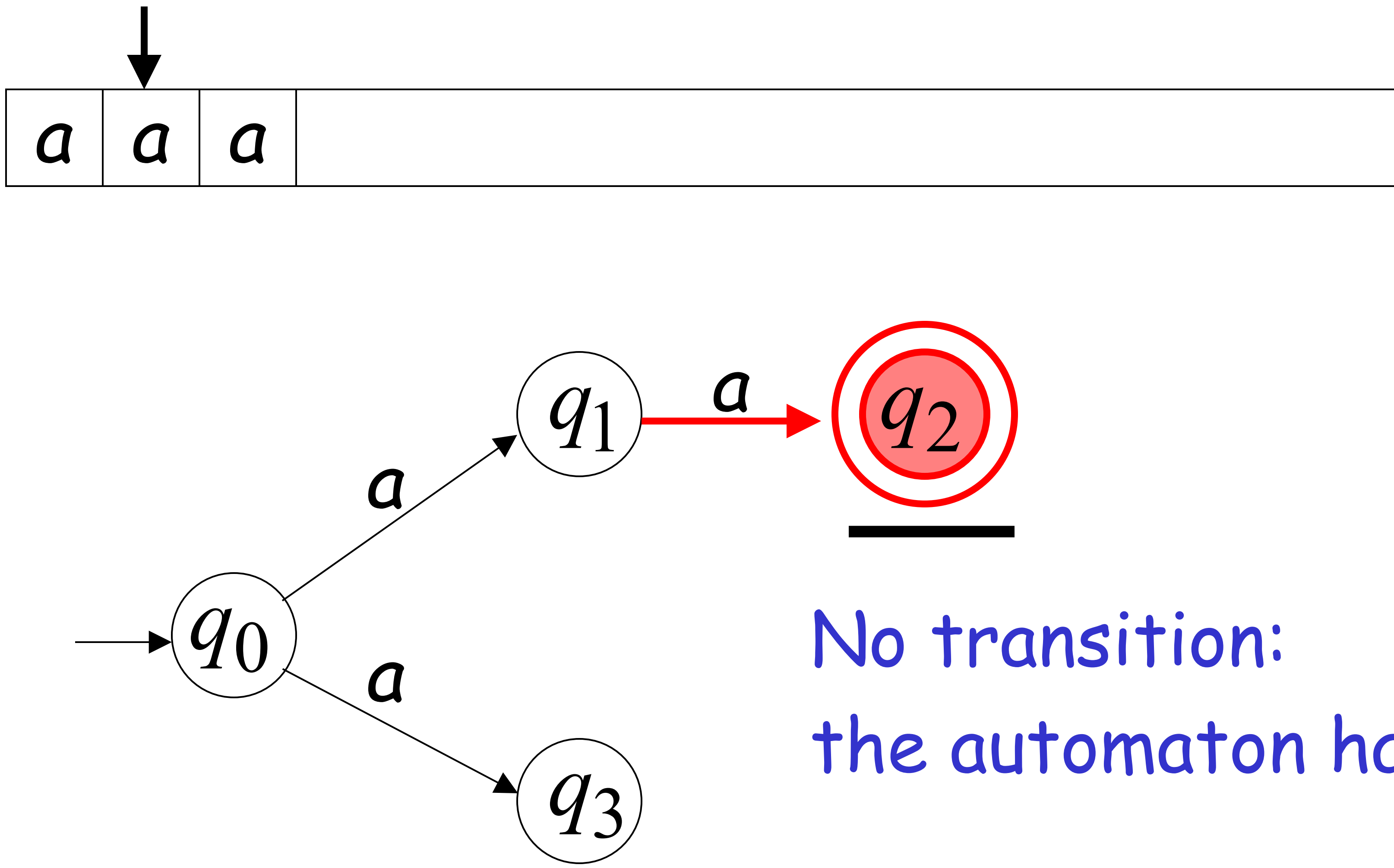
Rejection example



First Choice



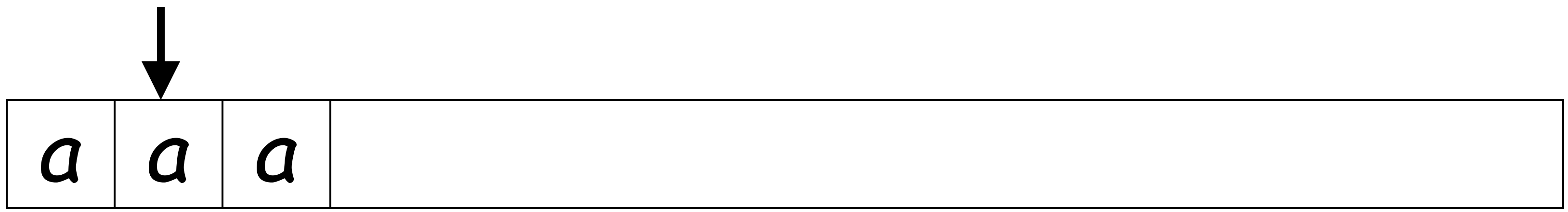
First Choice



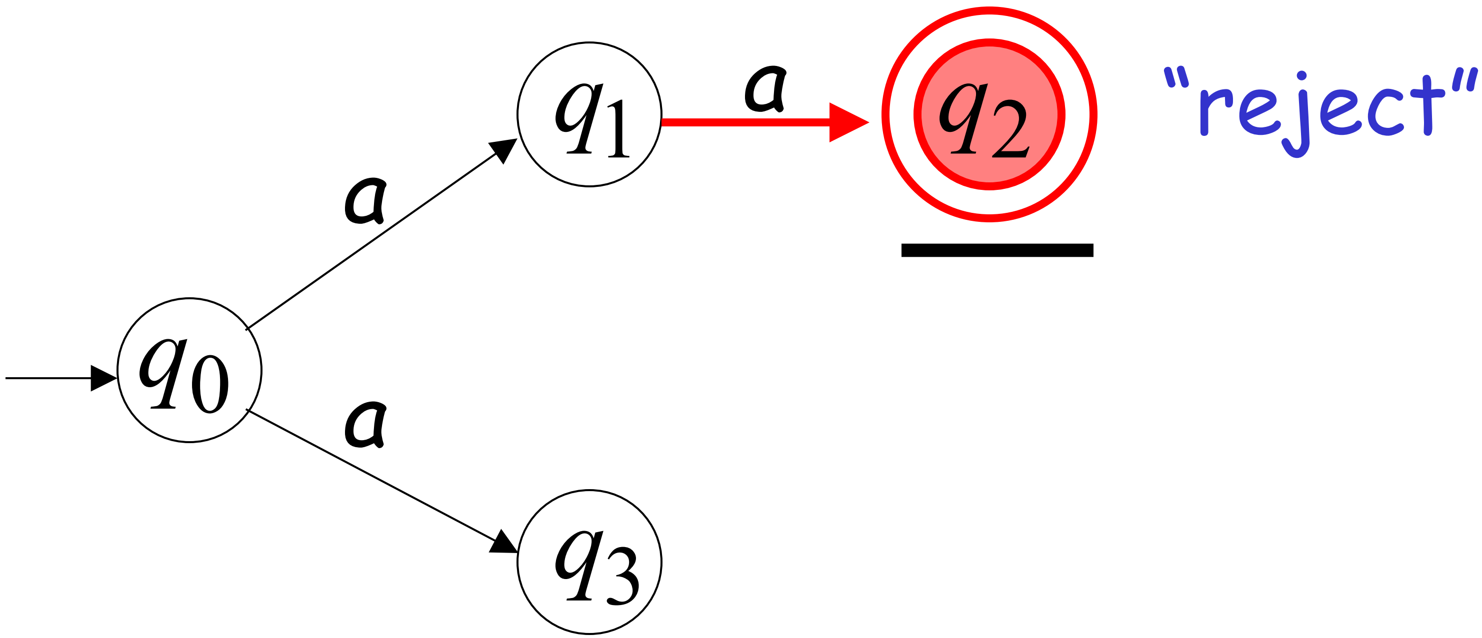
No transition:

the automaton hangs

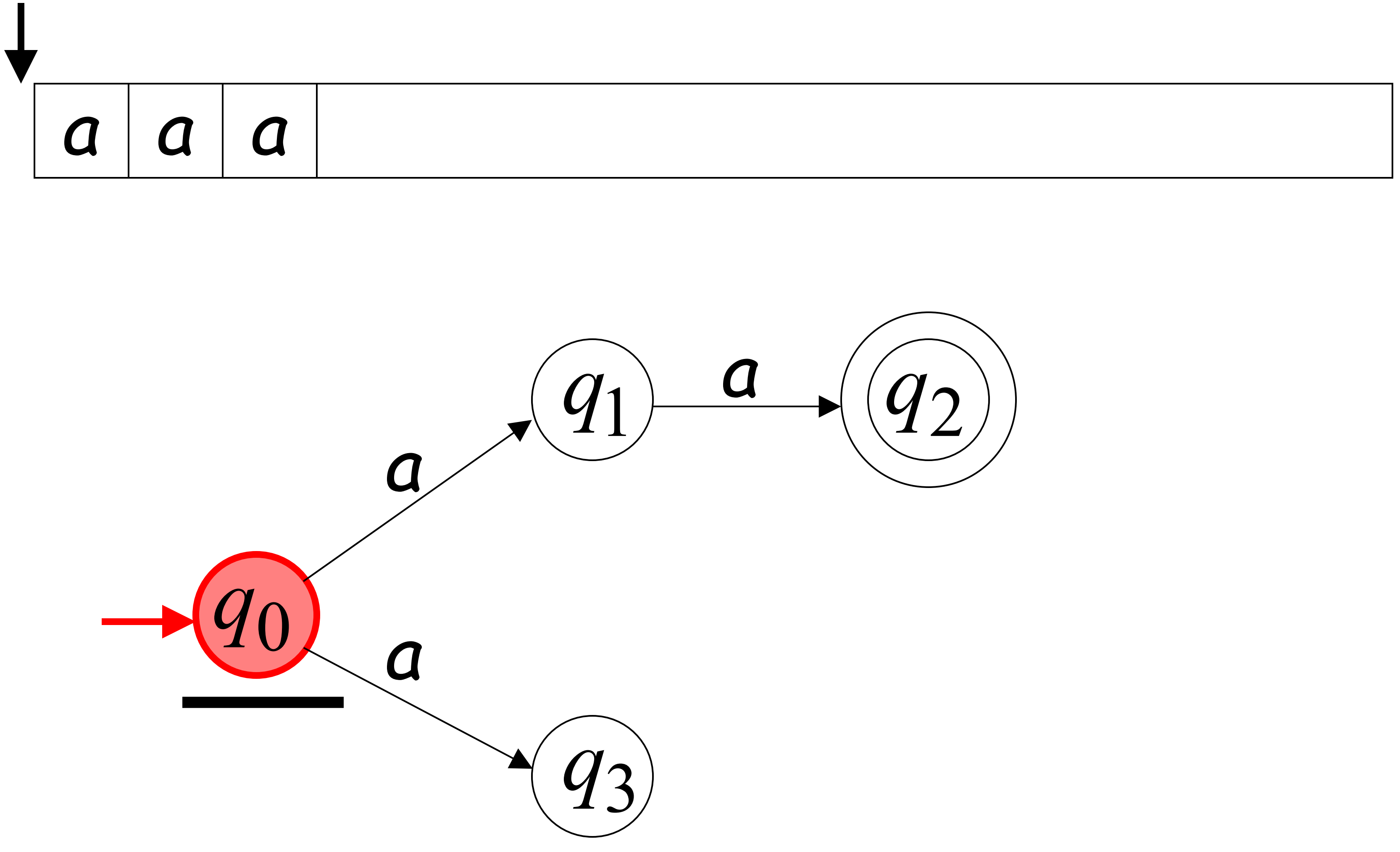
First Choice



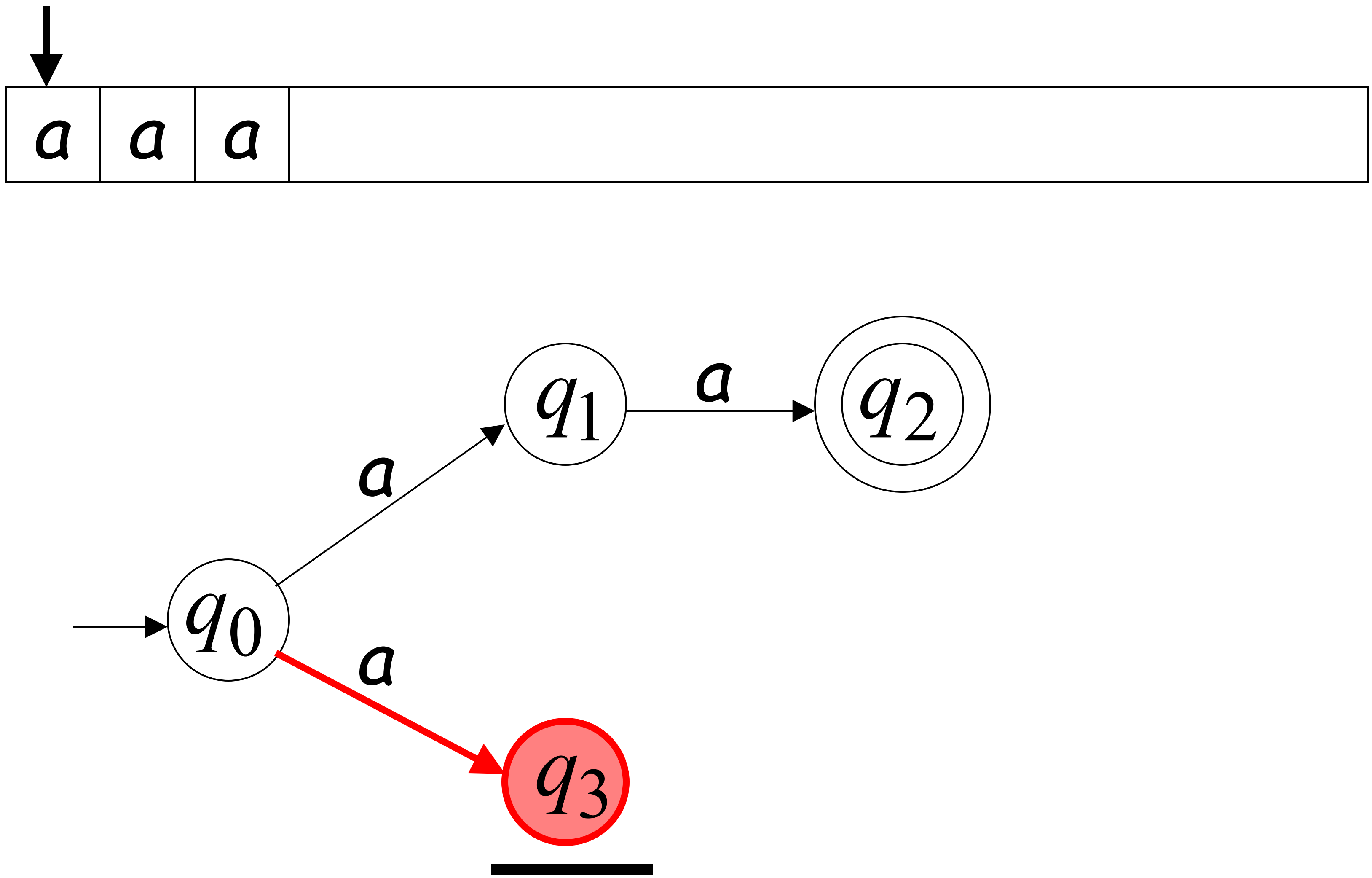
Input cannot be consumed



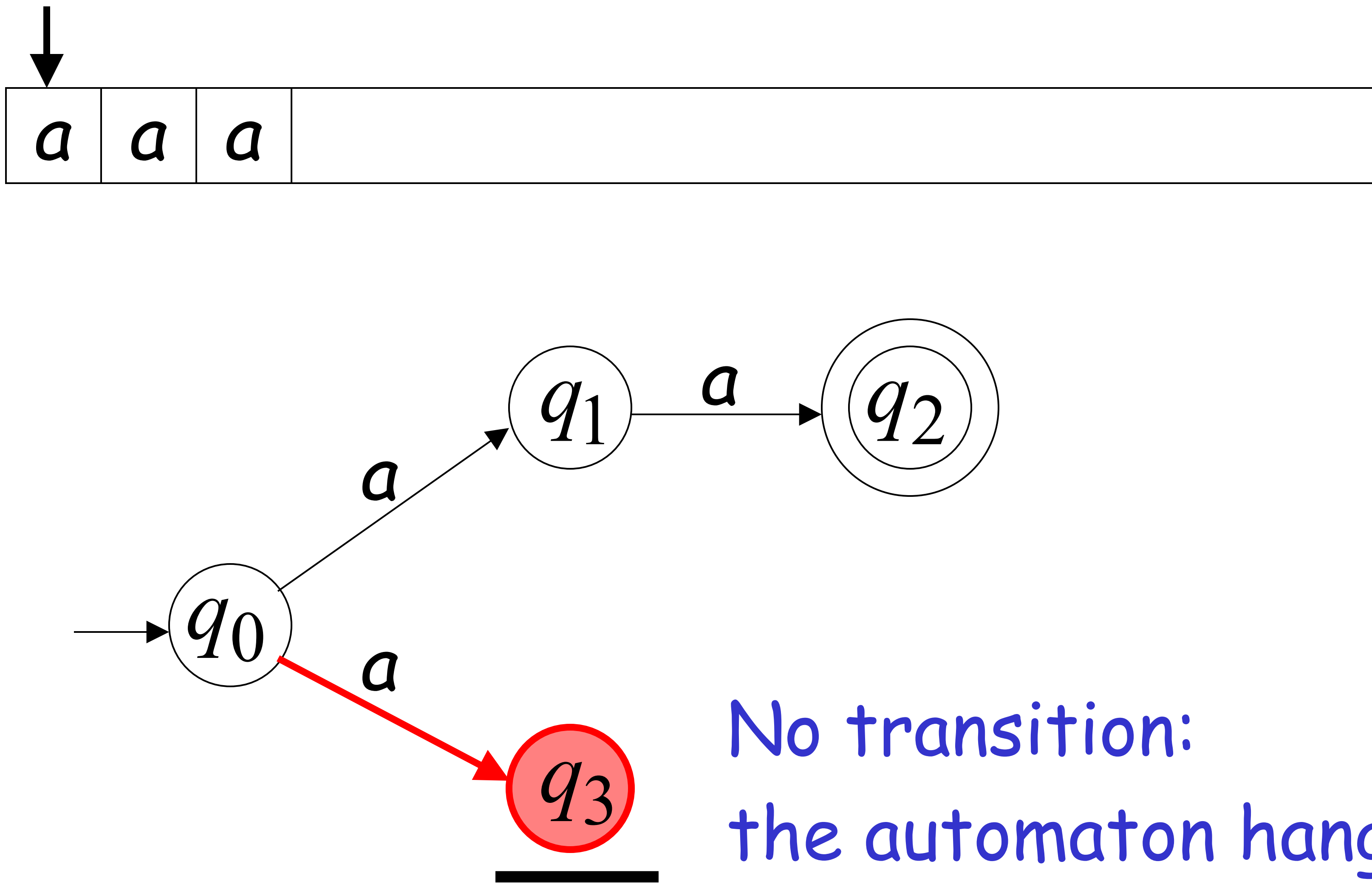
Second Choice



Second Choice

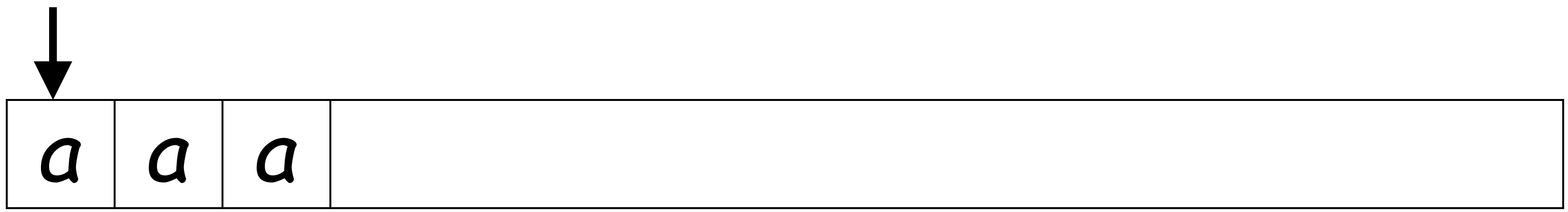


Second Choice

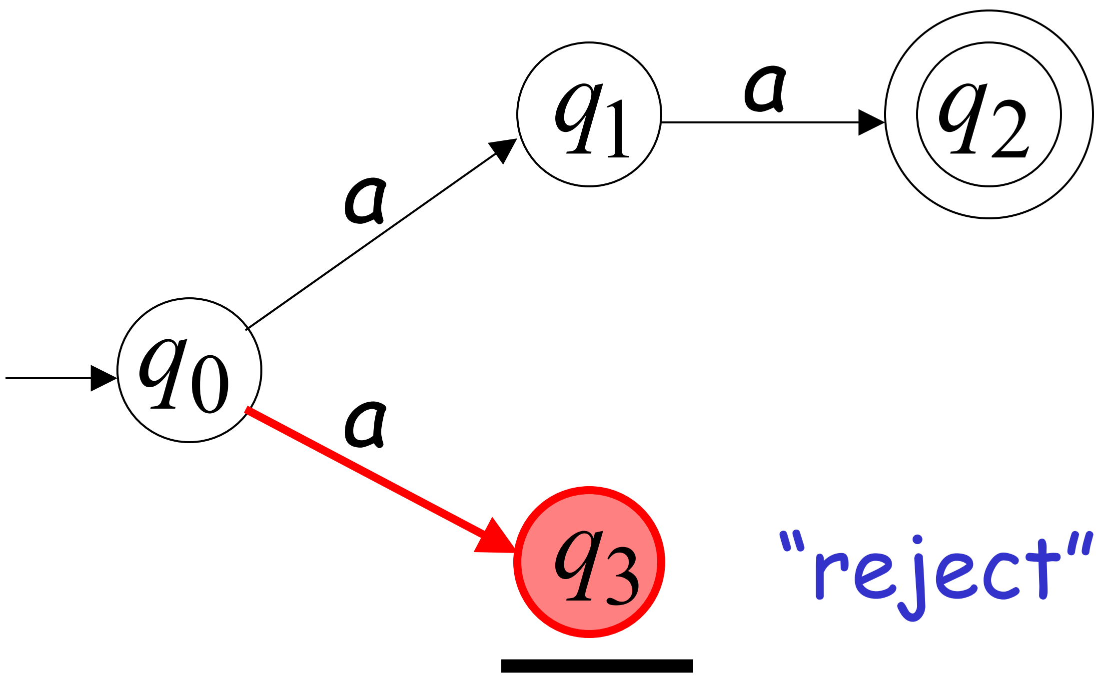


No transition:
the automaton hangs

Second Choice

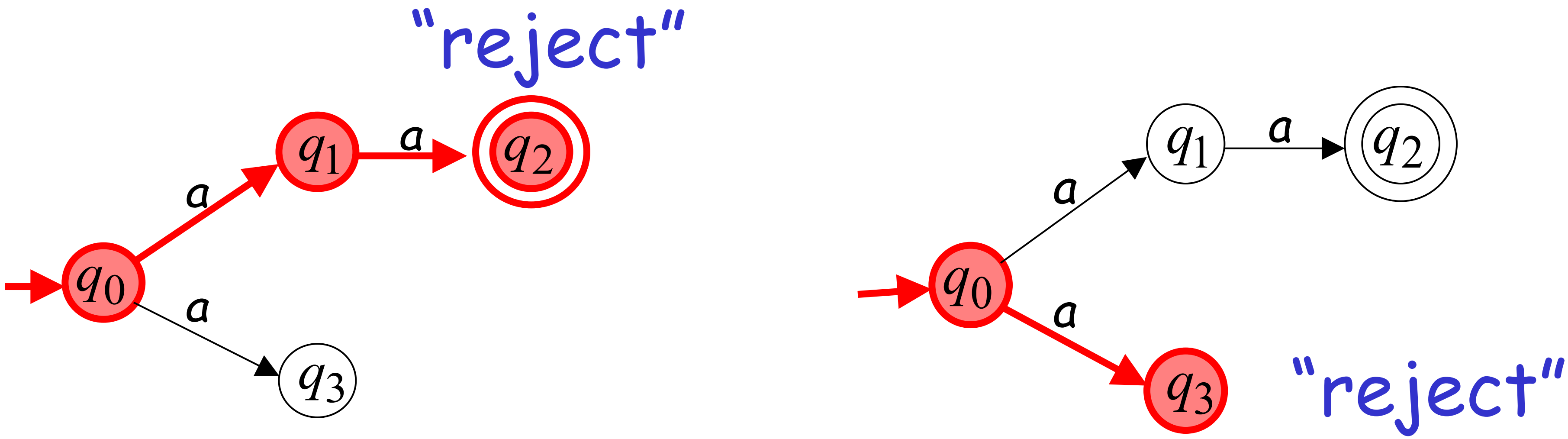


Input cannot be consumed



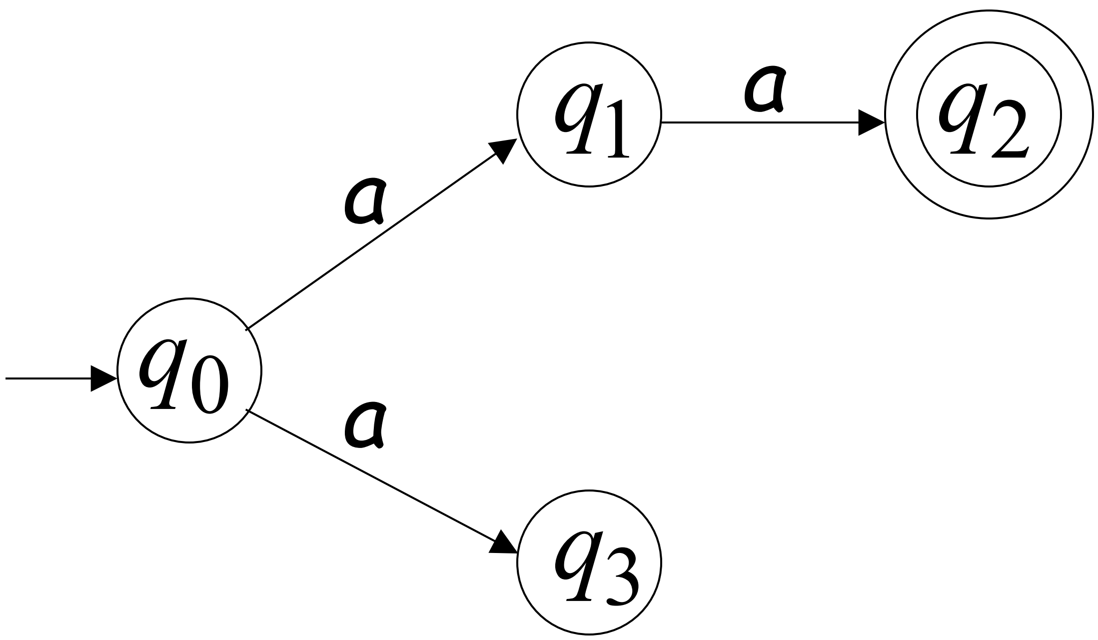
"reject"

aaa is rejected by the NFA:

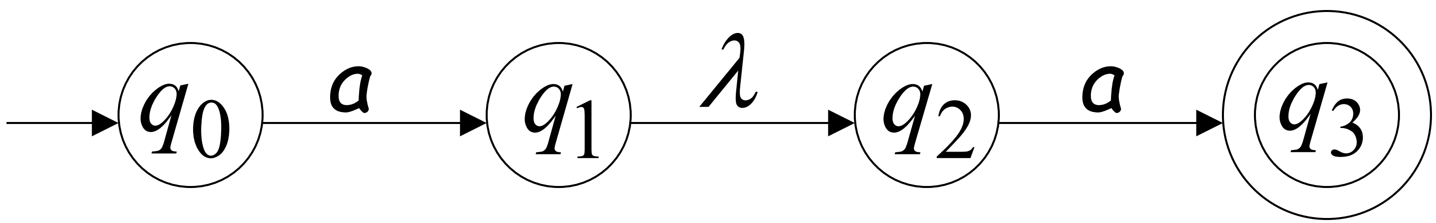


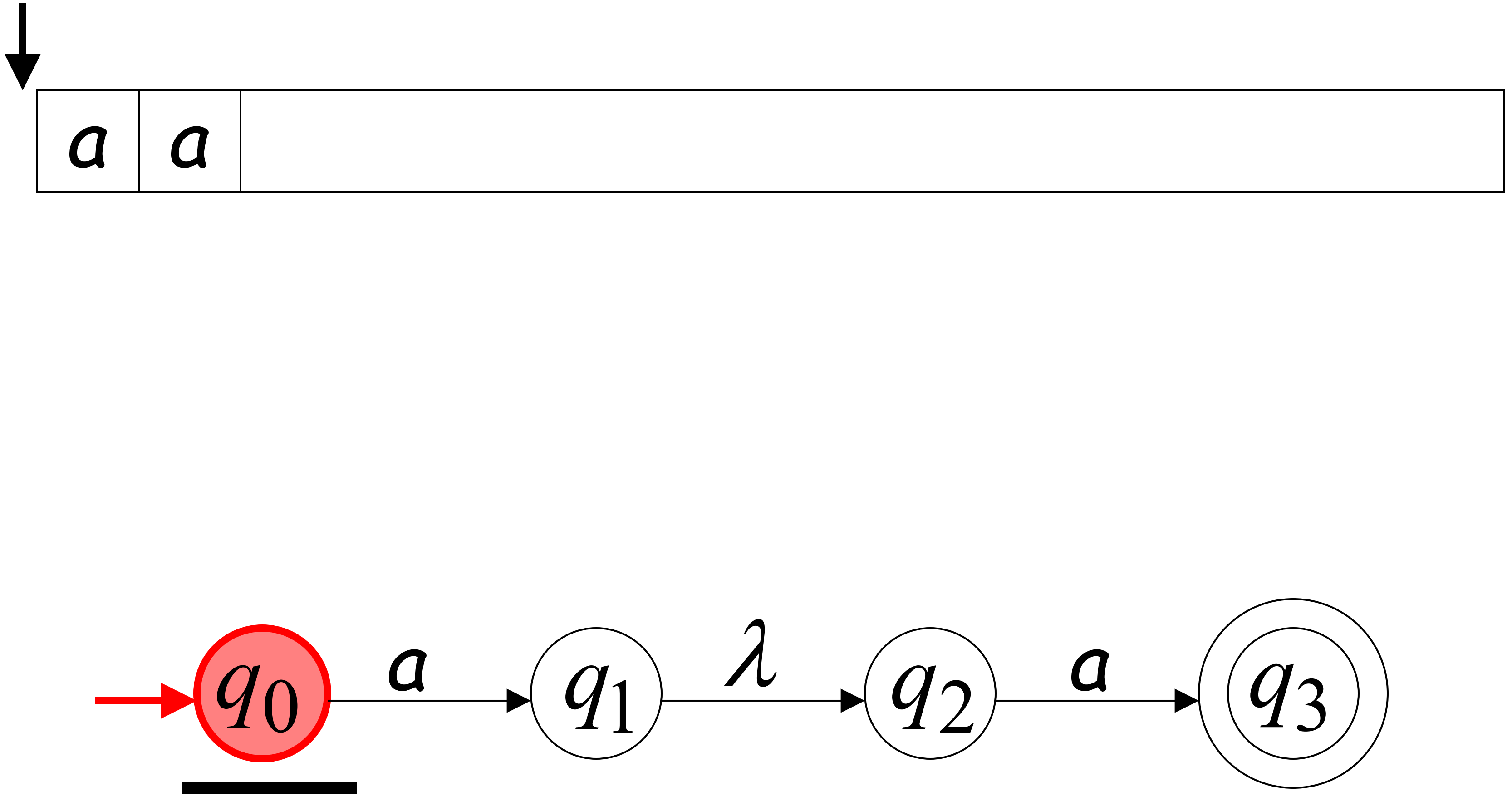
All possible computations lead to rejection

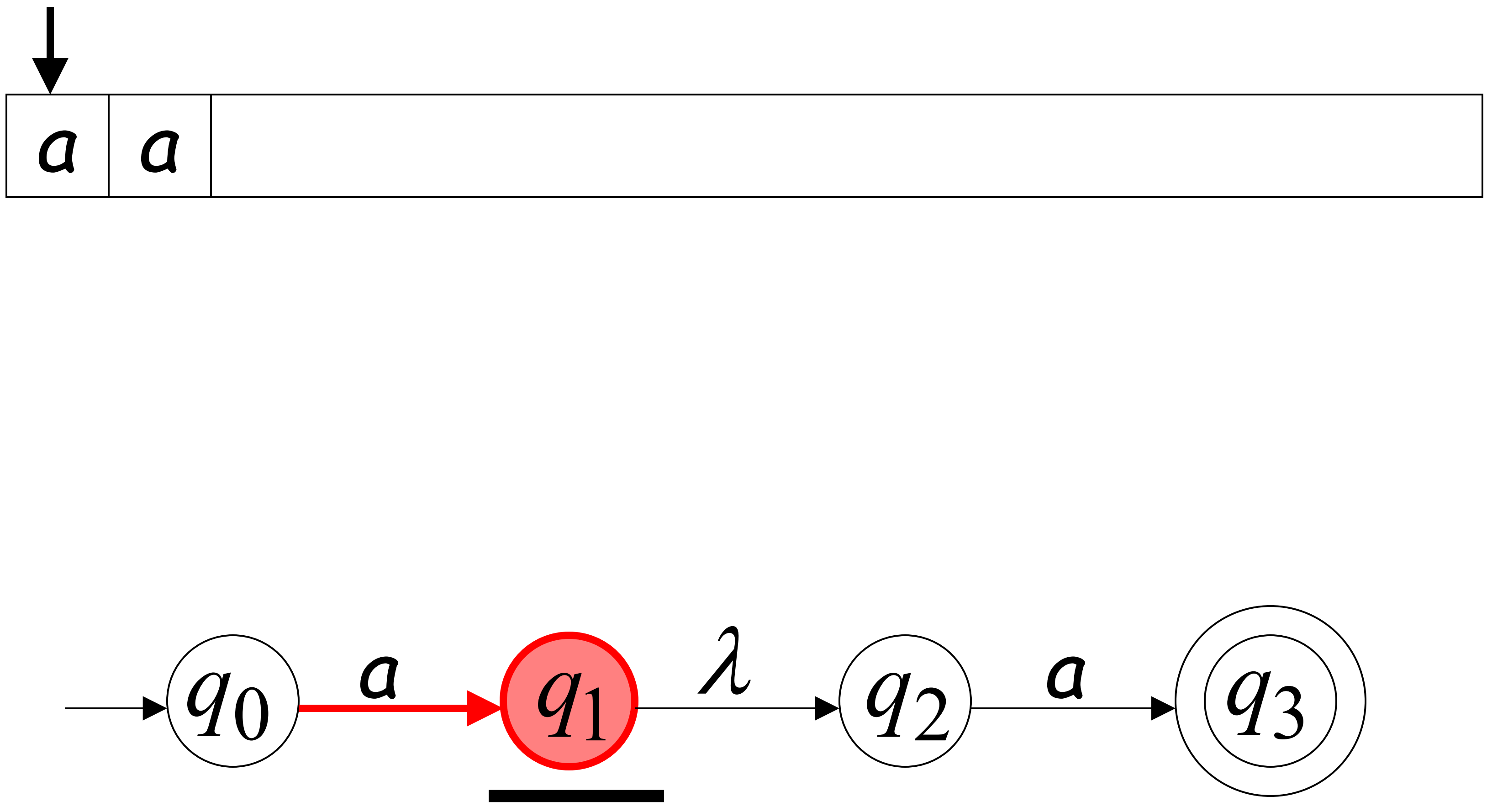
Language accepted: $L = \{aa\}$



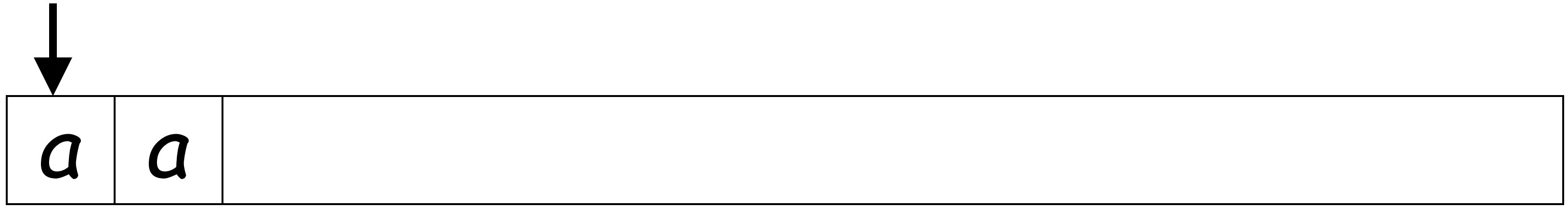
Lambda Transitions

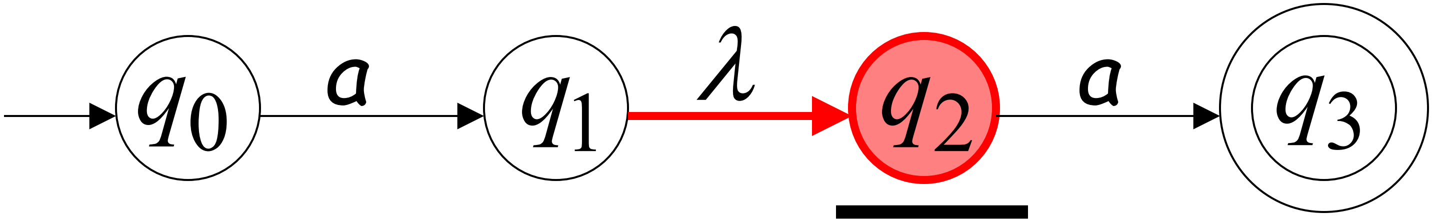


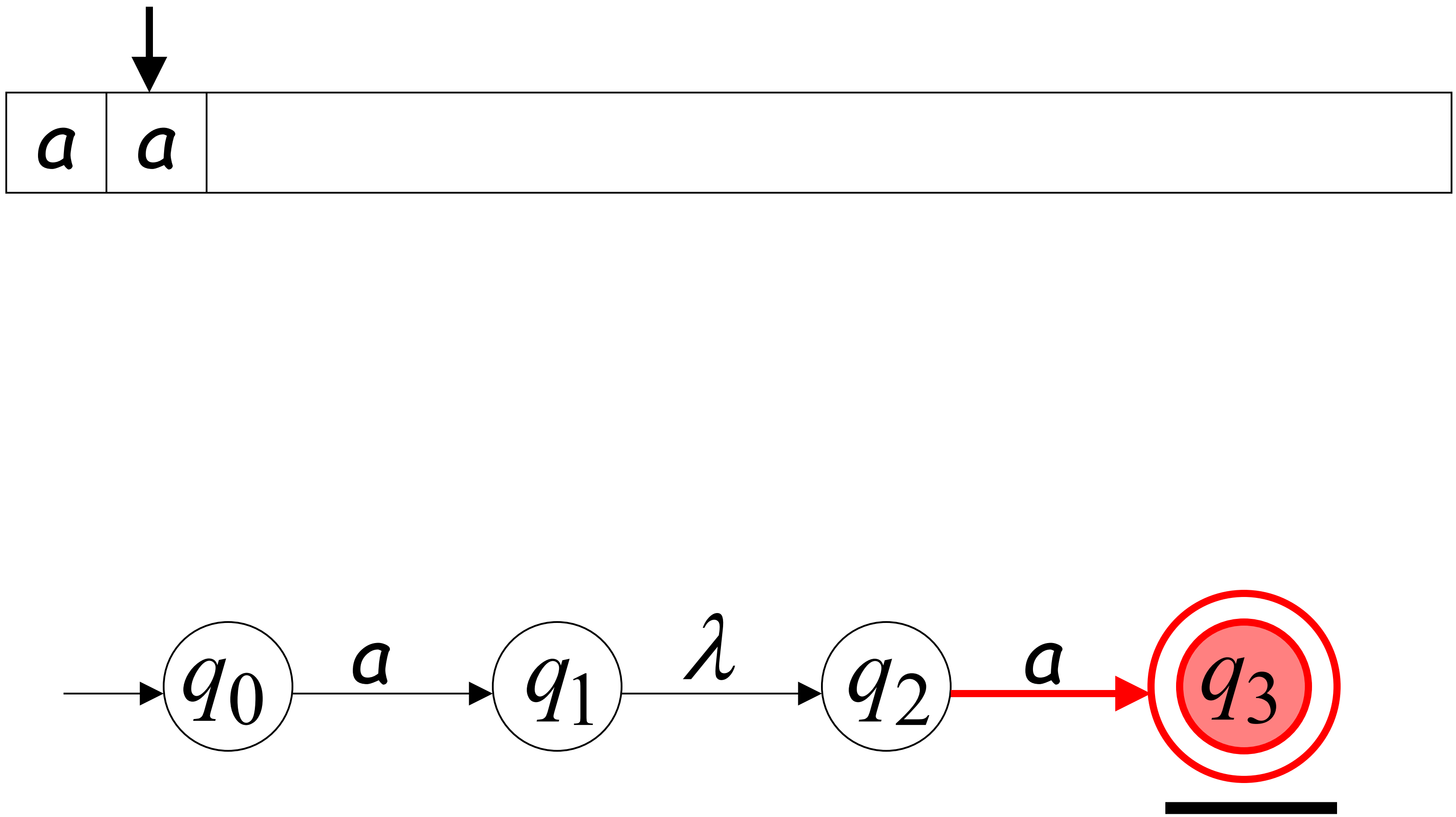

a
a
a
λ
a
q0
q1
q2
q3


a
a
q_0
a
q_1
λ
q_2
a
q_3

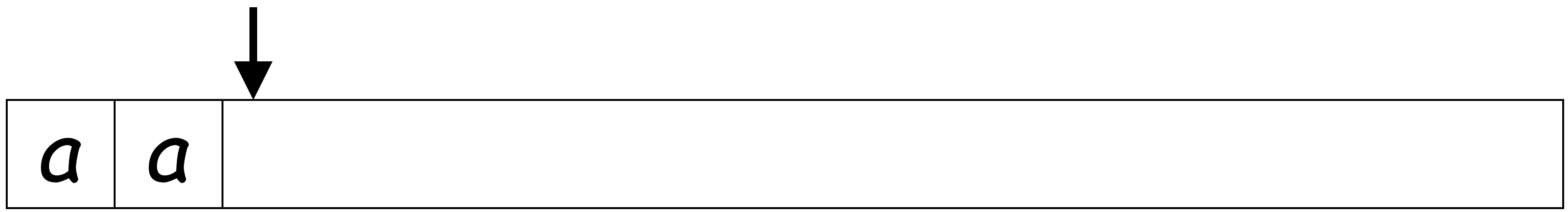
(read head does not move)



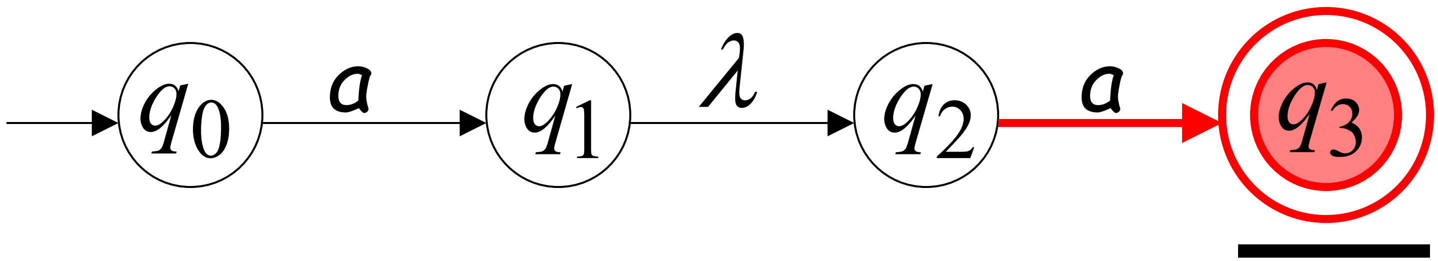



a
a
a
λ
a
q_0
q_1
q_2
q_3

all input is consumed

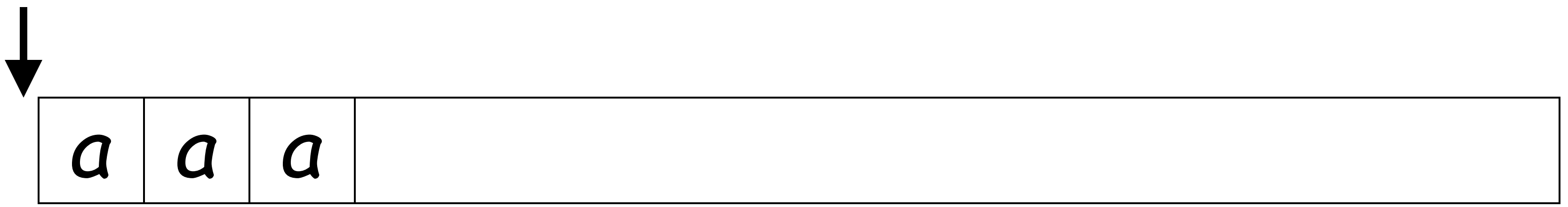


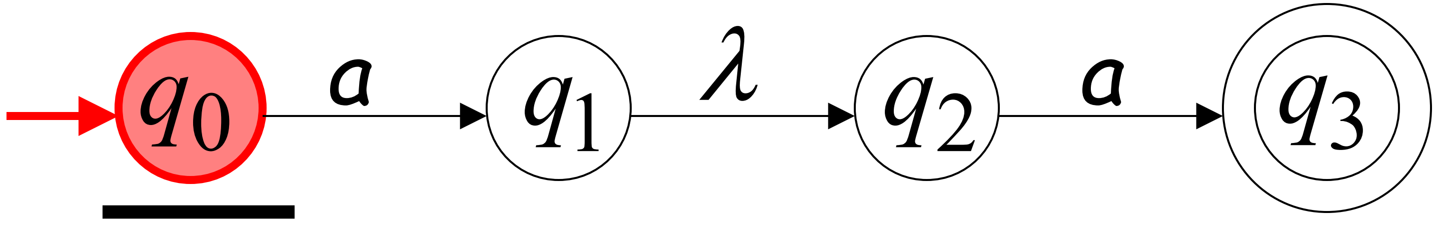
“accept”

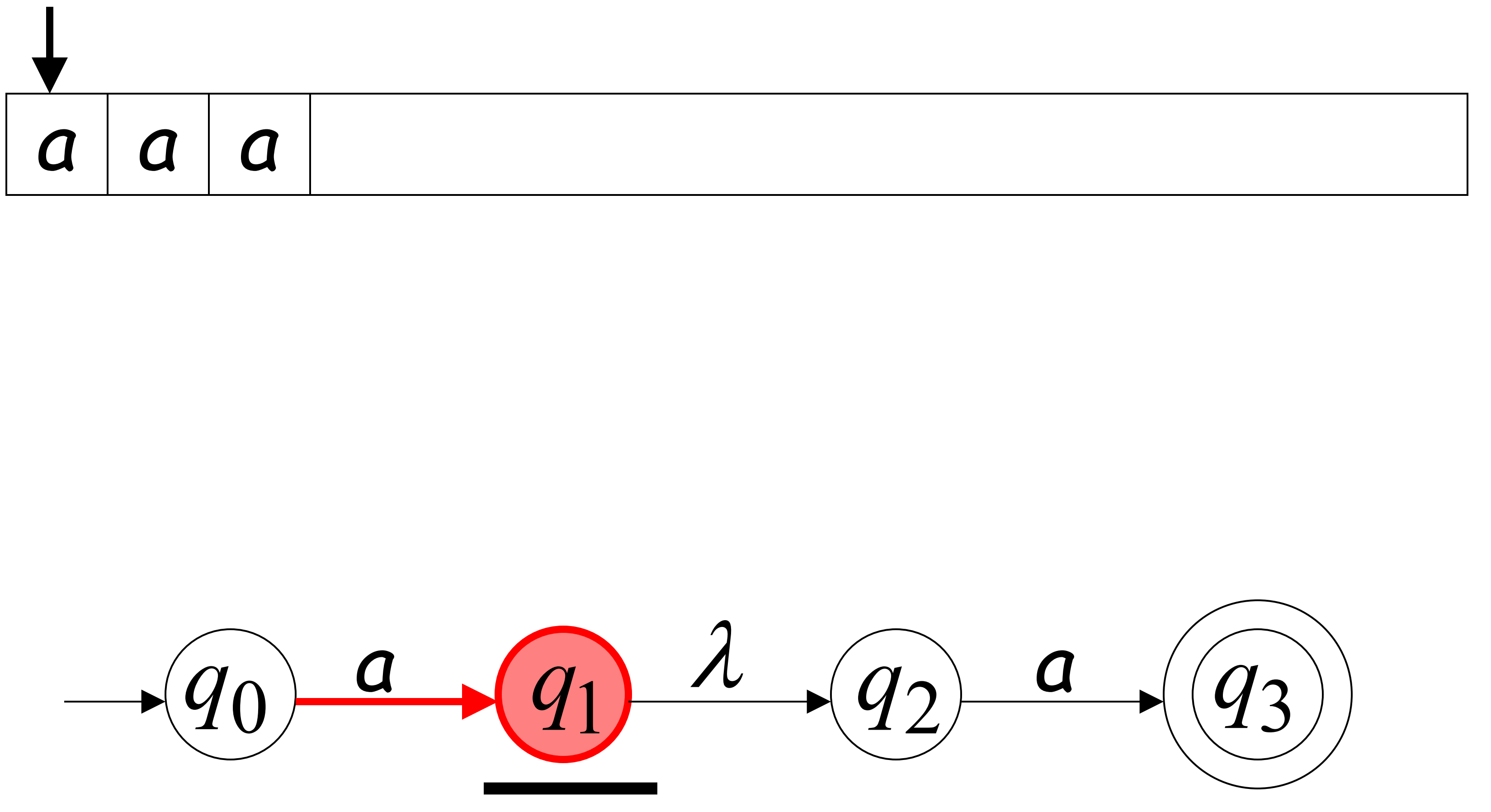


String aa is accepted

Rejection Example

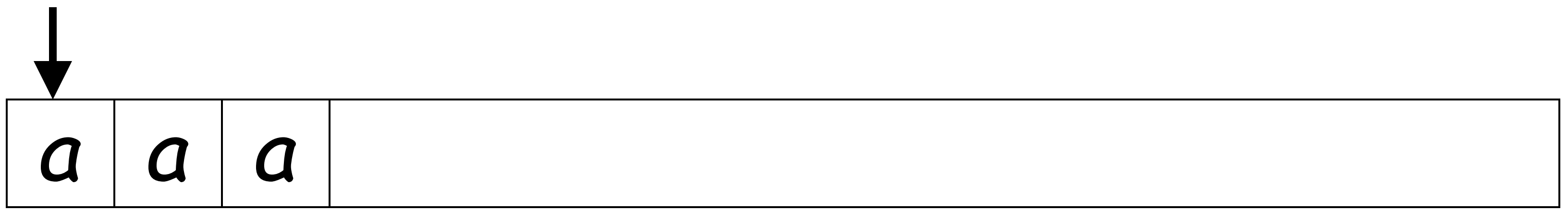


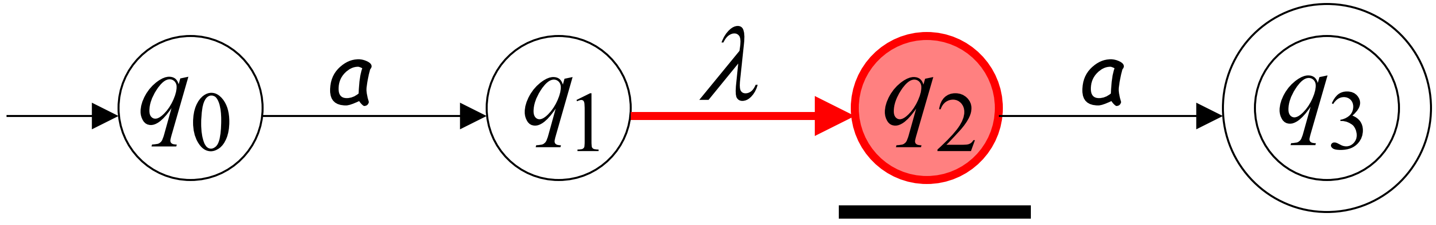


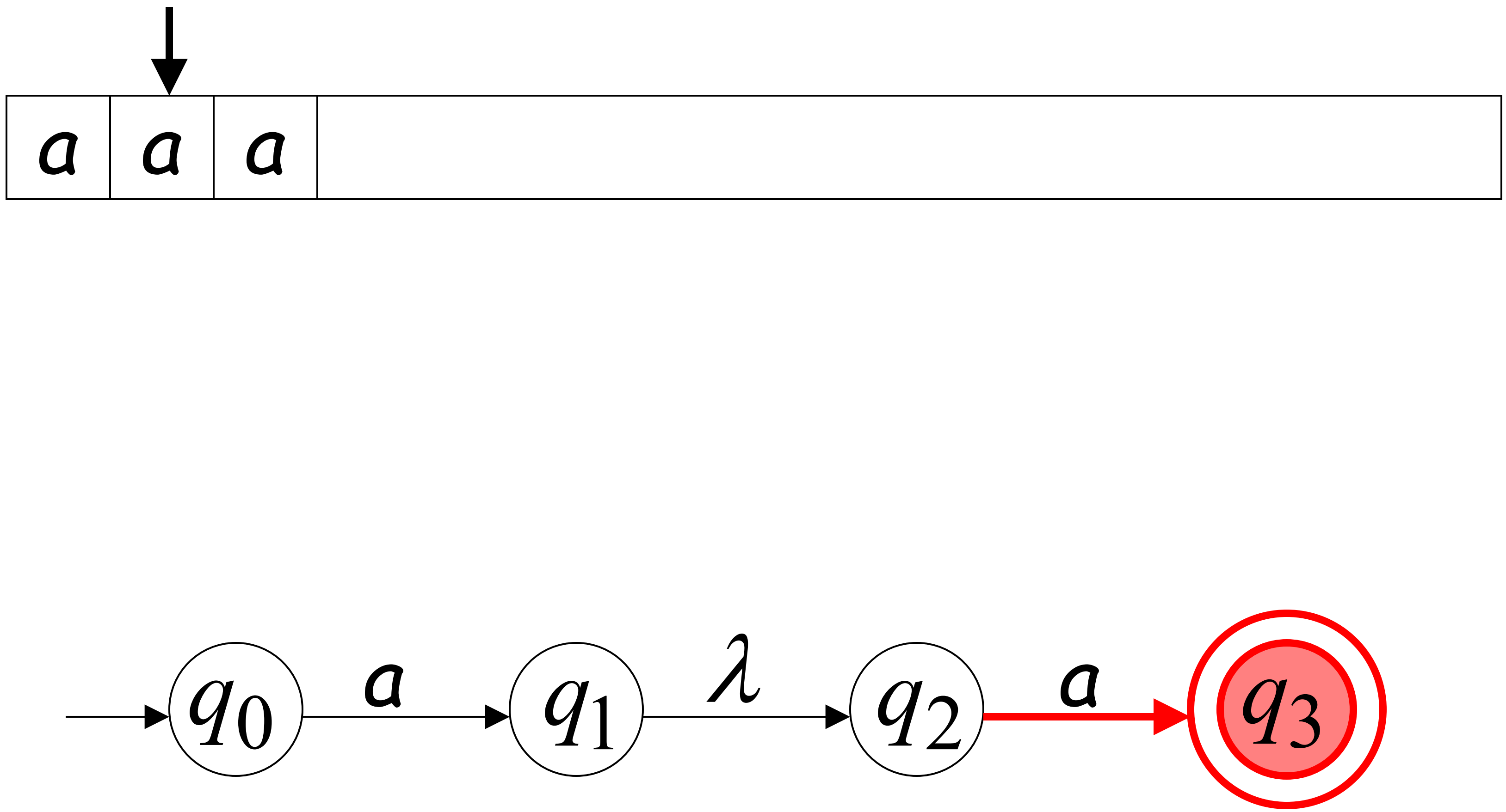


a
a
a
q_0
a
q_1
λ
q_2
a
q_3

(read head doesn't move)



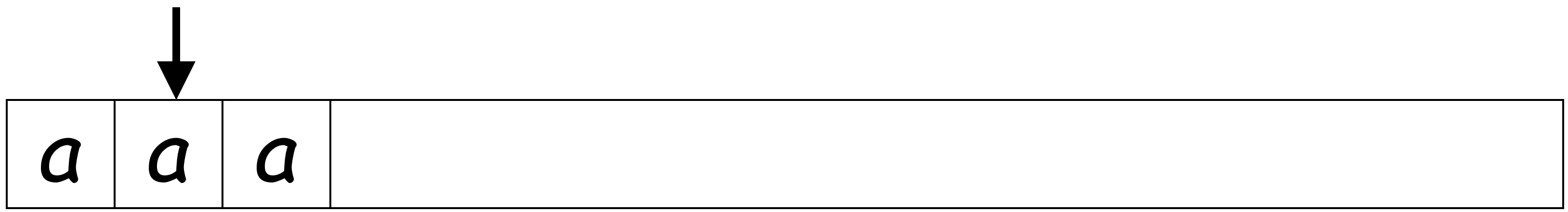




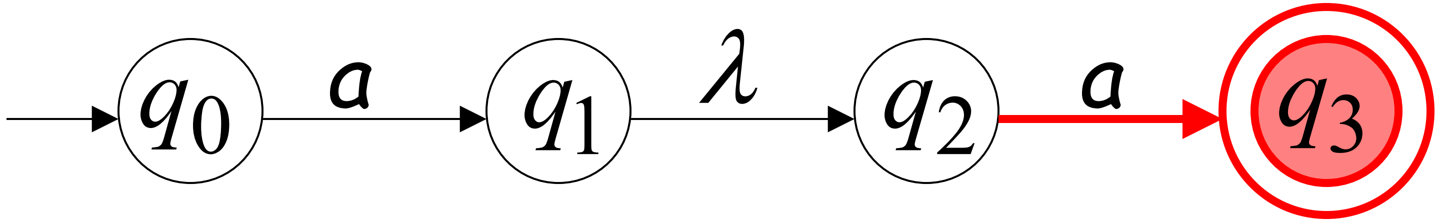
No transition:

the automaton hangs

Input cannot be consumed

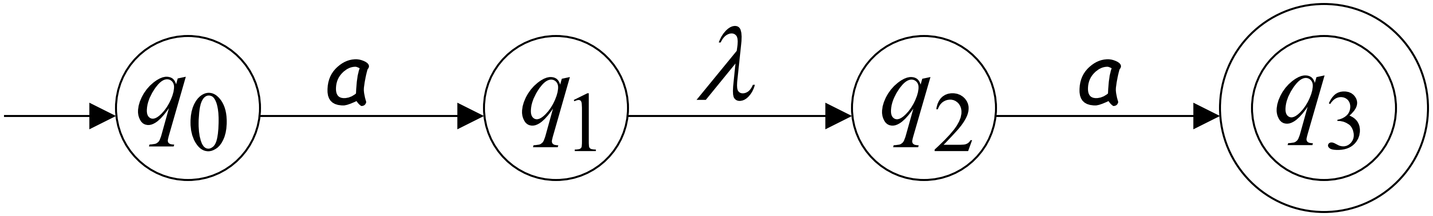


“reject”

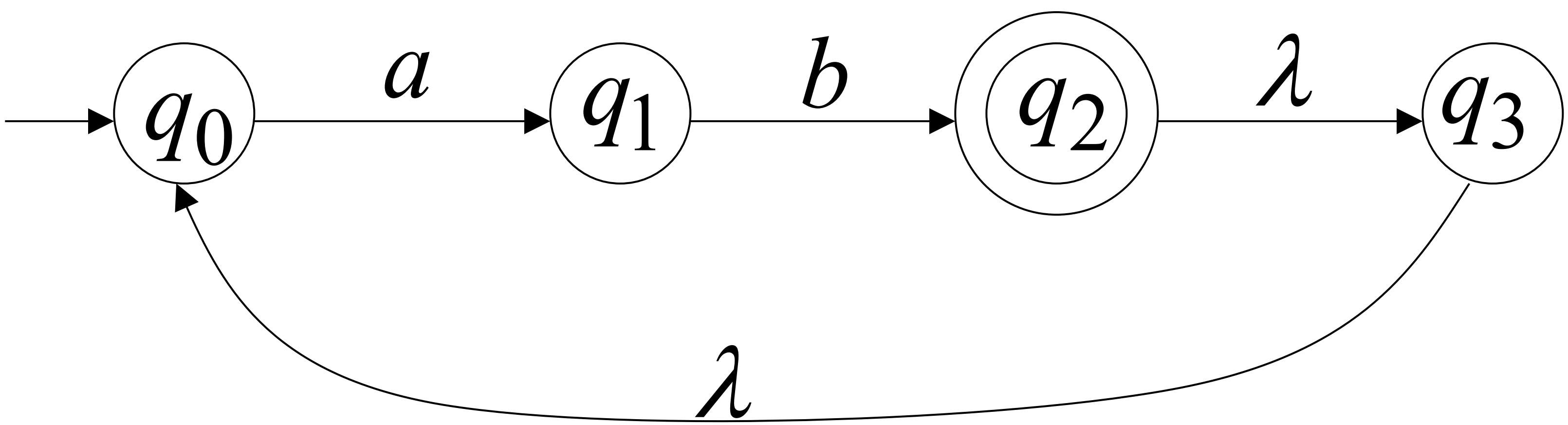


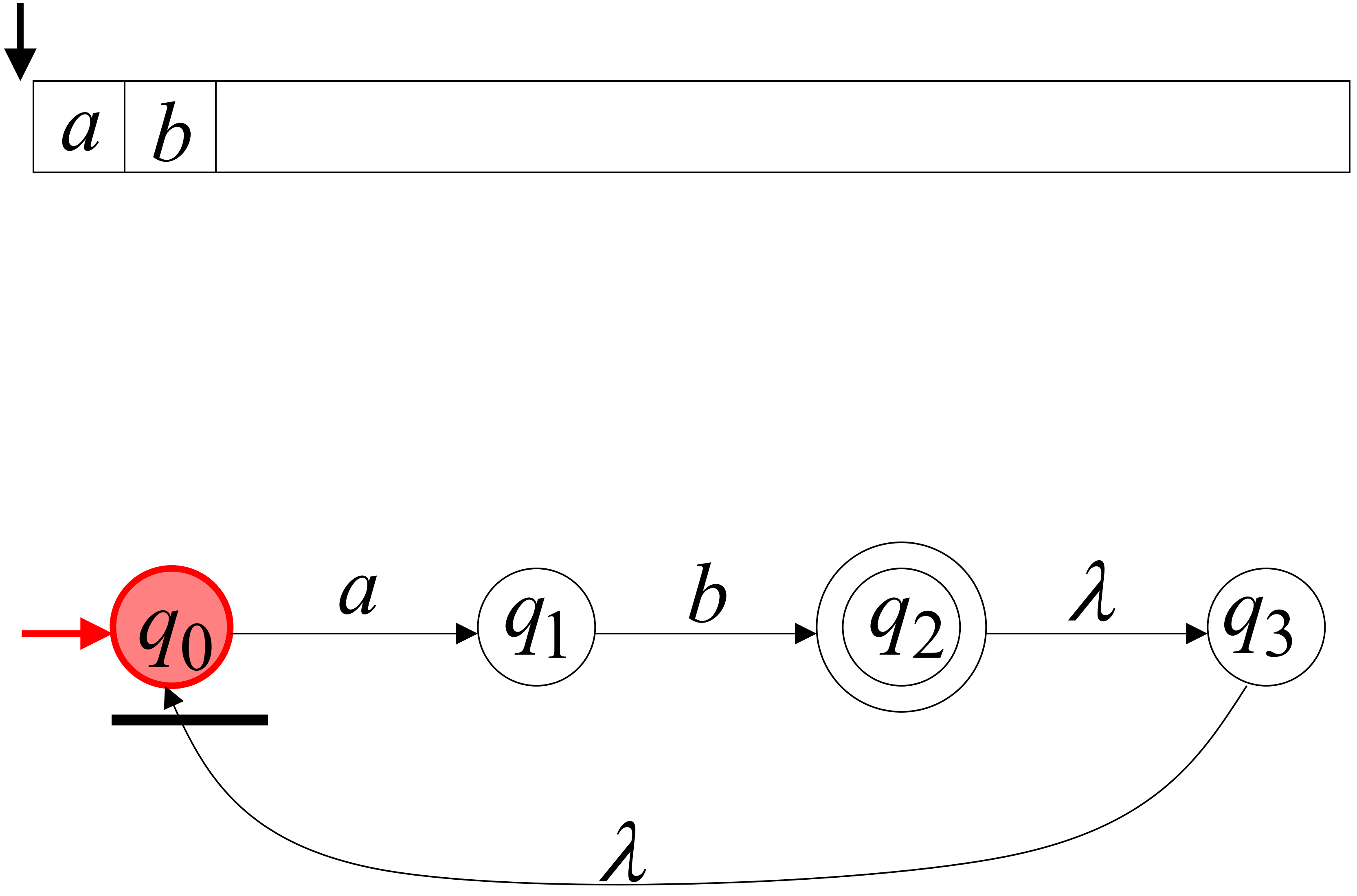
String aaa is rejected

Language accepted: $L = \{aa\}$

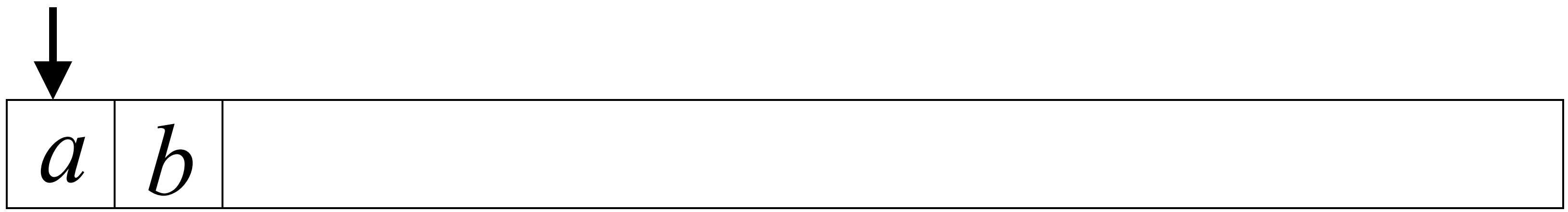


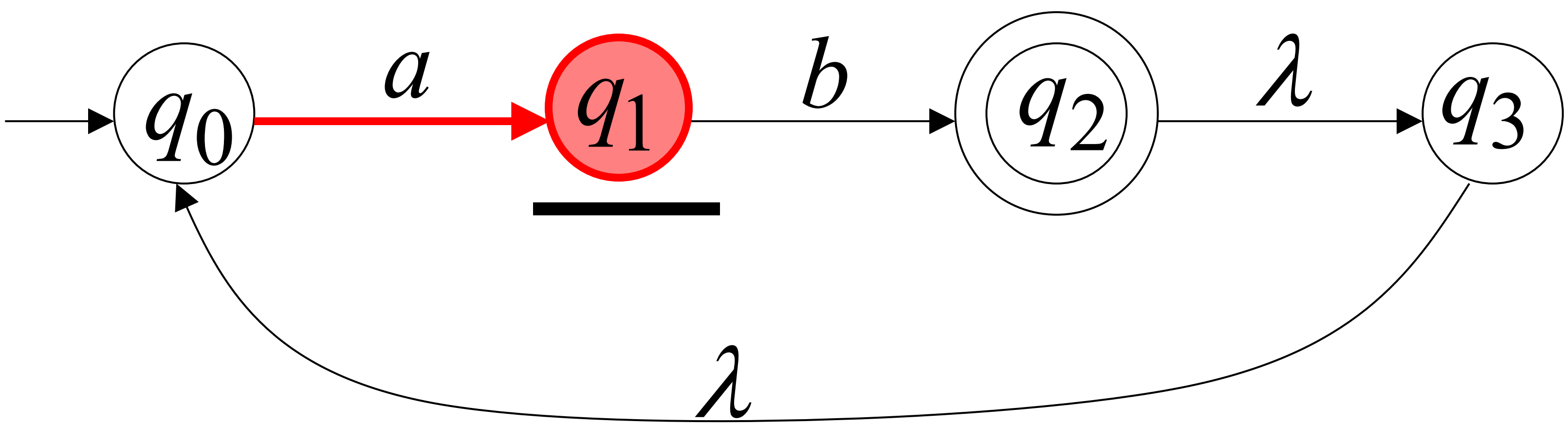
Another NFA Example

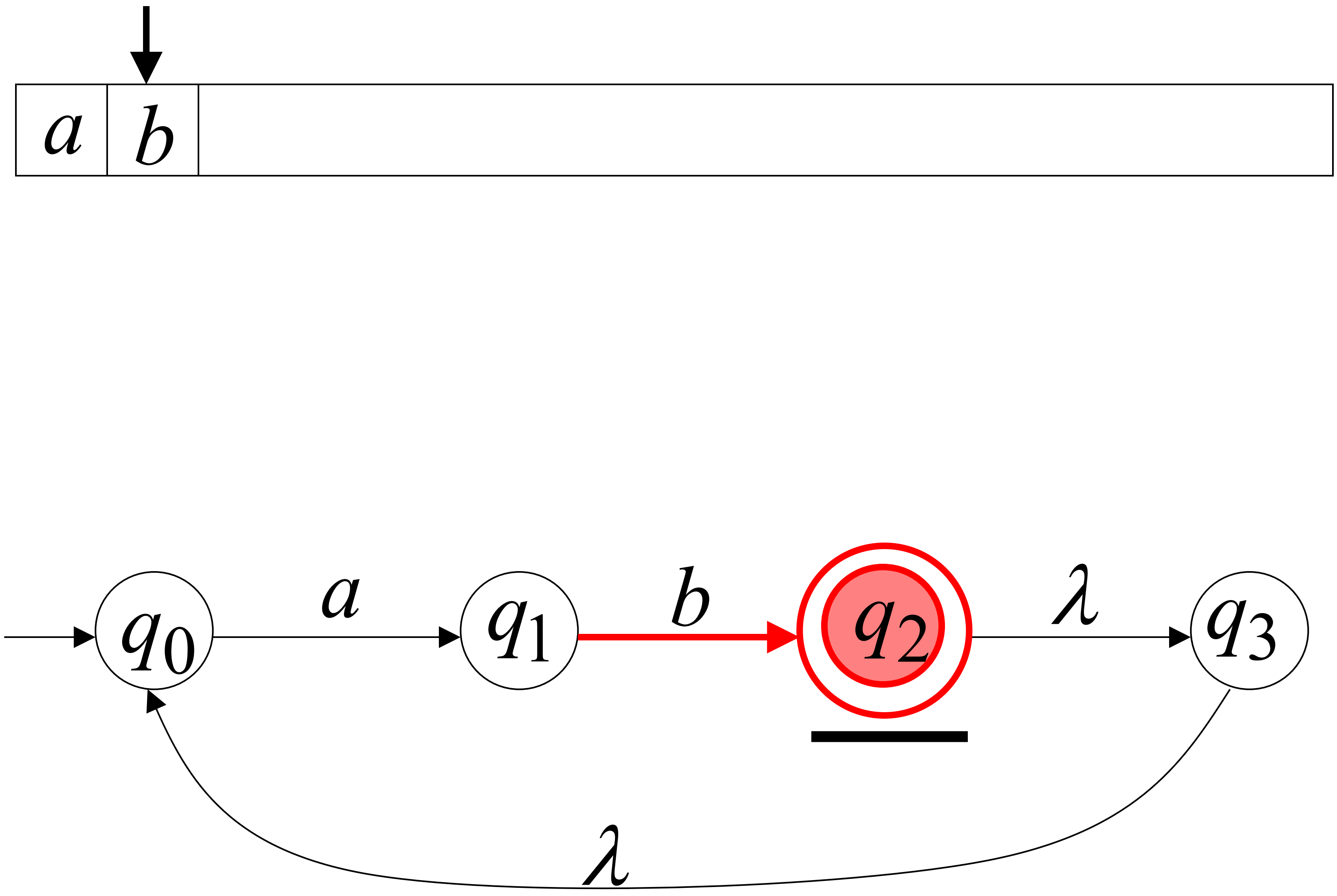




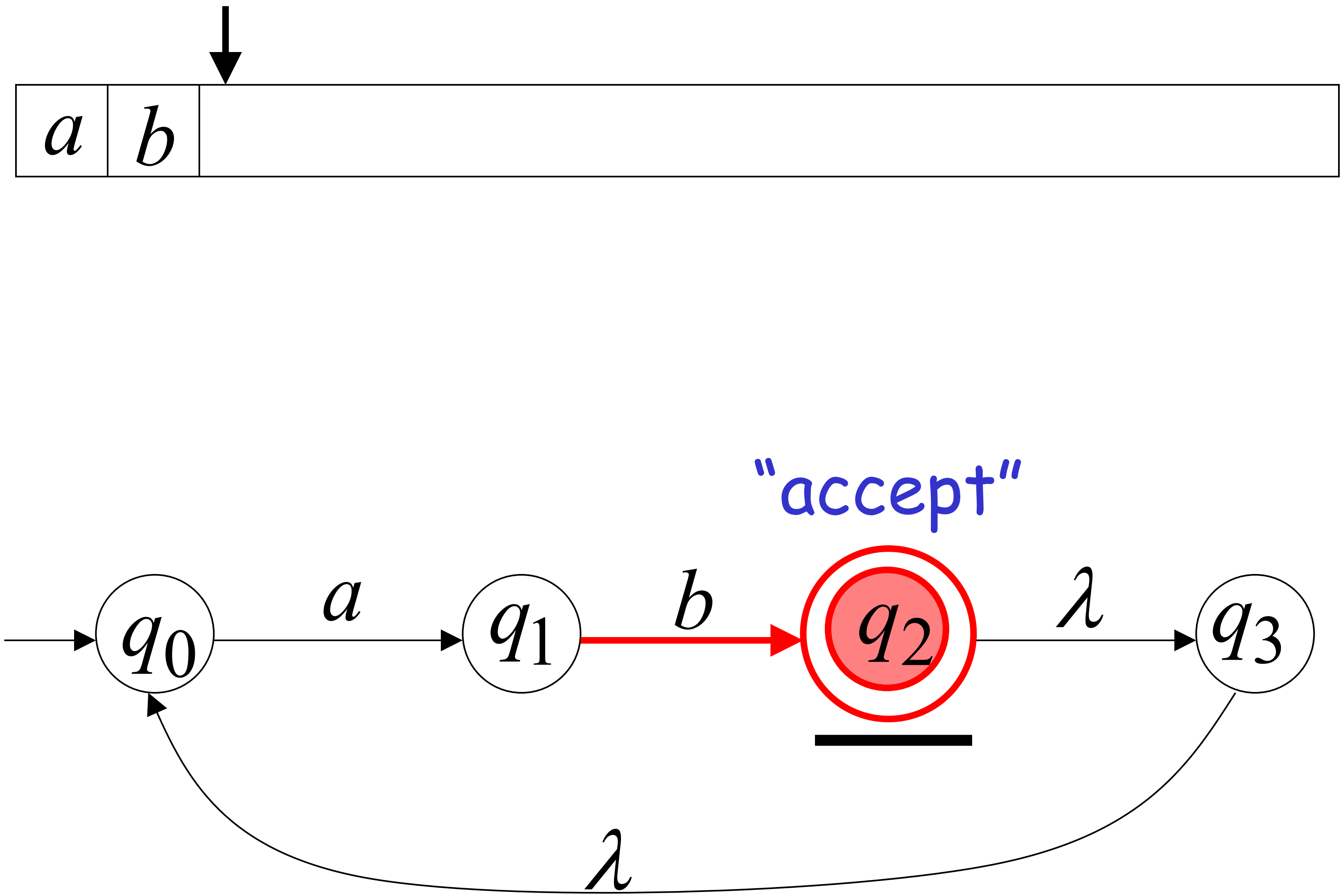
a
b
a
q0
q1
b
q2
λ
q3
λ


a
b


q_0
a
q_1
b
q_2
λ
q_3
λ

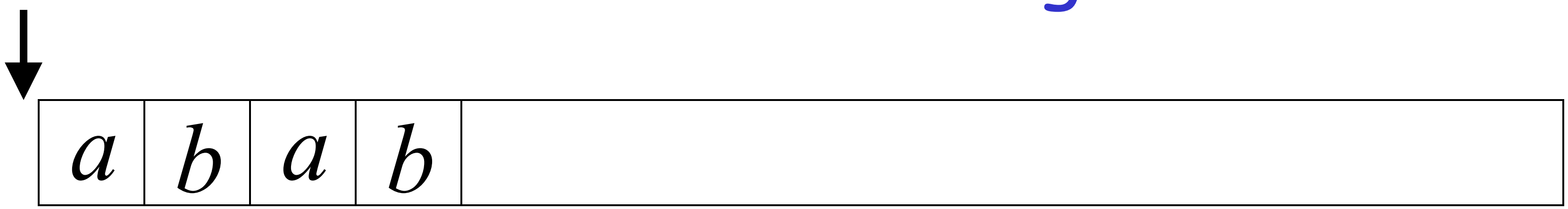


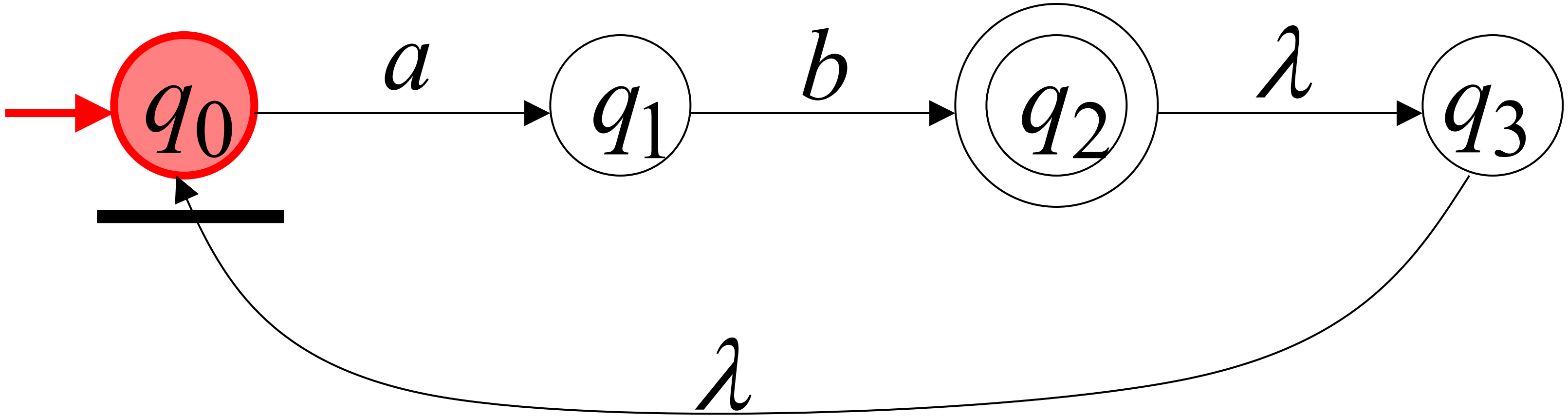
a
b
a
b
λ
λ
q0
q1
q2
q3

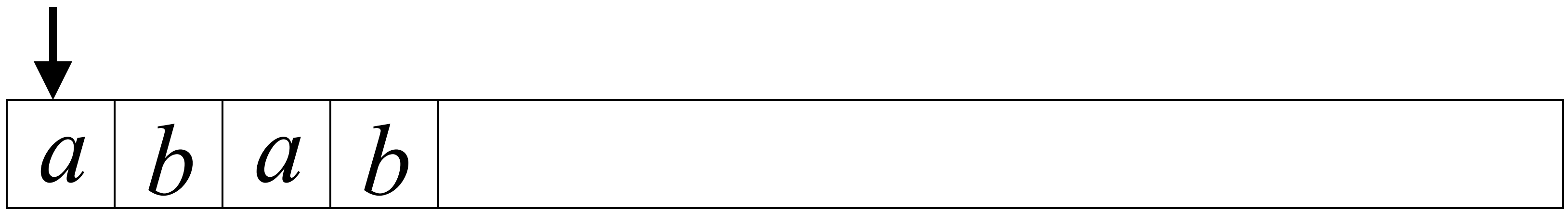


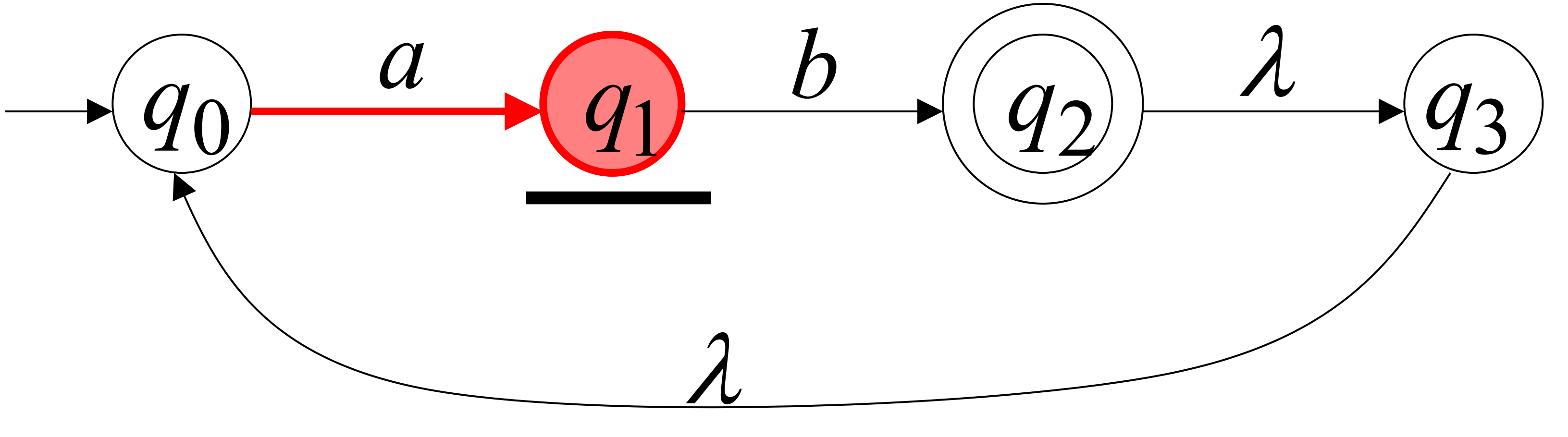
a
b
"accept"
q0
a
q1
b
q2
λ
q3
λ

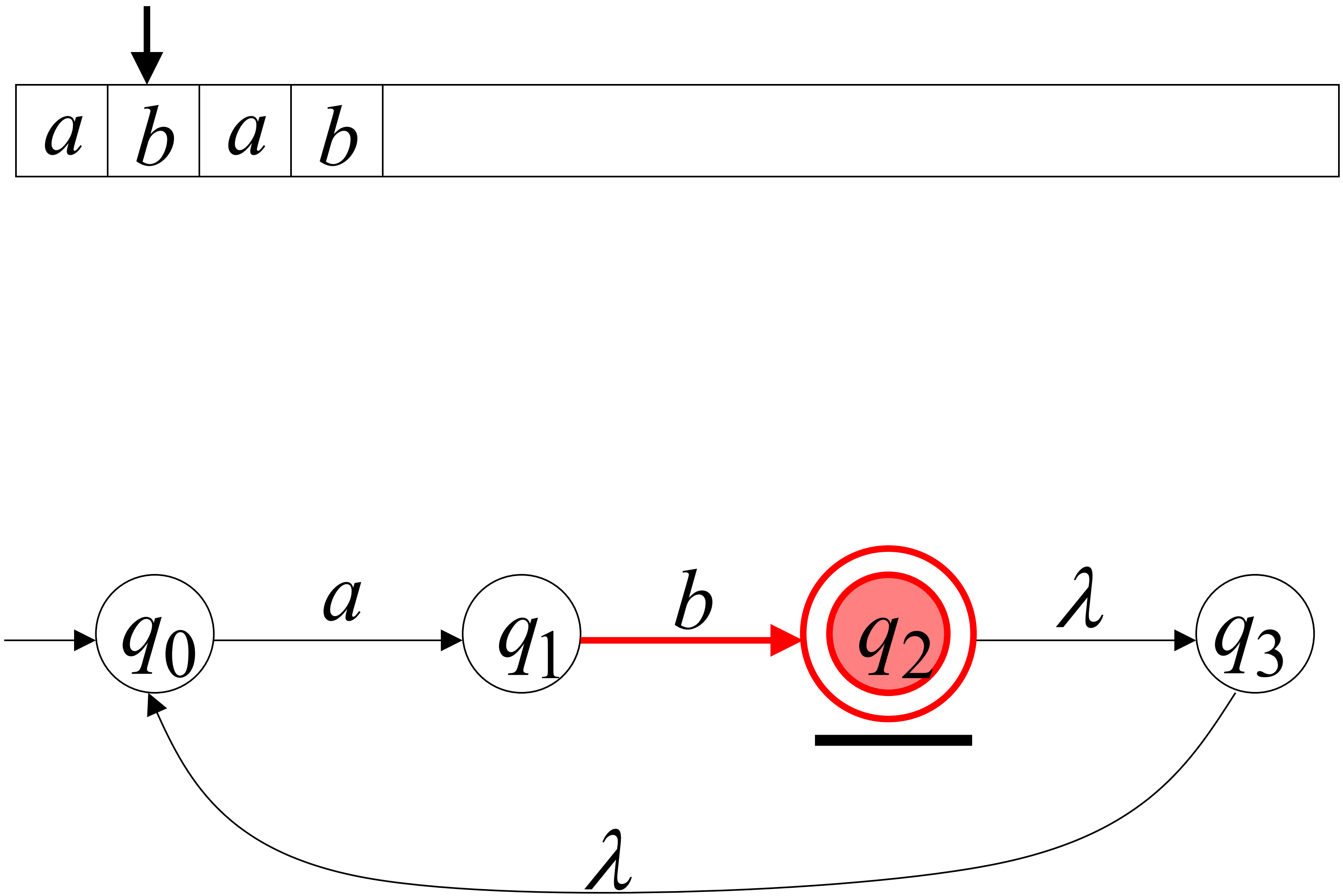
Another String



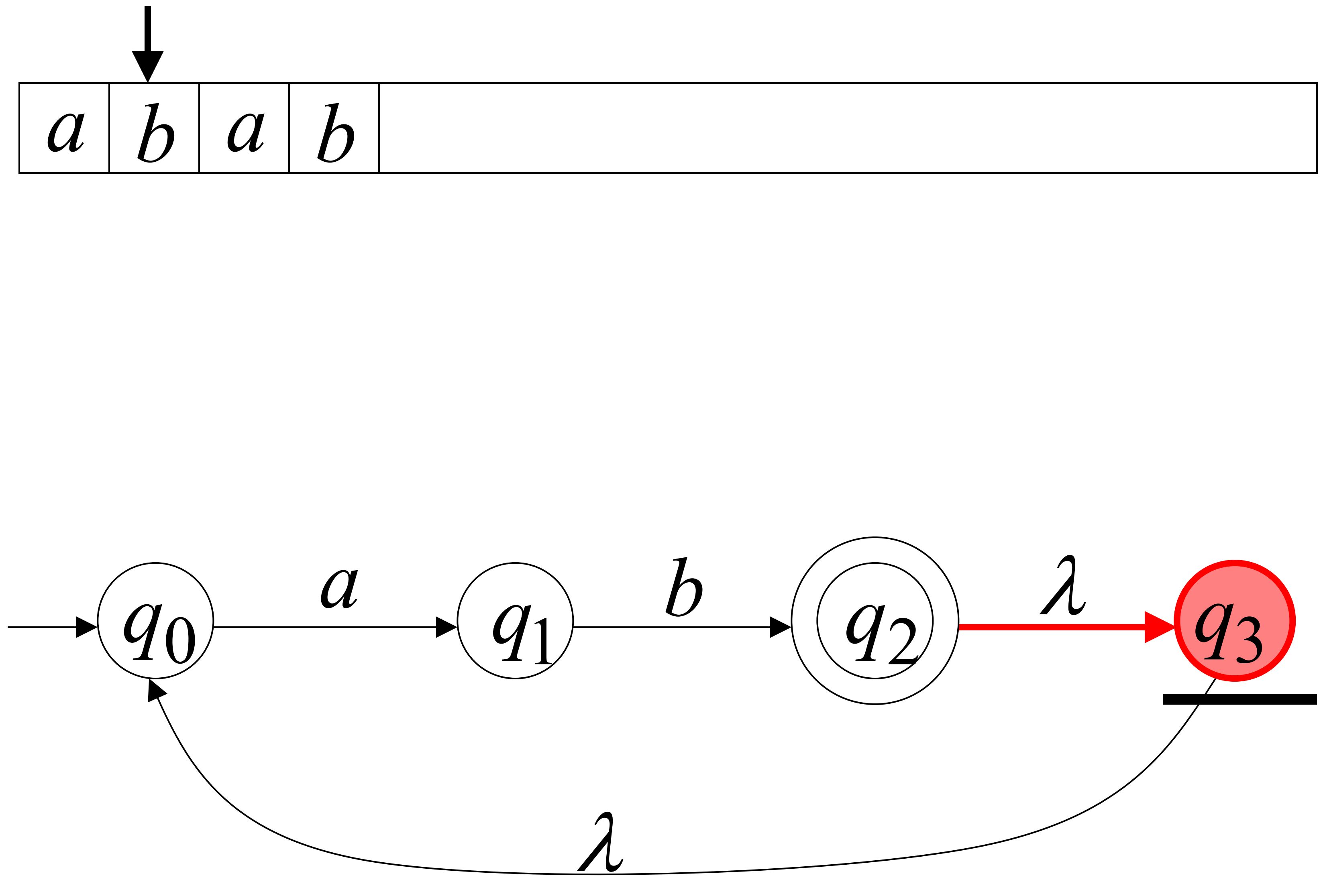



a
b
a
b

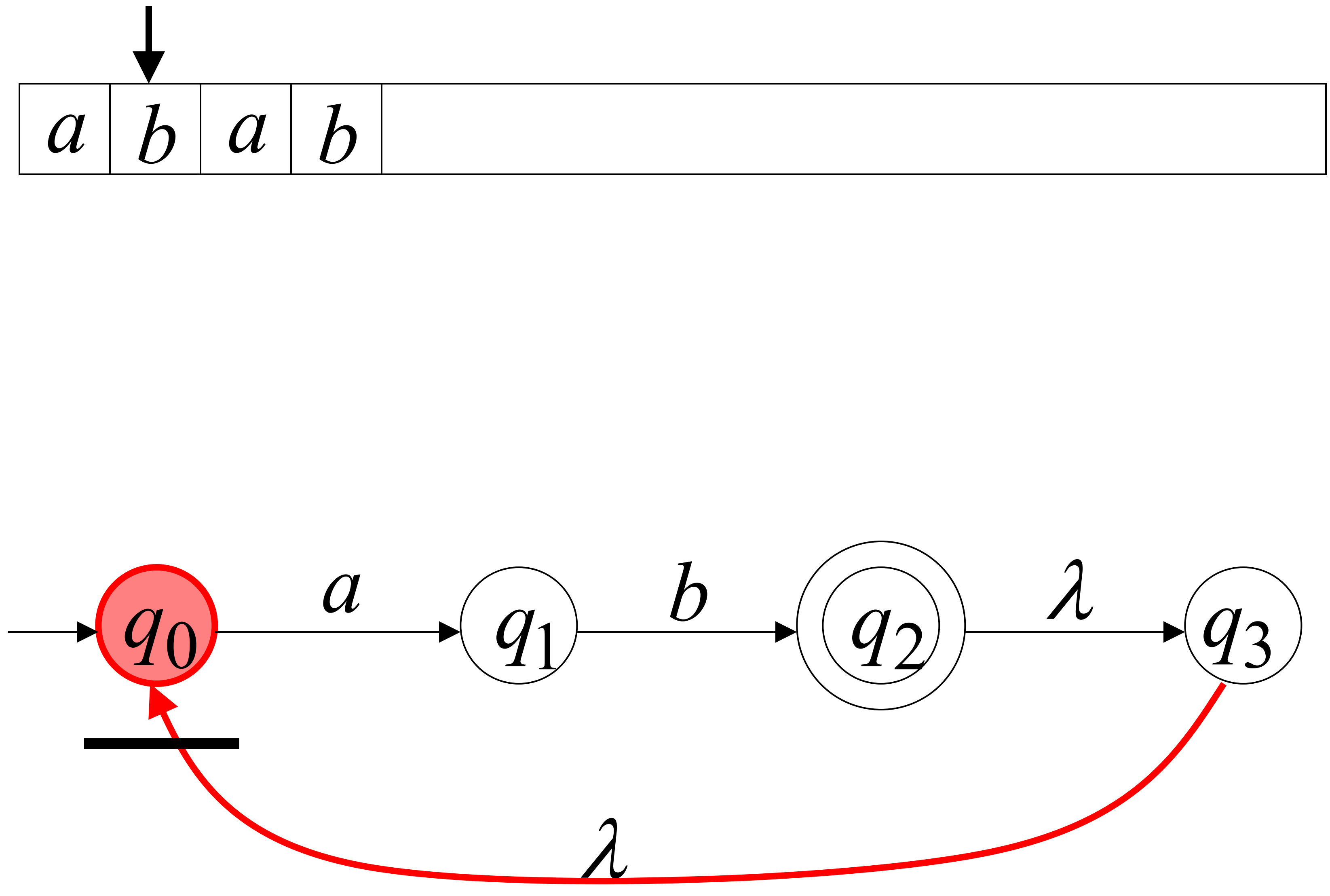

q0
a
q1
b
q2
λ
q3
λ



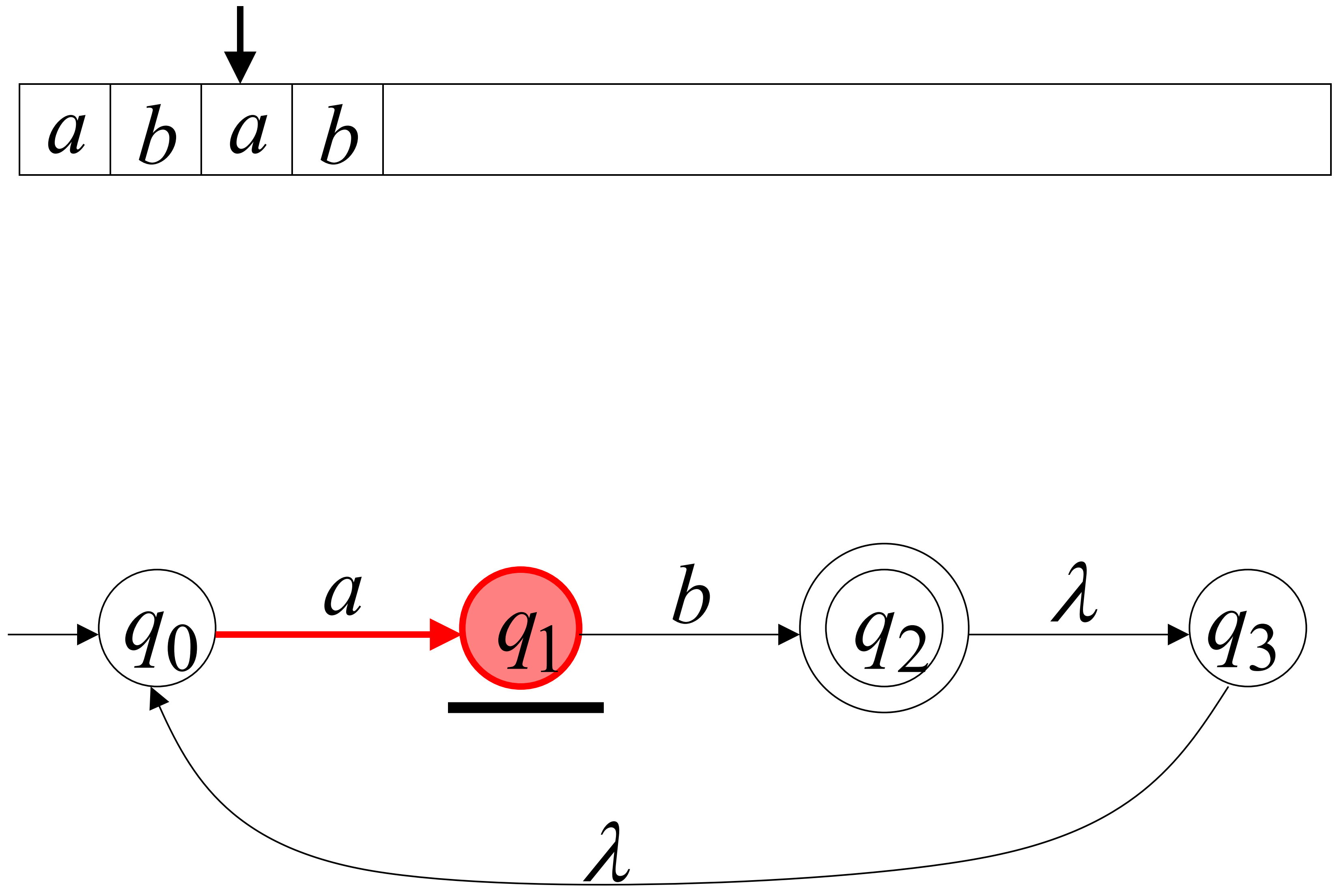
a
b
a
b
q_0
a
q_1
b
q_2
λ
q_3
λ



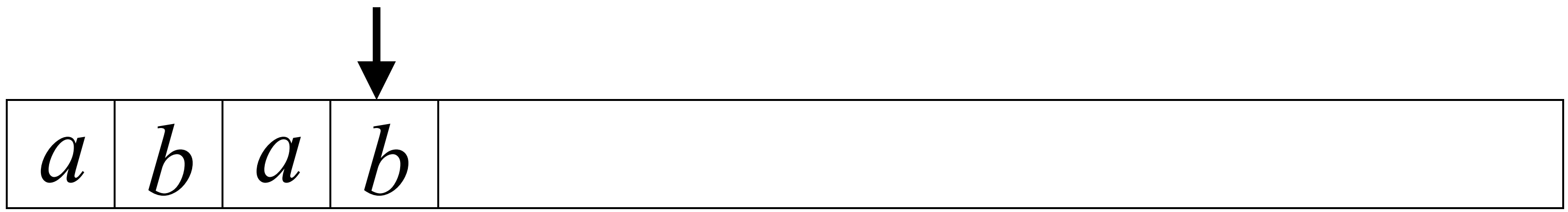
a
b
a
b
a
b
λ
λ
q0
q1
q2
q3

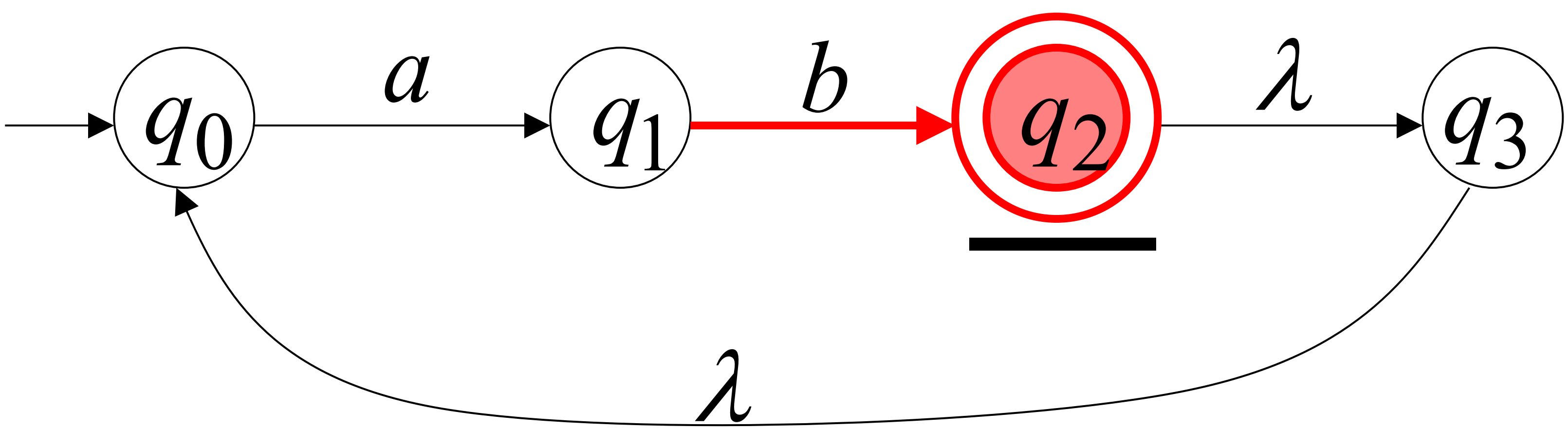


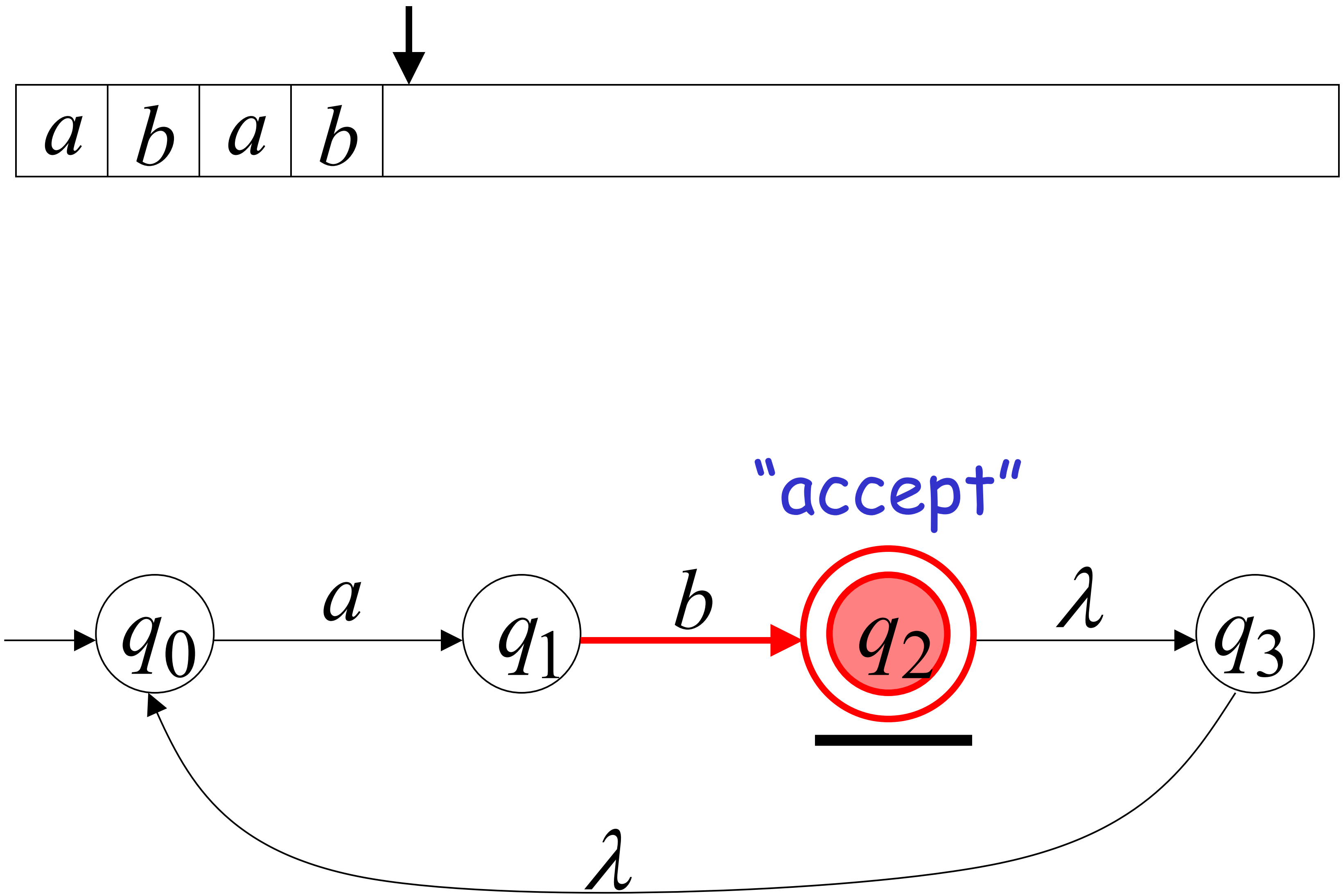
a
b
a
b
a
b
λ
λ
q_0
q_1
q_2
q_3



a
b
a
b
q_0
a
q_1
b
q_2
λ
q_3
λ


a
b
a
b


q_0
a
q_1
b
q_2
λ
q_3
λ

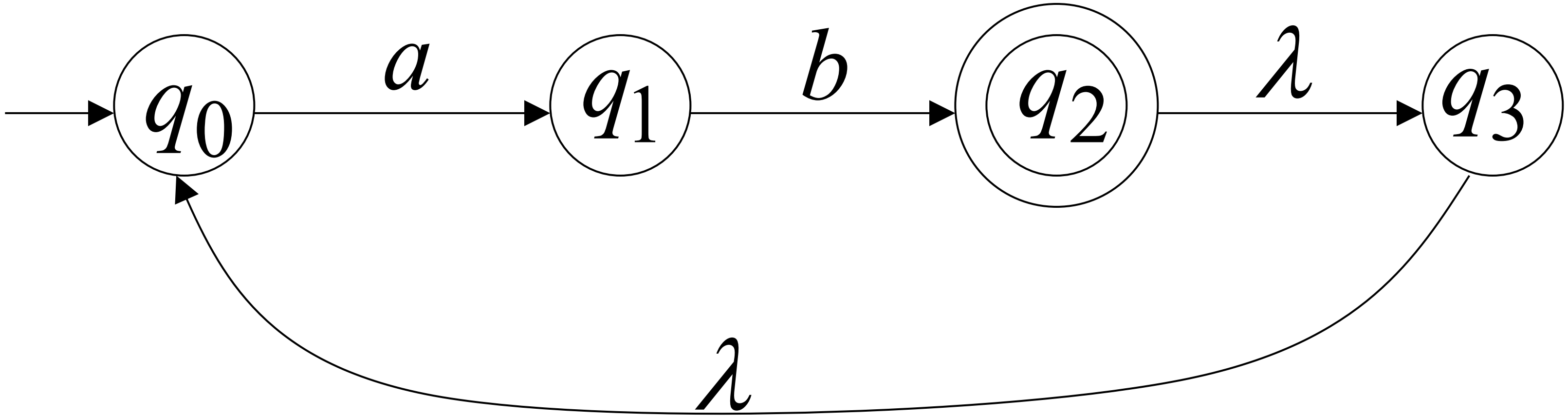


a
b
a
b
"accept"
q0
a
q1
b
q2
λ
q3
λ

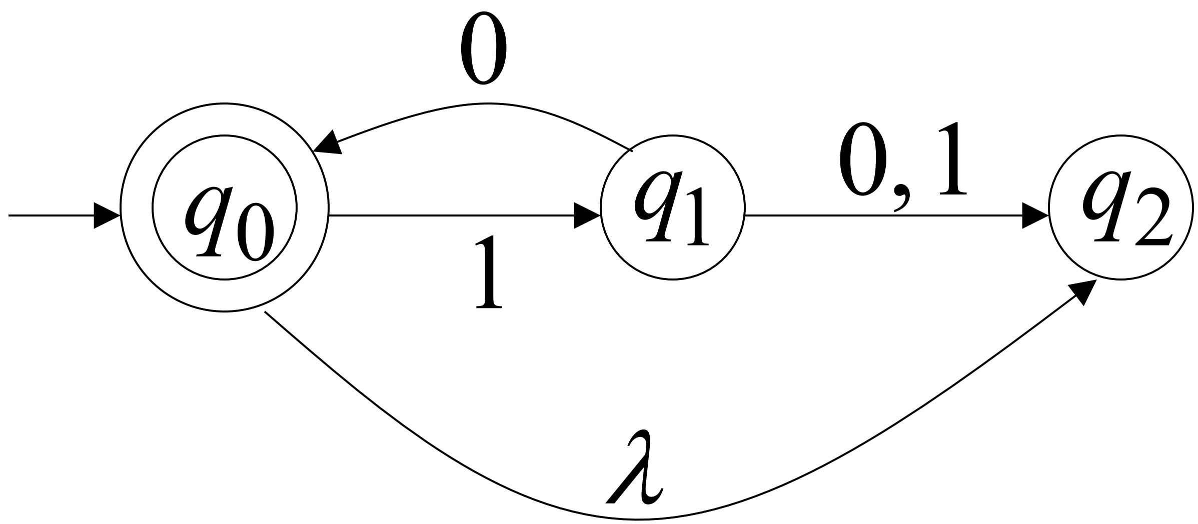
Language accepted

$$L = \{ab,\ abab,\ ababab,\ \ldots\}$$

$$= \{ab\}^+$$

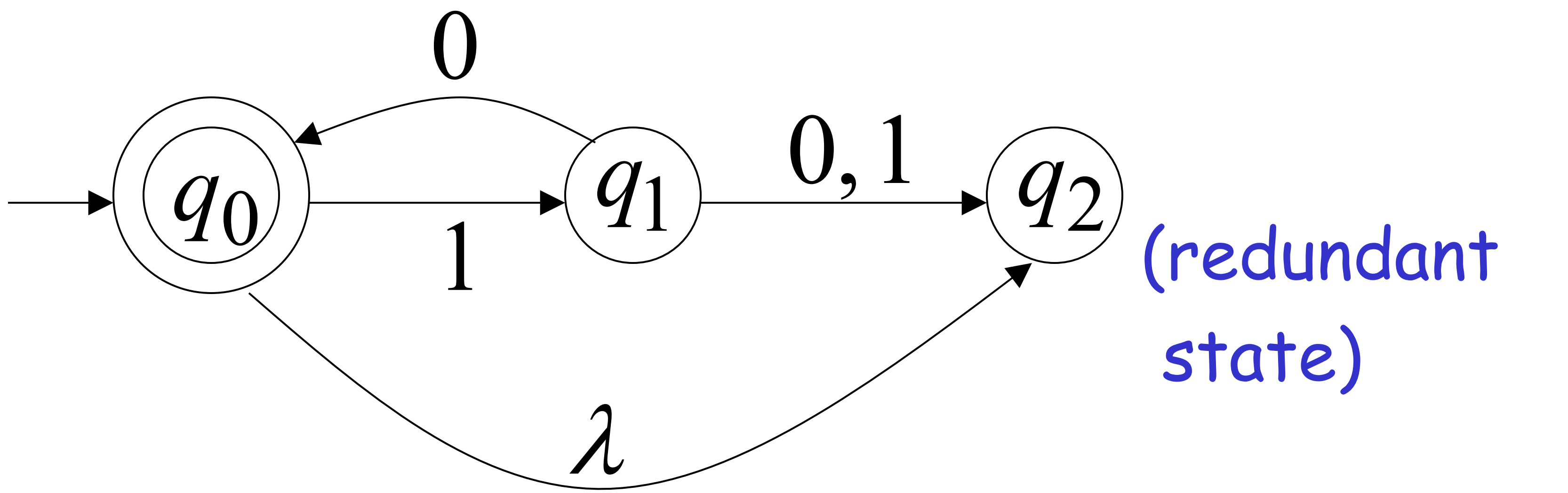


Another NFA Example



Language accepted

$$L(M) = \{\lambda,\ 10,\ 1010,\ 101010,\ \ldots\}$$
$$= \{10\}^*$$

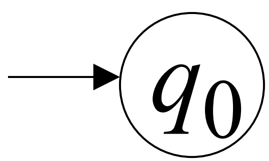


Remarks:

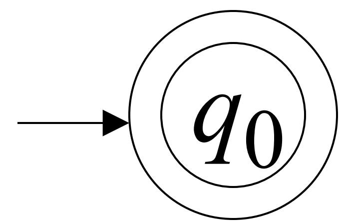
- The λ symbol never appears on the input tape

- Simple automata:

M_1



M_2

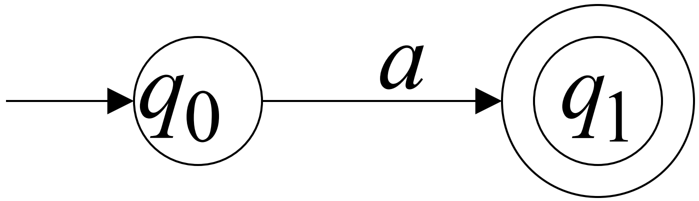


$L(M_1) = \{\}$

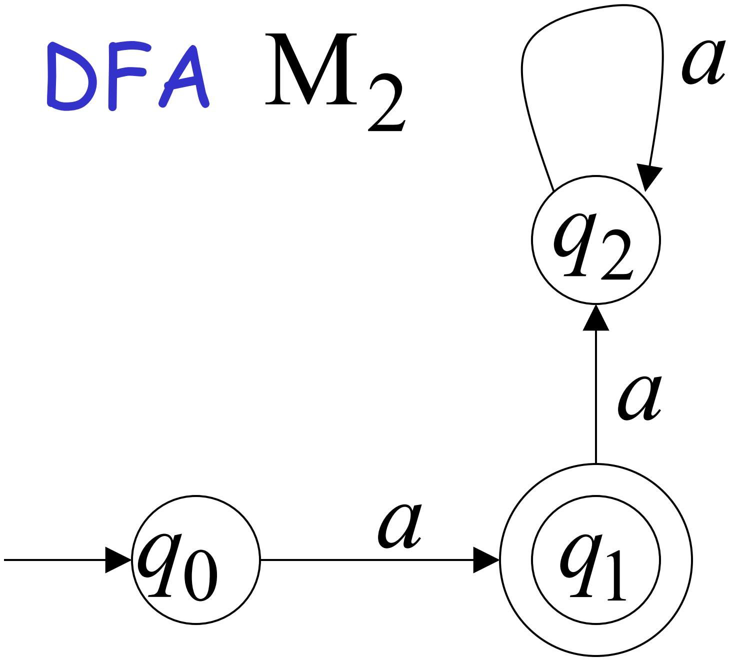
$L(M_2) = \{\lambda\}$

• NFAs are interesting because we can express languages easier than DFAs

NFA M_1



DFA M_2



$$L(M_1) = \{a\}$$

$$L(M_2) = \{a\}$$

Formal Definition of NFAs

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

$Q:$ Set of states, i.e. $\{q_0, q_1, q_2\}$

$\Sigma:$ Input aplhabet, i.e. $\{a, b\}$

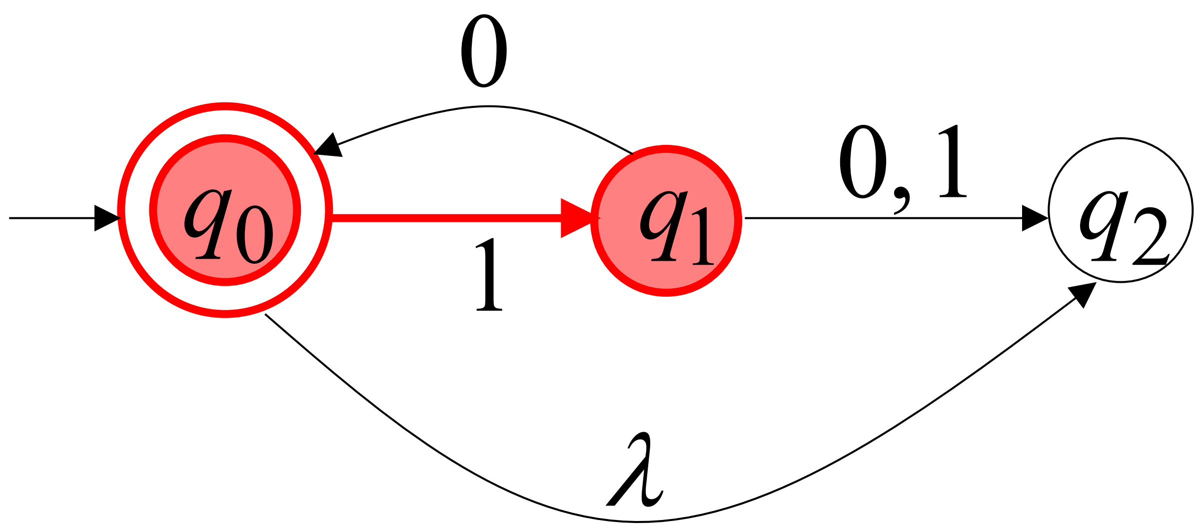
$\delta:$ Transition function

$q_0:$ Initial state

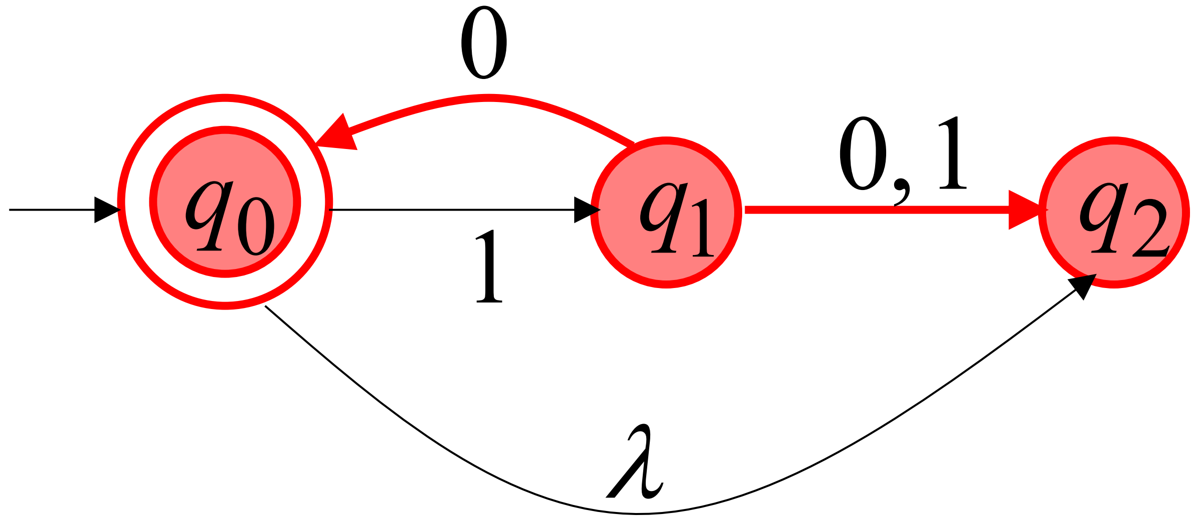
$F:$ Final states

Transition Function δ

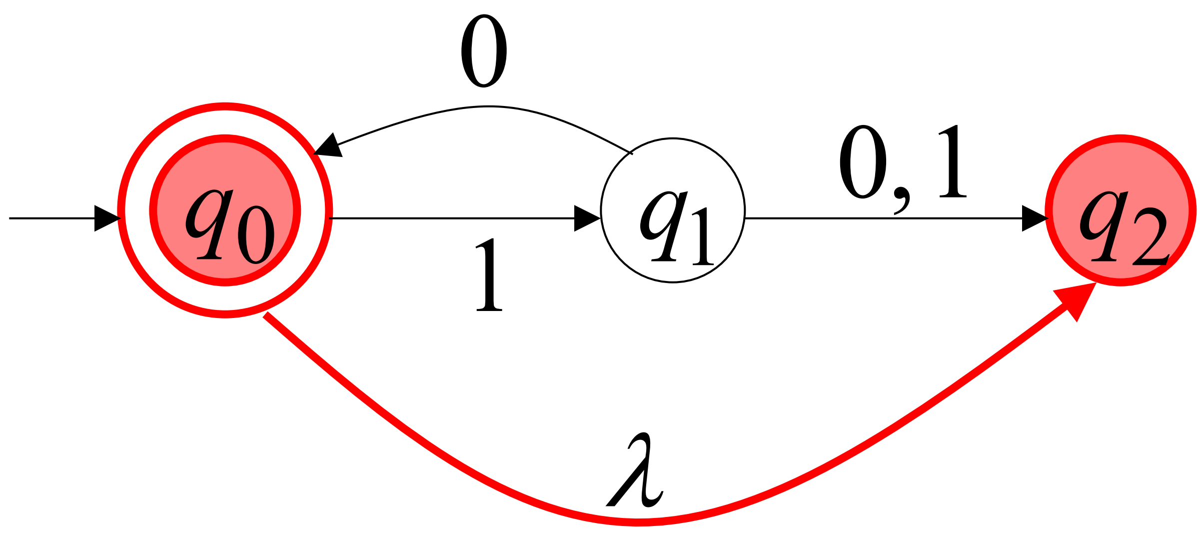
$$\delta(q_0, 1) = \{q_1\}$$



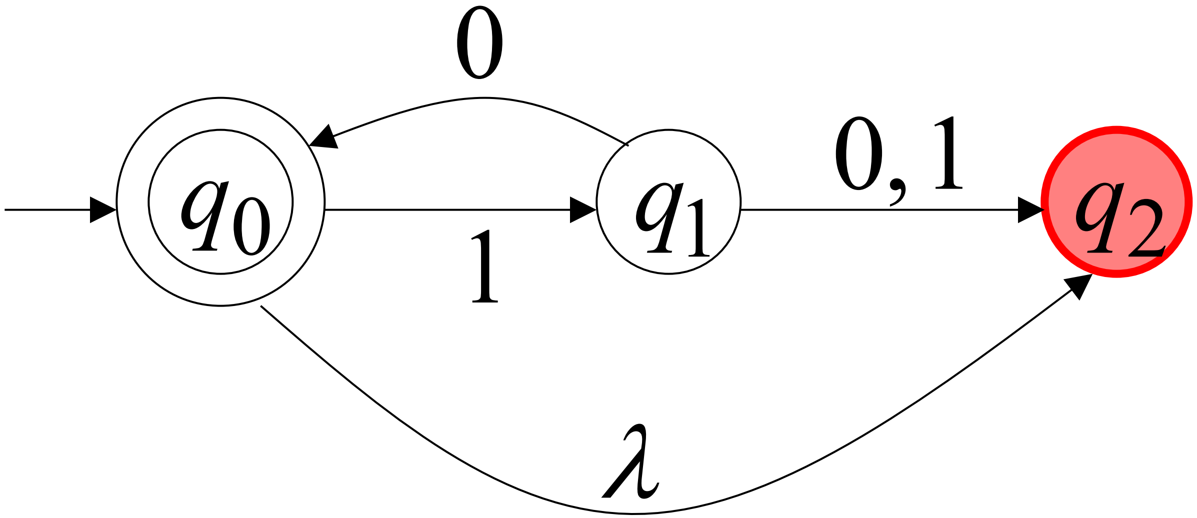
$$\delta(q_1, 0) = \{q_0, q_2\}$$



$$\delta(q_0, \lambda) = \{q_0, q_2\}$$

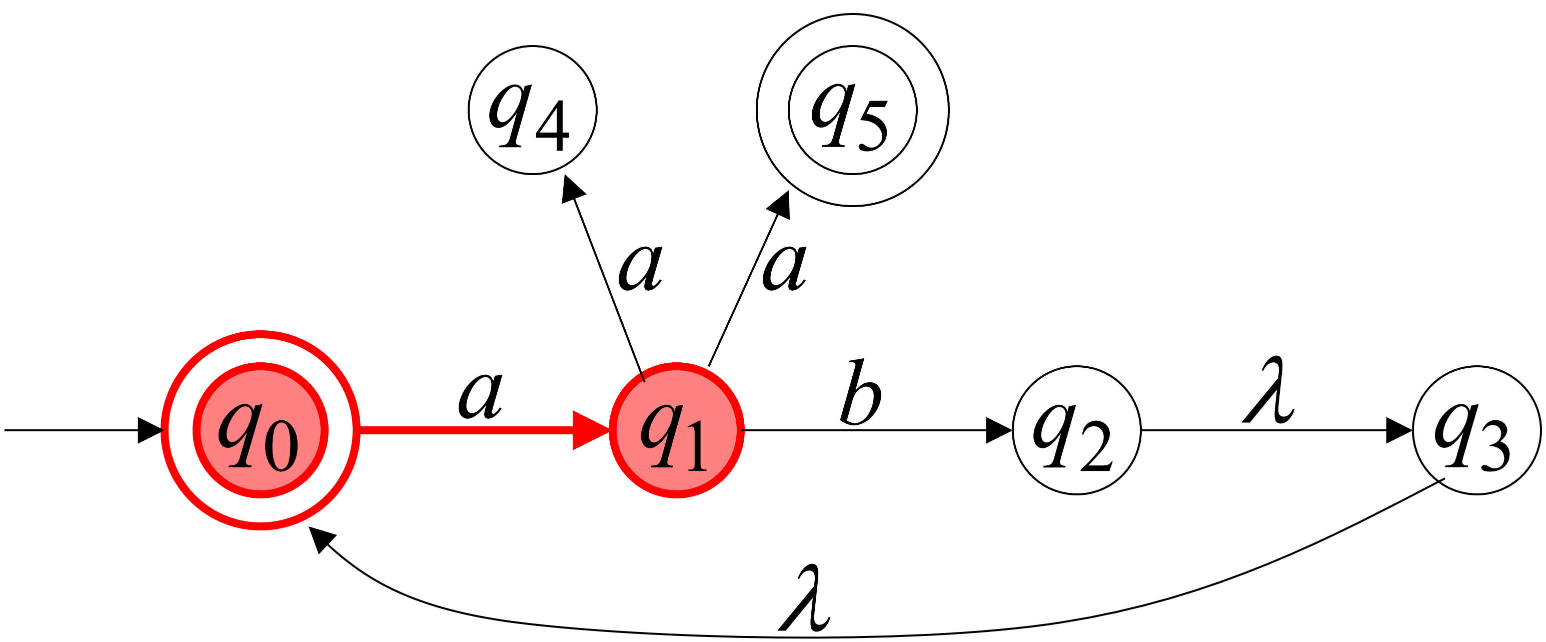


$$\delta(q_2, 1) = \varnothing$$

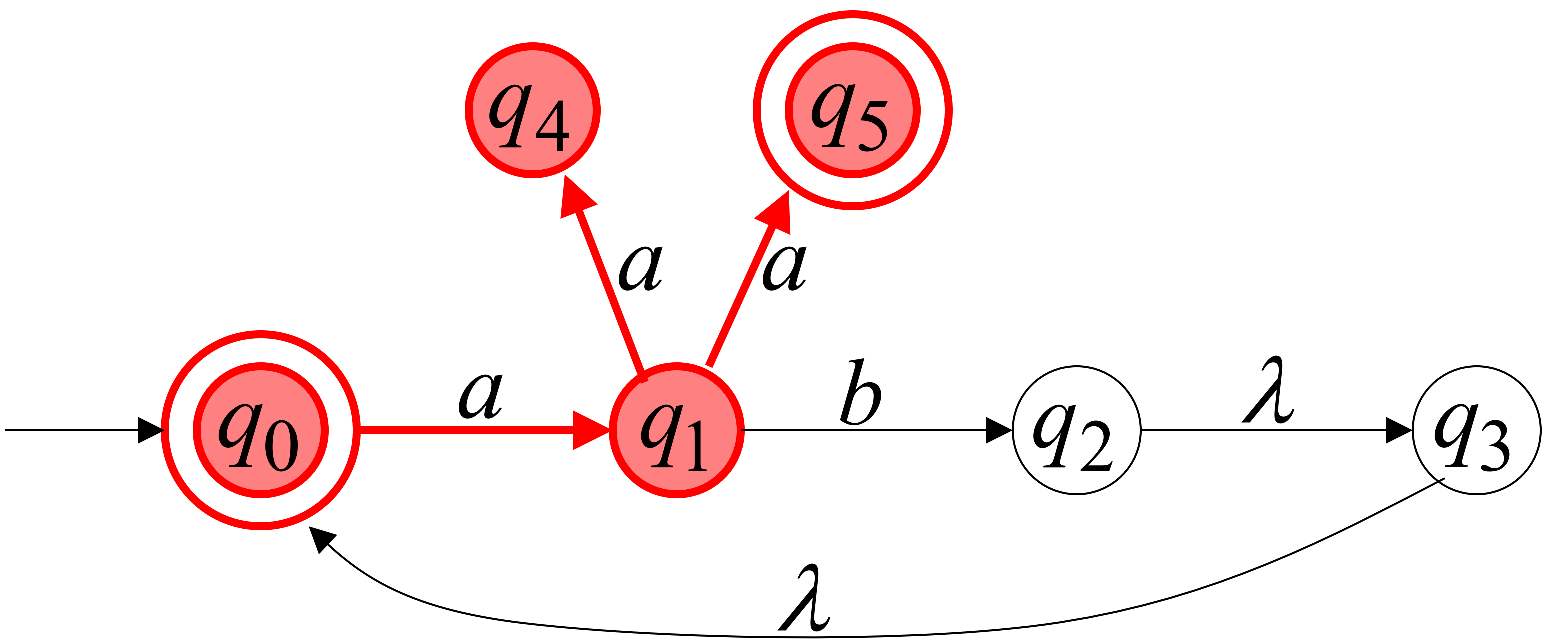


Extended Transition Function δ^*

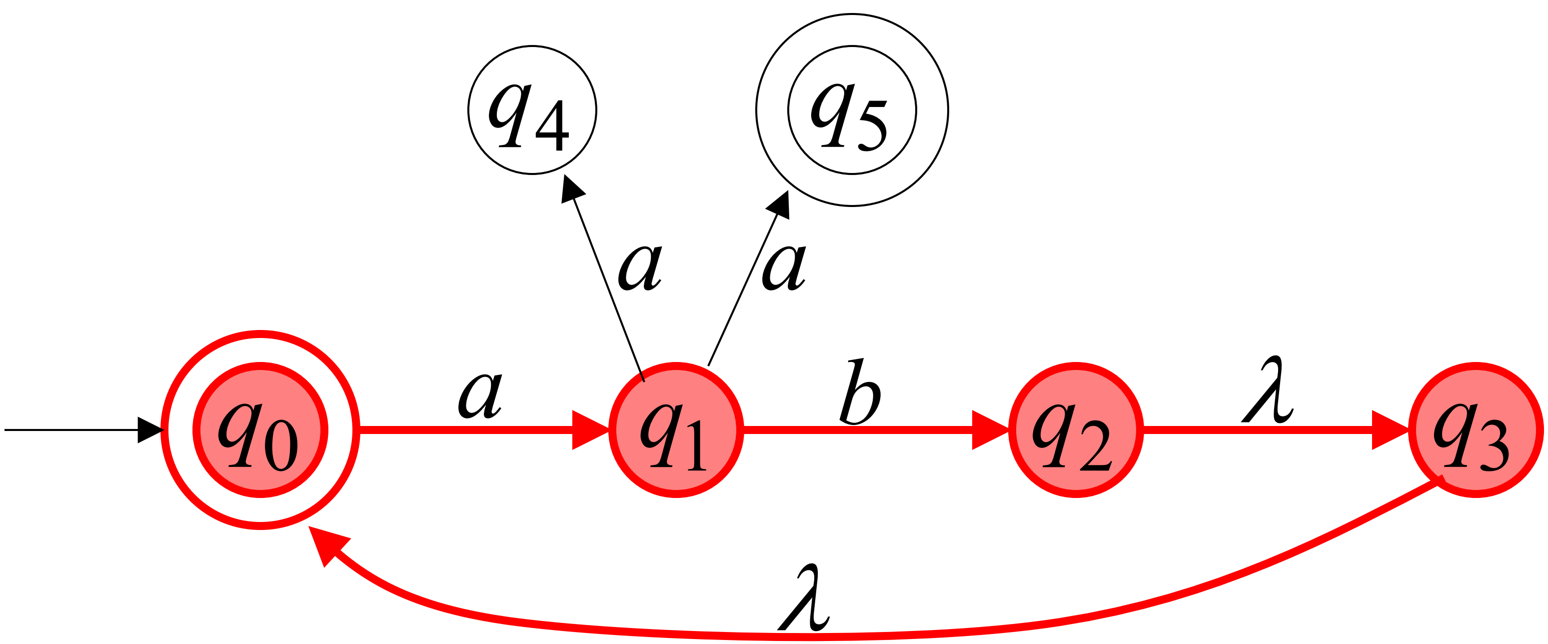
$$\delta^*(q_0, a) = \{q_1\}$$



$$\delta^*(q_0, aa) = \{q_4, q_5\}$$

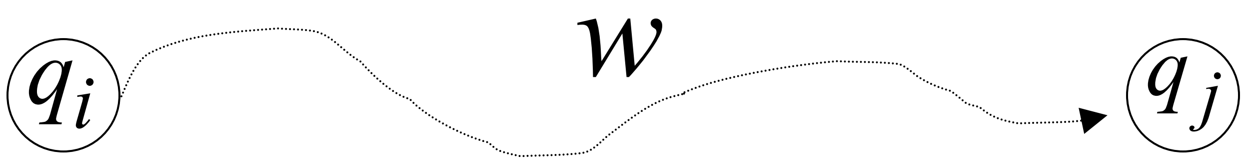


$$\delta^*(q_0, ab) = \{q_2, q_3, q_0\}$$

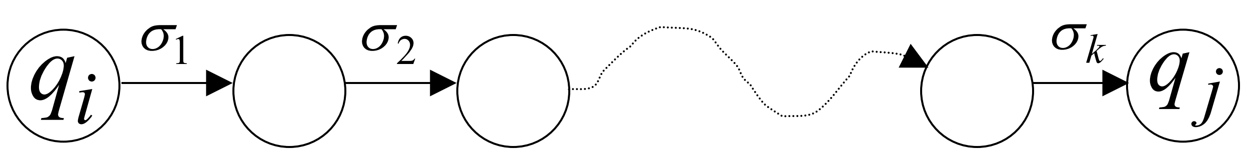


Formally

$q_j \in \delta^*(q_i, w)$: there is a walk from q_i to q_j with label w

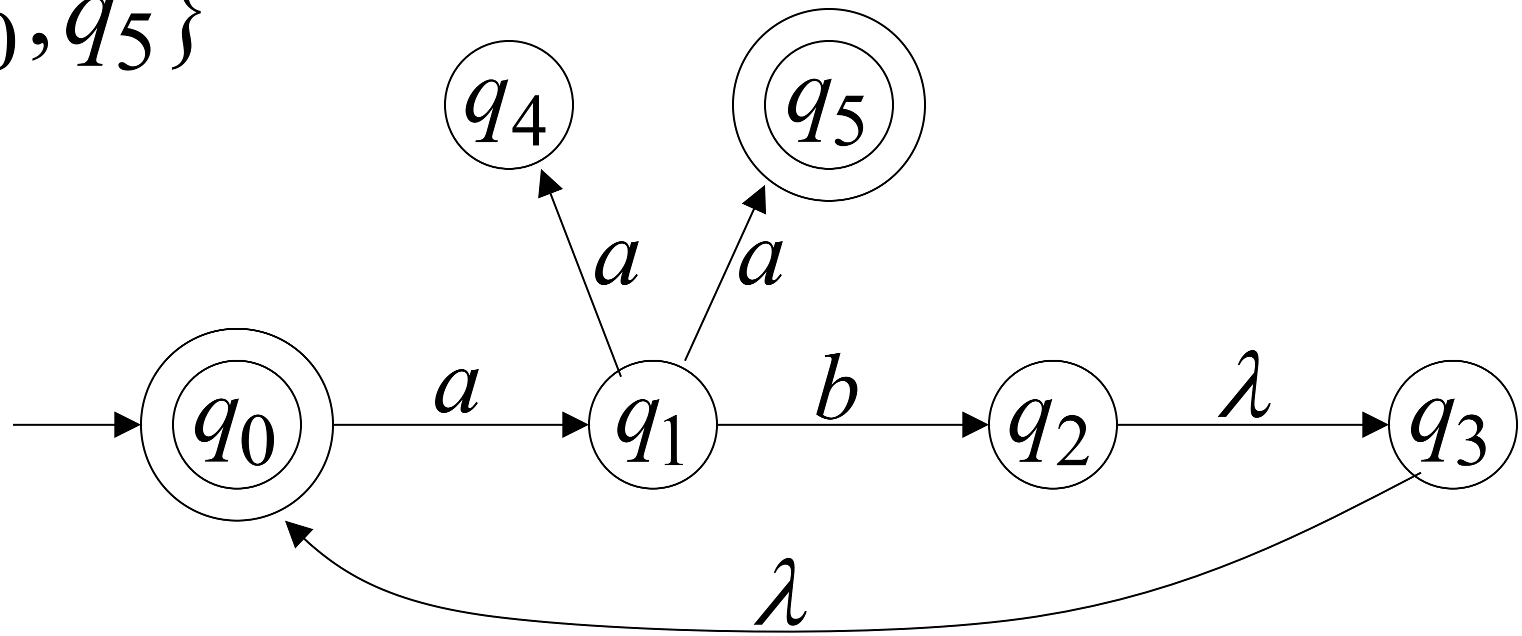


$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$



The Language of an NFA M

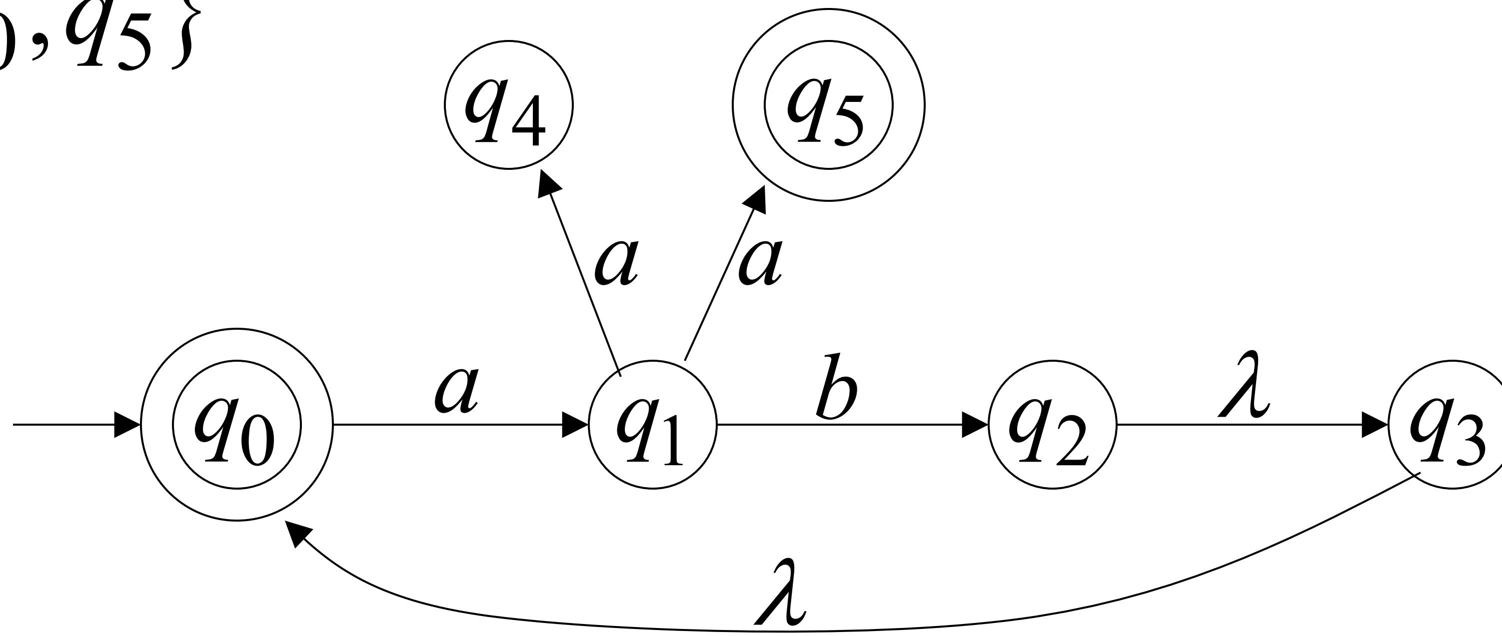
$F = \{q_0, q_5\}$



$\delta^*(q_0, aa) = \{q_4, \underline{q_5}\}$ $\quad aa \in L(M)$

$\searrow \in F$

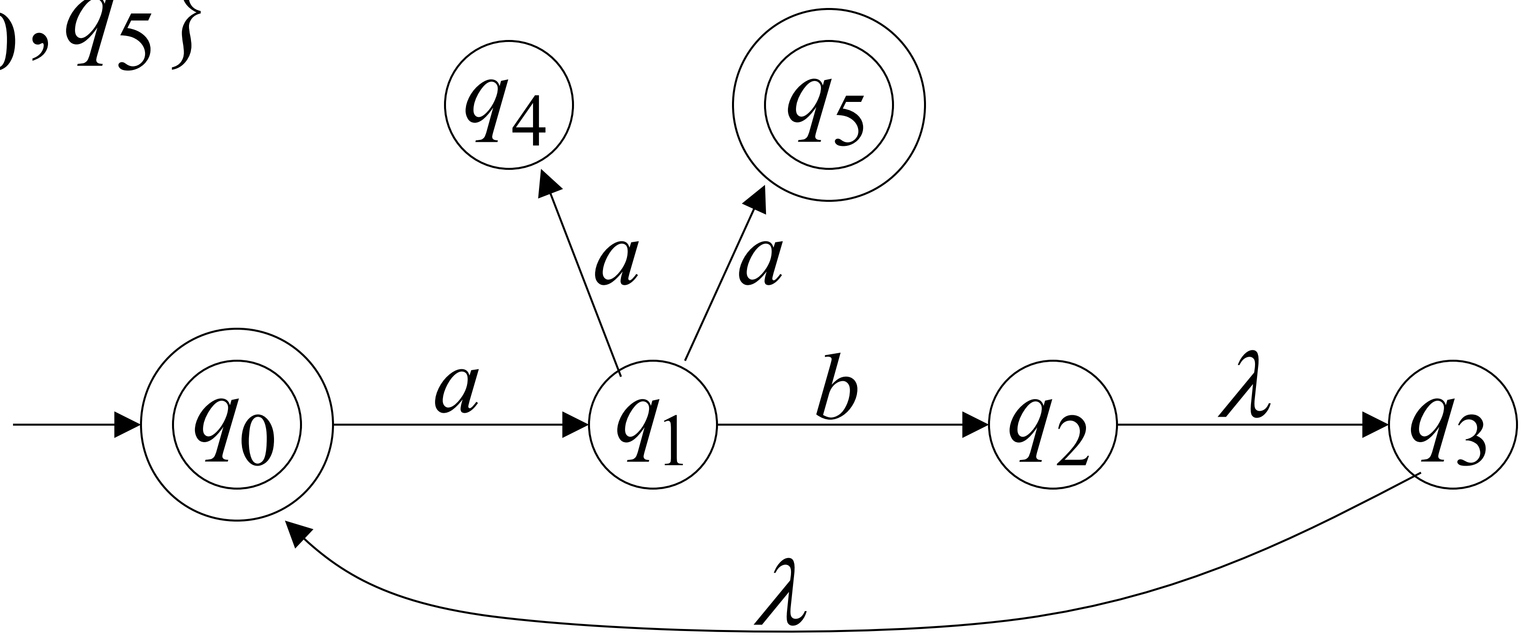
$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, ab) = \{q_2, q_3, \underline{q_0}\} \qquad ab \in L(M)$$

$$\searrow \in F$$

$$F = \{q_0, q_5\}$$

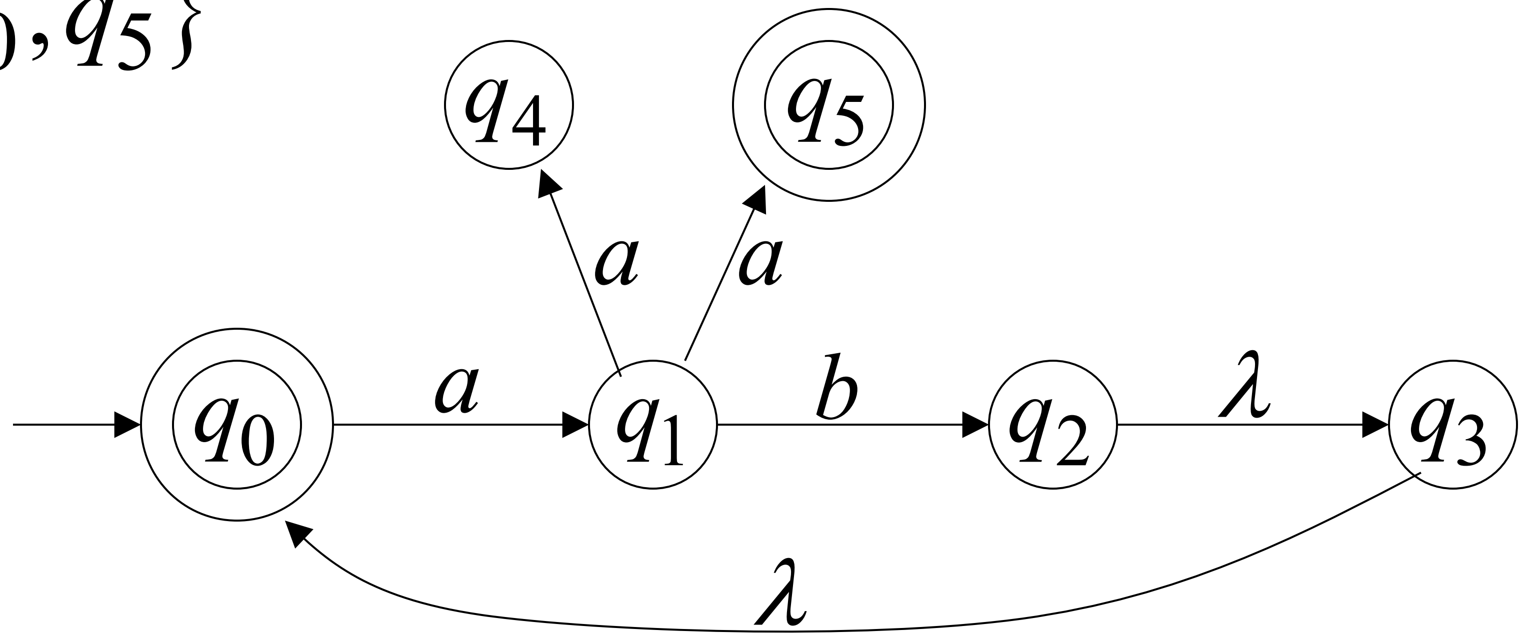


$$\delta^*(q_0, abaa) = \{q_4, \underline{q_5}\}$$

$\rightarrow \in F$

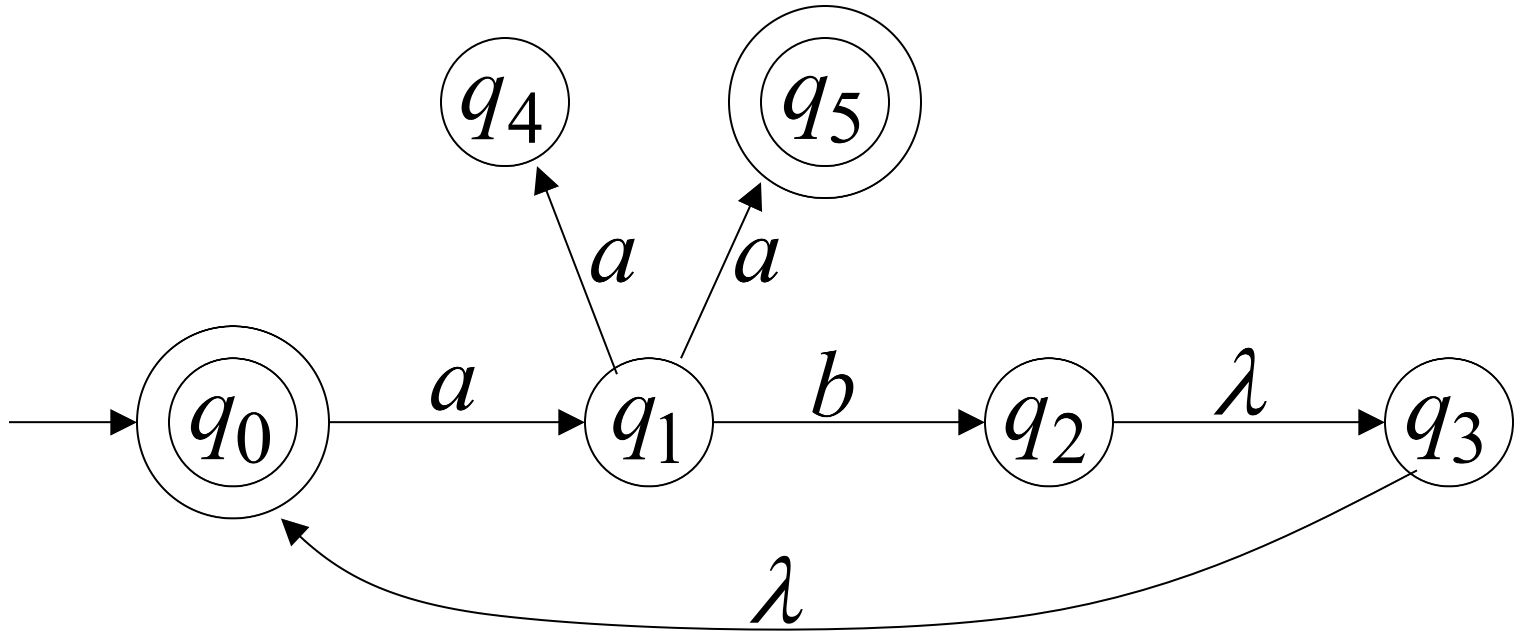
$$aaba \in L(M)$$

$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, aba) = \{q_1\} \qquad aba \notin L(M)$$

$$\notin F$$



$$L(M) = \{\lambda\} \cup \{ab\}^* \{aa\}$$

Formally

The language accepted by NFA M is:

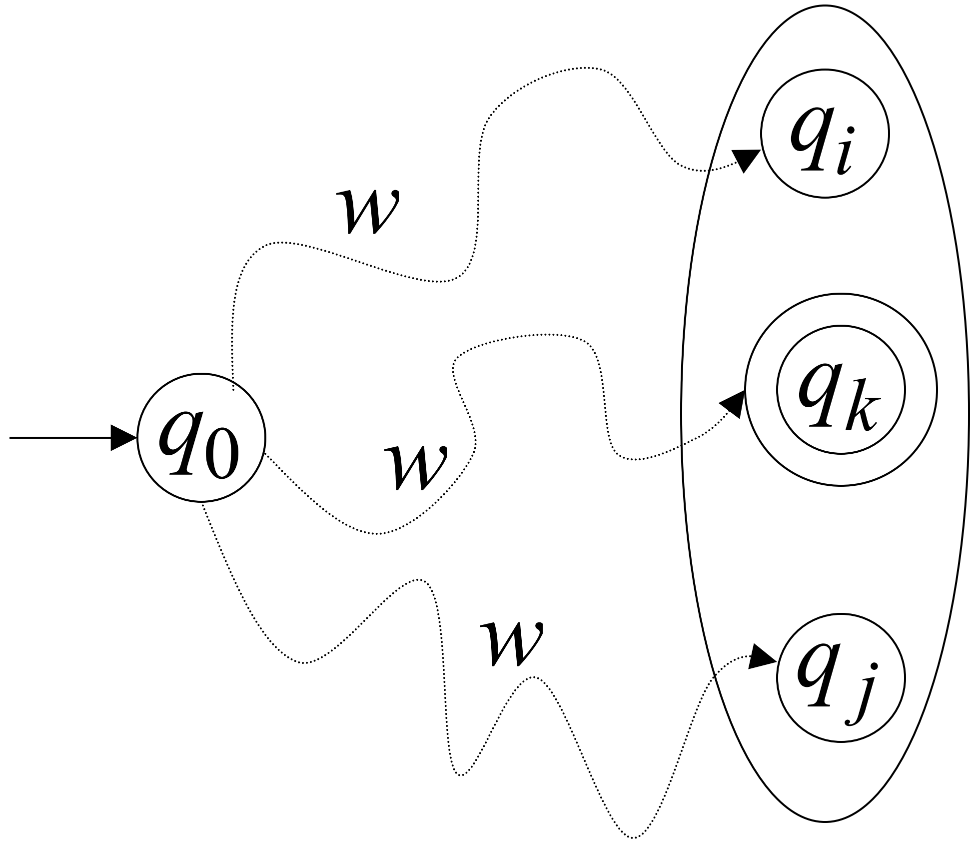
$$L(M) = \{w_1, w_2, w_3, \ldots\}$$

where $\delta^*(q_0, w_m) = \{q_i, q_j, \ldots, q_k, \ldots\}$

and there is some $q_k \in F$ (final state)

$w \in L(M)$

$\delta^*(q_0, w)$



$q_k \in F$

NFAs accept the Regular Languages

Equivalence of Machines

Definition for Automata:

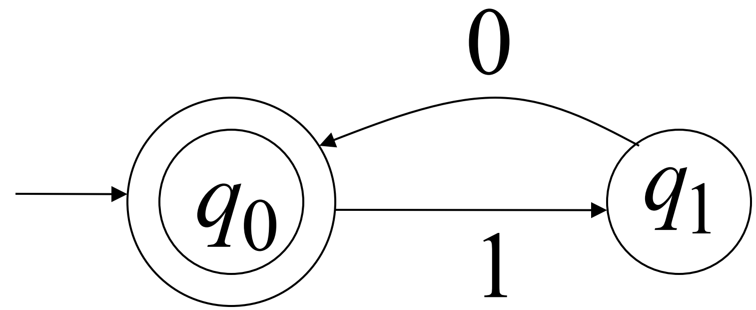
Machine M_1 is equivalent to machine M_2

if $L(M_1) = L(M_2)$

Example of equivalent machines

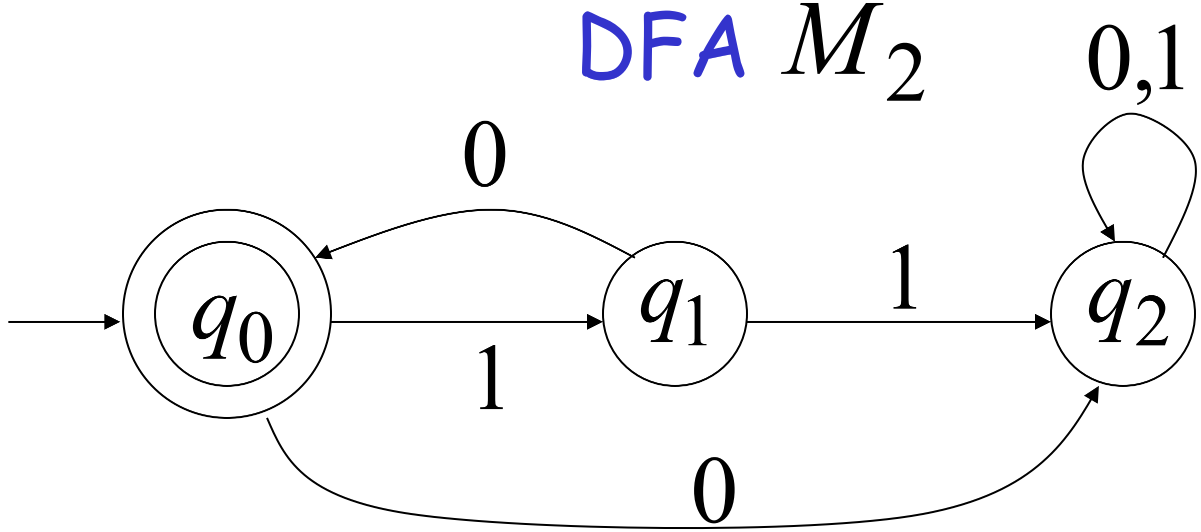
$$L(M_1) = \{10\}^*$$

NFA M_1



$$L(M_2) = \{10\}^*$$

DFA M_2



We will prove:

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Languages
accepted
by DFAs

NFAs and DFAs have the same computation power

Step 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Proof: Every DFA is trivially an NFA

$\Downarrow$

Any language L accepted by a DFA is also accepted by an NFA

Step 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

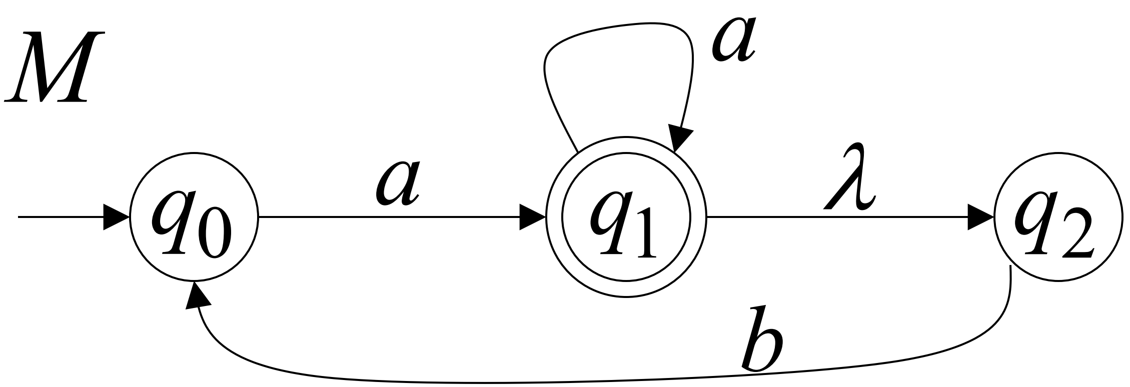
Proof: Any NFA can be converted to an equivalent DFA

$$\Downarrow$$

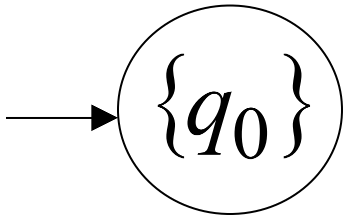
Any language L accepted by an NFA is also accepted by a DFA

Convert NFA to DFA

NFA M

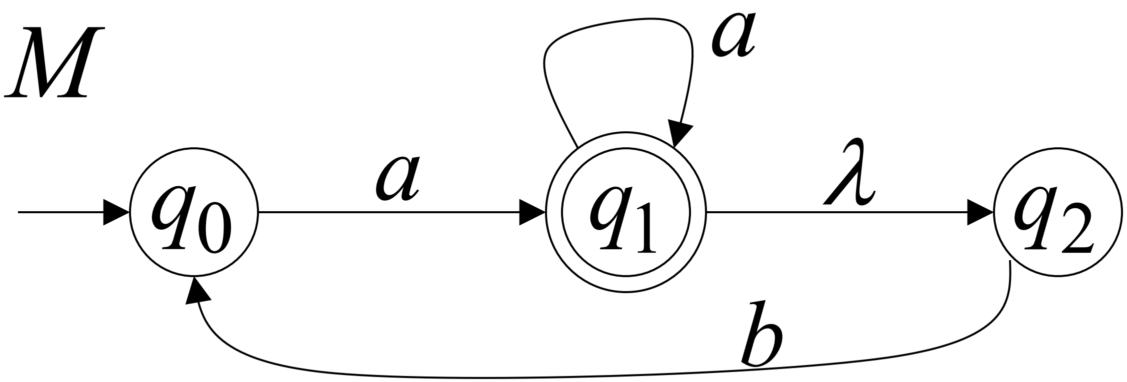


DFA M'

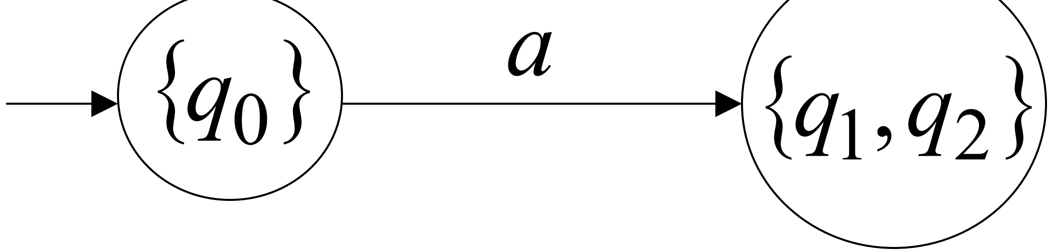


Convert NFA to DFA

NFA M

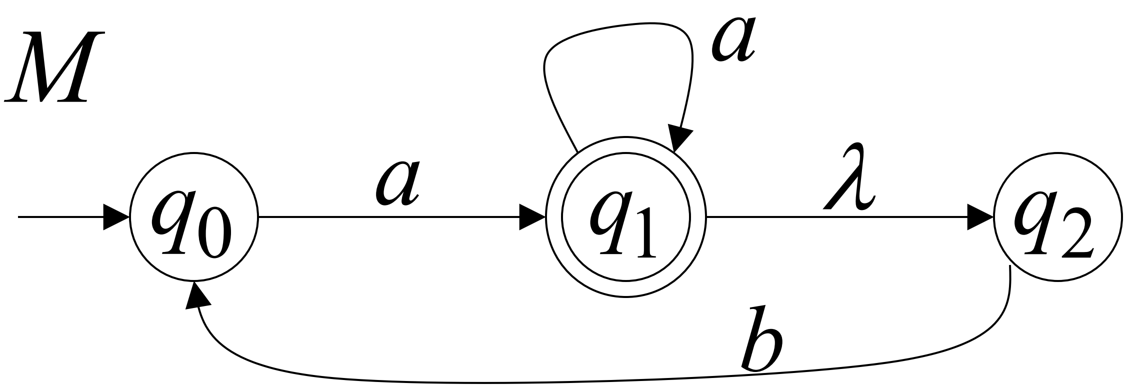


DFA M'

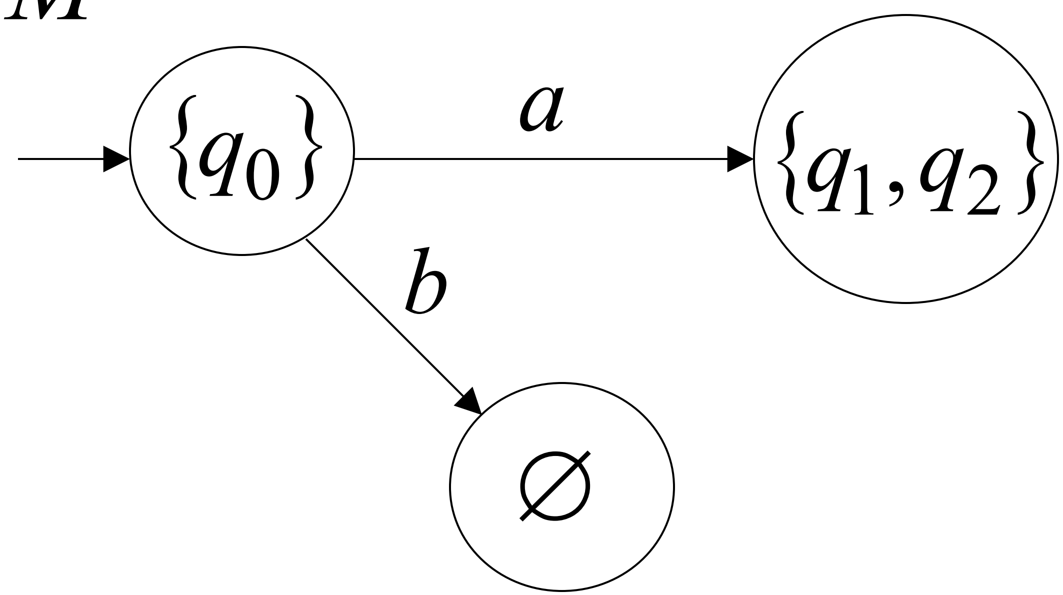


Convert NFA to DFA

NFA M

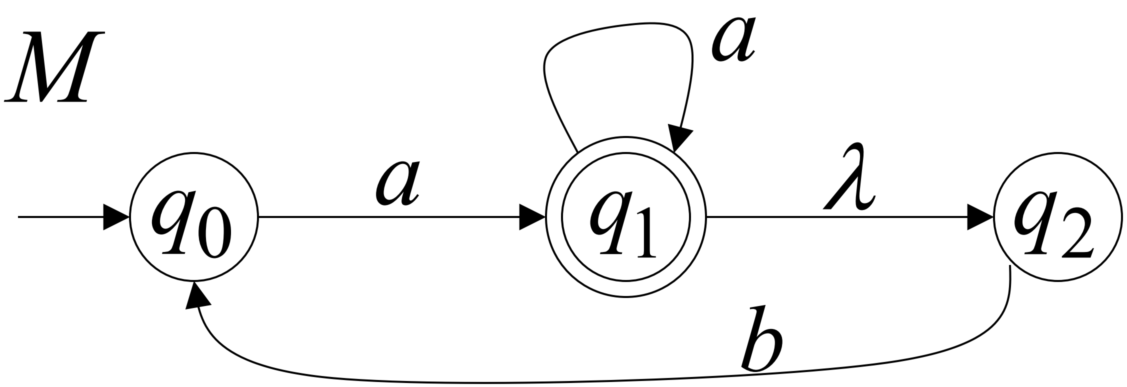


DFA M'

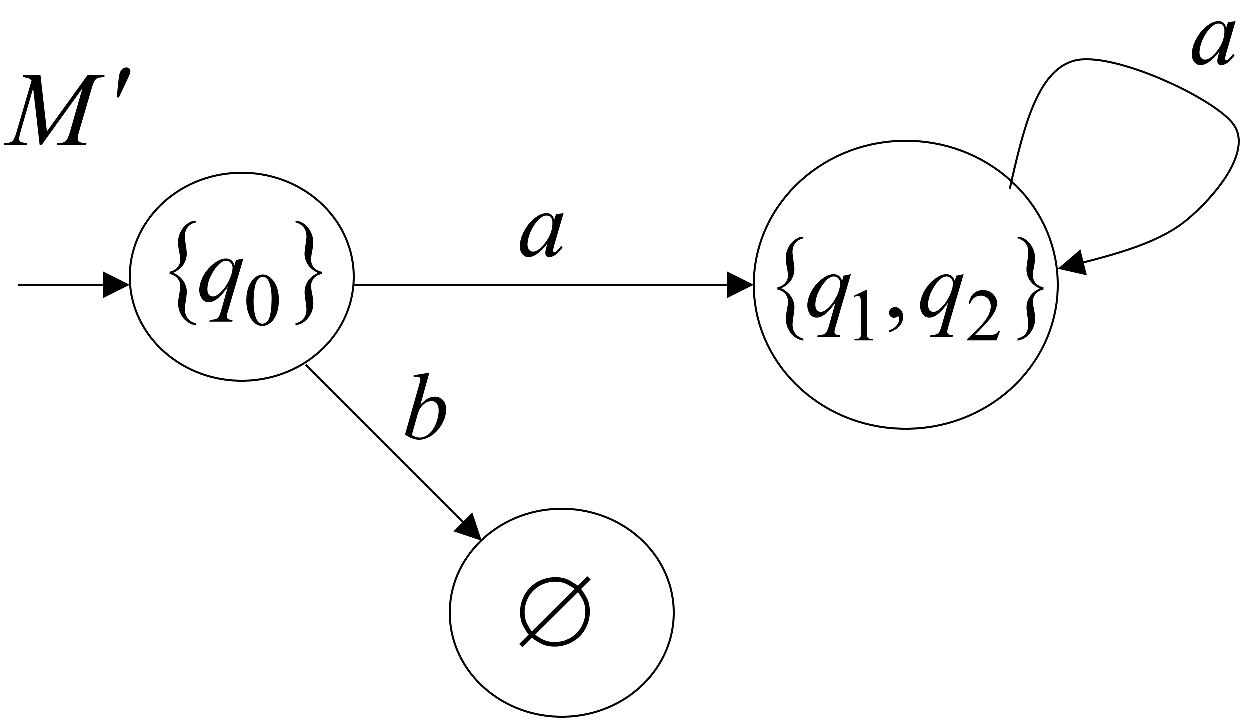


Convert NFA to DFA

NFA M

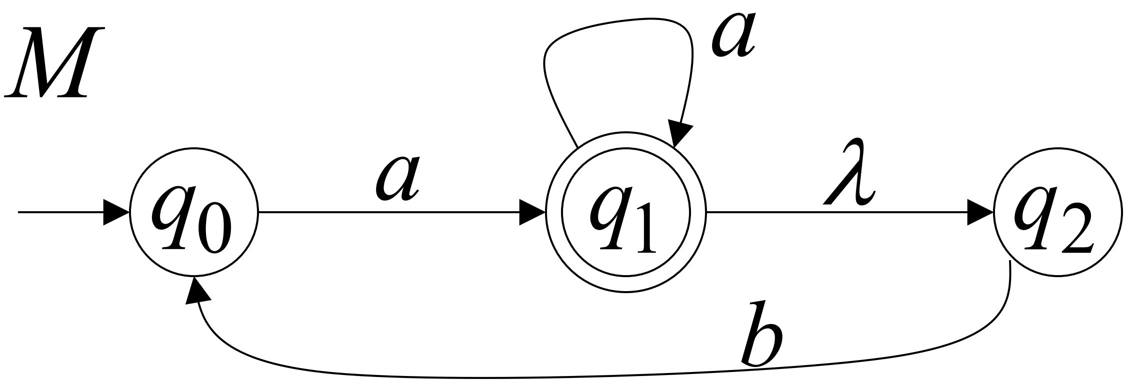


DFA M'

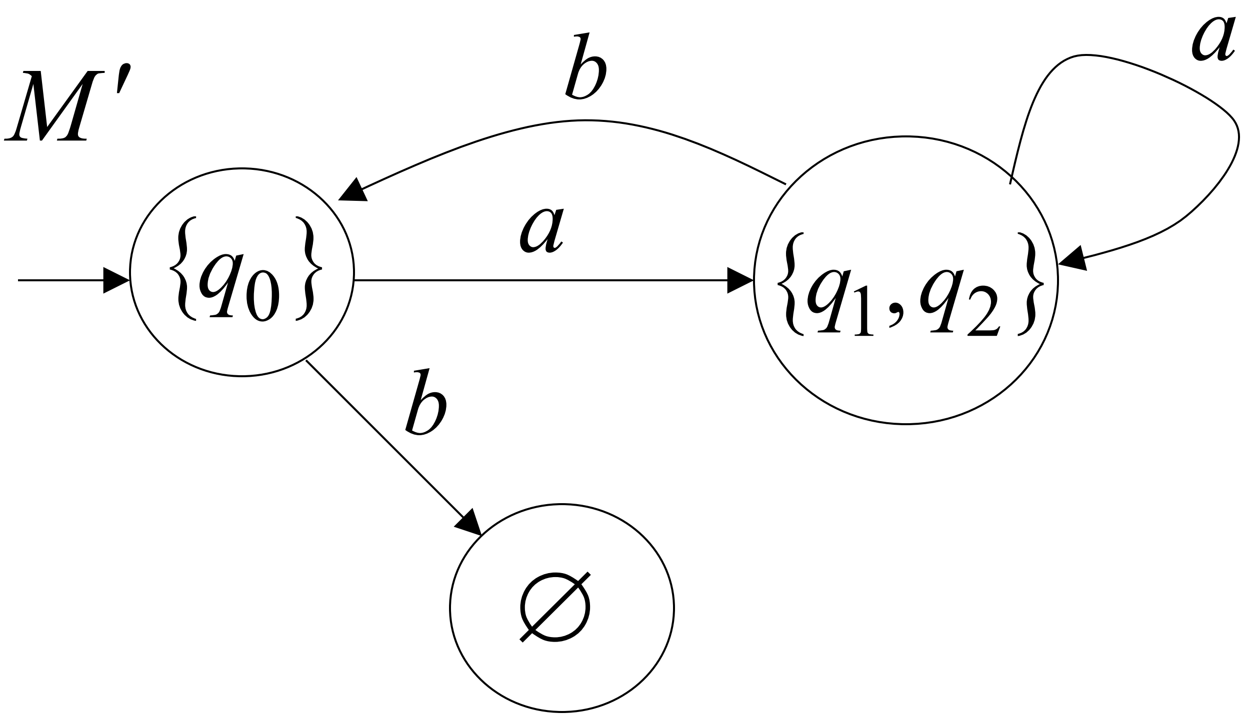


Convert NFA to DFA

NFA M

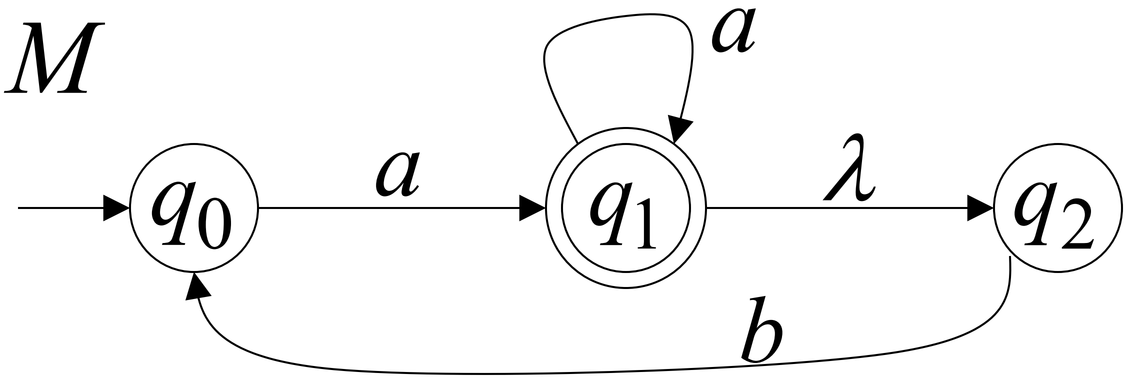


DFA M'

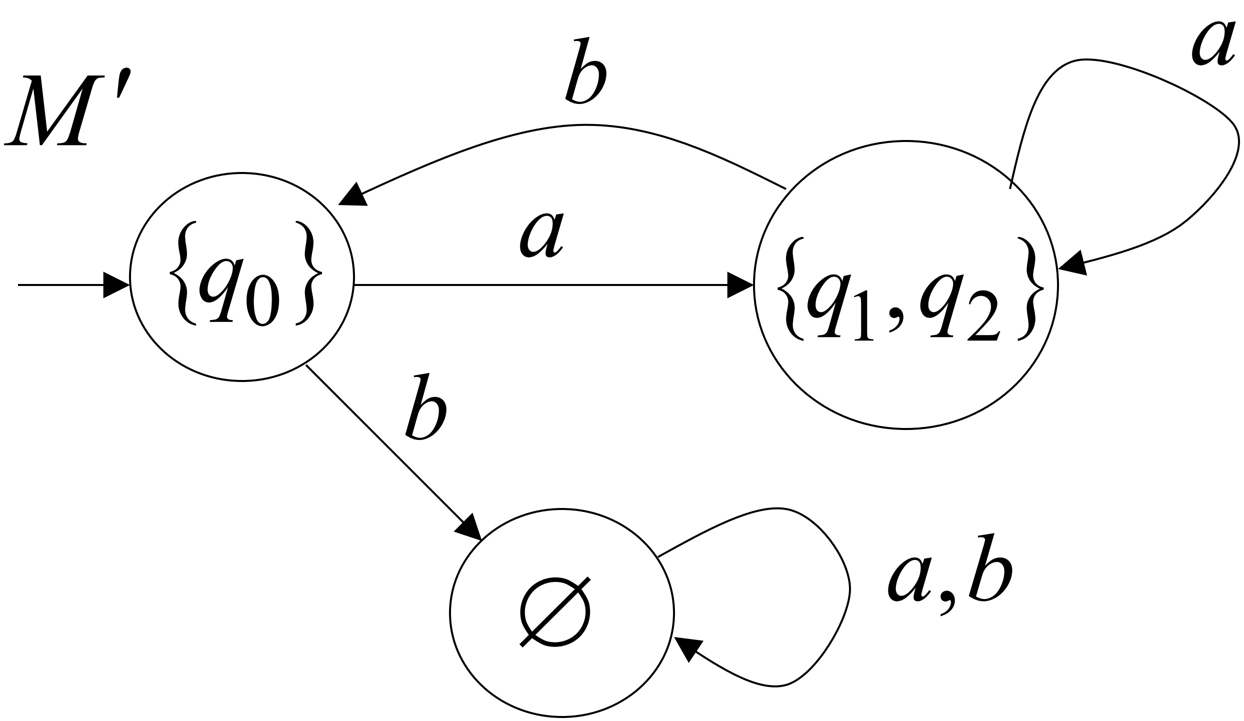


Convert NFA to DFA

NFA M

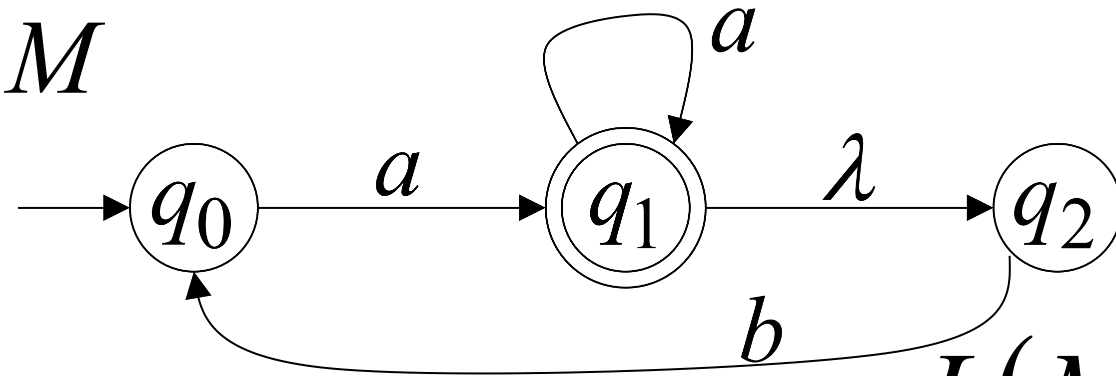


DFA M'



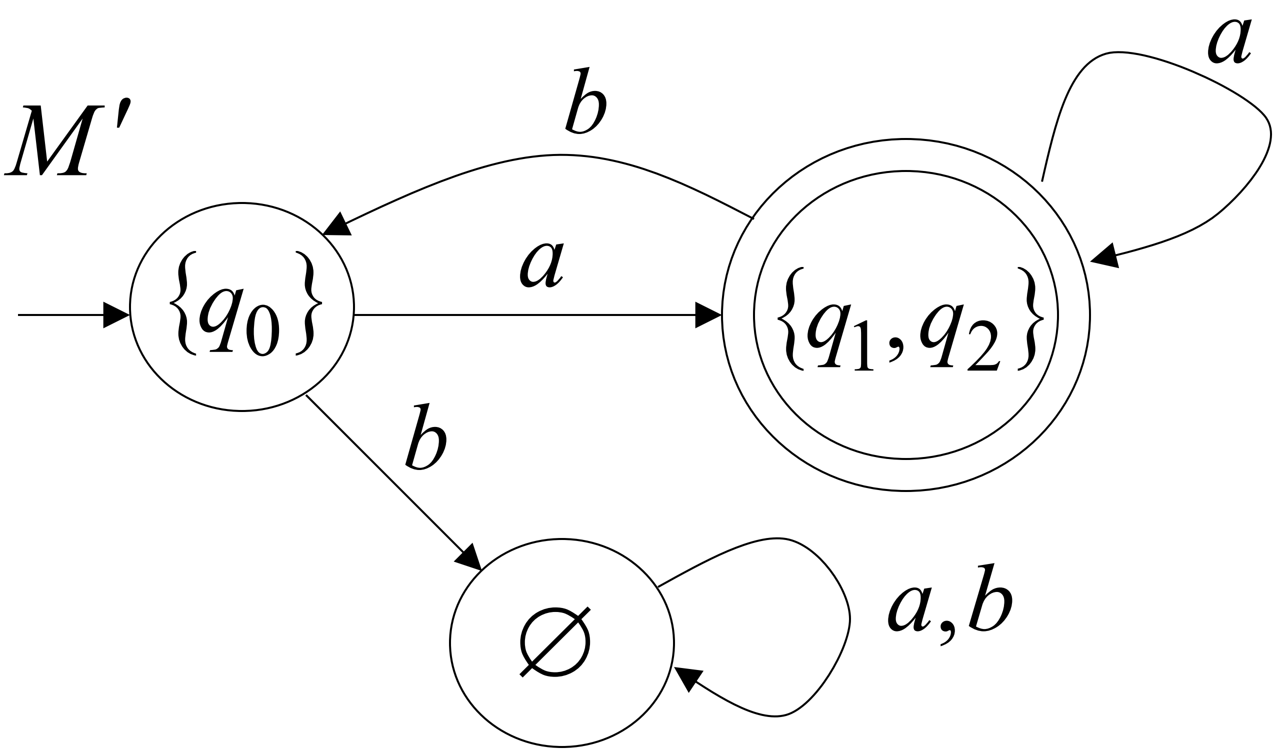
Convert NFA to DFA

NFA M



$$L(M) = L(M')$$

DFA M'



NFA to DFA: Remarks

We are given an NFA M

We want to convert it to an equivalent DFA M'

With $L(M) = L(M')$

If the NFA has states

$$q_0, q_1, q_2, \ldots$$

the DFA has states in the powerset

$$\emptyset, \{q_0\}, \{q_1\}, \{q_1, q_2\}, \{q_3, q_4, q_7\}, \ldots$$

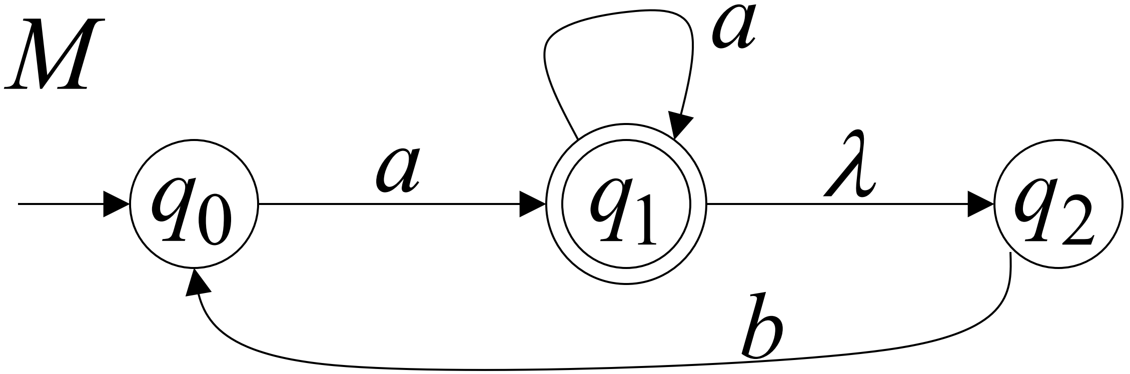
Procedure NFA to DFA

1. Initial state of NFA: q_0

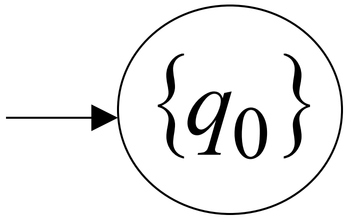
Initial state of DFA: $\{q_0\}$

Example

NFA M



DFA M'



Procedure NFA to DFA

2. For every DFA's state $\{q_i, q_j, \ldots, q_m\}$

Compute in the NFA

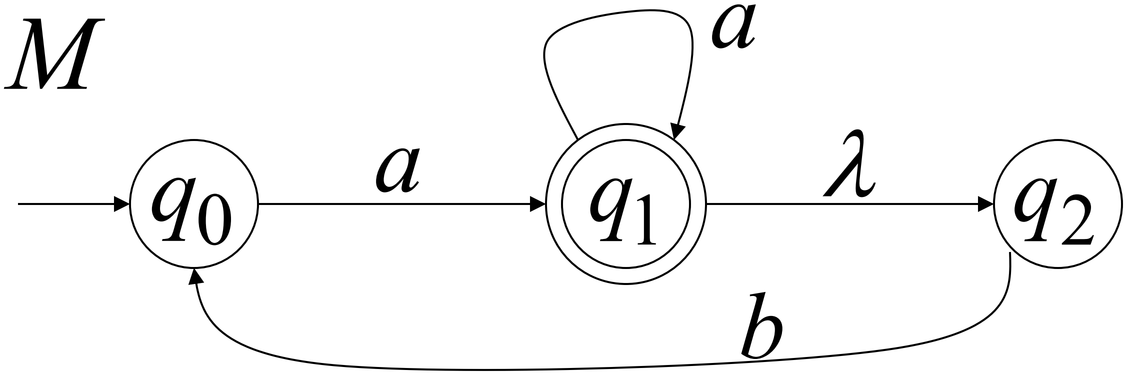
$$\left.\begin{array}{l} \delta^*(q_i, a), \\ \delta^*(q_j, a), \\ \ldots \end{array}\right\} = \{q'_i, q'_j, \ldots, q'_m\}$$

Add transition to DFA

$$\delta(\{q_i, q_j, \ldots, q_m\},\ a) = \{q'_i, q'_j, \ldots, q'_m\}$$

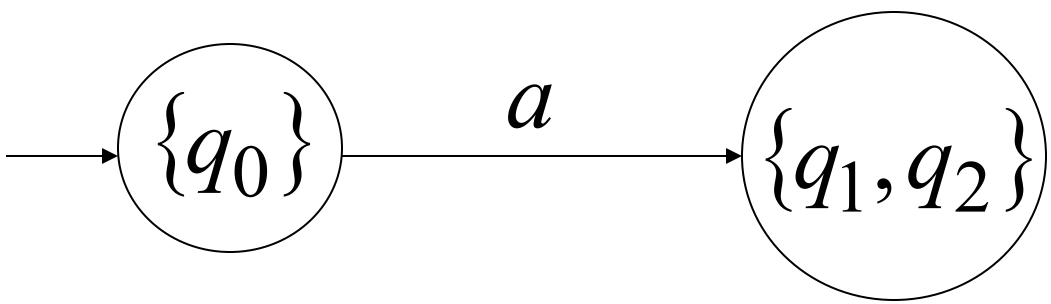
Exampe

NFA M



$$\delta^*(q_0, a) = \{q_1, q_2\}$$

DFA M'



$$\delta(\{q_0\}, a) = \{q_1, q_2\}$$

Procedure NFA to DFA

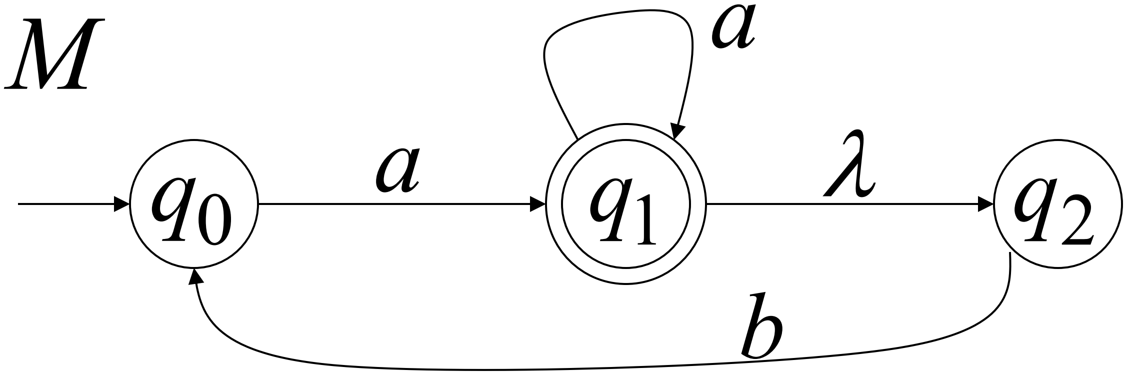
Repeat Step 2 for all letters in alphabet,

until

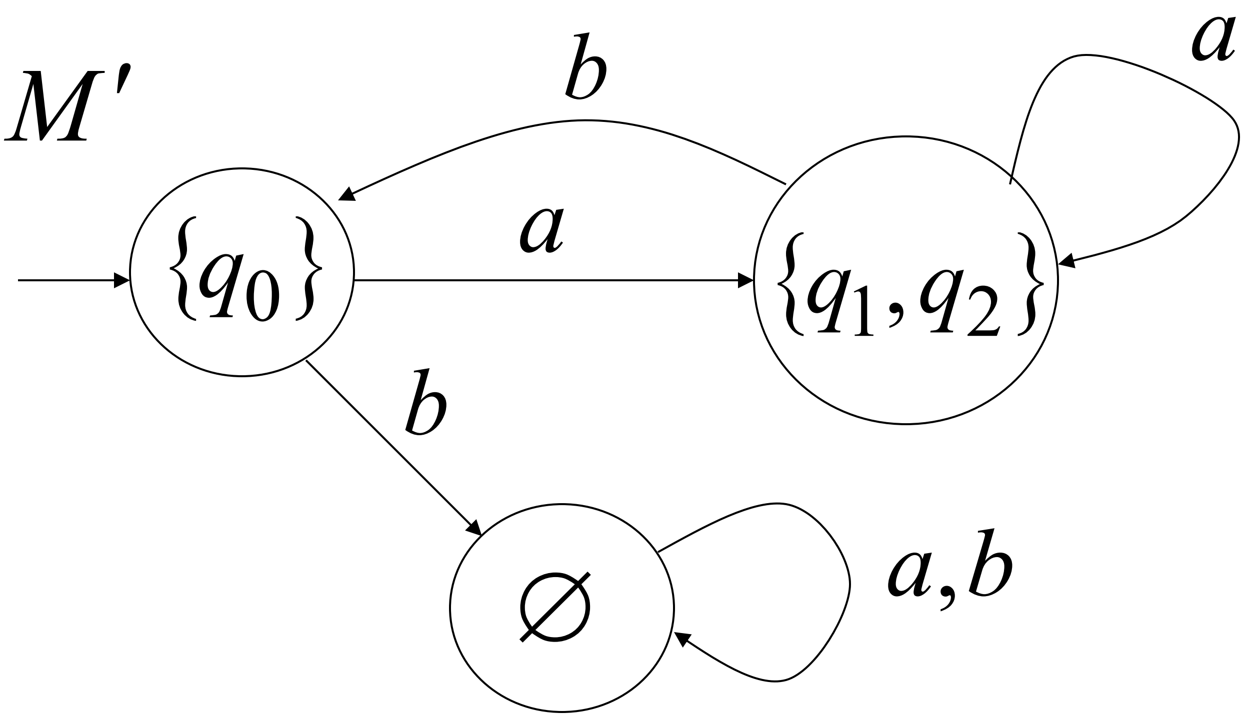
no more transitions can be added.

Example

NFA M



DFA M'



Procedure NFA to DFA

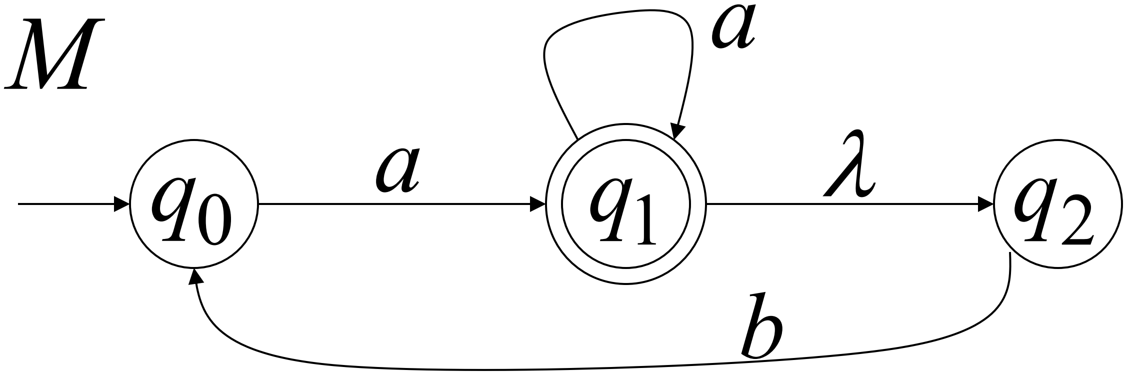
3. For any DFA state $\{q_i, q_j, \ldots, q_m\}$

If some q_j is a final state in the NFA

Then, $\{q_i, q_j, \ldots, q_m\}$
is a final state in the DFA

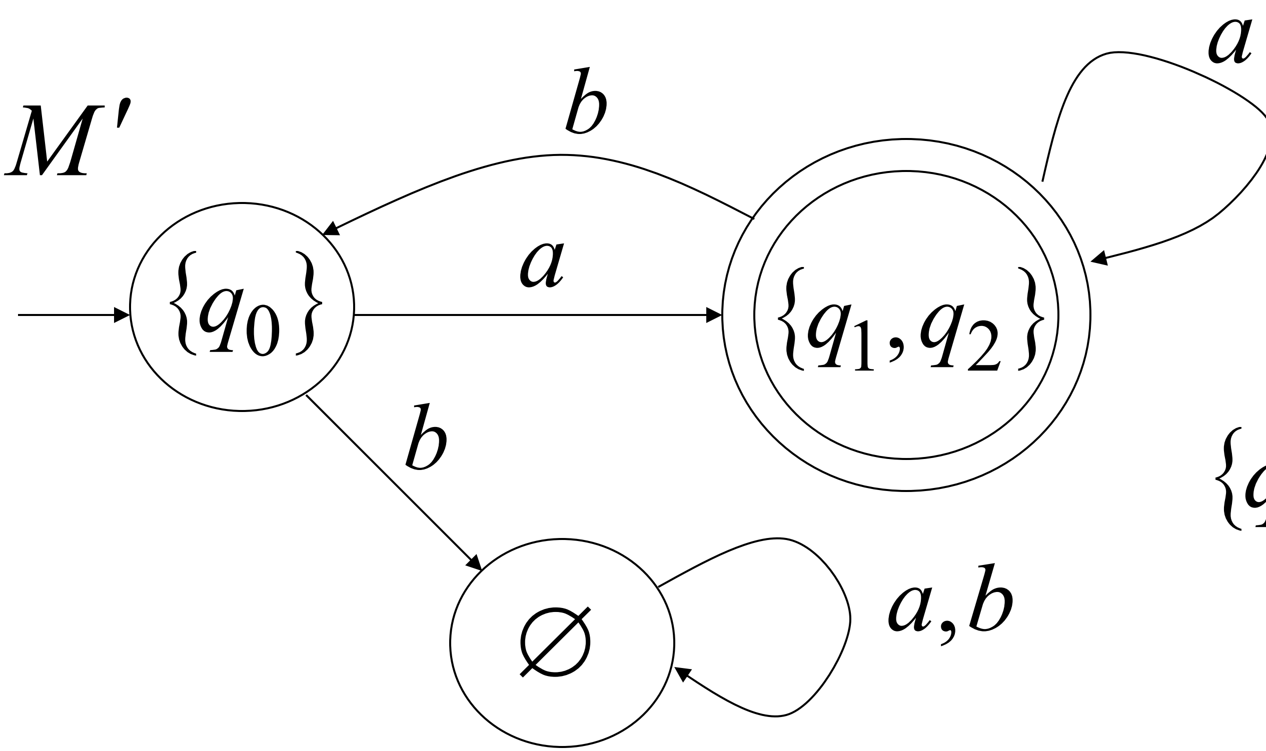
Example

NFA M



$q_1 \in F$

DFA M'



$\{q_1, q_2\} \in F'$

Theorem

Take NFA M

Apply procedure to obtain DFA M'

Then M and M' are equivalent :

$$L(M) = L(M')$$

Proof

$$L(M) = L(M')$$

$$\Updownarrow$$

$$L(M) \subseteq L(M') \quad \text{AND} \quad L(M) \supseteq L(M')$$

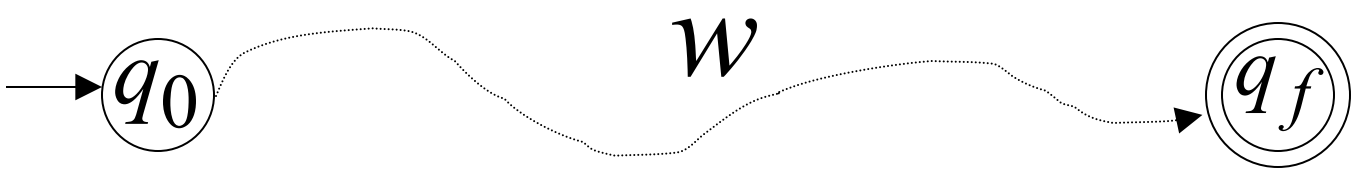
First we show: $L(M) \subseteq L(M')$

Take arbitrary: $w \in L(M)$

We will prove: $w \in L(M')$

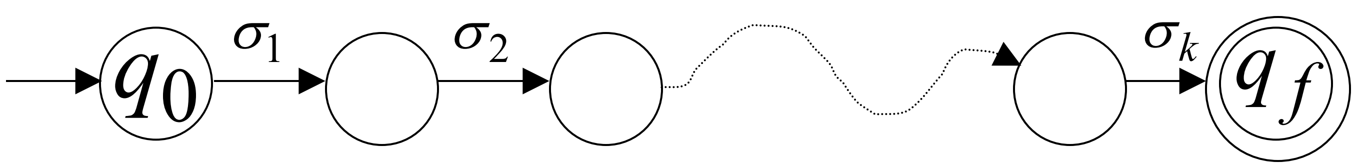
$$w \in L(M)$$

$M:$



$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$

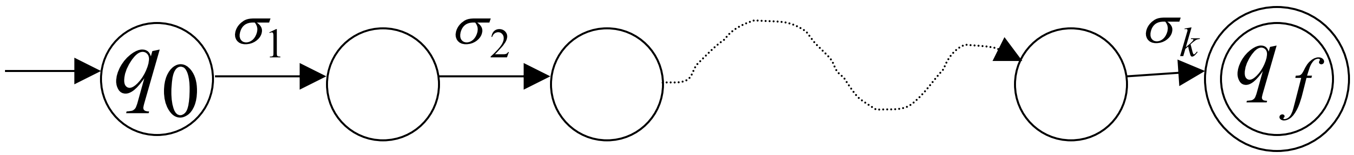
$M:$

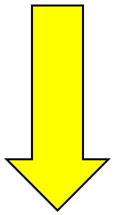


We will show that if $w \in L(M)$

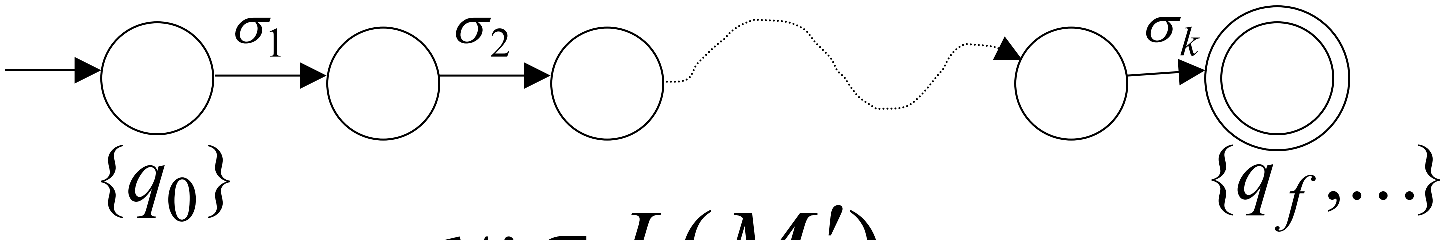
$w = \sigma_1 \sigma_2 \cdots \sigma_k$

$M:$





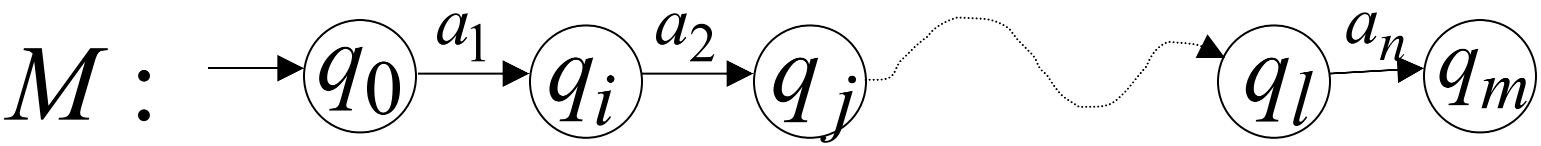
$M':$

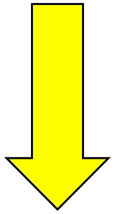


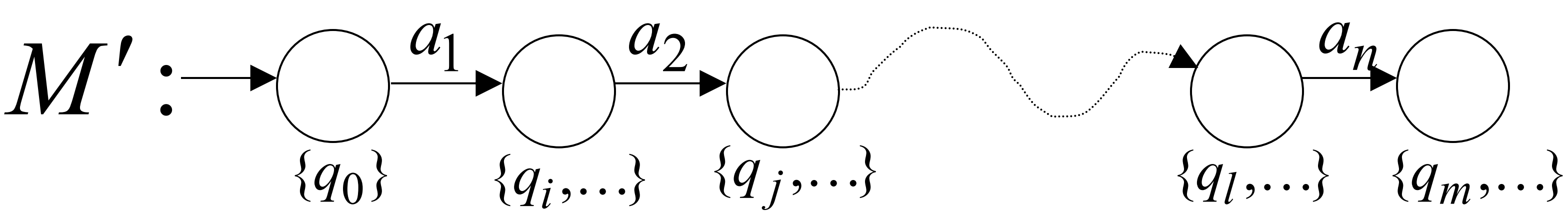
$w \in L(M')$

More generally, we will show that if in M:

(arbitrary string) $v = a_1 a_2 \cdots a_n$



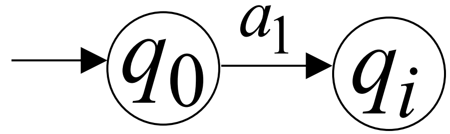




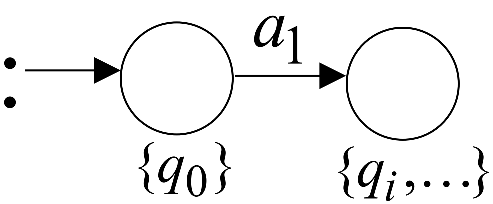
Proof by induction on $|v|$

Induction Basis: $v = a_1$

$M:$

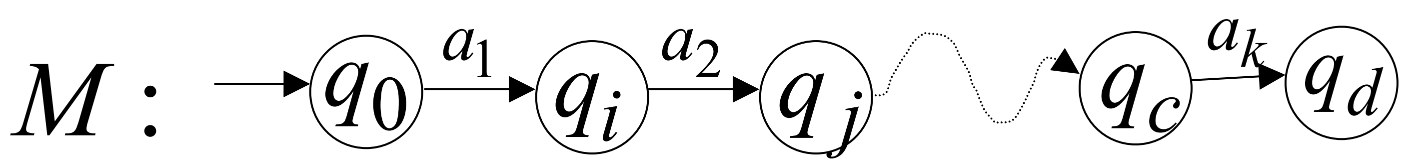


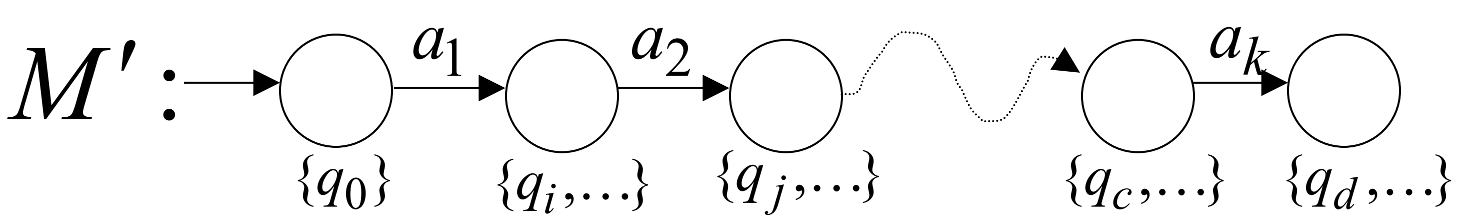
$M':$



Induction hypothesis: $1 \leq |v| \leq k$

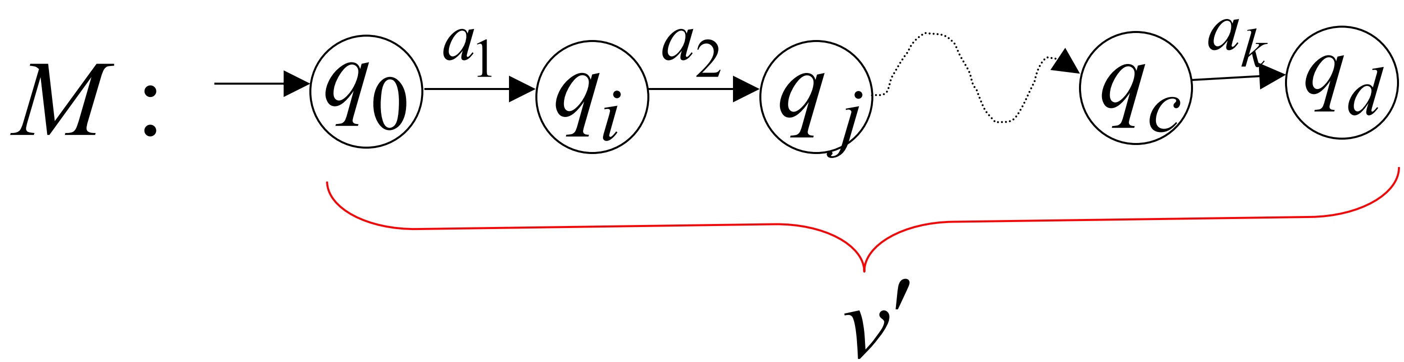
$$v = a_1 a_2 \cdots a_k$$

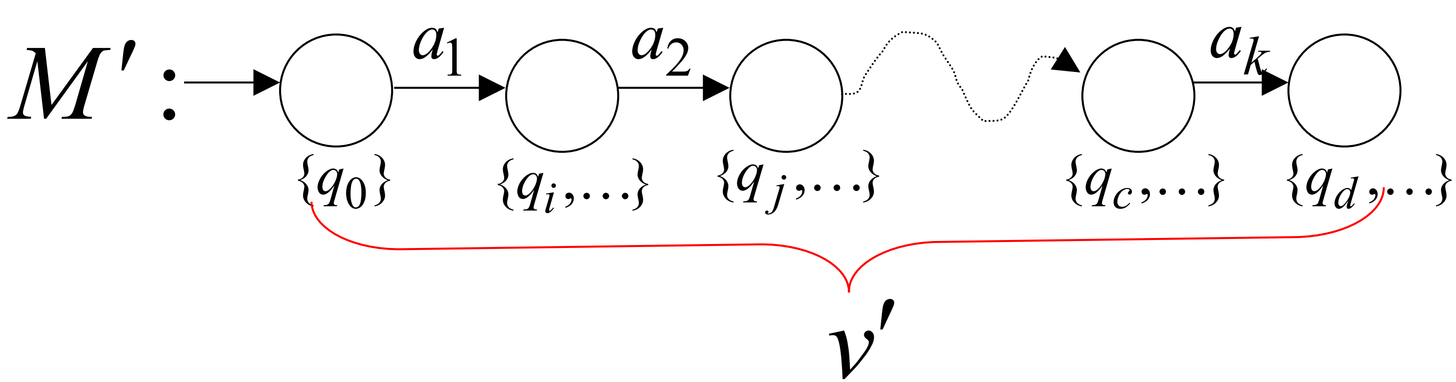




Induction Step: $|v| = k+1$

$$v = \underbrace{a_1 a_2 \cdots a_k}_{v'} a_{k+1} = v' a_{k+1}$$





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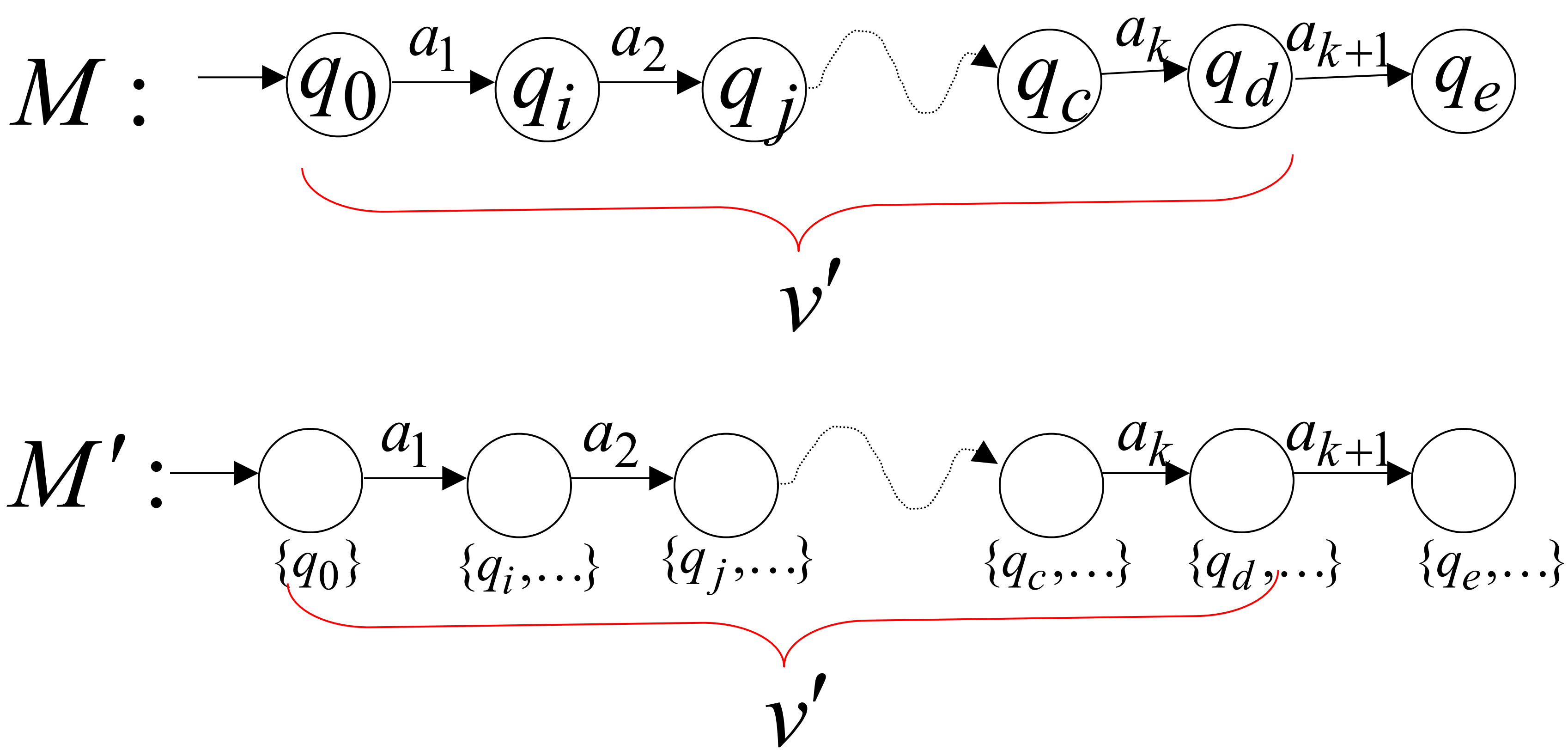
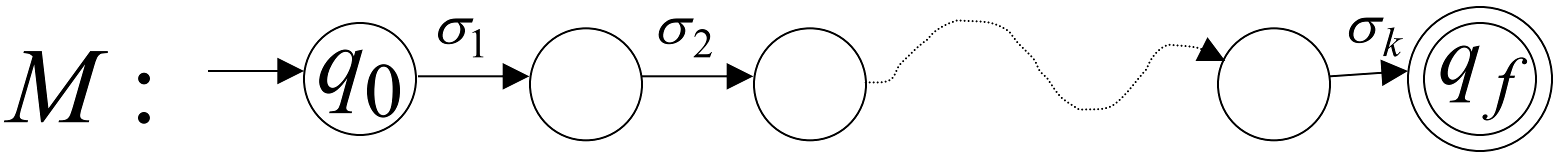
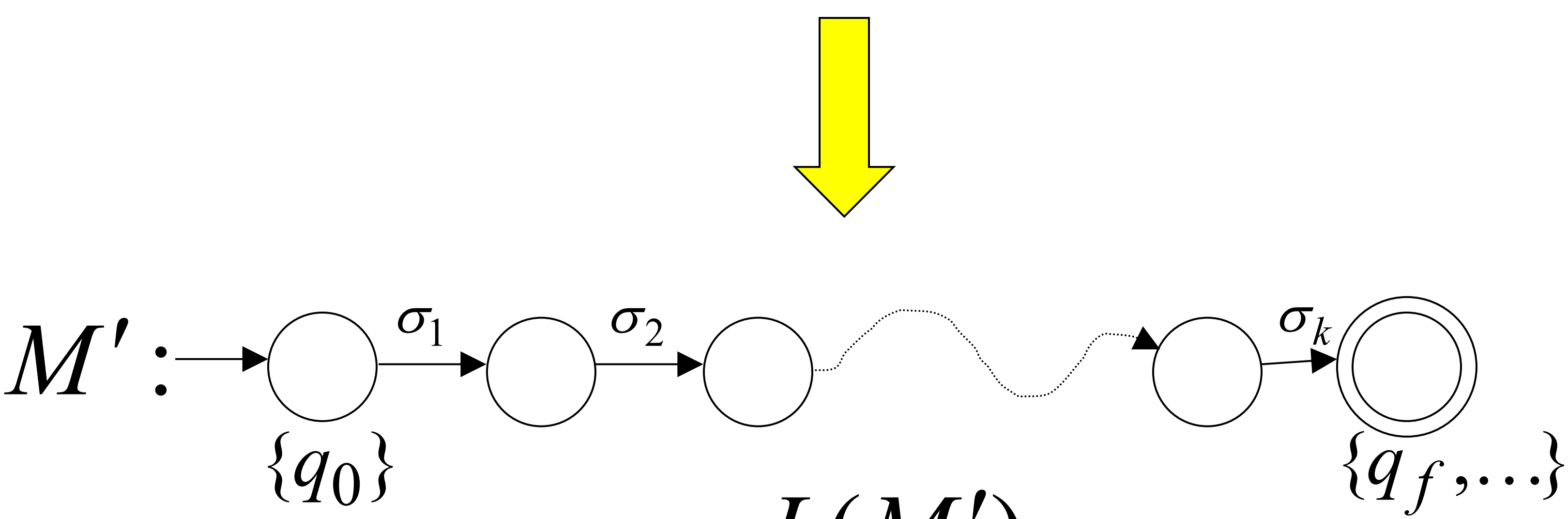


Therefore if $\quad w \in L(M)$

$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$





$$w \in L(M')$$

We have shown: $L(M) \subseteq L(M')$

We also need to show: $L(M) \supseteq L(M')$

(proof is similar)