Bose Condensates and Fermi Gases at Zero Temperature

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We consider the $T = 0$ ground state of mixtures of bosons and fermions. Applying the Thomas-Fermi approximation we show that, depending on the strength of boson-boson and boson-fermion interactions, the Fermi gas may constitute a "shell" around or a "core" inside the Bose condensate, or even both. [S0031-9007(98)05450-7]

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Bose-Einstein condensation has been observed in trapped dilute gases of rubidium [1], sodium [2], and lithium [3] atoms. Large efforts have been made to study many-body effects and macroscopic properties of the gases, which may be more transparently demonstrated here than in other many-body systems. One evident property of Bose-Einstein condensation is almost forgotten in the hectic current development of this field: A Bose condensate is very cold. When laser cooling a few years ago lead to temperatures in the μK range, these were announced as the lowest kinetic temperatures ever obtained. With the process of Bose-Einstein condensation we have now a mechanism which, due to the stimulated enhancement of the low-energy fraction of atoms, effectively cools large samples of atoms to temperatures in the nK range. Combining this effect with the sympathetic cooling mechanism, i.e., the exchange of energy due to elastic collisions between atoms of a cooled and a noncooled sample, it should be possible to cool a wider range of species to nK temperatures. This process has been observed in experiment [4], where Rb atoms in one internal state during forced evaporative cooling sympathetically cooled another internal state component of the gas, leading finally to two trapped condensates.

Inspired by this observation of successful sympathetic cooling/condensation we consider realizing the same process in alternative systems. As an example, in this Letter we consider the preparation of a degenerate Fermi gas by its interaction with a Bose condensate.

We consider atoms trapped in an external potential $V_{\text{ext}}(\vec{r})$. The atoms interact by elastic collisions, and we assume that the low kinetic energies of the atoms permit the replacement of their short range interaction potential by a delta function. In magnetic traps for atoms typically only one internal Zeeman state component will be trapped, hence neither spin degeneracy factors nor multiple Zeeman components of the same atoms will appear in the following.

Mean field theory and Thomas-Fermi approximation for bosons.—Self-consistent mean field theories, assuming that all N particles in a gas populate the same single particle wave function $\psi(\vec{r})$, lead to a nonlinear Schrödinger equation, the Gross-Pitaevskii equation, for $\psi(\vec{r})$,

$$\left[-\frac{\hbar^2}{2M}\nabla^2 + V_{\text{ext}}(\vec{r}) + Ng|\psi(\vec{r})|^2\right]\psi(\vec{r}) = \mu\psi(\vec{r}) . \tag{1}$$

The Thomas-Fermi approximation exploits the fact that the kinetic energy of the atoms is extremely low to simply neglect the kinetic energy operator in (1). One may then divide by the wave function and obtain the expression

$$n_B(\vec{r}) = N|\psi(\vec{r})|^2 = [\mu - V_{\text{ext}}(\vec{r})]/g . \tag{2}$$

Equation (2) is applied only when the right-hand side is positive, and μ has to be chosen so that the integral over space of $n_B(\vec{r})$ yields the number of particles in the gas.

In a harmonic oscillator potential

$$V_{\text{ext}}(\vec{r}) = \tfrac{1}{2}M(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2) , \tag{3}$$

μ can be determined analytically

$$\mu = \left[\left(\frac{M\overline{\omega}^2}{2}\right)^{3/2}\frac{15}{8\pi}Ng\right]^{2/5} , \tag{4}$$

where $\overline{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$. When N is not too small, the Thomas-Fermi approximation gives a quite good approximation to the exact distribution of particles and to the single particle energy [5].

Thomas-Fermi approximation for fermions.—Because of the Pauli exclusion principle, the atoms in a degenerate gas of fermions do not occupy a common single particle state vector. Hence we do not have the equivalent of the Gross-Pitaevskii equation for fermions. Instead, we describe the particles by classical positions and momenta, but we take into account the quantum mechanical result that a volume in phase space $d^3\vec{r}d^3\vec{k}$ can accommodate only $d^3\vec{r}d^3\vec{k}/(2\pi)^3$ fermions; i.e., if the local density is $n_F(\vec{r})$, the particles will have wave numbers within the interval $0 \leq k \leq k_F(\vec{r})$, where

$$\tfrac{4}{3}\pi k_F(\vec{r})^3 = (2\pi)^3 n_F(\vec{r}) . \tag{5}$$

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At the position $\vec{r}$ the most energetic fermion now has an energy equal to $\hbar^2 k_F(\vec{r})^2/(2M_F)$ plus the local position dependent potential. This energy must be position independent, since otherwise it would be favorable to move the atom to a region with a lower value of this quantity. Hence we can write

$$E_F = \frac{\hbar^2 k_F(\vec{r})^2}{2M_F} + V(\vec{r}). \tag{6}$$

This provides the Thomas-Fermi approximation for the fermion density, and it has been widely applied in atomic physics, nuclear physics, and astrophysics where the particles (electrons or nucleons) interact, but where closed equations for the spatial density can still be obtained. If there is no interaction among the atoms due to the suppression of s-wave scattering among identical fermions, we obtain the expression for the spatial density

$$n_F(\vec{r}) = \left\{\frac{2M}{\hbar^2}\left[E_F - V_{\text{ext}}(\vec{r})\right]\right\}^{3/2} \Big/ 6\pi^2. \tag{7}$$

As in (2) we have one free parameter, which must be identified so that the integral of the density yields the total number of particles. For a harmonic oscillator potential (3), to accommodate N fermions we must assign the value

$$E_F = \hbar\overline{\omega}(6N)^{1/3} \tag{8}$$

to the Fermi energy. Detailed analyses of the distributions of fermions also at nonvanishing temperatures and a discussion of the validity of the Thomas-Fermi approximation are presented in [6].

Coexisting Bose and Fermi components.—We shall now carry out a simple theoretical analysis of the situation with both a Bose condensate and a degenerate Fermi gas. To ease our notation we assume that the atoms are all trapped in the same external harmonic oscillator potential which, in turn, may be assumed to be isotropic, since for our static problem an anisotropy is trivially incorporated by providing physical quantities as isotropic functions of rescaled position coordinates $\tilde{x} = (\omega_x/\overline{\omega})x$, etc. We also assume that the atoms have the same mass M. The bosons interact among themselves with the coupling constant $g > 0$, and we assume a similar coupling among bosons and fermions characterized by the coupling constant $h > 0$, so that the equations for the densities of the two components are coupled,

$$\begin{gathered} V_{\text{ext}}(\vec{r}) + g n_B(\vec{r}) + h n_F(\vec{r}) = \mu, \\ \frac{\hbar^2}{2M}\left[6\pi^2 n_F(\vec{r})\right]^{2/3} + V_{\text{ext}}(\vec{r}) + h n_B(\vec{r}) = E_F. \end{gathered} \tag{9}$$

It is convenient to obtain the solutions to Eq. (9) by iterative insertion of one density distribution in the other equation and numerically searching for the energies μ and E_F yielding the desired number of particles. If there are only a few fermions, we may neglect them in the equation for the boson density, which becomes thus a quadratic function of position, given by Eqs. (2) and (3). The resulting simple equation for the fermions becomes

$$\frac{\hbar^2}{2M}\left[6\pi^2 n_F(\vec{r})\right]^{2/3} + \left(1 - \frac{h}{g}\right)V_{\text{ext}}(\vec{r}) + \frac{h}{g}\,\mu = E_F, \tag{10}$$

where the terms proportional with h are absent in regions with vanishing $n_B(\vec{r})$. The solution of this equation may serve as a guideline for the more complicated calculations.

If $h < g$ the fermions experience a potential minimum in the center of the trap, and if the number of fermions is small enough the entire distribution resides like a fermionic "core" within the Bose condensate. The spatial extent of a harmonically trapped Bose condensate and of a Fermi gas are obtained from the above expressions, and, e.g., for the experimental parameters of the MIT experiments on sodium [2], a condensate with about 10^6 atoms may enclose on the order of 10^3 fermions. If the outer part of the fermion distribution exceeds the limit of the Bose condensate, the atoms here experience a stronger confining potential. If $h = g$ the fermion density is a constant throughout the Bose condensate. If $h > g$ we have an inverted harmonic oscillator potential for the fermions near the origin, they are repelled from this region, and they localize near the edge of the condensate where their effective potential is minimum. The Bose condensate is in this case surrounded by a "shell" of fermions.

In Fig. 1 we show the distribution of 1000 fermions embedded in a Bose condensate with 10^6 bosonic atoms for different values of the boson-fermion interaction constant

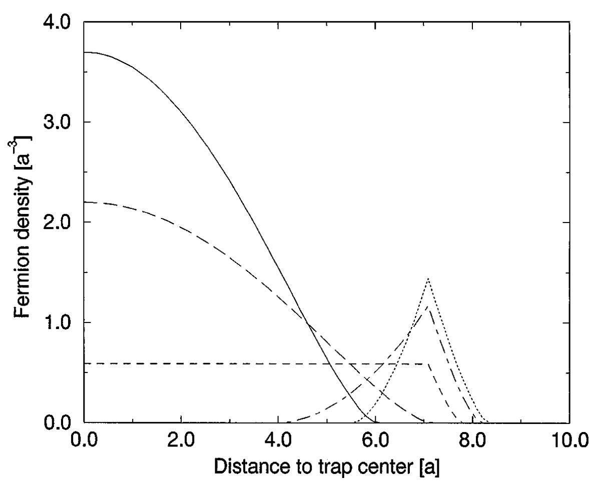

FIG. 1. Spatial distribution of 1000 fermions as a function of distance to the trap center. $g = 0.015\hbar\overline{\omega}a^3$, and the boson-fermion coupling takes the values, $h = 0$ (solid curve), $h = g/2$ (long-dashed curve), $h = g$ (dashed curve), $h = 3g/2$ (dot-dashed curve), and $h = 2g$ (dotted curve). The Bose-Einstein condensate component of 10^6 atoms is unaffected by the fermions and extends out to the distance $\sim 7a$, where cusps in the fermion distributions are visible.

h. We take the value $g = 0.015\hbar\overline{\omega}a^3$, where a is the oscillator ground state width, $a = \sqrt{\hbar/M\overline{\omega}}$. For this choice of parameters there is no noticeable difference between the solutions of Eqs. (9) and (10). We confirm the prediction that the fermions undergo a transition from being localized within the condensate to forming a shell around the Bose condensate. The areas under the curves are identical if the distribution functions are multiplied by a factor of $4\pi r^2$.

To identify the interesting regime where the fermions and bosons significantly overlap, and where their individual distributions in space are both strongly modified by the other component, we now assume equal numbers of bosons and fermions N, and from the Thomas-Fermi expressions for the individual distributions we infer the requirements on g and $\overline{\omega}$ for the clouds to have approximately the same size. The radius of the Bose condensate is $(15Ng/4\pi M\overline{\omega}^2)^{1/5}$, and the radius of the zero temperature Fermi gas is $(48N)^{1/6}\sqrt{\hbar/M\overline{\omega}}$. Equating these two numbers we get the constraint: $g/(\hbar\overline{\omega}a^3) \simeq 21.1N^{-1/6}$. This dimensionless quantity had the value 0.015 in our calculations above, in approximate agreement with the value in Na experiments [2]. To obtain the requested situation with 10^6 atoms we must have a 2 orders of magnitude increase corresponding to a substantially increased scattering length, provided, e.g., by another choice of atom like cesium which, however, still remains to be Bose condensed, or by modification of the scattering length by external fields [7]. In the dimension less units g is also larger in a steeper magnetic trap for the atoms. It is not necessary to insist that the number of fermions and bosons be identical, and we recall that only for simplicity of notation we assumed identical trapping potentials and masses for the particles. When these parameters are also allowed to vary, one may more easily enter the domain of interest.

We consider the case of 10^6 fermions and the same number of bosons with a coupling constant of $g = 2.11\hbar\overline{\omega}a^3$. We present in Fig. 2 the boson and fermion distributions for increasing values of the fermion boson coupling constant h. The results are quite remarkable, but they may be readily explained from our understanding of how the two density distributions effectively enter the effective potentials for each other: when h is increased, e.g., to the value of $0.75g$, the repulsion causes a spatial separation of the two species, and we observe how the bosons are excluded from the center of the trap. The fermion distribution is reduced in the region occupied by bosons, and when $h = g$, the fermions experience a constant potential in this region, and they are therefore uniformly distributed here. When h is further increased, the fermions are expelled outwards from this region, and the energy is minimum if part of the fermion distribution is located outside the Bose condensate.

Macroscopic change and collapse of distributions under variations of h.—It has been suggested to induce a dynamical evolution of Bose condensates by changing the sign of the scattering length (and of the coupling constant g) [8]. Here we have seen that if the boson-fermion coupling constant h can be changed around the value of g this should have a similar "macroscopic" effect on the physical system. The coupling h may also become negative, and one may question whether a collapse of the distributions will occur in this case, as it has been predicted for bicondensates [9]. For small negative values of h we see no effects, but for large values, our iterative solution of Eq. (9) leads to a diverging fermion density at the center of the trap. To investigate how this collapse appears we suggest a simple model, in which only a part of the boson density $(-h/g)n_F(\vec{r})$ following from the first line of (9) is inserted into the second line thus forming an equation for a fermion gas with attractive interactions,

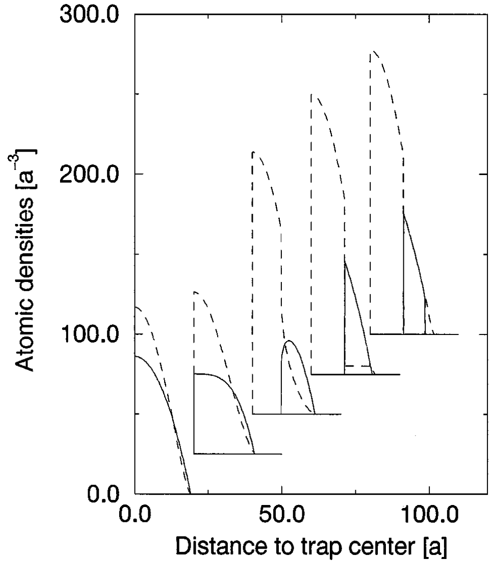

FIG. 2. Spatial distribution of 10^6 bosons (solid curves) and 10^6 fermions (dashed curves) as a function of distance to the trap center. $g = 2.11\hbar\overline{\omega}a^3$, and the boson-fermion coupling takes the values, $h = 0$, $g/2$, $3g/4$, g, and $5g/4$. The curves are displaced horizontally and vertically for visibility.

$$\frac{\hbar^2}{2M}[6\pi^2 n_F(\vec{r})]^{2/3} + V_{\mathrm{ext}}(\vec{r}) - \frac{h^2}{g}n_F(\vec{r}) = E_F\,. \quad (11)$$

For large values of the density $n_F(\vec{r})$, the linear dependence of the potential energy on density outweighs the $\frac{2}{3}$ power dependence of the kinetic energy and the collapsed state has the lowest energy. For a low density, the $\frac{2}{3}$ power dominates and a gradual redistribution of atoms towards the center of the trap is not favorable—the Fermi gas is (meta-)stable against the collapse. If, however, the total number of fermions is increased or if the coupling h^2/g is increased so that the density at the trap center reaches the crossing point of the linear and $\frac{2}{3}$ power law dependence, the collapse becomes inevitable.

At this point $E_F = 0$, and the quadratic potential allows a similarity transformation among the solutions to

Eq. (11) with $E_F = 0$: $\vec{r} \rightarrow \tilde{r} = q\vec{r}$, $n_F(\vec{r}) \rightarrow \tilde{n}_F(\tilde{r}) = q^3 n_F(q\vec{r})$, and $N \rightarrow \tilde{N} = q^6 N$ provided $(h^2/g) \rightarrow (\tilde{h}^2/g) = q^{-1}(h^2/g)$. That is, the critical value of the coupling parameter (h^2/g) scales as $N^{-1/6}$. A numerical solution of Eq. (11) confirms this behavior: the particle density at the origin and the value of E_F diverge when $(h^2/g) \sim 6.9\hbar\overline{\omega}a^3 N^{-1/6}$ [10]. In comparison, two Bose condensates with self-couplings g_1 and g_2 become unstable independently of the number of bosons if their coupling term $g_{12} < -\sqrt{g_1 g_2}$ [9].

The solution of Eq. (9) reveals a more complicated quantitative relationship between h, g, and N. The $N^{-1/6}$ power law dependence is not generally correct, but the qualitative behavior may be understood from our simple model. With the parameters chosen for Fig. 1, the collapse requires an unrealistically large negative value for h, but we do not exclude that the collapse may be investigated for other parameters.

It is noteworthy that the bosons and the fermions may have very different kinetic energies, but collision processes in which energy is transferred from a fast fermion to a slow boson are prevented because there is no unoccupied final state for the fermion. Removing fermions from the center of the trap, creating a hole in the Dirac sea, and allowing scattering processes where an amount of energy close to the Fermi energy can be transferred to one or several bosons point to an "Auger" type spectroscopy of the system.

Methods have been demonstrated for the "output coupling" of atoms from a condensate [11], and one might experimentally be able to extract the Bose "liquefier" after the cooling and thus obtain a situation with more fermions than bosons. Such a system may be interesting for its possible analogy with atomic nuclei and neutron stars, and a study of the BCS transition in the Fermi gas [12] is an interesting prospect of sympathetic cooling. Detailed theoretical studies along these lines have begun, and experiments have addressed the relevant scattering parameters for ^{6}Li [13].

One may also consider the possibilities of cooling other species than fermionic atoms, e.g., molecules or clusters. The prospects of freezing out rotational and vibrational excitations of such systems seem very attractive.

Finally, we recall that we applied the Thomas-Fermi approximation throughout and that corrections to our results should be expected, in particular, from the inclusion of the bosonic kinetic energy operator. As shown in [9], the clear separation of different particle components remains, but the distribution of atoms at the boundary are rounded off. The energy cost of introducing a boundary, discussed for bicondensates in [14], possibly makes the multilayered structure in Fig. 2 less favorable, and it may change the values of h where the separations in Fig. 2 occur. In order to reduce the boundary contribution to the energy, the fermion and boson components may also localize around different points in the trap, hence breaking the rotational symmetry of the problem. Such behavior has been predicted for two-dimensional bicondensates for certain critical values [15], and, in particular, in an elongated trap it is very likely that a separation of bosons and fermions into two well separated components is energetically preferable for large positive values of h. In order to address these possibilities quantitatively and to study also more carefully the stability of the system with attractive interactions, we are currently setting up calculations to solve self-consistently the Gross-Pitaevskii equation for the bosons and a Slater determinant Schrödinger equation for the fermions.

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