Zero-temperature phase diagram of binary boson-fermion mixtures

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We calculate the phase diagram for dilute mixtures of bosons and fermions at zero temperature. The linear stability conditions are derived and related to the effective boson-induced interaction between the fermions. We show that in equilibrium there are three possibilities: (a) a single uniform phase, (b) a purely fermionic phase coexisting with a purely bosonic one, and (c) a purely fermionic phase coexisting with a mixed phase.

I. INTRODUCTION

Recent developments in the trapping and cooling of atoms have made it possible to investigate the properties of dilute gases at very low temperatures where the bosonic or fermionic character of the atoms becomes crucial. Following the realization of Bose-Einstein condensation in a single-component gas of alkali atoms [1–3], Myatt *et al.* succeeded in trapping and cooling ^{87}Rb atoms in two different hyperfine states, thereby creating overlapping condensates in this boson-boson mixture [4].

Lately a lot of attention has also been given to fermions, especially in view of the possibility of achieving temperatures low enough to observe a BCS transition. Due to the vanishing *s*-wave cross section for identical fermions in the same spin state, evaporative cooling of a single species of fermions is ineffective. By using a mixture of two spin states DeMarco and Jin [5] recently succeeded in cooling the fermionic isotope ^{40}K to temperatures lower than the degeneracy temperature. Another possibility, however, is to trap fermions together with bosons and cool the latter, so that the fermions are cooled through their thermal contact with the bosons (so-called sympathetic cooling). Such fermion-boson mixtures form the topic of the present paper.

The miscibility properties of boson-boson mixtures in a trap have been discussed in several papers [6–8]. Trapped boson-fermion mixtures have been considered in Ref. [9] within the Thomas-Fermi approximation and in Ref. [10], where the separation of the components was studied numerically as a function of the interparticle interaction. The purpose of the present work is to carry out an analytical study of the miscibility of fermion-boson mixtures in the uniform case, in order to obtain insight into the types of phase boundaries that may occur in this system.

The article is organized as follows. We begin in Sec. II by studying linear stability of uniform mixtures and introduce the effective fermion-fermion interaction as mediated by the bosons. In Sec. III we determine the general conditions for phase equilibrium by considering the pressure and chemical potentials on each side of a phase boundary. We find that, depending on the total densities of the two components, as shown in Fig. 3 below, there are three possibilities, a uniform phase with bosons and fermions fully mixed, a purely fermionic phase coexisting with a purely bosonic one, and a purely fermionic phase coexisting with a mixed phase. How-

ever, a separation into two phases, each with different, non-zero, concentrations of bosons and fermions is never in equilibrium, nor is one with a purely bosonic phase and a mixed one. In Sec. IV we discuss the experimental implications of our results, using the fermion isotopes ^{6}Li, ^{40}K, and ^{84}Rb as examples.

II. LINEAR STABILITY AND INDUCED INTERACTIONS

The general conditions for stability of a binary mixture toward small changes in the concentrations of its components are obtained by considering the total energy $\mathcal{E}$ as a functional of the densities n_α and n_β of the two components:

$$\mathcal{E}= \int d\mathbf{r}\, E(n_\alpha(\mathbf{r}),n_\beta(\mathbf{r})). \tag{1}$$

We consider the change in total energy arising from small changes δn_α and δn_β in the concentrations of the two constituents. The first-order variation, $\delta\mathcal{E}$, must vanish, since the number of particles of each species is conserved,

$$\int d\mathbf{r}\, \delta n_i=0; \quad i=\alpha,\beta. \tag{2}$$

The second-order variation, $\delta^2\mathcal{E}$, is given by the quadratic form

$$\delta^2\mathcal{E}= \frac{1}{2}\int d\mathbf{r}\left(\frac{\partial^2 E}{\partial n_\alpha^2}(\delta n_\alpha)^2+\frac{\partial^2 E}{\partial n_\beta^2}(\delta n_\beta)^2\right.$$
$$\left.+2\frac{\partial^2 E}{\partial n_\alpha \partial n_\beta}\delta n_\alpha \delta n_\beta\right). \tag{3}$$

In carrying out this expansion we have implicitly assumed that the characteristic wavelength for the spatial variation of the densities is long compared to the microscopic lengths in the problem. As we shall see, for dilute mixtures of bosons and fermions the relevant length is the coherence length of the bosons. The derivative of the energy density with respect to the particle density, $\partial E/\partial n_i$, is the chemical potential μ_i for the species labeled i ($i=\alpha,\beta$). The quadratic form (3) is thus positive semidefinite, provided

$$\frac{\partial \mu_\alpha}{\partial n_\alpha} \geq 0, \quad \frac{\partial \mu_\beta}{\partial n_\beta} \geq 0, \tag{4}$$

and

$$\frac{\partial \mu_\alpha}{\partial n_\alpha} \frac{\partial \mu_\beta}{\partial n_\beta} - \frac{\partial \mu_\alpha}{\partial n_\beta} \frac{\partial \mu_\beta}{\partial n_\alpha} \geq 0. \tag{5}$$

Note that it is only necessary that one of the two conditions in Eq. (4) is satisfied, since the other one is automatically fulfilled when the condition (5) is.

For a dilute mixture of bosons with fermions in a single internal state, at zero temperature the energy functional has the form

$$\mathcal{E} = V(\tfrac{1}{2} n_B^2 U_{BB} + n_B n_F U_{BF} + \tfrac{3}{5} \epsilon_F n_F), \tag{6}$$

where V is the total volume. The first two terms are due to the boson-boson and boson-fermion interactions, respectively, with n_B and n_F denoting the boson and fermion densities. U_{BB} is the matrix element of the effective interaction for bosons with bosons and U_{BF} that for bosons with fermions. The last term represents the kinetic energy of the fermions. The fermion-fermion interaction energy is negligible, since (a) the s-wave scattering amplitude vanishes for fermions in the same spin state and (b) none of the higher partial waves contribute at the low temperatures we are considering. The chemical potentials are then seen to be

$$\mu_B = n_B U_{BB} + n_F U_{BF},$$

$$\mu_F = \epsilon_F + n_B U_{BF}. \tag{7}$$

Let us introduce the constant A through the definition

$$\epsilon_F = A n_F^{2/3}, \quad A = \frac{\hbar^2}{2 m_F} (6 \pi^2)^{2/3}, \tag{8}$$

where m_F denotes the fermion mass. The value of the numerical factor in A reflects the fact that we consider fermions in a definite spin state. According to Eq. (5) linear stability then requires

$$n_F^{1/3} \leq \frac{2}{3} \frac{A U_{BB}}{U_{BF}^2}. \tag{9}$$

If the interaction parameters are converted to scattering lengths a_{BB} and a_{BF} through the relations

$$U_{BB} = \frac{4 \pi \hbar^2 a_{BB}}{m_B}, \quad U_{BF} = \frac{4 \pi \hbar^2 a_{BF}}{m_{BF}}, \tag{10}$$

where m_B is the boson mass and $m_{BF} = 2 m_F m_B / (m_F + m_B)$ is twice the reduced mass of a boson-fermion pair, then the condition (9) becomes

$$n_F^{1/3} \leq \frac{(6 \pi^2)^{2/3}}{12 \pi} \frac{m_{BF}^2 a_{BB}}{m_B m_F a_{BF}^2}. \tag{11}$$

If the masses as well as the scattering lengths occurring in Eq. (11) are approximately equal in magnitude, then the stability condition requires the mean interfermion distance, which is approximately $n_F^{-1/3}$, to be greater than a scattering length. The trapped one-component gases that have so far been investigated experimentally are dilute in the sense that the mean interparticle distance is much greater than the scattering length. The stability condition (11) would therefore generally be expected to hold for dilute mixtures, unless the scattering length a_{BF} greatly exceeds a_{BB}. In this case the condition (11) may be violated at much lower fermion densities, given by $n_F^{1/3} a_{BB} \approx (a_{BB}/a_{BF})^2$ rather than $n_F^{1/3} a_{BB} \approx 1$.

It should be noted that the mean interfermion distance in the stability condition (11) formally plays the role of an effective fermion-fermion scattering length. This is apparent when we consider binary mixtures of bosons, labeled by α and β. In this case the energy functional takes the form

$$\mathcal{E} = V(\tfrac{1}{2} n_\alpha^2 U_{\alpha\alpha} + n_\alpha n_\beta U_{\alpha\beta} + \tfrac{1}{2} n_\beta^2 U_{\beta\beta}). \tag{12}$$

The condition (9) would then be replaced by

$$U_{\alpha\alpha} U_{\beta\beta} \geq U_{\alpha\beta}^2. \tag{13}$$

The conditions (9) and (13) assume the same form, apart from numerical factors, provided the combination $4 \pi \hbar^2 n_F^{-1/3}/m_F$ is interpreted as arising from a direct fermion-fermion interaction, while physically, of course, the term originates in the kinetic energy. The Fermi pressure thus behaves as a two-body interaction with a scattering length of the order of the particle separation.

Induced interactions

In a multicomponent system the effective interaction between atoms of one species is altered by the presence of the other components. As an example of this we return to the stability condition (5). Since the boson gas must be stable to density fluctuations when the fermions are uniform, the compressibility of the bosons must be positive, which means that $\partial \mu_B/\partial n_B > 0$. Also, $\partial \mu_B/\partial n_F = \partial \mu_F/\partial n_B = \partial^2 E/\partial n_B \partial n_F$. Thus the stability condition (5) may be written as

$$\frac{\partial \mu_F}{\partial n_F} - \left(\frac{\partial \mu_F}{\partial n_B} \right)^2 \frac{\partial n_B}{\partial \mu_B} \geq 0. \tag{14}$$

The first term here is the variation of the chemical potential of the fermions when the fermion density is changed, keeping the boson density fixed. The second term may be regarded as an induced interaction, due to the fact that one fermion tends to attract or repel bosons, depending on the sign of the boson-fermion interaction, and the change in the boson density changes the energy of a second fermion. The induced interaction is analogous to the phonon-induced attraction in metals. Note that this interaction is always attractive, irrespective of the sign of the boson-fermion interaction. By use of the chain rule, one may show that the stability condition is equivalent to
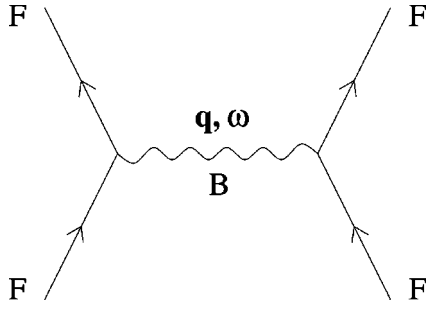
FIG. 1. Diagrammatic representation of the interaction between fermions induced by exchange of a boson density fluctuation.

$$\frac{\partial \mu_F}{\partial n_F}\bigg|_{\mu_B} \geqslant 0. \tag{15}$$

This way of looking at interactions has been exploited in the context of liquid mixtures of ^{3}He and ^{4}He [11]. The induced interaction contribution to the stability criterion corresponds to the diagram shown in Fig. 1, evaluated in the long-wavelength limit.

For a dilute gas, the induced interaction is easily evaluated using the expression (6) for the energy, since

$$\frac{\partial \mu_F}{\partial n_B} = U_{BF} \tag{16}$$

and

$$\frac{\partial \mu_B}{\partial n_B} = U_{BB}. \tag{17}$$

Thus the induced interaction is given by

$$U_{\text{ind}} = -\frac{U_{BF}^2}{U_{BB}}, \tag{18}$$

and it is equivalent to an effective reduction of the scattering length by an amount $(a_{BF}^2/a_{BB})(m_B m_F/m_{BF}^2)$.

At nonzero wave numbers and frequencies the effective interaction is the natural generalization of Eq. (18), the static long-wavelength response function $-\partial n_B/\partial \mu_B$ for the bosons being replaced by the density-density response function $\chi_B(q,\omega)$. Thus

$$U_{\text{ind}}(q,\omega) = U_{BF}^2 \chi_B(q,\omega). \tag{19}$$

In the Bogoliubov approximation the density-density response function is given by

$$\chi(q,\omega) = \frac{n_B q^2}{m_B(\omega^2 - \omega_q^2)}, \tag{20}$$

where the excitation frequencies are the Bogoliubov ones,

$$\hbar \omega_q = [\epsilon_q^0(\epsilon_q^0 + 2n_B U_{BB})]^{1/2}, \tag{21}$$

with $\epsilon_q^0 = \hbar^2 q^2/2m_B$ being the free boson energy. Thus the static induced interaction is

$$U_{\text{ind}}(q,0) = -U_{BF}^2 \frac{n_B}{n_B U_{BB} + (\hbar^2 q^2/4m_B)}. \tag{22}$$

In coordinate space this is a Yukawa, or screened Coulomb, interaction,

$$U_{\text{ind}}(r) = -\frac{m_B n_B U_{BF}^2}{\pi \hbar^2} \frac{e^{-\sqrt{2}r/\xi}}{r}, \tag{23}$$

where ξ is the coherence (healing) length for the bosons, given by

$$\xi^2 = \frac{\hbar^2}{2m_B n_B U_{BB}}. \tag{24}$$

The induced interaction thus has the interesting feature that at long wavelengths it gives rise to effects which are of the same order of magnitude as a typical bare interaction, if the boson-boson and boson-fermion interactions are of the same order of magnitude and boson and fermion masses are comparable. The reason for this is that even though the induced interaction involves two boson-fermion interactions, the density-density response function for the bosons at long wavelengths is inversely proportional to the boson-boson interaction. At wave numbers greater than the inverse of the coherence length for the bosons, the magnitude of the induced interaction is reduced, since the boson density-density response function has a magnitude $\sim 2n_B/\epsilon_q^0$. The induced interaction is thus strongest for momentum transfers smaller than $m_B s_B$, where $s_B = (n_B U_{BB}/m_B)^{1/2}$ is the sound speed in the boson gas. If bosons and fermions have comparable masses and densities, and the scattering lengths are comparable, for momentum transfers of order of the Fermi momentum the interaction will be of order $4\pi\hbar^2 k_F a^2/(2m)$. The induced interaction is therefore one power of the "diluteness parameter" $k_F a$ less than a typical direct interaction. For mixtures of two species of fermions (for example, two different hyperfine states) with bosons, the induced interaction is attractive, and therefore it will increase the transition temperature to a BCS superfluid state. Detailed calculations are necessary to determine how important the effect is quantitatively.

III. PHASE EQUILIBRIUM

The stability considerations given in the previous section are valid only for small changes. We now solve the general stability problem by considering the conditions under which mixtures with different concentrations of bosons and fermions may be in equilibrium with each other.

We consider a mixture of N_B bosons and N_F fermions in a box of volume V. These components may either mix uniformly or form distinct phases, which we label by the index i. If we ignore interpenetration effects, a possible phase-separated configuration may be described by the number of phases present, I, the bosonic and fermionic densities in each phase, $n_{B,i}$ and $n_{F,i}$, and the fractions of the total volume they occupy, v_i. Since the total number of particles is given,

the following relations must hold: $\Sigma_{i=1}^{I} n_{F,i} v_i = n_F = N_F/V$, $\Sigma_{i=1}^{I} n_{B,i} v_i = n_B = N_B/V$, and $\Sigma_i^I v_i = 1$. When $I=1$ we recover the case of a homogeneous mixture. Let us now turn our attention to the case $I=2$. The total energy is the sum of contributions due to boson-boson interactions, boson-fermion interactions, and the kinetic energy of the fermions,

$$\mathcal{E} = \sum_{i=1}^{2} \mathcal{E}_i = V \sum_{i=1}^{2} v_i E_i, \tag{25}$$

where

$$E_i = \tfrac{1}{2} n_{B,i}^2 U_{BB} + n_{B,i} n_{F,i} U_{BF} + \tfrac{3}{5} \epsilon_{F,i} n_{F,i}, \tag{26}$$

with the Fermi energies $\epsilon_{F,i}$ being given by

$$\epsilon_{F,i} = (6\pi^2)^{2/3} n_{F,i}^{2/3} \frac{\hbar^2}{2m_F}. \tag{27}$$

The pressure p_i in each phase is then found by differentiating the energy with respect to the volume $V_i = V v_i$ occupied by each phase,

$$p_i = -\frac{\partial \mathcal{E}_i}{\partial V_i} = \tfrac{1}{2} n_{B,i}^2 U_{BB} + n_{B,i} n_{F,i} U_{BF} + \tfrac{2}{5} \epsilon_{F,i} n_{F,i}, \tag{28}$$

using the volume dependence of the Fermi energy given by Eq. (27). The first requirement for equilibrium is that $p_1 = p_2$. The two other conditions involve the chemical potentials, which are given by

$$\mu_{B,i} = \frac{\partial E_i}{\partial n_{B,i}} = n_{B,i} U_{BB} + n_{F,i} U_{BF},$$
$$\tag{29}$$
$$\mu_{F,i} = \frac{\partial E_i}{\partial n_{F,i}} = \epsilon_{F,i} + n_{B,i} U_{BF}.$$

If the boson density $n_{B,i}$ is nonzero in both phases, then the chemical potentials $\mu_{B,i}$ must be equal. If the boson density vanishes in one phase, then the boson chemical potential in that phase must be higher than in the other one, in order for the system to be in equilibrium. The same is of course true for the fermions.

We remark that for a two-component system it is not possible to have three or more distinct phases in equilibrium. This is seen as follows. Consider a situation where there are I distinct mixed phases. The total number of conditions to be met is $3(I-1)$. If one wishes equilibrium to be possible over a range of parameters one may in addition fix another variable, such as the total pressure. The total number of conditions to be satisfied is then $3I-2$. However, the number of independent variables is just $2I$, corresponding to the boson and fermion densities for the I phases. It is then clear that solutions to the equations are only possible if $I \leq 2$. Similar arguments apply if some pure phases are present.

We thus consider a system consisting of two phases with densities $n_{B,1}, n_{F,1}$ and $n_{B,2}, n_{F,2}$, with the first one occupying a share v of the total volume. Due to the conditions on

the chemical potentials one has to distinguish four cases, which must be analyzed one by one.

(A) Two pure phases. The bosons and fermions are completely separated, corresponding to $n_{F,1}=0$, $n_{B,2}=0$ and $n_{B,1} \neq 0$, $n_{F,2} \neq 0$.

(B) A mixed phase and a purely fermionic one. The boson density vanishes in one region, corresponding to $n_{F,1} \neq 0$, $n_{B,2}=0$ and $n_{B,1} \neq 0$, $n_{F,2} \neq 0$.

(C) Two mixed phases. All densities $n_{B,1}$, $n_{F,1}$ and $n_{B,2}$, $n_{F,2}$ are different from zero, while $n_{B,1} \neq n_{B,2}$ and $n_{F,1} \neq n_{F,2}$.

(D) A mixed phase and a purely bosonic one. The fermion density vanishes in one region, corresponding to the conditions $n_{F,1}=0$, $n_{B,2} \neq 0$ and $n_{B,1} \neq 0$, $n_{F,2} \neq 0$.

First we consider cases A and B and identify the two regions of the n_B-n_F plane in which they can occur. We demonstrate below that these regions are in fact different and that in each of them the phase-separated configuration is energetically favored compared to the homogeneous one. Subsequently we prove that the cases C and D are not realizable.

A. Two pure phases

As mentioned previously, the pressures in the two phases must be the same. These are found from the general formula (28) by setting $n_{F,1}=0$ in the expression for p_1 and $n_{B,2}=0$ in the expression for p_2. The condition $p_1=p_2$ then becomes

$$\tfrac{1}{2} n_{B,1}^2 U_{BB} = \tfrac{2}{5} A n_{F,2}^{5/3}, \tag{30}$$

which shows that there is only one allowed value for $n_{B,1}$ for a given fermion density $n_{F,2}$.

Let us now look at the chemical potentials. If the chemical potential in the fermionic phase 2 has to be lower or at most equal to that in the bosonic phase 1, then it follows from Eq. (29) that

$$\epsilon_{F,2} \leq n_{B,1} U_{BF}. \tag{31}$$

Likewise, if the chemical potential in the bosonic phase has to be lower than in the fermionic one, then

$$n_{B,1} U_{BB} \leq n_{F,2} U_{BF}, \tag{32}$$

which shows that U_{BF} must be positive, since $U_{BB} \geq 0$ by Eq. (4). Inserting the value of $n_{B,1}$ from Eq. (30) into Eqs. (31) and (32), we obtain from Eq. (31) the condition

$$n_{F,2} \geq \left(\frac{5A U_{BB}}{4 U_{BF}^2} \right)^3, \tag{33}$$

while Eq. (32) yields

$$n_{F,2} \geq \left(\frac{4A U_{BB}}{5 U_{BF}^2} \right)^3, \tag{34}$$

which is less restrictive than Eq. (33). Hence, equilibrium of a pure boson phase and a pure fermion one is possible if and only if the fermion density satisfies Eq. (33) and $n_{B,1}$ is given by Eq. (30).

Now, since

$$n_B = v \, n_{B,1} \quad \text{and}$$
$$n_F = (1-v) \, n_{F,2}, \tag{35}$$

where v is the fraction of the total volume occupied by the bosons, we can find those total densities for which the system can completely separate by using the allowed values of $n_{F,2}$ and $n_{B,1}$ in Eq. (35). (These densities correspond to the points above the uppermost full line in the n_B vs n_F plot in Fig. 3 below.)

Let us now compare the energy of a phase-separated configuration with that of a mixed state with the same number of particles, as given by Eq. (35). According to Eq. (6) the energy of the mixed system is

$$\mathcal{E}_{\text{mix}} = V(\tfrac{1}{2} n_B^2 U_{BB} + n_B n_F U_{BF} + \tfrac{3}{5} n_F \epsilon_F), \tag{36}$$

where the densities n_B and n_F are given by Eq. (35). On the other hand, the energy of the phase-separated system with the same number of particles is

$$\mathcal{E}_{\text{sep}} = V[\tfrac{1}{2} n_{B,1}^2 v \, U_{BB} + \tfrac{3}{5} \epsilon_{F,2} n_{F,2}(1-v)]. \tag{37}$$

The energy difference $\mathcal{E}_{\text{mix}} - \mathcal{E}_{\text{sep}}$ is obtained by substituting for n_B and n_F from Eq. (35), with $n_{B,1}$ given by Eq. (30),

$$\frac{1}{V}(\mathcal{E}_{\text{mix}} - \mathcal{E}_{\text{sep}}) = -\tfrac{2}{5} v(1-v) A n_{F,2}^{5/3}$$
$$+ v(1-v) n_{F,2}^{11/6} U_{BF}(4A/5U_{BB})^{1/2}$$
$$- \tfrac{3}{5}(1-v)[1-(1-v)^{2/3}] A n_{F,2}^{5/3}. \tag{38}$$

Since by Eq. (33) we must have $n_{F,2}^{1/6} \geqslant \sqrt{5 A U_{BB}/4 U_{BF}^2}$, then

$$\frac{1}{V}(\mathcal{E}_{\text{mix}} - \mathcal{E}_{\text{sep}}) \geqslant \tfrac{3}{5} A n_{F,2}^{5/3}(1-v)^{5/3}[1-(1-v)^{1/3}] \geqslant 0 \tag{39}$$

for any v between 0 and 1, and the equality holds only at $v=0$ and $v=1$. So in the density ranges where equilibrium is possible, $\mathcal{E}_{\text{sep}}$ is always less than $\mathcal{E}_{\text{mix}}$, and phase separation is energetically favored compared to mixing.

B. A mixed phase and a purely fermionic one

We now consider a mixed phase in equilibrium with a pure fermion phase and let $n_{B,2}=0$. As we shall see in the following, this type of phase separation is stable in a different range of total densities than that for case A. The pressure equilibrium condition takes the form

$$\tfrac{1}{2} n_{B,1}^2 U_{BB} + n_{B,1} n_{F,1} U_{BF} + \tfrac{2}{5} \epsilon_{F,1} n_{F,1} = \tfrac{2}{5} \epsilon_{F,2} n_{F,2}. \tag{40}$$

In addition, we also have to impose the equality of the fermion chemical potentials in the two phases, which implies

$$n_{B,1} = \frac{A}{U_{BF}}(n_{F,2}^{2/3} - n_{F,1}^{2/3}). \tag{41}$$

First let us assume that U_{BF} is positive. Then Eq. (41) implies that $n_{F,2} \geqslant n_{F,1}$, and it allows us to eliminate $n_{B,1}$ from Eq. (40). The resulting equation is conveniently expressed in terms of the ratio between the fermion densities

$$x = \left(\frac{n_{F,1}}{n_{F,2}}\right)^{1/3} \tag{42}$$

and the dimensionless constant λ defined by

$$\lambda = \frac{A U_{BB}}{U_{BF}^2 n_{F,2}^{1/3}}. \tag{43}$$

Inserting $n_{B,1}$ from Eq. (41) into Eq. (40) we thus obtain

$$\frac{\lambda}{2}(1-x^2)^2 = \tfrac{2}{5} - x^3 + \tfrac{3}{5} x^5. \tag{44}$$

By solving this equation one obtains $n_{f,1}$ as a function of $n_{f,2}$. The problem is greatly simplified by noticing that $(1-x)^2$ is a common factor on both sides of Eq. (44). The value $x=1$ implies that the fermion density is the same everywhere, while the boson density vanishes everywhere according to Eq. (41). The solution $x=1$ can therefore be rejected, and Eq. (44) becomes the cubic equation

$$6x^3 + (12-5\lambda)x^2 + (8-10\lambda)x + 4 - 5\lambda = 0. \tag{45}$$

Since we have assumed that U_{BF} is positive or zero, which, according to Eq. (41), implies that $n_{F,1} \leqslant n_{F,2}$, the physical range of x lies between 0 and 1. Evidently $\lambda = 4/5$ yields the solution $x=0$, while $x=1$ is a solution provided $\lambda = 3/2$. The cubic equation (45) has one positive root in the range $0 < x < 1$, when λ lies in the range

$$\tfrac{4}{5} < \lambda < \tfrac{3}{2}. \tag{46}$$

According to Eq. (43) this implies that the fermion density $n_{F,2}$ is bounded from below as well as from above. We conclude that to each value of $n_{F,2}$ lying within the range given in Eq. (46) there corresponds one value of $n_{F,1}$ and, from Eq. (41), one value of $n_{B,1}$. In Fig. 2 we plot the values of $n_{F,1}$ and $n_{B,1}$ found for each $n_{F,2}$.

When a mixture is separated into two phases with one allowed set of $n_{F,2}$, $n_{F,1}$, and $n_{B,1}$, it is straightforward to show that the boson chemical potential is always less in the mixed phase than in the purely fermionic one, but for brevity we omit the proof.

Just as we did in the previous section, we can deduce the set of all total densities that could undergo a phase separation of this type. Since
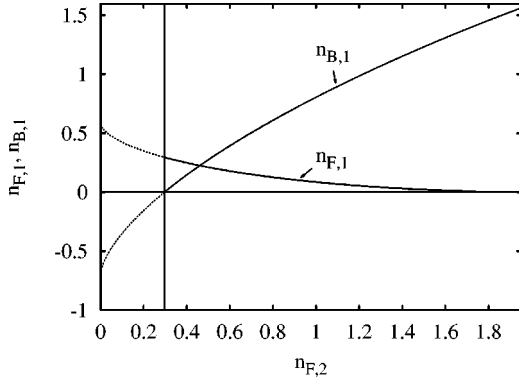
FIG. 2. Densities of fermions ($n_{F,1}$) and bosons ($n_{B,1}$) in a mixed phase in equilibrium with a pure fermion phase of density $n_{F,2}$. The fermion densities are in units of $A^3 U_{BB}^3 / U_{BF}^6$, and the boson densities in units of $A^3 U_{BB}^2 / U_{BF}^5$. The vertical line corresponds to the value $(2/3)^3$, for which $n_{F,1} = n_{F,2}$ and $n_{B,1} = 0$. The dashed parts of the curves are unphysical for $U_{BF} > 0$, since the boson density is negative. For negative U_{BF} they correspond to solutions with positive boson density, since the boson densities in the figure are then scaled to a negative quantity. However, for the mixed phase under these conditions $n_{F,1} > (2/3)^3 A^3 U_{BB}^3 / U_{BF}^6$ and it is unstable.

$$n_B = v \, n_{B,1} \quad \text{and}$$
$$n_F = v \, n_{F,1} + (1-v) \, n_{F,2}, \tag{47}$$

they are obtained by using allowed values for $n_{F,2}$, $n_{F,1}$, and $n_{B,1}$, and varying v from zero to 1. These densities correspond to the points in the region between the upper and lower full curves in Fig. 3. The lower line is obtained by letting $v = 1$ and is therefore just the set of allowed $n_{B,1}$ and $n_{F,1}$. The upper one is found by setting $n_{F,2} = (5/4)^3 (A U_{BB} / U_{BF}^2)^3$, the maximum value according to Eq. (46), to which correspond $n_{F,1} = 0$ and $n_{B,1} = (5/4)^2 (A^3 U_{BB}^2 / U_{BF}^5)$, and varying v. It should be noted that this upper curve is exactly the one above which complete separation into a pure boson phase and a pure fermion
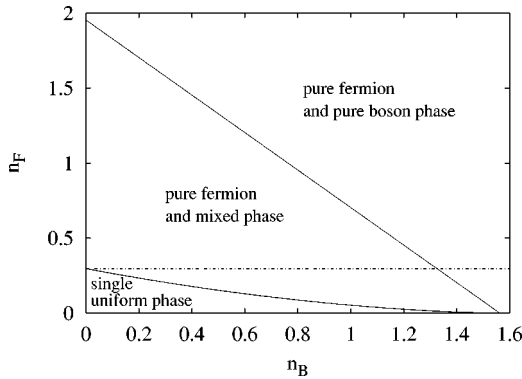

FIG. 3. Phase diagram of uniform boson-fermion mixtures. The total fermion density n_F is given in units of $A^3 U_{BB}^3 / U_{BF}^6$, and the total boson density n_B in units of $A^3 U_{BB}^2 / U_{BF}^5$. The horizontal line corresponds to the value $n_F = (2/3)^3 A^3 U_{BB}^3 / U_{BF}^6$, above which the uniform system is unstable to small fluctuations.

one could take place. This proves the statement that cases A and B are stable for different values of total densities, which in view of the proof for the energy of case A, also demonstrates that above the upper line of Fig. 3 the system will be completely phase separated. In the present case we can show also that the phase-separated configuration has lower energy than the corresponding mixed one. The proof proceeds as in case A and we shall therefore omit it.

One more comment is due at this point. In the linear stability analysis we found that the homogeneous system would become unstable at densities violating Eq. (9), but this value of the fermion density is just where the lower curve in Fig. 3 meets the vertical axis.

Finally, let us consider the situation when U_{BF} is negative. In this case the fermion density in phase 1 must be greater than that in phase 2, $n_{F,1} > n_{F,2}$, according to Eq. (41). We find again that the density ratio $x = n_{F,1} / n_{F,2}$ must satisfy Eq. (45), but now x lies in the region between 1 and infinity. By solving the cubic equation (45) it is readily shown that the values obtained for $n_{F,1}$ are now *greater* than that which is required to ensure linear stability in phase 1 (compare Fig. 2). The phase-separated configuration is thus unstable for $U_{BF} < 0$. The attractive boson-fermion interaction in phase 1 leads to a collapse of the configuration with a mixed and a purely fermionic phase, presumably to a state where the bosons and fermions are clumped together and where the total energy is no longer given by Eq. (6).

With this our knowledge of the phase diagram is complete, since the cases C and D cannot be realized, as proven below.

C. Two mixed phases

We shall now demonstrate that the cases C and D cannot be realized. First we consider two phases with different nonzero boson and fermion densities.

The condition for equality of pressures is

$$\frac{1}{2}(n_{B,1}^2 - n_{B,2}^2) U_{BB} + (n_{B,1} n_{F,1} - n_{B,2} n_{F,2}) U_{BF}$$
$$+ \frac{2}{5}(\epsilon_{F,1} n_{F,1} - \epsilon_{F,2} n_{F,2}) = 0, \tag{48}$$

and for equality of the chemical potentials

$$(n_{B,1} - n_{B,2}) U_{BB} = (n_{F,2} - n_{F,1}) U_{BF} \tag{49}$$

for bosons and

$$(n_{B,1} - n_{B,2}) U_{BF} = A(n_{F,2}^{2/3} - n_{F,1}^{2/3}) \tag{50}$$

for fermions. Combining Eq. (49) with Eq. (50) and using the definitions (42) and (43) we get

$$1 - x^3 = \lambda(1 - x^2). \tag{51}$$

Substituting Eq. (49) into Eq. (48) and rearranging terms, we obtain furthermore

$$(1 - x^6) = \frac{4}{5}\lambda(1 - x^5). \tag{52}$$

By writing $(1-x^6)=(1+x^3)(1-x^3)$ and inserting $1-x^3$ from Eq. (51) into Eq. (52), we obtain

$$-\tfrac{1}{5}x^5+\tfrac{1}{5}+x^3-x^2=0. \tag{53}$$

The left hand side of Eq. (53) has a triple root at $x=1$, and Eq. (53) can therefore be written as

$$\tfrac{1}{5}(x^2+3x+1)(x-1)^3=0, \tag{54}$$

which clearly shows, since x is positive, that the only solution is the trivial one $x=1$. We have thus proven that two mixed phases cannot be in equilibrium with each other.

D. A mixed phase and a purely bosonic one

Finally we consider the case when one of the phases, say phase 1, is free of fermions and prove that equilibrium is impossible. We observe that Eqs. (48) and (49) are valid also in this case, if we put $n_{F,1}=0$. The same is true also for Eq. (52), derived by combining them, which upon the use of $x=0$ becomes

$$\lambda=\tfrac{5}{4}. \tag{55}$$

According to the definition (43) of λ, the system can therefore be in equilibrium for only one particular value of the fermion density. However, the fermion chemical potential in region 1 must be greater than that in region 2,

$$n_{B,1}U_{BF}\geq An_{F,2}^{2/3}+n_{B,2}U_{BF}. \tag{56}$$

Substituting for $n_{B,1}-n_{B,2}$, as derived from Eq. (49) with $n_{F,1}=0$, we obtain

$$n_{F,2}\frac{U_{BF}^2}{U_{BB}}\geq An_{F,2}^{2/3}, \tag{57}$$

which according to the definition (43) implies $\lambda\leq 1$. But the solution (55) does not satisfy this condition and must therefore be discarded. We conclude that case D also cannot be realized physically.

IV. DISCUSSION AND CONCLUSIONS

As we have emphasized, the properties of boson-fermion mixtures are of interest because such mixtures allow sympathetic cooling of fermions, and because the boson-induced fermion-fermion interaction, being attractive, favors pairing of fermions, thus raising the BCS transition temperature. In this concluding section we discuss the implications of our results for actual experimental conditions.

Our main findings are summarized in Fig. 3. This phase diagram shows that fermions and bosons under most circumstances will form a mixed phase, unless the interaction parameters assume quite large values. We shall now quantify this statement, using lithium, potassium, and rubidium atoms as concrete examples. From the results obtained above, e.g., the stability condition (33), it is apparent that the value of the dimensionless fermion density $\tilde{n}_F=n_F(U_{BF}^2/AU_{BB})^3$ plays a key role in determining whether the fermions and bosons

mix or phase-separate into either a purely fermionic phase together with a mixed phase or a purely fermionic phase together with a purely bosonic phase (cf. Fig. 3). The corresponding dimensionless value of the boson density according to Eq. (30) is given by $n_B(U_{BF}^5/A^3U_{BB}^2)$.

Let us therefore look at the values of the actual densities for which these dimensionless densities are equal to 1. A mixture of ^{7}Li (boson) and ^{6}Li (fermion) has recently been produced by Mewes *et al.* [12]. With their setup ^{7}Li can be trapped in the hyperfine state $|1,-1\rangle$ which, contrary to the $|2,2\rangle$ state used in [3], has a positive scattering length ≈ 10 a.u. (see [13]). For a ^{7}Li-^{6}Li collision, on the other hand, the scattering length is ≈ 38 a.u. [14]. From this we deduce that $\tilde{n}_F=1$ corresponds to the density $n_F=9.3\times 10^{23}$ m^{-3}, which is several orders of magnitude higher than the densities that can presently be realized experimentally. When expressed in terms of the boson-boson scattering length a_{BB}, the density is seen to be $n_F=(1.2\times 10^{-4})a_{BB}^{-3}$, indicating the smallness of the gas parameter $n_Fa_{BB}^3$.

Other interesting mixtures involve the potassium isotopes ^{39}K, ^{40}K, and ^{41}K, of which the first and the last are bosons. In these cases, using the values for the scattering lengths given in [15], we predict that phase separation effects would take place for densities exceeding a_{BB}^{-3}, which not only are much higher than can be experimentally realized, but also fall outside the range of applicability of the dilute gas approximation.

A possible candidate for phase separation does, however, exist among the rubidium isotopes. From the scattering lengths given in [16] we find that for ^{87}Rb and ^{84}Rb mixtures the phase separation could take place at reasonable densities. Using the estimates $a_{BB}\approx 100$ a.u. and $a_{BF}\approx 550$ a.u., we have $\tilde{n}_F=1$ when $n_F=(8.1\times 10^{-6})a_{BB}^{-3}=(7.1\times 10^{19})$ m^{-3}, which is of the same order as the densities already realized in experiments with Rb isotopes.

It is also illuminating to compare our findings for the uniform case with previous calculations [9,17] for mixtures in a trap. Due to the Fermi pressure, the fermion density in a trap is usually much smaller than the boson density, and it is therefore a good approximation to assume that the bosonic cloud is unaffected by the fermions. The fermions, on the other hand, experience an effective potential that may be attractive or repulsive inside the boson cloud, depending on the potential parameters and the ratio U_{BF}/U_{BB} of the interaction parameters [9,17]. If for simplicity the two trapping potentials are taken to be isotropic with the same force constants, the effective potential is repulsive if $U_{BF}>U_{BB}$. In this case, the fermions are therefore expelled and form a shell outside the boson condensate. If the effective potential is attractive, corresponding to the condition $U_{BF}<U_{BB}$, the fermions will reside inside the boson cloud. From the discussion above, it is clear that in most cases the fermions inside the cloud will be fully mixed with the bosons. If the interaction parameters are appropriately chosen, one then enters the phase-separated regime of our phase diagram, as is the case in the example shown in Fig. 2 of [9], where a pure fermion phase forms at the center of the trap. The phase diagram that we have obtained here for the uniform case may thus serve

as a useful guide for understanding the general behavior of trapped mixtures.

Note added in proof. Recently, M. J. Bijlsma, B. A. Heringa, and H. T. C. Stoof, e-print cond-mat/9911084, have also considered the effective interaction between fermions in the presence of bosons and reached conclusions identical with ours. They also estimated changes in the transition temperature to the BCS state.

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