Formula Sheet

Physical Constants:

Quantity	Symbol, equation	Value
Speed of light	c	2.9979×10^8 m s^{-1}
Electron charge	e	1.602×10^{-19}C
Planck constant	h	6.626×10^{-34}Js
Planck constant, reduced	$\hbar = h/2\pi$	1.055×10^{-34} J s
Conversion constant	$\hbar c$	197.327MeVfm = 197.327eVnm
Electron mass	m_e	9.109×10^{-31} kg = 0.511MeV/c^2
Proton mass	m_p	1.673×10^{-27} kg^2 = 938.272MeV/c^2
Neutron mass	m_n	1.675×10^{-27} kg = 939.566MeV/c^2
Fine structure constant	$\alpha = e^2/\hbar c$	1/137.036
Classical electron radius	$r_e = e^2/m_e c^2$	2.818×10^{-15} m
Electron Compton wavelength	$\lambda = h/m_e c = r_e/\alpha$	2.426×10^{-12} m
Proton Compton wavelength	$\lambda = h/m_p c$	1.321×10^{-15} m
Bohr radius	$a_0 = r_e/\alpha^2$	0.529×10^{-10} m
Rydberg energy	$\mathcal{R} = m_e c^2 \alpha^2/2$	13.606eV^{-11}MeVT^{-1}
Bohr magneton	$\mu_B = e\hbar/2m_e$	5.788×10^{-11}
Nuclear magneton	$\mu_N = e\hbar/2m_p$	3.152×10^{-14}MeVT^{-1}
Avogadro number	N_A	6.022×10^{23} mol^{-1}
Boltzmann constant	k	1.381×10^{-23} J K^{-1}
		$= 8.617 \times 10^{-5}$eVK^{-1}
Gas constant	$R = N_A k$	8.31 J mol^{-1} K^{-1}
Gravitational constant	G	6.673×10^{-11} m^3 kg^{-1} s^{-2}
Permittivity of free space	$\epsilon_0 = 1/\mu_0 c^2$	8.854×10^{-12}Fm^{-1}
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ N A^{-2}

Conversion of units:

1fm = 10^{-15} m, 1fm = 10^{-15} m, 1 barn = 10^{-28} m^2 = 100fm^2, 1 atmosphere = 101325 Pa, Thermal energy at $T = 300$ K : $kT = [38.682]^{-1}$eV 0°C = 273.15 K, 1eV = 1.602×10^{-19} J, 1eV/c^2 = 1.783×10^{-36} kg

Properties of the Solution of 1D stationary Schrodinger Equation:

1. For 1D potential, all stationary solutions are non-degenerate.
2. Stationary square integrable solution exist only for $E > minV(x)$
3. If V(x) is real, then Ψ(x) can be taken to be real.
4. Eigenvalues of a Hermitian Hamiltonian are all real.
5. The eigenfunctions of a Hermitian operator form a complete orthogonal basis set, for smooth potentials.
6. 1D Schrodinger equation Solution is real up to an over all phase.
7. For a given 1D even potential the stationary states are either even or odd.
8. The wave function and its first order space derivative is continuous all over space and in particular at the boundaries of a finite potential.
9. At boundaries with Dirac delta function potential, the first space derivative of the wavefunction is discontinuous.
10. Physical solution should be finite all over space, no blow ups, in particular at infinity.
11. The number of nodes (zeros) of the eigenfunction increases by one unit as we move from the ground state (zero nodes) to higher excited states.
12. Bound states exist only for confining potential (classically between turning points of the potential).

The Origin of Quantum Physics:

$$E = h\omega \qquad p = \frac{h}{\lambda} = \hbar k \tag{1}$$

$$\lambda = \frac{h}{p} \qquad \lambda_C = \frac{h}{mc} \tag{2}$$

Blackbody Radiation:

Plank energy spectral density: $\rho(\nu) = \frac{8\pi h\nu^3}{c^3}\left[\frac{1}{\exp\left(\frac{h\nu}{k_B T}\right)-1}\right]$

Photoelectric Effect:

W is the work function of irradiated metal. V_s is the stopping potential.

$$K = {}^1\!/\!_2\, mv^2 = h\nu - W = \frac{hc}{\lambda} - W \;; \quad K = {}^1\!/\!_2\, mv^2 = |e|V_s \leftrightarrow V_s = \frac{K}{|e|}$$

$$\nu \geq \frac{W}{h} = \nu_{\min}; \quad \nu = \frac{1}{T} = \frac{c}{\lambda}; \quad \nu = \frac{\omega}{2\pi}$$

de Broglie Formula:

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

Bohr Hydrogen-like Atom:

$$E_n = -\frac{Z^2}{n^2}R \qquad R = 13.6\,\mathrm{eV}$$

$$r_n = \frac{a_0}{Z}n^2 \qquad a_0 = 0.53 \times 10^{-10}\,\mathrm{m}$$

$$h\nu = E_n - E_m = Z^2 R\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \qquad n > m$$

The Wave Function:

$\Psi(x,t)$ obeys Schrodinger's equation, and the normalization condition $\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$:

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx; \langle p\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi(x,t)dx; \langle Q(\hat{x},\hat{p})\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)Q\left(x, \frac{h}{i}\frac{\partial}{\partial x}\right)\Psi(x,t)dx \tag{3}$$

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = H\Psi(x,t); \qquad \Psi(x,t) = \psi(x)e^{-iEt/\hbar}; \qquad H\psi(x) = E\psi(x) \tag{4}$$

$$\rho(x,t) = |\Psi(x,t)|^2; \qquad \frac{\partial}{\partial t}\rho(\mathbf{x},t) + \nabla\cdot\mathbf{J}(\mathbf{x},t) = 0; \quad J(x,t) = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) \tag{5}$$

$$\mathbf{J}(\mathbf{x},t) = \frac{h}{2im}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \quad p = \frac{\hbar}{i}\nabla; \qquad [x_i, p_j] = i\hbar\;\delta_{i,j} \tag{6}$$

Hermitian conjugate $A^\dagger$ is defined by: $\int (A\psi(x))^*\psi(x), dx = \int \psi(x)A^\dagger\psi(x), dx$.

Fourier Transform

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int dk \Phi(k) e^{ikx}, \quad \Phi(k) = \frac{1}{\sqrt{2\pi}} \int dx \Psi(x) e^{-ikx} \quad \int dx |\Psi(x)|^2 = \int dk |\Phi(k)|^2 \tag{7}$$

$$\Psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \Phi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \Phi(k) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3x \Psi(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \quad \int d^x |\Psi(\mathbf{x})|^2 = \int d^k |\Phi(\mathbf{k})|^2 \tag{8}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx = \delta(k) \quad \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}} d^3x = \delta^{(3)}(k) \quad \Psi(x,t) = \frac{1}{(2\pi)^3} \int \phi(k) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t)}\, dk^3 \tag{9}$$

Wavepackets

$$v_{group} = \frac{d\omega}{dk};\ \Delta k \Delta x \simeq 1 \tag{10}$$

Complete Basis Set:

Given that $H\psi_n(x) = E_n\psi_n(x)$; $\int \phi_n^*(x)\phi_m(x)dx = \delta_{nm}$, where $\{\phi_n\}$ is a complete set, then:

$$\psi(x) = \sum_n c_n \phi_n(x); \quad C_n = \int \psi_n^*(x)\psi(x)\, dx \tag{11}$$

$$\int \psi^*(x)\psi(x)dx = \sum_n |c_n|^2 = 1 \tag{12}$$

$$E = \int \psi_n^*(x) H \psi_m(x) dx = \sum_n |c_n|^2 E_n \tag{13}$$

$$\Psi(x,0) = \psi(x) = \sum_n c_n \phi_n(x) \implies \Psi(x,t) = \sum_n c_n e^{-iE_n t/\hbar} \phi_n(x) \tag{14}$$

$$c_n = \int \phi_n^* \Psi(x,0) dx \tag{15}$$

Commutator Properties:

$$[A, A] = 0; \quad [A, B] = -[B, A]; \quad [A + B, C] = [A, C] + [B, C] \tag{16}$$

$$[AB, C] = [A, C]B + A[B, C]; \quad [A, BC] = [A, B]C + B[A, C]; \tag{17}$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{18}$$

Uncertainty Principle:

$$(\Delta Q)^2 = \langle Q^2 \rangle - \langle Q \rangle^2 = \langle (Q - \langle Q \rangle)^2 \rangle \qquad \Delta x \Delta p \geq \frac{\hbar}{2} \tag{19}$$

Where ΔQ is the uncertainity for the Hermitian Operator Q.

Operators:

For the operator $\hat{A}$, $\hat{A}\psi = a\psi$. a in an eigenvalue and ψ is an eigenfunction of A. Then, the following properties hold:

- $\hat{A}^n\psi = a^n\psi, \quad \hat{A}^{-1}\psi = a^{-1}\psi, \quad e^{i\hat{A}}\psi = e^{ia}\psi, \quad F(\hat{A})\psi = F(a)\psi$
- $\hat{A}^\dagger = A,\ \hat{A}\,|\phi_n\rangle = a_n\,|\phi_n\rangle \quad \implies a_n \in \mathbb{R}, \langle\phi_m|\phi_n\rangle = \delta_{mn}$
- If $\{\phi_n\}$ is a complete and orthonormal for a Hermitian operator, then the operator is diagonal in the eigenbasis, $\{\phi_n\}$, with eigenvalues,, $\{a_n\}$, as the diagonal elements. The basis set is unique iff there are no degenerate eigenvalues.
- If two Hermitian operators, $\hat{A}$ and $\hat{B}$, commute and have no degenerate eigenvalues. Then each eigenvector of $\hat{A}$ is also an eigenvector of $\hat{B}$. A common orthonormal basis can be made of the joint eigenvectors of $\hat{A}$ and $\hat{B}$.

1D Infinite Square Well:

$$H\psi_n(x) = E_n\psi_n(x) \qquad \int \psi_n^*(x)\psi_n(x)dx = \delta_{n,m} \tag{20}$$

$$\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right) \qquad \phi_n(x,t) = \phi_n(x)e^{-iE_nt/\hbar} \tag{21}$$

$$V(x) = \begin{cases} 0, 0 \le x \le a \\ \infty, \text{otherwise} \end{cases} \qquad E_n = \frac{\hbar^2k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2} \tag{22}$$

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n\phi_n(x)e^{-iE_nt/\hbar} \qquad c_n = \int_0^a \phi_n(x)\Psi(x,0)\,dx \tag{23}$$

Particle on a Ring:

$$\psi_\pm(\theta) = \frac{1}{\sqrt{2\pi}}\exp \pm i\frac{R\theta}{\hbar}\sqrt{2mE} = \frac{1}{\sqrt{2\pi}}e^{\pm ikx} \qquad x = R\theta; L = 2\pi R; k = \frac{2\pi n}{L} = \frac{n}{R} \tag{24}$$

$$\psi(\theta) = \frac{1}{\sqrt{2\pi}}\exp \pm in\theta \qquad E_n = \frac{n^2\hbar^2}{2mR^2}, \quad n = 0 \pm 1, \pm 2, \pm 3, ... \tag{25}$$

Harmonic Oscillator:

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m(\omega x)^2 \quad \left(\omega \equiv \sqrt{k/m}\right) \qquad E_n = \hbar\omega\left(n+\frac{1}{2}\right); n = 0, 1, 2, \ldots \tag{26}$$

$$H = \frac{1}{2m}[p^2 + (m\omega x)^2] = \hbar\omega\left(N + \frac{1}{2}\right) \qquad N = a_+a_- \quad (= a^\dagger a) \tag{27}$$

$$N\psi_n = n\psi_n \qquad N(a_+\psi_n) = [N, a_+]\psi_n \tag{28}$$

$$[N, a_\pm] = \pm a_\pm \qquad a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x) \tag{29}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \qquad p = i\sqrt{\frac{m\omega\hbar}{2}}(a_+ - a_-) \tag{30}$$

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1} \qquad a_-\psi_n = \sqrt{n}\psi_{n-1} \tag{31}$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \qquad \psi_n = \frac{1}{\sqrt{n!}}(a_+)^n\psi_0 \tag{32}$$

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x \qquad \mathcal{H}_n(\xi) = (-1)^n e^{\xi^2}\left(\frac{d}{d\xi}\right)^n e^{-\xi^2} \tag{33}$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\mathcal{H}_n(\xi)e^{-\xi^2/2} \qquad \int_{-\infty}^{+\infty}\mathcal{H}_n\mathcal{H}_m e^{-x^2}dx = 2^n n!\sqrt{\pi}\delta_{n,m} \tag{34}$$

Models of Dirac Delta Distribution $\delta(x)$:

$$\delta(x) = \lim_{\alpha\to\infty}\frac{\sin(\alpha x)}{\pi x};\ \delta(x) = \lim_{\epsilon\to 0^+}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}e^{-\epsilon|k|}\,dk = \lim_{\epsilon\to 0^+}\frac{\epsilon}{\pi(x^2+\epsilon^2)};\ \delta(x) = \lim_{\epsilon\to 0}\frac{\Theta(x+\epsilon)-\Theta(\epsilon)}{\epsilon} \tag{35}$$

where $\Theta(x)$ is Heaviside or step function.

Bound State of Single δ-Potential:

$$V = -\alpha\delta(x),\ \alpha > 0 \qquad \psi(x) = \sqrt{\frac{m\alpha}{\hbar^2}}\ e^{-\frac{m\alpha}{\hbar^2}|x|} \qquad E = -\frac{m\alpha^2}{2\hbar^2} \tag{36}$$

Scattering State:

$$V(x) = -\alpha\delta(x) \qquad \psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, x < 0 \\ Fe^{ikx}, x > 0 \end{cases} \tag{37}$$

$$T = \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2} \qquad R = \frac{|B|^2}{|A|^2} = \frac{\beta}{1+\beta} \qquad (\beta = m\alpha/\hbar^2 k) \tag{38}$$

Miscellaneous:

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}; \qquad \int_{-\infty}^{\infty} e^{-(ax^2+bx)}\,dx = e^{b^2/4a}\sqrt{\frac{\pi}{a}}; \qquad \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases} \tag{39}$$

Matrix Algebra:

Let A be a 2X2 matrix defined as: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then:

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}; \qquad |A| = ad - bc \tag{40}$$

Orbital Angular Momentum:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \tag{41}$$

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z, \quad \left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x, \quad \left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y. \tag{42}$$

$$\hat{L}^2 \equiv \hat{L}_x\hat{L}_x + \hat{L}_y\hat{L}_y + \hat{L}_z\hat{L}_z, \quad \left[\hat{L}^2, \hat{L}_i\right] = 0 \tag{43}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \tag{44}$$

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \tag{45}$$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}; \quad \hat{L}_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) = \hat{L}_x \pm i\hat{L}_y \tag{46}$$

General Momentum Operator

$$[J_x, J_y] = i\hbar J_z \qquad [J_z, J_x] = i\hbar J_y \qquad [J_y, J_z] = i\hbar J_x \tag{47}$$

$$J^2\,|jm\rangle = \hbar^2 j(j+1)\,|jm\rangle \qquad J_z\,|jm\rangle = \hbar m\,|jm\rangle \qquad J^2 = J_x^2 + J_y^2 + J_z^2 \tag{48}$$

$|jm\rangle$ are common eigenstates of the J^2, J_z;

Thus both J^2 and J_z are diagonal matrices for any fixed value of J, the dimension of these matrices is $(2J+1)\times(2J+1)$ since $m = -J, -J+1, \ldots, J-1, J$, (2J+1 Values). $J = \frac{1}{2} \implies 2J+1 = 2 \implies 2\times 2 matrices.$

$J = 1 \implies 2J+1 = 3 \implies 3\times 3 matrices.$

$$J_\pm = J_x \pm iJ_y \qquad \sqrt{J(J+1) - m(m\pm 1)}\,|jm\pm 1\rangle \tag{49}$$

so that $J_\pm$ are off diagonal matrices.

We oreder the eigenvectors for a fixed J-value as the following:

$|j,j\rangle \quad |j,j-1\rangle \quad |j,j-2\rangle \quad \ldots \quad |j,-j\rangle$

Spin Angular Momentum Operator

For $j=\frac{1}{2} \implies |\frac{1}{2},\frac{1}{2}\rangle\,;|\frac{1}{2},-\frac{1}{2}\rangle\, then:$

$$S_x=\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad S_y=\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad S_z=\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{50}$$

$$[S_x,S_y]=i\hbar S_z \qquad [S_z,S_x]=i\hbar S_y \qquad [S_y,S_z]=i\hbar S_x \tag{51}$$

$$S_+=S_x+iS_y \qquad S_-=S_x-iS_y \tag{52}$$

$$S_+=\frac{\hbar}{2}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix} \qquad S_-=\frac{\hbar}{2}\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix} \tag{53}$$

Spherical Harmonics:

$$Y_{\ell,m}(\theta,\phi)\equiv \mathcal{N}_{\ell,m}P_\ell^m(\cos\theta)e^{im\phi} \tag{54}$$

$$\hat{L}_z Y_{\ell m}=\hbar m Y_{\ell m} \tag{55}$$

$$\hat{L}^2 Y_{\ell m}=\hbar^2\ell(\ell+1)Y_{\ell m} \tag{56}$$

$$\int d\Omega Y^*_{\ell' m'}(\theta,\phi)Y_{\ell m}(\theta,\phi)=\delta_{\ell',\ell}\delta_{m',m}, \quad \int d\Omega=\int_0^{2\pi}d\phi\int_{-1}^{1}d(\cos\theta) \tag{57}$$

$$Y_{0,0}(\theta,\phi)=\frac{1}{\sqrt{4\pi}}; \quad Y_{1,\pm1}(\theta,\phi)=\mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi); \quad Y_{1,0}(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta \tag{58}$$

Central Potentials $V(\mathbf{r}) = V(r)$:

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(r,\theta,\phi) = E\psi(r,\theta,\phi) \tag{59}$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{L^2(\theta,\phi)}{\hbar^2 r^2} + V(r)\right]\psi(r,\theta,\phi) = E\psi(r,\theta,\phi) \tag{60}$$

$$\psi(r,\theta,\phi) = R(r)Y_{l,m} * (\theta\phi) \tag{61}$$

$$\psi(r,\theta,\phi) = \frac{u(r)}{r}Y_{\ell m}(\theta,\phi) \tag{62}$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right)u(r) = Eu(r) \tag{63}$$

$$u(r) \sim r^{\ell+1},\ \mathrm{as} r \to 0 \nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \tag{64}$$

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \tag{65}$$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi};\quad \hat{L}_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) = \hat{L}_x \pm i\hat{L}_y \tag{66}$$

$$\tag{67}$$

Hydrogen Atom (Z=1):

$$H = \frac{\mathbf{p}^2}{2m} - \frac{Ze^2}{r} \tag{68}$$

$$E_n = -\frac{Z^2e^2}{2a_0}\frac{1}{n^2},\quad a_0 = \frac{\hbar^2}{me^2} \simeq 0.529\times 10^{-10}\ \mathrm{m},\quad \frac{e^2}{2a_0} \simeq 13.6\mathrm{eV} \tag{69}$$

$$\psi_{n,\ell,m}(\vec{x}) = A\left(\frac{r}{a_0}\right)^{\ell}\left(\text{ Polynomial in } \frac{r}{a_0} \text{ of degree } n-(\ell+1)\right)e^{-\frac{Zr}{na_0}}Y_{\ell,m}(\theta,\phi) \tag{70}$$

$$n = 1,2,\ldots,\quad \ell = 0,1,\ldots,n-1,\quad m = -\ell,\ldots,\ell \tag{71}$$

$$\psi_{n,\ell,m}(\vec{x}) = \frac{u_{n\ell}(r)}{r}Y_{\ell,m}(\theta,\phi) \tag{72}$$

$$u_{1,0}(r) = \frac{2r}{a_0^{3/2}}\exp\left(-r/a_0\right) \tag{73}$$

$$u_{2,0}(r) = \frac{2r}{(2a_0)^{3/2}}\left(1-\frac{r}{2a_0}\right)\exp\left(-r/2a_0\right) \tag{74}$$

$$u_{2,1}(r) = \frac{1}{\sqrt{3}}\frac{1}{(2a_0)^{3/2}}\frac{r^2}{a_0}\exp\left(-r/2a_0\right) \tag{75}$$