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Sparse and robust mean–variance portfolio optimization problems

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HIGHLIGHTS

- We introduce a sparse mean–variance portfolio model.
- We propose two general sparse and robust portfolio models.
- Three empirical studies with real market data are proposed.

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ABSTRACT

Mean–variance portfolios have been criticized because of unsatisfying out-of-sample performance and the presence of extreme and unstable asset weights. The bad performance is caused by estimation errors in inputs parameters, that is the covariance matrix and the expected return vector, especially the expected return vector. This topic has attracted wide attention. In this paper, we aim to find better portfolio optimization model to reduce the undesired impact of parameter uncertainty and estimation errors of mean–variance portfolio model. Firstly, we introduce a sparse mean–variance portfolio model, and give some insight about sparsity. Secondly, we propose two sparse and robust portfolio models by using objective function regularization and robust optimization. Finally, three empirical studies are proposed with real market data.

1. Introduction

The mean–variance portfolio model originally proposed in Markowitz [1] indisputably constitutes the foundation for modern finance. Under the classical mean–variance framework, optimal portfolio weights are a function of two parameters: the expected returns vector and the inverse covariance matrix. Since the true values of these parameters are unknown in practice, investors traditionally estimate them by using historical data.

Some studies over the past years have shown that estimation errors lead to extremely poor out-of-sample portfolio performance. Green and Hollifield [2] argued that the extreme negative and positive weights are observed often as it entirely due to imprecise estimation of the inputs used to construct mean–variance efficient portfolios. Recently, DeMiguel et al. [3] evaluated the out-of-sample performance of the mean–variance model and found that none of existing portfolio selection consistently outperforms the naive $1/N$ portfolio. Such examples in Green and Hollifield [2],

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DeMiguel et al. [3] mean that the classical Markowitz portfolio selection model has to be modified when used in practice in order to make the resulting allocation depend less sensitively on the input vectors. Being aware of the importance of parameter uncertainty and estimation errors, various efforts have been made to modify the Markowitz mean–variance optimization problem to achieve reliability, stability, and robustness with respect to model and estimation errors.

The first approach aims to obtain better estimates of covariance matrix. Ledoit and Wolf [4,5] proposed replacing the sample covariance matrix with a weighted average of the sample covariance matrix and a low-variance target estimator, $\widehat{\Sigma}_{target}$. Jagannathan and Ma [6] imposed the no-short-sale constraint on the minimum-variance optimization problem and gave an insightful explanation and demonstration of why the constraints help. However, as to be shown in this article, the optimal no-short-sale portfolio is not diversified enough. DeMiguel et al. [7] provided a general framework for finding portfolios that perform well out-of-sample in the presence of estimation error. This framework relies on solving the traditional minimum-variance problem but subject to the additional constraint that the norm of the portfolio-weight vector be smaller than a given threshold. Brodie et al. [8] showed that imposing a penalty on the 1-norm of the asset weights vector (i.e. L_1-regularization) not only regularizes the problem, thereby improving the out-of-sample performance, but also allows to automatically select a subset of assets to invest in (promoting sparsity). The proposed approach recovers as special cases the no-short positions portfolios, but does allow for short positions in limited number. Recent examples with different norm of the asset weights vector applied in this framework are Fan et al. [9], Yen and Yen [10], and Fastrich et al. [11].

Another method to reduce the undesired impact of parameter uncertainty is robust optimization. During the last 2 decades, robust optimization methodology made good progress which refers to the modeling of optimization problems with data uncertainty (see [12–14]. In the framework of robust optimization, one tries to find a solution that is guaranteed to be satisfactory for most realizations of the uncertain parameters. In this setting, Lobo and Boyd [15], Goldfarb and Iyengar [16] studied the robust portfolio problem under the mean–variance framework. Instead of assuming precise information on the mean and the covariance matrix of asset returns, they introduced some types of uncertainties, such as polyhedral uncertainty, box uncertainty and ellipsoidal uncertainty, in the parameters in determining the mean and the covariance matrix, and they then transformed the problem into semidefinite programs (SDP) or second-order cone programs (SOCP), which can be efficiently solved by interior-point algorithms developed in recent years. Halldórsson and Tütüncu (2004) applied their interior-point method for saddle-point problems to the robust mean–variance portfolio selection under the box uncertainty of the elements in the mean vector and the covariance matrix. The robust optimization method also have been adopted on the other portfolio selection problems. For example, El Ghaoui et al. [17], Tütüncü and Koenig [18], Natarajan et al. [19], Zhu and Fukushima [20], Ruan and Fukushima [21], Dai and Wen [22,23], Romanko and Mausser [24], Dai and Wen [25], Dai et al. [26], Li et al. [27] and Pafka and Kondor [28].

In fact, the method of obtaining better estimates of covariance matrix mainly deals with the parameter uncertainty and estimation errors of covariance matrix. But it does not consider the parameter uncertainty and estimation errors of the mean returns. However, as pointed out by Merton [29], it is more difficult to estimate means than estimating covariances of asset returns in the mean–variance portfolio problem. Black and Litterman [30] showed that in the classical Markowitz mean–variance model, the portfolio decision is very sensitive to the mean returns and the covariance matrix, especially to the mean returns.

A natural question follows: can we find better models to reduce the undesired impact of parameter uncertainty and estimation errors of Markowitz mean–variance portfolio problem? The answer to this question clearly has important bearing on the debate about portfolio selection problems. Our objective is to address this question by two steps. Firstly, we introduce a sparse mean–variance portfolio model, and give some insight about it; Secondly, we propose some general sparse and robust portfolio models by using objective function regularization and robust optimization.

The rest of this paper is organized as follows. In the next section, we introduce a sparse mean–variance portfolio model. In Section 3, we present some sparse and robust mean–variance portfolio models. In Section 4, we report some empirical studies to test the proposed methods.

2. Sparse mean–variance portfolio model

In Markowitz portfolio selection problems, expected returns and covariance matrix are two inputs. For the empirical implementation, we can compute expectations and covariance matrix by sample. Let $r_t = (r_{1t}, r_{2t}, \ldots, r_{nt})^T \in R^n$ be the vector of asset returns at time $t (t = 1, \ldots, T)$, $E(r_t) = \mu$ and $\Sigma = E[(r_t - \mu)(r_t - \mu)^T]$ be the mean return vector and the covariance matrix of asset returns, where r_{it} is the return of asset i at time t. Since $E(r_t) = \mu$ and $\Sigma = E[(r_t - \mu)(r_t - \mu)^T]$, we can obtain

$$\mathbf{w}^T \Sigma \mathbf{w} = E[|\rho - \mathbf{w}^T r_t|^2] = \frac{1}{T} \|\rho e - R\mathbf{w}\|_2^2, \tag{2.1}$$

where R is $T \times n$ matrix of which row t equals r_t, that is, $R_{t,i} = (r_t)_i = r_{i,t}$.

If we replace expectations by sample averages, that is, $\widehat{\mu} = \frac{1}{T} \sum_{t=1}^{T} r_t$, then the mean–variance model can be expressed, in the statistical regression view, as follows

$$\min_{\mathbf{w} \in R^n} \quad \frac{1}{T} \|\rho e - R\mathbf{w}\|_2^2$$

$$
\begin{aligned}
s.t.\quad & \mathbf{w}^T\widehat{\mu} = \rho. \\
& \mathbf{w}^T\mathbf{e} = 1.
\end{aligned}
\tag{2.2}
$$

where $\|\cdot\|_2$ is the L_2 vector norm.

The problem (2.2) requires the solution of a constrained multivariate regression problem which may be quite challenging in practice, depending on the nature of the matrix R. If the condition number of R is small, the problem is numerically stable and easy to solve. However, If the condition number is large, a non-regularized numerical procedure will amplify the effects of noise, leading to an unstable and unreliable estimate of the vector $\mathbf{w}$.

To obtain meaningful, stable results for such ill-conditioned problems, one typically adopts a regularization procedure. Brodie et al. [8] succeed in applying the l_1-norm technique to the Markowitz mean–variance model to obtain the following sparse portfolios

$$
\begin{aligned}
\min_{\boldsymbol{w}\in R^n}\quad & \|\rho\mathbf{1}_T - \mathbf{R}\mathbf{w}\|_2^2 + \tau\|\mathbf{w}\|_1^1 \\
s.t.\quad & \mathbf{w}^T\widehat{\mu} = \rho, \\
& \mathbf{w}^T\mathbf{1}_n = 1,
\end{aligned}
\tag{2.3}
$$

where $\widehat{\mu} = \frac{1}{T}\sum_{t=1}^{T} r_t$, define $\mathbf{R}$ as the $T \times n$ matrix of which row t equals r_t^T, and $\|\mathbf{w}\|_1^1 = \sum_{i=1}^{n}|w_i|$, and τ is a parameter that allows us to adjust the relative importance of the L_1 penalization in our optimization. In the above portfolio optimization model, the constraint on short selling is not considered. because increasing the parameter τ can result in on short selling.

Adding an L_1 penalty to the objective function in (2.1) has several useful consequences. (1) It promotes sparsity. As we know, sparsity can play a key role in the task of selecting investment portfolios: investors frequently want to be able to limit the number of positions. (2) It stabilizes the problem. By imposing a penalty on the size of the coefficients of $\mathbf{w}$ in an appropriate way, we can reduce the sensitivity of the optimization to the possible coefficients between the assets, especially, if the condition number of R is large. (3) It regulates the amount of shorting in the portfolio designed by the optimization process. By decreasing τ in the L_1-penalized objective function, one relaxes the constraint without removing it completely. As it then no longer imposes positivity absolutely, but still penalizes overly large negative weights.

3. Sparse and robust portfolio selection

Although the L_1-regularization mean–variance portfolio model (2.3) not only can prevent an overfitting of the estimates of covariance matrix, specifically, when the condition number of R is large, but also can obtain a sparse portfolio. It do not consider the parameter uncertainty and estimation errors of the mean return vector. However, as pointed out by Black and Litterman [30], in the classical Markowitz mean–variance model, the portfolio decision is very sensitive to the mean and the covariance matrix, especially to the mean. Chopra and Ziemba [31] showed that small changes in the input parameters can result in large changes in the optimal portfolio allocation.

The expected returns μ in constraint (2.3) are assumed to be exactly known. In fact, the expected assets' returns μ are uncertain. It is well known that it is more difficult to estimate means than covariances of asset returns and also that errors in estimates of means have a larger impact on portfolio weights than errors in estimates of covariances (see Merton [29], DeMiguel et al. [7]). One way to address this issue is to consider a robust version of the expected portfolio returns. More specifically, we propose the following robust version of the constraint on the expected portfolio returns

$$
\min_{\mu\in U}\quad \mathbf{w}^T\mu = \rho, \tag{3.1}
$$

where ρ denotes the worst-case required expected return specified by the investor.

The above discussions motivate how to achieve both sparsity and robust in a portfolio choice problem. We propose a new sparse and robust portfolio optimization model, which can be represented as follows

$$
\begin{aligned}
\min_{\boldsymbol{w}\in R^n}\quad & \|\rho\mathbf{1}_T - \mathbf{R}\mathbf{w}\|_2^2 + \tau\|\mathbf{w}\|_1^1 \\
s.t.\quad & \min_{\mu\in U}\quad \mathbf{w}^T\mu = \rho, \\
& \mathbf{w}^T\mathbf{1} = 1,
\end{aligned}
\tag{3.2}
$$

Where $\|\mathbf{w}\|_1^1 = \sum_{i=1}^{n}|w_i| \le \delta$, τ is a regularization parameter that allows us to adjust the relative importance of the L_1 penalization in our optimization. Model (3.2) is not ready for application because of the minimize operation involved in the constraints. In the following, we will investigate a tractable robust formulation of the constraint on the expected return (3.1) which belongs to different uncertainty sets.

3.1. Box uncertainty set

Suppose that μ belongs to a box uncertainty set, that is,

$$
\mathcal{U}_B = \{\mu : \mu = \mu_0 + \xi, |\xi_i| \le \delta_i, \quad i = 1, \dots, n\}, \tag{3.3}
$$

where μ_0 denotes the nominal value of μ. In this paper, we select $\widehat{\mu} = \frac{1}{T}\sum_{t=1}^{T} r_t$ as μ_0. The worst case mean return of a fixed portfolio $\boldsymbol{w}$ is given by

$$\min_{\{\mu \in \mathcal{U}_B\}} \quad \boldsymbol{w}^T \widehat{\mu} = \mu_0^T \boldsymbol{w} - \delta^T |\boldsymbol{w}|. \tag{3.4}$$

Adding an auxiliary vector α, the robust counterpart of (3.1) is equivalent to

$$\begin{aligned} &\mu_0^T \boldsymbol{w} - \delta^T \alpha = \rho, \\ &\alpha_i \geq w_i, i = 1, \ldots, n, \\ &\alpha_i \geq -w_i, i = 1, \ldots, n. \end{aligned} \tag{3.5}$$

which are evidently linear constraints.

Therefore, under box uncertainty set, the sparse and robust Markowitz mean–variance portfolio selection can be written as

$$\begin{aligned} \min_{\boldsymbol{w} \in R^n} \quad & \|\rho \mathbf{1}_T - \mathbf{R}\mathbf{w}\|_2^2 + \tau \|\mathbf{w}\|_1^1 \\ s.t. \quad & \mu_0^T \boldsymbol{w} - \delta^T \alpha = \rho, \\ & \alpha_i \geq w_i, i = 1, \ldots, n, \\ & \alpha_i \geq -w_i, i = 1, \ldots, n, \\ & \mathbf{w}^T \mathbf{1} = 1. \end{aligned} \tag{3.6}$$

where $\mu_0 = \widehat{\mu} = \frac{1}{T}\sum_{t=1}^{T} r_t$, $\|\mathbf{w}\|_1^1 = \sum_{i=1}^{n} |w_i| \leq \delta$, τ is a regularization parameter. Following Brodie et al. [8], in the above portfolio optimization model, the constraint on short selling is not considered. Soyster [32] proposed a linear optimization model to construct a solution that is feasible for all data that belong in a box uncertainty set. The resulting model produces solutions that are some conservative in the sense that we give up some of optimality for the nominal problem in order to ensure robustness.

3.2. Ellipsoidal uncertainty set

A significant step forward for developing a theory for robust optimization was taken independently by El Ghaoui et al. [12] and Ben-Tal and Nemirovski [13,14]. To address the issue of overconservatism, these papers proposed less conservative models by considering uncertain linear problems with ellipsoidal uncertainty set $\mathcal{U}$, that is,

$$\mu \in \mathcal{U}_E = \{\mu : \mu = \mu_0 + Pu, \|u\|_2 \leq 1\}, \tag{3.7}$$

where μ_0 denotes the nominal value of μ, and $\mathrm{P} = \mathrm{P}^T \in \mathrm{R}^{n \times n}$ is the scaling matrix of the ellipsoid.

Suppose that μ belongs to an ellipsoidal uncertainty set (3.7). Then, (3.1) can be expressed as

$$\min_{\{\mu \in \mathcal{U}_E\}} \quad \boldsymbol{w}^T \mu = \boldsymbol{x}^T \mu_0 - \|P\boldsymbol{w}\|_2 = \rho_w, \tag{3.8}$$

which is evidently a second-order cone constraint.

Under ellipsoidal uncertainty set (3.7), the sparse and robust Markowitz mean–variance portfolio selection optimization can be expressed as

$$\begin{aligned} \min_{\boldsymbol{w} \in R^n} \quad & \|\rho \mathbf{1}_T - \mathbf{R}\mathbf{w}\|_2^2 + \tau \|\mathbf{w}\|_1^1 \\ s.t. \quad & \boldsymbol{w}^T \mu_0 - \|P\boldsymbol{w}\|_2 = \rho, \\ & \mathbf{w}^T \mathbf{1} = 1. \end{aligned} \tag{3.9}$$

where $\mu_0 = \widehat{\mu} = \frac{1}{T}\sum_{t=1}^{T} r_t$, $\|\mathbf{w}\|_1^1 = \sum_{i=1}^{n} |w_i| \leq \delta$, τ is a regularization parameter. In the above sparse and robust Markowitz mean–variance portfolio selection optimization model, the constraint on short selling is not considered. As increasing the parameter τ can result in on short selling.

It is obvious that the robust counterpart of optimization problem (3.9) becomes an second-order cone programming (SOCP) which can be solved efficiently by interior-point methods developed in recent years. Moreover, solving the above model also is a easy task for which, standard software solution exists, for example, optimization package cvx [33]. Comparing robust mean–variance portfolio model (3.9) to (3.6), as ellipsoidal uncertainty can reduce the degree of conservatism of box uncertainty, the robust mean–variance portfolio model (3.9) strategy we can be excepted to yield the better result.

4. Empirical studies

In this section, we apply the methodologies described above to construct optimal portfolios and evaluate their out-of-sample performance by employing real market data.

Table 1
List of data sets.

No.	Data set	Number of stock	Time period	Source
1	FF-100	100	12/1992-12/2016	K. French
2	FF-48	48	12/1992-12/2016	K. French
3	500 CRSP	500	04/1992-04/2016	CRSP
4	100 CRSP	100	04/1992-04/2016	CRSP

4.1. Data and models

In our empirical studies, the tested portfolio models have the following meanings:

- "1/N" stands for equally-weighted (1/N) portfolio [3].
- "MINC" stands for minimum-variance portfolio with shortsales constrained [6].
- "MV1R" stands for L_1 regularization mean–variance portfolio model (2.3).
- "MVSRB" stands for sparse and robust mean–variance portfolio model under box uncertainty set (3.6).
- "MVSRE" stands for sparse and robust mean–variance portfolio model under ellipsoidal uncertainty set (3.9).

We compute the optimal solutions of the above models by optimization package cvx [33], which can be taken from the website http://cvxr.com/cvx/download/.

In our empirical study, we use four historical returns of exchange-traded assets to test the out-of-sample performance of proposed models. The list of the data used in our empirical studies is described in Table 1.

Table 1 lists the various data sets used for the evaluation of the portfolio performance, their abbreviations, the number of assets that each data set comprises, the time period over which we use data from each particular data set, and the data sources.

The one hundred and forty-eight Fama French portfolios, FF-100 and FF-48, are taken from Ken French's website

http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

and represent different cuts of the US stock market.

The data set of 500CRSP is constructed in a way that is similar to Jagannathan and Ma [6], DeMiguel et al. [7], Zhang and Liu [34], Shen et al. [35] and Dai and Wen [25] with monthly rebalancing: in April of each year we randomly select 500 assets among all assets in the CRSP data set for which there is return data for the previous 120 months as well as for the next 12 months. We then consider these randomly selected 500 assets as our asset universe for the next 12 months. The 100CRSP is constructed in a similar way as 500CRSP expect that we randomly select 100 assets among all assets in the CRSP data set.

4.2. Performance measures

We compare the performance of the proposed portfolios to the portfolios in the literature using the following three criteria: (1) Out-of-sample portfolio variance; (2) Out of-sample portfolio Sharpe ratio; (3) Portfolio turnover (trading volume).

We use the following "rolling-horizon" procedure for the comparison.

- Firstly, we choose a window over which to perform the estimation. We denote the length of the estimation window by $\tau < T$, where T is the total number of returns in the data set. For the FF-100, FF-48, 500 CRSP and 100 CRSP in our experiments, we use an estimation window of $\tau = 120$ data points, which for monthly data corresponds to 10 years.
- Secondly, using the return data over the estimation window, τ, Σ. We compute the various portfolios.
- Thirdly, we repeat this "rolling-window" procedure for the next month by including the (return) data for the next month and dropping the (return) data for the earliest month. We continue doing this until the end of the data set is reached.

At the end of this process, we have generated $T-\tau$ portfolio-weight vectors for each strategy i, that is, $\mathbf{w}_t^i, t = \tau, \tau+1, \ldots, T-1$ and for each strategy i. Holding the portfolio $\mathbf{w}_t^i$ for one month gives the out-of-sample return at time $t+1 : r_{t+1}^i = \mathbf{w}_t^{i'} \mathbf{r}_{t+1}$, where $\mathbf{r}_{t+1}$ denotes the asset returns.

We use the time series of returns and weights for each strategy to compute the out-of-sample variance ($\widehat{\sigma}$), Sharpe ratio (SR), and turnover(TR):

$$(\widehat{\sigma}^i) = \frac{1}{T-\tau-1}\sum_{t=\tau}^{T-1}(w_t^{i'} r_{t+1} - \widehat{\mu}^i)^2, \quad (\widehat{\mu}^i) = \frac{1}{T-\tau}\sum_{t=\tau}^{T-1}(w_t^{i'} r_{t+1}), \tag{4.1}$$

Table 2
Computational weights for FF48 data set at the first rolling-horizon.

No.	MINC	MV1R	MVSRB	MVSRE	No.	MINC	MV1R	MVSRB	MVSRE
1	0.0566	0.0582	0.0597	0.0621	25	0	0	0	−0.0232
2	0.1686	0.1633	0.1575	0.1458	26	0.0624	0.0635	0.0671	0.0837
3	0	0	0	0	27	0.0306	0.0317	0.0326	0.0334
4	0.0139	0.0115	0.0078	0	28	0	0	0	0
5	0	0	0	0	29	0	−0.0017	−0.0028	−0.0042
6	0	0	0	0	30	0	0	0	0.0134
7	0	−0.0010	−0.0015	−0.0029	31	0.2908	0.2945	0.2970	0.2973
8	0	0	0	0	32	0	0.0005	0.0112	0.0373
9	0.1576	0.1610	0.1660	0.1762	33	0	0	0	0
10	0	0	0	0	34	0	0	0	0
11	0	0	0	0	35	0	0	0	0.0023
12	0	0.0069	0.0142	0.0394	36	0	0	0	0
13	0.0949	0.0924	0.0857	0.0639	37	0	0	0	0
14	0	0	0	0	38	0	0	0	0
15	0	0	0	0	39	0	0	0	0
16	0	−0.0045	−0.0170	−0.0397	40	0	0	0	0
17	0	0	0	0	41	0	0	0	0
18	0	0	0	−0.0089	42	0.1246	0.1326	0.1426	0.1702
19	0	−0.0087	−0.0202	−0.0461	43	0	0	0	0
20	0	0	0	0	44	0	0	0	0
21	0	0	0	0	45	0	0	0	0
22	0	0	0	0	46	0	0	0	0
23	0	0	0	0	47	0	0	0	0
24	0	0	0	0	48	0	0	0	0

Table 3
Portfolio variances of all considered portfolio strategies.

Data set	500 CRSP	100 CRSP	FF-100	FF48
MVSRB	0.00055	0.00115	0.00102	0.00118
MVSRE	0.00052	0.00108	0.00108	0.00118
MV1R	0.00051	0.00112	0.00105	0.00113
1/N	0.00125	0.00152	0.00135	0.00146
MINC	0.00094	0.00139	0.00142	0.00141

$$(\widehat{SR}^i) = \frac{\widehat{\mu}^i}{\widehat{\sigma}^i}, \tag{4.2}$$

$$(\widehat{TR}^i) = \frac{1}{T-\tau-1}\sum_{t=\tau}^{T-1}\sum_{j=1}^{N}|w^i_{j,t+1} - w^i_{j,t+}|. \tag{4.3}$$

where in the definition of the out-of-sample variance ($\widehat{\sigma}$), sharpe ratio (SR), and turnover (TR):

- $w^i_{j,t}$ denotes the portfolio weight in asset j chosen at time t under strategy i,
- $w^i_{j,t+}$ the portfolio weight before rebalancing but at $t+1$,
- $w^i_{j,t+1}$ the desired portfolio weight at time $t+1$ (after rebalancing),

4.3. Out-of-sample evaluation of the proposed portfolios

In this subsection, we compare across four different data sets (listed in Table 1) the out-of-sample empirical performance of the proposed portfolios to some portfolios from the existing literature using three performance metrics: the out-of-sample portfolio variance, the out-of-sample sharpe ratio, and turnover.

Before carrying out out-of-sample evaluation of the proposed portfolios, we will show the computational weights for each strategy for the first rolling-horizon except for 1/N strategy. Because of the limitation of space of paper, we only give the computational weights for FF48 data set (see Table 2).

4.3.1. Discussion of the out-of-sample variances

Table 3 shows the out-of-sample variances of all considered portfolio strategies with four different data sets.

Assessing the variances within the tested strategies, we find that MVSRB and MVSRE typically have lower out-of-sample variances than the other portfolios except for the MV1R. Meanwhile, the 1/N, and minimum-variance portfolio

Table 4
Out-of-sample Sharpe ratio of the portfolio strategy.

Data set	500 CRSP	100 CRSP	FF-100	FF48
MVSRB	0.4202	0.4122	0.3580	0.3010
MVSRE	0.4438	0.4334	0.3724	0.3220
MV1R	0.4198	0.4080	0.3612	0.2985
1/N	0.3122	0.3388	0.2849	0.2556
MINC	0.3882	0.3649	0.3142	0.2712

Table 5
Turnover of the portfolio strategy.

Data set	500 CRSP	100 CRSP	FF-100	FF48
MVSRB	0.4115	0.3050	0.4123	0.2642
MVSRE	0.4156	0.3168	0.4024	0.2624
MV1R	0.4110	0.3148	0.3039	0.2625
1/N	0.0636	0.0452	0.0511	0.0326
MINC	0.3125	0.2025	0.2221	0.0822

with shortsales constrained (MINC) portfolios always achieve out-of-sample variances that are higher than the other portfolios, and the differences are statistically significant.

Comparing the variances of the L_1-regularization portfolio (MV1R) to the sparse and robust mean–variance portfolio model (MVSRB and MVSRE), we see that the MV1R has lower variances than the MVSRB and MVSRE for 500 CRSP and FF48 data sets.

Assessing the variances within the group of our developed strategies (MVSRB and MVSRE), we find that MVSRE yields a sight lower variance than MVSRB for all data sets except for FF-100.

4.3.2. Discussion of the out-of-sample sharpe ratios

Table 4 reports the out-of-sample Sharpe ratios for the different portfolios with four different data sets.

Assessing the Sharpe ratios within the tested strategies, we note that MVSRB, MVSRE and MV1R have higher Sharpe ratios than both the equally weighted (1/N) and minimum-variance portfolio with shortsales constrained portfolio (MINC) for all data sets, and the difference is substantial and significant on all considered data sets.

From Table 4, we note that MVSRE almost always attains higher Sharpe ratios than MVSRB and the differences of attained Sharpe ratios by MVSRE and MVSRB are significant for five data sets.

We also note that the sparse and robust mean–variance portfolio model (MVSRB and MVSRE) have higher Sharpe ratios than the solely L_1-regularization portfolio (MV1R) for most considered data sets. The differences between MVSRB and MV1R are not significant.

4.3.3. Discussion of turnover

Table 5 shows the turnover and short interest associated with each portfolio strategy with four different data sets.

From the definition of turnover in (4.3), the turnover represents the average monthly trading volume. For large turnover, if take the transaction fee into account, large turnover will wipe out the economic gains of portfolios.

From Table 5, we can see that the turnover of the sparse and robust mean–variance portfolio model (MVSRB and MVSRE) is comparatively similar with that of the L_1-regularization portfolio (MV1R), and MVSRE has lower turnover than MVSRB. Unsurprisingly, the long only portfolio strategies 1/N and MINC exhibit the lowest turnover of all portfolio strategies for all considered data sets.

5. Conclusion

In this work, we show that the performance of the mean–variance portfolio can be substantially improved by combining regularization methods with robust optimization. Our main contributions and findings can be summarized as follows.

Firstly, we introduce a sparse mean–variance portfolio model by adding an L_1-regularization term to the objective function of Markowitz mean–variance portfolio selection, and give some insight about sparsity.

Secondly, we propose two general sparse and robust portfolio models by using objective function regularization and robust optimization. One is under box uncertainty set, the other is under ellipsoidal uncertainty set. These sparse and robust portfolio models all can be converted into a easier form to solve.

Finally, three empirical studies with real market data show that the proposed strategies have better out-of-sample performance than some other strategies. Generally speaking, ellipsoidal uncertainty can reduce the degree of conservatism of box uncertainty. Hence, the MVSRB strategy we can rely upon to yield the better result so that it can guide our work.

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