Lecture 02: Cash-In-Advance Fully Dynamic Model

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Understanding Society

Outline for this Lecture

What have we seen so far

1. Macro data & AS-AD short-comings [CG 1]
2. Macro policy in practice [CG 6]
3. Discussion of assignment 0 [CG 1, 5]
4. Static CiA macroeconomic model [CG 1, 5]

What we will see today

1. Fully Dynamic CiA Model [CG 1, 4, 5]
2. Optimal Intertemporal Allocation [CG 1, 3, 4]
3. Steady-State Analysis [CG 3, 4]
4. Growth, Welfare, and Optimal Policy [CG 4, 5, 6]
5. Varying Money Velocity (if time permits) [CG 1, 4, 5]

Big Picture of the Lecture:

1. How do households optimally consume and allocate resources across time?
2. What role does money (optimally) play in resource allocation in the model?

Dynamic Cash-in-Advance Model

Model Setup - Cole (2020) chapter 3

Expanding the Time Horizon of our CiA Model

We now extend our model by creating a **genuine dynamic model!**

- We simply push out time by one period to the point at which we use a continuation payoff function, $V(M, B)$, to characterize outcomes.
- The household takes as given
  - → the price in the first and second periods, P_1 and P_2,
  - → and the productivity levels, Z_1 and Z_2,
  - → its initial money position M_1 and bond position B_1,
  - → the payoff from money and bonds going into the third period $V(M_3, B_3)$.
- The household is choosing
  - → consumptions C_1 and C_2,
  - → labor L_1 and L_2,
  - → money holdings M_2 and M_3,
  - → bond holdings B_2 and B_3.
- The government changes overall money supply through taxes and transfers of money in the asset market at the end of the period.
  - → When net transfers, $T_t > 0$, the household is receiving cash, and when the reverse is true, it is making a cash payment to the government.

Intertemporal Utility Maximization and the Continuation Function

The **household optimization problem** can be written as

$$\max_{\{C_t, L_t, M_{t+1}, B_{t+1}\}_{t=1,2}} u(C_1) - v(L_1) + \beta\left[u(C_2) - v(L_2)\right] + \beta^2 V(M_3, B_3)$$

subject to

$$\begin{aligned} M_t &\geq P_t C_t \quad \text{and} \\ P_t Z_t L_t + [M_t - P_t C_t] + B_t + T_t &\geq M_{t+1} + q_t B_{t+1} \quad \text{for } t = 1, 2. \end{aligned}$$

Difference to static model

The *real difference* in this version of the household's problem has to do with the impact of choosing M_2 and B_2.

The Intertemporal Lagrange Equation

The **household Lagrangian** is now given by

$$\begin{aligned}\mathcal{L} = & \max_{\{C_t, L_t, M_{t+1}, B_{t+1}\}_{t=1,2}} \min_{\{\lambda_t, \mu_t\}_{t=1,2}} \\ & u(C_1) - v(L_1) + \beta\left[u(C_2) - v(L_2)\right] + \beta^2 V(M_3, B_3) \\ & + \sum_{t=1,2} \lambda_t \{M_t - P_t C_t\} \\ & + \sum_{t=1,2} \mu_t \{P_t Z_t L_t + M_t - P_t C_t + B_t + T_t - M_{t+1} - q_t B_{t+1}\}.\end{aligned} \tag{1}$$

Intertemporal Discounting:

- Assume discounting of future utils by multiplying the next period's payoff by $\beta < 1$.
- Subsequently, we discount the payoff two periods ahead by β^2.
- This intertemporal problem can readily be pushed out to arbitrary T.

Dynamic Cash-in-Advance Model

First-Order Conditions - Cole (2020) chapter 3

Consumption and Labor FOCs

The **first-order conditions** (FOCs) for consumption and labor are still given by

$$\beta^{t-1}u'(C_t) - [\lambda_t + \mu_t]\, P_t \overset{!}{=} 0, \tag{2}$$

and

$$\beta^{t-1}v'(L_t) - \mu_t P_t Z_t \overset{!}{=} 0. \tag{3}$$

- Both FOCs are *"intra-temporal"* which means they describe behavior within a period.
- The intra-temporal FOCs are symmetric across all $t \in T$.

The Money FOC

Approach of deriving the money FOC:

- To derive the first-order condition for M_2 note that it shows up
  - → in the date $t = 1$ budget constraint as M_{t+1},, and
  - → in the date $t = 2$ cash-in-advance and budget constraints as M_t.
- Doing the differentiation correctly leads to

$$-\mu_t + \lambda_{t+1} + \mu_{t+1} \stackrel{!}{=} 0. \tag{4}$$

Interpretation of the money FOC:

- Cost of increasing money, M_{t+1}, is shadow price of the first period budget constraint, μ_t.
- The gain of increasing money holdings, M_{t+1}, is the sum of ...
  - → the shadow price of the second period cash-in-advance (CiA) constraint, λ_{t+1},
  - → and the shadow price of the second period budget constraint, μ_{t+1}.
- This condition now makes clear what stood behind the mystery term $V_1(M', B')$ in our static model of the previous chapter.

The Bond FOC

Approach of deriving the bond FOC:

- To derive the first-order condition for B_2 note that it shows up
  - → in the $t = 1$ budget constraint as B_{t+1}, and
  - → in the $t = 2$ budget constraint as B_t.
- When we differentiate correctly, we get the following expression

$$-\mu_t q_t + \mu_{t+1} \stackrel{!}{=} 0. \tag{5}$$

Interpretation of bond FOC:

- This expression highlights the difference between buying bonds vs. holding money:
  - → With bonds we get a price break to the extent that $q_t < 1$.
  - → But we don't get future service yield, λ_{t+1}, just future benefit of having more wealth, μ_{t+1}.
- This condition now makes clear what stood behind the mystery term $V_2(M', B')$ in our static model of the previous chapter.
- Note that $V_1(M', B') - V_2(M', B') = \lambda_{t+1} \geq 0$

A Tradeoff between Cash and Interest

- To get more insight into the gain from having more money in period 2, use the FOC for C_2:

$$\beta u'(C_2) = (\lambda_2 + \mu_2)P_2$$

- Note that the cost of second period consumption is the combination of the shadow prices of the cash-in-advance (CiA) and budget (BC) constraints.
- Using this expression we can rewrite the first-order condition for money:

$$\mu_1 = \frac{\beta u'(C_2)}{P_2}$$

Trade-Off between Money and Bonds

The household is trading off the benefit of being able to buy consumption tomorrow vs. the lower return on savings offered by money (again to the extent that $q_1 < 1$).

A Tradeoff between Consumption and Labor

- Starting from the FOC for L_1, we use the FOC for M_2 to replace μ_1, and then the FOC for C_2 to derive the **optimal labor-consumption condition:**

$$v'(L_1) = \mu_1 P_1 Z_1 = [\mu_2 + \lambda_2] P_1 Z_1$$

$$= \left[\frac{\beta u'(C_2)}{P_2}\right] P_1 Z_1 = \frac{P_1}{P_2} Z_1 \beta u'(C_2).$$

- If money is a bad asset, $\frac{P_2}{P_1} > q$, people will be discouraged from working.

$\Rightarrow$ This is one of the key inefficiency wedges that money has created in our model.

Money and the Intertemporal Tradeoff (1/2)

- The FOCs for C_1 and C_2 are given by

$$\begin{aligned} u'(C_1) &= [\mu_1 + \lambda_1] P_1, \\ \beta u'(C_2) &= [\mu_2 + \lambda_2] P_2. \end{aligned}$$

- The FOC for money says that $\mu_1 = \mu_2 + \lambda_2$. Making the substitution and rearranging:

$$\frac{u'(C_1)}{P_1} = [\mu_2 + \lambda_2 + \lambda_1], \tag{6}$$

$$\frac{\beta u'(C_2)}{P_2} = [\mu_2 + \lambda_2]. \tag{7}$$

- For the right-hand-side (rhs) of (6) and (7) to be equal, we need $\lambda_1 = 0$ or the shadow price of the CiA constraint to be 0.
- If $\lambda_1 \neq 0$, we strictly prefer to spend our money in the first period.

Money and the Intertemporal Tradeoff (2/2)

When will the shadow price of money be 0?

- Note that from our money and bond conditions

$$\begin{aligned} \mu_1 &= \mu_2 + \lambda_2 \text{ and} \\ \mu_1 q_1 &= \mu_2. \end{aligned}$$

- So, if $q_1 < 1$, these conditions say that $\lambda_2 > 0$.

Interpretation

The household will adjust the composition of its savings between money and bonds to ensure that the cash-in-advance constraint binds enough to offset the extra interest return that they get from holding bonds.

An Overview of the Model FOCs

Recursive structure of the model

So far, we only have modeled two periods. But if we extended the modeling out to an arbitrary period T, then we would get similar conditions.

First-order conditions (for any $t \in T$):

$$
\begin{aligned}
\text{Consumption:} \quad & \beta^{t-1} u'(C_t) = [\lambda_t + \mu_t] P_t \\
\text{Labor:} \quad & \beta^{t-1} v'(L_t) = \mu_t P_t Z_t \\
\text{Money:} \quad & \mu_t = \mu_{t+1} + \lambda_{t+1} \\
\text{Bonds:} \quad & \mu_t q_t = \mu_{t+1}
\end{aligned}
$$

When we solve for the steady state we will takes these as our conditions.

Dynamic Cash-in-Advance Model

General Equilibrium - Cole (2020) chapter 3

Closing the Model - Market Equilibrium

To *close the model*, we want to think about aggregate equilibrium variables.

- **Resource constraint:** Per capita output is equal to per capita consumption:

$$Z_t L_t = Y_t = C_t$$

- **Per capita money supply, $\bar{M}_t$:** Evolves over time because of net transfers:

$$\bar{M}_{t+1} = (1 + \tau_t)\bar{M}_t$$

- If we denote the **net growth rate of money** in period t by τ_t:

$$T_t = \tau_t \bar{M}_t.$$

- **Money market clearing:** The amount of money with which the household leaves the asset market must equal the supply:

$$M_{t+1} = \bar{M}_{t+1}.$$

Assumption - A Binding CiA Constraint

Assumption

In closing the model, let's assume that the cash-in-advance (CiA) constraint binds in any time period, $t \in T$.

It follows:

$$C_t = \frac{\bar{M}_t}{P_t}, \quad \text{or} \quad P_t = \frac{\bar{M}_t}{Z_t L_t}. \tag{8}$$

Properties:

- This is a simple velocity type equation, familiar from very old-school macro models.
- $M * v = P * Y$ where v is the velocity of money (here constant and equal to one).
- In our equations we can replace the price level with this simple relationship.

First-Order Conditions in the Closed Model

Next, we turn to our optimality conditions to finish closing the model.

- The consumption condition and the labor condition imply that:

$$\beta^{t-1} u'(Z_t L_t) = (\lambda_t + \mu_t) \frac{\bar{M}_t}{Z_t L_t} \tag{9}$$

$$\beta^{t-1} v'(L_t) = \mu_t Z_t \frac{\bar{M}_t}{Z_t L_t} \tag{10}$$

→ We have used the fact $C_t = Z_t L_t$ from our resource constraint, and replaced P_t in both equations above with the cash-in-advance constraint.

- The money condition and the bond condition imply that:

$$\mu_t = \mu_{t+1} + \lambda_{t+1} \tag{11}$$

$$\mu_t q_t = \mu_{t+1} \tag{12}$$

The Reduced Form Model (1/2)

We use the money condition along with our FOCs for consumption and labor to get that

$$\begin{aligned} \beta^{t-1} v'(L_t) &= [\mu_{t+1} + \lambda_{t+1}] \frac{\bar{M}_t}{L_t} \\ &= \left[\beta^t u'(Z_{t+1} L_{t+1}) \frac{Z_{t+1} L_{t+1}}{\bar{M}_{t+1}} \right] \frac{\bar{M}_t}{L_t} \\ &= \frac{1}{(1+\tau_t)} \frac{Z_{t+1} L_{t+1}}{L_t} \beta^t u'(Z_{t+1} L_{t+1}). \end{aligned} \tag{13}$$

- This is our **key dynamic equation** and it involves both L_t and L_{t+1}.
- However, these are the only endogenous variables in the equation.
- Hence, solving this equation will essentially solve our model.

Finally, we can determine the bond price off of the optimality conditions for labor and bonds

$$\begin{aligned}
\mu_t q_t &= \mu_{t+1} \\
\Leftrightarrow \frac{v'(L_t)L_t}{\bar{M}_t} q_t &= \frac{\beta v'(L_{t+1})L_{t+1}}{\bar{M}_{t+1}} \\
\Leftrightarrow q_t &= \frac{\beta}{(1+\tau_t)} \frac{v'(L_{t+1})L_{t+1}}{v'(L_t)L_t}. \qquad (14)
\end{aligned}$$

$\Rightarrow$ The unusual aspect of this equation is that we are pinning down the interest rate through the intertemporal tradeoff of working more today vs. tomorrow rather than consuming more today vs. tomorrow.

Dynamic Cash-in-Advance Model

Steady-State Analysis - Cole (2020) chapter 3

Definitions of the Steady-State

Deterministic Steady-State

- The economy is not exposed to any stochastic shocks at the moment.
- The agents do not expect any stochastic shocks in the future (no uncertainty).

$\Rightarrow$ The **deterministic steady-state** is given by the static equilibrium of the model economy.

Risky Steady-State

- The economy is not exposed to any stochastic shocks at the moment.
- Agents know about possible future shocks and take the uncertainty in their decisions into account.

$\Rightarrow$ The **risky steady-state** is given by the deterministic steady-state plus an uncertainty premium.

Calculating the Steady-State of the Dynamic CiA Model

We are going to shut down as much of the time variation as we can and solve for a *deterministic* steady-state equilibrium of our model.

- Assume that productivity is constant; i.e., $Z_t = Z$.
- Assume money supply grows at a constant rate τ.
- Suppose that labor, and hence consumption, are constant.
- If $L_t = L$, then our key dynamic equation becomes:

$$v'(L) \;=\; \frac{\beta}{(1+\tau)} * Z * u'(ZL)$$

- One can see from inspection that this equation is going to admit a unique solution in L:
  - $\rightarrow$ The left-hand-side (lhs) is increasing in L because $v'(L) > 0$ and $v''(L) > 0$.
  - $\rightarrow$ The right-hand-side (rhs) is decreasing in L because $u'(C) > 0$, but $u''(C) < 0$.

Utility Functional Forms for the Dynamic CiA Model

Before solving our model, we have to be explicit on the function forms of our preferences.

- A common assumption for consumption is that preferences are *constant-relative-risk-averse* (CRRA):

$$u(C) = \begin{cases} \frac{C^{1-\alpha}-1}{1-\alpha} & \text{if } \alpha \neq 1 \\ \log(C) & \text{o.w.} \end{cases} \tag{15}$$

  → The risk-aversion is defined by α.

  → Standard values for α are 1 or 2.

- A common assumption for labor is to assume a power utility form:

$$v(L) = \frac{L^{1+\gamma}}{1+\gamma}. \tag{16}$$

Frisch Elasticity of Labor Supply in the Dynamic CiA Model

Definition

The Frisch elasticity of labor asks how labor will change if we change the wage while holding fixed the marginal value of wealth (the penalty price of the budget constraint).

- From our first-order condition for labor, we get that

$$v'(L) = V'(W)PZ \qquad \text{or} \quad L^{\gamma} = V'(W)w.$$

  where PZ (marginal product of labor) is our stand-in for the nominal wage, w.

- Taking logs, holding fixed the marginal utility of wealth, and differentiating yields

$$\gamma d\log(L) = \log\left[V'(W)\right] + d\log(w) \implies \left.\frac{d\log(L)}{d\log(w)}\right|_{V'(W)} = \frac{1}{\gamma}.$$

- The **Frisch elasticity** is therefore $1/\gamma$. These's a lot of debate about this elasticity:
  - $\rightarrow$ Micro studies estimate it to be quite low (0 to 0.5).
  - $\rightarrow$ Macro studies generally estimate a significantly higher value (2 to 4).

Steady-State Labor with Functional Forms

Given our functional form assumption our steady state labor condition becomes:

$$L^{\gamma} = \frac{\beta}{(1+\tau)} * Z * (ZL)^{-\alpha}$$

Which implies a very simple analytic solution:

$$L = \left[\beta \frac{Z^{1-\alpha}}{(1+\tau)}\right]^{1/(\gamma+\alpha)}$$

- We can see how productivity Z raises labor and money growth τ lowers it.
- We can also see that the impact depends upon our preference parameters.

Steady-State Interest Rate with Functional Forms

- To fully solve the model steady-state, we have to derive the steady-state of q_t using (14).
- With the assumption that labor is constant in steady-state we get the following expression:

$$q = \frac{\beta}{(1+\tau)},$$

$\rightarrow$ Steady-state net growth rate of money, τ, increases steady-state nominal interest rate, $\frac{1}{q}$.

$\rightarrow$ Here: τ also describes the inflation rate in the economy.

Multipliers in the Steady State - Rendering the Model Stationary

In the steady state we get that

$$\beta^{t-1} v'(L) \frac{L}{\bar{M}_1 (1+\tau)^{t-1}} = \mu_t, \tag{17}$$

$$\beta^{t-1} u'(ZL) \frac{ZL}{\bar{M}_1 (1+\tau)^{t-1}} = (\lambda_t + \mu_t) \, . \tag{18}$$

- (17) implies that μ_t is shrinking at the rate $\frac{\beta^{t-1}}{(1+\tau)^{t-1}}$.
- It follows that (18) implies that λ_t is shrinking at the same rate.

$\Rightarrow$ But the *relative values* of the multipliers are staying constant.

Important insight for later: Our **multipliers are constant**, if we adjust them as follows:

$$\tilde{\mu}_t = \mu_t \frac{(1+\tau)^{t-1}}{\beta^{t-1}}$$

$$\tilde{\lambda}_t = \lambda_t \frac{(1+\tau)^{t-1}}{\beta^{t-1}}$$

Growth, Welfare, and Optimal Policy

Long-Run Analysis - Cole (2020) chapter 3

Optimal Policy - Friedman Rule (1/2)

- Consider a *social planner* who can ignore the cash-in-advance constraint (source of inefficiency),

$$\max_{L_1} u(Z_1 L_1) - v(L_1) + \beta V(\cdot),$$

where V is the continuation payoff to the social planner.

- The first-order condition for this problem is

$$u'(Z_1 L_1) Z_1 - v'(L_1) \;=\; 0.$$

- Comparing the social planner and (9)-(10), we see that they line up iff $\lambda_1 = 0$.
- If we look at the FOCs for money (11) and bonds (12), we see that $q = 1$.

⇒ **Friedman rule:** This means that the net nominal interest rate is being set to zero.

Optimal Policy - Friedman Rule (2/2)

- For $q = 1$, it must be the case that the steady-state is

$$\begin{aligned} q &= \frac{\beta}{(1+\tau)} \;=\; 1, \\ \Leftrightarrow \quad \tau &= \beta - 1. \end{aligned}$$

- This condition implies that $\tau < 0$: **Money supply must shrink!**
- But if the money supply is shrinking, prices will also be falling: **Deflation!**
- In fact, with L constant, it follows that in the optimum $P_{t+1} = \beta P_t$.
  - $\rightarrow$ The discount factor sets the optimal level of deflation and the optimal rate at which money must shrink.

What is the Cost of Inflation? (1/2)

Definition: Consumption Equivalents

The fraction of lifetime consumption that we would have to add or take away to make you just as well off.

- Denote by $L(\tau)$ the solution for inflation rate, τ. Then lifetime utility is given by

$$U(\tau) = \frac{1}{1-\beta}\left[\frac{(ZL(\tau))^{1-\alpha}-1}{1-\alpha} - \frac{L(\tau)^{1+\gamma}}{1+\gamma}\right].$$

- If we fix some particular τ_0 as our benchmark

$$U(\tau_0) = \frac{1}{1-\beta}\left[\frac{(\phi(\tau)ZL(\tau))^{1-\alpha}-1}{1-\alpha} - \frac{L(\tau)^{1+\gamma}}{1+\gamma}\right]$$

→ $\phi(\tau)$ indicates the difference in consumption goods between the actual inflation rate, τ, and the benchmark case, τ_0.

What is the Cost of Inflation? (2/2)

Solving for $\phi\left(\tau\right)$ leads to

$$\phi(\tau)^{1-\alpha} = \frac{\left\{U(\tau_0)(1-\beta) + \frac{L(\tau)^{1+\gamma}}{1+\gamma}\right\}(1-\alpha)+1}{(ZL(\tau))^{1-\alpha}}. \tag{19}$$

Once we compute $\phi(\tau)$ using this last expression, the consumption equivalent variation is just given by $\phi(\tau)-1$.

Accounting for Growth (1/3)

- So far, we have assumed that $Z_{t+1} = Z_t$, hence no growth in productivity.
- Assume now, that $Z_{t+1} = (1+g)Z_t$, hence *productivity grows* by g percent per period.
- Return to our fundamental equation (13) now modified to take account of growth in Z and normalized $Z_0 = 1$:

$$\begin{aligned} v'(L_t) &= \frac{\beta}{(1+\tau_t)} \frac{Z_{t+1} L_{t+1}}{L_t} u'(Z_{t+1} L_{t+1}). \\ &= \frac{\beta}{(1+\tau_t)} \frac{(1+g)^{t+1} L_{t+1}}{L_t} u'((1+g)^{t+1} L_{t+1}). \end{aligned}$$

- Unless the growth terms cancel on the rhs, the solution is not invariant to the level of Z. So L will be changing over time. *This is an issue for our model!*
- In order to have a steady state with L constant we must assume that $u(C) = log(C)$!
  - $\rightarrow$ The substitution effect of intertemporal allocation is equal to the income effect.

Accounting for Growth (2/3)

- When we specialize our utility function for consumption as a log-utility function, our fundamental equation becomes

$$\beta^{t-1} v'(L_t) = \frac{1}{(1+\tau_t)} \frac{Z_{t+1} L_{t+1}}{L_t} \beta^t \frac{1}{Z_{t+1} L_{t+1}}. \tag{20}$$

- This leads to a very simple expression for the equilibrium level of labor

$$L^{1+\gamma} = \frac{\beta}{1+\tau}.$$

- The price level (from the CiA constraint) is given by

$$P_t = \frac{M_1 (1+\tau)^{t-1}}{Z_1 (1+g)^{t-1}} = \frac{M_1}{Z_1} \left(\frac{1+\tau}{1+g} \right)^{t-1}$$

→ Inflation rate, $1+\pi = \frac{1+\tau}{1+g}$, depends on growth rate gap between money and output.

→ This is essentially a velocity of money equation.

Accounting for Growth (3/3)

- Since the labor condition is

$$\beta^{t-1} v'(L_t) \;=\; \mu_t Z_t \frac{\bar{M}_t}{Z_t L_t},$$

it follows that the steady-state interest rate is still

$$q \;=\; \frac{\beta}{1+\tau},$$

as the productivity level drops out.

Consumption Equivalence with Growth (1/3)

- To see how adding growth changes our calculation of the consumption equivalent variation, let us focus on the consumption term in the payoff. This becomes

$$\sum_{t=1}^{\infty}\beta^{t-1}log\left(ZL(\tau)(1+g)^{t-1}\right) = \frac{log(ZL(\tau))}{1-\beta}+\sum_{t=1}^{\infty}\beta^{t-1}(t-1)log(1+g)$$

- Then, note that

$$\frac{d}{d\beta}\sum_{t=1}^{\infty}\beta^{t-1} = \sum_{t=1}^{\infty}\frac{d}{d\beta}\beta^{t-1} = \sum_{t=1}^{\infty}(t-1)\beta^{t-2},$$

- While at the same time

$$\frac{d}{d\beta}\sum_{t=1}^{\infty}\beta^{t-1} = \frac{d}{d\beta}\frac{1}{1-\beta} = \frac{1}{(1-\beta)^2}.$$

Consumption Equivalence with Growth (2/3)

- Hence, it follows that our consumption payoff is

$$U(\tau) \;=\; \frac{log(ZL(\tau))}{1-\beta} + log(1+g)\frac{\beta}{(1-\beta)^2}.$$

- It follows that our lifetime utility is given by

$$U(\tau) \;=\; \frac{1}{1-\beta}\left[log(ZL(\tau)) - \frac{L(\tau)^{1+\gamma}}{1+\gamma}\right] + log(1+g)\frac{\beta}{(1-\beta)^2}.$$

- This expression implies that the impact of money growth and productivity growth comes in through two completely separate terms.
- Moreover, it means that our prior results on the cost of the money growth rate deviating from the optimal rate are unaffected by adding in productivity growth.

Consumption Equivalence with Growth (3/3)

$$U(\tau) \;=\; \frac{1}{1-\beta}\left[log(ZL(\tau)) - \frac{L(\tau)^{1+\gamma}}{1+\gamma}\right] + log(1+g)\frac{\beta}{(1-\beta)^2}.$$

- So how important can productivity growth be? Real risk-free interest rates are around 1-2 percent generally, so this suggests that an annual value for $\beta = 0.98$.
- Plugging everything in (accounting for the infinite future), we get that

$$\frac{\beta}{(1-\beta)^2} = \frac{0.98}{.02^2} = \frac{0.98}{.0004} = 2450$$

  which looks big.
- Productivity growth is an important determinant of welfare (compounding effect)!

Varying Velocity (If time permits)

Model Extension - Cole (2020) chapter 4

Setting up a Varying Velocity Model

The **velocity of money has been increasing** over time because of the increased sophistication in the payments system: Banks, electronic payments systems, FinTec, etc.

- Let's account for that in the model. Assume that the CiA constraint is given by

$$M_t \geq \kappa P_t C_t, \quad \kappa \in (0, 1]$$

- $\kappa = 1$ is our old model.
- The amount $(1 - \kappa) P_t C_t$ is settled up later in the asset market.
- When the CiA binds, we now get that

$$P = \frac{M}{\kappa ZL}.$$

- So money velocity is given by

$$v = \frac{PZL}{M} = \frac{M}{\kappa ZL} \frac{ZL}{M} = \frac{1}{\kappa}.$$

Optimization Problem of the Varying Velocity Model

- Here is our Lagrangian, adjusted for the change:

$$
\begin{aligned}
\mathcal{L} \;=\; & \max_{\{C_t, L_t, M_{t+1}, B_{t+1}\}_{t=1,2}} \; \min_{\{\lambda_t, \mu_t\}_{t=1,2}} \\
& u(C_1) - v(L_1) + \beta \left[u(C_2) - v(L_2)\right] + \beta^2 V(M_3, B_3) \\
& + \sum_{t=1,2} \lambda_t \left\{M_t - \kappa P_t C_t\right\} \\
& + \sum_{t=1,2} \mu_t \left\{P_t Z_t L_t + M_t - P_t C_t + B_t + T_t - M_{t+1} - q_t B_{t+1}\right\}.
\end{aligned}
$$

- The only change is w.r.t. the CiA constraint.
- This changes the FOC for consumption, which becomes

$$
\beta^{t-1} u'(C_t) = \left[\mu_t + \kappa \lambda_t\right] P_t.
$$

First-Order Conditions of the Varying Velocity Model

We still get the following first-order conditions (FOCs) for labor, money and bonds:

$$\begin{aligned}
\beta^{t-1} v'(L_t) &= \mu_t Z_t P_t \\
\mu_t &= \mu_{t+1} + \lambda_{t+1}, \\
\mu_t q_t &= \mu_{t+1},
\end{aligned}$$

Solving the Varying Velocity Model Further

- Unfortunately, things don't reduce quite as neatly once we conjecture that L is constant and use the fact that M grows at a constant rate.
- To keep things simple assume that Z is constant and $\bar{M}_0 = 1$, and hence we can rewrite these conditions as

$$\begin{aligned} \beta^{t-1}u'(ZL) &= [\mu_t + \kappa\lambda_t]\left(\frac{(1+\tau)^t}{\kappa ZL}\right) \\ \beta^{t-1}v'(L_t) &= \mu_t Z\left(\frac{(1+\tau)^t}{\kappa ZL}\right), \\ \mu_t &= \mu_{t+1} + \lambda_{t+1}. \end{aligned}$$

and our modified pricing equation.

⇒ Note that here too our multipliers won't be constant over time.

Appying a Change-in-Variables

- Make a "change in variables", to rewrite our equations in terms of stationary variables.
- Construct new multipliers

$$
\begin{aligned}
\tilde{\mu}_t &= \mu_t \left((1+\tau) \frac{1}{\beta} \right)^t \beta, \\
\tilde{\lambda}_t &= \lambda_t \left((1+\tau) \frac{1}{\beta} \right)^t \beta.
\end{aligned}
$$

- Using a change in variables along these lines will turn out to be a much more robust method of analyzing our models.

A Non-Linear System of Equations

- Rewrite the equations in terms of the new variables and cancel anything out to get

$$u'(ZL) = \left[\tilde{\mu}_t + \kappa\tilde{\lambda}_t\right]\left(\frac{1}{\kappa ZL}\right) \tag{21}$$

$$v'(L) = \tilde{\mu}_t Z\left(\frac{1}{\kappa ZL}\right), \tag{22}$$

$$\left(\frac{1+\tau}{\beta}\right)\tilde{\mu}_t = \tilde{\mu}_{t+1} + \tilde{\lambda}_{t+1}. \tag{23}$$

- We can guess that there is a constant solution where $\tilde{\mu}_t = \tilde{\mu}$ and $\tilde{\lambda}_t = \tilde{\lambda}$.
- However, system is too complicated to be solved directly (analytically).
- Instead, we could use a nonlinear equation solver to solve 3 equations in 3 unknowns.

Solving the Model: Brute Force

- Can use a brute force method for the system of equations because of its block structure.
- First, consider a grid on labor of $\mathbf{L} = [L_0, L_1, ..., L_N]$ values. The highest level of labor is the efficient level and the lowest is so low that it is very likely not a solution.
- Then use a for loop to sequentially compute the following:
  1. Use (22) to solve for $\tilde{\mu}_i$ given L_i.
  2. Use (21) to solve for $\tilde{\lambda}_i$ given $\tilde{\mu}_i$ and L_i.
  3. Given this, construct the deviations

$$ERR(i) = \left(\frac{1+\tau}{\beta}\right)\tilde{\mu}_i - \tilde{\mu}_i - \tilde{\lambda}_i.$$

  4. plot $ERR(i)$ and see where the zeros are.
- These are our solutions. If the plot is nice, there is only one solution.
- Once we know where the solution is, make our grid finer in this region.

Solving the Model: Non-Linear (Numeric) Solver

We can also just use a nonlinear equation solver!

- Step 1: Define the function to be solved. It will take the form $F(x)$ or $F(x, p)$ where p are some parameters we need to pass in too. (If we need to pass parameters, say in a loop, then we should define the "new function" $G(x) = F(x, p)$. The function returns a vector of zeros when x solves our equations.)
- Step 2: Come up with a good initial guess x_0 for x.
- Step 3: Create a function handle: $fun = @(x)\ F(x, p)$
- Step 4: $x = fsolve(fun, x_0)$ $\quad \rightarrow$ This solves for x such that $fun(x) = 0$
- Step 5: disp(x)

See Matlab for an implementation of this procedure.

Conclusion

Summarizing the Lecture

Can you summarize the three main aspects of the lecture?

Concluding Remarks

Big Picture of the Lecture:

1. How do households optimally consume and allocate resource across time?
2. What role does money (optimally) play in resource allocation in the model?

- We have developed a **fully dynamic CiA model:**
  - → Bonds vs money: Bonds pay interest, but money is necessary to buy goods.
  - → Money supply grows over time, rendering exisiting money less valuable.
  - → Production becomes more efficient over time leading to higher output.
- We have analyzed the **steady-state properties** of the model:
  - → CRRA utility functions and intertemporal elasticity of substitution.
  - → Labor supply depends on the **Frisch elasticity**.
- We have analyzed **optimal policy** and how to measure it:
  - → The **Friedman rule** implies a optimal zero nominal interest rate and deflation.
  - → Inflation reduces labor supply and thereby creates a real cost.
  - → Growth in productivity lowers inflation, even with positive money supply growth.

References

- Harold L. Cole (2020). Monetary and Fiscal Policy through a DSGE Lens. Oxford University Press.