Lecture 03: Shocks and Statistical Models

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Quantitative Dynamic Macroeconomics

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Understanding Society

Outline for this Lecture

What have we seen so far

1. Fully Dynamic CiA Model [CG 1, 4, 5]
2. Optimal Intertemporal Allocation [CG 1, 3, 4]
3. Steady-State Analysis [CG 3, 4]
4. Growth, Welfare, and Optimal Policy [CG 4, 5, 6]
5. Varying Money Velocity (if time permited) [CG 1, 4, 5]

What we will see today

1. Stochastic CiA Model [CG 1, 2, 3, 5]
2. Markov Process Computer Model [CG 3, 4]
3. AR(1) Process Computer Model [CG 3, 4]
4. Calculating Model Statistics [CG 2, 3, 4]

Big Picture of the Lecture:

1. How do random processes impact the results of the CiA model economy?
2. What role do expectations play in our CiA model?
3. How can we compare model statistics with the data?

Defining History States

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Making Things Random: Markov Processes (1/3)

We introduce stochastic (random) elements that drive the economy and lead to fluctuations of e.g. real GDP. We approach it as follows:

- Assume productivity growth rate $g_t > 0$ and money growth rate $\tau_t > 0$.
- Probability of change in growth rates given by first-order Markov process: $Pr\{g_{t+1}|g_t\}$
- Denote the state of the economy in period t by

$$s_t \;=\; \left(Z_{t-1}, \bar{M}_{t-1}, \tau_t, g_t\right).$$

$\rightarrow$ $Z_t = (1+g_t)Z_{t-1}$ and $\bar{M}_t = (1+\tau_t)\bar{M}_{t-1}$ just depends upon s_t.
$\rightarrow$ Need current shocks to forecast tomorrow.

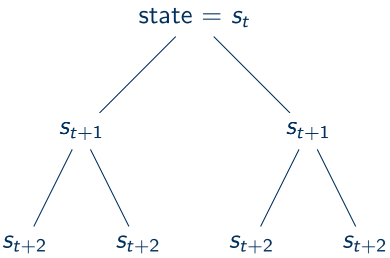
- The histroy of events is given by the history state:

$$s^t = \{s_1, s_2, \ldots, s_t\}.$$

$\rightarrow$ History is important for stock variables like wealth and capital.

Making Things Random: Markov Processes (2/3)

Figure 1: Event Tree



- **History state** gives unique path through event tree.
- $s^{t-1}(s^t)$ gives unique predecessor.
- $S^{t+1}(s^t)$ for set of successor or $s^{t+1} \succ s^t$ successor.

Making Things Random: Markov Processes (3/3)

We will need to talk about probabilities of different states and events:

- The unconditional probability of a given history state s^t is $Pr\{s^t\}$.
- The conditional probability of a history state tomorrow given one today is

$$\frac{Pr\{s^{t+1}\}}{Pr\{s^t\}}$$

  if $s^{t+1} \succ s^t$ and 0 otherwise.

The Stochastic Cash-in-Advance Model

Cole (2020) chapter 5

Stochastic CiA Model: Household Setup

Now, let us introduce Markov chains and state operators into our model! The **household payoff function** is the discounted probability weighted sum of our flow payoffs:

$$\sum_{t=1}^{T} \sum_{s^t} \beta^{t-1} \left[u(C(s^t)) - v(L(s^t)) \right] Pr\{s^t\}$$
$$+ \beta^T \sum_{s^{T+1}} V(M(s^T),\ B(s^T),\ s^{T+1}) Pr\{s^{T+1}\}$$
$$= \mathbb{E} \left\{ \begin{array}{c} \sum_{t=1}^{T} \beta^{t-1} \left[u(C(s^t)) - v(L(s^t)) \right] \\ +\beta^T V(M(s^T),\ B(s^T),\ s^{T+1}) \end{array} \right\}$$

- Our choice variables are: $C(s^t)$, $L(s^t)$, $M(s^t)$, $B(s^t)$.
- We inherit our past choices: $M(s^{t-1}(s^t))$, $B(s^{t-1}(s^t))$.
- We face prices: $P(s^t)$, $q(s^t)$.

Stochastic CiA Model: Lagrangian

The **Lagrangian** for this problem is kind of similar to what we had before:

$$
\begin{aligned}
\mathcal{L} \quad = \quad & \max_{\{L(s^t),\ C(s^t),\ M(s^t),\ B(s^t)\}_{t=1}^T} \ \min_{\{\mu(s^t),\ \lambda(s^t)\}_{t=1}^T} \\
& \mathbb{E}\left\{ \begin{array}{c} \sum_{t=1}^T \beta^{t-1}\left[u(C(s^t)) - v(L(s^t))\right] \\ +\beta^T V(M(s^T),\ B(s^T),\ s_{T+1}) \end{array} \right\} \\
& + \sum_{t=1}^T \sum_{s^t} \lambda(s^t)\left\{M(s^{t-1}(s^t)) - P(s^t)C(s^t)\right\} \\
& + \sum_{t=1}^T \sum_{s^t} \mu(s^t) \left\{ \begin{array}{c} P(s^t)Z(s^t)L(s^t) + \left[M(s^{t-1}(s^t)) - P(s^t)C(s^t)\right] \\ +B(s^{t-1}(s^t)) + T(s^t) - M(s^t) - q(s^t)B(s^t) \end{array} \right\}.
\end{aligned}
$$

- We have a CiA constraint and budget constraint for each history state, s^t.
- We have separate multipliers for each history state since they bind differentially.

Stochastic CiA Model: First-Order Conditions

The **FOCs of the stochastic CiA model** are as follows:

$$\mathcal{L}_{C(s^t)}: \quad \beta^{t-1} u'(C_t(s^t)) \Pr(s^t) - \left[\lambda(s^t) + \mu(s^t)\right] P(s^t) \overset{!}{=} 0 \tag{1}$$

$$\mathcal{L}_{L(s^t)}: \quad -\beta^{t-1} v'(L_t(s^t)) \Pr(s^t) + \mu(s^t) P(s^t) Z(s^t) \overset{!}{=} 0 \tag{2}$$

$$\mathcal{L}_{M(s^t)}: \quad -\mu(s^t) + \sum_{s^{t+1} \in S^{t+1}(s^t)} \left[\lambda(s^{t+1}) + \mu(s^{t+1})\right] \overset{!}{=} 0 \tag{3}$$

$$\mathcal{L}_{B(s^t)}: \quad -\mu(s^t) q(s^t) + \sum_{s^{t+1} \in S^{t+1}(s^t)} \mu(s^{t+1}) \overset{!}{=} 0 \tag{4}$$

$\rightarrow$ Because the impact of increasing money or bond stocks impacts all successor nodes, we get a summation term.

Stochastic CiA Model: Determining the Equilibrium

The steps here are the same: (1) impose market clearing + (2) cash-in-advance constraint:

- **Resource constraint:** $C_t = Z_t L_t$
- **Money market clearing:** $M_t = \bar{M}_t$
- **Binding CiA constraing:**

$$\implies P(s^t) = \frac{\bar{M}(s^{t-1}(s^t))}{Z(s^t)L(s^t)}. \tag{5}$$

- Set the preference assumptions as before:

$$\begin{aligned} u(C(s^t)) &= log(C(s^t)) \\ v(L(s^t)) &= \frac{L(s^t)^{1+\nu}}{1+\nu} \end{aligned}$$

Stochastic CiA Model: Change-in-Variables (1/3)

Impose a **change-in-variables** to render the model stationary! Start with the non-stationary consumption FOC (*log-utility* produces constant adjusted penalty prices across time):

$$
\begin{aligned}
\beta^{t-1}\,\mathrm{Pr}(s^t) &= \left[\lambda(s^t)+\mu(s^t)\right] P(s^t)C(s^t) \\
&= \left[\lambda(s^t)+\mu(s^t)\right] \bar{M}(s^{t-1}(s^t)) \\
\Leftrightarrow \quad \frac{\beta^{t-1}Pr(s^t)}{M(s^{t-1}(s^t))} &= \left[\lambda(s^t)+\mu(s^t)\right]
\end{aligned}
$$

We **render all penalty prices stationary** by applying the following transformation implied by the above condition:

$$
\begin{aligned}
\mu(s^t) &= \frac{\beta^{t-1}\tilde{\mu}(s^t)\,\mathrm{Pr}(s^t)}{\bar{M}(s^{t-1}(s^t))}, \\
\lambda(s^t) &= \frac{\beta^{t-1}\tilde{\lambda}(s^t)\,\mathrm{Pr}(s^t)}{\bar{M}(s^{t-1}(s^t))}.
\end{aligned}
$$

Using this change-in-variables, **rewrite our FOCs** as:

$$1 = \left[\tilde{\lambda}(s^t) + \tilde{\mu}(s^t)\right] \tag{6}$$

$$L(s^t)^\gamma = \tilde{\mu}(s^t)\left[\frac{1}{L(s^t)}\right] \tag{7}$$

$$\tilde{\mu}(s^t) = \frac{\beta}{(1+\tau(s^t))} \sum_{s^{t+1} \in S^{t+1}(s^t)} \underbrace{\left[\tilde{\mu}(s^{t+1}) + \tilde{\lambda}(s^{t+1})\right]}_{=1} \frac{Pr(s^{t+1})}{Pr(s^t)} \tag{8}$$

$$q(s^t) = \frac{\beta}{(1+\tau(s^t))} \sum_{s^{t+1} \in S^{t+1}(s^t)} \frac{\tilde{\mu}(s^{t+1})}{\tilde{\mu}(s^t)} \frac{Pr(s^{t+1})}{Pr(s^t)} \tag{9}$$

Stochastic CiA Model: Change-in-Variables (3/3)

Using this change-in-variables, **further rewrite our FOCs** as:

$$1 = \left[\tilde{\lambda}(s^t) + \tilde{\mu}(s^t)\right] \tag{10}$$

$$L(s^t)^\gamma = \tilde{\mu}(s^t)\left[\frac{1}{L(s^t)}\right] \tag{11}$$

$$\tilde{\mu}(s^t) = \frac{\beta}{(1+\tau(s^t))} \tag{12}$$

$$q(s^t) = \sum_{s^{t+1} \in S^{t+1}(s^t)} \frac{\beta}{(1+\tau(s^{t+1}))} \Pr(s_{t+1}|s_t) \tag{13}$$

- q_t is simply the real discount rate times the expected component of inflation coming from the expected increase in the money supply.
- Nothing in these equations reflects the past, and hence **everything is a function of the current state s_t only.**

Stochastic CiA Model: Forecasting the Future and Bond Prices

- In the bond price equation,

$$q(s_t) = \sum_{s_{t+1}} \frac{\beta}{(1+\tau(s_{t+1}))} \Pr(s_{t+1}|s_t)$$

  we are trying to forecast next period's growth rate from the current state.

- Really this is all about

$$\sum_{\tau_{t+1}} \frac{1}{1+\tau(s_{t+1})} Pr\{\tau_{t+1}|\tau_t, g_t\}.$$

- The g_t terms comes in just if there is any correlation between the two.
  - $\rightarrow$ Note: $1/E\{x\} \neq E\{1/x\}$ (Jensen's inequality).
- The labor equation is just analytic, so this one is going to be easy

$$L_t = \left[\frac{\beta}{(1+\tau_t)}\right]^{1/(1+\gamma)}.$$

Approach to Compute the Stochastic CiA Model

We take **two approaches to adding in stochastic shocks** and computing equilibria:

1. Use simple Markov chains which consist of a small set of possible realizations.
2. Use standard first-order autoregressive (AR(1)) processes with normal shocks.

In both cases, we will calibrate the shock processes, draw some shocks and use them to simulate our model.

Stochastic Computer Model with Markov Chains

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Markov Chains: Transition and Probabilities (1/2)

Assume that ...

- our random variable x can only take on a discrete set of values: $x \in X = \{x_1, ..., x_n\}$.
- the probability of going from x_i this period to x_j next period is simply given by $Q(i, j)$.
  - $\rightarrow$ Note: $\sum_{j=1}^{n} Q(i, j) = 1$ since this exhausts the possibilities.

We summarize all transition probabilities in a **transition matrix $\mathbb{Q}$:**

- The row number says where you are today.
- The column number is where you might go tomorrow.
- For example $x \in \{1, 2, 3\}$ where $x_1 > x_2 > x_3$. and

$$\mathbb{Q} = \begin{bmatrix} .3 & .4 & .3 \\ .2 & .45 & .35 \\ .1 & .5 & .4 \end{bmatrix}$$

$\rightarrow$ the first row gives the transition probabilities if today is i.

$\rightarrow$ and the columns are for going to $1, 2, 3$ respectively.

Markov Chains: Transition and Probabilities (2/2)

Let there be a vector of the **probabilities of being in each state** X at time t:

$$\mathbb{P}_t = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}^T$$

It follows that:

- The probability of being in each state next period is given by: $\mathbb{P}_{t+1} = \mathbb{P}_t \mathbb{Q}$
- This means that: $p_{t+1}(j) = \sum_i p_t(i) Q(i,j)$
- The *probability n-steps ahead* is given by: $\mathbb{P}_{t+n} = \mathbb{P}_t \mathbb{Q}^n$
- A *stationary distribution* is such that: $\mathbb{P} = \mathbb{P}\mathbb{Q}$
- If $Q(i,j) > 0 \; \forall i,j$, then $\mathbb{Q}$ has a unique stationary distribution and for any $\mathbb{P}_0$ it follows:

$$\mathbb{P} = lim_{n\to\infty} \mathbb{P}_0 \mathbb{Q}^n$$

Markov Chains: Defining the Productivity Growth Rate Process

Let us **calibrate our Markov processes.** Assume that ...

- the productivity growth rate shock has mean 0.03 and variance 0.002 (approx. US data).
- productivity growth rates are i.i.d.
- we want a two-state symmetric Markov chain with the above properties.

It follows that ...

- High growth rate, $g_h = 0.03 + a$, and low growth rate, $g_l = 0.03 - a$.
- Transition matrix: $\mathbb{Q} = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$
- Unique stationary distribution (due to symmetry) is $\mathbb{P} = [0.5, 0.5]$.

We can check the process by calculating the variance of g:

$$0.5[g_h - 0.03]^2 + 0.5[g_l - 0.03]^2 = a^2 = 0.002$$
$$\implies a = 0.002^{0.5} = 0.05$$

Markov Chains: Defining the Money Growth Rate Process

Next, **calibrate** inflation over the long haul, which is given by $\frac{1+\tau}{1+g}$. Assume that ...

- It is roughly 2 percent on average.
- It tends to be fairly persistent. So, let's assume $\mathbb{Q} = \begin{bmatrix} .75 & .25 \\ .25 & .75 \end{bmatrix}$.
- The unique stationary distribution (due to symmetry) is still $\mathbb{P} = [0.5, 0.5]$
- The mean of τ is 5 percent.
- Assume the variance is the same as g.

It follows that ...

- High growth rate: $\tau_h = 0.05 + a$
- Low growth rate: $\tau_l = 0.05 - a$
- Standard deviation: $a = 0.05$

Now we are ready to try and put this on the computer.

Markov Chains: Creating Shocks and Outcomes (1/2)

Generate a sequence of outcomes for random variables $\{g, \tau\}$ using the transition matrix.

1. Draw a uniform random variable:
   - $\rightarrow$ Go to state 0 if $x < Q(i, 0)$.
   - $\rightarrow$ Otherwise go to state 1, where i is the current state.

2. Given draws of $\{\tau_t\}$ compute the labor supply:

$$L(\tau_t) = \left[\frac{\beta}{1+\tau_t}\right]^{1/(1+\gamma)}$$

3. Compute the growth rate of output and prices, having already determined labor:

$$\begin{aligned}\frac{Y_t}{Y_{t-1}} &= (1+g_t)\frac{L_t}{L_{t-1}} \\ \frac{P_t}{P_{t-1}} &= \frac{1+\tau_t}{1+g_t}\frac{L_{t-1}}{L_t}\end{aligned}$$

4. Note that changes in relative labor will affect the volatility of the growth rates. We might want to revisit our calibration in light of this.

Markov Chains: Creating Shocks and Outcomes (2/2)

6. All that is left is the bond price, remember that

$$q(s_t) \;=\; \sum_{\tau_{t+1}} \frac{1}{1+\tau_{t+1}} Pr\{\tau_{t+1}|\tau_t, g_t\}$$

7. Conditional expectation of the growth rate of money at $t+1$ given state at t. There are two cases:

   → If $\tau = \tau_h$ then

$$\mathbb{E}\left[\frac{1}{1+\tau_{t+1}}\middle|\tau_h\right] = \frac{0.75}{1+\tau_h} + \frac{0.25}{1+\tau_l}$$

   → If $\tau = \tau_l$ then

$$\mathbb{E}\left[\frac{1}{1+\tau_{t+1}}\middle|\tau_l\right] = \frac{0.25}{1+\tau_h} + \frac{0.75}{1+\tau_l}$$

Stochastic Computer Model with AR(1)-Processes

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AR(1)-Process: Basic Setup (1/2)

- Our random variables are going to follow simple AR(1) processes in which

$$x_t = Ax_{t-1} + B + C\varepsilon_t,$$

- So long as the expected value of $\varepsilon_t = 0$,

$$\mathbb{E}\left\{x_t\right\} = AE\left\{x_{t-1}\right\} + B.$$

- Since this is an unconditional expectation, $\mathbb{E}\left\{x_t\right\} = \mathbb{E}\left\{x_{t-1}\right\}$, hence

$$\mathbb{E}\left\{x_t\right\} = \frac{B}{1-A},$$

AR(1)-Process: Basic Setup (2/2)

- Next, note that we can recursively substitute:

$$
\begin{aligned}
x_t &= B + C\varepsilon_t + A\{B + C\varepsilon_{t-1}\} + A^2\{B + C\varepsilon_{t-2}\} + \ldots \\
&= \frac{B}{1-A} + C\sum_{j=0}^{\infty} A^j \varepsilon_{t-j}
\end{aligned}
$$

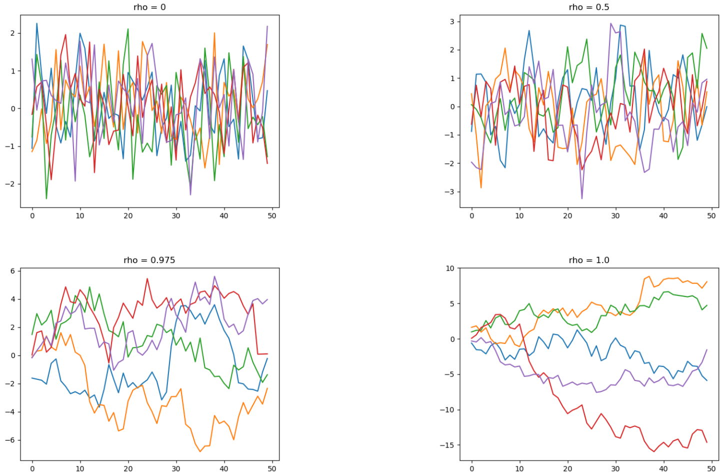
- This is because (geometric series):

$$
1 + A + A^2 + A^3 + \ldots = \frac{1}{1-A}
$$

- A is governing how persistent a shock is:
  - $\rightarrow$ First period $C\varepsilon_t$
  - $\rightarrow$ Second period $AC\varepsilon_t$
  - $\rightarrow$ Third period $A^2C\varepsilon_t$

AR(1)-Process: Examples

Here we simulate a simple AR(1): $x_t = \rho x_{t-1} + \epsilon_t$ for values of ρ.



What happens to the AR(1) process if $\rho > 1$?

AR(1)-Process in our Stochastic Model

- In our set-up, x_t will be a vector:

$$x_t = \begin{bmatrix} \tau_t \\ g_t \end{bmatrix}$$

- And our AR(1)-process is:

$$x_t = \begin{bmatrix} \rho_\tau & 0 \\ 0 & \rho_g \end{bmatrix} x_{t-1} + \begin{bmatrix} B_\tau \\ B_g \end{bmatrix} + \begin{bmatrix} \sigma_\tau & 0 \\ 0 & \sigma_g \end{bmatrix} \begin{bmatrix} \varepsilon_{\tau,t} \\ \varepsilon_{g,t} \end{bmatrix}$$

  $\rightarrow$ Zero entries on the off-diagonals indicate that both the process and shocks have no cross-correlations!

- So, the long-run means here is:

$$\begin{bmatrix} B_\tau/(1-\rho_\tau) \\ B_g/(1-\rho_g) \end{bmatrix}$$

AR(1)-Process: Transition and Probabilities (1/3)

In our theory we talk about $Pr\{s_{t+1}|s_t\}$. What does this mean?

- Start from our state vector $s_t = \left(Z_{t-1}, \bar{M}_{t-1}, \tau_t, g_t\right)$.
- The only state s_{t+1} that can follow must have $Z_t = Z_{t-1}(1+g_t)$ and $\bar{M}_t = (1+\tau_t)\bar{M}_{t-1}$:
  - $\rightarrow$ Either probability 0.
  - $\rightarrow$ Or $s_{t+1} = (Z_{t-1}(1+g_t), (1+\tau_t)\bar{M}_{t-1}, \tau_{t+1}, g_{t+1})$.
- However, there is a wide range of possible values for τ_{t+1} and g_{t+1}. We need to put probabilities on these possible outcomes.
- Once we do, the probability of

$$s_{t+1} = (Z_{t-1}(1+g_t), (1+\tau_t)\bar{M}_{t-1}, \tau_{t+1}, g_{t+1})$$

is simply

$$Pr\{s_{t+1}|s_t\} = Pr\{\tau_{t+1}, g_{t+1}|\tau_t, g_t\}$$

AR(1)-Process: Transition and Probabilities (2/3)

- For a given τ_{t+1} to follow from τ_t, it must be the case that the realized shock satisfies:

$$\tau_{t+1} \;=\; \rho_\tau \tau_t + B_\tau + \sigma_\tau \varepsilon_{\tau,t+1}$$

  $\rightarrow$ Hence, there is a unique shock associated with this realization.

- Denote the probability of that shock as:

$$\Pr(\varepsilon_{\tau,t+1}) \;=\; \Pr\left(\frac{\tau_{t+1} - \rho_\tau \tau_t - B_\tau}{\sigma_\tau}\right)$$

- Similarly if g_{t+1} is the realized growth rate, then the associated shock must satisfy:

$$g_{t+1} = \rho_g g_t + B_g + \sigma_g \varepsilon_{g,t+1}$$

  and hence the probability of that realization is:

$$\Pr\left(\varepsilon_{g,t+1}\right) = \Pr\left(\frac{g_{t+1} - \rho_g g_t - B_g}{\sigma_g}\right)$$

AR(1)-Process: Transition and Probabilities (3/3)

- Thus, the conditional probability of going from

$$s_t = \left(Z_{t-1}, \bar{M}_{t-1}, \tau_t, g_t\right)$$

to

$$s_{t+1} = \left(Z_{t-1}(1+g_t), (1+\tau_t)\bar{M}_{t-1}, \tau_{t+1}, g_{t+1}\right)$$

is given by:

$$\Pr\left(s_{t+1} | s_t\right) = \Pr\left(\frac{\tau_{t+1} - \rho_\tau \tau_t - B_\tau}{\sigma_\tau}\right) \Pr\left(\frac{g_{t+1} - \rho_g g_t - B_g}{\sigma_g}\right).$$

$\rightarrow$ This last bit follows from our independence assumption for our two shocks.

AR(1)-Process: Reduced Form Model

Our economy can be boiled down to two key equations.

1. The first is our labor supply condition:

$$L(s_t) = \left[\frac{\beta}{(1+\tau_t(s^t))}\right]^{\frac{1}{1+\gamma}} \tag{14}$$

2. The second equation is our interest rate condition:

$$q(s_t) = \sum_{s^{t+1}\in S^{t+1}(s^t)} \left[\frac{\beta}{(1+\tau_{t+1}(s^{t+1}))}\right] \Pr(s_{t+1}|s_t). \tag{15}$$

→ Approximating an integral can be done in clever ways, but we are not going to bother.

→ Instead, we draw enough ε_{t+1} shocks using a random number generator to get a representative distribution and use them in an approximation. So draw $\mathbb{C} = \{\varepsilon_1, ..., \varepsilon_N\}$.

→ Then just compute the implied update for each shock and take the appropriate mean

$$\mathbb{E}\left[\frac{1}{1+\tau_{t+1}}\middle|\tau_t\right] = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{1+\rho_\tau\tau_t + B_\tau + \sigma_\tau\varepsilon_i}\right]$$

.

Calculating Model Statistics

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Constructing Non-Stationary Variables

We've focused on the stationary random variables implied by our model. We can construct the nonstationary one too.

- We can construct the money supply recursively using the fact that

$$M_t = M_{t-1}(1+\tau_t)$$

  where we take $M_0 = 1$.

- We can do the same thing for productivity, Z_t:

$$Z_t = Z_{t-1}(1+g_t)$$

  where we take $Z_0 = 1$.

Calculating Growth Rates of the Variables

- Consumption is given by:

$$C_t = Z_t L_t$$

- The price level is given by:

$$P_t = \frac{M_{t-1}}{Z_t L_t}$$

With **nonstationary series** it is common to view them in growth rates:

- So if we have output Y_t and it grows at rate $1 + g_t$ then:

$$\frac{Y_t}{Y_{t-1}} = \frac{Y_{t-1}(1 + g_t)}{Y_{t-1}} = 1 + g_t$$

- Similarly, we can construct the inflation rate series as:

$$\frac{P_t}{P_{t-1}} = 1 + \pi_t$$

Calculating Growth Statistics

By calculating statistics about our growth processes, we can ...

- ask what is the mean growth rate:

$$\bar{g} = \frac{1}{N}\sum_{t=1}^{N} g_t$$

- ask what is the variance of our growth rate:

$$var(g) = \frac{1}{N}\sum_{t=1}^{N}(g_t - \bar{g})^2$$

- ask if output and inflation are correlated by computing:

$$corr(g,\pi) = \frac{\frac{1}{N}\sum_{t=1}^{N}(g_t - \bar{g})(\pi_t - \bar{\pi})}{\sqrt{var(g)var(\pi)}}$$

- Matlab has a bunch of statistical tools we can use like *mean*, *max*, *median*, *std*, and *corrcoef*.

Using Growth Statistics: Inflation and Output

Example

We can ask if systematic inflation is correlated with systematic growth in output?

Procedure:

1. Compute the average growth rate over some interval of time, e.g.:

$$
\begin{aligned}
(1+G_t) &= \left(\frac{Y_t}{Y_{t-3}}\right)^{1/3} = ((1+g_t)(1+g_{t-1})(1+g_{t-2}))^{1/3} \\
(1+\Pi_t) &= \left(\frac{P_t}{P_{t-3}}\right)^{1/3} = ((1+\pi_t)(1+\pi_{t-1})(1+\pi_{t-2}))^{1/3}
\end{aligned}
$$

   $\rightarrow$ This will give us the average growth rate of that interval.

2. Compute these average growth rates at each point in time t.
3. Check if these averages $1+G_t$ and $1+\Pi_t$ are correlated.

Sample Statistics vs. Analytical Statistics

We regularly compute all kinds of statistics in the data and from our models:

- How seriously should we take them?
- Will they change a lot if we just wait for some more data to come in?

One way to examine this is to **analytically derive the distribution** of our statistics.

- For example: If we have a sample of size n of normal random variables with mean μ and standard deviation σ, then ...
  - $\rightarrow$ the sample mean has expected value

$$\mathbb{E}\bar{X} = En^{-1}\sum_{t=1}^{n} X_t = \mu$$

  - $\rightarrow$ and variance

$$\mathbb{E}(\bar{X}-\mu)^2 = E\left[n^{-1}\sum_{t=1}^{n} X_t - \mu\right]^2 = E\left[n^{-2}\sum_{t=1}^{n}(X_t-\mu)^2\right] = \frac{\sigma^2}{n}$$

- This gives us an analytic expression for how accurate our sample mean is, given our sample size and the variance of the random variable.
- **But in general this is hard to do!**

Calculating Distributions: Monte Carlo Simulations

- An easy thing to do is to examine the distribution of a statistic in our model. Standard approach: **Monte Carlo simulation!**
  1. Draw different sequences of the shocks indexed by i, $\{\epsilon_t^i\}$.
  2. Build up the implied outcomes for each sequence.
  3. Compute the sample statistic for each sequence i, which we can denote by m_i.
  4. Examine the distribution of this sample statistic.

- This distribution gives us a lot of information about the accuracy of our estimate and is suggestive as to the accuracy of the data estimate as well.
  - $\rightarrow$ One way to think about accuracy is the standard deviation of m_i just as in our example with the sample mean of the normally distributed random variable.
  - $\rightarrow$ A more fundamental way to examine m_i is to plot the histogram. From this plot we can determine a 95% confidence interval for our model-based statistic.

Conclusions

Summarizing the Lecture

Can you summarize the three main aspects of the lecture?

Concluding Remarks

Big Picture of the Lecture:

1. How do random processes impact the results of the CiA model economy?
2. What role do expectations play in our CiA model?
3. How can we compare model statistics with the data?

- A stochastic model with **Markov processes:**
  - $\rightarrow$ The probability of future states can be described by the current state.
  - $\rightarrow$ Expectations can be formed by weighting future states with their probability.
  - $\rightarrow$ Otherwise, the model shows the same equations as before. Solutions can be easily derived.
- A stochastic model with **AR(1) processes:**
  - $\rightarrow$ A growth process depends on its realization in the previous period and a random shock.
  - $\rightarrow$ Any AR(1) process can be transformed into a Markov process.
  - $\rightarrow$ Any calculation of the AR(1) model is symmetric to the Markov model.
- Identifying **model statistics:**
  - $\rightarrow$ This allows to compare the model to the data!

References

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