Quantitative Dynamic Macroeconomics

– Assignment 01: A Three-Period CiA Model –

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Suppose there is an economy that exists for three periods, $t = 1, 2, 3$. Each producer-consumer household derives utility from consuming goods, c_t, and disutility from working, ℓ_t, in its own firm. Production is linear in labor, $y_{t,j} = Z_t \ell_{t,j}$, where Z_t is an exogenous productivity level. For simplicity, we assume that all goods have to be bought on the market, even the goods a household produces itself. Prices, p_t, are the same for all goods within a period. There is a cash-in-advance constraint, hence households have to hold money, M_t, at the beginning of a period to be able to buy goods. Alternatively, household can also buy one-period bonds, B_t, at a price q_t. The Lagrangian of the utility optimization problem of the household is as follows:

$$\mathcal{L} = \left\{ \max_{c_t, \ell_t, M_{t+1}, B_{t+1}} \right\}_{t=1,2,3} \left\{ \min_{\lambda_t} \right\}_{t=1,2,3} \sum_{t=1}^{3} \beta^{t-1} \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} - \phi \ell_t^{\nu} \right] \tag{1}$$

$$\begin{aligned} &+ \lambda_1 \left[M_1 - p_1 c_1 \right] \; + \; \mu_1 \left[p_{1,j} Z_1 \ell_1 + M_1 - p_1 c_1 + B_1 - q_1 B_2 - M_2 \right] \\ &+ \lambda_2 \left[M_2 - p_2 c_2 \right] \; + \; \mu_2 \left[p_{2,j} Z_2 \ell_2 + M_2 - p_2 c_2 + B_2 - q_2 B_3 - M_3 \right] \\ &+ \lambda_3 \left[M_3 - p_3 c_3 \right] \; + \; \mu_3 \left[p_{3,j} Z_3 \ell_3 + M_3 - p_3 c_3 + B_3 \right] \end{aligned} \tag{2}$$

The first set of equations numbered (1) is our objective function, and the second set of equations numbered (2) are our constraints. Some variables, like consumption and labor, only show up at one point in time. But other variables, like money and bonds, show up in two adjacent periods. So for these later variables, we have to be careful when we take the derivatives. Also, the time t and time $t + 1$ versions of our multipliers need not, and in general will not, be equal to each other.

Exercise 1 First-order conditions

Derive the first-order conditions for all relevant variables. Can you spot a pattern across time in the first-order conditions?

Exercise 2 Variables and economic interpretation

State the endogenous variables, exogenous variables, and penalty prices (Lagrange multipliers) of the model. How can we distinguish between endogenous and exogenous variables? How can we distinguish within the class of endogenous variables between control and non-control variables?

Exercise 3 Analytical solution of the model

Derive a solution to the three-period CiA model from above using the first-order conditions and the constraints. Assume that money supply growths by τ_t between two periods and that individual variables are equal to aggregate variables (representative household). Give a short interpretation of your results. *Hint: Work your way through the equations by backwards induction. This is, start in the last period and try to find a solution you can then plug in the period before.*