Quantitative Dynamic Macroeconomics

– Assignment 05: Capital in the CiA Model –

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The model in this assignment builds on the model derived in assignment 4. There are four changes: 1) All input factors are traded on competitive markets, hence a household employs labor and capital from the market instead of using only its own supply. 2) Production is given by a Cobb-Douglas function, $Y_t = \left(Z_t \bar{L}_t\right)^{1-\alpha} \bar{K}_t^{\alpha}$, where Z_t is the productivity level, $\bar{L}_t$ is labor traded on the input market, $\bar{K}_t$ is a capital stock traded on the input market, and $0 < \alpha < 1$ is a parameter. 3) The capital stock of each household changes over time accroding to the capital law of motion, $K_{t+1} = (1-\delta_K)K_t + X_t$, where $0 \leq \delta_K \leq 1$ is the capital depreciation rate, K_t is the individual capital stock, and X_t are investments into the capital stock. 4) There are no random shocks in this model, hence we can solve for the perfect foresight solution.

Each household chooses consumption today, C_t, labor supply today, L_t, money holdings next period, M_{t+1}, bond holdings next period, B_{t+1}, and the capital stock next period, K_{t+1}, to maximize utility given by

$$\mathbb{U}_t \;=\; \sum_{t=1,2} \beta^{t-1} \left[log(C_t) - \frac{\varphi}{1+\nu} L_t^{1+\nu} \right] + \beta^2 \mathbb{V}\left(M_3, B_3, K_3\right)$$

subject to the cash-in-advance constraint

$$\left(1-\delta_M\right) M_t \;=\; \bar{P}_t C_t,$$

and the budget constraint

$$\bar{P}_t w_t L_t + \bar{P}_t r_t K_t + \left(1-\delta_M\right) M_t + B_t + T_t \;=\; \bar{P}_t C_t + \bar{P}_t X_t + M_{t+1} + q_t B_{t+1},$$

and capital law of motion and the Cobb-Douglas production function as given above. The rest of the notation is as follows: $\bar{P}_t$ is the aggregate price level, M_t are cash holdings, B_t are one-period bonds, q_t is the cost of one bond, T_t are net government transfers, and $Z = 1$ is the normalized productivity level. $0 < \beta < 1$ is the period discount rate, and $\varphi, \nu > 0$ are labor parameters. δ_M is a constant money loss rate, and τ is a constant money growth rate. The money stock law of motion can then be described by $\bar{M}_{t+1} = (1+\tau)(1-\delta_M)\bar{M}_t$.

Assume that both labor and capital are traded on input factor markets and that both markets *are competitive, hence the real wage, w_t, and real rental rate, r_t, are given by their marginal produtivities.* In equilibrium, individual and aggregate variables are symmetric.

Exercise 1 An Economy with Money and Capital

a) Set up the Lagrangian and solve for the first-order conditions of the household.

b) Derive an appropriate change-in-variables and re-state the FOCs first to render the system of equations stationary. Then, describe how households optimally behave according to the first-order conditions and give economic intuition.

c) Summarize them as far as possible. Why is it not possible to find a closed form solution to the model as was the case in the previous versions of this model?

Exercise 2 Capital for Cash Assumption

Suppose that we also need to hold cash-in-advance to buy capital goods. *Hint: The only FOC that changes it the one for capital. However, be careful to use the new cash-in-advance constraint when substituting for it in the FOCs.*

a) Append the model from exercise 1 and derive the differences.

b) Describe the change in optimal household behavior according to the first-order conditions compared to exercise 1.