Lecture 06: Solving for the Dynamics of a DSGE Model

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Understanding Society

Outline for this Lecture

What have we seen so far

1. A model of money and capital [CG 1, 5]
2. Equilibrium tax distortions [CG 5, 6]
3. Optimal labor & capital taxation [CG 3, 6]

What we will see today

1. Stochastic CiA model with capital [CG 1, 5]
2. State-space and recursive methods [CG 1, 5]
3. Approximation & undet. coeff. [CG 3, 4]
4. Calibration & quantitative analysis [CG 4, 5]

Big Picture of the Lecture:

1. How can we solve intertemporal dynamic models?
2. What role do random shocks play in the solution?

The Complete CiA Dynamic Stochastic General Equilibrium Model

The Household Utility Maximization Problem

Each household maximizes utility by choosing $\{C_t, L_t, M_{t+1}, B_{t+1}, K_{t+1}\}_{t=1,2}$ such that

$$\max\ \mathbb{E}_t \left\{ \sum_{t=1,2} \beta^{t-1} \left[u(C_t) - v(L_t) \right] + \beta^2 \mathbb{V}\left(M_3, B_3, K_3 \right) \right\}$$

subject to

$$M_t \ \geq \ P_t C_t$$

$$\begin{aligned} & P_t \left[(Z_t L_t)^{1-\alpha} K_t^{\alpha} - \delta K_t \right] + \left[M_t - P_t C_t \right] + B_t + T_t \\ & \geq \ M_{t+1} + q_t B_{t+1} + P_t \left[K_{t+1} - K_t \right] \quad \forall t \leq 2 \end{aligned}$$

Model Setup: First-Order Conditions

The model is solved by its **FOCs and constraints** as follows:

$$
\begin{aligned}
1 &= \tilde{\lambda}_t + \tilde{\mu}_t && (1)\\
\tilde{\mu}_t &= \frac{\beta}{1+\tau_t} && (2)\\
L_t^{\gamma} &= \frac{\beta}{(1+\tau_t)\, C_t} Z_t^{1-\alpha} (1-\alpha)\, L_t^{-\alpha} K_t^{\alpha} && (3)\\
1 &= \mathbb{E}_t \left[\frac{\beta\,(1+\tau_t)\, C_t}{(1+\tau_{t+1})\, C_{t+1}} \left[(Z_{t+1} L_{t+1})^{1-\alpha} \alpha K_{t+1}^{\alpha-1} + 1 - \delta \right] \right] && (4)\\
C_t &= (Z_t L_t)^{1-\alpha} K_t^{\alpha} + (1-\delta) K_t - K_{t+1} && (5)
\end{aligned}
$$

- We use a change-in-variables as in previous lectures to derive the above conditions.
- The first two FOCs are determined only by parameters and exogenous variables, hence we can substitute them out in the last three equations.
- **Be mindful about the expectations operator in (4)!**

Model Setup: Fixed-Point Representation

Therefore, the three-equation block of the model that solves for the equilibrium is given by:

$$F^L(\cdot) = L_t^{\gamma} - \frac{\beta}{(1+\tau_t)\, C_t} Z_t^{1-\alpha} (1-\alpha)\, L_t^{-\alpha} K_t^{\alpha} \;=\; 0 \tag{6}$$

$$F^K(\cdot) = 1 - \mathbb{E}_t \left[\frac{\beta\,(1+\tau_t)\, C_t}{(1+\tau_{t+1})\, C_{t+1}} \left[\left(Z_{t+1} L_{t+1}\right)^{1-\alpha} \alpha K_{t+1}^{\alpha-1} + 1 - \delta \right] \right] \;=\; 0 \tag{7}$$

$$F^C(\cdot) = C_t - \left(Z_t L_t\right)^{1-\alpha} K_t^{\alpha} - (1-\delta) K_t + K_{t+1} \;=\; 0 \tag{8}$$

- (6)-(8) completely describes the model dynamics.
- The labor and consumption conditions depend only on current values.
- But the capital conditions has a much richer pattern of dependence.

$\Rightarrow$ **Issue of Solving the Model:** We are seeking a sequence of functions $\{C_t, L_t, K_{t+1}\}_{t=1}^{\infty}(s_t)$ that satisfy these three conditions at each point in time.

State Space Form and Recursive Methods

DSGE Models from a Mathematical Point of View

Any DSGE model can be represented by a function

$$f(X, U) \quad = \quad 0,$$

where

- X is a N x T matrix with N endogenous variables across T periods,
- U is a M x T matrix with M exogenous variables across T periods.

A model can have

- a finite number of endogenous variables, N, and exogenous variables, M,
- many periods, $t = 0, 1, 2, ..., T$, with possibly $T \to \infty$,
- many (possibly infinitely many) histories, $U_0, U_1, ..., U_t$.

Main Issue: We have to solve for a potentially very large number of unknowns!

Determining the Size of the Model

Size of the Problem:

$$(\#Variables, \quad \#Periods, \quad \#Histories)$$

- **No Uncertainty and Finite Number of Periods**
  - $\rightarrow$ N x T x 1 unknowns: Relatively small problem to solve!
  - $\rightarrow$ Solve for N x T (non-linear) equations and unknowns.
  - $\rightarrow$ See two-period model in assignment 0.
- **No Uncertainty and Infinite Number of Periods**
  - $\rightarrow$ N x ∞ x 1 unknowns: Impossible to solve!
  - $\rightarrow$ Well approximated by a large number of periods.
  - $\rightarrow$ See e.g. Cole (2020) chapter 19.4: Shooting algorithm.
- **Uncertainty and Infinite Number of Periods**
  - $\rightarrow$ N x ∞ x ∞ unknowns: Impossible to solve!
  - $\rightarrow$ Approximation requires large number of periods for every history (also a large number).
  - $\rightarrow$ Hence, difficult to solve! **Possible Solution?** $\Rightarrow$ Recursive Methods and Approximation!

Recursive Methods and the Policy Function (1/2)

- **Main Idea:** Break the model into smaller pieces (state-space form):

$$\mathbb{E}_t f\left(X_{t-1}, X_t, X_{t+1}, U_t, U_{t+1}\right) \quad = \quad 0$$

  - $\mathbb{E}_t$ is a rational expectation operator, X_t is a vector of model variables, and U_t is a vector of exogenous shocks.
  - Decisions at any point in time depend on the state of the economy, X_{t-1}, the exogenous shocks, U_t, and the expectation about the future, $(\mathbb{E}_t X_{t+1}; \mathbb{E}_t U_{t+1})$.
- Variables in period t depend only on the variables fully describing the history:

$$X_t \quad = g\left(X_{t-1}, U_t\right),$$

where $g(.)$ is a vector of unknown functions *(constant across time)*.

$\Rightarrow$ Hence, instead of solving for $\{X_t\}_{t=0}^{\infty}$, solve for a **vector of policy functions**, $\{g(.)\}_{n=1}^{N}$!

Recursive Methods and the Policy Function (2/2)

The Model in State-Space Form with Policy Functions:

$$\mathbb{E}_t f\left(X_{t-1}, g\left(X_{t-1}, U_t\right), g\left(g\left(X_{t-1}, U_t\right), U_{t+1}\right), U_t, U_{t+1}\right) \quad = \quad 0$$

Example - apply it to our model (see (6)-(8)):

$$F^L(\cdot) = (g_{L_t})^\gamma - \frac{\beta}{(1+\tau_t)(g_{C_t})} Z_t^{1-\alpha}(1-\alpha)(g_{L_t})^{-\alpha}(g_{K_t})^\alpha = 0 \tag{9}$$

$$F^K(\cdot) = 1 - \mathbb{E}_t\left[\frac{\beta(1+\tau_t)(g_{C_t})}{(1+\tau_{t+1})(g_{C_{t+1}})}\left[(Z_{t+1}(g_{L_{t+1}}))^{1-\alpha}\alpha(g_{K_{t+1}})^{\alpha-1} + 1 - \delta\right]\right] = 0 \tag{10}$$

$$F^C(\cdot) = (g_{C_t}) - (Z_t(g_{L_t}))^{1-\alpha} K_t^\alpha + (1-\delta)K_t - (g_{K_{t+1}}) = 0 \tag{11}$$

where

$$\begin{aligned}
L_t :=& \; g_{L_t} = g_L(s_t) = g_L(K_t, Z_t, \tau_t) \\
C_t :=& \; g_{C_t} = g_C(s_t) = g_C(K_t, Z_t, \tau_t) \\
K_{t+1} :=& \; g_{K_{t+1}} = g_K\{s_t\} = g_K(K_t, Z_t, \tau_t) \\
L_{t+1} :=& \; g_{L_{t+1}} = g_L(s_{t+1}(s_t)) = g_L(K_{t+1}, Z_{t+1}, \tau_{t+1}) = g_L(g_K(K_t, Z_t, \tau_t), Z_{t+1}, \tau_{t+1}) \\
C_{t+1} :=& \; g_{C_{t+1}} = g_C(s_{t+1}(s_t)) = g_C(K_{t+1}, Z_{t+1}, \tau_{t+1}) = g_C(g_K(K_t, Z_t, \tau_t), Z_{t+1}, \tau_{t+1})
\end{aligned}$$

$\rightarrow$ We stil need to define processes for the exogenous variables/shocks (e.g. AR(1))!

Solving for the Policy Function

How can we solve for the **policy functions** of the model economy?

- The policy function is a largely unknown function for us.
- It is a solution to system of (non-linear) first-order difference equations.

In this course: **Perturbation Method!** It requires 3 steps:

1. Calculating the **deterministic steady-state**:
   - $\rightarrow$ Known point of the system of first-order difference equations:
     Deterministic Steady-State: $f\left(\bar{X}, \bar{X}, \bar{X}, 0, 0\right) = 0$.
2. **Approximating** the dynamic system of equations:
   - $\rightarrow$ Approximation works well for small deviations around the deterministic steady-state.
   - $\rightarrow$ It relies on the differentiability of the system of equations.
   - $\rightarrow$ It can easily handle very large economic systems.
3. **Solving** for the first-order system of difference equations:
   - $\rightarrow$ Method of Undetermined Coefficients

Taylor Approximation:

Linearization of the System of Equations

First-Order Taylor Approximation (1/2)

Definition

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}(x_t - x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y_t - y_0)$$

- Given is the following equation $f(x,y) = g(x,y)$.
- Its steady-state is given by $f(x_0,y_0) = g(x_0,y_0)$.
- We calculate the Taylor approximation of both sides according to

$$\begin{aligned}\frac{\partial g(x_0,y_0)}{\partial x}(x_t - x_0) + \frac{\partial g(x_0,y_0)}{\partial y}(y_t - y_0) \\ \approx \frac{\partial f(x_0,y_0)}{\partial x}(x_t - x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y_t - y_0),\end{aligned}$$

where the steady-state on both sides cancels out.

First-Order Taylor Approximation (2/2)

In order to calculate percentage deviations from the steady-state, we use

$$\hat{x}_t = \frac{x_t - x_0}{x_0},$$

and plug it into the Taylor approximation

$$\frac{\partial g\left(x_0, y_0\right)}{\partial x} x_0 \hat{x}_t + \frac{\partial g\left(x_0, y_0\right)}{\partial y} y_0 \hat{y}_t \approx \frac{\partial f\left(x_0, y_0\right)}{\partial x} x_0 \hat{x}_t + \frac{\partial f\left(x_0, y_0\right)}{\partial y} y_0 \hat{y}_t.$$

Certainty Equivalence:

- The following condition holds in linear models: $\mathbb{E}\left[f\left(X\right)\right] = f\left(\mathbb{E}\left[X\right]\right)$.
- Only first moments (expected values) matter for the decisions.
- Expectations play no role in linear models: **Certainty Equivalence**!

Nth-Order Taylor Approximation

Definition

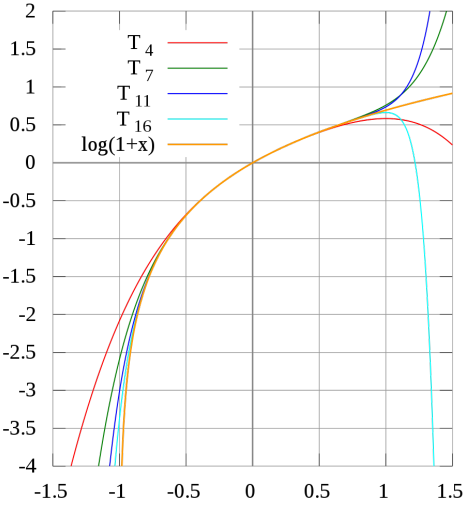
$$
\begin{aligned}
f(x) \approx\ & f(x_0) + \frac{f'(x_0)}{1!}(x_t - x_0) + \frac{f''(x_0)}{2!}(x_t - x_0)^2 \\
& + \frac{f'''(x_0)}{3!}(x_t - x_0)^3 + \cdots + \frac{f^N(x_0)}{N!}(x_t - x_0)^N
\end{aligned}
\tag{12}
$$

We need at least ...

- a second-order Taylor approximation to include (constant) uncertainty in the model.
- a third-order Taylor approximation to include conditional uncertainty in the model.

We can completely approximate any equation of order N by a Nth-order Taylor approximation.

What Degree of Approximation to Choose?



⇒ The function fit of the approximation does **NOT** increase monotonically in the order of the approximation!

Example: Linearization of our CiA DSGE Model (1/3)

Capital Law of Motion

- Original Equation:

$$K_{t+1} = (1-\delta) K_t + X_t$$

- Steady-State:

$$X = \delta K$$

- Linearized Equation:

$$\begin{aligned}
(K_{t+1} - K) &= (1-\delta)(K_t - K) + (X_t - X) \\
K \frac{K_{t+1} - K}{K} &= (1-\delta) K \frac{K_t - K}{K} + X \frac{X_t - X}{X} \\
K \hat{K}_{t+1} &= (1-\delta) K \hat{K}_t + X \hat{X}_t \\
\hat{K}_{t+1} &= (1-\delta) \hat{K}_t + \frac{X}{K} \hat{X}_t \\
\hat{K}_{t+1} &= (1-\delta) \hat{K}_t + \delta \hat{X}_t
\end{aligned}$$

Example: Linearization of our CiA DSGE Model (2/3)

Production Function

- Original Equation:

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}$$

- Steady-State:

$$Y = Z K^{\alpha} L^{1-\alpha}$$

- Linearized Equation:

$$\begin{aligned}
(Y_t - Y) &= K^{\alpha} L^{1-\alpha} (Z_t - Z) + Z K^{\alpha-1} L^{1-\alpha} \alpha (K_t - K) \\
&\quad + Z K^{\alpha} L^{-\alpha} (1-\alpha)(L_t - L) \\
Y \frac{Y_t - Y}{Y} &= Z K^{\alpha} L^{1-\alpha} \left[\frac{Z_t - Z}{Z} + \alpha \frac{K_t - K}{K} + (1-\alpha) \frac{L_t - L}{L} \right] \\
Y \hat{Y}_t &= Z K^{\alpha} L^{1-\alpha} \left[\hat{Z}_t + \alpha \hat{K}_t + (1-\alpha) \hat{L}_t \right] \\
\hat{Y}_t &= \hat{Z}_t + \alpha \hat{K}_t + (1-\alpha) \hat{L}_t
\end{aligned}$$

Example: Linearization of our CiA DSGE Model (3/3)

Linearized CiA DSGE Model

$$F^L(\cdot) \;=\; (\gamma+\alpha)\,\hat{L}_t - \Big[(1-\alpha)\hat{Z}_t + \alpha\hat{K}_t - \hat{C}_t - \hat{\tau}_t\Big] \;=\; 0 \tag{13}$$

$$F^K(\cdot) \;= \mathbb{E}_t\hat{C}_{t+1} - \hat{C}_t - \hat{\tau}_t + \mathbb{E}_t\hat{\tau}_{t+1} - \Sigma_1\mathbb{E}_t\left[\hat{Z}_{t+1} + \hat{L}_{t+1} - \hat{K}_{t+1}\right] \;=\; 0 \tag{14}$$

$$F^C(\cdot) \;= \hat{C}_t - (1+\delta\Sigma_2)\left[(1-\alpha)\left(\hat{Z}_t + \hat{L}_t\right) + \alpha\hat{K}_t\right] - \Sigma_2\left[(1-\delta)\hat{K}_t - \mathbb{E}_t\hat{K}_{t+1}\right] \;=\; 0 \tag{15}$$

where

$$\begin{aligned}
\Sigma_1 \;&=\; (1-\beta(1-\delta))\,(1-\alpha) \\
\Sigma_2 \;&=\; \frac{\alpha}{\frac{1-\beta}{\beta} + \delta(1-\alpha)}
\end{aligned}$$

Method of Undetermined Coefficients

Setup of the Method of Undetermined Coefficients (1/2)

Solution Approach:

- Take a **"guess"** of the solution of the system of difference equations.
- The solution must be **linear**, since the model is linear.
- Each **decision** on the control variables is based on state and exogenous stock variables.
- A guess to the solution takes the form

$$\hat{L}_t = g_L\left(\hat{K}_t, \hat{Z}_t, \hat{\tau}_t\right) = \psi_{L,K}\hat{K}_t + \psi_{L,Z}\hat{Z}_t + \psi_{L,\tau}\hat{\tau}_t \tag{16}$$

$$\hat{C}_t = g_C\left(\hat{K}_t, \hat{Z}_t, \hat{\tau}_t\right) = \psi_{C,K}\hat{K}_t + \psi_{C,Z}\hat{Z}_t + \psi_{C,\tau}\hat{\tau}_t \tag{17}$$

$$\hat{K}_{t+1} = g_K\left(\hat{K}_t, \hat{Z}_t, \hat{\tau}_t\right) = \psi_{K,K}\hat{K}_t + \psi_{K,Z}\hat{Z}_t + \psi_{K,\tau}\hat{\tau}_t \tag{18}$$

where $\psi_{L,K}, \psi_{L,Z}, \psi_{L,\tau}, \psi_{C,K}, \psi_{C,Z}, \psi_{C,\tau}, \psi_{K,K}, \psi_{K,Z}$, and $\psi_{K,\tau}$ are the **undetermined coefficients**. They depend on the parameters of the model (educated guess).

Setup of the Method of Undetermined Coefficients (2/2)

Expectations about future variables (symmetric for other variables and shock processes):

$$\mathbb{E}_t \hat{Z}_{t+1} = \mathbb{E}_t \left(\rho_Z \hat{Z}_t + \epsilon_{Z,t+1} \right) = \rho_Z \hat{Z}_t \tag{19}$$

$$\begin{aligned}
\mathbb{E}_t \hat{C}_{t+1} &= \mathbb{E}_t \left[\psi_{C,K} \hat{K}_{t+1} + \psi_{C,Z} \hat{Z}_{t+1} + \psi_{C,\tau} \hat{\tau}_{t+1} \right] \\
&= \psi_{C,K} \left(\psi_{K,K} \hat{K}_t + \psi_{K,Z} \hat{Z}_t + \psi_{K,\tau} \hat{\tau}_t \right) \\
&\quad + \psi_{C,Z} \rho_Z \hat{Z}_t + \mathbb{E}_t \hat{\epsilon}_{Z,t+1} + \psi_{C,\tau} \rho_\tau \hat{\tau}_t + \mathbb{E}_t \hat{\epsilon}_{\tau,t+1} \\
&= \psi_{C,K} \left(\psi_{K,K} \hat{K}_t + \psi_{K,Z} \hat{Z}_t + \psi_{K,\tau} \hat{\tau}_t \right) + \psi_{C,Z} \rho_Z \hat{Z}_t + \psi_{C,\tau} \rho_\tau \hat{\tau}_t
\end{aligned} \tag{20}$$

- **Constant policy functions:** Future decisions are made the same way as today, given expectations of future states and exogenous variables.
- **Iterative approach:** Expected future state variables are set by the decisions today.
- **Certainty Equivalence:** Only first moments (expected values) are part of the policy functions.
- **Uncertainty** (second moments) does not play any role!

Solving for the Undetermined Coefficients (1/5)

Substitute the policy functions for the (expected) control variables in the reduced-form model. The **labor FOC** is given by:

$$\begin{aligned} F^L(\cdot) \; = \; & (\gamma+\alpha)\left[\psi_{L,K}\hat{K}_t + \psi_{L,Z}\hat{Z}_t + \psi_{L,\tau}\hat{\tau}_t\right] \\ & + \left[\psi_{C,K}\hat{K}_t + \psi_{C,Z}\hat{Z}_t + \psi_{C,\tau}\hat{\tau}_t\right] \\ & - \alpha\hat{K}_t - (1-\alpha)\hat{Z}_t - \hat{\tau}_t \; = \; 0 \end{aligned}$$

Ordering the terms:

$$\begin{aligned} & \hat{K}_t\left[(\gamma+\alpha)\psi_{L,K} + \psi_{C,K} - \alpha\right] \\ & +\hat{Z}_t\left[(\gamma+\alpha)\psi_{L,Z} + \psi_{C,Z} - (1-\alpha)\right] \\ & +\hat{\tau}_t\left[(\gamma+\alpha)\psi_{L,\tau} + \psi_{C,\tau} - 1\right] \quad = \quad 0 \end{aligned} \tag{21}$$

Solving for the Undetermined Coefficients (2/5)

Substitute the policy functions for the (expected) control variables in the reduced-form model. The **capital FOC** is given by:

$$
\begin{aligned}
F^K(\cdot) \;=\; & \left[\psi_{C,K}\left(\psi_{K,K}\hat{K}_t + \psi_{K,Z}\hat{Z}_t + \psi_{K,\tau}\hat{\tau}_t\right) + \psi_{C,Z}\mathbb{E}_t\hat{Z}_{t+1} + \psi_{C,\tau}\mathbb{E}_t\hat{\tau}_{t+1}\right] \\
& - \left[\psi_{C,K}\hat{K}_t + \psi_{C,Z}\hat{Z}_t + \psi_{C,\tau}\hat{\tau}_t\right] - \hat{\tau}_t + \mathbb{E}_t\hat{\tau}_{t+1} \\
& - \Sigma_1\left[\mathbb{E}_t\hat{Z}_{t+1} + \psi_{L,K}\left(\psi_{K,K}\hat{K}_t + \psi_{K,Z}\hat{Z}_t + \psi_{K,\tau}\hat{\tau}_t\right) + \psi_{L,Z}\mathbb{E}_t\hat{Z}_{t+1} + \psi_{L,\tau}\mathbb{E}_t\hat{\tau}_{t+1}\right. \\
& \qquad \left. -\psi_{K,K}\hat{K}_t - \psi_{K,Z}\hat{Z}_t - \psi_{K,\tau}\hat{\tau}_t\right] \quad = \quad 0
\end{aligned}
$$

Ordering the terms:

$$
\begin{aligned}
& \hat{K}_t\left[\psi_{C,K}\left(\psi_{K,K}-1\right) - \Sigma_1\left(\psi_{L,K}-1\right)\psi_{K,K}\right] \\
& +\hat{Z}_t\left[\psi_{C,K}\psi_{K,Z} - \psi_{C,Z} - \Sigma_1\left(\psi_{L,K}\psi_{K,Z} - \psi_{K,Z}\right)\right] + \mathbb{E}_t\hat{Z}_{t+1}\left[\psi_{C,Z} - \Sigma_1\left(1+\psi_{L,Z}\right)\right] \\
& +\hat{\tau}_t\left[\psi_{C,K}\psi_{K,\tau} - \psi_{C,\tau} - 1 - \Sigma_1\left(\psi_{L,K}\psi_{K,\tau} - \psi_{K,\tau}\right)\right] + \mathbb{E}_t\hat{\tau}_{t+1}\left[\psi_{C,\tau} + 1 - \Sigma_1\psi_{L,\tau}\right] \quad = \quad 0
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\Leftrightarrow \quad & \hat{K}_t\left[\psi_{C,K}\left(\psi_{K,K}-1\right) - \Sigma_1\left(\psi_{L,K}-1\right)\psi_{K,K}\right] \\
& +\hat{Z}_t\left[\psi_{C,K}\psi_{K,Z} + \psi_{C,Z} - \Sigma_1\left(\psi_{L,K}\psi_{K,Z} - \psi_{K,Z}\right) + \rho_Z\left(\psi_{C,Z} - \Sigma_1\left(1+\psi_{L,Z}\right)\right)\right] \\
& +\hat{\tau}_t\left[\psi_{C,K}\psi_{K,\tau} - \psi_{C,\tau} - 1 - \Sigma_1\left(\psi_{L,K}\psi_{K,\tau} - \psi_{K,\tau}\right) + \rho_\tau\left(\psi_{C,\tau} + 1 - \Sigma_1\psi_{L,\tau}\right)\right] \quad = \quad 0
\end{aligned} \tag{23}
$$

Solving for the Undetermined Coefficients (3/5)

Substitute the policy functions for the (expected) control variables in the reduced-form model. The **resource constraint** is given by:

$$
\begin{aligned}
F^C(\cdot) \; &= \psi_{C,K}\hat{K}_t + \psi_{C,Z}\hat{Z}_t + \psi_{C,\tau}\hat{\tau}_t - (1+\delta\Sigma_2)(1-\alpha)\left[\hat{Z}_t + \psi_{L,K}\hat{K}_t + \psi_{L,Z}\hat{Z}_t + \psi_{L,\tau}\hat{\tau}_t + \frac{\alpha}{1-\alpha}\hat{K}_t\right] \\
&\quad + \Sigma_2\left[\psi_{K,K}\hat{K}_t + \psi_{K,Z}\hat{Z}_t + \psi_{K,\tau}\hat{\tau}_t\right] - \Sigma_2(1-\delta)\hat{K}_t \quad = \quad 0
\end{aligned}
$$

Ordering the terms:

$$
\begin{aligned}
&\hat{K}_t\left[\psi_{C,K} - (1+\delta\Sigma_2)\left[(1-\alpha)\psi_{L,K} + \alpha\right] + \Sigma_2\psi_{K,K} - \Sigma_2(1-\delta)\right] \\
&+\hat{Z}_t\left[\psi_{C,Z} - (1+\delta\Sigma_2)(1-\alpha)\left(1+\psi_{L,Z}\right) + \Sigma_2\psi_{K,Z}\right] \\
&+\hat{\tau}_t\left[\psi_{C,\tau} - (1+\delta\Sigma_2)(1-\alpha)\psi_{L,\tau} + \Sigma_2\psi_{K,\tau}\right] \quad = \quad 0
\end{aligned}
\tag{24}
$$

Solving for the Undetermined Coefficients (4/5)

In order for (21)-(24) to hold for all possible $\hat{K}_t, \hat{Z}_t, \hat{\tau}_t \forall t$, the coefficients in the square brackets of each variable have to be zero! The following **nine conditions** are derived:

$$(\gamma+\alpha)\,\psi_{L,K}+\psi_{C,K}-\alpha = 0 \tag{25}$$

$$\psi_{C,K}\,(\psi_{K,K}-1)-\Sigma_1\,(\psi_{L,K}-1)\,\psi_{K,K} = 0 \tag{26}$$

$$\psi_{C,K}-(1+\delta\Sigma_2)\,(1-\alpha)\psi_{L,K}+\Sigma_2\psi_{K,K}-\Sigma_2\,(1-\delta) = 0 \tag{27}$$

$$(\gamma+\alpha)\,\psi_{L,Z}+\psi_{C,Z}-(1-\alpha) = 0 \tag{28}$$

$$\psi_{C,K}\psi_{K,Z}+\psi_{C,Z}-\Sigma_1\,(\psi_{L,K}\psi_{K,Z}-\psi_{K,Z})+\rho_Z\,(\psi_{C,Z}-\Sigma_1\,(1+\psi_{L,Z})) = 0 \tag{29}$$

$$\psi_{C,Z}-(1+\delta\Sigma_2)\,(1-\alpha)\,(1+\psi_{L,Z})+\Sigma_2\psi_{K,Z} = 0 \tag{30}$$

$$(\gamma+\alpha)\,\psi_{L,\tau}+\psi_{C,\tau}-1 = 0 \tag{31}$$

$$\psi_{C,K}\psi_{K,\tau}-\psi_{C,\tau}-1-\Sigma_1\,(\psi_{L,K}\psi_{K,\tau}-\psi_{K,\tau})+\rho_\tau\,(\psi_{C,\tau}+1-\Sigma_1\psi_{L,\tau}) = 0 \tag{32}$$

$$\psi_{C,\tau}-(1+\delta\Sigma_2)\,(1-\alpha)\psi_{L,\tau}+\Sigma_2\psi_{K,\tau} = 0 \tag{33}$$

$\Rightarrow$ We solve for the (non-linear) system of nine equations and nine **undetermined coefficients**, $\psi_{L,K}, \psi_{L,Z}, \psi_{L,\tau}, \psi_{C,K}, \psi_{C,Z}, \psi_{C,\tau}, \psi_{K,K}, \psi_{K,Z}$, and $\psi_{K,\tau}$!

Solving for the Undetermined Coefficients (5/5)

We use (25)-(27) to solve for $\psi_{K,K}$:

(it enters quadratically as we have both K_t and K_{t+1} as states in the policy functions)

$$AA\psi_{K,K}^2 + BB\psi_{K,K} + CC = 0 \tag{34}$$

where

$$\begin{aligned}
AA &= \Sigma_2\left(\Sigma_1 + \gamma + \alpha\right) \\
BB &= (-1)\left\{\Sigma_2\left(1 - (1-\alpha)\delta\right)\left(\Sigma_1 + \gamma + \alpha\right) + \Sigma_2(\gamma + \alpha) + \left(\gamma + \alpha + (1-\alpha)(1 + \delta\Sigma_2)\right)\left(\Sigma_1 + \alpha\right)\right\} \\
CC &= \alpha\left(\gamma + \alpha + (1-\alpha)(1 + \delta\Sigma_2)\right) + \Sigma_2\left(1 - (1-\alpha)\delta\right)\left(\gamma + \alpha\right)
\end{aligned}$$

It follows

$$\left(\psi_{K,K}\right)_{1,2} = \frac{(-1)\,BB \pm \sqrt{BB^2 - 4AA \cdot CC}}{2AA}$$

Implications of $\psi_{K,K}$ for Stability

There are two solutions for (34). This allows for 3 different combinations:

- $(\psi_{K,K})_1 > 1$ and $(\psi_{K,K})_2 > 1$.
  Implications: Any exogenous shock drives $\hat{K}_t$ to $\pm\infty$. Unstable solution. $\rightarrow$ This root is ruled out by the **transversality condition!**

- $(\psi_{K,K})_1 < 1$ and $(\psi_{K,K})_2 < 1$.
  Implications: Stable solution. But there are two stable roots for capital in the consumption policy function. This leads to no unique root and self-fulfilling expectations. $\rightarrow$ **Indeterminacy!**

- $(\psi_{K,K})_1 < 1$ and $(\psi_{K,K})_2 > 1$.
  Implications: Stable solution. We can rule out the explosive root $\psi_{K,K} > 1$. Hence, we are left with one stable root that uniquely pins down consumption (determinacy). $\rightarrow$ **Saddle-path stability!**

Pinning Down the Solution

Given **saddle-path stability**, use the stable root to pin down the rest of the coefficients by (28) to (33). The policy functions give the **solution to the system of linear first-order difference equations**:

$$
\begin{aligned}
\hat{L}_t &= g_L\left(\hat{K}_t, \hat{Z}_t, \hat{\tau}_t\right) &= \psi_{L,K}\hat{K}_t + \psi_{L,Z}\hat{Z}_t + \psi_{L,\tau}\hat{\tau}_t \\
\hat{C}_t &= g_C\left(\hat{K}_t, \hat{Z}_t, \hat{\tau}_t\right) &= \psi_{C,K}\hat{K}_t + \psi_{C,Z}\hat{Z}_t + \psi_{C,\tau}\hat{\tau}_t \\
\hat{K}_{t+1} &= g_K\left(\hat{K}_t, \hat{Z}_t, \hat{\tau}_t\right) &= \psi_{K,K}\hat{K}_t + \psi_{K,Z}\hat{Z}_t + \psi_{K,\tau}\hat{\tau}_t
\end{aligned}
$$

The AR(1)-process of the stochastic shock is determined exogenously:

$$
\begin{aligned}
\hat{Z}_t &= \rho_Z\hat{Z}_{t-1} + \hat{\epsilon}_{Z,t} \\
\hat{\tau}_t &= \rho_\tau\hat{\tau}_{t-1} + \hat{\epsilon}_{\tau,t}
\end{aligned}
$$

The solutions to all other variables are given by the model equations:

$$
\begin{aligned}
\hat{X}_t &= \frac{1}{\delta}\left(\left(\psi_{K,K}\hat{K}_t + \psi_{K,Z}\hat{Z}_t + \psi_{K,\tau}\hat{\tau}_t\right) - (1-\delta)\hat{K}_t\right) \\
&= \psi_{X,K}\hat{K}_t + \psi_{X,Z}\hat{Z}_t + \psi_{X,\tau}\hat{\tau}_t \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
\hat{Y}_t &= (1-\alpha)\left[\hat{Z}_t + \psi_{L,K}\hat{K}_t + \psi_{L,Z}\hat{Z}_t + \psi_{L,\tau}\hat{\tau}_t\right] + \alpha\hat{K}_t \\
&= \psi_{Y,K}\hat{K}_t + \psi_{Y,Z}\hat{Z}_t + \psi_{Y,\tau}\hat{\tau}_t \qquad (36)
\end{aligned}
$$

Calibration and Quantitative Analysis

Calibrating the Model: Approach

Approach to parameterize a model:

- **Idea:** Set parameters of the model such that its simulation replicates aspects of the data.
- **Calibration:** Set parameters according to literature, data, etc.
- **Estimation:** Set parameters in an estimation to maximize likelihood of simulation in replicating the data *(not in this course)*.

How to calibrate our model?

- Use microeconomic studies about fundamental relationships within households, firms, etc.
- Target a set of moments in the data to be replicated in the model simulation:
  - $\rightarrow$ First moments: Target steady-state relationships of endogenous variables.
  - $\rightarrow$ Second moments: Target variance of endogenous variables in the data.

Calibrating the Model: Overview

Variable	Value	Variable	Value
β	0.98	σ_Z	0.009
γ	$\frac{1}{0.72}$	σ_τ	0.008
δ	0.02	ρ_Z	0.975
α	$\frac{1}{3}$	ρ_τ	0.48

Table 1: Calibration overview

- Set β by targeting the long-run mean real interest rate through the steady-state: $\beta = \frac{1}{1+r}$.
- Similarily, we can target the capital share in the data by $r \times K = \frac{\partial Y}{\partial K} \times K = \alpha$.
- Other parameters as e.g. γ do not have such a clear long-run mean representation in the data, hence microeconomic evidence from e.g. labor economics studies is used to set it.

Solving the Model: Step-by-Step Cookbook

Over the last six weeks, we have learned about each step to set up and **solve a DSGE model using perturbation methods.** Let's summarize step-by-step:

1. Set-up the **model framework** and solve for its first-order conditions:
   - $\rightarrow$ Take assumptions about households, firms, and markets.
   - $\rightarrow$ Use the Lagrange method to solve optimization under binding constraints.
   - $\rightarrow$ Impose equilibrium conditions through constraints and market clearing.
2. Solve the non-linear system of difference equations by the **perturbation method**:
   - $\rightarrow$ Solve for the deterministic steady-state of the model.
   - $\rightarrow$ Linearize the model around its deterministic steady-state.
   - $\rightarrow$ Derive the linear policy functions by solving for the undetermined coefficients of the model.
3. Initialize and **simulate the model** using random draws of shocks across time:
   - $\rightarrow$ Take assumptions about the random processes and draw shocks.
   - $\rightarrow$ Iterate policy functions forward using shocks and updating the states along the way.
4. Calculate statistics and display impulse response functions and time series data.

Conclusion

Summarizing the Lecture

Can you summarize the three main aspects of the lecture?

Concluding Remarks

Big Picture of the Lecture:

1. How can we solve intertemporal dynamic models?
2. What role do random shocks play in the solution?

- We substitute the problem of solving for infinitely many variables by solving for finitely many policy functions:
  - → **State-space model:** Decisions are based on the state of the economy today and expecations about the future. The decision rules are constant by **recursivity**.
  - → **Perturbation method:** We approximate the model equations by an easier functional form around a known point (deterministic steady-state).
- We approximate the non-linear system of model equations by a Taylor approximation:
  - → **Linearization:** Linear difference equations are easy to solve, but uncertainty drops out.
  - → Higher orders of approximation do not necessarily improve the the model fit.
- We **guess the functional form** for the policy functions and solve for its coefficients:
  - → A solution can only depend on the states of the economy.
  - → There is a stable and an unstable root of the economy.

References

- Harold L. Cole (2020). Monetary and Fiscal Policy through a DSGE Lens. Chapters 19 and 20. Oxford University Press.