New-Keynesian Phillips Curve

— Derivation and Recursive Framework —

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1 Set Up

In a standard New Keynesian business cycle framework prices are sticky and only adjust slowly to exogenous shocks. Every period there is a constant share of randomly selected firms which can adjust their prices. All other firms keep their prices from the last period. This approach follows the framework first introduced by Calvo (1983). The probability whether a single firm can adjust its prices in a given period is determined by a *Poisson distribution*, hence it is independent of the history of the firm.

Price index. The aggregate price level can be formulated by

$$P_t \quad = \quad \left[\int_{S(t)} p_{t-1}(i)^{1-\epsilon} di + (1-\omega)(p_t^*)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}, \tag{1}$$

where P_t is the average price level today, $p_{t-1}(i)$ is the price of a single firm i from last period, p_t^* is the price level chosen by price adjusting firms, ϵ is the elasticity of demand between differentiated goods, and ω is the constant share of firms that cannot adjust their prices today.

The price level is determined by two elements. First, the distribution of prices of all firms that cannot adjust their prices today. And second, the optimal price chosen by all firms that can adjust their prices today. Since the production function and demand schedule are symmetric across firms, all price adjusting firms choose the same price. We rewrite (1) as

$$P_t \quad = \quad \left[\omega P_{t-1}^{1-\epsilon} + (1-\omega)(p_t^*)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}, \tag{2}$$

where the aggregation of the non-adjusting firms follows from two assumptions. First, there is *mass one of identical firms*. Hence, there are infinitely many firms on the unit interval. And second, whether firms can adjust their prices or not in a given period is determined by a *Poisson distribution*. Hence, by the *law of large numbers*, it follows that the price distribution is equal across both groups of firms (before the adjusting firms choose a new price). The aggregate price level of non-adjusting firms is equal to the average price level of last period. We rewrite (2) by dividing through P_t

$$1 = (1-\omega)\left(\frac{p_t^*}{P_t}\right)^{1-\epsilon} + \omega\left(\frac{P_{t-1}}{P_t}\right)^{1-\epsilon} \tag{3}$$

$$\Leftrightarrow \frac{p_t^*}{P_t} = \left[\frac{1-\omega\pi_t^{\epsilon-1}}{1-\omega}\right]^{\frac{1}{1-\epsilon}}, \tag{4}$$

where $\pi_t = \frac{P_t}{P_{t-1}}$ is the gross inflation rate in period t.

Optimal price setting. Each firm that can adjust its price this period optimizes it by maximizing the present discounted value of future profits for the expected time this price is active - as long as the firm cannot re-adjust its prices in expectation. Hence, each firm solves

$$\max_{p_t(j)} E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i}\Pi_{t+i}(j) = \max_{p_t(j)} E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i}\left[\left(\frac{p_t(j)}{P_{t+i}}\right) - \varphi_{t+i}(j)\right] Y_{t+i}(j), \tag{5}$$

where $\beta_{t,t+i} = \beta^i\left(\frac{\lambda_{t+i}}{\lambda_t}\right)$ is the stochastic discount factor[1], $\Pi_{t+i}(j)$ is the profit function of firm j in period $t+i$, $Y_t(j) = \left(\frac{p_t(j)}{P_t}\right)^{-\epsilon} Y_t$ is the demand function for the good of firm j[2], $p_t(j)$ is the price set by firm j in period t, and $\varphi_{t+i}(j)$ is the real marginal cost of production of firm j in period $t+i$. Substituting for $Y_{t+i}(j)$ with the demand function one gets

$$\max_{p_t(j)} E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i}\left[\left(\frac{p_t(j)}{P_{t+i}}\right)^{1-\epsilon} - \varphi_{t+i}(j)\left(\frac{p_t(j)}{P_{t+i}}\right)^{-\epsilon}\right] Y_{t+i}, \tag{6}$$

which is a profit function depending on each firm's price and marginal cost only, while the aggregate price level and consumption are given.

2 First-Order Conditions

By solving (6) we receive the following first-order-condition

$$\frac{\partial \Pi_{t+i}(j)}{\partial p_t(j)} = E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i}\left[(1-\epsilon) + \epsilon\varphi_{t+i}(j)\left(\frac{p_t(j)}{P_{t+i}}\right)^{-1}\right]\left(\frac{p_t(j)}{P_{t+i}}\right)^{-\epsilon}\frac{Y_{t+i}}{P_{t+i}} = 0, \tag{7}$$

$$= E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i}\left[(1-\epsilon) + \epsilon\varphi_{t+i}(j)\frac{P_t}{p_t(j)}\frac{P_{t+i}}{P_t}\right]\left(\frac{P_t}{p_t(j)}\frac{P_{t+i}}{P_t}\right)^{\epsilon}\frac{Y_{t+i}}{P_{t+i}} = 0, \tag{8}$$

$$= \frac{1}{P_t}\left(\frac{P_t}{p_t(j)}\right)^{\epsilon} E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i}\left[(1-\epsilon) + \epsilon\varphi_{t+i}(j)\pi_{t,t+i}\frac{P_t}{p_t(j)}\right]\pi_{t,t+i}^{\epsilon}\frac{P_t}{P_{t+i}}Y_{t+i} = 0, \tag{9}$$

$$= -(\epsilon-1)E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i}\pi_{t,t+i}^{\epsilon-1}Y_{t+i} + \epsilon E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i}\pi_{t,t+i}^{\epsilon}Y_{t+i}\varphi_{t+i}(j)\frac{P_t}{p_t(j)} = 0, \tag{10}$$

$$= -(\epsilon-1)E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i}\pi_{t,t+i}^{\epsilon-1}Y_{t+i} + \epsilon E_t \sum_{i=0}^{\infty} \omega^i \beta_{t,t+i}\pi_{t,t+i}^{\frac{\epsilon}{1-\alpha}}Y_{t+i}\varphi_{t+i}\left(\frac{P_t}{p_t(j)}\right)^{1+\frac{\alpha\epsilon}{1-\alpha}} = 0, \tag{11}$$

[1]This firm discounting follows from the assumption that all firms are owned by a mutual fund which in turn is owned by households with equal shares. Hence, firms discount the future by the same factor as households.

[2]The demand function for a good of any firm follows from the first stage optimization of the households when firms are monopolistic competitors following Dixit and Stiglitz (1977).

where $\pi_{t,t+i} = \left(\frac{P_{t+i}}{P_t}\right)$ is the gross inflation rate across the time interval defined by i. (7) directly follows from taking the first derivative, (8) rewrites the aggregate prices as inflation rates, (9) takes all common terms that do not depend on the sum operator i out of the sum operator, (10) splits the terms inside the square bracket into two separate sum operators, and (11) uses the marginal cost function $\varphi_{t+i}(j) = \varphi_{t+i}\left(\frac{p_t(j)}{P_{t+i}}\right)^{\frac{-\alpha\epsilon}{1-\alpha}}$ (see Gali (2015), ch. 3 for the derivation).
Because the problem is symmetric for all firms that can re-adjust their prices, we can drop the firm index j, hence $p_t(j) = p_t^*$. Solving for $\frac{p_t^*}{P_t}$ we get

$$\left(\frac{p_t^*}{P_t}\right)^{1+\frac{\alpha\epsilon}{1-\alpha}} = \frac{\epsilon}{\epsilon-1}\frac{E_t\sum_{i=0}^{\infty}\omega^i\beta_{t,t+i}\pi_{t,t+i}^{\frac{\epsilon}{1-\alpha}}Y_{t+i}\varphi_{t+i}}{E_t\sum_{i=0}^{\infty}\omega^i\beta_{t,t+i}\pi_{t,t+i}^{\epsilon-1}Y_{t+i}}, \tag{12}$$

which is the optimal price set by all firms that can adjust their price in a given period. It is determined by the expected present value of all future costs divided by the expected present value of all future returns. Hence, it is determined by the inverse of the expected present value of price markups. Discounting is determined by the discount factor β and the average probability of this price still being active in a certain future period ω^i. Due to monopolistic competition, the firm aims to keep an average mark-up $\frac{\epsilon}{\epsilon-1}$ on its price across time.

3 Recursive Framework

The first-order condition contains two infinite sums. To solve the model both analytically and numerically, we need a more tractable version of this equation. By rewriting (12) recursively, we get a first-order difference equation which we can use in a canonical representation of the system. First, rewrite the equation by introducing two auxiliary variables, so each infinite sum is treated separately. Rewrite (12) as

$$\left(\frac{p_t^*}{P_t}\right)^{1+\frac{\alpha\epsilon}{1-\alpha}} = \frac{\epsilon}{\epsilon-1}\frac{Ax_t}{Ay_t}, \tag{13}$$

where $Ax_t = E_t\sum_{i=0}^{\infty}\omega^i\beta_{t,t+i}\pi_{t,t+i}^{\frac{\epsilon}{1-\alpha}}Y_{t+i}\varphi_{t+i}$ and $Ay_t = E_t\sum_{i=0}^{\infty}\omega^i\beta_{t,t+i}\pi_{t,t+i}^{\epsilon-1}Y_{t+i}$. Second, rewrite the infinite sums using continuation values in a recursive framework. I show how to rewrite the infinite sum of the first auxiliary variable, Ax_t, into a recursive framework. The approach for Ay_t is identical and left as an exercise. Start by writing out the infinite sum for some periods

$$Ax_t = Y_t\varphi_t + \omega E_t\beta_{t,t+1}\pi_{t,t+1}^{\frac{\epsilon}{1-\alpha}}Y_{t+1}\varphi_{t+1} + \omega^2 E_t\beta_{t,t+2}\pi_{t,t+2}^{\frac{\epsilon}{1-\alpha}}Y_{t+2}\varphi_{t+2} + ... \tag{14}$$

In a next step, shift Ax_t one period into the future

$$Ax_{t+1} = Y_{t+1}\varphi_{t+1} + \omega E_{t+1}\beta_{t+1,t+2}\pi_{t+1,t+2}^{\frac{\epsilon}{1-\alpha}}Y_{t+2}\varphi_{t+2} + \omega^2 E_{t+1}\beta_{t+1,t+3}\pi_{t+1,t+3}^{\frac{\epsilon}{1-\alpha}}Y_{t+3}\varphi_{t+3} + ... \tag{15}$$

Next, we manipulate (15) such that the terms for each time period $(t+1, t+2, t+3, ...)$ on the right-hand side are the same in (14) and (15). In the later equation we do not have period t, but we can neglect that for the moment. This approach results in

$$\omega\beta_{t,t+1}\pi_{t,t+1}^{\frac{\epsilon}{1-\alpha}}Ax_{t+1} = \omega\beta_{t,t+1}\pi_{t,t+1}^{\frac{\epsilon}{1-\alpha}}Y_{t+1}\varphi_{t+1} + \omega^2 E_{t+1}\beta_{t,t+2}\pi_{t,t+2}^{\frac{\epsilon}{1-\alpha}}Y_{t+2}\varphi_{t+2} + ... \tag{16}$$

where $\pi_{t,t+2} = \pi_{t+1,t+2} \cdot \pi_{t,t+1} = \frac{P_{t+2}}{P_{t+1}} \frac{P_{t+1}}{P_t}$ and $\beta_{t,t+2} = \beta_{t,t+1} \cdot \beta_{t+1,t+2} = \beta^2 [\frac{\lambda_{t+1}}{\lambda_t} \frac{\lambda_{t+2}}{\lambda_{t+1}}]$.
As a last step, correct for the expectation operator. Hence, multiply both sides of (16) by E_t. By the law of iterated expectations, $E_t \cdot E_{t+1} = E_t$, we get

$$\omega E_t \beta_{t,t+1} \pi_{t,t+1}^{\frac{\epsilon}{1-\alpha}} Ax_{t+1} = \omega E_t \beta_{t,t+1} \pi_{t,t+1}^{\frac{\epsilon}{1-\alpha}} Y_{t+1} \varphi_{t+1} + \omega^2 E_t \beta_{t,t+2} \pi_{t,t+2}^{\frac{\epsilon}{1-\alpha}} Y_{t+2} \varphi_{t+2} + ... \quad (17)$$

As the sum operator runs to infinity, the right-hand side of (17) and the right-hand side of (14) less the first term, $Y_t \varphi_t$, are identical. We can plug (17) into (14) and get

$$Ax_t = Y_t \varphi_t + \omega E_t \beta_{t,t+1} \pi_{t,t+1}^{\frac{\epsilon}{1-\alpha}} Ax_{t+1}. \quad (18)$$

The same procedure works for Ay_t, which results in

$$Ay_t = Y_t + \omega E_t \beta_{t,t+1} \pi_{t,t+1}^{\epsilon-1} Ay_{t+1}. \quad (19)$$

A framework where some variable in period t depends on itself in period $t+1$ is called a recursive framework. This type of framework is used extensively in business cycle theory to get more tractable frameworks and be able to calculate a solution for the system.

4 Linearization

For computational convenience, linearize the non-linear system around its deterministic steady-state by using a first-order Taylor approximation given by

$$h(x,y) \simeq h(\bar{x},\bar{y}) + h_x \frac{\bar{x}}{\bar{x}}(\bar{x},\bar{y})(x-\bar{x}) + h_y \frac{\bar{y}}{\bar{y}}(\bar{x},\bar{y})(y-\bar{y}) = h(\bar{x},\bar{y}) + h_x \bar{x}\hat{x} + h_y \bar{y}\hat{y}, \quad (20)$$

where $\bar{x}$ and $\bar{y}$ are the deterministic steady-states and $\hat{x} = \frac{x-\bar{x}}{\bar{x}}$ and $\hat{y} = \frac{y-\bar{y}}{\bar{y}}$ are percentage deviations from the respective deterministic steady-states[3]. For (13), (18), and (19) this approach results in

$$Ay = \frac{\epsilon}{\epsilon-1} Ax, \quad (21)$$

$$Ax = \frac{Y\varphi}{1-\omega\beta}, \quad (22)$$

$$Ay = \frac{Y}{1-\omega\beta}, \quad (23)$$

which can be solved for the steady-state real marginal costs $\varphi = \frac{\epsilon-1}{\epsilon}$. As a next step, linearize the system of equations around the deterministic steady-state by applying the first-order Taylor approximation to (13), (18), and (19), which results in

$$\left(1 + \frac{\alpha\epsilon}{1-\alpha}\right)\left(\hat{p}_t^* - \hat{P}_t\right) = \hat{A}x_t - \hat{A}y_t, \quad (24)$$

$$\hat{A}x_t = (1-\omega\beta)(\hat{Y}_t + \hat{\varphi}_t) + \omega\beta(\hat{\beta}_{t,t+1} + \frac{\epsilon}{1-\alpha}\hat{\pi}_{t+1} + \hat{A}x_{t+1}), \quad (25)$$

$$\hat{A}y_t = (1-\omega\beta)\hat{Y}_t + \omega\beta(\hat{\beta}_{t,t+1} + (\epsilon-1)\hat{\pi}_{t+1} + \hat{A}y_{t+1}), \quad (26)$$

[3] To calculate the deterministic steady-state in a time-discrete system without any trend variable, simply drop all time indexes and solve the system of equations for the endogenous variables.

where (22) and (23) are used to simplify the notation of steady-state variables in the linearized equations. Additionally, linearize the aggregate price level index (2) around its deterministic steady-state. It is given by

$$\hat{p}_t^* - \hat{P}_t \quad = \quad \frac{\omega}{1-\omega}\hat{\pi}_t. \tag{27}$$

5 The New-Keynesian Phillips Curve (NKPC)

Finally, let's put everything back together. First, plug (25) and (26) into (24), hence

$$\begin{aligned}\left(1+\frac{\alpha\epsilon}{1-\alpha}\right)\left(\hat{p}_t^* - \hat{P}_t\right) &= (1-\omega\beta)\left(\hat{Y}_t + \hat{\varphi}_t\right) + \omega\beta\left(\hat{\beta}_{t,t+1} + \frac{\epsilon}{1-\alpha}\hat{\pi}_{t+1} + \hat{A}x_{t+1}\right) \\ &\quad - (1-\omega\beta)\,\hat{Y}_t - \omega\beta\left(\hat{\beta}_{t,t+1} + (\epsilon-1)\hat{\pi}_{t+1} + \hat{A}y_{t+1}\right)\end{aligned}, \tag{28}$$

$$\quad = (1-\omega\beta)\,\hat{\varphi}_t + \omega\beta\left(\left(\frac{\epsilon}{1-\alpha} + (\epsilon-1)\right)\hat{\pi}_{t+1} + \hat{A}x_{t+1} - \hat{A}y_{t+1}\right). \tag{29}$$

Replace $(\hat{A}x_{t+1} - \hat{A}y_{t+1})$ by $\left(1+\frac{\alpha\epsilon}{1-\alpha}\right)(\hat{p}_{t+1}^* - \hat{P}_{t+1})$ (using (24) forwarded one period) to get

$$\left(1+\frac{\alpha\epsilon}{1-\alpha}\right)\left(\hat{p}_t^* - \hat{P}_t\right) = (1-\omega\beta)\hat{\varphi}_t + \omega\beta\left(\left(\frac{\epsilon}{1-\alpha} + (\epsilon-1)\right)\hat{\pi}_{t+1} + \left(1+\frac{\alpha\epsilon}{1-\alpha}\right)\left(\hat{p}_{t+1}^* - \hat{P}_{t+1}\right)\right). \tag{30}$$

As a last step, use (27) to solve for the inflation rate

$$\begin{aligned}\left(1+\frac{\alpha\epsilon}{1-\alpha}\right)\frac{\omega}{1-\omega}\hat{\pi}_t &= (1-\omega\beta)\,\hat{\varphi}_t + \omega\beta\left(1+\frac{\alpha\epsilon}{1-\alpha}\right)\left(\hat{\pi}_{t+1} + \frac{\omega}{1-\omega}\hat{\pi}_{t+1}\right) \\ \Leftrightarrow \hat{\pi}_t &= \kappa\hat{\varphi}_t + \beta E_t\hat{\pi}_{t+1},\end{aligned} \tag{31}$$

where $\kappa = \frac{(1-\omega)(1-\omega\beta)}{\omega}\frac{1-\alpha}{1+\alpha(\epsilon-1)}$. Overall, in a New-Keynesian framework with Calvo pricing, inflation today depends on the deviations of real marginal costs from their steady-state today and expected inflation tomorrow. The importance of either effect depends on the price stickiness ω and the household discount rate β. We can iterate forward by substituting for $\hat{\pi}_{t+1}$,$\hat{\pi}_{t+2}$, ...,

$$\hat{\pi}_t \quad = \quad E_t\sum_{i=0}^{\infty}\beta^i\kappa\hat{\varphi}_{t+i}, \tag{32}$$

which results in a representation of inflation as the expected present value of all future deviations of real marginal costs from their steady-state. The impact of real marginal costs on inflation is determined by the slope of the NKPC given by κ.

References

Calvo, G.A., 1983. Staggered prices in a utility-maximizing framework. Journal of monetary economics 12, 383–398.

Dixit, A.K., Stiglitz, J.E., 1977. Monopolistic competition and optimum product diversity. The American economic review 67, 297–308.

Galí, J., 2015. Monetary policy, inflation, and the business cycle: An introduction to the new Keynesian framework and its applications. Second edition ed., Princeton University Press, Princeton, Oxford.

Schmitt-Grohé, S., Uribe, M., 2007. Optimal simple and implementable monetary and fiscal rules. Journal of monetary economics 54, 1702–1725.

Walsh, C.E., 2010. Monetary theory and policy. 3. ed. ed., MIT Press, Cambridge, Mass.