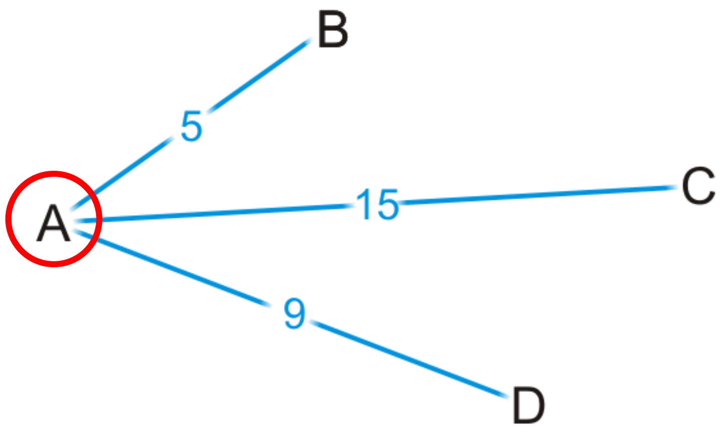
Dijkstra’s algorithm

Strategy

Suppose you are at vertex A

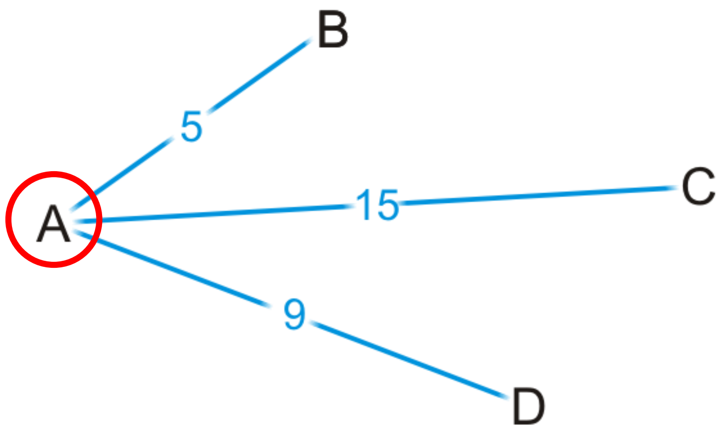
- You are aware of all vertices adjacent to it
- This information is either in an adjacency list or adjacency matrix



Strategy

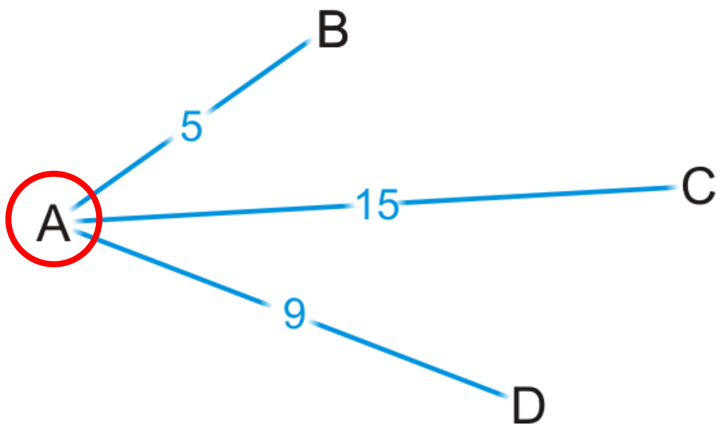
Is 5 the shortest distance to B via the edge (A, B)?

- Why or why not?



Strategy

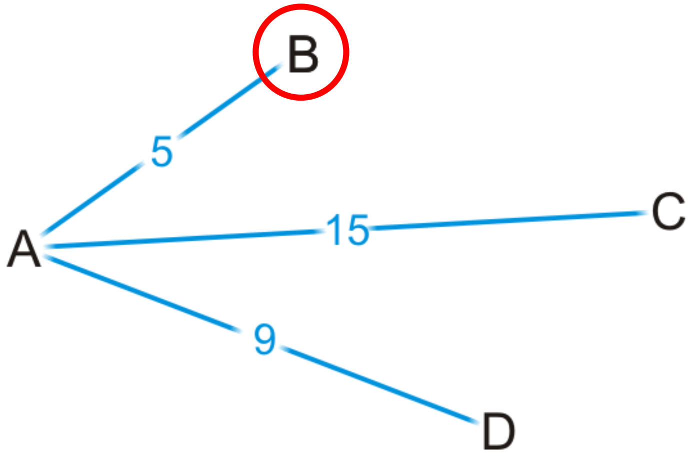
Are you guaranteed that the shortest path to C is (A, C), or that (A, D) is the shortest path to vertex D?



Strategy

We accept that (A, B) is the shortest path to vertex B from A

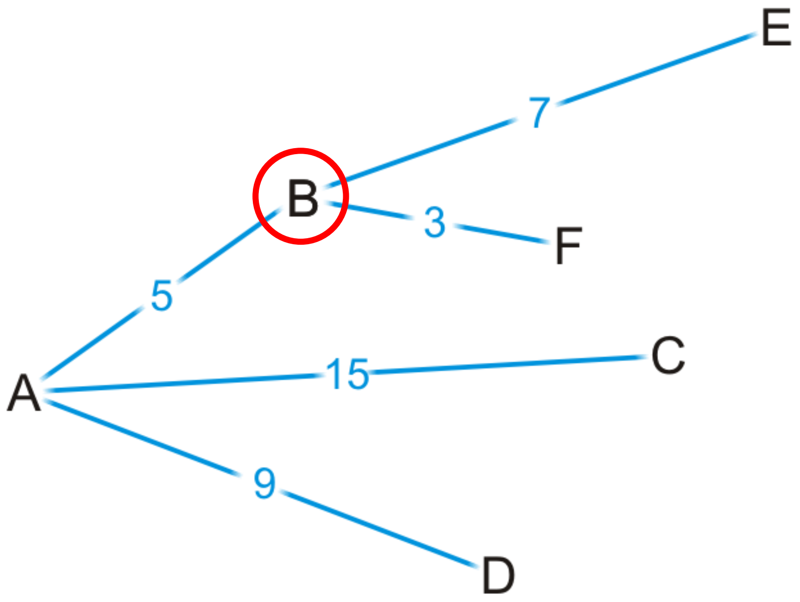
- Let’s see where we can go from B



Strategy

By some simple arithmetic, we can determine that

- There is a path (A, B, E) of length 5 + 7 = 12
- There is a path (A, B, F) of length 5 + 3 = 8

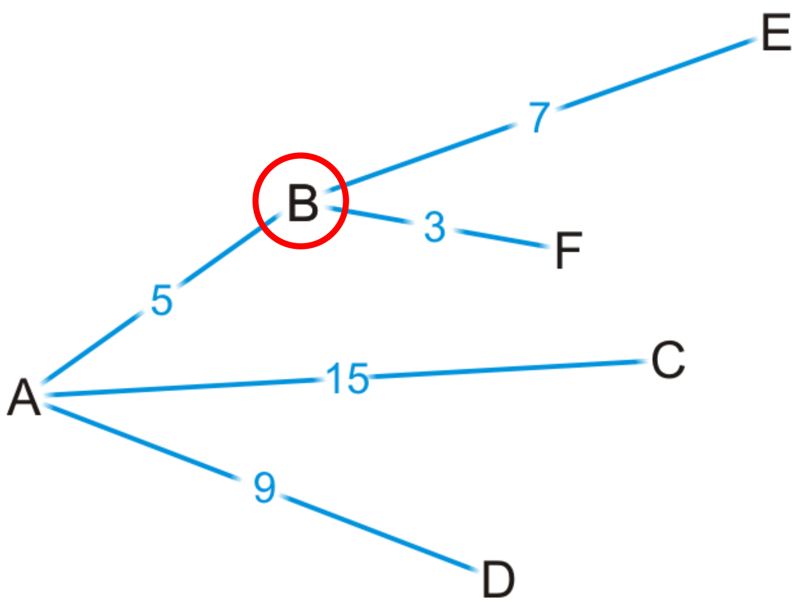


Strategy

Is (A, B, F) is the shortest path from vertex A to F?

- Why or why not?

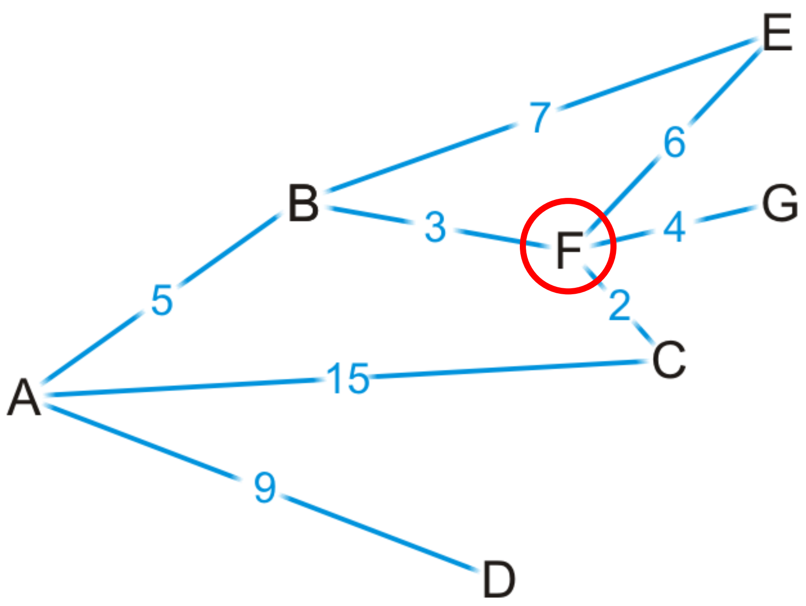
Are we guaranteed that any other path we are currently aware of is also going to be the shortest path?



Strategy

Okay, let's visit vertex F

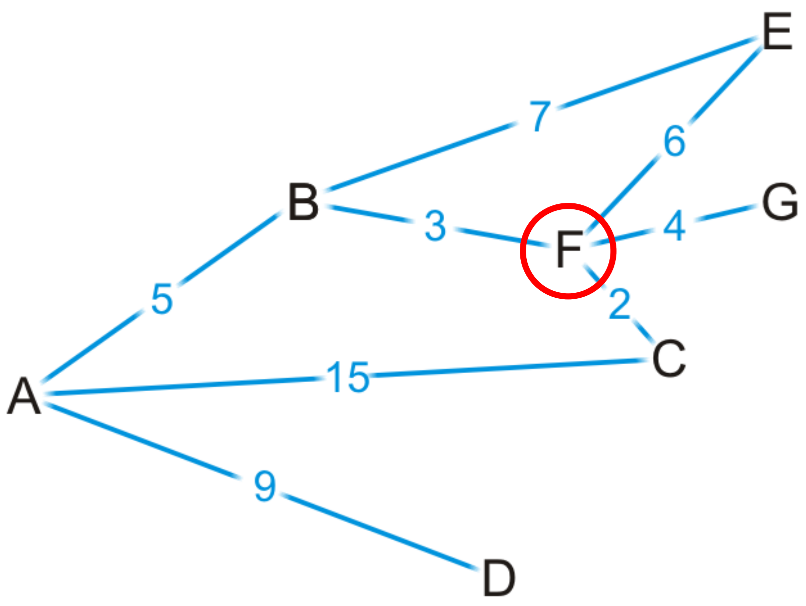
- We know the shortest path is (A, B, F) and it's of length 8



Strategy

There are three edges exiting vertex F, so we have paths:

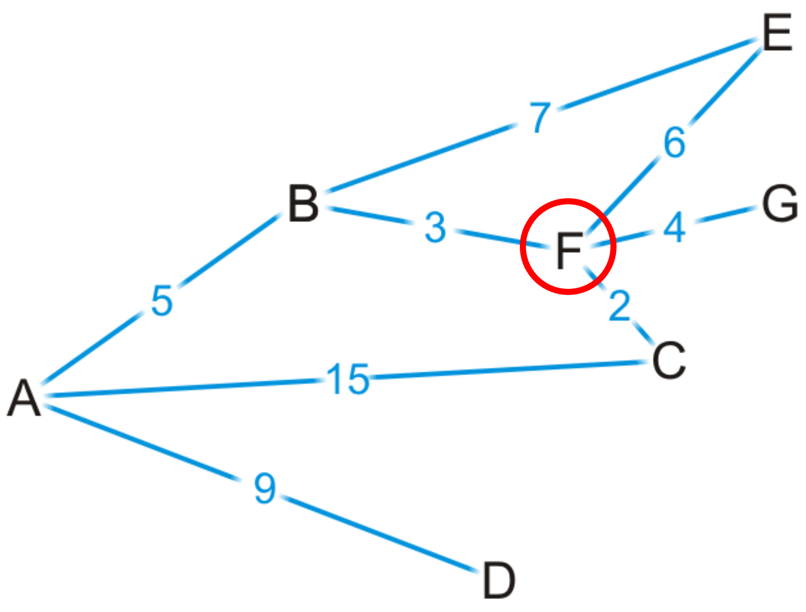
- (A, B, F, E) of length 8 + 6 = 14
- (A, B, F, G) of length 8 + 4 = 12
- (A, B, F, C) of length 8 + 2 = 10



Strategy

By observation:

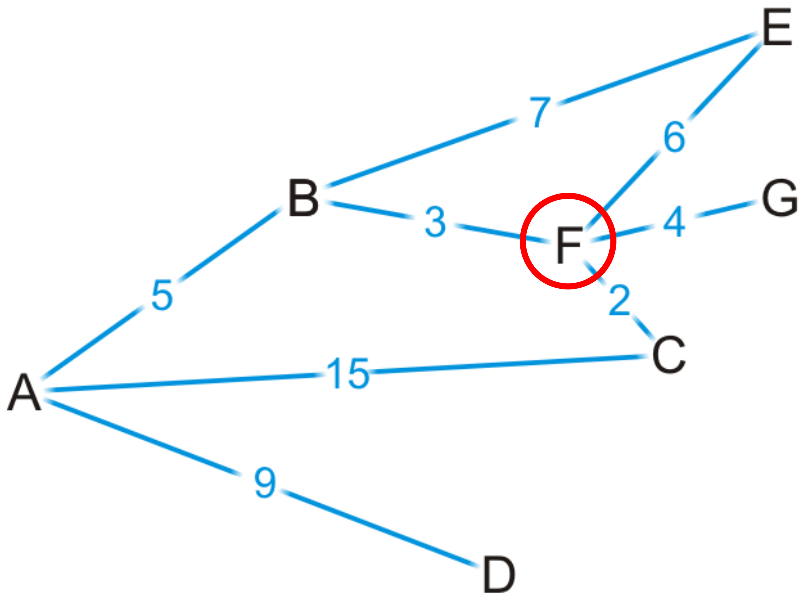
- The path (A, B, F, E) is longer than (A, B, E)
- The path (A, B, F, C) is shorter than the path (A, C)



Strategy

At this point, we've discovered the shortest paths to:

- Vertex B: (A, B) of length 5
- Vertex F: (A, B, F) of length 8



Dijkstra’s algorithm

Dijkstra’s algorithm solves the single-source shortest path problem

- It is very similar to Prim’s algorithm
- Assumption: all the weights are positive

Like Prim’s algorithm,

We initially don’t know the distance to any vertex except the initial vertex

We require an array of distances, all initialized to infinity except for the source vertex, which is initialized to 0

Each time we visit a vertex, we will examine all adjacent vertices

We need to track visited vertices—a Boolean table of size $|V|$

We need an array of previous vertices, all initialized to null

Dijkstra's algorithm

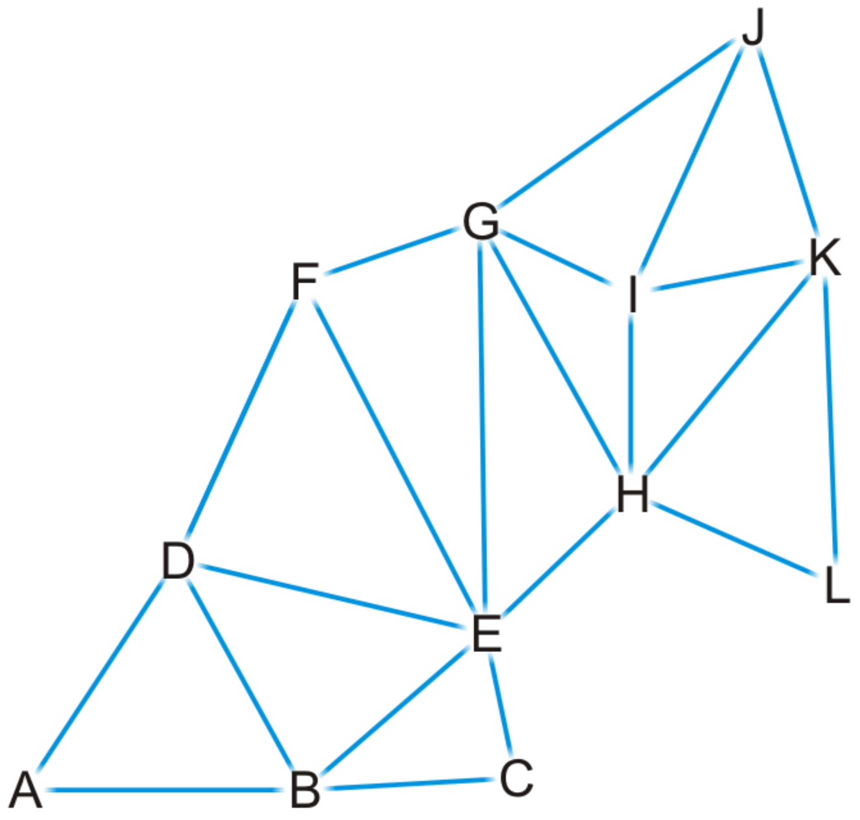
Thus, we will iterate $|V|$ times:

- Find that unvisited vertex $\boldsymbol{v}$ that has a minimum distance to it
- Mark it as having been visited
- Consider every adjacent vertex $\boldsymbol{w}$ that is unvisited:
  - Is the distance to $\boldsymbol{v}$ plus the weight of the edge $(\boldsymbol{v}, \boldsymbol{w})$ less than our currently known shortest distance to $\boldsymbol{w}$
  - If so, update the shortest distance to $\boldsymbol{w}$ and record $\boldsymbol{v}$ as the previous pointer

- Continue iterating until all vertices are visited or all remaining vertices have a distance to them of infinity

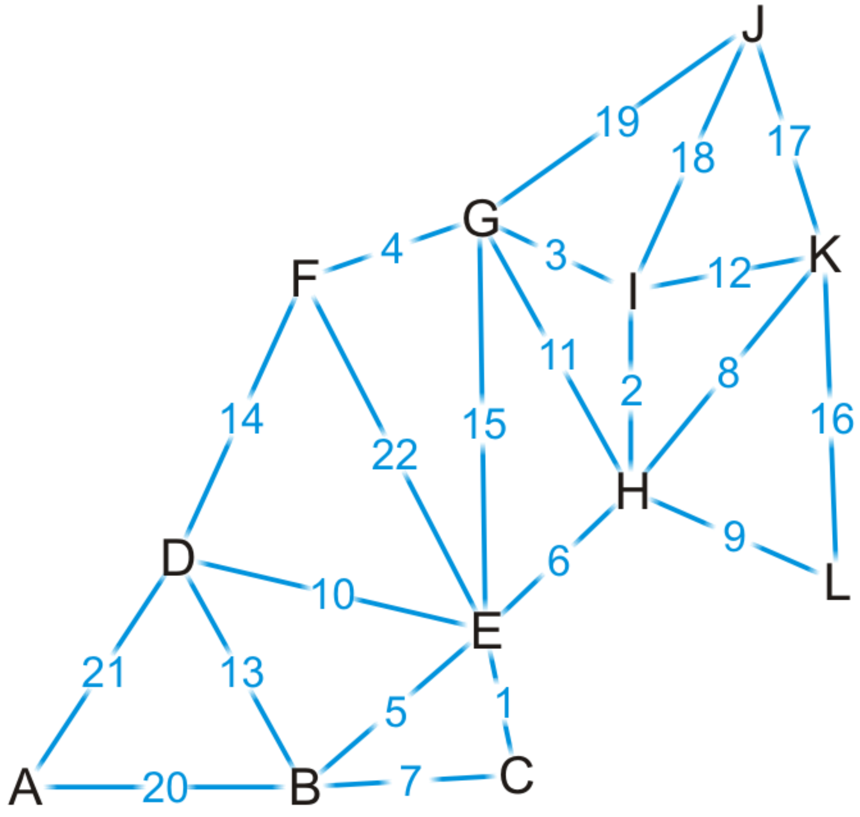
Example

Here is our abstract representation



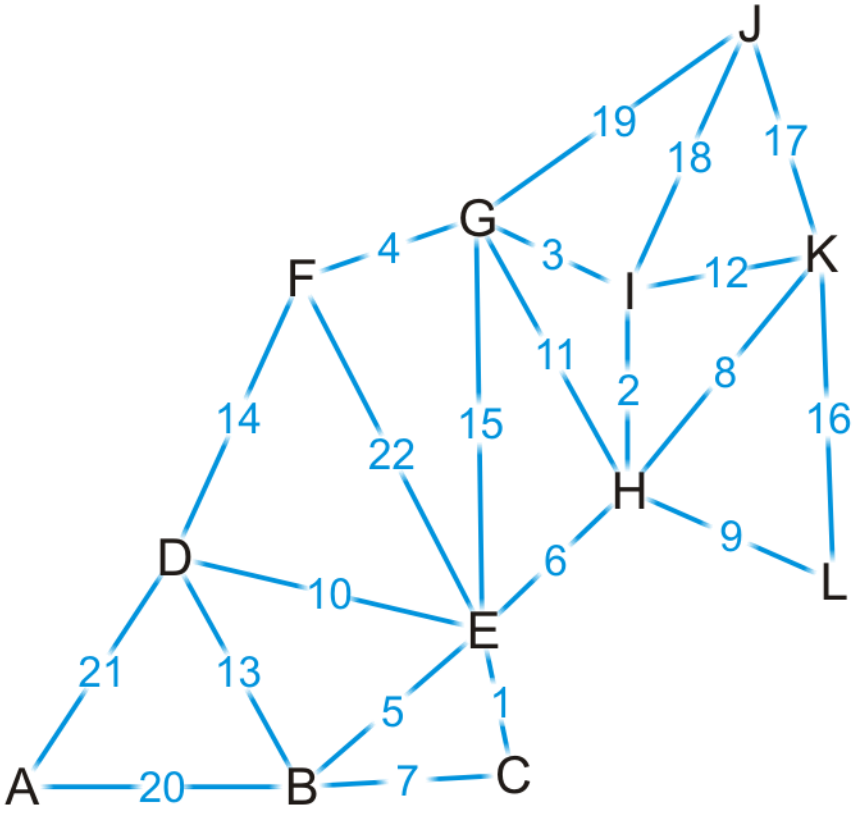
Example

Let us give a weight to each of the edges



Example

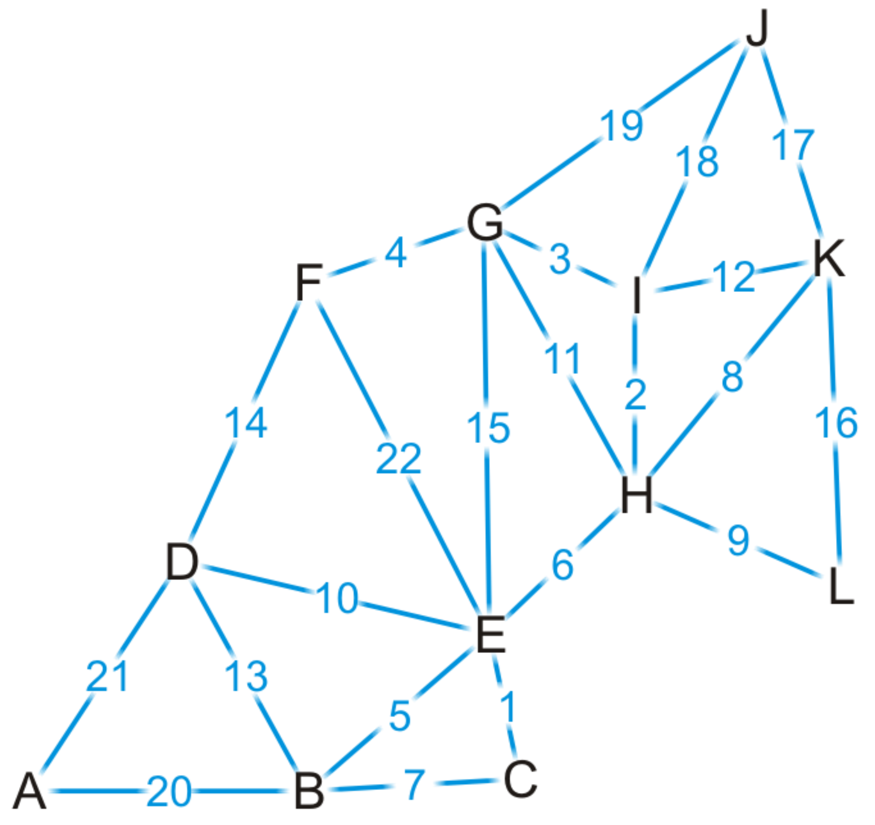
Find the shortest distance from Kamchatka (K) to every other region



Example

We set up our table

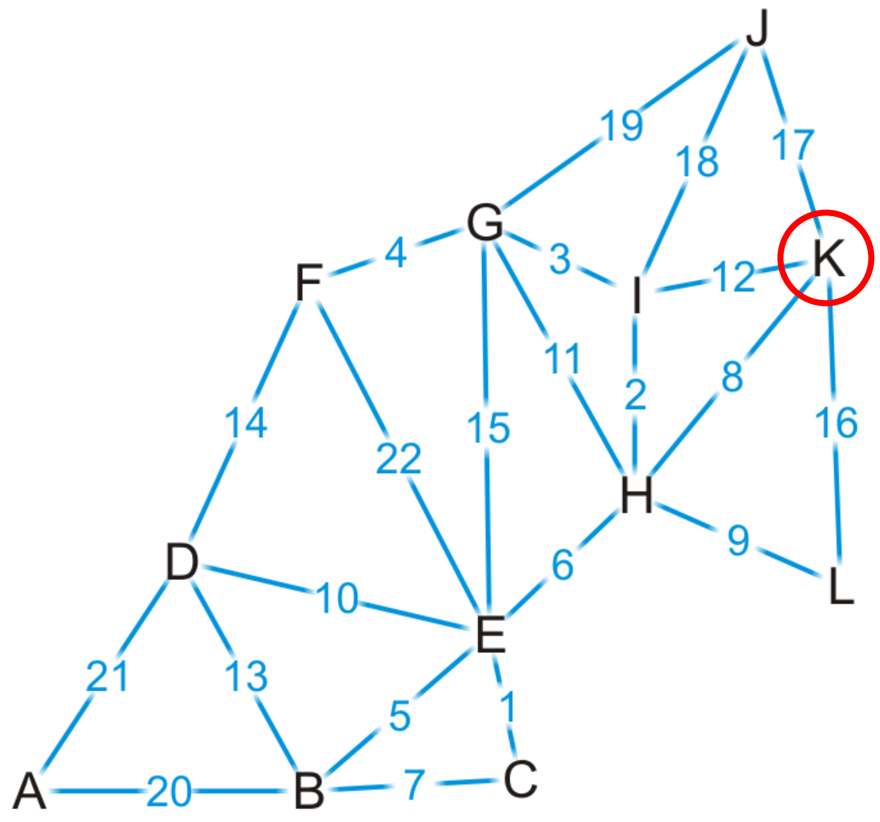
- Which unvisited vertex has the minimum distance to it?



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	∞	Ø
F	F	∞	Ø
G	F	∞	Ø
H	F	∞	Ø
I	F	∞	Ø
J	F	∞	Ø
K	F	0	Ø
L	F	∞	Ø

Example

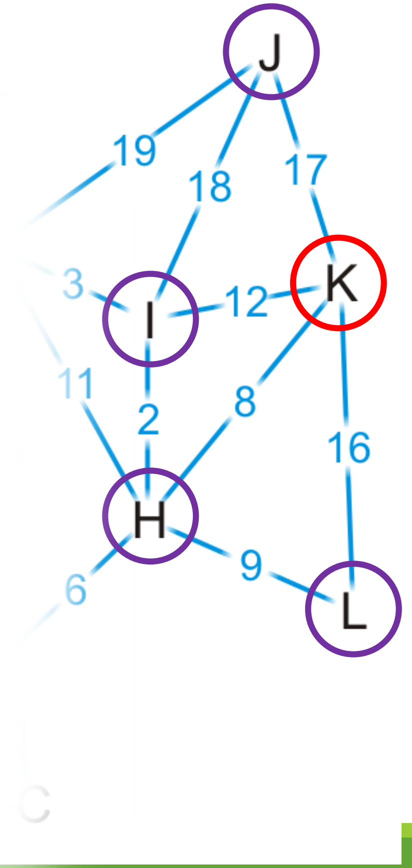
We visit vertex K



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	∞	Ø
F	F	∞	Ø
G	F	∞	Ø
H	F	∞	Ø
I	F	∞	Ø
J	F	∞	Ø
K	**T**	**0**	**Ø**
L	F	∞	Ø

Example

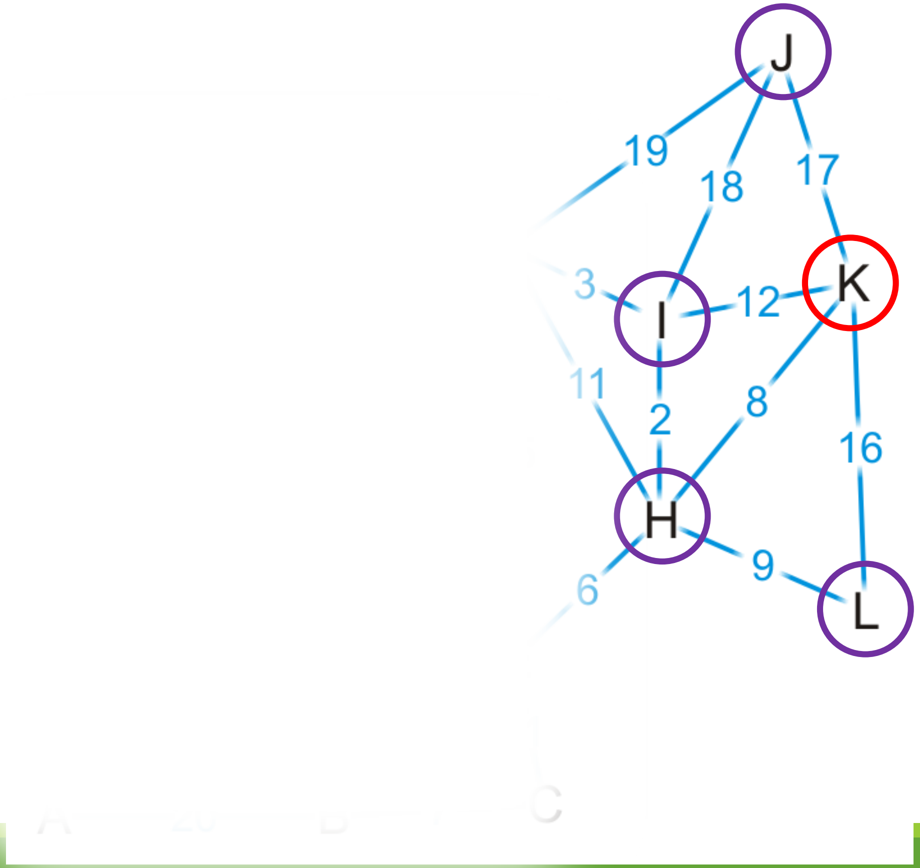
Vertex K has four neighbors: H, I, J and L



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	∞	Ø
F	F	∞	Ø
G	F	∞	Ø
H	F	∞	Ø
I	F	∞	Ø
J	F	∞	Ø
K	T	0	Ø
L	F	∞	Ø

Example

We have now found at least one path to each of these vertices

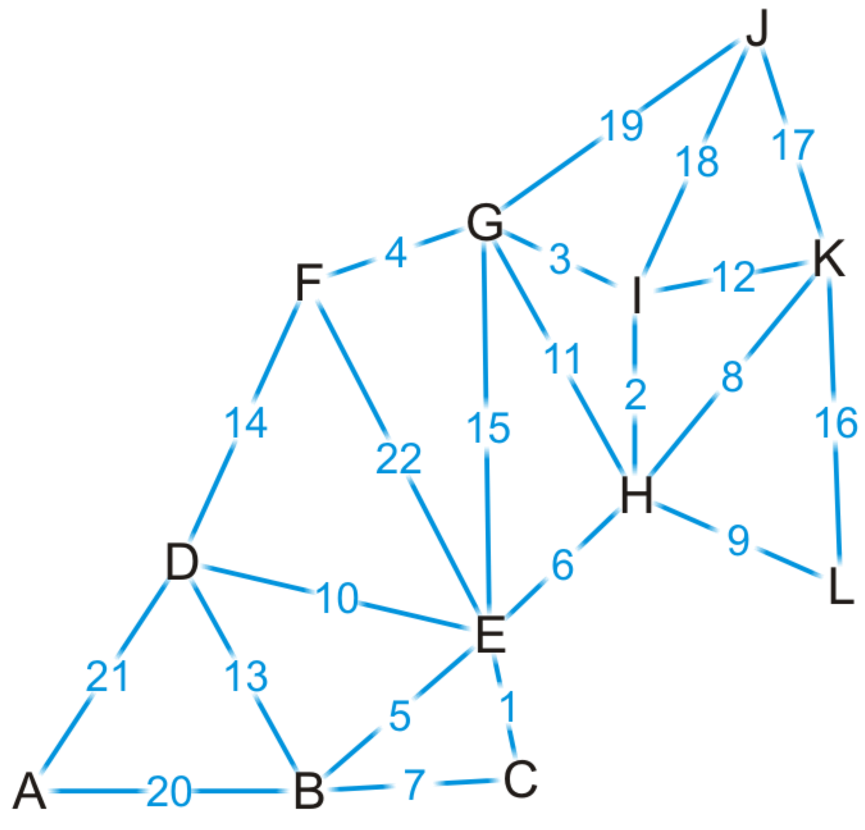


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	∞	Ø
F	F	∞	Ø
G	F	∞	Ø
H	F	8	K
I	F	12	K
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

We're finished with vertex K

- To which vertex are we now guaranteed we have the shortest path?

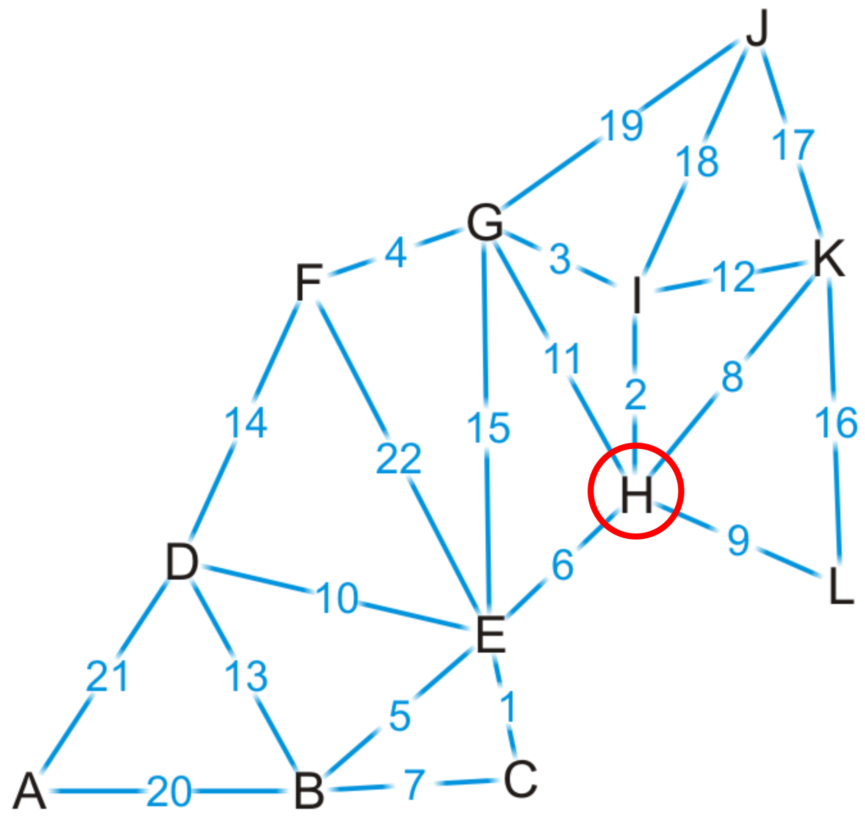


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	∞	Ø
F	F	∞	Ø
G	F	∞	Ø
H	F	8	K
I	F	12	K
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

We visit vertex H: the shortest path is (K, H) of length 8

- Vertex H has four unvisited neighbors: E, G, I, L



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	∞	Ø
F	F	∞	Ø
G	F	∞	Ø
H	**T**	**8**	**K**
I	F	12	K
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

Consider these paths:

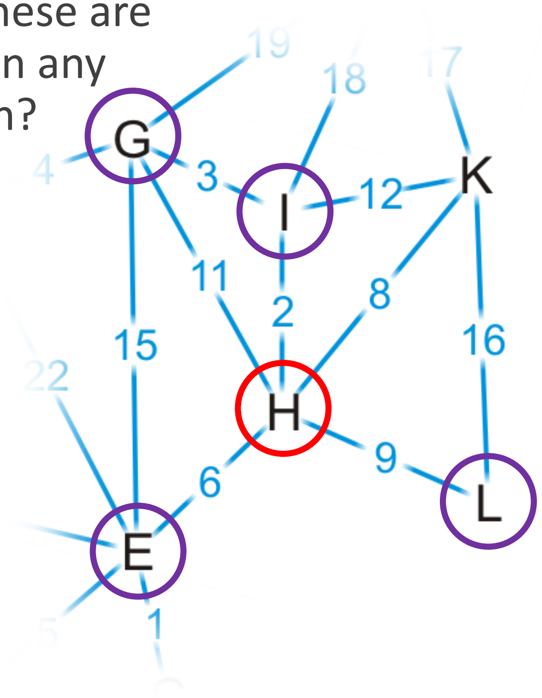
(K, H, E) of length 8 + 6 = 14

(K, H, I) of length 8 + 2 = 10

(K, H, G) of length 8 + 11 = 19

(K, H, L) of length 8 + 9 = 17

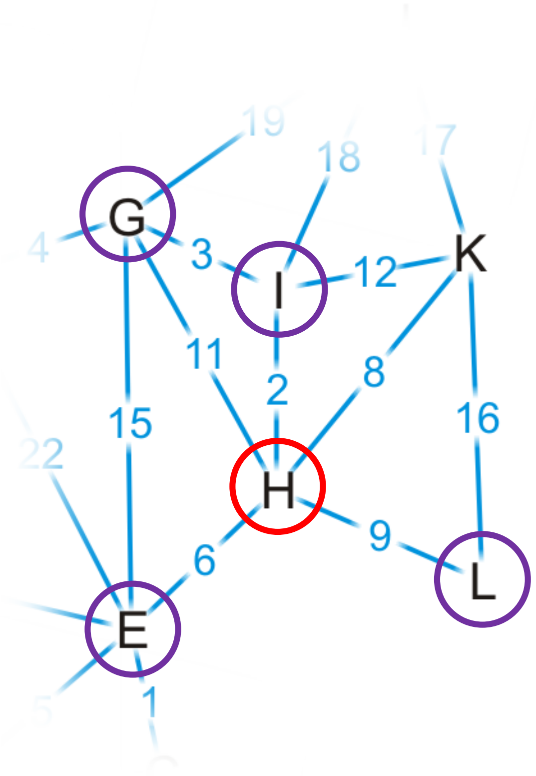
- Which of these are shorter than any known path?



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	∞	Ø
F	F	∞	Ø
G	F	∞	Ø
H	**T**	**8**	**K**
I	F	12	K
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

We already have a shorter path (K, L), but we update the other three

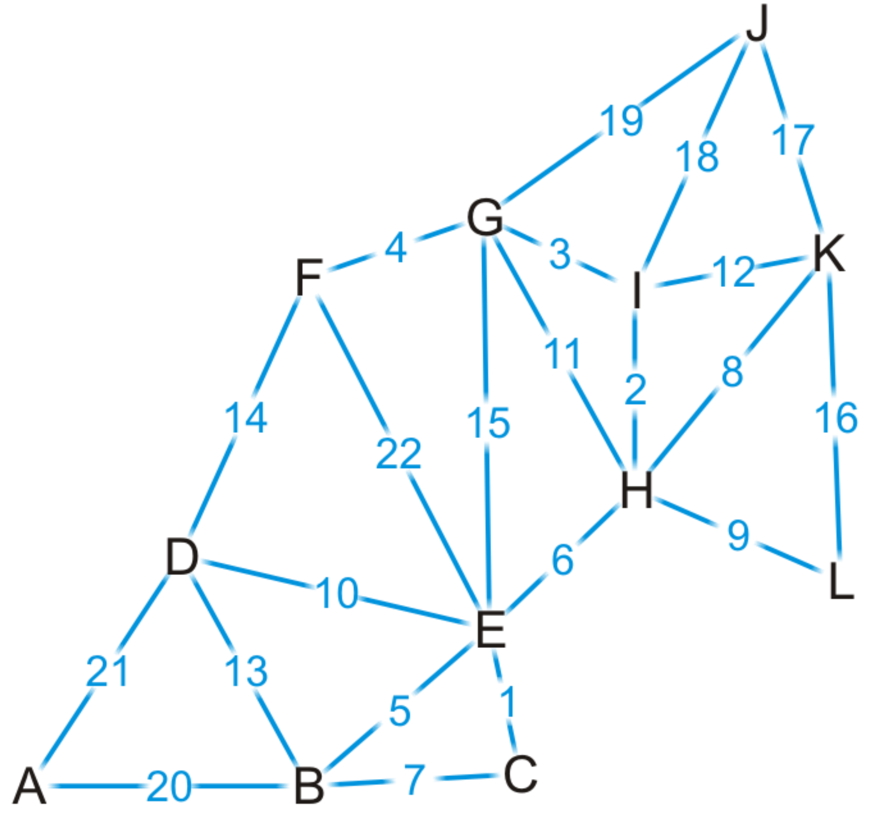


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	**14**	**H**
F	F	∞	Ø
G	F	**19**	**H**
H	**T**	**8**	**K**
I	F	**10**	**H**
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

We are finished with vertex H

- Which vertex do we visit next?

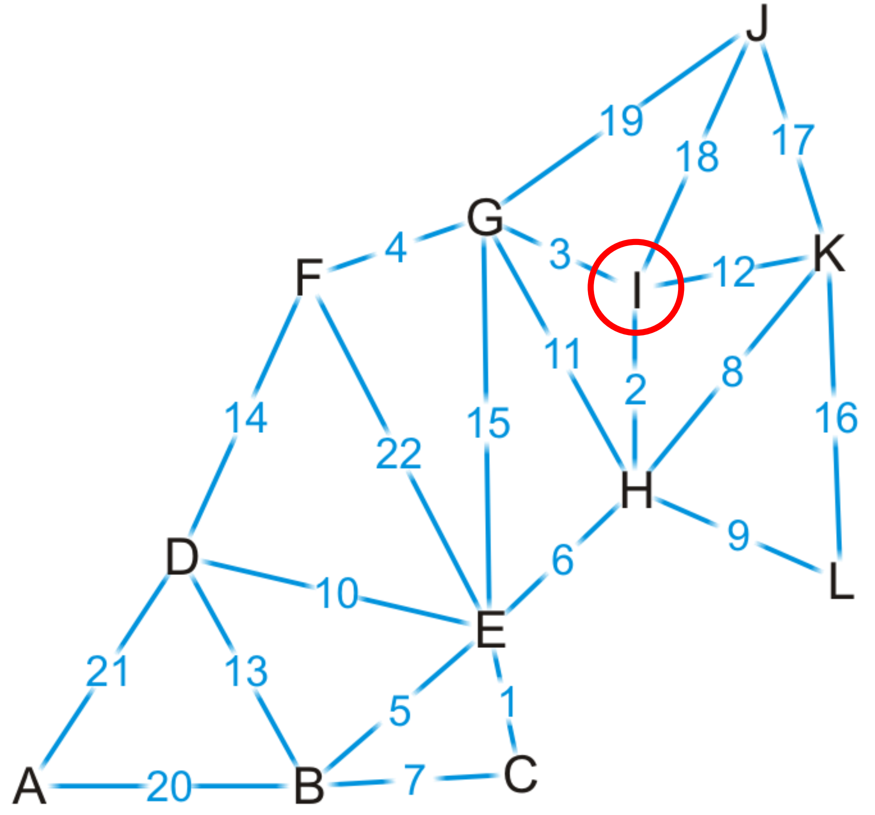


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	14	H
F	F	∞	Ø
G	F	19	H
H	T	8	K
I	F	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

The path (K, H, I) is the shortest path from K to I of length 10

- Vertex I has two unvisited neighbors: G and J

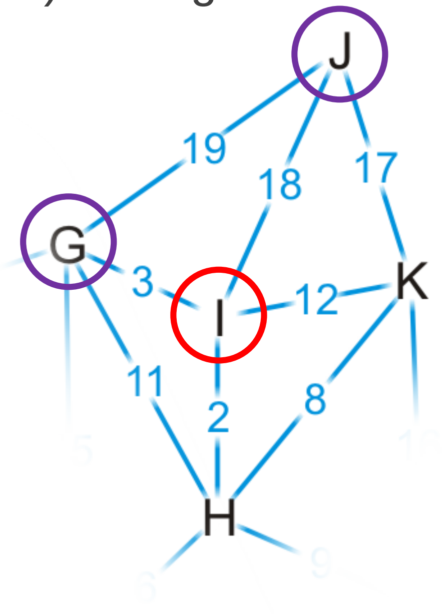


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	14	H
F	F	∞	Ø
G	F	19	H
H	T	8	K
I	**T**	**10**	**H**
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

Consider these paths:

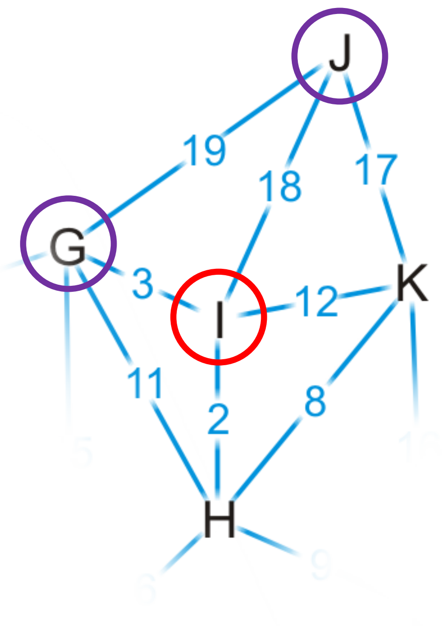
(K, H, I, G) of length 10 + 3 = 13 (K, H, I, J) of length 10 + 18 = 28



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	14	H
F	F	∞	Ø
G	F	19	H
H	T	8	K
I	**T**	**10**	**H**
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

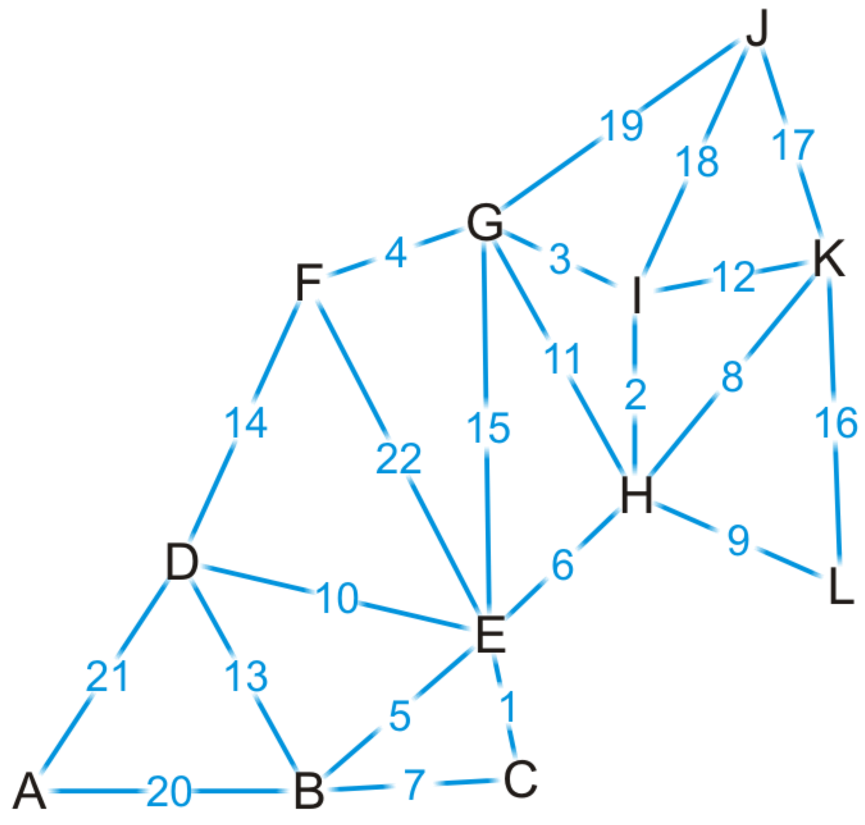
We have discovered a shorter path to vertex G, but (K, J) is still the shortest known path to vertex J



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	14	H
F	F	∞	Ø
G	F	**13**	**I**
H	T	8	K
I	**T**	**10**	**H**
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

Which vertex can we visit next?

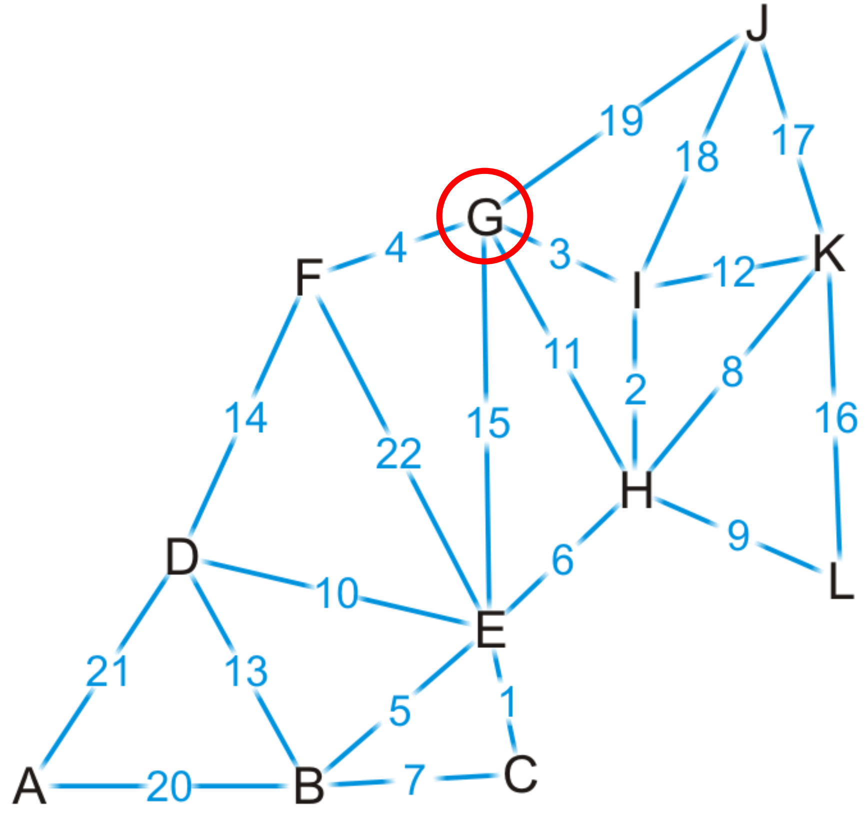


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	14	H
F	F	∞	Ø
G	F	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

The path (K, H, I, G) is the shortest path from K to G of length 13

- Vertex G has three unvisited neighbors: E, F and J



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	14	H
F	F	∞	Ø
G	**T**	**13**	**I**
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

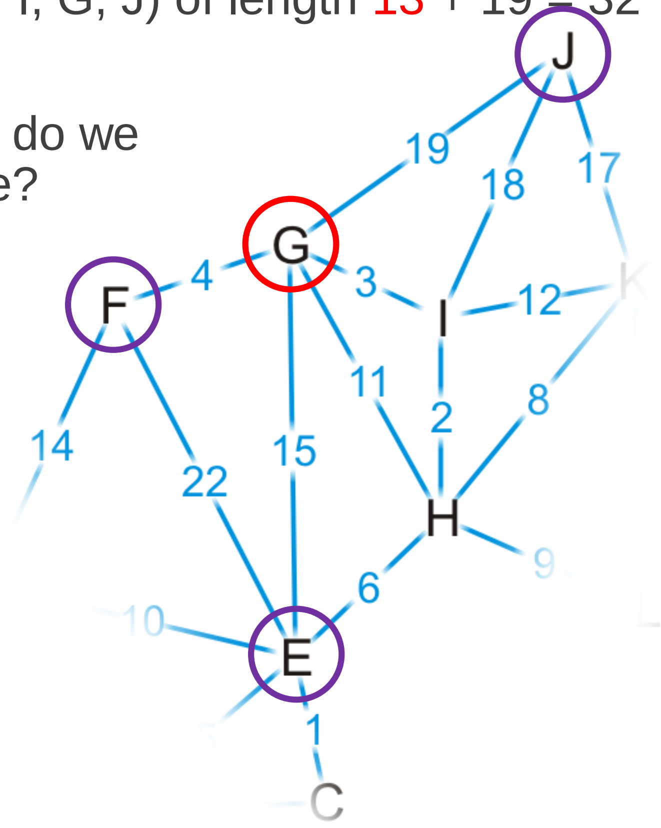
Consider these paths:

(K, H, I, G, E) of length 13 + 15 = 28

(K, H, I, G, J) of length 13 + 19 = 32

(K, H, I, G, F) of length 13 + 4 = 17

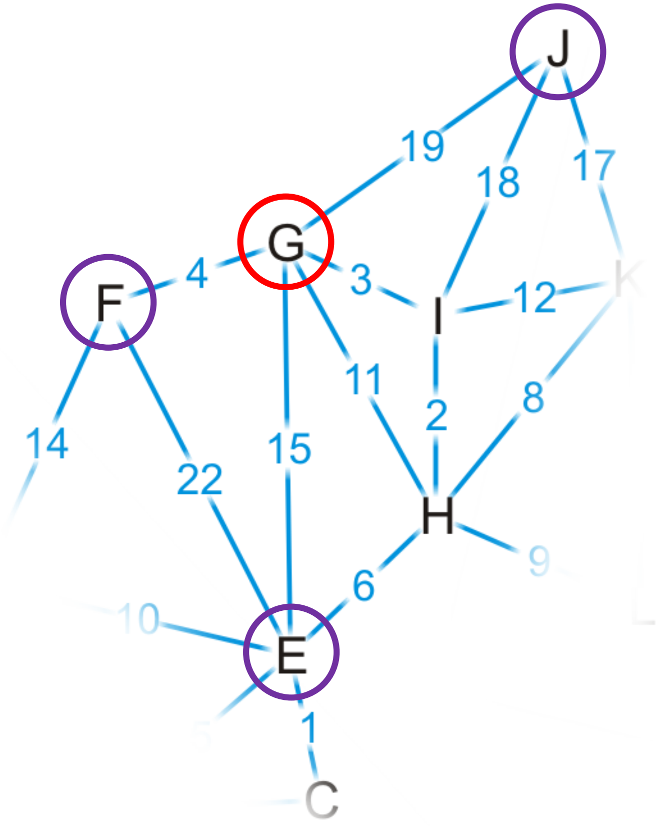
- Which do we update?



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	14	H
F	F	∞	Ø
G	**T**	**13**	**I**
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

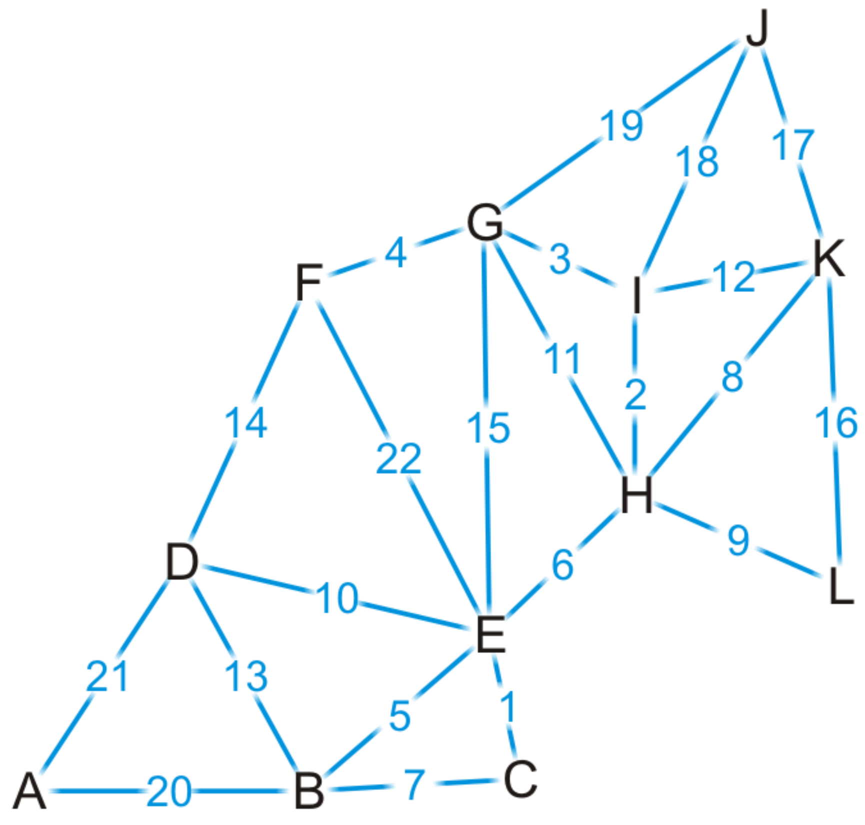
We have now found a path to vertex F



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	14	H
F	F	**17**	**G**
G	**T**	**13**	**I**
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

Where do we visit next?

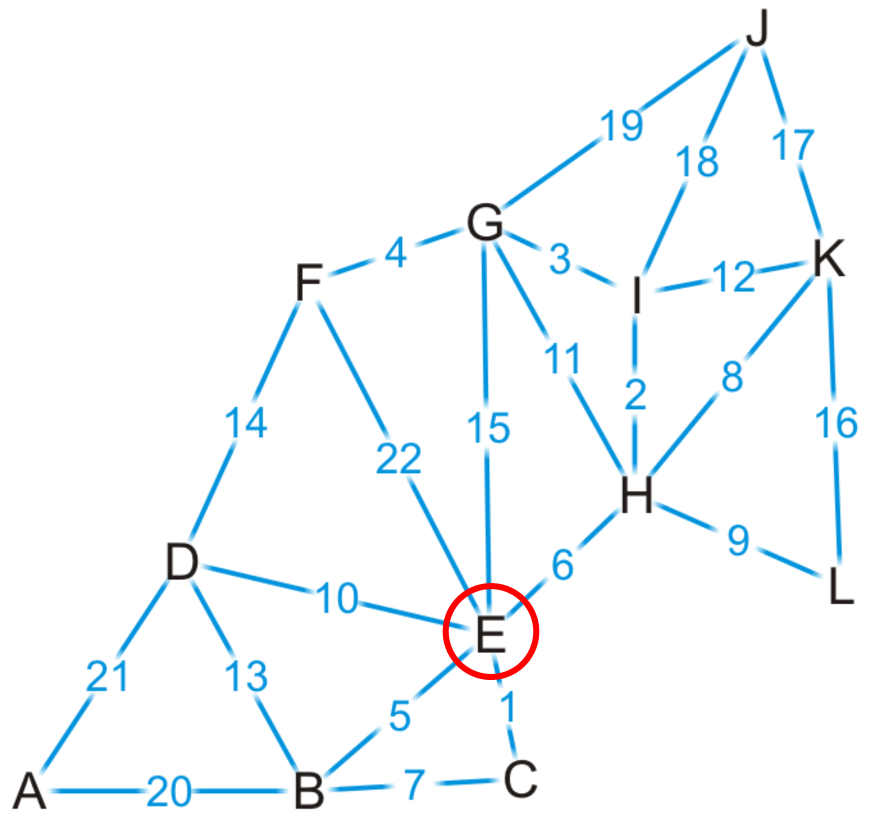


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	F	14	H
F	F	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

The path (K, H, E) is the shortest path from K to E of length 14

- Vertex G has four unvisited neighbors: B, C, D and F

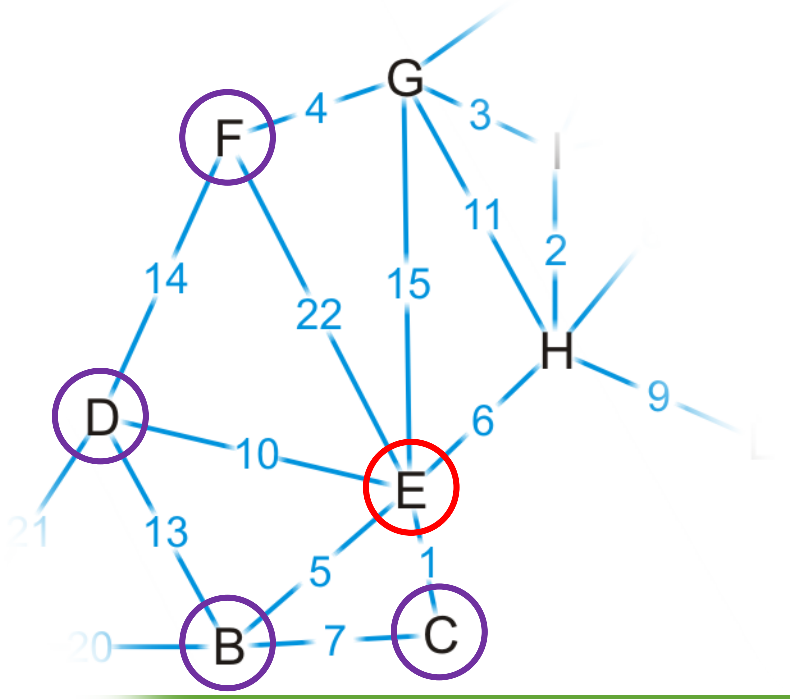


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	**T**	**14**	**H**
F	F	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

The path (K, H, E) is the shortest path from K to E of length 14

- Vertex G has four unvisited neighbors: B, C, D and F



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	**T**	**14**	**H**
F	F	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

Consider these paths:

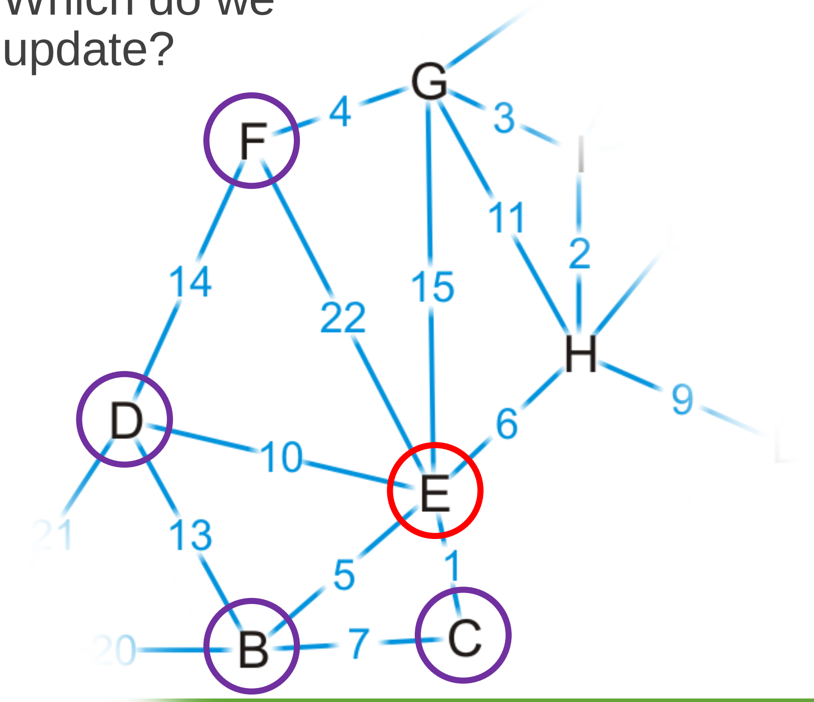
(K, H, E, B) of length 14 + 5 = 19

(K, H, E, D) of length 14 + 10 = 24

(K, H, E, C) of length 14 + 1 = 15

(K, H, E, F) of length 14 + 22 = 36

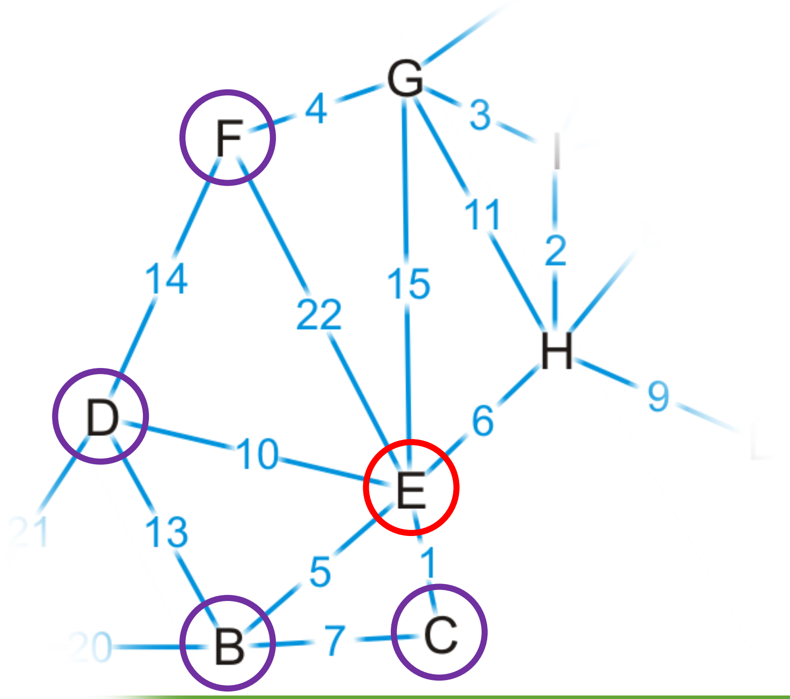
- Which do we update?



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	∞	Ø
C	F	∞	Ø
D	F	∞	Ø
E	**T**	**14**	**H**
F	F	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

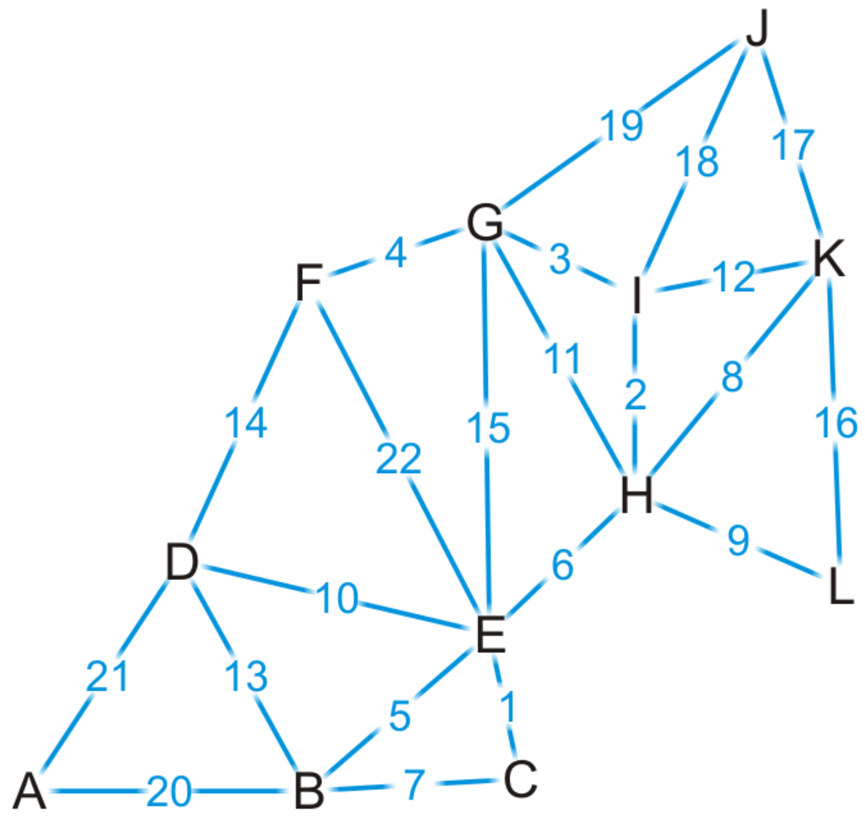
We've discovered paths to vertices B, C, D



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	**19**	**E**
C	F	**15**	**E**
D	F	**24**	**E**
E	**T**	**14**	**H**
F	F	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

Which vertex is next?

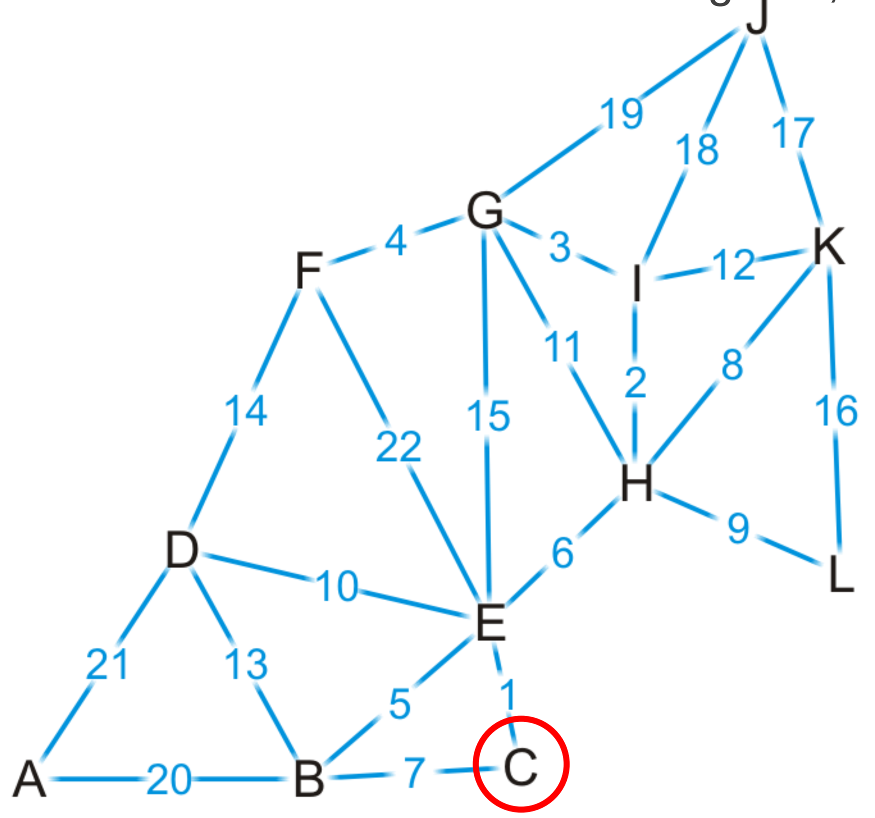


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	19	E
C	F	15	E
D	F	24	E
E	T	14	H
F	F	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

We've found that the path (K, H, E, C) of length 15 is the shortest path from K to C

- Vertex C has one unvisited neighbor, B

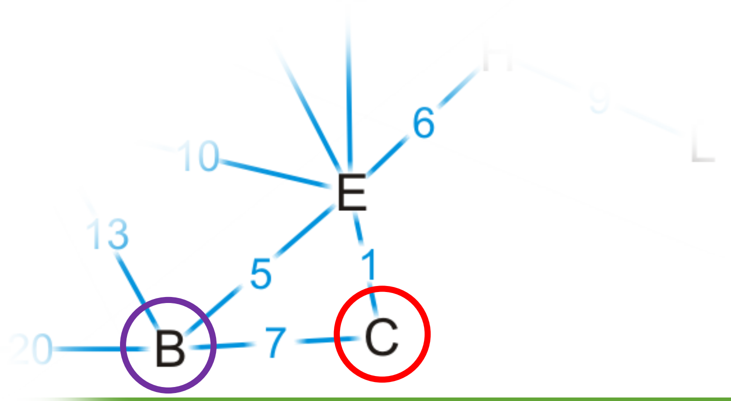


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	19	E
C	**T**	**15**	**E**
D	F	24	E
E	T	14	H
F	F	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

The path (K, H, E, C, B) is of length 15 + 7 = 22

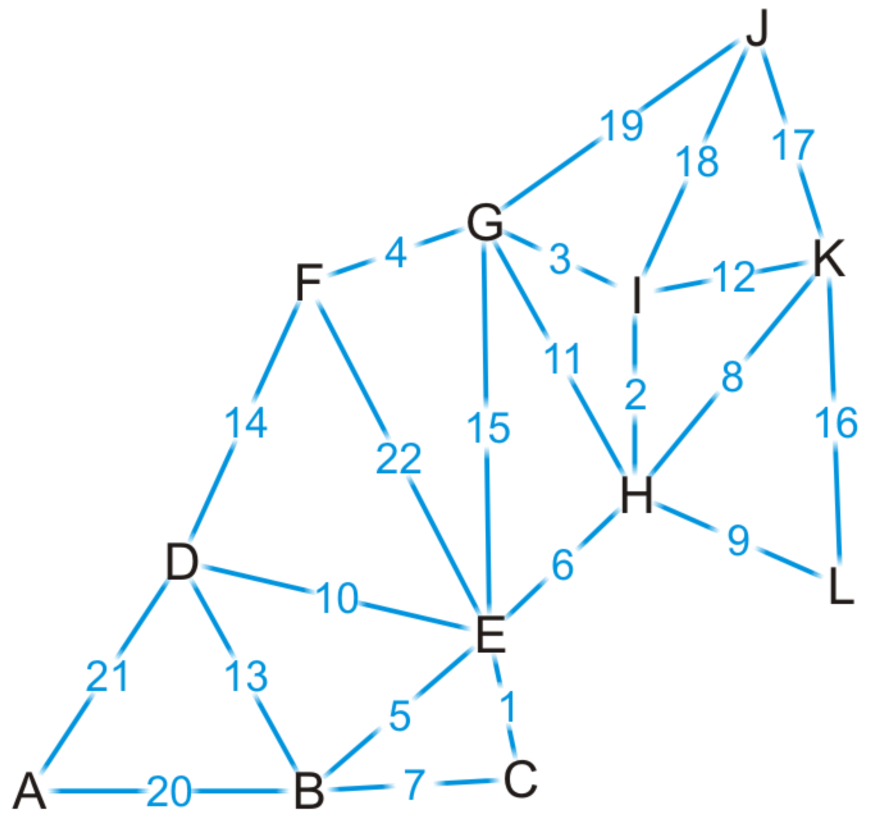
- We have already discovered a shorter path through vertex E



Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	19	E
C	**T**	**15**	**E**
D	F	24	E
E	T	14	H
F	F	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

Where to next?

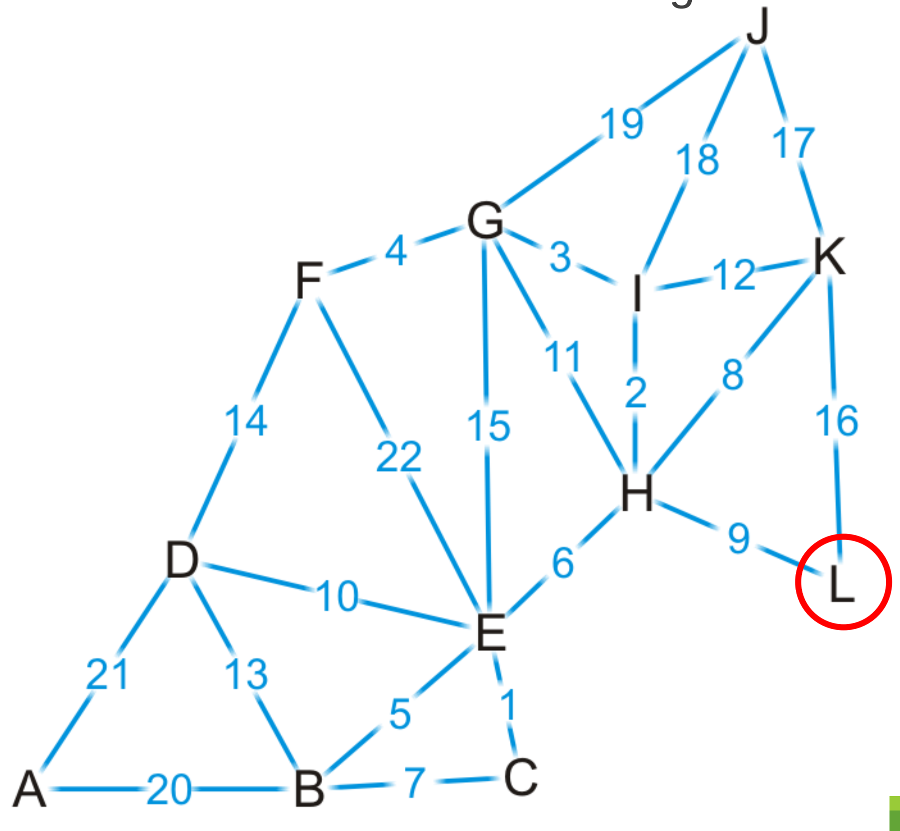


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	19	E
C	T	15	E
D	F	24	E
E	T	14	H
F	F	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	F	16	K

Example

We now know that (K, L) is the shortest path between these two points

- Vertex L has no unvisited neighbors

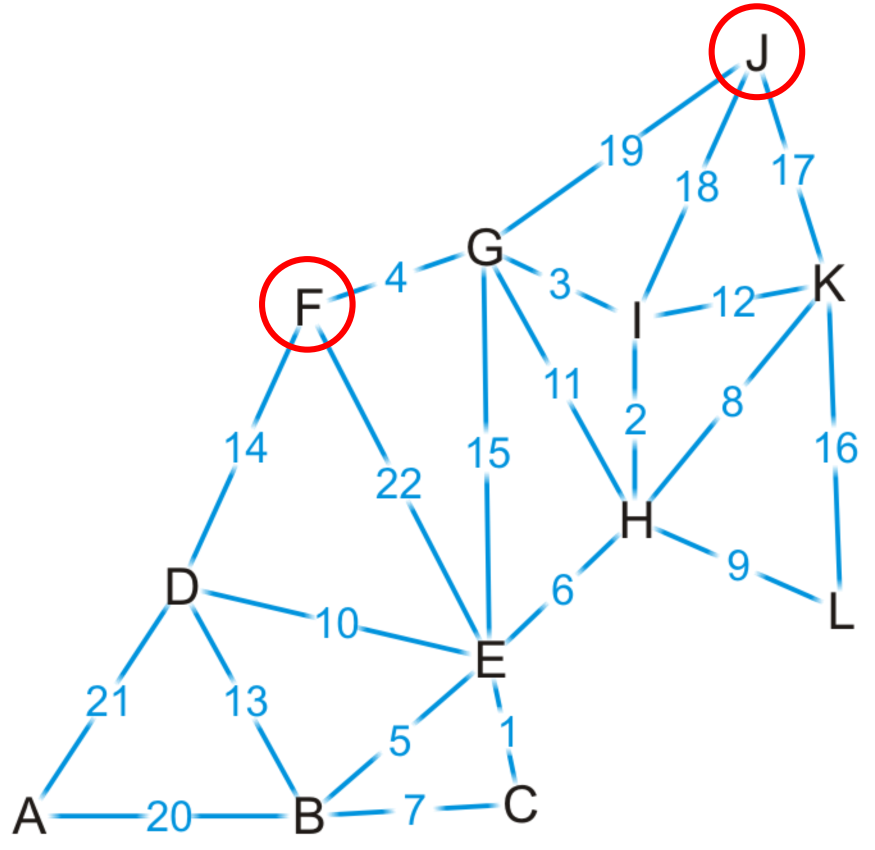


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	19	E
C	T	15	E
D	F	24	E
E	T	14	H
F	F	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	**T**	**16**	**K**

Example

Where to next?

- Does it matter if we visit vertex F first or vertex J first?

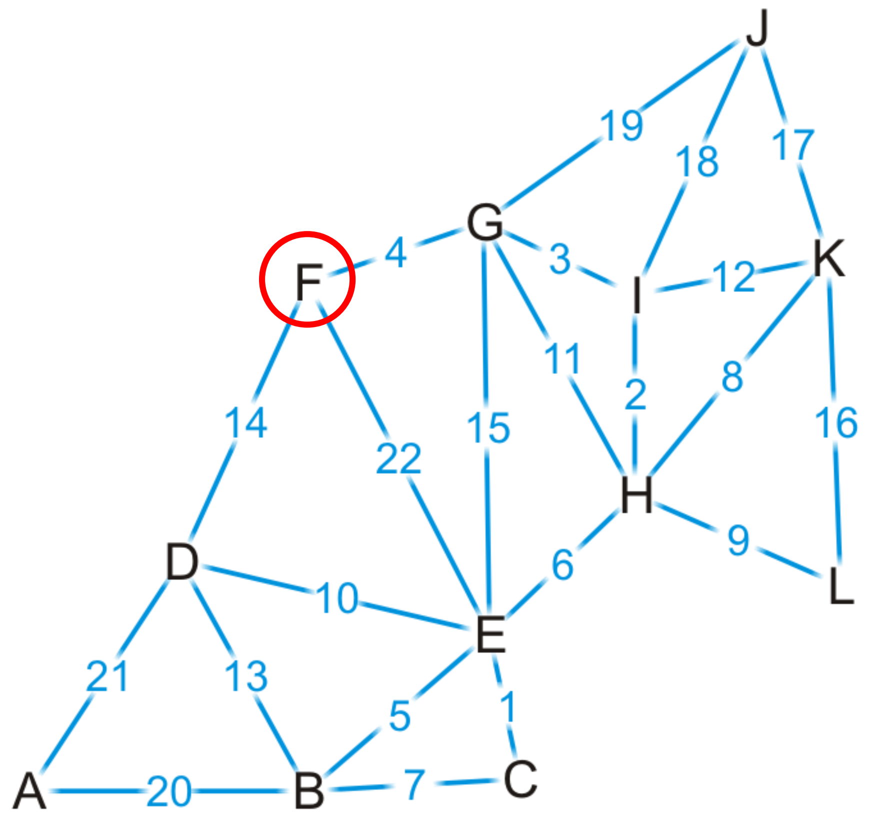


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	19	E
C	T	15	E
D	F	24	E
E	T	14	H
F	F	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	T	16	K

Example

Let's visit vertex F first

- It has one unvisited neighbor, vertex D

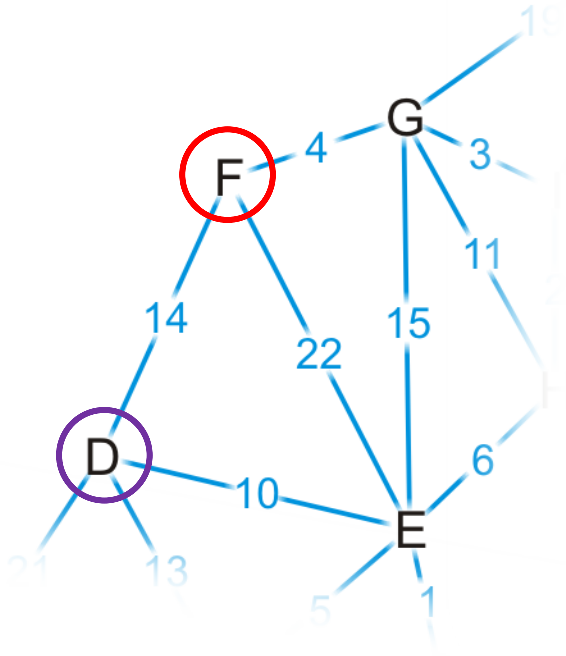


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	19	E
C	T	15	E
D	F	24	E
E	T	14	H
F	**T**	**17**	**G**
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	T	16	K

Example

The path (K, H, I, G, F, D) is of length 17 + 14 = 31

- This is longer than the path we've already discovered

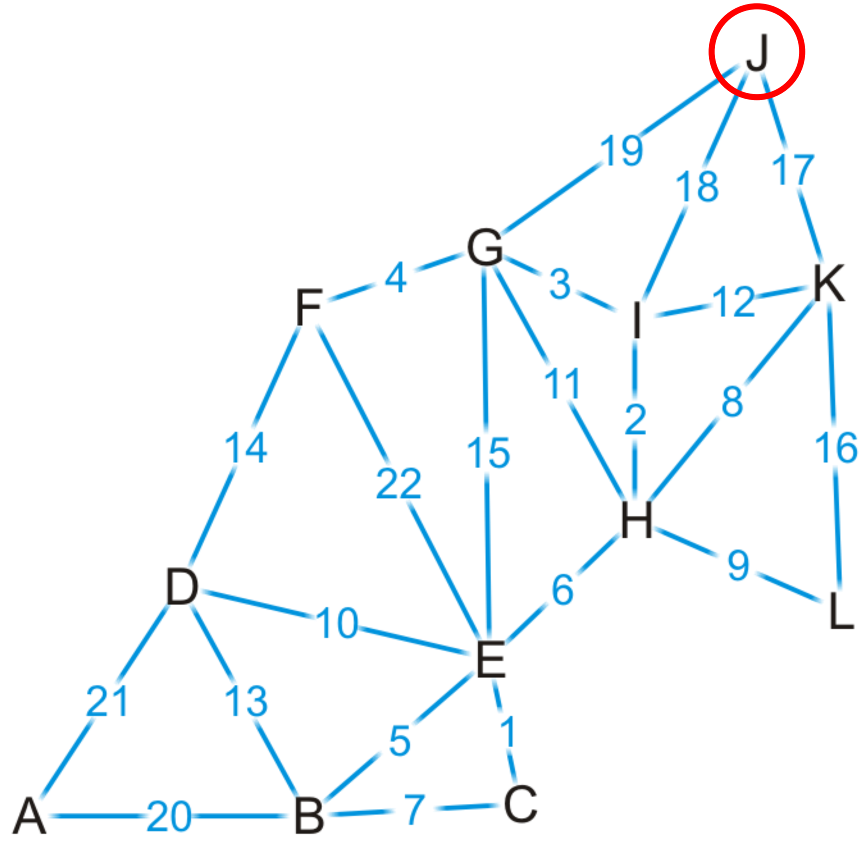


Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	19	E
C	T	15	E
D	F	24	E
E	T	14	H
F	T	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	F	17	K
K	T	0	Ø
L	T	16	K

Example

Now we visit vertex J

- It has no unvisited neighbors



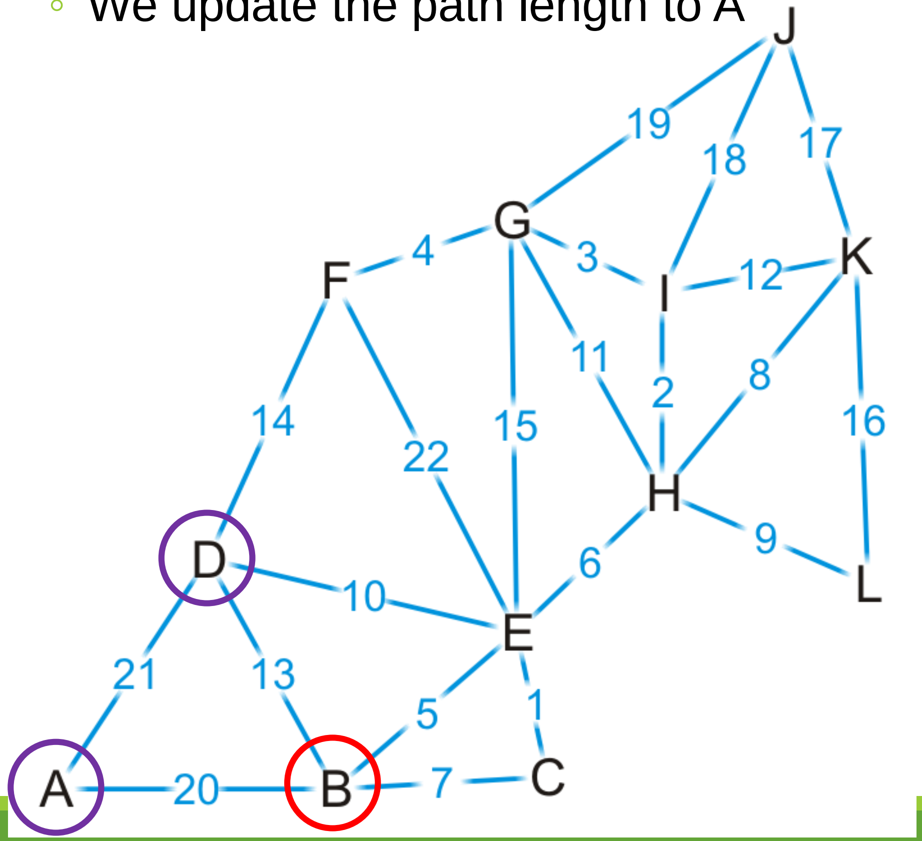
Vertex	Visited	Distance	Previous
A	F	∞	Ø
B	F	19	E
C	T	15	E
D	F	24	E
E	T	14	H
F	T	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	**T**	**17**	**K**
K	T	0	Ø
L	T	16	K

Example

Next we visit vertex B, which has two unvisited neighbors:

(K, H, E, B, A) of length 19 + 20 = 39 (K, H, E, B, D) of length 19 + 13 = 32

- We update the path length to A

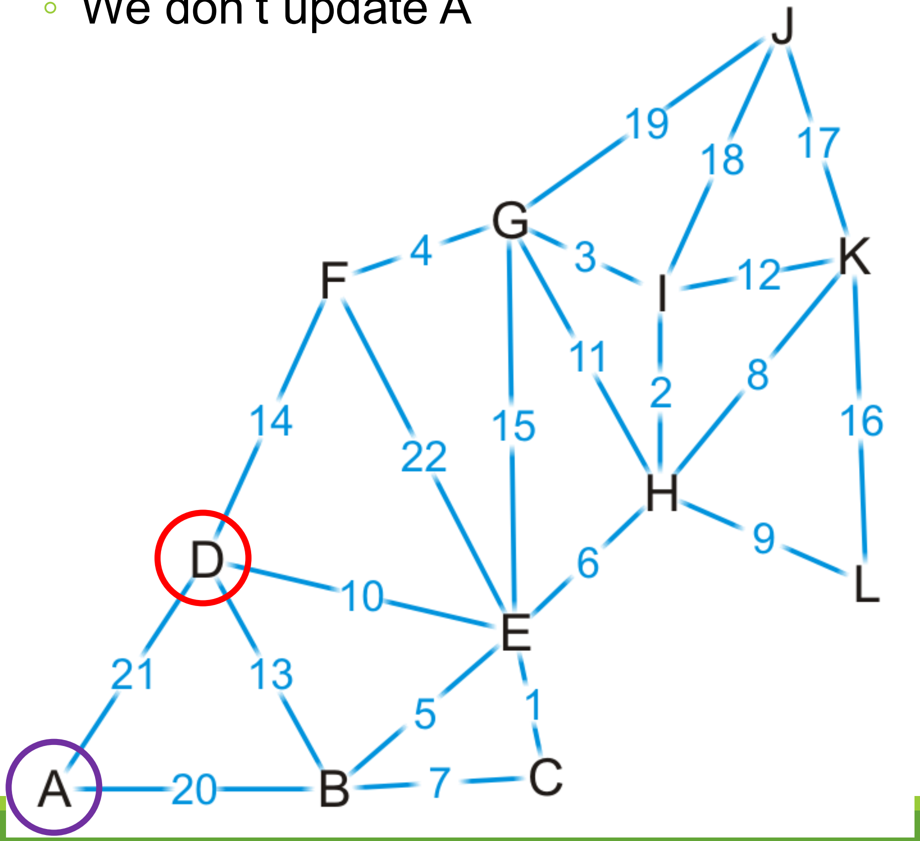


Vertex	Visited	Distance	Previous
A	F	**39**	**B**
B	**T**	**19**	**E**
C	T	15	E
D	F	24	E
E	T	14	H
F	T	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	T	17	K
K	T	0	Ø
L	T	16	K

Example

Next we visit vertex D

- The path (K, H, E, D, A) is of length 24 + 21 = 45
- We don’t update A

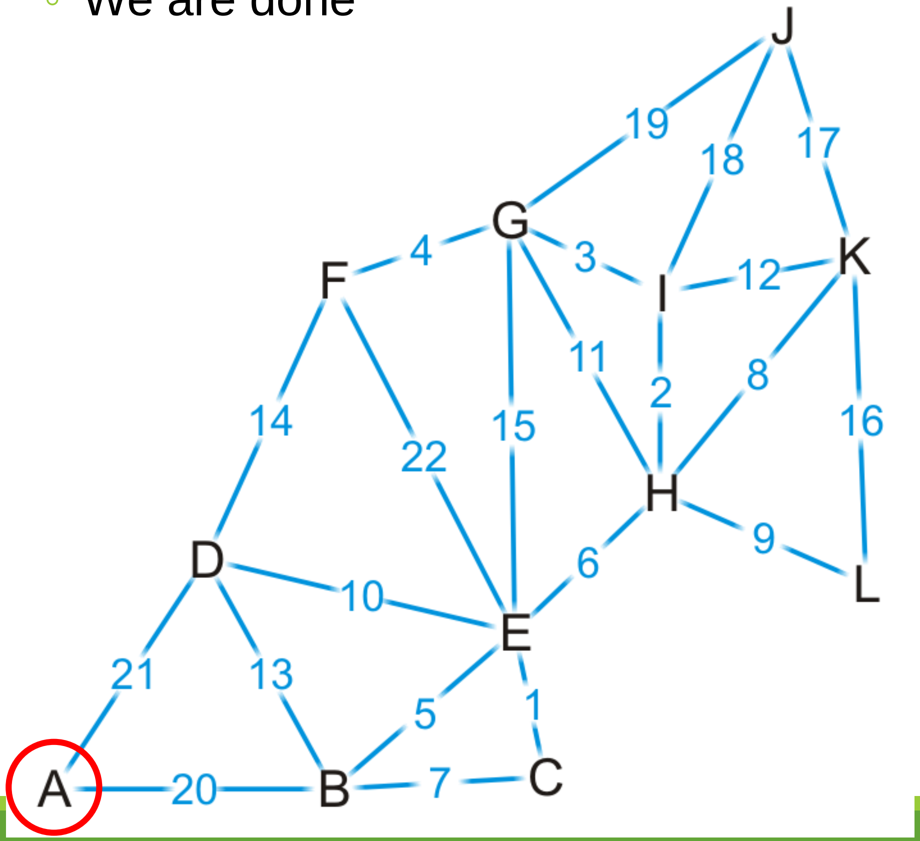


Vertex	Visited	Distance	Previous
A	F	39	B
B	T	19	E
C	T	15	E
D	**T**	**24**	**E**
E	T	14	H
F	T	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	T	17	K
K	T	0	Ø
L	T	16	K

Example

Finally, we visit vertex A

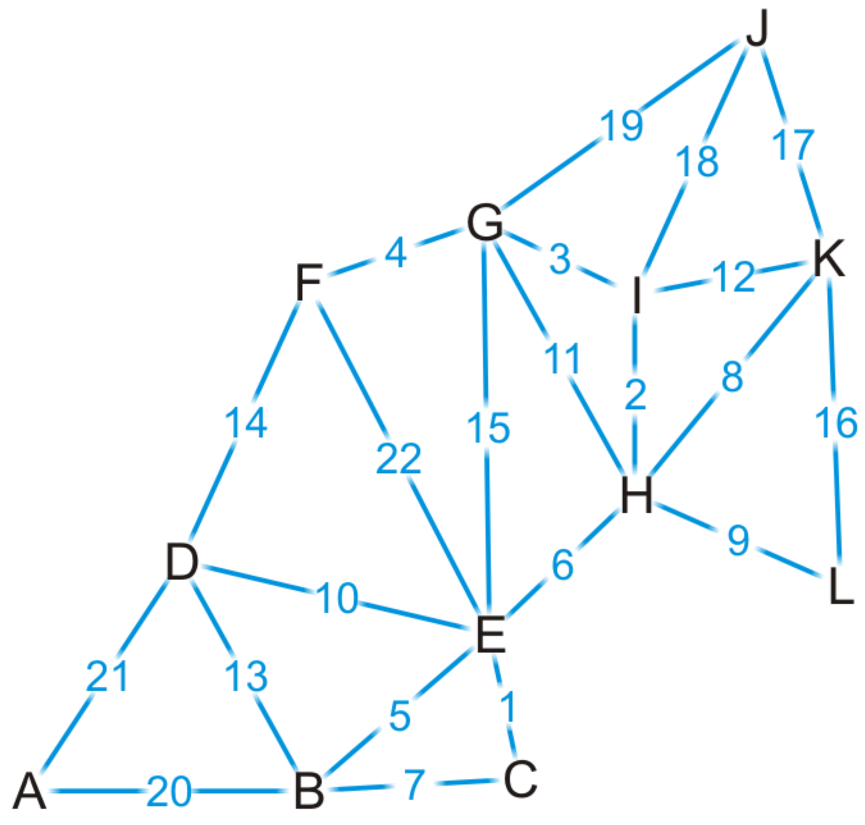
- It has no unvisited neighbors and there are no unvisited vertices left
- We are done



Vertex	Visited	Distance	Previous
A	**T**	**39**	**B**
B	T	19	E
C	T	15	E
D	T	24	E
E	T	14	H
F	T	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	T	17	K
K	T	0	Ø
L	T	16	K

Example

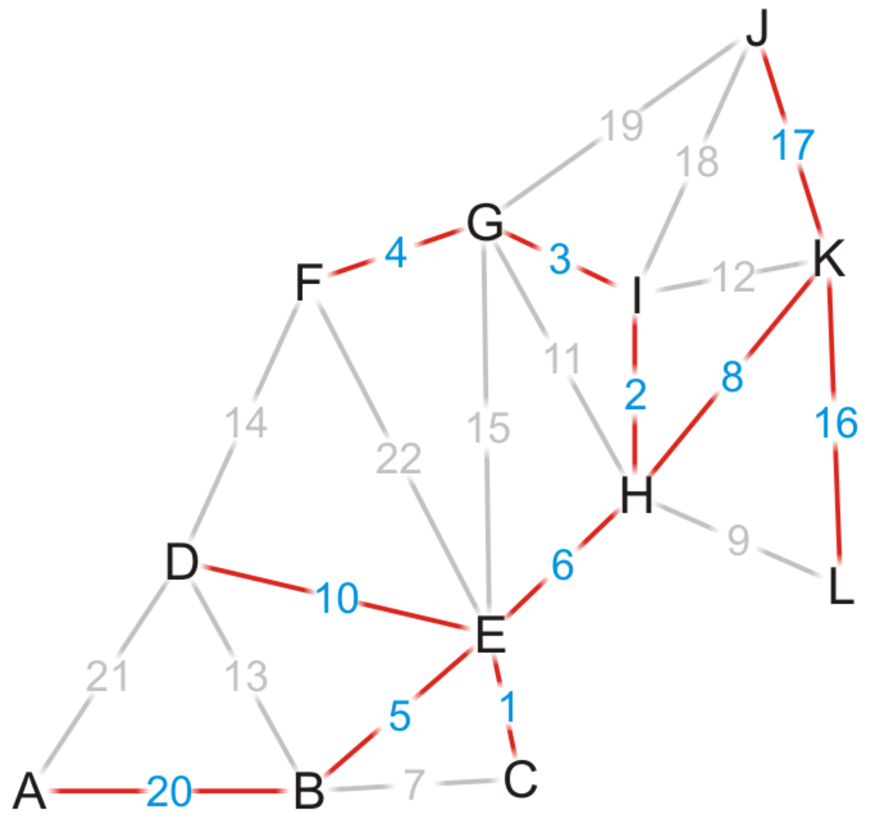
Thus, we have found the shortest path from vertex K to each of the other vertices



Vertex	Visited	Distance	Previous
A	T	39	B
B	T	19	E
C	T	15	E
D	T	24	E
E	T	14	H
F	T	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	T	17	K
K	T	0	Ø
L	T	16	K

Example

Using the *previous* pointers, we can reconstruct the paths

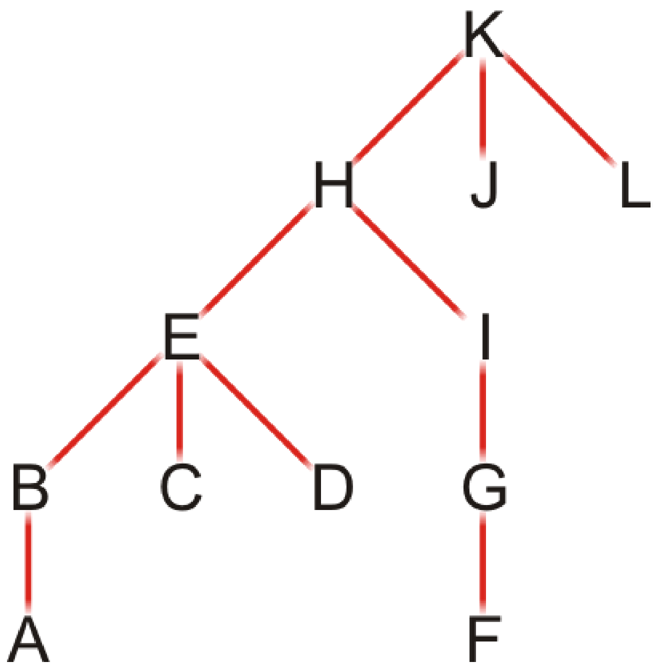


Vertex	Visited	Distance	Previous
A	T	39	B
B	T	19	E
C	T	15	E
D	T	24	E
E	T	14	H
F	T	17	G
G	T	13	I
H	T	8	K
I	T	10	H
J	T	17	K
K	T	0	Ø
L	T	16	K

Example

Note that this table defines a rooted parental tree

- The source vertex K is at the root
- The previous pointer is the *parent* of the vertex in the tree



Vertex	Previous
A	B
B	E
C	E
D	E
E	H
F	G
G	I
H	K
I	H
J	K
K	Ø
L	K

Comments on Dijkstra's algorithm

Questions:

- What if at some point, all unvisited vertices have a distance ∞? Diconnected graph
- What if we just want to find the shortest path between vertices v_j and v_k?
- Does the algorithm change if we have a directed graph?

Implementation and analysis

The initialization requires $\Theta(|V|)$ memory and run time

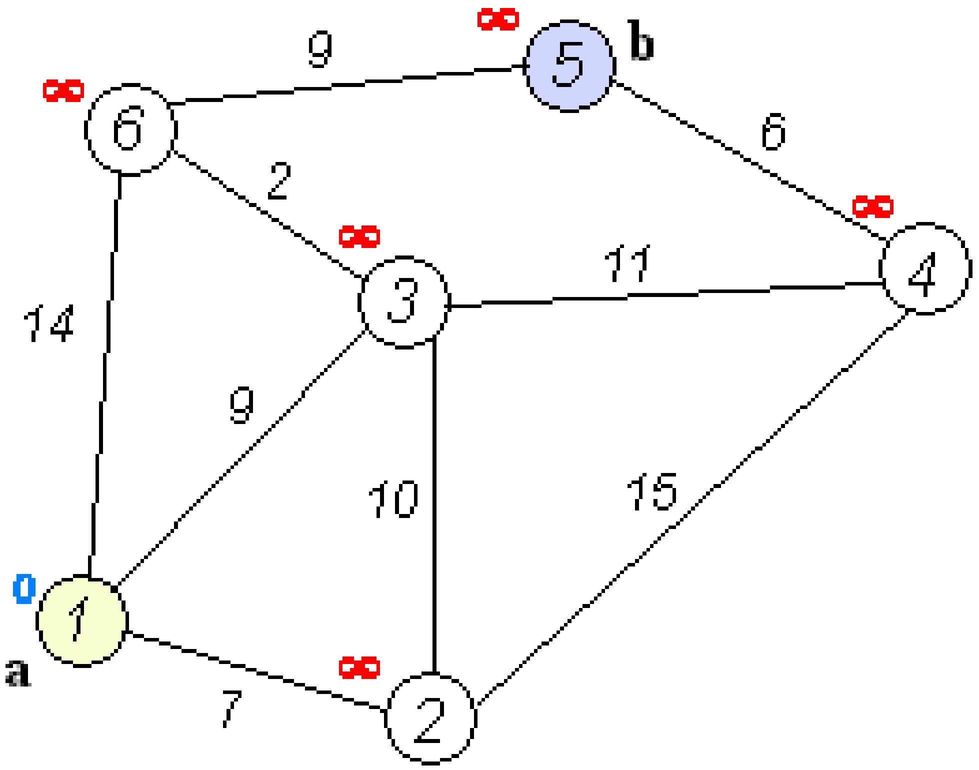
We iterate $|V| - 1$ times, each time finding next closest vertex to the source

- Iterating through the table requires is $\Theta(|V|)$ time
- Each time we find a vertex, we must check all of its neighbors
- With an adjacency matrix, the run time is $\Theta(|V|(|V| + |V|)) = \Theta(|V|^2)$
- With an adjacency list, the run time is $\Theta(|V|^2 + |E|) = \Theta(|V|^2)$ as $|E| = O(|V|^2)$

Can we do better?

- Recall, we only need the closest vertex
- How about a priority queue?
  - Assume we are using a binary heap
  - We will have to update the heap structure—this requires additional work
- the total run time is $O(|V| \ln(|V|) + |E| \ln(|V|)) = O(|E| \ln(|V|))$

Another Example



Practice Example