Department of Economics

Master of Science in Applied Economics & Finance

November 2011

Master's Thesis

Comparison of Value-at-Risk Estimates from GARCH Models

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Pages: 75
Characters: 145000

Acknowledgements

Executive Summary

This thesis has the objective to compare Value-at-Risk estimates from selected GARCH models. It will start with a theoretical part which familiarizes the reader with Value-at-Risk and its main concepts. Known models will be explained and applied, time-series analysis with theories and tests for financial time-series are discussed. For evaluating different Value-at-Risk estimates backtests will be described and discussed.

The empirical part of this research starts with the selection process for the preferred GARCH (p, q) model, on the time-series of S&P500 Total Return index ranging from 20.07.01-19.07.11 (10 years of data).

The selected model is a restricted GARCH (1, 2) model which is evaluated against the GARCH (1, 1) model and the Riskmetrics model. These models were evaluated at the 5% and 1% Value-at-Risk levels for three sample periods. One being the Full sample, another the Before Crisis sample and lastly the With Crisis sample.

The results of the evaluations for the models showed that it is hard to select one of them, as the best model for the time-series. The GARCH (1, 1) performed best on the Full sample, while the Riskmetrics performed best on the Before Crisis sample and the GARCH (1, 2) performed best on the With Crisis sample.

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1 Introduction

With financial markets being abnormally turbulent in recent years, the task for financial firms to manage risk is even more compelling. One of the main factors of managing financial risk is the calculation of Value-at-Risk. This project explains the concept of Value-at-Risk. The development of Value-at-Risk will be briefly discussed and the fundamentals of the concept. They need to be considered carefully before applying a Value-at-Risk model to a portfolio, despite which model is chosen. Four of the most well known models will be shown, described and applied to the S&P500 Total Return index, which will be used throughout this paper for application and evaluation of Value-at-Risk models.

Statistical analysis of financial time-series will be shown in this paper since it is important to know the common behavior of financial time-series while working with them.

For Value-at-Risk it is not enough to simply choose a model and apply it on the portfolio at hand. One needs to evaluate the model and preferably compare its performance to other models. Since one model might fit well to certain time-series, but fail when it is applied to a different set of time- series. The evaluation process for Value-at-Risk estimations is called backtesting, the methods for it will be described and applied.

The empirical research will consist of analysis on the time-series for the S&P500 Total Return index with data ranging back ten years, selecting Value-at-Risk models to apply on the time-series and then compare them.

1.1 Objective

This thesis contributes to empirical studies, familiarizes the reader with the fundamental concepts and backtests of Value-at-Risk and the theories behind them. The empirical research is where Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models will be applied to time-series, they compared to each other and a chosen benchmark model. The main objective is to see which of these models predicts Value-at-Risk most accurately on the past ten years of the S&P500 Total Return index.

1.2   Methodology & Delimitations

The theories that will be applied in this study will be done by the use of Microsoft Excel, with the help of Visual Basic Application (VBA), for computations that Excel's built in functions cannot easily handle. In some cases computations will be executed in MATLAB a well known mathematical and statistical software. It should be noted that a lot of time was spent on writing, debugging and running the VBA codes, and the worksheets attached to the enclosed CD.

Although Value-at-Risk is sometimes referred to as a measurement for normal market behaviours, since it does very seldom manage to mitigate risks during abnormal market behaviours, the financial crisis of late 2008 will be covered in the research.

Furthermore the model selection process for this thesis will try to find the best model from the GARCH (p, q) models, hybrids or extensions from this model will not be considered

Since there are many statistical tests discussed and applied in this paper, it should be noted that there is always a possibility of type I or type II errors with hypothesis tests. A type I error is when a test rejects a true null-hypothesis and a type II error is when a test fails to reject a false null hypothesis.

With relation to the mathematics and statistics, proofs for functions and formulae presented will not be shown, it is beyond the scope of this thesis.

The assumption for normality will hold throughout this paper.

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2   Value-at-Risk

"Risk is like fire: if controlled it will help you, if uncontrolled it will rise up and destroy you."

Theodore Roosevelt

Value-at-Risk is a measure of the risk an entity is facing by its operations. A number that states how much can be lost after a period of time. This number is derived from calculations made by e.g. the risk manager who is trying to get a glimpse into the magnitude of potential losses of the current position. While constructing Value-at-Risk selecting the method used for the calculation is of upmost importance. This can lead to a strong number if done properly and if not much thought is put into the selection process of a method it is more likely to lead to a weaker Value-at-Risk estimate.

In this chapter the development of Value-at-Risk, fundamentals and some of the most known methods will be discussed.

2.1   Development

During the late 1970s and 1980s many major financial institutions began working on models that were designed to measure and aggregate risks throughout their whole operations. At this time there was a high demand for such models since there had been a fair amount of financial crisis right before and during this timespan. The most well known of these and still popular today is the Riskmetrics system, which was introduced by JP Morgan. It is said that the manager of JP Morgan, Dennis Weatherstone, asked his staff to write a daily one page report that would account for risks and potential losses over the next 24 hours, for the banks total trading portfolio. The staff at JP Morgan then developed a risk measurement that shows the maximum likely loss over the next trading day in one number, it was called Value at Risk (Dowd, 1998).

Value-at-Risk was derived from the breakthrough research that Harry Markowitz made on portfolio theory in the 1950s. He later received the Nobel Memorial Prize in Economic Sciences for his contribution. Markowitz's research confirmed that the standard deviations and expected returns of portfolios could be estimated within the framework of the normal distribution and could tell a story about the risk involved. While smaller standard deviation exhibits less risk but consequently often reduces the expected return, making a trade-off

between risk and return. Markowitz's research motivated portfolio managers to diversify their portfolios and calculate correlations, standard deviations and expected returns between securities within their current position (Jorion, 2007).

When the staff at JP Morgan tried to solve the problem they concentrated on the left or negative side of the normal distribution while they constructed Value-at-Risk. In October 1994 JP Morgan decided to make the Riskmetrics system public. This lead many small software providers to include it in new software packages or implement it in its existing software, numerous financial institutions made use of it (Dowd, 1998).

There have been many other methods developed and few will be included in later sections, however JP Morgan's Riskmetrics seems to be a good benchmark for evaluating other models and will be used as such for the purpose of this research.

2.2   Fundamentals

There are three parameters that always have to be kept in mind and stated when Value-at-Risk is computed, these parameters are more of a specification for the Value-at-Risk. They will be described in sections 2.2.2-2.2.4 and sections 2.2.5-2.2.6 will describe how the regulators want these specifications to be. The first section will explain why it is important to use market values when calculating Value-at-Risk.

2.2.1   Mark to Market

One of the crucial parts of risk management is to mark to market. That means that they need to think about daily market changes instead of using accrual accounting methods. They observe how securities rise or fall instead of waiting until they materialize. With accrual methods losses were often revealed only when securities matured. If they would not have waited and the security had lost value, they could have acted earlier and prevented further loss. Also when marking to market the investor can see volatilities and take account for them (Dowd, 1998).

In this research the assumption will be made that the investor reinvests all dividends allocated in the portfolio and therefore it will consist of the S&P500 Total Return index instead of the normal S&P index, since the Total Return index adjusts the index rate for dividends. An

investor that compares his portfolio profits to a market portfolio that is not adjusted to dividends could think that he has beaten the market, where he really has not, since he has benefitted from dividend payments.

2.2.2 Holding Period

Usually the holding period is chosen to be one day or one month, but there can be varieties in holding periods based on the type of portfolio. The main factors that affect the choice of holding period are four. First is the liquidity of the portfolio, if it is in a market that has very frequent trading, shorter holding period would be efficient, because then it is easy to get rid of assets in the portfolio or exchange assets almost instantly. However if it is hard to get counterparty for trades in the market, like in many Over The Counter (OTC) markets a longer holding period is usually chosen since the market is less liquid.

The other three factors all suggest shorter holding periods, one of them is for the normal approximation to be more efficient, with a short holding period the normal approximation usually makes a better fit to the position. Another reason is to adjust for changes in the portfolio. If the chosen holding period is one month a manager would want to change his position if it is showing losses, but with a one day holding period changes in exposures can be made more frequently and therefore reduce the risk of an unwanted loss.

Shorter holding periods are better for valuation purposes, for example with one day holding period it does not take a long time until the sample that is forecasted from is large enough, with one year giving around 250 observations. However with a holding period of one month it would take almost 21 years to get 250 observations and since the first observation would have happened more than two decades ago, changes in the macro environment (politics, technology, currencies etc.) are likely to make the observation obsolete (Dowd, 1998).

2.2.3 Observation Period

The choice of observation period is important, it depends on which model is being used, for the Riskmetrics model which will be explained in section 2.3.4, the observation period does not matter as long as enough observations are included for all the related weights to have an impact on the outcome. However with models such as the Variance Covariance model or the

Historical Simulation model (explained in section 2.3.1) the observation period is of high importance, since all observations are given equal weight. The observation period simply specifies what range of previous observations should be used when calculating the Value-at-Risk number. So an observation period of 250 would use the last 250 observations to obtain Value-at-Risk.

2.2.4 Confidence Level

The confidence level explains how often the observed results or portfolio returns should exceed the Value-at-Risk number. Usually confidence levels ranging from 90% to 99% are used and most often 95% or 99%. For the 99% confidence level you would expect the observed loss to exceed the forecasted Value-at-Risk number once every hundred observations. For evaluation purposes the 95% confidence level is more convenient because exceedance from the forecasted Value-at-Risk number should happen more frequently. Institutions use Value-at-Risk to specify capital requirements, then a risk averse institution would prefer to set the confidence level at higher rate, closer to or even exceeding the 99% level. Since then the capital requirements would be higher and less risk of not being able to cover losses in its exposures. Figure 1 shows how different confidence levels are calculated if the normality assumption is used.
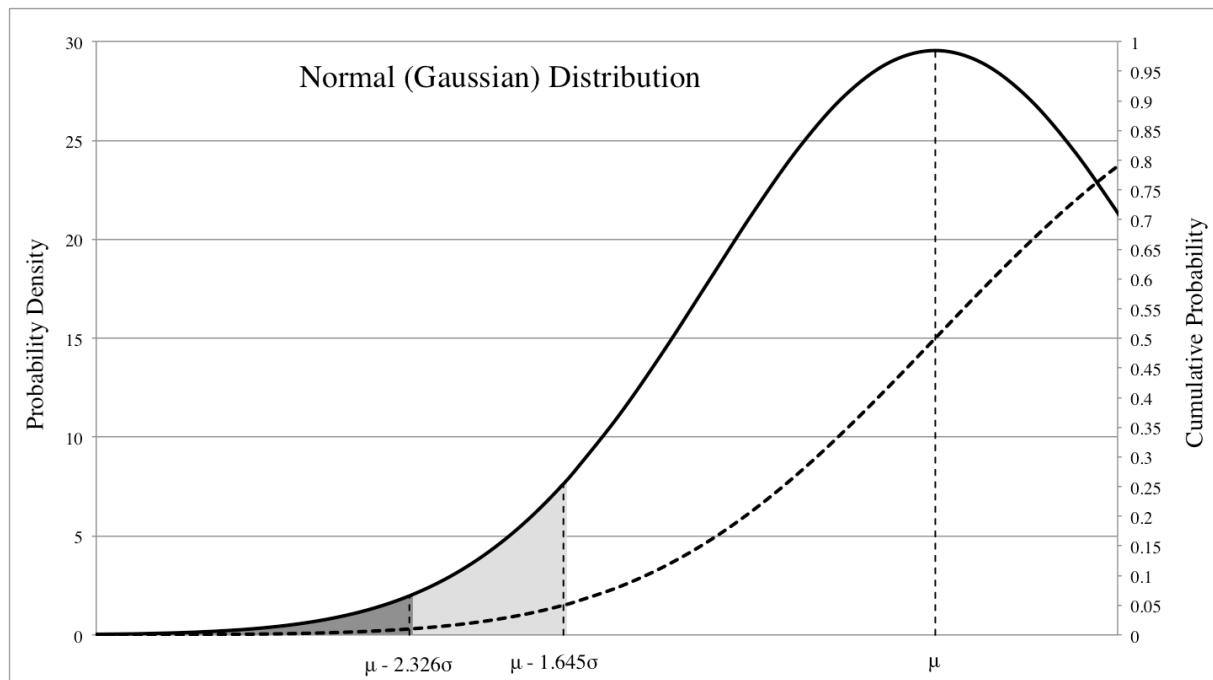
Figure 1: Normal distribution and confidence levels for Value-at-Risk[i]

One of the conveniences from using the normal assumption is that it is very easy to construct Value-at-Risk at different confidence levels. For example if a bank wants to compare its Value-at-Risk number to another bank, but they do not publish their Value-at-Risk numbers at the same confidence level. It can simply adjust the number to whatever confidence level it chooses to, in order for that to work the normality assumption must be used in both cases.

Confidence level	99%	95%
Value-at-Risk	$\mu - 2{,}326 \times \sigma$	$\mu - 1{,}645 \times \sigma$

As seen above the only variables needed are the mean, standard deviation and the confidence level, the numbers 2,326 and 1,645 are the coefficients for the 99% and 95% confidence levels respectively. After the confidence level has been chosen and Value-at-Risk achieved the whole value of the underlying portfolio is multiplied by the Value-at-Risk number to see how much it is in the given currency. When financial institutions make the normality assumption they sometimes make the assumption as $N(0, \sigma^2)$ the mean returns are assumed to be zero. It can be debated if the assumption for zero mean should be used, but the mean for daily observations is usually a little bit above zero. For the risk manager using a holding period of daily returns the assumption of zero mean can be justified. As for longer horizons, it would not be wise to make the zero mean assumption (Dowd, 1998).

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[i] This graph can be found in the Excel worksheet normaldistributiongraph.xlsm on the enclosed CD.

Being a convenient way for Value-at-Risk forecasting, the normal assumption has its pitfalls. One of them is that securities tend to have fatter tails than accounted for by the normal distribution. Meaning that Value-at-Risk models using the normal assumption might underestimate the Value-at-Risk. Other approaches have been used such as changing the distribution to e.g. the Student-T, which can account for fatter tails or by using the Historical Simulation method, which does not assume for any kind of distribution (Dowd, 1998).

2.2.5   CESR Regulation for UCITS

The Committee of European Securities Regulators (CESR) have directives for Undertakings for the Collective Investment in Transferable Securities (UCITS), those directives account for funds that are operating in countries within the European Union. There are also domestic regulations, which are different from country to country within the EU, but CESR regulations are used as a base point.

With regards to the publication of Value-at-Risk for funds that are specified as UCITS the base regulation enforces the calculation of Value-at-Risk and specifies what the maximum Value-at-Risk value should be, it is stated that it should never surpass 20% when the parameters discussed earlier in this section are at a predefined value. According to CESR guidelines set in July 2010 the confidence level should be set at 99%, holding period equal to one month (20 business days) and observation period at least one year (250 business days) unless external factors suggest a shorter period e.g. for a drastic change in price volatilities during an extreme market condition. As mentioned in this section, to have a holding period of a month can be troublesome since observations might be too few. That is why CESR have allowed for rescaling of the Value-at-Risk number, if UCITS are using 95% confidence level, they can rescale it according to the normal distribution. Also if they are using a holding period of less than 20 business days they are allowed to rescale the Value-at-Risk number using the square root of time rule. This rule can be shown as follows:

$$VaR\ (x\ days) \approx \frac{\sqrt{x}}{\sqrt{t}} \times VaR(t\ days)$$

If it is needed to rescale a daily horizon up to 20 days as CESR sets for the maximum Value-at-Risk, x is simply set to a value of 20 while t is set to a value of 1 (Committee of European Securities Regulators, 2010).

2.2.6   BASEL Regulation

The Basel Committee on Banking Supervision (BCBS) set guidelines and standards for banks so that minimum requirements of BCBS are met.

Basel sets minimum requirements for quantitative standards of Value-at-Risk for banks in their revision to the Basel II market risk framework. It states that Value-at-Risk must be calculated on a daily basis, it must have a confidence level of 99% and the holding period set at 10 days. The observation period should be a minimum of 250 days. The observation period can be less than a year if unusual conditions are in the market increasing the price volatility. As with the CESR guidelines this does not mean that banks should use exactly those parameters when estimating Value-at-Risk for their internal risk management. They are also allowed to rescale from e.g. the normal distribution and by using the square root of time rule (Basel Committee on Banking Supervision, 2011).

2.3   Popular Methods[i]

In this section the models that are most popular or common in the literature and in practice will be explained, the ones that are most often mentioned in discussions about Value-at-Risk models are the Historical Simulation, Variance Covariance and Monte Carlo Simulation. Those models can be described as the core models since many models have been created based on one or more of these three. The Riskmetrics model will be explained as well, since it will be used as a benchmark in this research.

2.3.1   Historical Simulation

Historical Simulation ranks the losses and gains from the worst and upwards. For illustration a Historical Simulation distribution will be shown for the S&P500 Total Return index. Figure 2 shows a frequency histogram for daily returns of the S&P500 Total Return index. The darker bars in the histogram represent the worst 5% of daily returns.

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[i] Computation of all the graphs presented in section 2.3 can be found in the Excel worksheet popularmethods.xlsm on the enclosed CD.
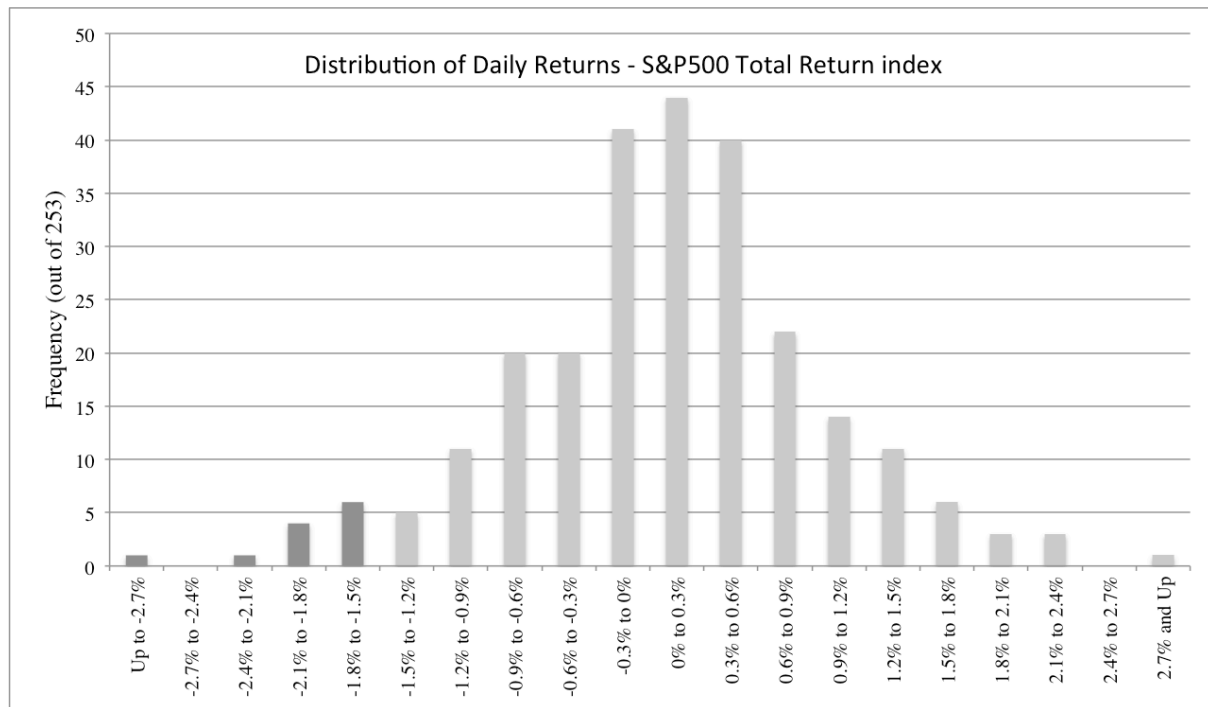
Figure 2: Histogram of daily returns, S&P500 TR index (20.7.10-19.7.11)

In order to find the Value-at-Risk based on the Historical Simulation the observations should first be ranked from worst to best, then use the number as Value-at-Risk that corresponds to the percentile of the distribution based on the chosen confidence level. To calculate the 95% Value-at-Risk for a one-year observation period of the S&P500 Total Return index, which gives 253 observations, when the daily returns have been ranked the position of the 5th percentile is found by multiplying 0,05 with the number of observations. In the case of the S&P500 Total Return index sample used in Figure 2, that number is 12,65, so the 5% Value-at-Risk (with a confidence level of 95%) is based on the 13th number out of the ranking order. For the sample the 5% Value-at-Risk for the next day is 1.4626% so based on the Historical Simulation there is a 5% chance of losing 1.4626% or more for this particular portfolio in the next day.

Advantages of the Historical Simulation would be how easy it is to implement and explain to others. It does not depend on any kind of assumption for the distribution of returns, therefore it does not come with the problem of accounting for fat-tails of the distribution, as tends to happen with the normal distribution. The Historical Simulation has disadvantages since it bases its prediction on past returns. Each observation has an equal weight. An observation that happened recently therefore has the same weight as an observation from a later period. A left-tailed outlier will affect the results of the historical simulation until it is dropped out of the window given by the observation period (Dowd, 1998).

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2.3.2   Variance Covariance

The Variance Covariance approach assumes for the normal distribution, few of its features were explained in section 2.2.4. It simply approximates the returns from the observation window to the normal distribution. The only parameters needed for the Variance Covariance method are average returns and standard deviation of returns. The standard deviation is multiplied by the confidence coefficient determined by the confidence level chosen. When this approach is used on a portfolio the correlation and covariance between the assets in the portfolio are calculated and from them the standard deviation is achieved. The following formula shows how the portfolio variance is calculated.

$$\sigma_p^2 = \sum_i \omega_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} \omega_i \omega_j \rho_{ij} \sigma_i \sigma_j$$

Where $\omega_{i/j}$ represents the weights of asset i or j and ρ_{ij} represents the correlation coefficient between the returns on assets i and j. The portfolio standard deviation is the square root of the portfolio variance (Dowd, 1998). As with the Historical Simulation the Variance Covariance method is used on the S&P Total Return index and the following graph shows the results.
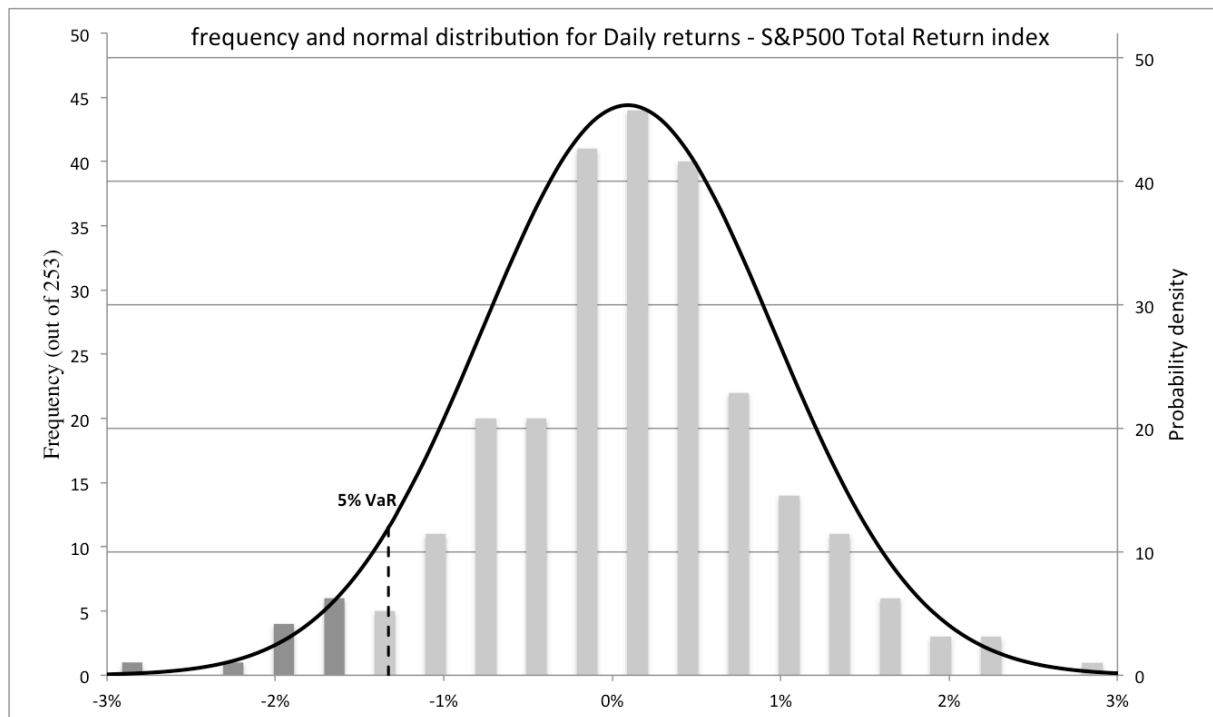
Figure 3: Histogram & normal distr. of daily returns, S&P500 TR index (20.7.10-19.7.11)

The 5% Value-at-Risk is lower than the one from the Historical Simulation or at 1,3290%. The frequency distribution is in Figure 3 to see if the actual distribution deviates from the normal distribution. The frequency distribution does not seem too far away, although not fully correlated with the normal distribution. The Jarque-Bera test for normality, checks if the sample distribution has skewness and kurtosis that match the normal distribution. It is asymptotically chi-square distributed and has two degrees of freedom, making the critical rate 5,99 at the 95% confidence level. The test statistic is calculated as follows:

$$JB = \frac{n}{6}\left(S^2 + \frac{1}{4}(K - 3)^2\right)$$

Where JB represents the Jarque-Bera test statistic, n represents number of observations, S represents skewness and K represents kurtosis (Jarque & Bera, 1980). The Jarque-Bera test statistic on the one year sample for S&P500 Total Return index gave a value of 8,1201 rejecting the null hypothesis of normality at the 95% confidence level. The assumption could not be rejected at the 99% confidence level since the critical value is lower than the observed statistic.

Advantages of the Variance Covariance approach are how easy it is to implement and fetch other important statistical details from the parameters it estimates. It can be troublesome to explain this method to someone that has not been trained using the normal distribution and therefore will not understand its core concepts. The assumption for the normal distribution does not always represent the financial instruments it is modeling since those instruments often tend to have fatter tails than the normal distribution accounts for (Dowd, 1998).

2.3.3   Monte Carlo Simulation

The idea behind the Monte Carlo Simulation is to simulate random processes affecting the prices of financial instruments. With each simulation predicting a value for the portfolio after the chosen horizon period. For the Monte Carlo Simulation to be more efficient, a high number of simulations are needed. If the amount of simulations is large enough the simulation has converged to the portfolio's "true" unknown distribution. At the beginning of the process a model must be chosen and its parameters estimated. Then a hypothetical price path is simulated that depends on the model and the random numbers generated (Dowd, 1998).

A Monte Carlo Simulation was run on the same observation period as with previous models. This simulation is based on Geometric Brownian Motion, which has been widely used to simulate stock prices. Price paths are simulated for the next day after the latest observation in the sample. Parameters used in the simulation are estimates from the sample, the estimate for the standard deviation and the mean. Then the representation of the time interval t is chosen, in this simulation it represents a minute. There are 1440 minutes in a day resulting in 1440 random processes for each return forecast. The model is as follows:

$$P_t = P_{t-1} e^{\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma W_t}$$

Where P is the price of the asset, μ is the mean return and W_t is a geometric Wiener process, which is a random number generated from the normal distribution. The mean and the standard deviation need to be converted to the time interval, since the time interval is a minute in order to acquire the correct parameters the daily mean from the sample is divided by 1440 and the daily standard deviation is divided by the square root of 1440 (using the square root of time rule). This model simply describes a random walk with a drift. The reason for subtracting half the variance from the mean is to account for eroding returns (Croce, 2009). The simulation was run 1000 times gaining 1000 different returns for the next day, it did not fully converge, which will take a lot more simulations to achieve, but it is enough to show how the Monte Carlo Simulation works. Figure 4 presents the frequency distribution of the obtained returns.
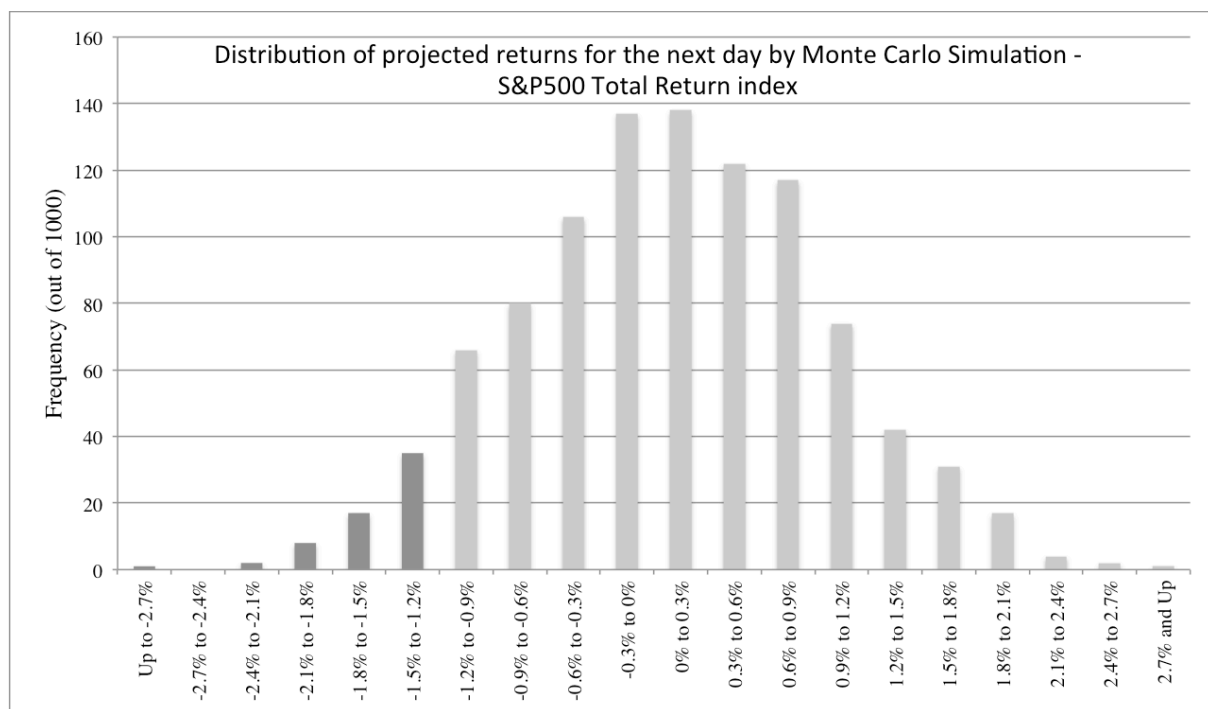
Figure 4: Histogram of returns by Monte Carlo, S&P500 TR index (20.7.10-19.7.11)

Since there were 1000 observations, in order to find the 5% Value-at-Risk based on the Monte Carlo Simulation the resulting observations are sorted from lowest to highest and the 51st represents the 5% Value-at-Risk, for the Monte Carlo Simulation that number is 1,3114%. Since the random process was based on the normal distribution the frequency distribution is near normal, in fact the Jarque-Bera test statistic is 0,5295. It significantly accepts the hypothesis for a normal distribution and with convergence the test statistic should be even lower.

There are many advantages to the Monte Carlo Simulation since almost any model can be used and simulated. One disadvantage is its time consuming nature i.e. it is computationally intensive to generate all the simulations needed. Another disadvantage is how problematic it is to describe to others, even more difficult than with the Variance Covariance approach.

2.3.4   Riskmetrics

As with all models previously discussed (except for the Historical Simulation) the Riskmetrics model measures volatility of the sample data. There is one added feature; it does not give equal weights to all the observations within the sample period like the Variance Covariance model. It uses Exponentially Weighted Moving Average (EWMA). Which adds a decay factor to the model, it determines how the different weights are distributed among the sample observations. JP Morgan set a value for the decay factor, being 0.94 for daily and 0.99 for monthly holding periods. Historical time-series of stock prices are known to show volatility clustering, meaning that volatilities tend to cluster together, making periods with high and low volatilities. The Riskmetrics model can be shown as follows:

$$\sigma_t^2 = r_{t-1}^2(1-\lambda) + \lambda\,\sigma_{t-1}^2$$

Where lambda represents the decay factor. Figure 5 shows how the weights change with different lambda values.
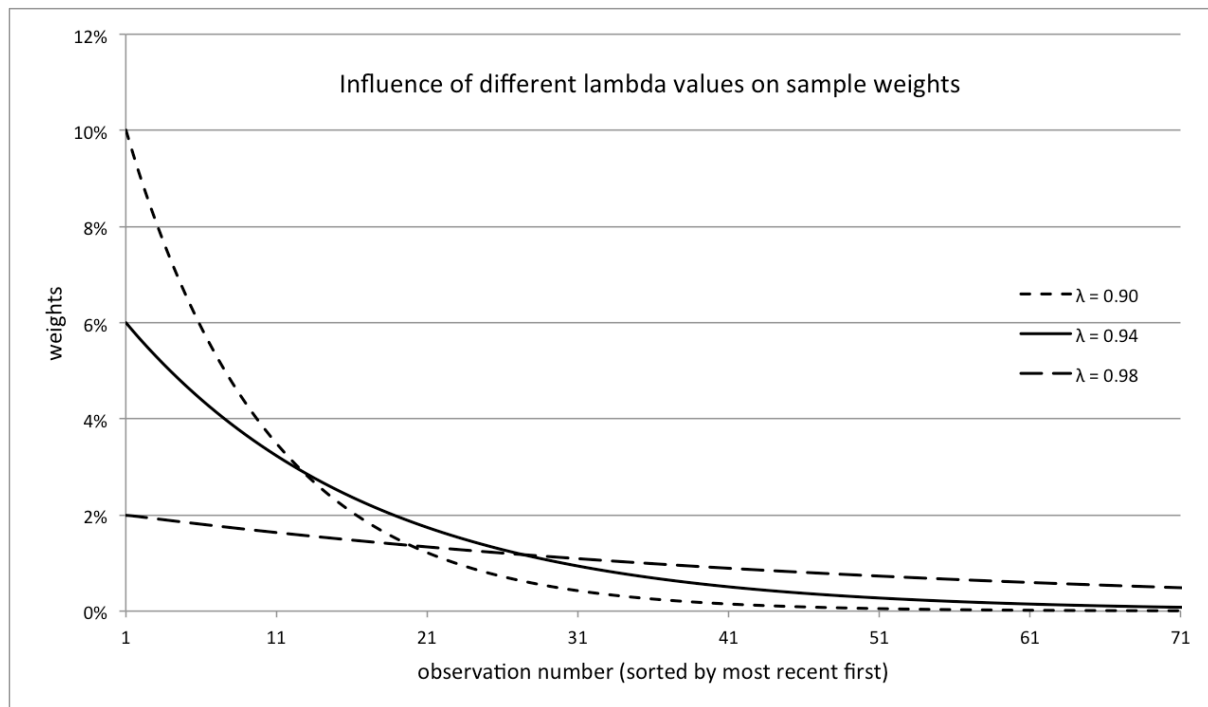
Figure 5: Graph showing how different lambda values affect sample weights

The weights of each return diminishes exponentially over time, since the lambda value controls the weights, it therefore decides the size of the observation window. The result from calculating log (0.001)/log (λ) shows at which previous period the accumulated weights are up to 99,9% of the variance calculated by the model. For a lambda value of 0.94 that number would be 111,64 making the 112th observation as the latest to be included for 99,9% of the weights. As with a lambda value of 0.97 that number would move up to 227 days (Mina & Xiao, 2001). The Riskmetrics method was applied to the S&P Total Return index (20.07.10-19.07.11). It gave 1,5733% Value-at-Risk at the 5% level.

The Riskmetrics model is actually an IGARCH (1,1) process with no constant, GARCH models will be discussed in the next section and the features of the Riskmetrics model help to understand the different GARCH models.

From all the models, 5% Value-at-Risk was calculated with each giving different results.

3 Time-Series Analysis

All the models described in the previous section can be placed under the umbrella of time-series analysis. For time-series analysis in the literature for econometric research the Autoregressive Conditional Heteroskedasticity (ARCH) and Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models are frequently discussed. Robert Engle pioneered the ARCH model in 1982 (which he in 2003 shared the Nobel Price in Economics for) and in 1986 Tim Bollerslev introduced the GARCH model. The Riskmetrics model can easily fit into that group as well, being a slightly modified version of the IGARCH (1, 1) model.

ARCH and GARCH models measure volatility. They are based on least squares model, it assumes that expected values of all error terms squared should be the same at any point in time. In other words, the basic least squares model assumes for homoscedasticity. When a sample does not have the variances of the error terms equal, it is suffering from heteroskedasticity. The assumption for homoscedasticity is the focus of the ARCH/GARCH models and the models are constructed in such a way that they take account for heteroskedasticity. If not, the estimated confidence intervals would be too narrow. Financial time-series tend to exhibit volatility clustering. Which is a combination of heteroskedasticity and autocorrelation. The variances of the error terms can be different from each other. Whereas they can be high during one period and low during another, when this happens there is heteroskedasticity (Engle R. , 2001).
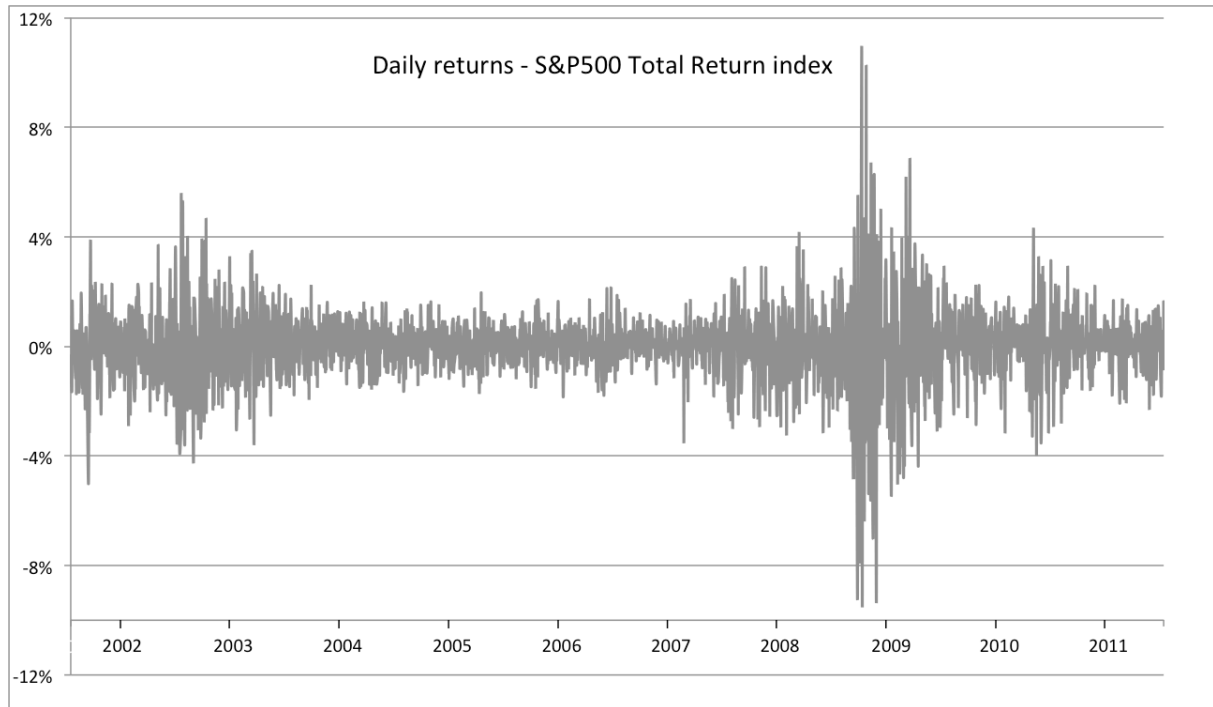
Figure 6: Line graph displaying daily returns, S&P500 TR index (20.7.1-19.7.11)[i]

The figure above shows how daily returns for the S&P500 Total Return index have changed throughout the period 20.7.1-19.7.11 consisting of 10 years in total. From the graph volatility clustering is evident, as some periods show high volatility while others show low volatility. As can be seen on figure 6, the most recent financial crisis from late 2008 presented exceptionally high variances, making the markets behave like roller coasters.

This exact timespan (10 years from the 19th of June 2011) will be used for all the model estimations and related calculations in this research.

At the beginning of the process it is good to check how the sample data behaves, by constructing correlograms for the returns. The correlogram shows the correlations between the returns and the lagged returns referred to as autocorrelation, a lag of one would in our sample show the correlation between the daily returns and the daily returns one day back in time. A lag of two would go two days back in time etc. The calculations for each lag in the correlogram are shown here:

$$r_k = \frac{\sum_{t=1}^{n-k} r_t r_{t+k}}{\sum_{t=1}^{n-1} r_t^2} = \frac{\hat{\gamma}_k}{\hat{\gamma}_0}$$

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[i] Computation of the graph can be found in the Excel worksheet volatilityclustering.xlsm on the enclosed CD.

Where r represents the returns, k represents the lag length, t represents time and n represents the sample size. It is same as the formula for correlation between sample a and b, the main difference is that the mean is not subtracted because it is assumed to be zero. The correlogram can be used to see if there is any evidence of autocorrelation present in the data. A confidence interval can be generated for every lag of the correlogram, if the autocorrelation coefficient exceeds the confidence band, the null-hypothesis of no autocorrelation at that particular lag and beyond is rejected. To calculate the confidence band the sample standard error (SE) is found and to get the 95% confidence limit, it is multiplied by two since the 95% confidence limit is two standard errors from the mean (Enders, 1995). The SE statistic is as follows:

$$SE = \frac{1}{\sqrt{n}}$$

To find the 95% upper and lower confidence bands ±2SE is calculated, where the upper band uses a plus sign and the lower a minus sign. Figure 7 represents the correlogram showing 100 lags for the sample.
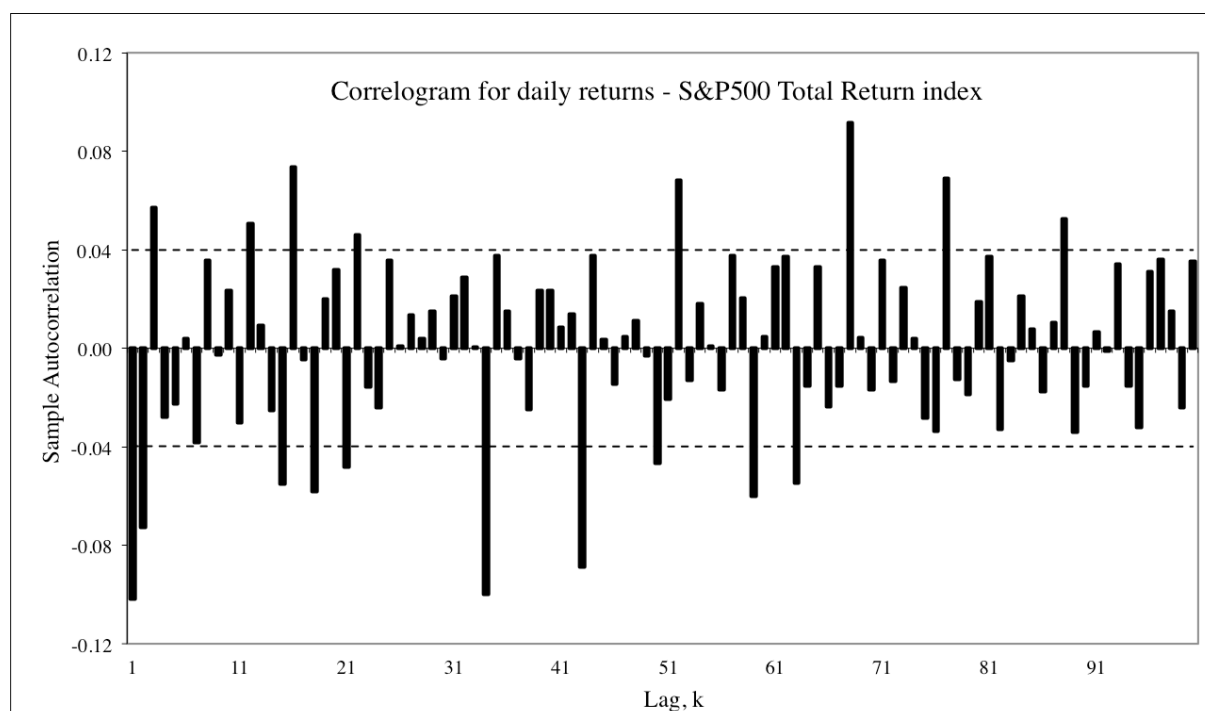
Figure 7: Correlogram for daily returns, S&P500 TR index (20.7.1-19.7.11)[i]

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[i] Computation of the correlogram can be found in the Excel worksheet autocorrelationandheteroskedasticity.xlsm on the enclosed CD.

The dotted lines in figure 7 show the upper and lower 95% confidence bands. The first three lags exceed the confidence limit, they do it as well at some of the higher order lags. By that it cannot be stated that the returns are independent of each other.

G. M. Ljung and G. E. P. Box made the Ljung-Box Q test in 1978, it is an hypothesis test which checks if there is any autocorrelation present up to a predefined lag. It follows the chi-square distribution and the null hypothesis states that there is no autocorrelation. If the null hypothesis is rejected at least one autocorrelation is not zero. The Q statistics can be shown as follows:

$$Q_k = n(n+2) \sum_{j=1}^{k} \frac{r_j^2}{n-j}$$

Where r_j represents the autocorrelation at lag j, n represents the sample size and k represents the lag which the Q statistics is testing up to. The test is asymptotically chi-square distributed and has k degrees of freedom. If the Q statistics is above the critical value, the null hypothesis can be rejected at the chosen confidence level (Ljung & Box, 1978). The values of k can affect the Q_k statistic, simulation studies have shown that choosing $k \approx \ln(n)$ gives the test a better power, usually there are several test statistics shown. (Tsay, 2005) The Ljung-Box Q test was applied on the 10 years of data from the S&P Total Return index, giving 2512 observations. By calculation of $\ln(n)$ the result is 7,83. Making a k value of 8 predicted to give the best power. The test was therefore run with lagged values of 6, 7, 8, 9 and 10. Below the test results are shown:

Ljung-Box Q test

k	Q_k	$X^2_{(1-0,05), k}$	P-value
6	50,7593	12,592	0,0000***
7	54,4201	14,067	0,0000***
8	57,6169	15,507	0,0000***
9	57,6338	16,919	0,0000***
10	58,9992	18,307	0,0000***

Table 1: Ljung-Box Q test for daily returns, S&P500 TR index (20.7.1-19.7.11)[i]

Note: * p < .05; ** p < .01; *** p < .001

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[i] Computation of the Ljung-Box Q test can be found in the excel worksheet autocorrelationandheteroskedasticity.xlsm on the enclosed CD.

As visualized in figure 7, the null hypothesis is significantly rejected and therefore there is autocorrelation in the data.

It is also interesting to see how the correlogram behaves when correlations are calculated between the squared returns and their lagged values. Since the expected daily return is zero, the unconditional variance can be expressed as:

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{t=1}^{n} r_t^2$$

A correlogram of the squared returns is used to see how the autocorrelations of the variances are. Figure 8 shows a correlogram for the squared returns.
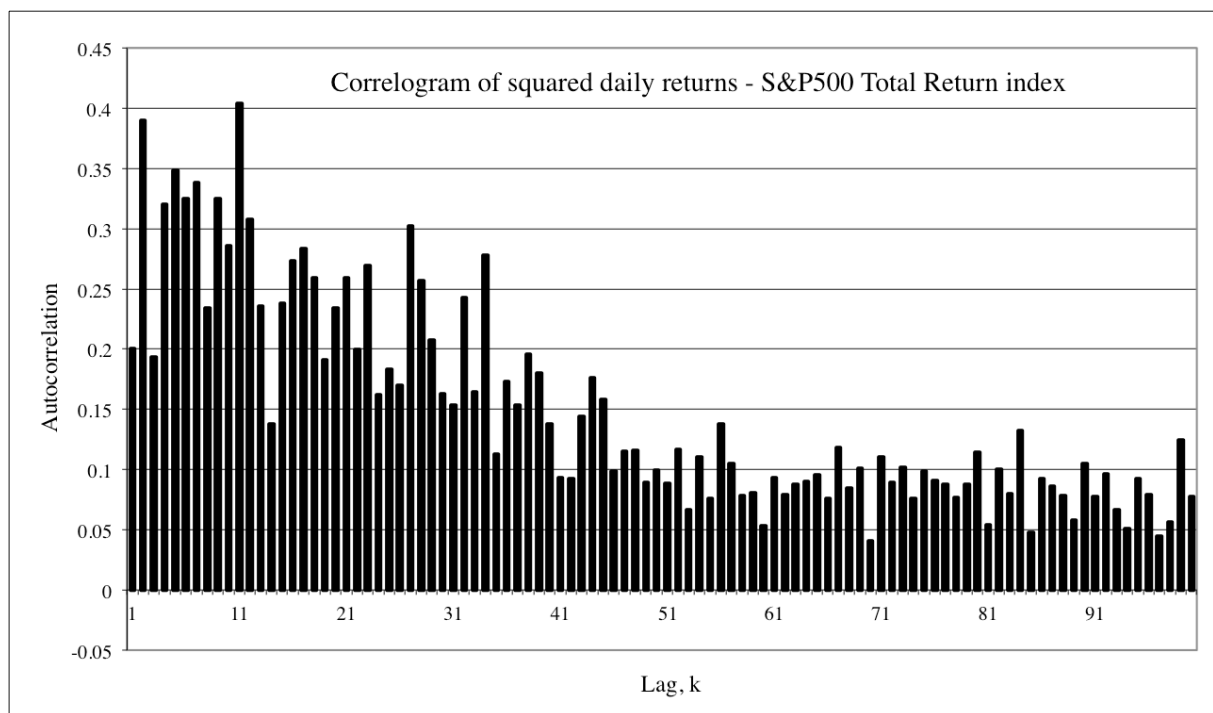
Figure 8: Correlogram of squared daily returns, S&P500 TR index (20.7.1-19.7.11)[i]

The correlogram for the squared returns is different from the one on the returns, the autocorrelations are all positive and much higher. By looking at it an assumption for autocorrelations to be zero at all lags is significantly rejected. It is also interesting to see that the correlations tend to decline with increasing lag lengths. This is a clear signal for volatility clustering and therefore heteroskedasticity since with volatility clustering correlations are expected to decline when lags move further away in time.

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[i] Computation of the correlogram can be found in the worksheet autocorrelation.xlsm

The outcome of tests for heteroskedasticity and autocorrelation determines if data can be said to be stationary, an important factor of time-series analysis. If the residuals have the same variance and mean over time, and show the same autocorrelations at same lag orders over time. The residuals are considered to be wide sense stationary. Stationarity is important for time-series models to be reliable.

The following shows what is meant with the residuals having the same autocorrelations at same lag orders over time:

$$r_k = \frac{\sum_{t=1}^{n-k} r_t r_{t+k}}{\sum_{t=1}^{n-1} r_t^2} = \frac{\hat{\gamma}_k}{\hat{\gamma}_0} = \frac{\hat{\gamma}_{k-s}}{\hat{\gamma}_{0-s}}$$

3.1   Iterated Cumulative Sums of Squares (ICSS) Algorithm

In 1994 Inclan and Tiao proposed the iterated cumulative sums of squares (ICSS) algorithm. It detects variance changes in a given sample and then finds breaking points, which determines at what time the variance changed. The algorithm works sufficiently for data sets of 200 observations and higher. This algorithm can help to identify the variance changes and can prove helpful for financial time-series, since there is heteroskedasticity present in many of them. There are other methods that aim at identifying the variance shifts of observations however many of them require vast computational power, which the ICSS algorithm does not. Making it a good algorithm for VBA in Excel. This approach uses cumulative sums of squares in order to search for variance change points with a systematic effort on different parts of the sample. An algorithm that iteratively finds possible change points and tests if these points are valid as change points in the sample.

The algorithm takes a step-by-step approach.

Step 1: First t_1 is set to equal one, $t_2 = 2$ and up to $t_T = T$, where T would be the latest observation in the sample.

Step 2: Calculate the absolute value of D_k for every k up to T. In order to do that, first the cumulative sum of squares needs to be calculated for every k up to T, as shown below:

$$C_k = \sum_{t-1}^{k} a_t^2$$

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Where a_t^2 represents a squared observation at time t. By plugging in the values for C_k in the formula below, D_k is obtained:

$$D_k = \frac{C_k}{C_T} - \frac{k}{T}, \qquad k = 1, 2, \dots, T \text{ with } D_0 = D_T = 0$$

The value of D_k is made absolute $|D_k|$. Then the observation is found that has the maximum number for $|D_k|$ and is used to test if it is considered a change point:

$$M(t_1:T) = \max_{t_1 \leq k \leq T} \sqrt{\frac{T - t_1 + 1}{2}} \times |D_k(a[t_1:T])|$$

If $M(t_1:T)$ is higher than the critical value of 1.358, which is at the 95% confidence interval, it could possibly be a changing point at $k(a[t_1:T])$ and the algorithm proceeds to the next step. If the value is less than the critical value the algorithm stops because there is no evidence of a variance change in the sample.

Step 3a: t_2 is set equal to $k(a[t_1:T])$ and $D_k(a[t_1:t_2])$ is evaluated. The cumulative sums of squares are only applied on the beginning of the series and up to point t_2. Then the same procedures are repeated with this part of the sample as was in step 2, if $M(t_1:t_2)$ is higher than 1.358 there is a new change point and step 3a should be repeated until $M(t_1:t_2)$ is less than the critical value. When that occurs, there is no evident variance change between the points t_1 and t_2 and the first point of change is set to $k_{first} = t_2$ and we proceed to step 3b.

Step 3b: Now the search for change points is continued on the next part of the sample, which starts from the change point found in step 2 to the end of the series. A new value is defined for t_1 let $t_1 = k(a[t_1:T]) + 1$. Then $D_k(a[t_1:T])$ is evaluated for a possible change point and step 3b is repeated until $M(t_1:T)$ is less than the critical value. In that case k_{last} is set to equal $t_1 - 1$.

Step 3c: The values of k_{first} and k_{last} are checked and if $k_{first} = k_{last}$ there is only one change point and the algorithm stops. When $k_{first} < k_{last}$, both values are saved as possible change points and steps 2 and 3 are repeated with those points as starting and ending points of the series i.e. $t_1 = k_{first} + 1$ and $T = k_{last}$. When Steps 3a and 3b are repeated they can result in one or two more change points. $\widehat{N}_T$ is set to the number of change points found.

Step 4: When the possible change points found are two or more they get sorted in increasing order and cp is denoted as a vector of all the possible change points that have been found. The lowest and the highest values are defined as $cp_0 = 0$ and $cp_{N_T+1} = T$. Each possible change point is checked by calculation of:

$$D_k\big(a[cp_{j-1} + 1 : cp_{j+1}]\big), \qquad j = 1, 2, \dots, \widehat{N}_T$$

If $M\big(cp_{j-1} + 1 : cp_{j+1}\big)$ is more than the critical value then the point is kept and if not it is eliminated. Step 4 is then repeated until the number of change points stays the same and the points found are close to those found on the previous pass. When that happens the algorithm is considered to have converged (Inclan & Tiao, 1994).

The ICSS algorithm was applied on the returns series for the S&P500 TR index (20.07.10-19.07.11) and found seven change points with convergence after nine iterations. The change points (shown in figure 9) are on the following dates 14.06.02, 25.07.03, 09.07.07, 12.09.08, 02.12.08, 01.06.09 and 07.09.10.
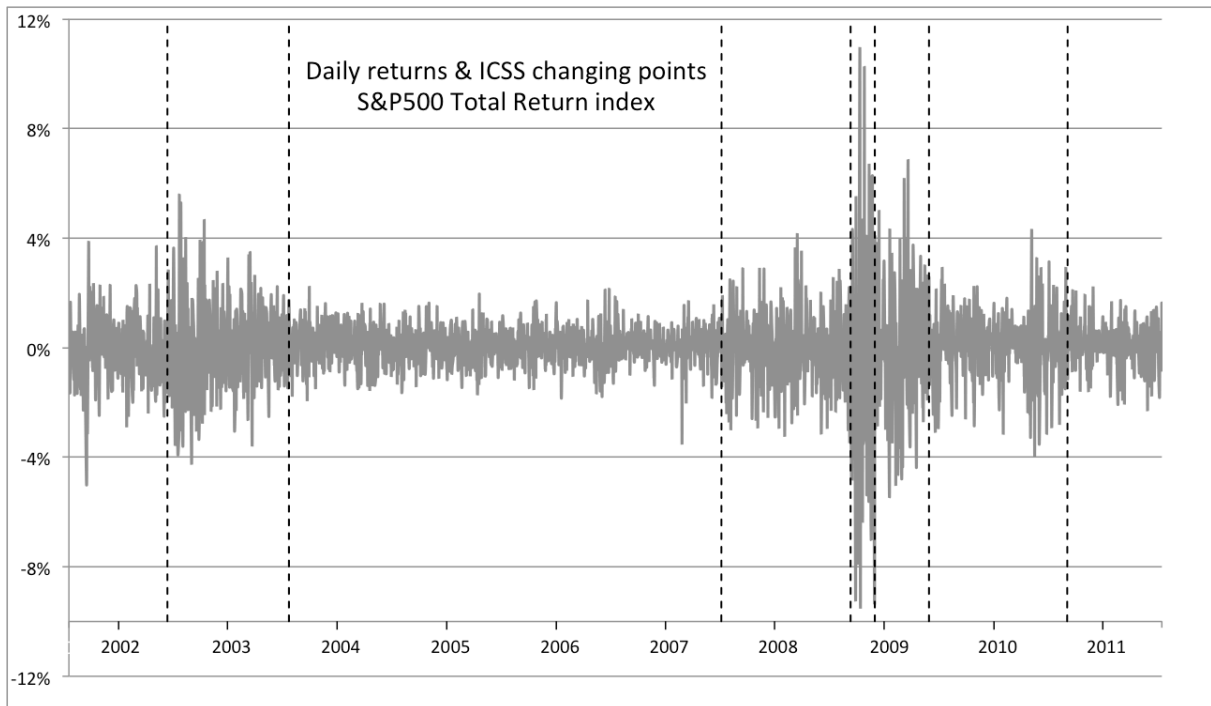
Figure 9: Daily returns and ICSS change points, S&P500 TR index (20.7.1-19.7.11)[i]

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[i] Computation of this graph and the VBA code for the ICSS algorithm can be found in the Excel worksheet ICSScalculation.xlsm on the enclosed CD.

By visually examining the change points, it seems that the ICSS algorithm adequately represents where the variances start to change. It is interesting to see how many change points are near each other, in and around the start of the financial crisis in late 2008.

3.2   Autoregressive Conditional Heteroskedasticity (ARCH)

The ARCH model introduced by Engle in 1982 is designed to tackle the problems of volatility clustering in time-series therefore making a time-series model that takes account for heteroskedasticity in the data. It is described as ARCH (q) where q is the number of lagged values of r^2 used in the model, stated as the order of the ARCH process. The ARCH model can be shown as:

$$\sigma_{t+1}^2 = \omega + \sum_{i=1}^{q} \alpha_i r_{t+1-i}^2$$

Where ω and α_i are estimated parameters and σ_{t+1}^2 is the conditional (changing) variance.

The error terms, denoted by r_{t+1} are determined by a stochastic value $\mathcal{Z}_{t+1}$ and a time dependent conditional standard deviation σ_{t+1}.

$$r_{t+1} = \sigma_{t+1}\mathcal{Z}_{t+1}$$

Where $\mathcal{Z}_t \sim N(0,1)$ and σ_{t+1} comes from the original model.

To ensure non-negative volatility, all the alpha values are required to fit this condition (Engle R. F., 1982):

$$\omega > 0 \ and \ \alpha_1, \cdots, \alpha_q \geq 0$$

For σ_{t+1}^2 to ensure wide sense stationarity the parameters are also required to have the following constraints:

$$\alpha_1 + \alpha_2 + \cdots + \alpha_q < 1$$

When those conditions are satisfied the unconditional (long term) variance becomes existent and is calculated as (Hamilton, 1994):

$$\sigma^2 = E(\sigma_t^2) = \frac{\omega}{1 - \alpha_1 - \alpha_2 - \cdots - \alpha_q}$$

The recursive form of the ARCH (q) model, which in many cases is easier to comprehend is written as:

$$\sigma_{t+1}^2 = \omega + \alpha_1 r_{t+1-1}^2 + \alpha_2 r_{t+1-2}^2 + \cdots + \alpha_q r_{t+1-q}^2$$

The simplest ARCH process would be defined as ARCH (1) and is written as:

$$\sigma_{t+1}^2 = \omega + \alpha_1 r_t^2$$

For the parameter estimation a regression is needed, based on the maximum likelihood function. When the likelihood function is maximized based on the sample data and constraints, the parameters are at the optimal level. The function to be maximized is l, shown below:

$$l = \frac{1}{T} \sum_{t=1}^{T} l_t$$

$$l_t = -\frac{1}{2} \log \sigma_t^2 - \frac{1}{2} \frac{r_t^2}{\sigma_t^2}$$

Determined by the order of q lags, the ARCH process accounts for volatility clustering since it gives weights to the nearest squared returns. However for the ARCH model to converge it usually needs the order of the ARCH (q) process to be set at a high value. Making the estimation difficult since when q increases, the number of parameters increases. One of the solutions to this problem is to make the lagged residuals have linearly decreasing weights, similar to the Riskmetrics model.

$$\sigma_{t+1}^2 = \omega + \alpha \sum_{i=1}^{q} w_i r_{t+1-i}^2$$

Where w_i represents the weights given to each residual used in the model. The weights are calculated using this formula, making the sum of all weights equal to one (Engle R. F., 1982).

$$w_i = \frac{2(q + 1 - i)}{q(q + 1)}$$

Using this method only two parameters need to be estimated (ω and α) making the regression computationally less intensive. Bollerslev noted that an arbitrary lag structure like this estimates a totally free lag distribution and therefore does in many cases lead to a violation of

the non-negativity constraints. In 1986 Bollerslev presented the GARCH model, described in the next section (Bollerslev, 1986).

3.3   Generalized Autoregressive Conditional Heteroskedasticity (GARCH)

The difficulties with the ARCH process noted by Bollerslev are estimates of a totally free lag distribution created by the ARCH process, where the number of lags could be high, which could lead to a violation of the non-negativity constraints. He presented an extension of the ARCH model denoted as GARCH (p, q) where p represents the order of the GARCH elements and q represents the order of the ARCH elements. The model looks like this:

$$\sigma_{t+1}^2 = \omega + \sum_{i=1}^{q} \alpha_i r_{t+1-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t+1-j}^2$$

Where ω, α_i and β_j are parameters to be estimated. Last part of the explanatory variables is the extension created by a GARCH model. Where lagged conditional variance is a dependent variable on the conditional variance. To ensure non-negative volatility the following constraints are added:

$$p \geq 0, \quad q > 0,$$
$$\omega > 0, \quad \alpha_i \geq 0, \quad i = 1,\ldots,q,$$
$$\beta_j \geq 0, \quad j = 1,\ldots,p.$$

If p is equal to zero, the GARCH model is reduced to an ARCH (q) model (Bollerslev, 1986).

For wide sense stationarity and the variance to be mean reverting, constraint to be added is:

$$\alpha_i + \beta_j < 1$$

When the variance is mean reverting it converges to the unconditional variance when high number of forecast steps is used.

The unconditional variance can be calculated as (Engle R. , 2001):

$$\sigma^2 = \frac{\omega}{1 - \alpha_1 - \alpha_2 - \cdots - \alpha_q - \beta_1 - \beta_2 - \cdots - \beta_p}$$

The model takes the recursive form:

$$\sigma_{t+1}^2 = \omega + \alpha_1 r_{t+1-1}^2 + \alpha_2 r_{t+1-2}^2 + \cdots + \alpha_q r_{t+1-q}^2 + \beta_1 \sigma_{t+1-1}^2 + \beta_2 \sigma_{t+1-2}^2 + \cdots + \beta_p \sigma_{t+1-p}^2$$

A special feature of the GARCH model is that it acts as an infinite order ARCH model (ARCH (∞)) making it preferable by many since the parameters to be estimated are much fewer and yield similar results.

The GARCH (1,1), which is most commonly used on financial time-series, is shown as:

$$\sigma_{t+1}^2 = \omega + \alpha r_t^2 + \beta \sigma_t^2$$

Estimation of parameters for GARCH (p, q) model is same as with ARCH (q) model (Bollerslev, 1986). Using the same likelihood function shown in section 3.2.

The GARCH model is an important part of this research, where the preferred GARCH (p, q) model is found by model selection process introduced in section 5.1 and evaluated in section 5.6. This leads to backtesting a set of evaluation methods for Value-at-Risk. Defined and explained in Chapter 4.

4 Backtests

Backtesting is an important part of the Value-at-Risk model evaluation process. It takes the values that have been calculated by the selected model and tests if the model has been accurate enough to justify its use on a given portfolio.

The tests are often put in two sets of groups, unconditional coverage and independence. Unconditional coverage counts the frequencies of violations. A violation is when the actual return exceeds the Value-at-Risk number for that date. If the Value-at-Risk level were 5%, from a sample of 100 Value-at-Risk estimates against actual return observations, it would expect five of them to be violations. The test for independence makes assumption for the observations to be independent of each other, based on that, when a violation happens for two or more consecutive days there might be a problem with the model.

Six backtests will be described in the following sections.

4.1 Basel Traffic Light Approach

The Basel committee designed a test that banks are regulated to use, it is the Basel Traffic Light approach, which is an unconditional test. It is set to account for 1% Value-at-Risk with a holding period of 10 days and an observation period of one year (250 days) for internal processes of banks. The test can be done with other values as well. The approach is based on the cumulative binomial distribution and can be shown as:

$$P[T \le v] = \sum_{t=0}^{v} \binom{n}{t} p^t (1-p)^{n-t}, \quad \binom{n}{t} = \frac{n!}{t!\,(n-t)!}$$

Where $P[T \le v]$ represents the probability of violations to be less than or equal to violations observed, n represents the number of trials (observation period), v represents the observed violations, t represents cumulative probabilities up to the value of t and p represents the probability of failure for each trial (p = 0.05 for 5% Value-at-Risk).

The Basel Committee then set out zones that models could fit in, three zones in all: green zone, yellow zone and red zone. The cumulative probability gives the probability of a correct model showing fewer or the same amount of violations as have been observed. When the cumulative probabilities range up to 95% the model is in the green zone, the yellow zone

begins when the cumulative probability exceeds 95% and the red zone begins when the cumulative probability exceeds 99,99%. A model in the green zone is considered to be sound. If it is in the yellow zone, some investigation is required and probably changes. The red zone makes the model insufficient and change of operations is needed (Nieppola, 2009). There is one fault with this model, even though the financial regulators might not care, since it is made from their perspective. It fails to take into account if violations are too few. It does not matter what the observation period is or what the level of confidence is set to, the Basel Traffic Light approach always displays the model in the green zone when there are zero violations up to the yellow zone threshold level. A model with zero violations is most likely overestimating Value-at-Risk and that should alarm the risk manager.

4.2 Kupiec Test

Kupiec suggested a test in 1995. It is an unconditional coverage test and it measures whether the number of violations is consistent with the chosen confidence level. The number of exceptions follows the binomial distribution and it is a hypothesis test, where the null-hypothesis is:

$$H_0: p = \hat{p} = \frac{x}{T}$$

Where p represents the violation rate from the chosen Value-at-Risk level, $\hat{p}$ represents the observed violation rate and x represents the number of observed violations. T is the number of observations. It is conducted as a likelihood-ratio (LR) test and can be formulated as:

$$LR_{UC} = 2 \ln \left(\frac{\hat{p}^x (1 - \hat{p})^{T-x}}{p^x (1 - p)^{T-x}} \right), \qquad \hat{p} = \frac{x}{T}$$

The test is asymptotically chi-square distributed with one degree of freedom, for a confidence level of 95% the critical value is 3,84. If the LR_{UC} statistic exceeds the critical value, the null hypothesis is rejected and therefore the model seems inaccurate (Néri, 2005).

4.3 Christoffersen Test

Christoffersen test is an independence test and explained in Christoffersen & Pelletier (2004). It is a likelihood-ratio test like the Kupiec test. It collects data for violations and if they happen subsequently. The results from this data collection are used to create the test results. The observations can have two values as shown below.

$$I_t = \begin{cases} 1, & if\ violation\ occurs \\ 0, & if\ no\ violation\ occurs \end{cases}$$

A violation occurs when the actual returns exceed the projected Value-at-Risk number $(r_t < -VaR_t)$. Those results are categorized in the following manner: if there was a violation followed by non violation, a non violation followed by a violation, a non violation followed by a non violation and a violation followed by a violation, as shown below:

	$I_{t-1} = 0$	$I_{t-1} = 1$	
$I_t = 0$	n_{00}	n_{10}	$n_{00} + n_{10}$
$I_t = 1$	n_{01}	n_{11}	$n_{01} + n_{11}$
	$n_{00} + n_{01}$	$n_{10} + n_{11}$	N

From the table above, values for π_0 and π_1 are calculated where they represent the sample probabilities of a violation occurring conditional on if there was or was not a violation in the previous day. π is then also calculated and represents the violation rate, as shown below:

$$\pi_0 = \frac{n_{01}}{n_{00} + n_{01}}, \quad \pi_1 = \frac{n_{11}}{n_{10} + n_{11}} \quad and \quad \pi = \frac{n_{01} + n_{11}}{n_{00} + n_{01} + n_{10} + n_{11}}$$

The likelihood-ratio test is calculated under the null-hypothesis that all violations should be independent of each other. The null-hypothesis takes the form:

$$H_0: \pi_0 = \pi_1$$

The likelihood statistics is calculated as:

$$LR_{ind} = -2 \ln\left(\frac{(1 - \pi)^{n_{00}+n_{10}}\pi^{n_{01}+n_{11}}}{(1 - \pi_0)^{n_{00}}\pi_0^{n_{01}}(1 - \pi_1)^{n_{10}}\pi_1^{n_{11}}}\right)$$

Which is asymptotically chi-square distributed with one degree of freedom, making the critical value at 95% confidence interval 3,84. If the test statistics is above that value the null-hypothesis is rejected and the model is thought to have independence problems. A test statistic under the critical rate assumes the model to be sound (Nieppola, 2009).

4.4 Test for Conditional Coverage

Christoffersen created a joint test consisting of the previously described tests for unconditional coverage and for independence. The test statistic is as follows:

$$LR_{cc} = LR_{UC} + LR_{ind}$$

Kupiec test for unconditional coverage and Christoffersen test for independence are summed to get the test results for conditional coverage. The test for conditional coverage is also asymptotically chi-square distributed and has two degrees of freedom. The critical value at 95% confidence level is 5,99. Making the model pass the test if its test statistic is under the critical value (Christoffersen & Pelletier, 2004).

4.5 Pearson's Q Test

Although the previously discussed backtests can all be considered to be adequate, there is one thing they are lacking. They don't test simultaneously the predicted Value-at-Risk for multiple confidence levels, that can be done with the Pearson's Q test for goodness of fit. To run the test, first the chosen confidence levels that are tested simultaneously should be specified, and separated into bins. If it is for instance decided to test Value-at-Risk at 95% and 99% confidence levels, the bins should be [0.00, 0.01], [0.01, 0.05] and [0.05, 1.00]. Where all the violations over the 99% confidence level are in the [0.00, 0.01] bin, violations in between the 99% and 95% confidence levels are in the [0.01, 0.05] bin and the rest (returns that are lower than the predicted 5% Value-at-Risk loss) are in the [0.05, 1.00] bin. The Q test is then computed according to the following formula:

$$Q = \sum_{i=1}^{k} \frac{\left(N_{(l_i,u_i)} - N(u_i - l_i)\right)^2}{N(u_i - l_i)}$$

Where $N_{(l_i,u_i)}$ represents the number of observations in the ith bin and N represents the total number of observations in the whole test. u_i and l_i represent the upper and lower bounds of each bin. The test is asymptotically chi-square distributed with k-1 degrees of freedom. If the test would be conducted on the 1% and 5% Value-at-Risk levels the critical value will be 5,99 (given 95% confidence) and if the test results are above that value, the model is rejected (Campbell, 2005).

4.6   Lopez's Loss Function

Lopez proposed a magnitude loss function in 1999. It is neither an independence test nor a test for unconditional coverage. Looking at the losses made when there is a violation and by how much the losses exceed the projected Value-at-Risk number. Every observation gets a value, similar to the independence test where every violation got a value of one otherwise zero. For observations with no violation the value is set at zero, alternatively when there is a violation the value is one plus the magnitude of the exceedance squared. As shown below:

$$L(VaR_t(\alpha), r_t) = \begin{cases} 1 + \left(r_t + VaR_t(\alpha)\right)^2, & if\ violation\ occurs \\ 0, & if\ no\ violation\ occurs \end{cases}$$

Where $VaR_t(\alpha)$ represents the projected Value-at-Risk number for time t, α represents the chosen confidence level and r_t is the observed return at time t.

To get a single value, the results are calculated using the following formula:

$$\hat{L} = \frac{1}{T} \sum_{t=1}^{T} L(VaR_t(\alpha), r_t)$$

There are some shortfalls with the Lopez Loss function, since the actual distribution of the returns is unknown. It is hard to know how much exceedance is actually appropriate and how much leads to a rejection of the model. The Loss function is good though for comparing different models and to see if there are some extraordinary exceedances prevailing (Lopez, 1999).

5 Empirical Research

This research will use the S&P500 Total Return index, ranging from 20.07.01-19.07.11 making a total of 10 years and 2512 observations. Returns in the data that showed zero were taken out (holidays). Additionally, one week has been removed because of closed markets. It is the week right after the September 11th attacks on the World Trade Center in the United States.

The returns are calculated continuously, following the given description:

$$r_t = \ln\left(\frac{S_t}{S_{t-1}}\right)$$

Where r_t represents the returns at time t and S represents the index rate at the given time.

The preferred GARCH (p, q) model will be found based on the model selection process. The most widely used model GARCH (1, 1) will be evaluated in this research regardless of what model will be selected. The models will be compared to the Riskmetrics model with a lambda value of 0,94.

Calculations will be in Excel and VBA. Additionally MATLAB mathematical and statistical software will be utilized. The VBA codes are available in appendix A.

The next section will go into the model selection process, which determines how different GARCH models perform against each other.

5.1 Model Selection Process

There will be two types of tests performed on the returns series to see which GARCH model is considered most fitting on the time-series. The tests are Akaike Information Criterion (AIC) and Schwartz Bayesian Criterion (SBC). The preferred model will show the lowest values of these criterions. The main difference between the criterions is that SBC penalizes models more when number of parameters increases. The AIC is calculated as:

$$AIC = -2L\left(\sum \hat{a}_t^2\right) + 2K$$

Where K is the number of free parameters to be estimated in the model and $L(\sum \hat{a}_t^2)$ is the likelihood function. For a GARCH (1, 1) model the K would equal 3 since omega, beta and alpha are estimated. The SBC criterion is similar and can be shown as:

$$SBC = -2L\left(\sum \hat{a}_t^2\right) + K \ln(n)$$

Where n represents the number of residuals in the model, the data has 2512. The AIC and SBC were calculated for GARCH model with the least possible parameters and up to GARCH (3, 3) the results are shown in the table below:

AIC and SBC statistics for GARCH models

Model	AIC	SBC
GARCH (0,1)	-14776,786	-14759,299
GARCH (1,1)	-15834,083	-15810,768
GARCH (2,1)	-15832,083	-15802,939
GARCH (1,2)	-15862,874	-15833,730
GARCH (2,2)	-15860,873	-15825,900
GARCH (3,1)	-15830,085	-15795,112
GARCH (3,2)	-15858,874	-15818,072
GARCH (1,3)	-15861,634	-15826,661
GARCH (2,3)	-15859,721	-15818,919
GARCH (3,3)	-15857,721	-15811,090

Table 2: AIC and SBC statistics for various GARCH models[i]

The strongest model based on both the AIC and SBC is the GARCH (1, 2) model, which can be expressed as:

$$\sigma_{t+1}^2 = \omega + \alpha_1 r_t^2 + \alpha_2 r_{t-1}^2 + \beta \sigma_t^2$$

To further examine the GARCH (1, 2), parameter estimations need to be calculated with regression statistics, the results are shown in table 3:

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[i] The mathematical and statistical software MATLAB was used to attain the AIC and SBC statistics.

Estimated parameters for the GARCH (1, 2) model

Parameter	Value	Standard Error	T-Statistic	P-value
ω	1,8984e-006	3,2028e-007	5,9272	0,0000***
α_1	0	0,015413	0,0000	1,0000
α_2	0,1035	0,018373	5,6331	0,0000***
β	0,885	0,012099	72,9396	0,0000***
Likelihood	7936,4370398			

Table 3: Estimated parameters for the GARCH (1, 2) model[i]

Note: * p < .05; ** p < .01; *** p < .001

It is interesting to see that the value for α_1 is zero and the T-statistic is zero making the p-value go up to 1, accepting the null-hypothesis for the estimate to be zero and therefore not contributing to the model, which is quite surprising, one would think that the first lagged residual will have some explanatory power. When the estimated parameters are inputted the model looks as follows:

$$\sigma_{t+1}^2 = 0,0000018984 + 0 \times r_t^2 + 0,1035 \times r_{t-1}^2 + 0,885 \times \sigma_t^2$$

Here the parameter values are shown in the model, to explain their functions. ω is the conditional variance constant, the conditional variance being what the GARCH model is measuring. α_1 is the coefficient related to the lagged residuals squared, α_1 would then represent the weight that is put on the latest observation squared. α_2 represents the weight that is put on the second latest observation squared and so forth. β represents the coefficient related to the lagged conditional variance. Its value represents the weight that is put on the latest lagged conditional variance. Since there is only one beta to be estimated in the GARCH (1, 2) model it is given the parameter β. If there would be more than one beta to be estimated, as is in GARCH (p > 1, q) models. The coefficient or weight for the first lagged conditional variance would get the parameter β_1, the second β_2 and so forth.

Since this model can be explained as GARCH (1, 1) model with an extra ARCH parameter the GARCH (1, 1) model is nested within it. The GARCH (1, 1) model is usually the preferred model for financial time-series making it interesting to compare those two by the

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[i] The mathematical and statistical software MATLAB was used to estimate the parameters for the GARCH (1, 2), restricted GARCH (1, 2) and the GARCH (1, 1) models.

Likelihood Ratio Test. In order to do that, first the parameter estimation for the GARCH (1, 1) is calculated, primarily to get the likelihood. Shown in table 4.

Estimated parameters for the GARCH (1, 1) model

Parameter	Value	Standard Error	T-Statistic	P-value
ω	1,2207e-006	2,1058e-007	2,8115	0,0050**
α	0,075626	0,0079575	9,5037	0,0000***
β	0,9155	0,008567	106,8640	0,0000***
Likelihood	7921,0415508			

Table 4: Estimated parameters for the GARCH (1, 1) model

Note: * p < .05; ** p < .01; *** p < .001

Based on the estimation results, all the p-values are very low, rejecting the null-hypothesis for the estimations to be zero, making the model a good fit. The model can be shown as:

$$\sigma_{t+1}^2 = 0{,}0000012207 + 0{,}075626 \times r_t^2 + 0{,}9155 \times \sigma_t^2$$

This model has already been compared to the GARCH (1, 2) model with the AIC and SBC where the GARCH (1, 2) resulted in lower values, making it seem to be a better fit. To further test the models against each other the Likelihood Ratio Test is conducted, it can be shown as:

$$2[l_U - l_R] \approx x^2(m)$$

Where l_u is the value of the likelihood function for the unrestricted model, l_R is the value of the likelihood function for the restricted model and m is the number of restrictions put on the unrestricted model. The null-hypothesis is that the restricted model is the "true" model. The test is asymptotically chis-square distributed with m degrees of freedom (Hamilton, 1994).

Plotting the likelihood results, shown in tables 3 and 4, from the unrestricted (GARCH (1, 1)) and the restricted (GARCH (1, 2)) model. We get a test result of 30,8 significantly rejecting null-hypothesis of GARCH (1, 1) model being the "true" model and therefore strengthening results from the AIC and SBC. The critical rate at 95% confidence interval was 3,814 given one degree of freedom, since the restricted model reduced the unrestricted model by one parameter.

From the estimation of the GARCH (1, 2) model in table 3 it was interesting to see that the estimated parameter for the weights on the first lagged residual was zero and very significantly so, according to its p-value. Because of that, a new estimation was run where

this parameter was dropped that could lead to a stronger model. The new model will be called restricted GARCH (1, 2). The regressions statistics, AIC and SBC for the new model are shown below:

Estimated parameters for the restricted GARCH (1, 2) model

Parameter	Value	Standard Error	T-Statistic	P-value
ω	1,9054e-006	3,197e-007	5,9600	0,0000***
α_2	0,10339	0,011168	9,2581	0,0000***
β	0,88238	0,012059	73,1692	0,0000***
Likelihood	7936,4377438			
AIC	-15864,875			
SBC	-15841,560			

Table 5: Estimated parameters for the restricted GARCH (1, 2) model

Note: * p < .05; ** p < .01; *** p < .001

All the estimated parameters are significant with more than 99,9% confidence.

Both the AIC and SBC penalize models more when the number of free parameters increases, that contributes to why they show better results for the restricted GARCH (1, 2) model than with unrestricted. According to the AIC and SBC values the restricted GARCH (1, 2) model is more fitting to the sample than the GARCH (1, 2) model, when all the parameter estimates are inputted, the model looks as follows:

$$\sigma_{t+1}^2 = 0,0000019054 + 0,10339 \times r_{t-1}^2 + 0,88238 \times \sigma_t^2$$

The most interesting thing about this model specification is that the only squared residual that is weighted to attain the conditional variance, is the second lagged residual squared. The most recent one is left out of the equation.

To further test this model against the GARCH (1, 2) the Likelihood Ratio Test was conducted where this is the restricted model and GARCH (1, 2) is the unrestricted model. It is interesting to compare the likelihoods, for the restricted GARCH (1, 2) the likelihood is higher than for the GARCH (1, 2) model. Usually the likelihood of the model that is nested within the other has a lower likelihood but here it is not the case. The Likelihood ratio test was conducted on these two models giving a test result of -0,0014; with a 95% critical value of 3,814 the null-hypothesis is very significantly accepted for the restricted model to be the "true" model.

The restricted GARCH (1, 2) model will therefore be selected. As previously mentioned, even though the GARCH (1, 1) model was not chosen in the model selection process it will be used and evaluated against the restricted GARCH (1, 2), and the Riskmetrics model.

In the next two sections the Ljung-Box Q test and the test for ARCH effects introduced by Engle will be conducted on the residuals of the GARCH (1, 1) and the restricted GARCH (1, 2) models.

5.2   Ljung-Box Q Test

The Ljung-Box Q test has already been performed on the original sample, it is interesting to see how the autocorrelations with the residuals of a given model perform under the same test. This hypothesis test, which tests if there is any autocorrelation up to a predefined lag was performed on the GARCH (1, 1) and the restricted GARCH (1, 2) models. The null-hypothesis states that there is no autocorrelation in the residuals and if the null-hypothesis is rejected, if at least one autocorrelation is not zero (Ljung & Box, 1978). The formula used is the same as before:

$$Q_k = n(n+2) \sum_{j=1}^{k} r_j^2 \, / \, (n-j)$$

Where r_j^2 represents autocorrelations for the residuals at lag j. The test is asymptotically chi-square distributed with k degrees of freedom. Table 6 shows the results for the residuals from the GARCH (1, 1) model for the same lags as was used on the returns in section 6 (lags 6, 7, 8, 9 and 10).

Ljung-Box Q test for the GARCH (1, 1) model

k (lag)	Q_k	$X^2_{(1-0,05), k}$	p-value
6	13,539	12,592	0,0352*
7	15,766	14,067	0,0273*
8	17,662	15,507	0,0239*
9	18,447	16,919	0,0303*
10	29,945	18,307	0,0027**

Table 6: Ljung-Box Q test for the GARCH (1, 1) model[i]

Note: * p < .05; ** p < .01; *** p < .001

The Ljung-Box Q test rejects the null-hypothesis for no autocorrelation with 95% confidence at lags 6, 7, 8 and 9. It furthermore rejects the null-hypothesis with 99% confidence at lag 10. These results confirm that there is evidence for autocorrelation in the residuals of the GARCH (1, 1) model.

The test was performed on the residuals from the restricted GARCH (1, 2) model using the same lags, the results are shown in table 6.

Ljung-Box Q test for the restricted GARCH (1, 2) model

k (lag)	Q_k	$X^2_{(1-0,05), k}$	p-value
6	2,706	12,592	0,8448
7	4,081	14,067	0,7704
8	4,638	15,507	0,7955
9	5,293	16,919	0,8080
10	9,560	18,307	0,4799

Table 7: Ljung-Box Q test for the restricted GARCH (1, 2) model

Results for the restricted GARCH (1, 2) model show more promising results by accepting the null-hypothesis at all of the chosen lags with high significance.

5.3   ARCH Test

The ARCH test checks if the residuals defined as u_t exhibit time-varying heteroskedasticity. The test is based on the Lagrange multiplier principle, where first the regression of the model

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[i] The mathematical and statistical software MATLAB was used to attain the Ljung-Box Q results for the GARCH (1, 1) and the restricted GARCH (1, 2) models

is calculated, in order to get the residuals. When the model specification is complete and the residuals attained, $\hat{u}_t^2$ is regressed on a constant and m of its own lagged values:

$$\hat{u}_t^2 = \omega + \alpha_1\hat{u}_{t-1}^2 + \alpha_2\hat{u}_{t-2}^2 + \cdots + \alpha_m^2\hat{u}_{t-m}^2 + e_t$$

For t = 1, 2, …, T. The statistics can be reviewed under the chi-square distribution with m degrees of freedom with the null-hypothesis being that there is no heteroskedasticity present (Hamilton, 1994). There is a problem with choosing the m value that shows the most powerful test estimate. However often the ARCH tests in literature show test statistics up to around 15 for m. The ARCH test results for m as even numbers starting from 2 and up to 20 for the GARCH (1, 1) model are shown in table 8.

ARCH test for the GARCH (1, 1) model			
m (lag)	Statistic	$X^2_{(1-0,05),\,m}$	p-value
2	10,901	5,992	0,0043**
4	12,527	9,488	0,0138*
6	13,639	12,592	0,0339*
8	17,912	15,507	0,0219*
10	27,836	18,307	0,0019**
12	28,881	21,026	0,0041**
14	29,040	23,685	0,0103*
16	30,772	26,296	0,0144*
18	30,922	28,869	0,0294*
20	32,269	31,410	0,0405*

Table 8: ARCH test for the GARCH (1, 1) model[i]

Note: * p < .05; ** p < .01; *** p < .001

Based on the ARCH test, the GARCH (1, 1) model does not eliminate heteroskedasticity in its residuals. For all chosen lag lengths the null hypothesis is rejected with at least 95% confidence interval and lag lengths 2, 10, 12 reject the null hypothesis with 99% confidence interval.

The ARCH test results for the restricted GARCH (1, 2) model are shown in table 9.

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[i] The mathematical and statistical software MATLAB was used to attain the ARCH test results for the GARCH (1, 1) and the restricted GARCH (1, 2) models

ARCH test for the restricted GARCH (1, 2) model

m (lag)	Statistic	$X^2_{(1-0,05),\ m}$	p-value
2	1,653	5,992	0,4377
4	1,720	9,488	0,7871
6	2,705	12,592	0,8448
8	4,603	15,507	0,7991
10	9,906	18,307	0,4488
12	11,092	21,026	0,5210
14	11,371	23,685	0,6567
16	12,530	26,296	0,7068
18	12,545	28,869	0,8179
20	14,309	31,410	0,8145

Table 9: ARCH test for the restricted GARCH (1, 2) model

The null-hypothesis is accepted at all lag levels, so the restricted GARCH (1, 2) model does not show any signs of heterskedasticity in its residuals.

These results, from both the Ljung-Box Q test and the ARCH test indicate that the restricted GARCH (1, 2) model is a better fit for the time-series, than the GARCH (1, 1) model. Its residuals show no evident sign of autocorrelation or heterskedasticity whereas the GARCH (1, 1) model does so in both cases.

5.4   Parameter Estimation[i]

The estimation of the parameters for the GARCH (1, 1) model and the GARCH (1, 2) models used a rolling parameter estimation window. Throughout the returns series, for each day there will be a new value estimated for the parameters based on window size, which "rolls" up the time-series. There is a VBA algorithm that calculates these estimates automatically. It is called the Nelder-Mead algorithm, which minimizes a function containing more than one parameter to be estimated. Since the likelihood function for the GARCH models needs to be maximized, the objective function in the algorithm was defined as the negative of that

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[i] Parameter estimation for the Riskmetrics model and all rolling window sizes for the GARCH models can be found in Excel worksheets in the sub-folder "variances" on the enclosed CD. It is recommended to either set calculation preferences in Excel to "manual" before opening the files or to disable macros when opening the files. Otherwise Excel might start recalculating the whole worksheet, which might take a few hours.

function. The equality constraints for the GARCH models were added. The VBA code for this algorithm is from the book Option Pricing Models & Volatility (Rouah & Vainberg, 2007).

The main problem with using rolling parameter estimation is to find the optimal window size. It might vary depending on the state of the time-series since financial time-series exhibit volatility clustering. If a drastic change in volatility happens and before that volatility had been near constant, then the model estimation could rely too heavily on how the volatilities had been in the previous period.

There is an idea to use the ICSS algorithm to find the volatility change points and then adjust the model accordingly for every new point. The ICSS algorithm is applied on a small sub-sample at the beginning of the S&P500 Total Return index sample and run several times. For each run the range is incremented by one day, until a new change point is discovered. For each change point, the date for which the range has gone up to is marked and checked how far away that date was from the change point. That date will then represent the time when a new change point is found in practice by applying the ICSS algorithm. If it is close enough to the change point, the model can be adapted accordingly. Otherwise too much time has passed for an adjustment of the model to adequately cover the change of variance. This test found that the time between the change points and the dates they were realized on could range from ≈ 30 days up to ≈ 300 days. Therefore, it cannot be assumed that a change point will be realized around the time it takes place, by the use of the ICSS algorithm. That's why no adjustments will be made to the models.

Because of how difficult it is to find the optimal window size for a rolling GARCH model, the models were estimated with multiple window sizes: 250, 500, 750, 1000, 1250 and 1500. Resulting in six different estimates for each of the GARCH models. The Riskmetrics model was set with a lambda value of 0,94 making consideration for window sizes unnecessary.

5.5   Backtest Samples

For backtesting purposes the data is split into two sub-samples, each of them is backtested along with the full sample. The full sample of the S&P500 Total Return index, ranging from 20.07.10 to 19.07.11 giving 2512 observations in total. The financial crisis of late 2008 is in the sample, this was a good opportunity to test how the different models performed in the midst of the crisis, to see how the models perform in turbulent times. Since the ICSS

algorithm had already been applied on the Full sample, there is information present on the variance change points in the Full sample. Let the change points be defined as with the ICSS algorithm, a total of seven change points in the Full sample, so the first change point would be defined cp_1 second cp_2 and so forth up to the last cp_7. Furthermore, cp_0 is the start of the sample and cp_8 is the end of the sample.

By examining the change points it is evident that five of them are relatively close to each other around the time of the financial crisis, these points range from cp_3 to cp_7. The first cp_3 and the last cp_7 mark the boundaries of the sample used for backtesting the financial crisis. That sample will start from cp_3 and go up to cp_7 minus one day. Giving a sample range from 09.07.07 to 03.09.10, a total of 797 observations.

The last sample used for backtesting purposes will be the part that did not show as turbulent times and could be considered normal. This sample starts from cp_0 and ends at the start of the crisis sample at cp_3 minus one day. This sample ranges from 20.07.01 to 06.07.07 giving a total of 1496 observations.

The three samples will be referred to as "Full", "With Crisis" and "Before Crisis" from now on. For illustration of where the backtesting samples fit within the full sample of the S&P500 Total Return index, figure 10 has been presented below.
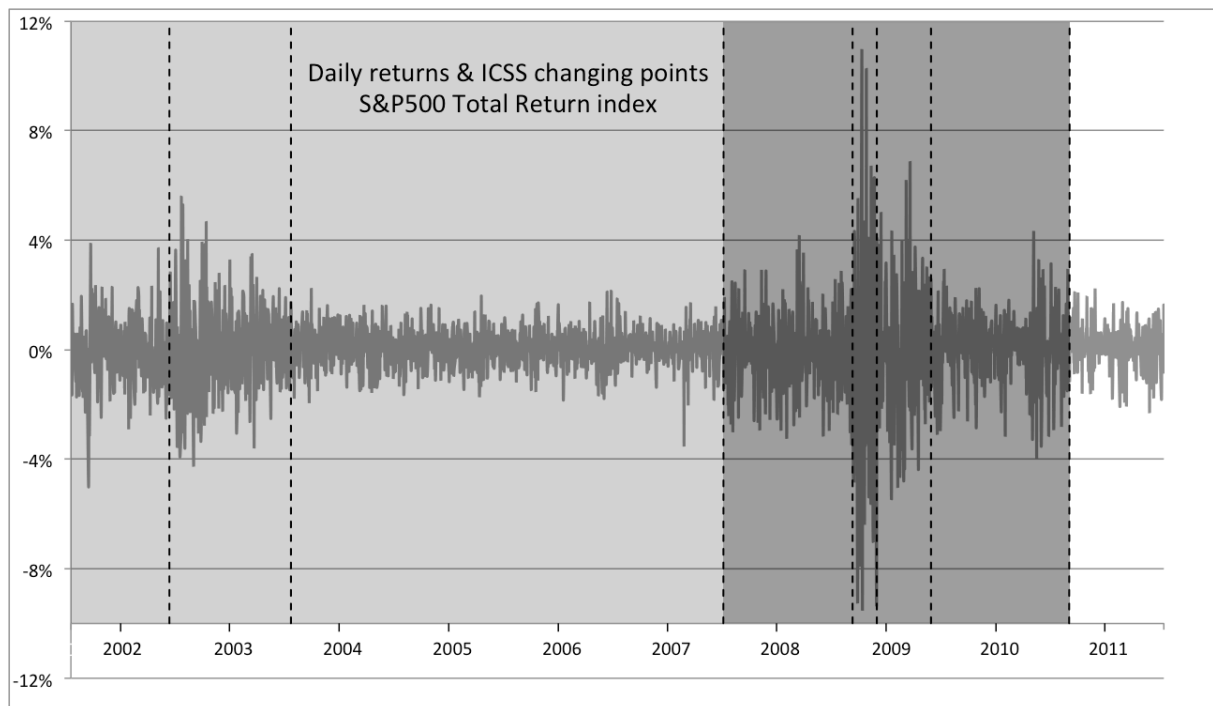

Figure 10: Sample ranges used for baktesting, S&P500 TR index (20.7.1-19.7.11)

It is the graph shown in section 3.1, with the areas for each sample dimmed differently. The darkest part represents the With Crisis sample, the slightly dimmed part represents the Before Crisis sample and all areas (dark grey, light grey and white) put together represents the Full sample.

5.6 Backtest Results[i]

The backtests were performed on the various models used in this research (on 5% and 1% Value-at-Risk levels). First the different window sizes of the GARCH (1, 1) and the GARCH (1, 2) model, were compared to each other and then the window size for each model that performed best according to the backtests was chosen. This was done for each of the models to have only one set of estimations for comparison. Among the GARCH (1, 1) the window size that performed best was of 1500 and for the restricted GARCH (1, 2) estimations it was 500. In this section the results will therefore be shown for the rolling GARCH (1, 1) with a window size of 1500, the rolling restricted GARCH (1, 2) with a window size of 500 and the Riskmetrics model.

Statistics from all tests except for the Basel Traffic Light approach are shown. That is because of similarities between the Basel Traffic Light approach and the test for unconditional coverage and the fact that the test for unconditional coverage does penalize models with too few violations where the Basel Traffic Light does not. The results from the tests are shown based on what part of the sample is tested for, starting with the Full sample. Furthermore, the results from the Lopez's loss function are presented in section 5.6.4 where the results are accompanied with graphs.

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[i] All backtests can be found in Excel worksheets in the sub-folder "backtests" on the enclosed CD.

5.6.1 Full

Table 10 shows the test results for the three models when the Full sample is used.

Backtest results for the Full sample

	Rolling GARCH (1, 1)		Rolling GARCH (1, 2)[i]		Riskmetrics	
Observations	2512					
VaR level	5%	1%	5%	1%	5%	1%
Violations	133	44	137	47	146	48
Failure rate	5,295%	1,752%	5,454%	1,871%	5,812%	1,911%
LR_{uc}	0,45064	11,70992	1,05930	15,32250	3,32216	16,61467
LR_{ind}	0,18002	1,56896	0,03396	1,10093	0,03188	0,00755
LR_{cc}	0,63065	13,27889	1,09326	16,42342	3,35404	16,62223
Pearson's Q test	15,52462		20,20541		21,07534	

Table 10: Backtest results for the Full sample

By looking at the number of violations and failure rates for each graph for the 1% and 5% Value-at-Risk predictions. The rolling GARCH (1, 1) has the failure rate closest to 5% and 1% with 133 and 44 violations, a failure rate of 5,295% and 1,752% respectively.

The test for unconditional coverage is lowest for the rolling GARCH (1, 1) with a value of 0,45064 at the 5% Value-at-Risk level and 11,70992 at the 1% level. The 95% confidence critical rate is 3,842 so the model cannot be rejected at the 5% Value-at-Risk level, the test results are also under the critical rate for the 5% Value-at-Risk level. However all the models fail to fall under the critical level at the 1% Value-at-Risk level.

In the test for independence, the Riskmetrics model shows the lowest value at the 5% and the 1% Value-at-Risk levels, showing values of 0,03188 and 0,00755 respectively. The critical level is the same as with the test for unconditional coverage and it scores well under that rate at both 1% and 5% Value-at-Risk levels. Both the other models do also show low values for the test for independence and fall under the critical rate.

In the test for conditional coverage, the rolling GARCH (1, 1) gives the best results for both Value-at-Risk levels, 0,63065 for the 5% Value-at-Risk level and 13,27889 1% making it fall under the critical value of 5,992 at the 5% Value-at-Risk level but goes over it at the 1%

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[i] In this table and the following tables the rolling restricted GARCH (1, 2) model will be named Rolling GARCH (1, 2).

Value-at-Risk level and is therefore rejected. The other two models are rejected at the 1% Value-at-Risk level and pass the test at the 5% Value-at-Risk level.

The model showing the best results for the Pearson's Q test, although not sufficient with 5,992 as the critical rate, is the rolling GARCH (1, 1) model with a value of 15,5246.

Based on these results, the model that performed best for the Full sample was the rolling GARCH (1, 1) model, performing well at the 5% Value-at-Risk level. However it did not perform as well at the 1% Value-at-Risk level, seeing rejection in the test for unconditional and conditional coverage, and it was rejected in the Pearson's Q test. Despite that it showed the best results at the 1% Value-at-Risk level. The other two models did pass all the tests at the 5% Value-at-Risk level.

5.6.2 Before Crisis

Table 11 shows the test results for the three models when the Before Crisis sample is used.

Backtest results for the Before Crisis sample

	Rolling GARCH (1, 1)		Rolling GARCH (1, 2)		Riskmetrics	
Observations	1496					
VaR level	5%	1%	5%	1%	5%	1%
Violations	62	11	64	21	75	19
Failure rate	4,144%	0,740%	4,280%	1,404%	5,01%	1,270%
LR_{uc}	2,44218	1,16592	1,72221	2,18865	0,00056	1,01527
LR_{ind}	2,00209	0,16296	1,67103	4,53910	2,52788	0,48884
LR_{cc}	4,44427	1,32888	3,39324	6,72775	2,52844	1,50411
Pearson's Q test	2,46943		7,25975		1.33746	

Table 11: Backtest results for the Before Crisis sample

The numbers of violations and failure rates seem to be closest to the 5% and 1% levels for the Riskmetrics model. Giving a failure rate of 5,01% with 75 violations and 1,270% with 19 violations for the 5% and 1% Value-at-Risk levels respectively. The rolling GARCH (1, 1) model shows a failure rate of 0,740% at the 1% Value-at-Risk level, which has a similar closeness to the 1% as the Riskmetrics.

The test for unconditional coverage gives the best scores for the Riskmetrics at both Value-at-Risk levels, 0,00056 for the 5% level and 1,01527 for the 1% level. All the models pass the test for unconditional coverage at both Value-at-Risk levels.

In the test for independence the rolling GARCH (1, 1) shows the best results, giving a value of 2,00209 for the 5% Value-at-Risk level and 0,16296 for the 1% Value-at-Risk level. The other models pass the test for independence except for the rolling restricted GARCH (1, 2) at the 1% Value-at-Risk level, it measures 4,53910 exceeding the critical rate of 3,842.

The test for conditional coverage shows the best results for the Riskmetrics model at both levels, giving a value of 2,52844 and 1,50411 for the 5% and 1% Value-at-Risk levels respectively. The other models pass the test for conditional coverage at all levels except for the rolling restricted GARCH (1, 2) model at the 1% Value-at-Risk level, showing a value of 6,72775, which exceeds the critical level set at 5,992. This is largely because of the poor test statistics from the test for independence.

The Pearson's Q test shows the lowest value for the Riskmetrics model, being 1,33746. Passing the test where the critical rate is set at 5,992. The Rolling GARCH (1, 1) did pass the test, however the rolling restricted GARCH (1, 2) did not pass the Pearson's Q test with a value of 7,25975.

Based on the results, the model that performed best for the Before Crisis sample was the Riskmetrics model. The models passed all tests, except for the rolling restricted GARCH (1, 2) which did not pass test for independence and conditional coverage for Value-at-Risk level of 1%. Additionally it faced rejection in Pearson's Q test.

5.6.3   With Crisis

Table 12 shows the test results for the three models when the With Crisis sample is used.

Backtest results for the With Crisis sample

	Rolling GARCH (1, 1)		Rolling GARCH (1, 2)		Riskmetrics	
Observations	797					
VaR level	5%	1%	5%	1%	5%	1%
Violations	60	30	62	23	57	26
Failure rate	7,528%	3,764%	7,779%	2,886%	7,152%	3,262%
LR$_{uc}$	9,34773	36,09166	11,16388	18,97938	6,89532	25,84060
LR$_{ind}$	4,47312	2,34740	2,41118	1,36713	3,70733	0,02755
LR$_{cc}$	13,82085	38,43907	13,57506	20,34651	10,60264	25,86815
Pearson's Q test	61,54058		30,58205		41,20082	

Table 12: Backtest results for the With Crisis sample

By looking at the number of violations and failure rates the Riskmetrics model has a failure rate closest to 5% at the 5% Value-at-Risk level with a value of 7,152% and 57 violations. For the 1% Value-at-Risk level the rolling restricted GARCH (1, 2) exhibits a failure rate closest to 1% giving a value of 2,886% and 23 violations. All the failure rates are relatively high, suggesting that all the models have projected Value-at-Risk measures that underestimate the risk at hand.

In the test for unconditional coverage all the models are in the rejection region while the Riskmetrics model shows the lowest value for the 5% Value-at-Risk level being 6,89532 and the rolling restricted GARCH (1, 2) shows the lowest value for the 1% Value-at-Risk level, being 18,97938.

In the test for independence the rolling restricted GARCH (1, 2) shows the best result for the 5% Value-at-Risk level, a value of 2,41118 and the Riskmetrics model shows the best test result for the 1% Value-at-Risk level, at 0,02755. All the models pass the test for independence except for the rolling GARCH (1, 1) at the 5% Value-at-Risk level, showing a test result of 4,47312.

The test for conditional coverage rejects all the models at both levels. With the Riskmetrics model showing the lowest value for the 5% Value-at-Risk level at 10,60264 and the rolling

restricted GARCH (1, 2) shows the lowest value for the 1% Value-at-Risk level, which is 13,57506.

The Pearson's Q test rejects all the models where the rolling restricted GARCH (1, 2) shows the lowest value of 30,58205.

From these results, none of the models managed to mitigate risk for the period with the financial crisis. This is a main pitfall of Value-at-Risk models, it is hard to find a model that efficiently forecasts risk during turbulent times. This comparison shows that the rolling restricted GARCH (1, 2) model performed best for the With Crisis sample.

5.6.4   Lopez's Loss Function

This section shows the results for the Lopez's loss function for the three models using the Full sample[i]. The results from the Lopez's loss function are in table 13.

Lopez's loss function results for the full sample

	Rolling GARCH (1, 1)		Rolling GARCH (1, 2)		Riskmetrics	
Observations	2512					
VaR level	5%	1%	5%	1%	5%	1%
Loss function	0,10694	0,03171	0,09946	0,02947	0,10876	0,03323

Table 13: Lopez's loss function results for the Full sample

The rolling restricted GARCH (1, 2) shows the lowest values both for the 5% and 1% Value-at-Risk levels. This would definitely be preferred by the regulators, however a financial firm set to maximize profits might not necessarily choose the model that shows the lowest loss function because that could mean that the Value-at-Risk band is wider and therefore more capital would be reserved to make up for potential losses.

Sections 5.6.4.1-5.6.4.3 show graphically by how much each violation exceeded the projected Value-at-Risk at both confidence levels and the distribution of violations, with each section presenting one of the models, the sample used is the Full sample[ii].

---

[i] The Lopez's Loss function for all the models and sample parts can be found in Excel worksheets in the sub-folder "backtests" on the enclosed CD.

[ii] Computation of these graphs can be found in the Excel worksheet violationexceedancegraphs.xlsm on the enclosed CD.

5.6.4.1 Violation Exceedance for Rolling GARCH (1, 1)

Figures 11 and 12 show graphically how much each violation exceeded the projected confidence level and the distribution of violations. Figure 11 shows the violations for the 5% and figure 12 for the 1% Value-at-Risk levels.
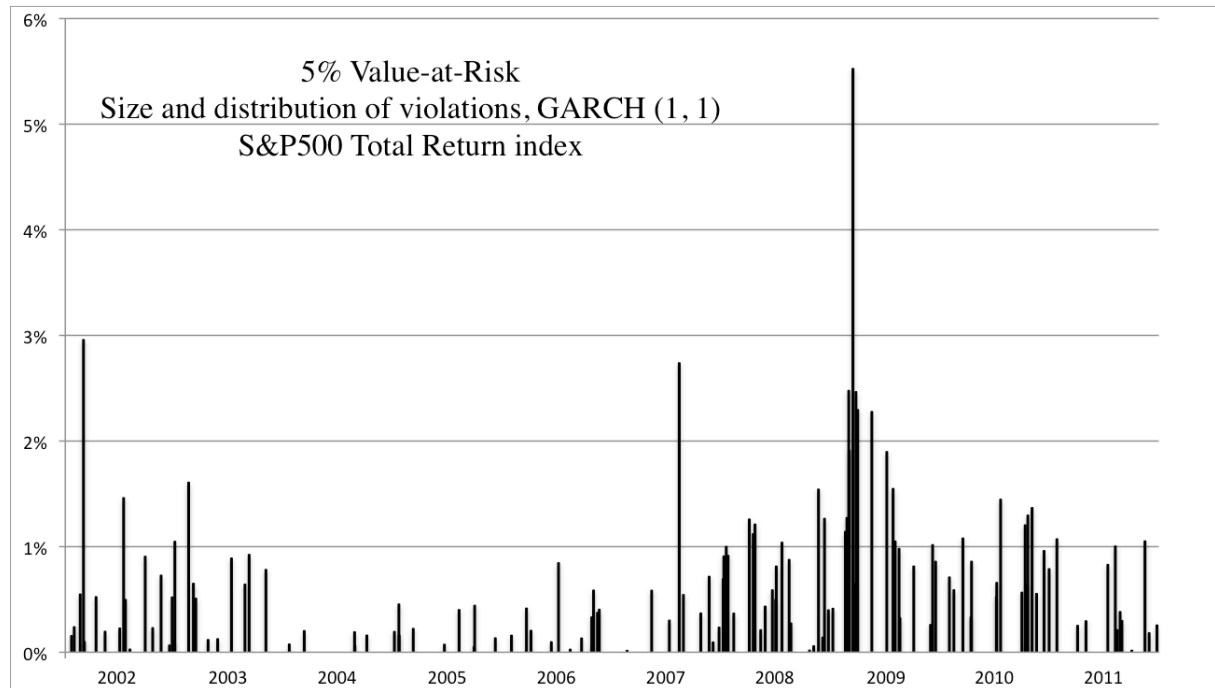

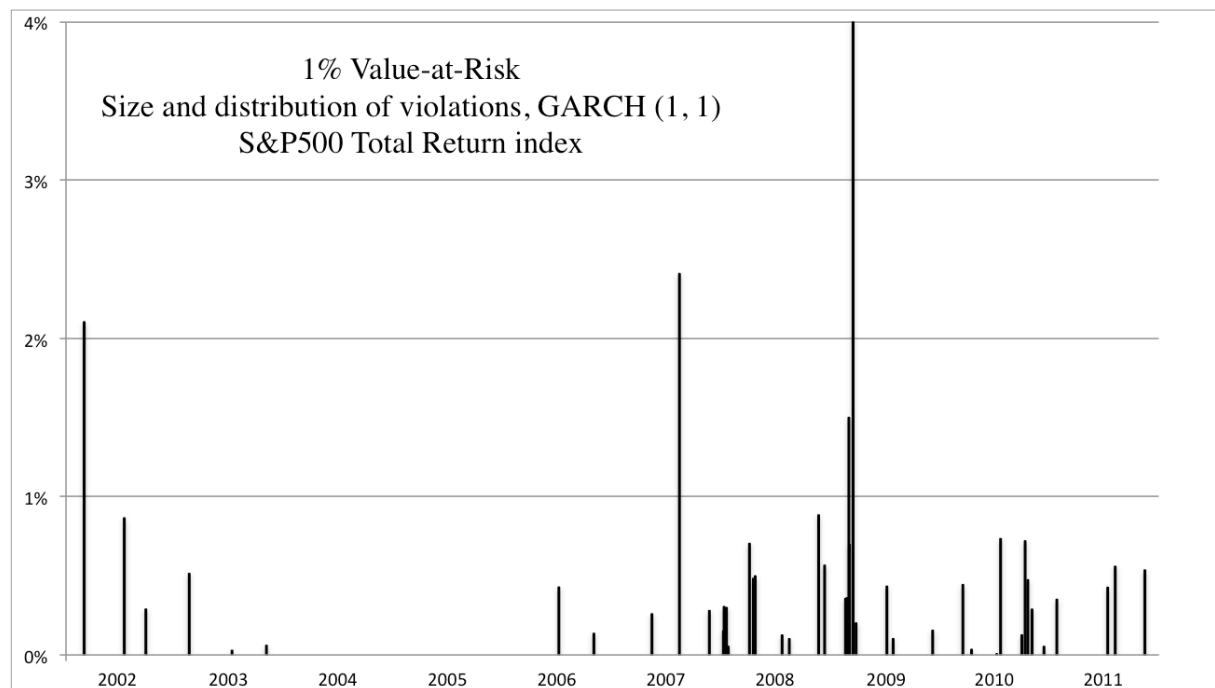**Figure 11: Size and distribution of violations (5% VaR), rolling GARCH (1, 1)**

Figure 12: Size and distribution of violations (1% VaR), rolling GARCH (1, 1)

The 5% Value-at-Risk level graph for the rolling GARCH (1, 1) has a total of 133 violations and shows that the violations are clustered together in and around the financial crisis. One particular violation is much higher than all the others. That violation appeared on the 29th of September 2008, then the S&P Total Return index fell by 9,1957%, which exceeded the projected Value-at-Risk number by 5,5196%.

The 1% Value-at-Risk level graph has a total of 44 violations, the highest one being at the same date as for the 5% level, at 3,9966%. It is interesting to see that there were no violations observed in 2004 and 2005 (exact timespan: 19.05.03-20.01.06) making the model overestimate risk during that period.

5.6.4.2   Violation Exceedance for Rolling Restricted GARCH (1, 2)

Figures 13 and 14 show graphically how much each violation exceeded the projected confidence level and the distribution of violations. Figure 13 shows the violations for the 5% and figure 14 for the 1% Value-at-Risk levels.
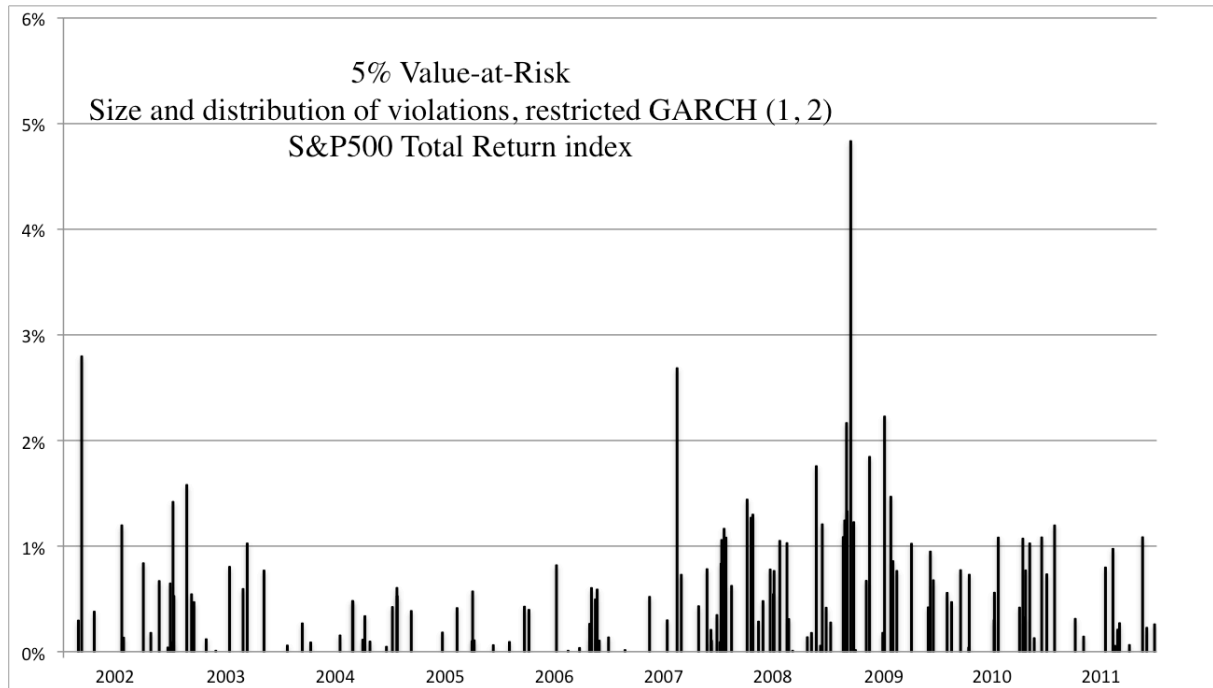

Figure 13: Size and distribution of violations (5% VaR), rolling restricted GARCH (1, 2)
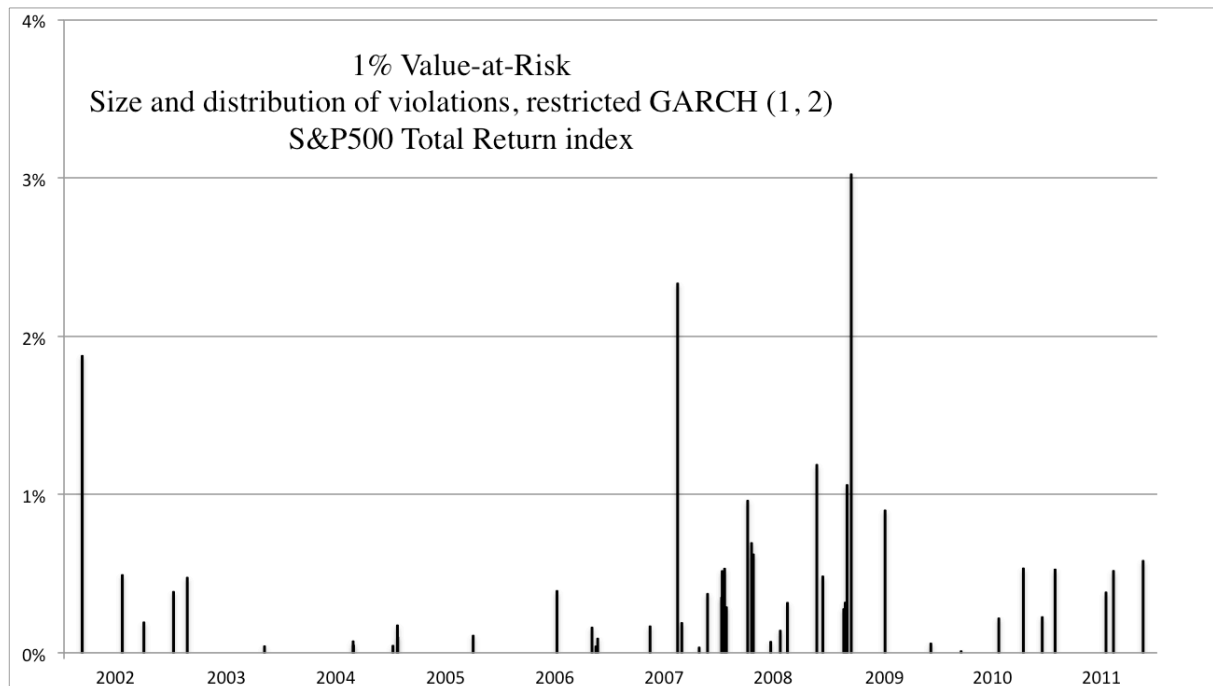

Figure 14: Size and distribution of violations (1% VaR), rolling restricted GARCH (1, 2)

The 5% Value-at-Risk level graph for the rolling GARCH (1, 1) shows no striking difference when it is compared to the graph for the rolling restricted GARCH (1, 2) model. The violations are clustered in and around the financial crisis in both cases three exceedances are highest at the same dates. The rolling restricted GARCH (1, 2) has 137 violations. The highest exceedance is 4,8309% above the projected Value-at-Risk number. It does have lower exceedances than the rolling GARCH (1, 1) as the Lopez's loss function suggested.

The 1% Value-at-Risk level graph has a total of 47 violations, the highest exceedance is at 3,0225% and the exceedances are lower than for the rolling GARCH (1, 1) model. Violations are observed in 2004 and 2005 making it not overestimate the risk during that period as much the rolling GARCH (1, 1) did.

5.6.4.3   Violation Exceedance for Riskmetrics

Figures 15 and 16 show graphically how much each violation exceeded the projected confidence level and the distribution of violations. Figure 13 shows the violations for the 5% and figure 14 for the 1% Value-at-Risk levels.
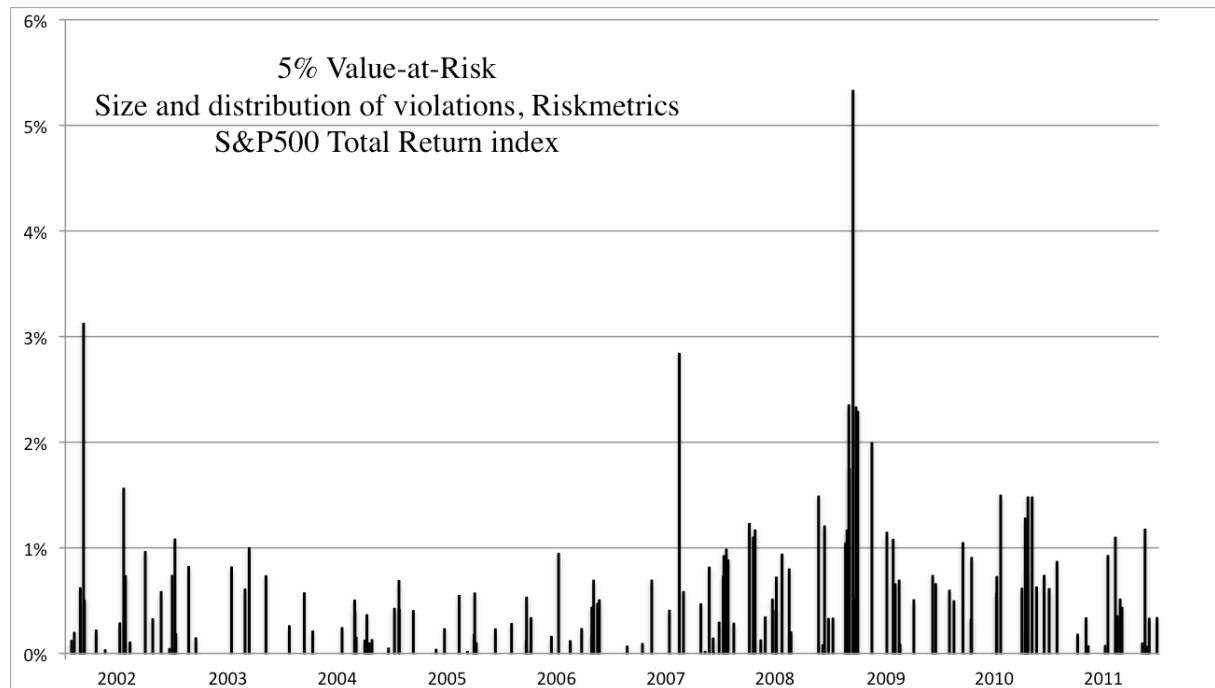

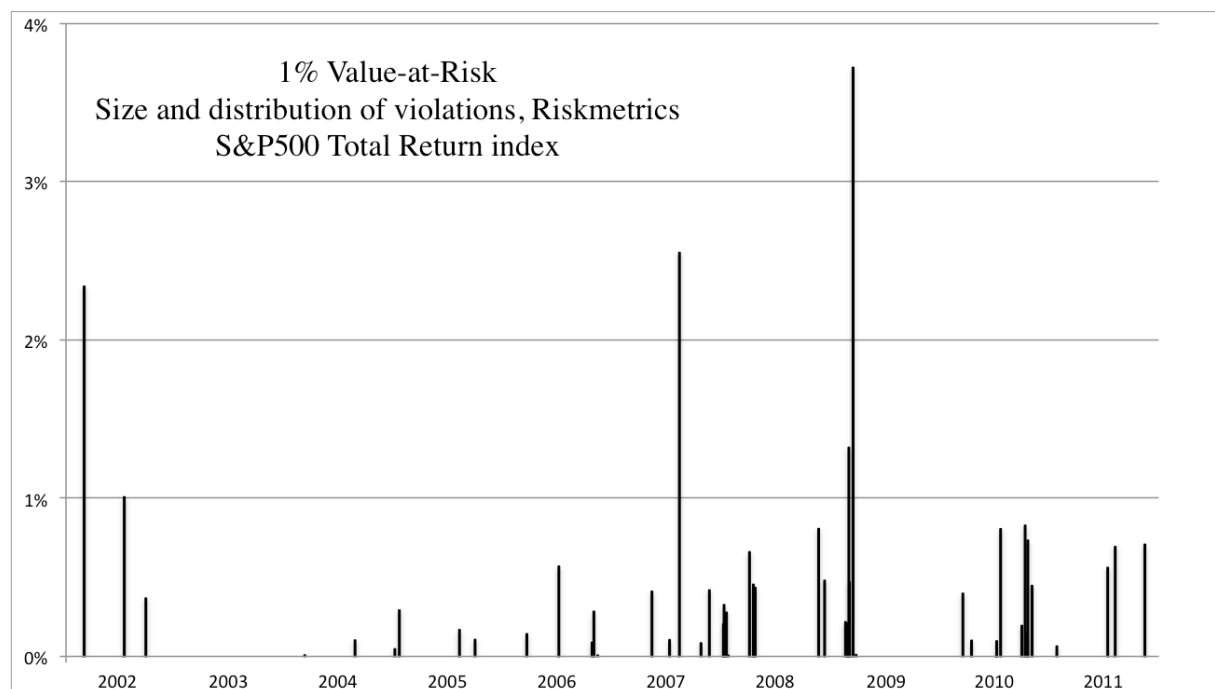**Figure 15: Size and distribution of violations (5% VaR), Riskmetrics**

Figure 16: Size and distribution of violations (1% VaR), Riskmetrics

The 5% Value-at-Risk level graph for the Riskmetrics model is similar, violations are clustered together in and around the financial crisis and three of the highest exceedances are at the same dates as with the other two models. The highest exceedance is at 5,3241% and there are a total of 146 violations.

The 1% Value-at-Risk level graph has a total of 48 violations, the highest exceedance is at 3,7201%. There is a period of around one and a half year (11.04.02-24.09.03) where no violations are observed, during which risk is overestimated.

All the models show similar patterns of distribution of violations and sizes at the 5% Value-at-Risk level. At the 1% Value-at-Risk level the rolling GARCH (1, 1) and the Riskmetrics model overestimated risk during certain periods. It should not be surprising that all the models show similar results for the period since all the models come from the same family of time-series models, the GARCH family.

6   Discussion

It was interesting to discover that the restricted GARCH (1, 2) model performed best under the model selection process, showing the lowest values for both the AIC and SBC. The residuals from this model show no sign of autocorrelation or heteroskedasticity, where the residuals from the GARCH (1, 1) model did. However, this did not lead to the restricted GARCH (1, 2) model outperforming the GARCH (1, 1) model when they were evaluated with the backtests. The reason might be that since the AIC and SBC check for the fitness of the model for the whole distribution and the backtests check a part of the distribution, the GARCH (1, 1) model was better fitting for the left tail of the distribution. The sample included the financial crisis of late 2008 that could also have been a factor in why the restricted GARCH (1, 2) had lower AIC and SBC results than the GARCH (1, 1). The restricted GARCH (1, 2) did outperform the GARCH (1, 1) for the With Crisis sample backtests.

It is evident that the models did not perform well for the 1% Value-at-Risk level, this can be due to the frequently observed fat tails in actual distribution of financial time-series and that the normal assumption does not take account for those fat tails. Another distribution can be used and compared with the normal distribution to see if it performs better at the 1% Value-at-Risk level. The reason is likely that the conditional variance was abnormally high during the financial crisis and therefore the absolute value of returns was high as well. Making the negative returns surpass the 1% Value-at-Risk level more frequently.

It is problematic to find an optimal window size for GARCH models and different window sizes show different estimates, this can be considered as a pitfall for studies when different models are compared. Since it is relatively easy to manipulate data by changing window sizes and show results that might be more fitting to the goals of the research.

To make an attempt to figure out why the restricted GARCH (1, 2) outperformed the GARCH (1, 1) model in the model selection process. An assumption was made for the returns at time t to be dependent on whether the returns are negative or positive at time t-1. The full sample was split in two sub samples. One consisting of returns where the previous return had been negative and the other consisting of returns where the previous return had been positive. The preferred models were found from those sub-samples, according to the model selection process and both sub-samples had the lowest AIC and SBC for the GARCH (1, 1) model.

The residuals for the GARCH (1, 1) model of both sub-samples did not show any sign of autocorrelation or heteroskedasticity either, with implementation of the Ljung-Box Q test and the ARCH test. Then the GARCH variances were calculated for the time-series, where a positive return used the estimates from the sub-sample consisting of returns with positive previous returns and a negative return used the estimates from the other sub-sample. The model estimated from both sub-samples was a rolling GARCH (1, 1) model with a window size of 500. When the variances had been calculated the backtesting procedures were performed and this model adjustment compared to the models used in the empirical research. The model did not perform well enough according to the backtests and therefore the initial assumption was weakened and the test statistics not included in the empirical research.

There is a vast amount of different models that can be tested, as the family of time-series models only gets larger with time. It would be interesting to see results from other models.

7 Conclusion

The theoretical part of this thesis described how Value-at-Risk was developed and the theories behind it. Fundamentals of the procedures were explained, the importance of marking to market and the three parameters that needs to be specified for every model. Furthermore BCBS and CESR regulations on how financial institutions should specify these parameters were briefly described.

Known methods were described and applied to a part of the time-series to demonstrate the differences between them.

Time-series analysis was introduced with explanation of tests that help examining the data. The ICSS algorithm was explained and applied to find volatility change points. Furthermore the ARCH (q) and GARCH (p, q) models were introduced and their properties.

The theories behind well known backtests, their differences and drawbacks are included in the thesis.

The empirical research examines the ten year sample of S&P500 Total Returns index (20.07.01-19.07.11) where a model selection process for the preferred GARCH (p, q) is applied finding the restricted GARCH (1, 2) to be the best fit. The residuals from that model along with the GARCH (1, 1) model were tested for serial autocorrelation and heteroskedasticity, the results from these tests show evidence for autocorrelation and heteroskedasticity in the residuals from the GARCH (1, 1) model while there is no evidence of those factors in the residuals from the restricted GARCH (1, 2) models.

These models were given a rolling parameter estimation based on different window sizes and they were backtested according to the tests described in the theoretical part. Eventually a window size of 1500 was selected for the rolling GARCH (1, 1) and a window size of 500 for the rolling restricted GARCH (1, 2). The Value-at-Risk estimates from these models were used for evaluation against the Riskmetrics model. Where three samples were used: the entire sample, before the financial crisis and with the financial crisis.

The results from the evaluation indicated that the rolling GARCH (1, 1) performed best for the Full sample, the Riskmetrics performed best for the Before Crisis sample and the rolling GARCH (1, 2) performed best for the With Crisis sample.

The Lopez loss function was applied on the models used for the evaluation and resulting in the rolling restricted GARCH (1, 2) showing the lowest value, the loss function calculates the average exceedance level, which is contributed to when there is a violation. Along with the results from the loss function, exceedance graphs with distribution of violations for all the models at 5% and 1% Value-at-Risk levels were shown. They were similar for all the models at the 5% level, however the 1% level showed that for certain periods both the rolling GARCH (1, 1) and the Riskmetrics model had no violations, resulting in overestimation of the risk.

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9   Appendix A: VBA Codes

The codes are all designed for the VBA module to be set at

Option Base 1

It needs to be written at the start of the module.

When the results are given in arrays you need to press CMD + SHIFT + ENTER if you are using a mac and CTRL + SHIFT + ENTER if you are using a pc. All the codes are fully functional in the worksheets where they were applied found on the enclosed CD. To view them you click the Developer tab in Excel and in there you click on Editor, then the relevant module is selected. All the codes were created by the author except for the Nelder-Mead algorithm attained from the enclosed CD from Rouah, F. D., & Vainberg, G. (2007). Following are the codes used in this research.

9.1   ICSS Algorithm

To use this code you need to select the cells where the results should appear (they appear as an array, the size of it depending on the number of iterations and change points found). Then write: "=ICSSalgorithm([insert selected range])" in the formula tab. The selected range should be the returns data, which you are testing for, sorted by the most recent return at the top of the range. It gives results as numbers, 1 representing the bottom observation, 2 representing the observation above etc. Each column carrying the results represents iteration and the change points found during that iteration, 1_{st} column being the first iteration, 2_{nd} column the second iteration etc. The code is as follows:

```
Function ICSSalgorithm(returns As Range)

Dim rflipped() As Variant

Dim T
Dim M
Dim start
Dim a
Dim i
Dim j
Dim k
Dim middle

a = 1
```

```
start = 1
T = returns.Rows.count
M = returns.Rows.count

Dim count()
ReDim count(M)

Dim count2()
ReDim count2(M)

ReDim rflipped(M)
Dim rflipped2()

Dim c()
ReDim c(T)

Dim x()
ReDim x(T)

   For i = 1 To T
      rflipped(i) = returns.Cells(T - (i - 1))
   Next i

Dim Tlast
Tlast = M
Dim Tfirst
Tfirst = 0
Dim Tupper
Tupper = M

Dim returnarray()
ReDim returnarray(21, 50)

j = 1
Do

ReDim rflipped2((T - start) + 1)
   For i = 1 To ((T - start) + 1)
      rflipped2(i) = rflipped(start - 1 + i)
   Next i

ReDim count((T - start) + 1)
   For i = 1 To ((T - start) + 1)
      If i = 1 Then
      count(i) = 1
      Else
      count(i) = 1 + count(i - 1)
      End If
   Next i

ReDim count2((T - start) + 1)
   For i = 1 To ((T - start) + 1)
      If i = 1 Then
      count2(i) = start
      Else
      count2(i) = 1 + count2(i - 1)
      End If
   Next i

ReDim c((T - start) + 1)
   For i = 1 To ((T - start) + 1)
      If i = 1 Then
      c(i) = rflipped2(i) ^ 2
```

```vba
    Else
    c(i) = rflipped2(i) ^ 2 + c(i - 1)
    End If
  Next i

ReDim x((T - start) + 1)
  For i = 1 To ((T - start) + 1)
    x(i) = Abs(c(i) / c((T - start) + 1) - i / ((T - start) + 1))
  Next i


  If Sqr(((T - start) + 1) / 2) * Application.Max(x) > 1.358 And (a Mod 3 = 1) Then
  returnarray(1, j) = Application.Index(count2, Application.Match(Application.Max(x), x, 0))
  T = Application.Index(count, Application.Match(Application.Max(x), x, 0)) + start - 1
  Tupper = Application.Max(count) + start - 1
  middle = T
  a = a + 1
  Tfirst = T
  Tlast = T
  ElseIf Sqr(((T - start) + 1) / 2) * Application.Max(x) < 1.358 And (a Mod 3 = 1) Then
  returnarray(1, j) = 0
  a = a + 1
  start = T
  ElseIf Sqr(((T - start) + 1) / 2) * Application.Max(x) > 1.358 And (a Mod 3 = 2) Then
  returnarray(1, j) = Application.Index(count2, Application.Match(Application.Max(x), x, 0))
  T = Application.Index(count, Application.Match(Application.Max(x), x, 0)) + start - 1
  Tfirst = T
  ElseIf Sqr(((T - start) + 1) / 2) * Application.Max(x) < 1.358 And (a Mod 3 = 2) Then
  a = a + 1
  returnarray(1, j) = 0
  start = middle + 1
  T = Tupper
  ElseIf Sqr(((T - start) + 1) / 2) * Application.Max(x) > 1.358 And (a Mod 3 = 0) Then
  returnarray(1, j) = Application.Index(count2, Application.Match(Application.Max(x), x, 0))
  start = Application.Index(count, Application.Match(Application.Max(x), x, 0)) + start
  Tlast = start - 1
  ElseIf Sqr(((T - start) + 1) / 2) * Application.Max(x) < 1.358 And (a Mod 3 = 0) Then
  a = a + 1
  returnarray(1, j) = 0
  start = Tfirst + 1
  T = Tlast
  End If

j = j + 1
Loop Until start > T - 2

Dim temp As Variant

sorted = False
Do While Not sorted
sorted = True
For i = 1 To 49
  If returnarray(1, i) > returnarray(1, i + 1) Then
  temp = returnarray(1, i + 1)
  returnarray(1, i + 1) = returnarray(1, i)
  returnarray(1, i) = temp
  sorted = False
  End If
Next i
Loop

For i = 1 To 49
  If returnarray(1, i) = returnarray(1, i + 1) Then
```

```
   returnarray(1, i) = 0
   End If
Next i


sorted = False
Do While Not sorted
sorted = True
For i = 1 To 49
   If returnarray(1, i) > returnarray(1, i + 1) Then
   temp = returnarray(1, i + 1)
   returnarray(1, i + 1) = returnarray(1, i)
   returnarray(1, i) = temp
   sorted = False
   End If
Next i
Loop


Do While returnarray(1, 1) = 0
For i = 1 To 49
   If returnarray(1, i) = 0 Then
   temp = returnarray(1, i + 1)
   returnarray(1, i + 1) = returnarray(1, i)
   returnarray(1, i) = temp
   End If
Next i
Loop


Dim NT
NT = 0
Do
NT = NT + 1
Loop Until returnarray(1, NT) = 0
NT = NT - 1


ReDim Preserve returnarray(21, NT)


For k = 1 To 20
For j = 1 To NT


If j = 1 And (NT > 1) Then
start = 1
T = returnarray(k, j + 1)
ElseIf j = NT And (NT > 1) Then
start = returnarray(k, j - 1) + 1
T = M
ElseIf j = 1 And (NT = 1) Then
start = 1
T = M
Else
start = returnarray(k, j - 1) + 1
T = returnarray(k, j + 1)
End If


ReDim count((T - start) + 1)
   For i = 1 To ((T - start) + 1)
     If i = 1 Then
     count(i) = 1
     Else
     count(i) = 1 + count(i - 1)
     End If
   Next i


ReDim rflipped2((T - start) + 1)
```

```vba
   For i = 1 To ((T - start) + 1)
      rflipped2(i) = rflipped(start - 1 + i)
   Next i

ReDim c((T - start) + 1)
   For i = 1 To ((T - start) + 1)
      If i = 1 Then
      c(i) = rflipped2(i) ^ 2
      Else
      c(i) = rflipped2(i) ^ 2 + c(i - 1)
      End If
   Next i

ReDim x((T - start) + 1)
   For i = 1 To ((T - start) + 1)
      x(i) = Abs(c(i) / c((T - start) + 1) - i / ((T - start) + 1))
   Next i

   If Sqr(((T - start) + 1) / 2) * Application.Max(x) > 1.358 And (j = 1) Then
   returnarray(k + 1, j) = Application.Index(count, Application.Match(Application.Max(x), x, 0))
   ElseIf Sqr(((T - start) + 1) / 2) * Application.Max(x) > 1.358 And (j > 1) Then
   returnarray(k + 1, j) = Application.Index(count, Application.Match(Application.Max(x), x, 0)) + returnarray(k, j - 1)
   Else
   returnarray(k + 1, j) = 0
   End If

Next j

sorted = False
Do While Not sorted
sorted = True

For i = 1 To NT - 1
   If returnarray(k + 1, i) > returnarray(k + 1, i + 1) Then
   temp = returnarray(k + 1, i + 1)
   returnarray(k + 1, i + 1) = returnarray(k + 1, i)
   returnarray(k + 1, i) = temp
   sorted = False
   End If
Next i
Loop

For i = 1 To NT - 1
   If returnarray(k + 1, i) = returnarray(k + 1, i + 1) Then
   returnarray(k + 1, i) = 0
   End If
Next i

sorted = False
Do While Not sorted
sorted = True

For i = 1 To NT - 1
   If returnarray(k + 1, i) > returnarray(k + 1, i + 1) Then
   temp = returnarray(k + 1, i + 1)
   returnarray(k + 1, i + 1) = returnarray(k + 1, i)
   returnarray(k + 1, i) = temp
   sorted = False
   End If
Next i
Loop

Do While returnarray(k + 1, 1) = 0
```

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```
For i = 1 To NT - 1
   If returnarray(k + 1, i) = 0 Then
   temp = returnarray(k + 1, i + 1)
   returnarray(k + 1, i + 1) = returnarray(k + 1, i)
   returnarray(k + 1, i) = temp
   End If
Next i
Loop

For i = 1 To NT
   If returnarray(k + 1, i) = 0 Then
   NT = 0
   Do
   NT = NT + 1
   Loop Until returnarray(k + 1, NT) = 0
   NT = NT - 1
   End If
Next i

Next k

ICSSalgorithm = Application.Transpose(returnarray)

End Function
```

9.2 Nelder-Mead Algorithm

To make the estimations for the GARCH (1, 1) model you select the cells where the results should be in an array of four cells from top to bottom. And write: "=GARCHparams([insert selected range],[insert range with estimation of parameters to start from])" in the formula bar. The selected range should be the returns data, which you are estimating the parameters for, sorted by the most recent return at the top of the range. The range with estimation of parameters to start from, are specified values for omega, alpha and beta, where omega is at the top and beta at the bottom. These values tell the algorithm at what value it should start the process of finding the maximum log-likelihood. If they are very close to the actual values, the parameter estimation process should not take as much computational time.

For the restricted GARCH (1, 2) model you do the same as with the GARCH (1, 1) except the name of the function is GARCH12params, so write: "=GARCH12params([insert selected range],[insert range with estimation of parameters to start from])" in the formula bar. The specified values would then be for omega, alpha2 and beta. The code is as follows:

```
Function BubSortRows(passVec)
   Dim tmpVec() As Double, temp() As Double

   uVec = passVec
   rownum = UBound(uVec, 1)
   colnum = UBound(uVec, 2)

   ReDim tmpVec(rownum, colnum) As Double
   ReDim temp(colnum) As Double


   For i = rownum - 1 To 1 Step -1
      For j = 1 To i
         If (uVec(j, 1) > uVec(j + 1, 1)) Then
            For k = 1 To colnum
               temp(k) = uVec(j + 1, k)
               uVec(j + 1, k) = uVec(j, k)
               uVec(j, k) = temp(k)
            Next k
         End If

      Next j
   Next i

   BubSortRows = uVec
End Function


Function NelderMead(fname As String, rets, startParams)
   Dim resMatrix() As Double
   Dim x1() As Double, xn() As Double, xw() As Double, xbar() As Double, xr() As Double, xe() As Double, xc() As Double, xcc()
   As Double
```

```vba
Dim funRes() As Double, passParams() As Double
MAXFUN = 1000
TOL = 1e-10
rho = 1
Xi = 2
gam = 0.5
sigma = 0.5


paramnum = Application.Count(startParams)
ReDim resmat(paramnum + 1, paramnum + 1) As Double
ReDim x1(paramnum) As Double, xn(paramnum) As Double, xw(paramnum) As Double, xbar(paramnum) As Double,
xr(paramnum) As Double, xe(paramnum) As Double,
xc(paramnum) As Double, xcc(paramnum) As Double
ReDim funRes(paramnum + 1) As Double, passParams(paramnum)
For i = 1 To paramnum
    resmat(1, i + 1) = startParams(i)
Next i
resmat(1, 1) = Run(fname, rets, startParams)

For j = 1 To paramnum
   For i = 1 To paramnum
     If (i = j) Then
        If (startParams(i) = 0) Then
           resmat(j + 1, i + 1) = 0.05
        Else
           resmat(j + 1, i + 1) = startParams(i) * 1.05
        End If
     Else
        resmat(j + 1, i + 1) = startParams(i)
     End If
     passParams(i) = resmat(j + 1, i + 1)
   Next i
   resmat(j + 1, 1) = Run(fname, rets, passParams)
Next j

For lnum = 1 To MAXFUN

   resmat = BubSortRows(resmat)
   If (Abs(resmat(1, 1) - resmat(paramnum + 1, 1)) < TOL) Then
      Exit For
   End If

   f1 = resmat(1, 1)

   For i = 1 To paramnum
      x1(i) = resmat(1, i + 1)
   Next i
   fn = resmat(paramnum, 1)

   For i = 1 To paramnum
      xn(i) = resmat(paramnum, i + 1)
   Next i
   fw = resmat(paramnum + 1, 1)
   For i = 1 To paramnum
      xw(i) = resmat(paramnum + 1, i + 1)
   Next i

   For i = 1 To paramnum
      xbar(i) = 0
      For j = 1 To paramnum
         xbar(i) = xbar(i) + resmat(j, i + 1)
      Next j
      xbar(i) = xbar(i) / paramnum
```

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```
Next i

For i = 1 To paramnum
   xr(i) = xbar(i) + rho * (xbar(i) - xw(i))
Next i

fr = Run(fname, rets, xr)

shrink = 0
If ((fr >= f1) And (fr < fn)) Then
   newpoint = xr
   newf = fr
ElseIf (fr < f1) Then
   'calculate expansion point
   For i = 1 To paramnum
      xe(i) = xbar(i) + Xi * (xr(i) - xbar(i))
   Next i
   fe = Run(fname, rets, xe)
   If (fe < fr) Then
      newpoint = xe
      newf = fe
   Else
      newpoint = xr
      newf = fr
   End If
ElseIf (fr >= fn) Then
   If ((fr >= fn) And (fr < fw)) Then
      For i = 1 To paramnum
         xc(i) = xbar(i) + gam * (xr(i) - xbar(i))
      Next i
      fc = Run(fname, rets, xc)
      If (fc <= fr) Then
         newpoint = xc
         newf = fc
      Else
         shrink = 1
      End If
   Else
      For i = 1 To paramnum
         xcc(i) = xbar(i) - gam * (xbar(i) - xw(i))
      Next i
      fcc = Run(fname, rets, xcc)
      If (fcc < fw) Then
         newpoint = xcc
         newf = fcc
      Else
         shrink = 1
      End If
   End If

End If
If (shrink = 1) Then
   For scnt = 2 To paramnum + 1
      For i = 1 To paramnum
         resmat(scnt, i + 1) = x1(i) + sigma * (resmat(scnt, i + 1) - x1(1))
         passParams(i) = resmat(scnt, i + 1)
      Next i
      resmat(scnt, 1) = Run(fname, rets, passParams)
   Next scnt
Else
   For i = 1 To paramnum
      resmat(paramnum + 1, i + 1) = newpoint(i)
   Next i
```

```
        resmat(paramnum + 1, 1) = newf
      End If

  Next lnum
  If (lnum = MAXFUN + 1) Then
    MsgBox "Maximum Iteration (" & MAXFUN & ") exceeded"
  End If
  resmat = BubSortRows(resmat)
  For i = 1 To paramnum + 1
    funRes(i) = resmat(1, i)
  Next i
  funRes(1) = funRes(1)
  NelderMead = Application.Transpose(funRes)

End Function


Function GARCHMLE(rets, startParams)
Dim VARt() As Double
n = Application.Count(rets)
ReDim VARt(n) As Double
omega = startParams(1)
alpha = startParams(2)
beta = startParams(3)

If ((omega < 0) Or (alpha < 0) Or (beta < 0)) Or (alpha + beta > 0.99999) Then
  GARCHMLE = -9999
Else

VARt(n) = Application.Var(rets)
GARCHMLE = -Log(VARt(n)) - (rets(n) ^ 2 / VARt(n))
For cnt = n - 1 To 1 Step -1
  VARt(cnt) = omega + alpha * rets(cnt + 1) ^ 2 + beta * VARt(cnt + 1)
  GARCHMLE = GARCHMLE - Log(VARt(cnt)) - (rets(cnt) ^ 2 / VARt(cnt))
Next cnt
End If
  GARCHMLE = -GARCHMLE
End Function


Function GARCHMLE12(rets, startParams)
Dim VARt() As Double
n = Application.Count(rets)
ReDim VARt(n) As Double
omega = startParams(1)
alpha2 = startParams(2)
beta = startParams(3)

If ((omega < 0) Or (alpha2 < 0) Or (beta < 0)) Or (alpha2 + beta > 0.99999) Then
  GARCHMLE12 = -9999
Else

VARt(n) = Application.Var(rets)
GARCHMLE12 = -Log(VARt(n)) - (rets(n) ^ 2 / VARt(n))
For cnt = n - 1 To 1 Step -1
  VARt(cnt) = omega + alpha2 * rets(cnt + 2) ^ 2 + beta * VARt(cnt + 1)
  GARCHMLE12 = GARCHMLE12 - Log(VARt(cnt)) - (rets(cnt) ^ 2 / VARt(cnt))
Next cnt
End If
  GARCHMLE12 = -GARCHMLE12
End Function
```

```
Function GARCHparams(rets, startParams)
    GARCHparams = NelderMead("GARCHMLE", rets, startParams)
End Function
```

```
Function GARCH12params(rets, startParams)
    GARCH12params = NelderMead("GARCHMLE12", rets, startParams)
End Function
```

9.3 Variance Forecasts

To make a forecast for the variance from GARCH (1, 1) models you select a cell and write: "=GARCHVarForecast([insert selected range], [insert omega value], [insert alpha value], [insert beta value])" in the formula bar. The selected range should be the returns data, which you are forecasting the next period variance for, sorted by the most recent return at the top of the range. The omega, alpha and beta values are the estimated values of the parameters from the GARCH (1, 1) model.

To make a forecast for the variance from restricted GARCH (1, 2) models you select a cell and write: "=GARCH12VarForecast([insert selected range], [insert omega value], [insert alpha2 value], [insert beta value])" in the formula bar. The selected range should be the returns data, which you are forecasting the next period variance for, sorted by the most recent return at the top of the range. The omega, alpha2 and beta values are the estimated values of the parameters from the restricted GARCH (1, 2) model.

To make a forecast for the variance from Riskmetrics models you select a cell and write: "=riskmetricVariance([insert selected range], [insert lambda value]" in the formula bar. The selected range should be the returns data, which you are forecasting the next period variance for, sorted by the most recent return at the top of the range. The lambda value is the chosen lambda value for the forecast. The code is as follows:

```
Function GARCHVarForecast(rng As Range, omega As Double, alpha As Double, beta As Double)

Dim flipped() As Variant

Dim k As Integer
Dim i As Integer

Dim n
Dim x

n = rng.Rows.Count

ReDim flipped(n)

  For k = 1 To n
      flipped(k) = rng.Cells(n - (k - 1))
  Next k

ReDim x(n + 1) As Double

For i = 1 To n + 1
  If i = 1 Then
  x(i) = omega
```

```
   Else
   x(i) = omega + alpha * flipped(i - 1) ^ 2 + beta * x(i - 1)
   End If
Next i

GARCHVarForecast = x(n + 1)

End Function


Function GARCH12VarForecast(rng As Range, omega As Double, alpha2 As Double, beta As Double)

Dim flipped() As Variant

Dim k As Integer
Dim i As Integer

Dim n
Dim x

n = rng.Rows.Count

ReDim flipped(n)

   For k = 1 To n
        flipped(k) = rng.Cells(n - (k - 1))
   Next k

ReDim x(n + 1) As Double

For i = 1 To n + 1
   If i = 1 Then
   x(i) = omega
   ElseIf i = 2 Then
   x(i) = omega + beta * x(i - 1)
   Else
   x(i) = omega + alpha2 * flipped(i - 2) ^ 2 + beta * x(i - 1)
   End If
Next i

GARCH12VarForecast = x(n + 1)

End Function


Function riskmetricVariance(rng As Range, lambda)

Dim i
Dim n

n = rng.Rows.Count

Dim x()
ReDim x(n)

For i = 1 To n
   x(i) = (1 - lambda) * lambda ^ (i - 1) * rng.Rows(i) ^ 2
Next i

riskmetricVariance = Application.Sum(x)

End Function
```

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