Information Theory and Gambling/Economics

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1 Kelly criterion

1.1 Introduction

Kelly's strategy, emphasizing on long term bet rather than maximizing profit from each bet, is discovered by John Larry Kelly. What a gambler needs to do is to find out the optimal betting fraction of the current bankroll at each iteration of the game [10]. More importantly, by using the Kelly's criterion, the gambler will not only maximize the long-term rate of growth, but also never lose all his/her bankroll in a infinite repeatedly game, thus it has undoubtable advantage in long-term investment. [16]

1.2 Variety of the Kelly criterion

1.2.1 The Original Kelly Criterion

Kelly derived this criterion by using Shannon's paper, which frequently used the information theory terminologies, "G" as the exponential rate of growth of the gambler's bankroll, "N" is the number of bets, "X_0" stands for the initial bankroll and "X_N" represents the gambler's bankroll. [15] Then "G" is given by.

$$G = \lim_{N\to\infty} \frac{1}{N} \log \frac{X_N}{X_0} \text{ [10]} \tag{1}$$

As a simple case of a noise channel, where each symbol was transmitted with a success probability of "p" and a failure probability "q". And we assume that a gambler betted his or her entire bankroll.

1.2.2 Variant of the Kelly criterion with fractional odds

In this section, we will discuss fractional odds of O-to-1, which is very similar to the original Kelly's criterion. [19]

$$G = p \ln p(1+O) + q \ln \frac{q(O+1)}{O} \text{ [10]} \tag{2}$$

Considering the fraction odds of O to 1, in the fraction odds system, a gambler will win or lose the money he or she originally betted. X_0 is defined as current bankroll. The bet size is the amount of $f * X_0$ and the amount returned is $O * f * X_0$ if a gambler wins the current play. If the gambler loses the current play, he/she loses the amount of the bet size of $f * X_0$.

1.3 The Kelly criterion and Deficiencies In The Markowitz Theory of Portfolio Selection

How to maximize gain from apportion funds among investments has endlessly puzzled economists. Markowitz' work on portfolio selection appeared, it became the standard reference.

Markowitz considers situations in which there are r alternative and, in general correlated, investments, with the gain per unit invested of $X_1, ..., X_r$, respectively. To select a portfolio is to apportion our resources so that f_i is placed in the ith investment. Markowitz' basic idea is that a portfolio is better if it has higher expectation and at least as small a variance or if it has at least as great an expectation and has a lower variance. If two portfolios have the same expectation and variance, neither is preferable. As the f_i range over all possible admissible values, the set of portfolios is generated. Typically the assumptions on the f_i are $\sum f_i = 1$, and $f_i \geq 0$ for $i = 1, ..., r$. [15]

From Markowitz point of view, the invester should always choose an efficient portfolio. But there is a obvious deficiency that if E_i and $\sigma_i^2, i = 1, 2,$ are the expectation and variance of portfolios 1 and 2, then if $E_1 < E_2$ and $\sigma_1^2 < \sigma_2^2$, the theory cannot choose between the portfolios.

2 Optimal Gambling

2.1 Gambling and Side Information

Considering the problem of finding a betting strategy for an infinite sequence of wagers where the optimality criterion is the minimization of the expected exit time of wealth from an interval. We add the side constraint that the right boundary is hit first with at least some specified probability. The optimal strategy is derived for a diffusion approximation [8].

2.1.1 Horse Race

House race is an example for understand how information theory applies to gambling.Assume that m horses run in a race. Let the ith horse win with probability p_i . If horse i wins, the payoff is o_i for 1. We assume that the gambler distributes all of his wealth across the horses. Let b_i be the fraction of the gamblers wealth invested in horse i, where $b_i \geq 0$ and $\sum b_i = 1$. Then if horse i wins the race, the gambler will receive σ_i times the amount of wealth bet on horse i. All the other bets are lost. Thus, at the end of the race, the gambler will have multiplied his wealth by a factor $b_i\sigma_i$ if horse

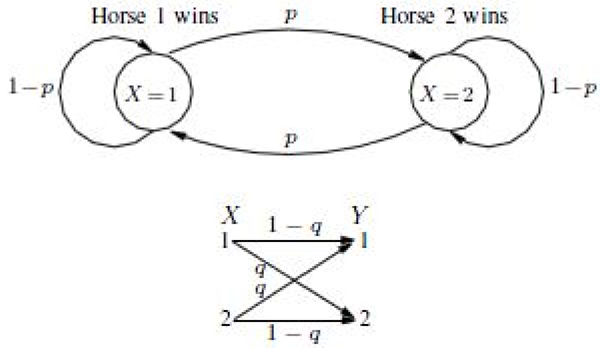

Figure 1: Example of a horse race

i wins, and this will happen with probability p_i.Let b_0 be the proportion of wealth held out as cash, and $b_1, b_2, ..., b_m$ be the proportions bet on the various horses.

$$S(X) = b_0 + b_X o(X)$$

According to the real life, we assume $\sum \frac{1}{o_i} > 1$. The organizers of the race track take a cut of all the bets. In this case it is optimal to bet only some of the money and leave the rest aside as cash. Proportional gambling is no longer log-optimal. A parametric form for the optimal strategy can be found using KuhnCTucker conditions; it has a simple water-filling interpretation [6].

2.1.2 Side Information

Sometimes the gambler will have some information which relevant to the outcome of the gamble, such as the performance of the horse, the rank of the horse. To utilize this information, what should we do? The side information will increase in wealth, which means to increase the doubling rate. The increase ΔW in doubling rate due to side information Y for a horse race X is

$$\Delta W = I(X;Y) \text{ [6]}$$

So, the mutual information $I(X;Y)$ is the increase of the doubling rate. And we can see that independent side information does not increase the doubling rate.

Consider the horse race process depicted in Figure 1, where two horses are racing and the winning horse X_i behaves as a Markov process. A horse that won will win again with probability $1-p$ and lose with probability p. At time zero, we assume that both horses have probability $\frac{1}{2}$ of winning. The side information Y_i at time i is a noisy observation of the horse race outcome X_i. It has probability $1-q$ of being equal to X_i, and probability q of being different from X_i [17]. For this example, the increase in growth rate due to side information as n goes to infinity is

$$\Delta W = h(p * q) - h(q) \text{ [18]}$$

If the side information is known with some lookahead $k \in 0, 1, ...$, that is, if the gambler knows Y^{i+k} at time i, then the increase in growth rate is given by

$$\Delta W = \lim_{n\to\infty} \frac{1}{n} I(Y^{n+k} \to X^n) = H(Y_{k+1}|Y^k, X_0) - H(Y_1|X_1)$$

2.2 Kelly criterion and Gambling

From the first section "Kelly criterion", we know that it is risking a fixed fraction of one's gambling capital each time when faced with a series of comparable favorable bets, is known to be optimal under several criteria. Here is an example to see how kelly criterion performance index betting,for detail to see [9].

Horse	Win Probability	Spread
1	0.25	(12,14)
2	0.05	(2,4)
3	0.20	(17,19)
4	0.05	(3,5)
5	0.20	(19,21)
6	0.15	(25,27)
7	0.10	(15,17)

Suppose that thewinner scores 65, the second scores 35 and all other score 0. Let Table 1 give the win probabilities p_i, and the spreads, and $p_{ij} = p_i p_j/(1-p_i)$ for the probabilities for the two horses. The optimal bet uses the way from [9]. the optimal growth rate G = 0.1354. Although $\sum(u_i+v_i) > 1$, even the least favorable outcome does not exhaust the bettor's capital.

3 Optimal Investment

There is a research about sports betting markets, for details, to see [7]. We are not only want to know how to gambling, but also how to investment. Investing in the stock market can be considered as a continuous gambling game with a positive, one year excepted return equal to the average of historical annual returns over a sufficiently long time span. To an investor, there are more complicate, because of many factors such as taxes, and time-varying purchasing power of money. Hence, time is a very important factor. [12]

3.1 Stock Market and Side Information

3.1.1 Introduction

A stock market is represented as a vector of stocks $X = (X_1, X_2, ..., X_m)$, $X_i \geq 0, i = 1, 2, ..., m$, where m is the number of stocks and the price relative X_i is the ratio of the price at the end of the day to the price at the beginning of the day. So typically, X_i is near 1. For example, $X_i = 1.03$ means that the ith stock went up 3 percent that day [14].

3.1.2 Growth Rate and Side Information

In the stock market, one normally reinvests every day, so that the wealth at the end of n days is the product of factors, one for each day of the market. The behavior of the product is determined not by the expected value but by the expected logarithm. This leads us to define the growth rate as follows: The increase $\triangle W$ in growth rate due to side information Y is bounded by

$$\triangle W \leq I(X;Y)$$

The growth rate of a stock market portfolio **b** with respect to a stock distribution $F(x)$ is defined as

$$W(b,F) = \int \log b^t x dF(X) = E(\log b^t x)$$

The capital return S from investment portfolio b is

$$S = b^t X = \sum_{i=1}^{m} b_i X_i \text{ [2]}$$

Hence, how to chose right b is the key of investment. A portfolio b^*. That achieves the maximum of $W(b,F)$ is called a log-optimal portfolio or growth optimal portfolio. There is no optimality of this procedure with respect to other obvious investment goals, and no choice procedure among the efficient portfolios is provided [15]. we will discuss several way to find b in following sections.

3.2 Log-Optimal Portfolio

3.2.1 Kuhn-Trucker Characterization of Log-Optimal Portfolio

As we talked about from the beginning, For the long term run, using the strategy will not let you lose all your money. the Kuhn.Tucker conditions that characterize the log-optimum b. [6] are equivalent to the following necessary and sufficient conditions:

$$E(\frac{X_i}{b^{*t}X})\begin{cases} = 1, & \text{if } b_i > 0 \text{ ;} \\ \leq 1, & \text{if } b_i = 0 \text{ .} \end{cases}$$

So the the proportion of wealth in stock i expected at the end of the day is the same as the proportion invested in stock i at the beginning of the day. [2]

3.2.2 Asymptotic of Log-Optimal Portfolio

Log-Optimal Portfolio is a better way of invest than other investor who uses a causal investment strategy. Let S_n^* be the wealth after n days using the log-optimal strategy b on i.i.d. stocks, and let S_n be the wealth using a causal portfolio strategy bi. Then

$$E \log S_n^* = nW^* \geq E \log S_n$$

Let $X_1, X_2, ..., X_n$ be a sequence of i.i.d. stock vectors drawn according to $F(x)$. Let $S_n^* = \prod_{i=1}^n b^{*t} X_i$, where b^* is the log-optimal portfolio, and let $S_n = \prod_{i=1}^n bX_i$ be the wealth resulting from any other causal portfolio. Then

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{S_n}{S_n^*} \leq 0 with probability 1.$$

Shows that b^* maximizes the expected log wealth, and that the resulting wealth S_n^* is equal to 2^{nW^*} to first order in the exponent, with high probability.

3.2.3 Competitive Optimality of Log-Optimal Portfolio

Let S^* be the wealth at the end of one period of investment in a stock market X with the log-optimal portfolio, and let S be the wealth induced by any other portfolio. Let U^* be a random variable independent of X uniformly distributed on [0, 2], and let V be any other random variable independent of X and U^* with $V \geq 0$ and $EV = 1$ [3].

Here U^* and V correspond to initial fair randomization of the initial wealth. This exchange of initial wealth $S_0 = 1$ for fair wealth U^* can be achieved in practice by placing a fair bet. The effect of the fair randomization is to randomize small differences, so that only the significant deviations of the ratio S/S^* affect the probability of winning. Then

$$Pr(VS \geq U^* S^*) \leq \frac{1}{2}$$

This is perhaps due to the information theoretical nature of the Kelly criterion, and it highlights the relevance of information theoretical methods in the analysis of financial problems [20].

3.3 Universal Portfolios

Instead of considering the particular case of portfolios, let's put it in general. We consider a sequential portfolio selection procedure for investing in the stock market with the goal of performing as well as if we knew the empirical distribution of future market performance. We will be able to show that there is a "universal" portfolio strategy $\widehat{b}_k$, where $\widehat{b}_k$ is based only on the past $x_1, x_2, ..., \widehat{x}_{k-1}$ [4], that will perform asymptotically as well as the best constant balanced portfolio based on foreknowledge of the sequence of price relatives. At first it may seem surprising that the portfolio $\widehat{b}_k$ should depend on the past, because the future has no relationship to the past. Indeed the stock sequence is arbitrary, and a malicious nature can structure future x_k 's to take advantage of past beliefs as expressed in the portfolio $\widehat{b}_k$. Nonetheless the resulting wealth can be made to track S^*_n.

The proposed universal adaptive portfolio strategy is the performance weighted strategy specified by

$$S_k(x^n) = \prod_{i=1}^{n} \widehat{b}_i^t(x^{i-1})x_i \text{ [1]}$$

3.3.1 Finite-Horizon Universal Portfolios

We now present the main result, a theorem characterizing the extent to which the best constant rebalanced portfolio computed in hindsight can be tracked in the worst case. Our analysis is best expressed in terms of a contest between an investor. The investor, wishing to protect himself from this worst case, selects that non-anticipating investment strategy $\widehat{b}_i(\cdot)$ which maximizes the worst-case ratio of wealths.

$$\max_{\widehat{b}} \min_{x^n} \frac{\widehat{S}_n(x^n)}{S_n^*(x^n)} = V_n \text{ [13]}$$

where

$$V_n = \frac{1}{(\sum_{n_1+\ldots+n_m=n} C^n_{n_1+\ldots+n_m} 2^{-nH(\frac{n_1}{n},\ldots,\frac{n_m}{n})})}$$

The strategy achieving the maximum, depends on the horizon n. For m = 2 stocks, the causal portfolio strategy $\widehat{b}_i^t(x^{i-1})$ achieves wealth $\widehat{S}_n(x^n)$ such that

$$\frac{\widehat{S}_n(x^n)}{S_n^*(x^n)} \geq V_n \geq \frac{1}{2\sqrt{n+1}}$$

for all market sequence x^n.

3.3.2 Why Portfolios works

The main idea of the portfolio algorithm is quite simple. [11] The idea is to give an amount $d\mathbf{b}/\int_B d\mathbf{b}$ [5] at exponential rate $W(\mathbf{b}, F_n)$ and pool the wealth at the end. Of course, all dividing and repooling is done "on paper" at time k, resulting in $\widehat{b}_k$ Since the average of exponentials has, under suitable smoothness conditions, the same asymptotic exponential growth rate as the maximum, one achieves almost as much as the wealth S_n^* achieved by the best constant rebalanced portfolio. The trap to be avoided is to put a mass distribution on the market distributions $F(x)$. It seems that this cannot be done in a satisfactory way.

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