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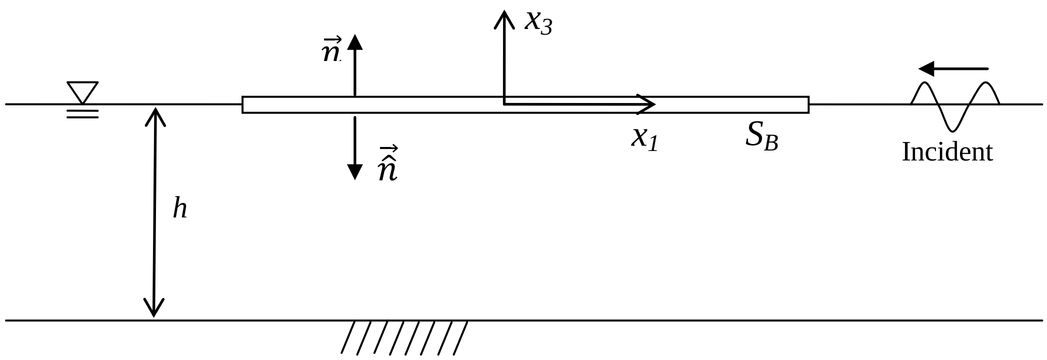
https://sites.google.com/site/modelingnsimulation/

Hydroelastic analysis of two dimensional floating structure



Green function

○ For finite depth case :

$$G(x_1,x_3;\xi_1,\xi_3) = \ln\sqrt{(x_1-\xi_1)^2+(x_3-\xi_3)^2} + \ln\sqrt{(x_1-\xi_1)^2+(2h+x_3+\xi_3)^2} - 2\ln h$$
$$-2P.V.\int_0^\infty \left[\frac{(K+\tau)e^{-\tau h}\cosh\tau(\xi_3+h)\cosh\tau(x_3+h)\cos\tau|x_1-\xi_1|}{\tau(\tau\sinh\tau h - K\cosh\tau h)} + \frac{e^{-\tau h}}{\tau}\right]d\tau \tag{1}$$
$$+2\pi i\frac{(K+k_0)e^{-k_0 h}\cosh k_0(\xi_3+h)\cosh k_0(x_3+h)\cos k_0|x_1-\xi_1|}{k_0(\sinh k_0 h + k_0 h\cosh k_0 h - Kh\sinh k_0 h)}$$

where $h =$ water depth

$x_i,\ \xi_i =$ spatial point, source point

$P.V. =$ Cauchy principle value

$K = \omega^2/g$, $k_0\sinh k_0 h - K\cosh k_0 h = 0$: dispersion relation

$g =$ acceleration of gravity

$\omega =$ angular frequency

○ In series representation $(x_3 = 0)$:

$$G(x_1;\xi_1) = 2\pi\frac{iLke^{-ik|x_1-\xi_1|}}{\frac{h}{L}\left(L^2k^2-\left(\frac{L\omega^2}{g}\right)^2\right)+\frac{L\omega}{g}} - 2\pi\sum_{n=1}^{\infty}\frac{L\bar{k}_n e^{-\bar{k}_n|x_1-\xi_1|}}{\frac{h}{L}\left(L^2\bar{k}_n^{\,2}+\left(\frac{L\omega^2}{g}\right)^2\right)-\frac{L\omega}{g}} \tag{2}$$

where $k =$ wave number

$L =$ structure length

$K + \bar{k}_n \tan \bar{k}_n h = 0,\ \bar{k}_n = ik_n$

$\bar{k}_n = \dfrac{\pi n - \theta_n}{h}$: roots of dispersion equation

$\theta_n = \tan^{-1}\left[\dfrac{\omega^2 h / g}{\pi n - \theta_n}\right]$

○ For infinite depth case :

$$G(x_1, x_3; \xi_1, \xi_3) = \ln\sqrt{(x_1-\xi_1)^2 + (x_3-\xi_3)^2} + P.V.\int_0^\infty \left[\frac{(K+\tau)e^{\tau(x_3+\xi_3)}\cos\tau\left|x_1-\xi_1\right|}{\tau(K-\tau)} - \frac{e^{-\tau}}{\tau}\right] d\tau + 2\pi i e^{K(x_3+\xi_3)}\cos K\left|x_1-\xi_1\right| \tag{3}$$

Boundary integral equation

○ For $\vec{x}$ on S_B $(x_3 = 0)$:

$$2\pi\phi(\vec{x}) - 2\pi\phi_I(\vec{x}) = \int_{S_B}\left[\phi(\vec{\xi})\frac{\partial G(\vec{x},\vec{\xi})}{\partial n} - G(\vec{x},\vec{\xi})\frac{\partial \phi(\vec{\xi})}{\partial n}\right] dS_B \tag{4}$$

where $\phi_I = i\dfrac{ga}{\omega}\dfrac{\cosh[k(x_3+h)]}{\cosh kh}e^{ikx}$: potential of incident wave

$a =$ amplitude of incident wave

Using the body boundary condition and free surface boundary condition

$$\frac{\partial \phi}{\partial n} = i\omega u_3\ ,$$

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial x_3} = \frac{\omega^2}{g}G\ , \tag{5}$$

equation (4) can be rewritten

$$2\pi\phi(\vec{x}) - 2\pi\phi_I(\vec{x}) = \int_{S_B}\left[\phi(\vec{\xi})\frac{\omega^2}{g} - i\omega u_3\right] G(\vec{x},\vec{\xi})\ dS_B\ . \tag{6}$$

The linearized Bernoulli equation is

$$\frac{\partial \Phi}{\partial t} + \frac{P}{\rho_w} + g x_3 = 0\ , \tag{7}$$

where $\rho_w =$ density of fluid.

For the steady state problem, we use

$$x_3 = U_3\ ,\quad U_3 = u_3 e^{i\omega t}\ ,\quad \Phi = \phi e^{i\omega t}\ ,\quad P = p e^{i\omega t}\ , \tag{8}$$

and then equation (7) becomes

$$i\omega\phi + \frac{p}{\rho_w} + g u_3 = 0\ ,\quad \phi = -\frac{p}{i\omega\rho_w} - \frac{g u_3}{i\omega}\ ,\quad u_3 = -\frac{i\omega}{g}\phi - \frac{p}{\rho_w g}\ . \tag{9}$$

Using equation (9) in equation (6) gives

$$2\pi\left(\frac{i}{\omega\rho_w}p(\vec{x}) + \frac{ig}{\omega}u_3(\vec{x})\right) - 2\pi\phi_I(\vec{x}) = \int_{S_B}\frac{i\omega}{\rho_w g}p(\vec{\xi})G(\vec{x},\vec{\xi})dS_B\ . \tag{10}$$

Since $\vec{x} = (x_1,\ 0,\ 0)^T$ and $\vec{\xi} = (\xi_1,\ 0,\ 0)^T$. Equation (10) can be rewritten

$$2\pi\left(\frac{i}{\omega\rho_w}p(x_1) + \frac{ig}{\omega}u_3(x_1)\right) - 2\pi\phi_I(x_1) = \int_{S_B}\frac{i\omega}{\rho_w g}p(\xi_1)G(x_1,\xi_1)dS_B\ . \tag{11}$$

Variational formulation

○ Structure part (δu_3 = virtual displacement) :

$$-\int_V \omega^2 \rho_s u_3(\vec{x})\delta u_3(\vec{x})\, dV + \int_V \sigma(\vec{x})\delta\varepsilon(\vec{x})\, dV = \int_{S_B} p(\vec{x})\delta u_3(\vec{x})\, dS_B \tag{12}$$

where $\rho_s =$ density of structure

$\sigma(\vec{x}) =$ stress

$\varepsilon(\vec{x}) =$ strain

In the Euler-Bernoulli beam theory,

$$\begin{aligned} &\sigma(\vec{x}) \longrightarrow \sigma_{xx}(x_1) = E\varepsilon_{xx}(x_1)\ , \\ &\varepsilon(\vec{x}) \longrightarrow \varepsilon_{xx}(x_1) = -x_3 \frac{\partial^2 u_3(x_1)}{\partial {x_1}^2}\ , \end{aligned} \tag{13}$$

where $E =$ modulus of elasticity.

Using equation (13) in equation (12) gives

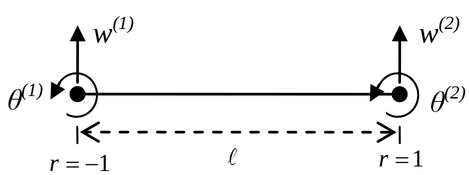
$$-\int_V \omega^2 \rho_s u_3(x_1)\delta u_3(x_1)\, dV + \int_V E x_3^2 \frac{\partial^2 u_3(x_1)}{\partial {x_1}^2}\frac{\partial^2 \delta u_3(x_1)}{\partial {x_1}^2}\, dV = \int_{S_B} p(x_1)\delta u_3(x_1)\, dS_B\ . \tag{14}$$

○ Fluid part (δp = virtual pressure) :

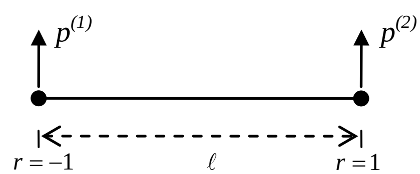
$$\begin{aligned} \int_{S_B} 2\pi \frac{i}{\omega\rho_w} p(x_1)\delta p(x_1) dS_B + \int_{S_B} 2\pi \frac{gi}{\omega} u_3(x_1)\delta p(x_1) dS_B - \int_{S_B} 2\pi\phi_I(x_1)\delta p(x_1) dS_B \\ = \int_{S_B} \frac{i\omega}{\rho_w g}\int_{S_B} p(\xi_1)G(x_1,\xi_1)\delta p(x_1) dS_B dS_B \end{aligned} \tag{15}$$

Discretization

- ✓ Unknown DOF : $\hat{\bar{p}}$ = pressure, $\hat{\bar{u}}_3$ = displacement
- ✓ Structure : 2-node Euler-Bernoulli beam element (2 DOF per node)
- ✓ Fluid : 2-node fluid element (1 DOF per node)



< 2-node Euler-Bernoulli beam element >



< 2-node fluid element >

○ Hermitian cubic interpolation function ($\mathbf{H}^e_{\mathbf{u}}$) :

$$\mathbf{H}^e_{\mathbf{u}} = \begin{bmatrix} h_{w^{(1)}} & h_{\theta^{(1)}} & h_{w^{(2)}} & h_{\theta^{(2)}} \end{bmatrix} \tag{16}$$

where $h_{w^{(1)}} = \frac{1}{4}(1-r)^2(2+r)$, $h_{\theta^{(1)}} = \frac{1}{8}\ell(1-r)^2(1+r)$, $h_{w^{(2)}} = \frac{1}{4}(1+r)^2(2-r)$, $h_{\theta^{(2)}} = \frac{1}{8}\ell(1+r)^2(1-r)$

The superscript e denotes the element.

○ 2-node interpolation function ($\mathbf{H}^e$) :

$$\mathbf{H}^e = \begin{bmatrix} h^{(1)} & h^{(2)} \end{bmatrix} \tag{17}$$

where $h^{(1)} = \frac{1}{2}(1-r)$, $h^{(2)} = \frac{1}{2}(1+r)$

○ Interpolation scheme :

$$x_1 = \mathbf{H}^e \hat{x}_1^e \,, \quad \hat{x}_1^e = \begin{bmatrix} x_1^{(1)} & x_1^{(2)} \end{bmatrix}^T$$

$$\xi_1 = \mathbf{H}^e \hat{\xi}_1^e \,, \quad \hat{\xi}_1^e = \begin{bmatrix} \xi_1^{(1)} & \xi_1^{(2)} \end{bmatrix}^T \tag{18}$$

Note that $\xi_1^{(i)}$ are identical to $x_1^{(i)}$. The unknown DOF can be defined

$$p = \mathbf{H}\vec{\hat{p}} = \sum_{e=1}^{N} \mathbf{H}^e \hat{p}^e, \quad \hat{p}^e = \begin{bmatrix} p^{(1)} & p^{(2)} \end{bmatrix}^T,$$

$$u_3 = \mathbf{H_u}\vec{\hat{u}}_3 = \sum_{e=1}^{N} \mathbf{H_u^e}\, \hat{u}_3^e \,, \quad \hat{u}_3^e = \begin{bmatrix} w^{(1)} & \theta^{(1)} & w^{(2)} & \theta^{(2)} \end{bmatrix}^T. \tag{19}$$

We can discretize equation (14) and (15) using equations (16) ~ (19).

○ Structure part :

$$\left(\omega^2 \rho_s H \int_{-1}^{1} \mathbf{H_u^T H_u} \det \mathbf{J} dr - \frac{EI}{B} \int_{-1}^{1} \left(\frac{\partial^2 \mathbf{H_u}}{\partial r^2} \right)^T \left(\frac{\partial^2 \mathbf{H_u}}{\partial r^2} \right) \det \mathbf{J} dr \right) \vec{\hat{u}}_3 + \left(\int_{-1}^{1} \mathbf{H_u^T H} \det \mathbf{J} dr \right) \vec{\hat{p}} = 0 \tag{20}$$

where H = structure thickness

B = structure width

I = moment of inertia

$\det \mathbf{J}$ = determinant of Jacobian matrix (= $\ell / 2$)

○ Fluid part :

$$\left(\int_{-1}^{1} \mathbf{H}^T \mathbf{H_u} \det \mathbf{J} dr \right) \vec{\hat{u}}_3 + \left(\frac{1}{\rho_w g} \int_{-1}^{1} \mathbf{H}^T \mathbf{H} \det \mathbf{J} dr \right) \vec{\hat{p}}$$
$$- \left(\frac{\omega^2}{2\pi \rho_w g^2} \int_{-1}^{1} \mathbf{H}^T(s) \left(\int_{-1}^{1} \mathbf{H}(r) G(x_1, \xi_1) \det \mathbf{J} dr \right) \det \mathbf{J} ds \right) \vec{\hat{p}} = a \int_{-1}^{1} \mathbf{H}^T \exp(x_1) \det \mathbf{J} dr \tag{21}$$

Final matrix form

We can obtain the final matrix form using equations (20) and (21) :

$$\begin{bmatrix} \mathbf{S_M} - \mathbf{S_K} & \mathbf{S_C} \\ \mathbf{F_C} & \mathbf{F_M} - \mathbf{F_G} \end{bmatrix} \begin{bmatrix} \hat{\vec{u}}_3 \\ \hat{\vec{p}} \end{bmatrix} = \begin{bmatrix} \hat{\vec{0}} \\ \hat{\vec{F}}_I \end{bmatrix}, \tag{22}$$

where $\mathbf{S_M} = \omega^2 \rho_s H \int_{-1}^{1} \mathbf{H_u}^T \mathbf{H_u} \det \mathbf{J} dr$,

$$\mathbf{S_K} = \frac{EI}{B} \int_{-1}^{1} \left(\frac{\partial^2 \mathbf{H_u}}{\partial r^2} \right)^T \left(\frac{\partial^2 \mathbf{H_u}}{\partial r^2} \right) \det \mathbf{J} dr \ ,$$

$$\mathbf{S_C} = \mathbf{F_C}^T = \int_{-1}^{1} \mathbf{H_u}^T \mathbf{H} \det \mathbf{J} dr \ ,$$

$$\mathbf{F_M} = \frac{1}{\rho_w g} \int_{-1}^{1} \mathbf{H}^T \mathbf{H} \det \mathbf{J} dr \ ,$$

$$\mathbf{F_G} = \frac{\omega^2}{2\pi \rho_w g^2} \int_{-1}^{1} \mathbf{H}^T(s) \left(\int_{-1}^{1} \mathbf{H}(r) G(x_1, \xi_1) \det \mathbf{J} dr \right) \det \mathbf{J} ds \ ,$$

$$\hat{\vec{F}}_I = a \int_{-1}^{1} \mathbf{H}^T \exp(x_1) \det \mathbf{J} dr \ .$$

Note that the matrix form in the left hand side of equation (22) is symmetric. Now we can calculate hydrodynamic pressure and displacement. The total pressure can be divided

$$\hat{\vec{p}} = \hat{\vec{p}}_d + \hat{\vec{p}}_s + \hat{\vec{p}}_r \ , \tag{23}$$

where $\hat{\vec{p}}_d$ = diffraction pressure,

$\hat{\vec{p}}_s$ = hydrostatic pressure,

$\hat{\vec{p}}_r$ = radiation pressure.

Since diffraction effect happens when the waves encounter the fixed structure, we set $\hat{\vec{u}}_3 = 0$ in equation (22) and obtain the diffraction pressure

$$\hat{\vec{p}}_d = \left(\mathbf{F_M} - \mathbf{F_G} \right)^{-1} \hat{\vec{F}}_I \ . \tag{24}$$

The hydrostatic pressure can be calculated

$$\vec{\hat{p}}_s = -\rho_w \, g \, \vec{\hat{u}}_3 \;, \tag{25}$$

and the radiation pressure is

$$\vec{\hat{p}}_r = \vec{\hat{p}} - \vec{\hat{p}}_d - \vec{\hat{p}}_s \;. \tag{26}$$

Complex response

The nodal solution $\vec{\hat{u}}_3$ consists of complex numbers and the steady state response is

$$\vec{\hat{U}}_3 = \vec{\hat{u}}_3 e^{\,i\omega t} , \tag{27}$$

where $\vec{\hat{u}}_3 = \mathrm{re}\left(\vec{\hat{u}}_3\right) + i \ \mathrm{im}\left(\vec{\hat{u}}_3\right)$,

$$e^{i\omega t} = \cos\omega t + i \sin\omega t \;.$$

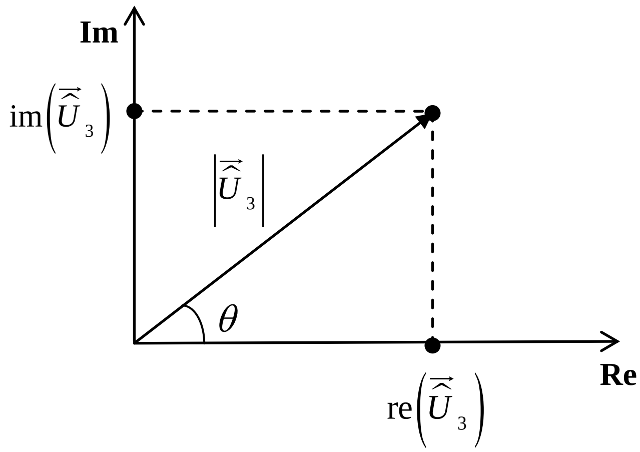
Equation (27) can be rewritten

$$\vec{\hat{U}}_3 = \mathrm{re}\left(\vec{\hat{U}}_3\right) + i \ \mathrm{im}\left(\vec{\hat{U}}_3\right), \tag{28}$$

where $\mathrm{re}\left(\vec{\hat{U}}_3\right) = \mathrm{re}\left(\vec{\hat{u}}_3\right)\cos\omega t - \mathrm{im}\left(\vec{\hat{u}}_3\right)\sin\omega t$,

$$\mathrm{im}\left(\vec{\hat{U}}_3\right) = \mathrm{im}\left(\vec{\hat{u}}_3\right)\cos\omega t + \mathrm{re}\left(\vec{\hat{u}}_3\right)\sin\omega t \;.$$

$\vec{\hat{U}}_3$ is called the complex response. It is understood that the actual steady state motion will be given by either the real part of $\vec{\hat{U}}_3$ or the imaginary part of $\vec{\hat{U}}_3$ depending on ωt.



< Complex response representation >

The magnitude and phase angle of $\vec{\hat{U}}_3$ are

$$\left|\vec{\hat{U}}_3\right| = \sqrt{\left(\mathrm{re}\left(\vec{\hat{U}}_3\right)\right)^2 + \left(\mathrm{im}\left(\vec{\hat{U}}_3\right)\right)^2} \;,$$

$$\theta = \tan^{-1}\frac{\mathrm{im}\left(\vec{\hat{U}}_3\right)}{\mathrm{re}\left(\vec{\hat{U}}_3\right)} \;. \tag{29}$$

Note that the actual steady state motion of $\vec{\hat{U}}_3$ will be maximum when the phase θ is zero and it is same with the magnitude of $\vec{\hat{U}}_3$. Mathematically, the magnitude of $\vec{\hat{U}}_3$ is identical to magnitude of $\vec{\hat{u}}_3$:

$$\left|\vec{\hat{U}}_3\right| = \sqrt{\left(\mathrm{re}\left(\vec{\hat{u}}_3\right)\right)^2 + \left(\mathrm{im}\left(\vec{\hat{u}}_3\right)\right)^2} \;. \tag{30}$$

Example problem

In this section, we report an example. The structure and fluid properties are summarized in Tables 1 and 2. Using these properties, we can calculate the dimensionless parameters. The definitions and quantities of the dimensionless parameters are summarized in Table 3.

Fig 1 shows the real, imaginary and absolute values of displacement. Fig 2 shows the hydrodynamic pressure and each pressure component .

<Table 1> Structure properties

	Notation (unit)	Quantity
Structure length	L (m)	10
Structure width	B (m)	1
Structure thickness	H (m)	1
Modulus of elasticity	E (N/m^2)	5.79×10^{3}
Moment of inertia	I (m^4)	8.33×10^{-2}
Density of structure	ρ_s (kg/m^3)	8.57
Draft	d (m)	8.36×10^{-3}

<Table 2> Fluid properties

	Notation (unit)	Quantity
Density of fluid	ρ_w (kg/m^3)	1025
Water depth	h (m)	1.1
Wave length	λ (m)	3.176
Wave number	k (rad/m)	1.979
Angular frequency	ω (rad/sec)	4.347
Amplitude of incident wave	a (m)	0.001

<Table 3> dimensionless parameters

Definition	Quantity
$\delta = \frac{h}{L}$	0.11
$\overline{K} = kL\tanh kL\delta$	19.2823
$\beta^4 = \frac{EI}{\rho_w g L^4}$	4.8026×10^{-6}

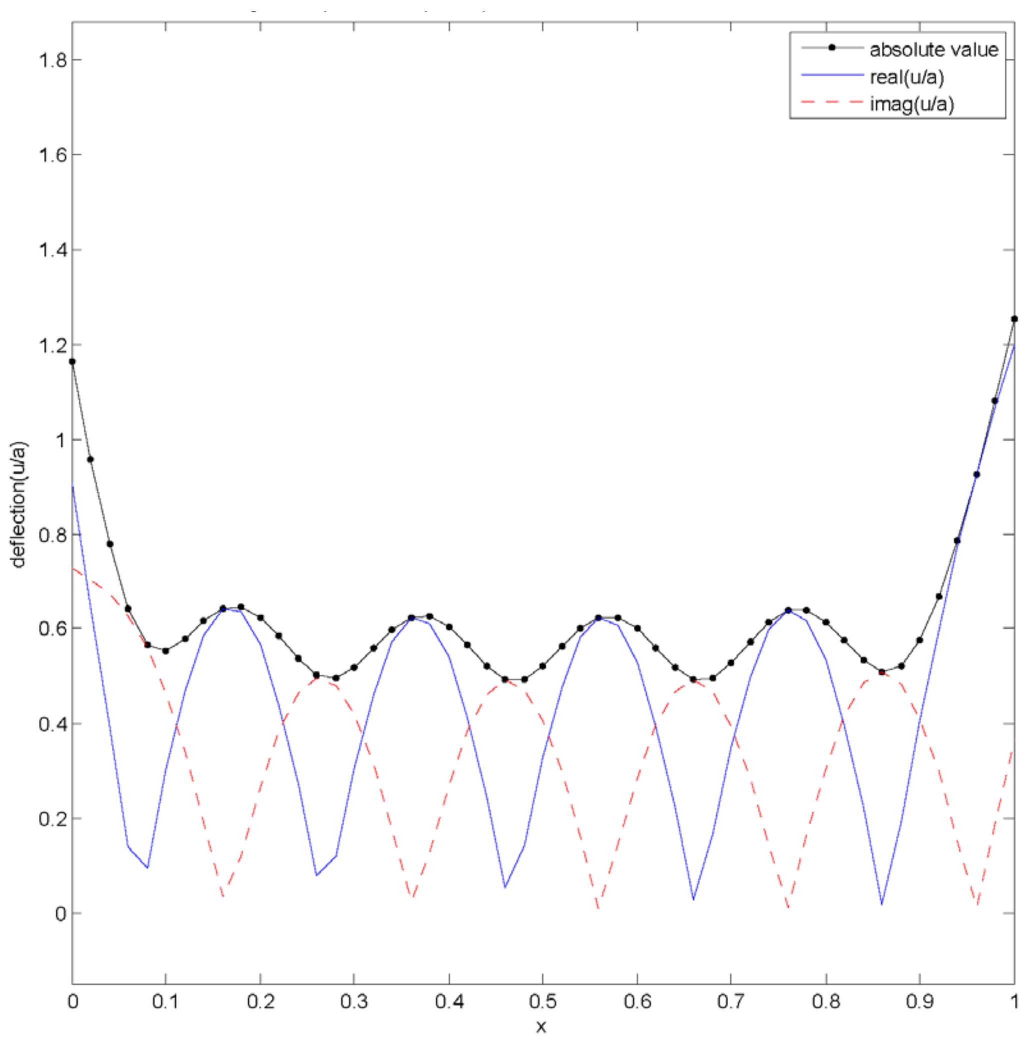


Fig 1. Displacement (heave)

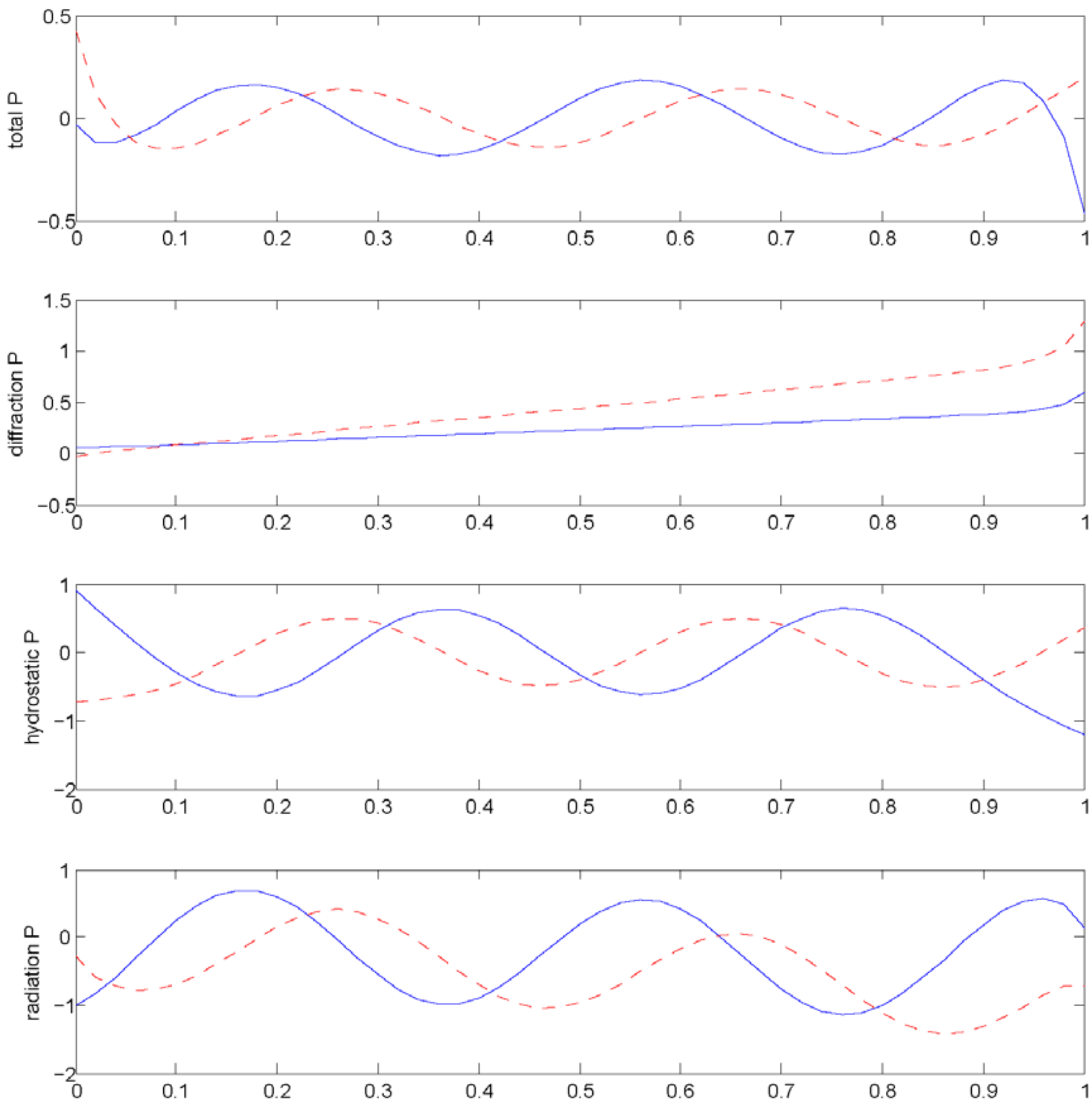


Fig 2. Total pressure and each pressure component

Homework #6

due date : December 6, 2011

You can download the MATLAB code for the hydroelastic analysis of floating beam problems at the course website. However, the code does not have a completed function which calculates the coupling matrix $\mathbf{S_c}$ (or $\mathbf{F_c}^T$) in equation (22) as shown in Fig 1.

```
% A function that computes the A_ele
function A_ele = A_ele(span_xbar)

[                                        ]

    r = Gauss_p(i);

[                                        ]

End
```

Fig 1. A function to calculate the coupling matrix

(1) Fill in the blank of the MATLAB file : A_ele.m.

(2) Calculate and plot the displacement $\vec{\hat{u}}_3$ and hydrodynamic pressure $\vec{\hat{p}}$ for the example problem in the lecture note.

(3) The hydroelastic response of floating beam structures depends on the dimensionless stiffness parameter β^4. Plot the displacement $\vec{\hat{u}}_3$ for the different β^4 given in the Table 1 and discuss the results.

Table 1. Dimensionless stiffness parameter β^4

	Case 1	Case 2	Case 3
$\beta^4 = \dfrac{EI}{\rho_w g L^4}$	4.8026×10^{-6}	4.8026×10^{-4}	4.8026×10^{-2}