Threshold Exceedances and Records of Continuous Time Random Maxima

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Abstract

Extreme Value theory deals with the observation of extreme events which are the maxima of a sequence of observations and admits that these events occur at regular intervals in time. Recently a new theory called Continuous Time Random Maxima gets much regard which can be thought of as a generalized extreme value theory. Instead of looking at events at fixed, regular time-points, this theory assumes random waiting times between the observations. This theory provides a model for bursty events, where the waiting times between the observations are heavy tailed.

Keywords: CTRM; exceedances; extreme value statistics; bursts.

1 Introduction

Extreme Value theory deals with the observation of extreme events which are the maxima of a sequence of observations and admits that these events occur at regular intervals in time. Recently a new theory called Continuous Time Random Maxima gets much regard, which can be thought of as a generalized extreme value theory. Instead of looking at events at fixed, regular time-points, this theory assumes random waiting times between the observations.

2 CTRMs and their scaling limits

2.1 CTRMs

Let $(J, W), (J_1, W_1), (J_2, W_2), \dots$ be i.i.d. bivariate random vectors on $\mathbb{R} \times (0, \infty)$. The components J and W represent an event magnitude and a waiting time, respectively. We first set up some notation:

Definition 1. *We write $S(n)$ and $M(n)$ for the* ***cumulative sum of the waiting times*** *$(W_i)_{i\in\mathbb{N}}$ and the* ***cumulative maximum of the magnitudes*** *$(J_i)_{i\in\mathbb{N}}$, more precisely*

$$S(t) = \sum_{i=1}^{\lfloor t \rfloor} W_i, \qquad M(t) = \bigvee_{i=1}^{\lfloor t \rfloor} J_i \tag{2}$$

for the cumulative sum of the waiting times and the cumulative maximum of the magnitudes. The renewal process associated with S is

$$N(t) = \max\{n \in \mathbb{N} : S(n) \leq t\}. \tag{3}$$

Finally the process

$$V(t) = M\left(N(t)\right) = \bigvee_{k=1}^{N(t)} J_k, \quad t \geq 0. \tag{4}$$

*is called a **CTRM process (Continuous Time Random Maxima)** and moreover the process*

$$\tilde{V}(t) = M\left(N(t)+1\right) = \bigvee_{k=1}^{N(t)+1} J_k, \quad t \geq 0. \tag{5}$$

*an **OCTRM (Oracle Continuous Time Random Maxima)** .*

The conceptual difference between the CTRM and the OCTRM is that for the CTRM, the waiting time W_k precedes the magnitude J_k, whereas for the OCTRM it succeeds it. In other words, for the CTRM we have $W_1, J_1, W_2, J_2, W_3, \ldots$ whereas for the OCTRM, we have $J_1, W_1, J_2, W_2, J_3, \ldots$.

2.2 Rescaling

Let $c > 0$ be a scaling parameter, and let $a(c), b(c)$ and $d(c)$ be deterministic scaling functions, defining rescaled waiting times and magnitudes as follows:

$$J^{(c)} \stackrel{d}{=} \frac{J - d(c)}{a(c)}, \qquad W^{(c)} \stackrel{d}{=} \frac{W}{b(c)} \tag{6}$$

where $\stackrel{d}{=}$ denotes equality in distribution. Starting from $W^{(c)}$ and $J^{(c)}$ rather than W and J, we thus define rescaled versions $S^{(c)}, M^{(c)}, N^{(c)}, V^{(c)}$ and $\tilde{V}^{(c)}$ of the stochastic processes introduced above.

Throughout, we assume that the distribution of J is continuous. It is then well-known in extreme value theory that there exist $a(c)$ and $d(c)$ such that as $c \to \infty$, $M^{(c)}(c)$ converges weakly to a random variable A with a Generalized Extreme Value (GEV) distribution:

$$M^{(c)}(c) \xrightarrow{d} A, \quad \mathbb{P}(A \leq z) = G(z) = \exp\left(-[1+\xi z]^{-1/\xi}\right), \quad 1 + \xi z > 0. \tag{7}$$

Equation (7) is the so-called van-Mises representation for Extreme Value distributions. It comprises the Fréchet ($\xi > 0$), Weibull ($\xi < 0$), and Gumbel ($\xi = 0$) distributional families. We write GEV(ξ, μ, σ) for the probability distribution of the random variable $\sigma A + \mu$.

The extremal limit theorem allows for an extension to a functional limit:

$$M^{(c)}(ct) \xrightarrow{d} A(t), \quad c \to \infty. \tag{8}$$

The convergence is in J_1-topology, the strongest of the topologies defined by Skorokhod in XX and which is sometimes also called Skorokhod topology. The limit process $A(t)$ is an F-extremal process, with finite-dimensional distributions given by

$$\mathbb{P}(A(t_i) \le x_i, 1 \le i \le d) = G(\wedge_{i=1}^d x_i)^{t_1} G(\wedge_{i=2}^d x_i)^{t_2-t_1} \cdot \ldots \cdot G(x_d)^{t_d-t_{d-1}}.$$

We study the case where the waiting times W have a heavy tail with parameter $\beta \in (0,1)$, i.e.

$$\mathbb{P}(W > t) \sim L(t)t^{-\beta}, \quad t \uparrow \infty$$

for some slowly varying function $L(t)$.[1] The W_k are then said to lie in the domain of attraction of a stable law, meaning that

$$S^{(c)}(c) \xrightarrow{d} D, \quad c \to \infty \tag{9}$$

exists, for some $b(c)$ which varies regularly at ∞ with exponent $1/\beta$. The limit D is then a positively skewed stable random variable, defined via its Laplace transform $\mathbb{E}[\exp(-sD)] = \exp(-s^\beta)$. As for the maximum, the following functional limit theorem holds for the sum:

$$S^{(c)}(ct) \xrightarrow{d} D(t), \quad c \to \infty \tag{10}$$

with convergence in the Skorokhod J_1 topology. The limit $D(t)$ is a stable subordinator, i.e. an increasing Lévy process with Laplace transform $\exp(-ts^\beta)$. In the following we will exclude the case that $D(t)$ is not strictly increasing, in equality the case, that $D(t)$ is compound Poisson.

It is well known (see e.g. [MS04]) that the renewal process then satisfies the functional limit

$$N^{(c)}(t)/\tilde{b}(c) \xrightarrow{d} E(t) = \inf\{r : D(r) > t\}, \quad c \to \infty \tag{11}$$

for a scaling function $\tilde{b}(c)$ which is asymptotically inverse to $b(c)$, in the sense of [Sen76, p.20]:

$$b(\tilde{b}(c)) \sim c \sim \tilde{b}(b(c)). \tag{12}$$

Note that $\tilde{b} \in \mathrm{RV}_\infty(\beta)$ [MS04]. The limit process $E(t)$ is called the *inverse* stable subordinator [MS13].

Finally, we can give the functional limit of the CTRM and OCTRM processes. For the OCTRM, we have

$$\lim_{c\to\infty} \tilde{V}^{(c)}(t) = \tilde{V}(t) := A \circ E(t), \tag{13}$$

where $\circ$ denotes stochastic process composition. For the CTRM, we have

$$\lim_{c\to\infty} V^{(c)}(t) = V(t) := (A_- \circ E)_+(t), \tag{14}$$

where A_- denotes the version of A with left-continuous sample paths, and where the composition is re-cast to be right-continuous; see [HS17].

[1] We write $f(t) \sim g(t)$ if their quotient converges to 1.

3 Threshold events: time and magnitude

Definition 15. *Write* $\mathbf{U}$ *for the interior of the support of the GEV distribution from* (7)*. Let* $u \in \mathbf{U}$ *be a threshold, and write* $\tau^{(c)}(u) := \min\{n : J_n^{(c)} > u\}$ *for the index of the first threshold exceedance. Then*

$$Y^{(c)}(u) := J^{(c)}_{\tau^{(c)}(u)}$$

is called the ***threshold event magnitude****. The random variables*

$$T^{(c)}(u) = \sum_{n=1}^{\tau^{(c)}(u)} W_n^{(c)} \quad \textit{resp.} \quad T'^{(c)}(u) = \sum_{n=1}^{\tau^{(c)}(u)-1} W_n^{(c)} \tag{16}$$

are then called the ***threshold event time*** *for the CTRM resp. the OCTRM.*

(Note that $J^{(c)} \leq u$ iff $J \leq a(c)u + d(c)$, and that $a(c)u + d(c) \uparrow x_R$ as $c \to \infty$.)

In the following we define with $\mathbb{D}_u(E)$ the subset of all functions $\alpha \in \mathbb{D}(E)$ which are unbounded from above. Furthermore $\mathbb{D}_\uparrow(E)$ resp. $\mathbb{D}_{\uparrow\uparrow}(E)$ is the subset of all functions $\alpha \in \mathbb{D}(E)$ with α monotone increasing resp. strictly increasing. Additionally we define $\mathbb{D}_{u,\uparrow}(E) := \mathbb{D}_u(E) \cap \mathbb{D}_\uparrow(E)$ and $\mathbb{D}_{u,\uparrow\uparrow}(E) := \mathbb{D}_u(E) \cap \mathbb{D}_{\uparrow\uparrow}(E)$. The right and left continuous inverse of a cadlag process $\alpha \in D_u$ are defined by

$$\alpha^{-1}(t) := \inf\{s : \alpha(s) > t\} \qquad \text{und} \qquad \alpha^{\leftarrow}(t) := \inf\{s : \alpha(s) \geq t\}.$$

respectively. Notice that

$$\tau^{(c)}(u) = \inf\left\{t : M^{(c)}(t) > u\right\}$$

is the right continuous inverse of the partial maxima process, also called the first hitting time of the partial maxima process. The processes $T^{(c)}$ and $T'^{(c)}$ are the right continuous inverses of the CTRM rsp. the OCTRM.

Theorem 17. *At scale c, define the stochastic processes*

$$Y^{(c)} := \{Y^{(c)}(u)\}_{u\in\mathbf{U}}, \qquad T^{(c)} := \{T^{(c)}(u)\}_{u\in\mathbf{U}}, \qquad T'^{(c)} := \{T'^{(c)}(u)\}_{u\in\mathbf{U}}.$$

These processes have the following limits:

1. $\lim_{c\to\infty} Y^{(c)} = Y := \{A \circ A^{-1}(u)\}_{u\in\mathbf{U}}$
2. $\lim_{c\to\infty} T^{(c)} = T := \{D \circ A^{-1}(u)\}_{u\in\mathbf{U}}$
3. $\lim_{c\to\infty} T'^{(c)} = T' := \{(D_- \circ A_-^{-1})_+(u)\}_{u\in\mathbf{U}}$

where 1. holds with respect to the J_1 *topology and 2.&3. hold with respect to the* M_1 *topology.*

Proof. 1. We view $(A^{(c)}, D^{(c)})$ and (A, D) as random elements on the Skorokhod space Ω of all right-continuous sample paths with left-hand limits in $\mathbb{R} \times \mathbb{R}$, defined on the open interval $(0, \infty)$. Write $\mathbb{P}^{(c)}$ and $\mathbb{P}$ for their probability distributions on Ω, respectively. Let $S \subset \Omega$ be the subset of sample paths (a, d) whose first component a is non-decreasing and has a discrete range $\mathcal{R}(a) := \{a(t) : t > 0\}$. By discrete we mean that for each $r \in \mathcal{R}(a)$ there exists $\varepsilon > 0$ such that $(r - \varepsilon, r + \varepsilon) \cap \mathcal{R}(a) = \emptyset$. Since $A^{(c)}$ and A are maximal resp. extremal processes, we have $\mathbb{P}^{(c)}(S) = \mathbb{P}(S) = 1$ (see e.g. [Res13, Prop 4.1 & Prop 4.8]). Moreover, by [Res13, Prop 4.20] we have that $\mathbb{P}^{(c)} \to \mathbb{P}$, weakly with respect to the Skorokhod J_1 topology. By the continuous mapping theorem, it then suffices to show that the mapping

$$\Phi : S \ni a \mapsto a \circ a^{-1} \in S$$

is J_1-continuous. Let a_n be a sequence in S such that $a_n \to a \in S$ with respect to J_1. Let $t > 0$, and let t_n be any sequence such that $t_n \to t$. By [EK05, Th 3.6.5], it suffices to show the following three statements:

1. $\Phi(a_n)(t_n) \to \{\Phi(a)(t), \Phi(a)(t-)\}$ [2]
2. If $\Phi(a_n)(t_n) \to \Phi(a)(t)$ and $s_n \geq t$ and $s_n \to t$, then $\Phi(a_n)(s_n) \to \Phi(a)(t)$.
3. If $\Phi(a_n)(t_n) \to \Phi(a)(t-)$ and $s_n \leq t$ and $s_n \to t$, then $\Phi(a_n)(s_n) \to \Phi(a)(t-)$.

Now write $r(t) = \inf\{\mathcal{R}(a) \cap (t, \infty)\}$ and $l(t) = \sup\{\mathcal{R}(a) \cap (0, t]\}$, and define $r_n(t_n)$ and $l_n(t_n)$ similarly using a_n and t_n. Note that $\Phi(a)(t) = r(t)$ and $\Phi(a_n)(t_n) = r_n(t_n)$. If $t \notin \mathcal{R}(a)$, then

$$\mathcal{R}(a) \ni l(t) < t < r(t) \in \mathcal{R}(a).$$

Since $a_n \to a$ in J_1, it follows that $\mathcal{R}(a_n) \to \mathcal{R}(a)$ in the Hausdorff metric. But then $\Phi(a_n)(t_n) \to r(t) = \Phi(a)(t)$, and all three assertions follow, as $t_n \to t$ was chosen arbitrarily. If on the other hand $t \in \mathcal{R}(a)$, then

$$\mathcal{R}(a) \ni l(t) = t = \Phi(a)(t-) < \Phi(a)(t) = r(t) \in \mathcal{R}(a).$$

Again since $\mathcal{R}(a_n) \to \mathcal{R}(a)$ in the Hausdorff metric, it follows that $\Phi(a_n)(t_n) \to \{l(t), r(t)\}$, showing the first statement. For the second statement, if $s_n \geq t_n$, and $\Phi(a_n)(t_n) \to \Phi(a)(t)$, by a_n being non-decreasing it cannot be that $\Phi(a_n)(s_n) \to \Phi(a)(t-)$. The third statement can be shown with a similar contradiction.
2.&3. If we show that

$$T = \{D \circ A^{-1}(u)\}_{u \in \mathbf{U}} = \{V^{-1}(u)\}_{u \in \mathbf{U}} \tag{18}$$

$$T' = \{(D_- \circ A_-^{-1})_+(u)\}_{u \in \mathbf{U}} = \{\tilde{V}^{-1}(u)\}_{u \in \mathbf{U}} \tag{19}$$

the convergence 2. and 3. in the M_1-topology follows with Theorem 13.6.3 in [Whi01]. We therefore first show (18). Due to readability we define $\beta := \{A(t)\}_{t>0}$ and $\sigma := \{D(t)\}_{t>0}$, and so $\sigma^{-1}(t) = \{E(t)\}_{t>0}$. It is $\beta \in D_{\uparrow,u}$ and $\sigma \in D_{\uparrow\uparrow,u}$. Furthermore $\sigma^{-1} = \sigma^{\leftarrow}$, since σ^{-1} is continuous (see Lemma 13.6.5 in [Whi01]). Equation(18) is fulfilled, if we show that for all $x_0 < a < x_F$

$$((\beta^- \circ \sigma^{\leftarrow})^+)^{-1}(a) = \sigma(\beta^{-1}(a)). \tag{20}$$

[2] This means that $\Phi(a_n)(t_n)$ has at most two possible limit points, $\Phi(a)(t)$ or $\Phi(a)(t-)$.

Since $(\beta^- \circ \sigma^{\leftarrow})^+ \in D_{\uparrow,u}$ und $(x^{-1})^{-1} = x$ for $x \in D_{\uparrow,u}$ (see Corollary 13.6.1 in [Whi01]) is (20) equivalent to

$$(\beta^- \circ \sigma^{\leftarrow})^+(a) = (\sigma \circ \beta^{-1})^{-1}(a).$$

This is again $x^{\leftarrow}(t) = x^{-1}(t-)$ equivalent to

$$(\beta^- \circ \sigma^{\leftarrow})(a) = (\sigma \circ \beta^{-1})^{\leftarrow}(a).$$

The last equality is true since due to $x^{\leftarrow}(t) \leq s \Leftrightarrow x(s) \geq t$ for $x \in D_{\uparrow,u}$ (see Lemma 13.6.3. in [Whi01]) and $(x^{-1})^{\leftarrow} = x^-$ for $x \in \mathbb{D}_{\uparrow,u}$, it follows that

$$\begin{aligned}(\sigma \circ \beta^{-1})^{\leftarrow}(a) &= \inf\left\{s \in \mathbf{U} : \sigma(\beta^{-1}(s)) \geq a\right\} \\ &= \inf\left\{s \in \mathbf{U} : \sigma^{\leftarrow}(a) \leq \beta^{-1}(s)\right\} \\ &= \beta^-(\sigma^{\leftarrow}(a)).\end{aligned}$$

Equation (19) is fulfilled if we show that for all $x_0 < a < x_F$ and $s \geq 0$

$$(\beta \circ \sigma^{-1})^{-1} = (\sigma_- \circ \beta^{\leftarrow})^+.$$

However, this is equivalent to

$$(\beta \circ \sigma^{-1})^{\leftarrow} = \sigma_- \circ \beta^{\leftarrow},$$

which is true because

$$\begin{aligned}(\beta \circ \sigma^{-1})^{\leftarrow}(a) &= \inf\left\{s \geq 0 : (\beta \circ \sigma^{-1})(s) \geq a\right\} \\ &= \inf\left\{s \geq 0 : \beta^{\leftarrow}(a) \leq \sigma^{-1}(s)\right\} \\ &= (\sigma^{-1})^{\leftarrow}(\beta^{\leftarrow}(a))\end{aligned}$$

and the fact that $(\sigma^{-1})^{\leftarrow} = \sigma_-$. Hence the assertion 3. follows.

□

4 Joint distribution of threshold event and threshold crossing time

Define the potential measure, or expected occupation time, of the bivariate Markov process $(A(u), D(u))$:

$$U(B) = \mathbb{E}\left[\int_0^\infty \mathbf{1}\{(A_u, D_u) \in B\}\, du\right]$$

Define the bivariate tail function $\overline{\nu}(x,t)$ via

$$\overline{\nu}(x,t) = \lim_{c\to\infty} c\mathbb{P}[J^{(c)} > x, W^{(c)} > t] \tag{21}$$

Theorem 22.

$$\mathbb{P}[T(u) > t, Y(u) > y] = \iint_{x\leq u, t'\in[0,t)} \overline{\nu}(y, t-t')U(dx, dt') \tag{23}$$

5 Semi-Markov property of CTRMs

The maximal process $M = \{M(n)\}_{n\in\mathbb{N}}$ and the extremal process $A = \{A(t)\}_{t>0}$ are well understood, see e.g. [Res13, Chapter 4]. Both are Markov processes with piecewise constant non-decreasing sample paths, in discrete time (M) resp. continuous time (A). The sample paths of A are right-continuous. The jump times (i.e. points of increase) τ_n are called *record times*, and their values at the record times are called *records*. At a record x, the holding time for M follows a $\mathrm{Geo}(1-F(x))$ distribution, where $F(x)$ is the cumulative distribution of magnitudes. For A, the holding time is $\mathrm{Exp}(-\log G(x))$ distributed, where $G(x)$ is the underlying GEV distribution.[3] The probability distribution of the next record y is simply given by

$$\mathbb{P}[M(\tau_{n+1}) > y | M(\tau_n) = x] = 1 \wedge \frac{1-F(y)}{1-F(x)}, \tag{24}$$

$$\mathbb{P}[A(\tau_{n+1}) > y | A(\tau_n) = x] = 1 \wedge \frac{-\log G(y)}{-\log G(x)}. \tag{25}$$

Finally, even more can be said about the structure of records: the set of all records is a Poisson Point process (PPP) with mean measure $R(dx) = d(-\log(1-F(x)))$ for M and $d(-\log(-\log G(x)))$ for A. In the special case $F(x) = 1 - e^{-x}$ resp. where $G(x)$ is the Gumbel distribution ($\xi = 0$), the mean measure is Lebesgue measure (on $(0,\infty)$ resp. $\mathbb{R}$).

Paths of the CTRM process $V^{(c)}$ and the CTRM limit process V result from time changes of $M^{(c)}$ and A. Hence the set of points traversed by them, i.e. the set of records, follows the same probability law, and the only difference lies in the distribution of the holding times.

Proposition 26. *The CTRM process $V^{(c)}$, the OCTRM process $\tilde{V}^{(c)}$ and their scaling limits V resp. $\tilde{V}$, are all Semi-Markov processes. Their holding times at a record x are, respectively,*

$$T^{(c)}(x), \quad W^{(c)}(x) + T'^{(c)}(x), \quad T(x), \quad W(x) + T'(x), \tag{27}$$

where $W^{(c)}(x)$ resp. $W(x)$ is an independent random variable with density

$$t \mapsto \frac{\nu^{(c)}(x,t)}{\int\limits_{[0,\infty)} \nu^{(c)}(x,t')\,dt'} \qquad \textit{resp.} \qquad t \mapsto \frac{\nu(x,t)}{\int\limits_{[0,\infty)} \nu(x,t')\,dt'} \tag{28}$$

Proof. As mentioned above, the sequence of records is a Markov chain, and it only remains to show that the holding time $\tau_{n+1} - \tau_n$ only depends on the current record $V(\tau_n)$, that is,

$$\mathbb{P}[\tau_{n+1} - \tau_n | \tau_1, \ldots, \tau_n, V(\tau_1), \ldots V(\tau_n)] = \mathbb{P}[\tau_{n+1} - \tau_n | V(\tau_n)]. \tag{29}$$

For the CTRM, if $x = V^{(c)}(\tau_n) = J_k^{(c)}$ is a record, then the record time is $\tau_n = W_1^{(c)} + \ldots + W_k^{(c)}$. It marks the beginning of the waiting time $W_{k+1}^{(c)}$, a magnitude $J_{k+1}^{(c)}$, and so

[3] In general, extremal processes can be defined using any continuous distribution F, but the limit (8) results in the GEV distribution G.

on. Hence the CTRM is renewed at τ_n, and the time until the next record $y = V^{(c)}(\tau_{n+1})$, i.e. the holding time at x, is equal in distribution to $T^{(c)}(x)$.
For the OCTRM, and a record as above, the record time is $\tau_n = W_1^{(c)} + \ldots + W_{k-1}^{(c)}$. This shows that the OCTRM is renewed only after the following waiting time $W_k^{(c)}$. The holding time is hence equal in distribution to $W(x)+T'^{(c)}(x)$, where $W(x)$ has the density (28)
For the CTRM limit, let τ be a record time, i.e. $V(\tau) = x$ and $V(t) < x$ if $t < \tau$. Then for any $\varepsilon > 0$, $\tau = \inf\{t : V(t) > x - \varepsilon\} = \inf\{t : (A_- \circ E)_+(t) > x - \varepsilon\} = \inf\{t : A_- \circ E(t) > x - \varepsilon\}$. By construction, E must be right-increasing at τ, i.e. $E(t) > E(\tau)$ if $t > \tau$. If it wasn't, we would have $V(\tau) = A_- \circ E(\tau)$, which is the value of A just before it reaches x, in cotradition to $V(\tau) = x$. Thus $\tau = D(u)$ for some $u \geq 0$, which means that τ is a renewal point (compare [Ber99]). The time until an exceedance of x is hence just $T(x)$.
For the OCTRM limit, let x be a record and τ a record time, i.e. $\tilde{V}(\tau) = A \circ E(t) = x$ and $\tilde{V}(\tau - \varepsilon) < x$. Then $E(\tau)$ is a jump time of A, i.e. $u = E(\tau)$ marks the first occurrence of an event $J_u \geq x$. Since we do not assume the uncoupled case, D may also have a jump $W_u > 0$ at u. This means that E is constant on the interval $[\tau, \tau + W_u]$, and the next renewal time after τ is $\tau + W_u$. Given the event $J_u = x$, the distribution of $W_u | J_u = x$ has the density (28). Then the time until the next record y, i.e. the holding time at x, is equal in distribution to $W(x) + T'(x)$. □

Theorem 30. *Suppose F_J is continuous, and let*

$$R(dx) = d(-\log(1 - F(x))), \quad S(dx) = d(-\log(-\log G(x))). \tag{31}$$

Then

1. $\{(V^{(c)}(\tau_n), \tau_{n+1} - \tau_n), n \geq 1\}$ *are the points of a bivariate Poisson random measure on* $(x_l, x_0) \times (0, \infty)$ *with mean measure*
$$\mu^*(dx, dy) = R(dx)\mathbb{P}(T^{(c)}(x) \in dy), \tag{32}$$
2. $\{(\tilde{V}^{(c)}(\tau_n), \tau_{n+1} - \tau_n), n \geq 1\}$ *are the points of a bivariate Poisson random measure on* $(x_l, x_0) \times (0, \infty)$ *with mean measure*
$$\mu^*(dx, dy) = R(dx)\mathbb{P}(W^{(c)}(x) + T^{(c)}(x) \in dy), \tag{33}$$
3. $\{(V(\tau_n), \tau_{n+1} - \tau_n), n \geq 1\}$ *are the points of a bivariate Poisson random measure on* $\mathbf{U} \times (0, \infty)$ *with mean measure*
$$\mu^*(dx, dy) = S(dx)\mathbb{P}(T(x) \in dy), \tag{34}$$
4. $\{(\tilde{V}(\tau_n), \tau_{n+1} - \tau_n), n \geq 1\}$ *are the points of a bivariate Poisson random measure on* $\mathbf{U} \times (0, \infty)$ *with mean measure*
$$\mu^*(dx, dy) = S(dx)\mathbb{P}(W(x) + T'(x) \in dy). \tag{35}$$

Proof. By [Res13, Prop 4.1(iii)] and the observation that $V^{(c)}$ and $\tilde{V}^{(c)}$ are time changes of maximal processes, The records $V^{(c)}(\tau_n)$ resp. $\tilde{V}^{(c)}(\tau_n)$ form a PPP with mean measure $R(dx)$. Similarly, by [Res13, Prop 4.8(iii)], the records $V(\tau_n)$ and $\tilde{V}(\tau_n)$ form a PPP with mean measure $S(dx)$. By Proposition 26, the holding times $\tau_{n+1} - \tau_n$ are conditionally independent given their record $V^{(c)}(\tau_n), \tilde{V}^{(c)}(\tau_n)$, etc. Applying [Res13, Prop 3.8] to the kernels $K(x, dy) = \mathbb{P}(T^{(c)}(x) \in dy)$, etc., marks each record with its corresponding holding time, and proves the theorem. □

6 Governing Equations of CTRM limits

7 Conclusion

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