RELATION BETWEEN TERMS OF SEQUENCES AND INTEGRAL POWERS OF METALLIC RATIOS

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Abstract

Among several interesting sequences that occur in mathematics, sequences whose successive terms converging to specific numbers called metallic ratios are very special having plenty of applications in branches of science, engineering and nature. In this paper, I will introduce the general sequence corresponding to metallic ratios and obtain interesting relationship between the terms and its integral powers.

1 Introduction

It is well known that the ratio of successive terms of Fibonacci sequence converges to Golden Ratio. We can generalize the Fibonacci sequence in a natural way so that the ratio of successive terms converges to specific real numbers called Metallic Ratios. In particular, the golden, silver and bronze ratios are special cases of these metallic ratios. In this paper, I will prove some interesting theorems relating terms of the sequence that are defined recursively and integral powers of metallic ratios.

Definition 1.1. Let k be a natural number. The terms of metallic ratio of order k sequence is defined recursively by

$$M_{n+2} = kM_{n+1} + M_n, \text{ for } n \geq 1 \tag{1}$$

with initial conditions $M_0 = 0, M_1 = 1, M_2 = k$.

The terms of the metallic ratio of order k sequence as defined in equation (1) are given by

$$0, 1, k, k^2+1, k^3+2k, k^4+3k^2+k, k^5+4k^3+k^2+2k, k^6+5k^4+k^3+5k^2+k, \ldots \tag{2}$$

1.1 Metallic ratio of order k

Through the recurrence relation defined in equation (1), we can solve for explicitly. Using the shift operator, the recurrence relation yield the quadratic equation $m^2 - km - 1 = 0$. The two real roots of this quadratic equation are given by $m = \dfrac{k \pm \sqrt{k^2+4}}{2}$. The positive value among these two roots is defined as the metallic ratio of order k denoted by ρ_k. Thus,

$$\rho_k = \frac{k + \sqrt{k^2+4}}{2}. \tag{3}$$

Since the sum of two roots is k, the other root is

$$k - \rho_k = \frac{k - \sqrt{k^2+4}}{2}. \tag{4}$$

1.2 Special cases of metallic ratio

(i) If $k = 1$, then from equation (3)

$$\rho_1 = \frac{1+\sqrt{5}}{2} \tag{5}$$

is called the golden ratio.

(ii) If $k = 2$, then from equation (3)

$$\rho_2 = 1 + \sqrt{2} \tag{6}$$

is called the silver ratio.

(iii) If $k = 3$, then from equation (3)

$$\rho_3 = \frac{3+\sqrt{13}}{2} \tag{7}$$

is called the bronze ratio.

The numbers give by equation (5),(6) and (7) form the metallic ratios of first second and third orders, respectively.

Lemma 1.2. *If $\rho_k = \dfrac{k^2+4}{2}$ is the metallic ratio of order k, then*

$$\rho_k - \frac{1}{\rho_k} = k$$

Proof. From equation (3) and (4), we know that $\rho_k = \frac{k+\sqrt{k^2+4}}{2}$ and $k - \rho_k = \frac{k-\sqrt{k^2+4}}{2}$ are the two real roots of $m^2 - km - 1 = 0$. We have

$$\begin{aligned}
\frac{1}{\rho_k} &= \frac{1}{\frac{k+\sqrt{k^2+4}}{2}} \\
&= \frac{2}{k+\sqrt{k^2+4}} \\
&= \frac{2(k-\sqrt{k^2+4})}{(k+\sqrt{k^2+4})(k-\sqrt{k^2+4})} \\
&= \frac{2(k-\sqrt{k^2+4})}{k^2-(k^2+4)} \\
&= \frac{2(k-\sqrt{k^2+4})}{k^2-k^2-4} \\
&= \frac{2(k-\sqrt{k^2+4})}{-4} \\
&= \frac{k-\sqrt{k^2+4}}{-2} \\
&= -\frac{k-\sqrt{k^2+4}}{2} \\
&= -(k-\rho_k) \\
&= \rho_k - k,
\end{aligned}$$

hence

$$\begin{aligned}
\frac{1}{\rho_k} + k &= \rho_k \\
k &= \rho_k - \frac{1}{k}
\end{aligned}$$

□

2 Preliminary

Properties of real number ([1])

Let a, b, and c represent real numbers.

1. Closure Properties

$$\begin{aligned}
a+b &\quad \text{is a real number.} \\
ab &\quad \text{is a real number.}
\end{aligned}$$

2. Commutative Properties

$$a + b = b + a.$$
$$ab = ba.$$

3. Associative Properties

$$(a + b) + c = (a + b) + c.$$
$$(ab)c = a(bc).$$

4. Identity Properties

   There exist a unique real number 0 such that

$$a + 0 = a \quad \text{and} \quad 0 + a = a.$$

   There exist a unique real number 1 such that

$$a \cdot 1 = a \quad \text{and} \quad 1 \cdot a = a.$$

5. Inverse Properties

   There exist a unique real number $-a$ such that

$$a + (-a) = 0 \quad \text{and} \quad -a + a = 0.$$

   if $a \neq 0$, there exists a unique real numbers $\frac{1}{a}$ such that

$$a \cdot \frac{1}{a} = 1 \quad \text{and} \quad \frac{1}{a} \cdot a = 1.$$

6. Distributive Properties

$$a(b + c) = ab + ac$$
$$a(b - c) = ab - ac$$

Properties of fractions([2])

Let a, b, c and d be real numbers, variables or algebraic expressions such that $b \neq 0$ and $d \neq 0$.

1. Equivalent fractions:

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if } ad = bc.$$

2. Rules of signs:
$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b} \quad \text{and} \quad \frac{-a}{-b} = \frac{a}{b}.$$
3. Generate equivalent fractions:
$$\frac{a}{b} = \frac{ac}{bc}, \quad c \neq 0.$$
4. Add or subtract with like denominators:
$$\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}.$$
5. Add or subtract with unlike denominators:
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$
6. Divide fractions:
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}, \quad c \neq 0.$$

Properties of integer exponents ([2])

For n and m be integers and a and b are real numbers.

1. $a^m a^n = a^{m+n}$.
2. $(a^n)^m = a^{mn}$.
3. $(ab)^m = a^m b^m$.
4. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$.
5. $\frac{a^m}{a^n} = \begin{cases} a^{m-n} \\ \frac{1}{a^{n-m}}, a \neq 0. \end{cases}$

Special products ([3])

Let a and b be real numbers, variables, or algebraic expressions.

1. Sum and difference of same terms
$$(a+b)(a-b) = a^2 - b^2.$$
2. Square of a Binomial
$$(a+b)^2 = a^2 + 2ab + b^2.$$
$$(a-b)^2 = a^2 - 2ab + b^2.$$
3. Cube of Binomial
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

Quadratic formula

Theorem 2.1 ([4]). *If $p(x) = ax^2 + bx + c = 0, a \neq 0$ and $0 \leq b^2 - 4ac$. Then The two real roots of p are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Principle of Strong Mathematical Induction

Theorem 2.2 ([5]). *To prove that $P(n)$ is true for all positive integer n, Where $P(n)$ is a propositional function, we complete two steps:*

Basis step : We verify that the proposition $P(1)$ is true.

Inductive step: We show that the conditional statement $[P(1) \wedge P(2) \wedge \ldots, P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

3 Main Results

We will consider $k = 1$ in equation (1) and (2). Then the recurrence relation would become

$$M_{n+2} = M_{n+1} + M_n, \text{ for } n \geq 1 \tag{8}$$

where $M_0 = 0, M_1 = 1, M_2 = 1$.

From equation (2) we get Fibonacci sequence whose terms are given by

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

It is well known that (see [6]) the ratio of successive terms of the Fibonacci sequence approaches the golden ratio $\rho_1 = \varphi = \dfrac{1+\sqrt{5}}{2}$. We will now prove the following theorem.

Theorem 3.1. *If $\{M_n\}$ is the Fibonacci sequence as defined on equation (8), then for any positive integer n, we have*

$$\varphi^n + \left(-\frac{1}{\varphi}\right)^n = M_{n-1} + M_{n+1}. \tag{9}$$

Proof. We prove by using principle of strong mathematical induction on $n \in \mathbb{N}$.

Let $P(n) : \varphi^n + \left(-\dfrac{1}{\varphi}\right)^n = M_{n-1} + M_{n+1}$.

(i) To show that $P(1)$ is true, that is

$$\varphi - \frac{1}{\varphi} = M_0 + M_2.$$

From Lemma 1.2 and $k = 1$, we have $\rho_1 - \dfrac{1}{\rho_1} = 1$.

Since $\varphi = \rho_1$ and $M_0 = 0, M_2 = 1$then

$$\begin{aligned}\varphi - \frac{1}{\varphi} &= 1\\ &= 0 + 1\\ &= M_0 + M_2.\end{aligned}$$

Thus $P(1)$ is true.

(ii) For $r \geq 1$. We assume that $P(r-1)$ and $P(r)$ are true, that is

$$\varphi^{r-1} + \left(-\frac{1}{\varphi}\right)^{r-1} = M_{r-2} + M_r,$$

and

$$\varphi^{r} + \left(-\frac{1}{\varphi}\right)^{r} = M_{r-1} + M_{r+1},$$

To show that $P(r+1)$ is true, that is

$$\varphi^{r+1} + \left(-\frac{1}{\varphi}\right)^{r+1} = M_r + M_{r+2}.$$

Consider

$$\begin{aligned}M_r + M_{r+2} &= M_{r-1} + M_{r-2} + M_{r+1} + M_r\\ &= M_{r-2} + M_r + M_{r-1} + M_{r+1}\\ &= \varphi^{r-1} + \left(-\frac{1}{\varphi}\right)^{r-1} + \varphi^r + \left(-\frac{1}{\varphi}\right)^{r}\\ &= \varphi^{r-1} + \varphi^r + \left(-\frac{1}{\varphi}\right)^{r-1} + \left(-\frac{1}{\varphi}\right)^{r}\\ &= \varphi^{r+1}(\varphi^{-2} + \varphi^{-1}) + \left(-\frac{1}{\varphi}\right)^{r+1}\left[\left(-\frac{1}{\varphi}\right)^{-2} + \left(-\frac{1}{\varphi}\right)^{-1}\right]\\ &= \varphi^{r+1}\left(\frac{1}{\varphi^2} + \frac{1}{\varphi}\right) + \left(-\frac{1}{\varphi}\right)^{r+1}(\varphi^2 - \varphi)\\ &= \varphi^{r+1} \cdot \frac{1+\varphi}{\varphi^2} + \left(-\frac{1}{\varphi}\right)^{r+1}(\varphi^2 - \varphi).\end{aligned}$$

Since φ is the root of $m^2 - m - 1 = 0$, we obtain $\varphi^2 - \varphi - 1 = 0$, then $\varphi^2 = \varphi + 1$, thus

$$\begin{aligned}M_r + M_{r+2} &= \varphi^{r+1} \cdot \frac{1+\varphi}{\varphi+1} + \left(-\frac{1}{\varphi}\right)^{r+1}(\varphi + 1 - \varphi)\\ &= \varphi^{r+1} + \left(-\frac{1}{\varphi}\right)^{r+1}.\end{aligned}$$

Thus $P(r+1)$ is true.

By principle of strong mathematical induction, we obtain

$$\varphi^n + \left(-\frac{1}{\varphi}\right)^n = M_{n-1} + M_{n+1}, \text{ for all } n \in \mathbb{N}.$$

□

We will consider $k = 2$ in equation (1) and (2). Then the recurrence relation would become

$$M_{n+2} = 2M_{n+1} + M_n. \text{ for } n \geq 1 \tag{10}$$

where $M_0 = 0, M_1 = 1, M_2 = 2$.

From equation (2) we get sequence whose terms are given by

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \ldots$$

It can be show that (see [6]) the ratio of successive terms of this sequence approaches the silver ratio $\rho_2 = \lambda = 1 + \sqrt{2}$. We will now prove the following theorem.

Theorem 3.2. *If* $\{M_n\}$ *is the sequence as defined on equation* (10)*, then for any positive integer* n*, we have*

$$\lambda^n + \left(-\frac{1}{\lambda}\right)^n = M_{n-1} + M_{n+1}. \tag{11}$$

Proof. We prove by using principle of strong mathematical induction on $n \in \mathbb{N}$.

Let $P(n) : \lambda^n + \left(-\frac{1}{\lambda}\right)^n = M_{n-1} + M_{n+1}$.

(i) To show that $P(1)$ is true, that is

$$\lambda - \frac{1}{\lambda} = M_0 + M_2.$$

From Lemma 1.2 and $k = 2$, we have $\rho_2 - \dfrac{1}{\rho_2} = 2$.

Since $\lambda = \rho_2$ and $M_0 = 0, M_2 = 2$, then

$$\begin{aligned} \lambda - \frac{1}{\lambda} &= 2 \\ &= 0 + 2 \\ &= M_0 + M_2, \end{aligned}$$

thus $P(1)$ is true.

(ii) For $r \geq 1$. We assume that $P(r-1)$ and $P(r)$ are true, that is

$$\lambda^{r-1} + \left(-\frac{1}{\lambda}\right)^{r-1} = M_{r-2} + M_r,$$

and

$$\lambda^{r} + \left(-\frac{1}{\lambda}\right)^{r} = M_{r-1} + M_{r+1},$$

respectively.

To show that $P(r+1)$ is true, that is

$$\lambda^{r+1} + \left(-\frac{1}{\lambda}\right)^{r+1} = M_r + M_{r+2}.$$

Consider

$$\begin{aligned}
M_r + M_{r+2} &= 2M_{r-1} + M_{r-2} + 2M_{r+1} + M_r \\
&= M_{r-2} + M_r + 2M_{r-1} + 2M_{r+1} \\
&= M_{r-2} + M_r + 2(M_{r-1} + M_{r+1}) \\
&= \lambda^{r-1} + \left(-\frac{1}{\lambda}\right)^{r-1} + 2\left[\lambda^r + \left(-\frac{1}{\lambda}\right)^{r}\right] \\
&= \lambda^{r-1} + \left(-\frac{1}{\lambda}\right)^{r-1} + 2\lambda^r + 2\left(-\frac{1}{\lambda}\right)^{r} \\
&= \lambda^{r-1} + 2\lambda^r + \left(-\frac{1}{\lambda}\right)^{r-1} + 2\left(-\frac{1}{\lambda}\right)^{r} \\
&= \lambda^{r+1}(\lambda^{-2} + 2\lambda^{-1}) + \left(-\frac{1}{\lambda}\right)^{r+1}\left[\left(-\frac{1}{\lambda}\right)^{-2} + 2\left(-\frac{1}{\lambda}\right)^{-1}\right] \\
&= \lambda^{r+1}\left(\frac{1}{\lambda^2} + \frac{2}{\lambda}\right) + \left(-\frac{1}{\lambda}\right)^{r+1}(\lambda^2 - 2\lambda) \\
&= \lambda^{r+1} \cdot \frac{1+2\lambda}{\lambda^2} + \left(-\frac{1}{\lambda}\right)^{r+1}(\lambda^2 - 2\lambda).
\end{aligned}$$

Since λ is the root of $m^2 - 2m - 1 = 0$, we obtain $\lambda^2 - 2\lambda - 1 = 0$, then $\lambda^2 = 2\lambda + 1$, thus

$$\begin{aligned}
M_r + M_{r+2} &= \lambda^{r+1} \cdot \frac{1+2\lambda}{2\lambda+1} + \left(-\frac{1}{\lambda}\right)^{r+1}(2\lambda + 1 - 2\lambda) \\
&= \lambda^{r+1} + \left(-\frac{1}{\lambda}\right)^{r+1}.
\end{aligned}$$

Thus $P(r+1)$ is true.

By principle of strong mathematical induction, we obtain

$$\lambda^n + \left(-\frac{1}{\lambda}\right)^n = M_{n-1} + M_{n+1}, \text{ for all } n \in \mathbb{N}.$$

$\square$

We will consider $k = 3$ in equation (1) and (2). Then the recurrence relation would become

$$M_{n+2} = 3M_{n+1} + M_n. \text{ for } n \geq 1 \tag{12}$$

where $M_0 = 0, M_1 = 1, M_2 = 3$.

From equation (2) we get sequence whose terms are given by

$$0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, \ldots$$

It can be show that (see [6]) the ratio of successive terms of this sequence approaches the silver ratio $\rho_3 = \mu = \dfrac{3+\sqrt{13}}{2}$. We will now prove the following theorem.

Theorem 3.3. *If* $\{M_n\}$ *is the sequence as defined on equation* (12)*, then for any positive integer* n*, we have*

$$\mu^n + \left(-\frac{1}{\mu}\right)^n = M_{n-1} + M_{n+1}. \text{ for } n \geq 1 \tag{13}$$

Proof. We prove by using principle of mathematical induction on $n \in \mathbb{N}$.

Let $P(n) : \mu^n + \left(-\frac{1}{\mu}\right)^n = M_{n-1} + M_{n+1}$.

(i) To show that $P(1)$ is true, that is

$$\mu - \frac{1}{\mu} = M_0 + M_2.$$

From Lemma 1.2 and $k = 3$, we have $\rho_3 - \dfrac{1}{\rho_3} = 3$.

Since $\mu = \rho_3$ and $M_0 = 0, M_2 = 3$, then

$$\begin{aligned} \mu - \frac{1}{\mu} &= 3 \\ &= 0 + 3 \\ &= M_0 + M_2, \end{aligned}$$

thus $P(1)$ is true.

(ii) For $r \geq 1$. We assume that $P(r-1)$ and $P(r)$ are true, that is

$$\mu^{r-1} + \left(-\frac{1}{\mu}\right)^{r-1} = M_{r-2} + M_r,$$

and

$$\mu^{r} + \left(-\frac{1}{\mu}\right)^{r} = M_{r-1} + M_{r+1},$$

repectively.

To show that $P(r+1)$ is true, that is

$$\mu^{r+1} + \left(-\frac{1}{\mu}\right)^{r+1} = M_{r} + M_{r+2}.$$

Consider

$$\begin{aligned}
M_r + M_{r+2} &= 3M_{r-1} + M_{r-2} + 3M_{r+1} + M_r \\
&= M_{r-2} + M_r + 3M_{r-1} + 3M_{r+1} \\
&= M_{r-2} + M_r + 3(M_{r-1} + M_{r+1}) \\
&= \mu^{r-1} + \left(-\frac{1}{\mu}\right)^{r-1} + 3\left[\mu^r + \left(-\frac{1}{\mu}\right)^{r}\right] \\
&= \mu^{r-1} + \left(-\frac{1}{\mu}\right)^{r-1} + 3\mu^r + 3\left(-\frac{1}{\mu}\right)^{r} \\
&= \mu^{r-1} + 3\mu^r + \left(-\frac{1}{\mu}\right)^{r-1} + 3\left(-\frac{1}{\mu}\right)^{r} \\
&= \mu^{r+1}(\mu^{-2} + 3\mu^{-1}) + \left(-\frac{1}{\mu}\right)^{r+1}\left[\left(-\frac{1}{\mu}\right)^{-2} + 3\left(-\frac{1}{\mu}\right)^{-1}\right] \\
&= \mu^{r+1}\left(\frac{1}{\mu^2} + \frac{3}{\mu}\right) + \left(-\frac{1}{\mu}\right)^{r+1}(\mu^2 - 3\mu) \\
&= \mu^{r+1}\cdot\frac{1+3\mu}{\mu^2} + \left(-\frac{1}{\mu}\right)^{r+1}(\mu^2 - 3\mu).
\end{aligned}$$

Since μ is the root of $m^2 - 3m - 1 = 0$, we obtain $\mu^2 - 3\mu - 1 = 0$, then $\mu^2 = 3\mu + 1$, thus

$$\begin{aligned}
M_r + M_{r+2} &= \mu^{r+1}\cdot\frac{1+3\mu}{3\mu+1} + \left(-\frac{1}{\mu}\right)^{r+1}(3\mu + 1 - 3\mu) \\
&= \mu^{r+1} + \left(-\frac{1}{\mu}\right)^{r+1}.
\end{aligned}$$

Thus $P(r+1)$ is true.

By principle of strong mathematical induction, we obtain

$$\mu^n + \left(-\frac{1}{\mu}\right)^n = M_{n-1} + M_{n+1}, \text{ for all } n \in \mathbb{N}.$$

□

Next, I will obtain a general result connecting terms of the metallic ratio sequence defined in equation (1) and (2). Then the recurrence relation of general metallic ratio sequence is given by

$$M_{n+2} = kM_{n+1} + M_n.$$

where $M_0 = 0, M_1 = 1, M_2 = k$.

From equation (2), we get a sequence whose terms are given by

$$0, 1, k, k^2+1, k^3+2k, k^4+3k^2+k, k^5+4k^3+k^2+2k, k^6+5k^4+k^3+5k^2+k, \ldots$$

It can be shown that (see [6]) the ratio of successive terms of this sequence approaches the metallic ratio of order k given by $\rho_k = \dfrac{k+\sqrt{k^2+4}}{2}$. We will now prove the following theorem.

Theorem 3.4. *If $\{M_n\}$ is the sequence as defined on equation (1), then for any positive integer n, we have*

$$\rho_k^n + \left(-\frac{1}{\rho_k}\right)^n = M_{n-1} + M_{n+1}. \tag{14}$$

Proof. We prove by using principle of strong mathematical induction on $n \in \mathbb{N}$

Let $P(n) : \rho_k^n + \left(-\frac{1}{\rho_k}\right)^n = M_{n-1} + M_{n+1}$.

(i) To show that $P(1)$ is true, that is

$$\rho_k - \frac{1}{\rho_k} = M_0 + M_2.$$

From Lemma 1.2, we have $\rho_k - \frac{1}{\rho_k} = k$ and $M_0 = 0, M_2 = k$, then

$$\begin{aligned} \rho_k - \frac{1}{\rho_k} &= k \\ &= 0 + k \\ &= M_0 + M_2, \end{aligned}$$

thus $P(1)$ is true.

(ii) For $r \geq 1$. We assume that $P(r-1)$ and $P(r)$ are true, that is

$$\rho_k^{r-1} + \left(-\frac{1}{\rho_k}\right)^{r-1} = M_{r-2} + M_r,$$

and

$$\rho_k^r + \left(-\frac{1}{\rho_k}\right)^r = M_{r-1} + M_{r+1},$$

respectively.

To show that $P(r+1)$ is true, that is

$$\rho_k^{r+1} + \left(-\frac{1}{\rho_k}\right)^{r+1} = M_r + M_{r+2}.$$

Consider

$$\begin{aligned}
M_r + M_{r+2} &= kM_{r-1} + M_{r-2} + kM_{r+1} + M_r \\
&= M_{r-2} + M_r + kM_{r-1} + kM_{r+1} \\
&= M_{r-2} + M_r + k(M_{r-1} + M_{r+1}) \\
&= \rho_k^{r-1} + \left(-\frac{1}{\rho_k}\right)^{r-1} + k\left[\rho_k^r + \left(-\frac{1}{\rho_k}\right)^r\right] \\
&= \rho_k^{r-1} + \left(-\frac{1}{\rho_k}\right)^{r-1} + k\rho_k^r + k\left(-\frac{1}{\rho_k}\right)^r \\
&= \rho_k^{r-1} + k\rho_k^r + \left(-\frac{1}{\rho_k}\right)^{r-1} + k\left(-\frac{1}{\rho_k}\right)^r \\
&= \rho_k^{r+1}(\rho_k^{-2} + k\rho_k^{-1}) + \left(-\frac{1}{\rho_k}\right)^{r+1}\left[\left(-\frac{1}{\rho_k}\right)^{-2} + k\left(-\frac{1}{\rho_k}\right)^{-1}\right] \\
&= \rho_k^{r+1}\left(\frac{1}{\rho_k^2} + \frac{k}{\rho_k}\right) + \left(-\frac{1}{\rho_k}\right)^{r+1}(\rho_k^2 - k\rho_k) \\
&= \rho_k^{r+1} \cdot \frac{1 + k\rho_k}{\rho_k^2} + \left(-\frac{1}{\rho_k}\right)^{r+1}(\rho_k^2 - k\rho_k).
\end{aligned}$$

Since ρ_k is the root of $m^2 - km - 1 = 0$, we obtain $\rho_k^2 - k\rho_k - 1 = 0$, then $\rho_k^2 = k\rho_k + 1$, thus

$$\begin{aligned}
M_r + M_{r+2} &= \rho_k^{r+1} \cdot \frac{1 + k\rho_k}{k\rho_k + 1} + \left(-\frac{1}{\rho_k}\right)^{r+1}(k\rho_k + 1 - k\rho_k) \\
&= \rho_k^{r+1} + \left(-\frac{1}{\rho_k}\right)^{r+1}.
\end{aligned}$$

Thus $P(r+1)$ is true.

By principle of strong mathematical induction, we obtain

$$\rho_k^n + \left(-\frac{1}{\rho_k}\right)^n = M_{n-1} + M_{n+1}, \text{ for all } n \in \mathbb{N}.$$

□

4 Conclusion

Introducing golden, silver and bronze ratios through general metallic ratio sequence as defined in equation (1) and (2) I had established three interesting theorems 3.1, 3.2 and 3.3 respectively. I had established the same result for general metallic ratio of order k. It is amusing to notice that all metallic ratios satisfy similar relation regarding the integral powers and alternate terms of the sequence. This result may pave way for proving other similar results regarding metallic ratios.

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