RELATION BETWEEN TERMS OF SEQUENCES AND INTEGRAL POWERS OF METALLIC RATIOS

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Abstract

Among several interesting sequences that occur in mathematics, sequences whose successive terms converging to specific numbers called metallic ratios are very special having plenty of applications in branches of science, engineering and nature. In this paper, I will introduce the general sequence corresponding to metallic ratios and obtain interesting relationship between the terms and its integral powers.

1 Introduction

It is well known that the ratio of successive terms of Fibonacci sequence converges to Golden Ratio. We can generalize the Fibonacci sequence in a natural way so that the ratio of successive terms converges to specific real numbers called Metallic Ratios. In particular, the golden, silver and bronze ratios are special cases of these metallic ratios. In this paper, I will prove some interesting theorems relating terms of the sequence that are defined recursively and integral powers of metallic ratios.

Definition 1.1. Let k be a natural number. The terms of metallic ratio of order k sequence is defined recursively by

$$M_{n+2} = kM_{n+1} + M_n \tag{1}$$

for $n \geq 1$ with initial condition $M_0 = 0, M_1 = 1, M_2 = k$.

The terms of the metallic ratio of order k sequence as defined in (1) are given by

$$0, 1, k, k^2 + 1, k^3 + 2k, k^4 + 3k^2 + k, k^5 + 4k^3 + k^2 + 2k, k^6 + 5k^4 + k^3 + 5k^2 + k, \dots \tag{2}$$

1.1 Metallic ratio of order k

Through the recurrence relation defined in (1), We can solve for explicitly. Using the shift operator, the recurrence relation yield the quadratic equation $m^2 - km - 1 = 0$. The two real roots of this quadratic equation are given by $m = \dfrac{kx \pm \sqrt{k^2+4}}{2}$. The positive value among these two roots is defined as the metallic ratio of order k denoted by ρ_k. Thus,

$$\rho_k = \frac{k + \sqrt{k^2+4}}{2} \tag{3}$$

Since the sum of two roots is k, the other root is

$$k - \rho_k = \frac{k - \sqrt{k^2+4}}{2} \tag{4}$$

1.2 Special cases of metallic ratio

(i) If $k = 1$, then from (3)

$$\rho_1 = \frac{1 + \sqrt{5}}{2} \tag{5}$$

is called the golden ratio.

(ii) If $k = 2$, then from (3)

$$\rho_2 = 1 + \sqrt{2} \tag{6}$$

is called the silver ratio.

(iii) If $k = 3$, then from (3)

$$\rho_3 = \frac{3 + \sqrt{13}}{2} \tag{7}$$

is called the bronze ratio.

The numbers give by (5),(6) and (7) form the metallic ratios of first second and third orders.

Lemma 1.2. If $\rho_k = \dfrac{k^2+4}{2}$ is the metallic ratio of order k, then

$$\rho_k - \frac{1}{\rho_k} = k$$

Proof. From (3) and (4) we know that $\rho_k = \dfrac{k+\sqrt{k^2+4}}{2}, k-\rho_k = \dfrac{k-\sqrt{k^2+4}}{2}$ are the two real roots of $m^2 - km - 1 = 0$. Now taking reciprocal of ρ_k we get

$$\begin{aligned}
\frac{1}{\rho_k} &= \frac{1}{\dfrac{k+\sqrt{k^2+4}}{2}} \\
&= \frac{2}{k+\sqrt{k^2+4}} \\
&= \frac{2}{k+\sqrt{k^2+4}} \times \frac{k-\sqrt{k^2+4}}{k-\sqrt{k^2+4}} \\
&= \frac{2(k-\sqrt{k^2+4})}{k^2-(k^2+4)} \\
&= \frac{2(k-\sqrt{k^2+4})}{k^2-k^2-4} \\
&= \frac{2(k-\sqrt{k^2+4})}{-4} \\
&= \frac{k-\sqrt{k^2+4}}{-2} \\
&= -\left(\frac{k-\sqrt{k^2+4}}{2}\right) \\
&= -(k-\rho) \\
&= \rho_k - k
\end{aligned}$$

Hence $\rho_k - \dfrac{1}{\rho_k} = k$. This completes the proof. □

2 Preliminary

2.1 Properties of Real number ([1])

Let a, b, and c represent real numbers.

1. **Closure Properties**

$$\begin{aligned}
a+b &\quad \text{is a real number.} \\
ab &\quad \text{is a real number.}
\end{aligned}$$

2. **Commutative Properties**

$$a + b = b + a$$
$$ab = ba$$

3. **Associative Properties**

$$(a + b) + c = (a + b) + c$$
$$(ab)c = a(bc)$$

4. **Identity Properties**

   There exist a unique real number 0 such that

$$a + 0 = a. \quad \text{and} \quad 0 + a = a.$$

   There exist a unique real number 1 such that

$$a \cdot 1 = a. \quad \text{and} \quad 1 \cdot a = a.$$

5. **Inverse Properties**

   There exist a unique real number $-a$ such that

$$a + (-a) = 0. \quad \text{and} \quad -a + a = 0.$$

   if $a \neq 0$, there exists a unique real number $\frac{1}{a}$ such that

$$a \cdot \frac{1}{a} = 1. \quad \text{and} \quad \frac{1}{a} \cdot a = 1.$$

6. **Distributive Properties**

$$a(b + c) = ab + ac$$
$$a(b - c) = ab - ac$$

Theorem 2.1 (Fraction Properties [2])**.** *For all real number $a, b, c, d,$ and k (division by 0 excluded):*

1. $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.

2. $\frac{ka}{kb} = \frac{a}{b}$.

3. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

4. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

5. $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

6. $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$.

7. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

Theorem 2.2. Properties of Integer Exponents([2]).

For n and m integers and a and b real numbers:

1. $a^m a^n = a^{m+n}$.

2. $(a^n)^m = a^{mn}$.

3. $(ab)^m = a^m b^m$.

4. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad b \neq 0$.

5. $\frac{a^m}{a^n} = \begin{cases} a^{m-n} \\ \frac{1}{a^{n-m}} & a \neq 0 \end{cases}$.

Special Products ([3])

Let a and b be real numbers, variables, or algebraic expressions.

1. **Sum and Difference of Same Terms**
   $(a+b)(a-b) = a^2 - b^2$

2. **Square of a Binomial**
   $(a+b)^2 = a^2 + 2ab + b^2$
   $(a-b)^2 = a^2 - 2ab + b^2$

3. **Cube of Binomial**
   $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
   $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

2.2 Quadratic formula

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2.3 Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for integer n, and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:

1. $P(a), P(a+1), \ldots, and P(b)$ are all true. **(basis step)**
2. For any integer $k \geq b$, if $P(i)$ is true for all integer i from a through k then $P(k+1)$ is true **(inductive step)**

Then the statement

$$\text{for all integer } n \geq a, \quad P(n)$$

is true. (The supposition that $P(i)$ is true for all integers i from a through k is called the **inductive hypothesis**. Another way to state the inductive hypothesis is to say that $P(a), P(a+1), \ldots, P(k)$ are all true.)

3 Main Results

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RELATION WITH RESPECT TO GOLDEN RAITO

We will consider $k = 1$ in (1) and (2). Then the recurrence relation would become

$$M_{n+1} = M_{n+1} + M_n. \tag{8}$$

From (2) we get Fibonacci sequence whose terms are given by
$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

It is well known that (see [4]) the ratio of successive terms of the Fibonacci sequence approaches the golden ratio $\rho_1 = \varphi = \dfrac{1+\sqrt{5}}{2}$. We will now prove the following theorem.

Theorem 3.1. If $\{M_n\}$ is the Fibonacci sequence as defined in (8) then for any positive integer n we have

$$\varphi^n + \left(-\frac{1}{\varphi}\right)^n = M_{n-1} + M_{n+1}. \tag{9}$$

Proof. $\varphi^n + \left(-\dfrac{1}{\varphi}\right)^n = M_{n-1} + M_{n+1}.$ for all $n \in \mathbb{N}$

Let P(n) : $\varphi^n + \left(-\dfrac{1}{\varphi}\right)^n = M_{n-1} + M_{n+1}.$

(i) Show P(1) is true

That is $\varphi - \dfrac{1}{\varphi} = M_0 + M_2$

Consider

$$\varphi - \frac{1}{\varphi} = \rho_1 - \frac{1}{\rho_1}$$

Since (1.2) from lemma. $\rho_k - \dfrac{1}{\rho_k} = k$. Thus we obtain $\rho_1 - \dfrac{1}{\rho_1} = 1$.

$$\begin{aligned} \varphi - \frac{1}{\varphi} &= 1 \\ &= o + 1 \\ &= M_0 + M_2 \end{aligned}$$

$\therefore$ P(1) is true

(ii) Assume P$(r-1)$ and P(r) is true

That is

$$\begin{aligned} \varphi^{r-1} + \left(-\frac{1}{\varphi}\right)^{r-1} &= M_{r-2} + M_r \quad , \\ \varphi^{r} + \left(-\frac{1}{\varphi}\right)^{r} &= M_{r-1} + M_{r+1} \end{aligned}$$

Show P$(r+1)$ is true

That is show

$$\varphi^{r+1} + \left(-\frac{1}{\varphi}\right)^{r+1} = M_r + M_{r+2}$$

Consider

$$\begin{aligned} M_r + M_{r+2} &= M_{r-1} + M_{r-2} + M_{r+1} + M_r \\ &= M_{r-2} + M_r + M_{r-1} + M_{r+1} \\ &= \varphi^{r-1} + \left(-\frac{1}{\varphi}\right)^{r-1} + \varphi^r + \left(-\frac{1}{\varphi}\right)^{r} \\ &= \varphi^{r-1} + \varphi^r + \left(-\frac{1}{\varphi}\right)^{r-1} + \left(-\frac{1}{\varphi}\right)^{r} \\ &= \varphi^{r+1}(\varphi^{-2} + \varphi^{-1}) + \left(-\frac{1}{\varphi}\right)^{r+1} \left[\left(-\frac{1}{\varphi}\right)^{-2} + \left(-\frac{1}{\varphi}\right)^{-1}\right] \\ &= \varphi^{r+1}\left(\frac{1}{\varphi^2} + \frac{1}{\varphi}\right) + \left(-\frac{1}{\varphi}\right)^{r+1} [(-\varphi^2) + (-\varphi)] \\ &= \varphi^{r+1}\left(\frac{1+\varphi}{\varphi^2}\right) + \left(-\frac{1}{\varphi}\right)^{r+1} (\varphi^2 - \varphi) \end{aligned}$$

Since φ is the root of $m^2 - m - 1 = 0$. Thus $\varphi^2 - \varphi - 1 = 0$.
We obtain $\varphi^2 = 1 + \varphi$. and $\varphi^2 - \varphi = 1$.

$$\begin{aligned} M_r + M_{r+2} &= \varphi^{r+1}\left(\frac{\varphi^2}{\varphi^2}\right) + \left(-\frac{1}{\varphi}\right)^{r+1} \\ &= \varphi^{r+1} + \left(-\frac{1}{\varphi}\right)^{r+1} \end{aligned}$$

$\therefore$ P$(r+1)$ is true.
So by strong mathematical induction principle $\varphi^n + \left(-\frac{1}{\varphi}\right)^n = M_{n-1} + M_{n+1}$ hold for all $n \in \mathbb{N}$. □

RELATION WITH RESPECT TO SILVER RATIO

We will consider $k = 2$ in (1) and (2). Then the recurrence relation would become

$$M_{n+1} = 2M_{n+1} + M_n. \tag{10}$$

From (2) we get Fibonacci sequence whose terms are given by
$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \ldots$$

It can be show that (see [4]) the ratio of successive terms of this sequence approaches the silver ratio $\rho_2 = \lambda = 1 + \sqrt{2}$. We will now prove the following theorem.

Theorem 3.2. If $\{M_n\}$ is the Fibonacci sequence as defined in (10) in reccurence then for any positive integer n we have

$$\lambda^n + \left(-\frac{1}{\lambda}\right)^n = M_{n-1} + M_{n+1}. \tag{11}$$

Proof. $\lambda^n + \left(-\frac{1}{\lambda}\right)^n = M_{n-1} + M_{n+1}$. for all $n \in \mathbb{N}$

Let P(n) : $\lambda^n + \left(-\frac{1}{\lambda}\right)^n = M_{n-1} + M_{n+1}$.

(i) Show P(1) is true
That is $\lambda - \frac{1}{\lambda} = M_0 + M_2$
Consider

$$\lambda - \frac{1}{\lambda} = \rho_2 - \frac{1}{\rho_2}$$

Since (1.2) from lemma. $\rho_k - \dfrac{1}{\rho_k} = k$. Thus we obtain $\rho_2 - \dfrac{1}{\rho_2} = 2$.

$$\begin{aligned}\lambda - \frac{1}{\lambda} &= 2 \\ &= o + 2 \\ &= M_0 + M_2\end{aligned}$$

$\therefore$ P(1) is true

(ii) Assume P$(r-1)$ and P(r) is true

That is

$$\lambda^{r-1} + \left(-\frac{1}{\lambda}\right)^{r-1} = M_{r-2} + M_r \quad ,$$
$$\lambda^{r} + \left(-\frac{1}{\lambda}\right)^{r} = M_{r-1} + M_{r+1}$$

Show P$(r+1)$ is true

That is show

$$\lambda^{r+1} + \left(-\frac{1}{\lambda}\right)^{r+1} = M_r + M_{r+2}$$

Consider

$$\begin{aligned}M_r + M_{r+2} &= 2M_{r-1} + M_{r-2} + 2M_{r+1} + M_r \\ &= M_{r-2} + M_r + 2M_{r-1} + 2M_{r+1} \\ &= M_{r-2} + M_r + 2(M_{r-1} + M_{r+1}) \\ &= \lambda^{r-1} + \left(-\frac{1}{\lambda}\right)^{r-1} + 2\left[\lambda^r + \left(-\frac{1}{\lambda}\right)^{r}\right] \\ &= \lambda^{r-1} + \left(-\frac{1}{\lambda}\right)^{r-1} + 2\lambda^r + 2\left(-\frac{1}{\lambda}\right)^{r} \\ &= \lambda^{r-1} + 2\lambda^r + \left(-\frac{1}{\lambda}\right)^{r-1} + 2\left(-\frac{1}{\lambda}\right)^{r} \\ &= \lambda^{r+1}(\lambda^{-2} + 2\lambda^{-1}) + \left(-\frac{1}{\lambda}\right)^{r+1}\left[\left(-\frac{1}{\lambda}\right)^{-2} + 2\left(-\frac{1}{\lambda}\right)^{-1}\right] \\ &= \lambda^{r+1}\left(\frac{1}{\lambda^2} + \frac{2}{\lambda}\right) + \left(-\frac{1}{\lambda}\right)^{r+1}[(-\lambda^2) + 2(-\lambda)] \\ &= \lambda^{r+1}\left(\frac{1+2\lambda}{\lambda^2}\right) + \left(-\frac{1}{\lambda}\right)^{r+1}(\lambda^2 - 2\lambda)\end{aligned}$$

Since λ is the root of $m^2 - 2m - 1 = 0$. Thus $\lambda^2 - 2\lambda - 1 = 0$.
We obtain $\lambda^2 = 1 + 2\lambda$. and $\lambda^2 - 2\lambda = 1$.

$$
\begin{aligned}
M_r + M_{r+2} &= \lambda^{r+1}\left(\frac{\lambda^2}{\lambda^2}\right) + \left(-\frac{1}{\lambda}\right)^{r+1} \\
&= \lambda^{r+1} + \left(-\frac{1}{\lambda}\right)^{r+1}
\end{aligned}
$$

$\therefore$ P$(r+1)$ is true.

So by strong mathematical induction principle $\lambda^n + \left(-\frac{1}{\lambda}\right)^n = M_{n-1} + M_{n+1}$.
hold for all $n \in \mathbb{N}$. $\square$

RELATION WITH RESPECT TO BRONZE RATIO

We will consider $k = 3$ in (1) and (2). Then the recurrence relation would become

$$
M_{n+1} = 3M_{n+1} + M_n. \tag{12}
$$

From (2) we get Fibonacci sequence whose terms are given by
$0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, \ldots$

It can be show that (see [4]) the ratio of successive terms of this sequence approaches the bronze ratio $\rho_3 = \mu = \frac{3+\sqrt{13}}{2}$. We will now prove the following theorem.

Theorem 3.3. If $\{M_n\}$ is the Fibonacci sequence as defined in (12) then for any positive integer n we have

$$
\mu^n + \left(-\frac{1}{\mu}\right)^n = M_{n-1} + M_{n+1}. \tag{13}
$$

Proof. $\mu^n + \left(-\frac{1}{\mu}\right)^n = M_{n-1} + M_{n+1}$. for all $n \in \mathbb{N}$

Let P(n) : $\mu^n + \left(-\frac{1}{\mu}\right)^n = M_{n-1} + M_{n+1}$.

(i) Show P(1) is true
That is $\mu - \frac{1}{\mu} = M_0 + M_2$
Consider

$$
\mu - \frac{1}{\mu} = \rho_3 - \frac{1}{\rho_3}
$$

Since (1.2) from lemma. $\rho_k - \frac{1}{\rho_k} = k$. Thus we obtain $\rho_3 - \frac{1}{\rho_3} = 3$.

$$
\begin{aligned}
\mu - \frac{1}{\mu} &= 3 \\
&= o + 3 \\
&= M_0 + M_2
\end{aligned}
$$

$\therefore$ P(1) is true

(ii) Assume P$(r-1)$ and P(r) is true

That is

$$\mu^{r-1} + \left(-\frac{1}{\mu}\right)^{r-1} = M_{r-2} + M_r \quad ,$$
$$\mu^{r} + \left(-\frac{1}{\mu}\right)^{r} = M_{r-1} + M_{r+1}$$

Show P$(r+1)$ is true

That is show

$$\mu^{r+1} + \left(-\frac{1}{\mu}\right)^{r+1} = M_r + M_{r+2}$$

Consider

$$\begin{aligned}
M_r + M_{r+2} &= 3M_{r-1} + M_{r-2} + 3M_{r+1} + M_r \\
&= M_{r-2} + M_r + 3M_{r-1} + 3M_{r+1} \\
&= M_{r-2} + M_r + 3(M_{r-1} + M_{r+1}) \\
&= \mu^{r-1} + \left(-\frac{1}{\mu}\right)^{r-1} + 3\left[\mu^r + \left(-\frac{1}{\mu}\right)^{r}\right] \\
&= \mu^{r-1} + \left(-\frac{1}{\mu}\right)^{r-1} + 3\mu^r + 3\left(-\frac{1}{\mu}\right)^{r} \\
&= \mu^{r-1} + 3\mu^r + \left(-\frac{1}{\mu}\right)^{r-1} + 3\left(-\frac{1}{\mu}\right)^{r} \\
&= \mu^{r+1}(\mu^{-2} + 3\mu^{-1}) + \left(-\frac{1}{\mu}\right)^{r+1}\left[\left(-\frac{1}{\mu}\right)^{-2} + 3\left(-\frac{1}{\mu}\right)^{-1}\right] \\
&= \mu^{r+1}\left(\frac{1}{\mu^2} + \frac{3}{\mu}\right) + \left(-\frac{1}{\mu}\right)^{r+1}[(-\mu^2) + 3(-\mu)] \\
&= \mu^{r+1}\left(\frac{1+3\mu}{\mu^2}\right) + \left(-\frac{1}{\mu}\right)^{r+1}(\mu^2 - 3\mu)
\end{aligned}$$

Since μ is the root of $m^2 - 3m - 1 = 0$. Thus $\mu^2 - 3\mu - 1 = 0$.
We obtain $\mu^2 = 1 + 3\mu$. and $\mu^2 - 3\mu = 1$.

$$\begin{aligned}
M_r + M_{r+2} &= \mu^{r+1}\left(\frac{\mu^2}{\mu^2}\right) + \left(-\frac{1}{\mu}\right)^{r+1} \\
&= \mu^{r+1} + \left(-\frac{1}{\mu}\right)^{r+1}
\end{aligned}$$

$\therefore$ P$(r+1)$ is true.

So by strong mathematical induction principle $\mu^n + \left(-\frac{1}{\mu}\right)^n = M_{n-1} + M_{n+1}$.
hold for all $n \in \mathbb{N}$. $\square$

RELATION WITH RESPECT TO GENERAL METALLIC RATIO

In this section, I will obtain a general result connecting terms of the metallic ratio sequence defined in (1) and (2). Then the recurrence relation of general metallic ratio sequence is given by

$$M_{n+2} = kM_{n+1} + M_n.$$

From (2) we get a sequence whose terms are given by
$0, 1, k, k^2+1, k^3+2k, k^4+3k^2+k, k^5+4k^3+k^2+2k, k^6+5k^4+k^3+5k^2+k, \dots$

It can be shown that (see [4]) the ratio of successive terms of this sequence approaches the metallic ratio of order k given by $\rho_k = \dfrac{k+\sqrt{k^2+4}}{2}$. We will now prove the following theorem.

Theorem 3.4. If $\{M_n\}$ is the Fibonacci sequence as defined in (1) then for any positive integer n we have

$$(\rho_k)^n + \left(-\frac{1}{\rho_k}\right)^n = M_{n-1} + M_{n+1}. \tag{14}$$

Proof. $(\rho_k)^n + \left(-\frac{1}{\rho_k}\right)^n = M_{n-1} + M_{n+1}$. for all $n \in \mathbb{N}$

Let $\mathrm{P}(n)$: $(\rho_k)^n + \left(-\frac{1}{\rho_k}\right)^n = M_{n-1} + M_{n+1}$.

(i) Show P(1) is true
That is $\rho_k - \dfrac{1}{\rho_k} = M_0 + M_2$
Since (1.2) from lemma. We obtain $\rho_k - \dfrac{1}{\rho_k} = k$.
Consider

$$\begin{aligned}\rho_k - \frac{1}{\rho_k} &= k\\ &= o + k\\ &= M_0 + M_2\end{aligned}$$

$\therefore$ P(1) is true

(ii) Assume $\mathrm{P}(r-1)$ and $\mathrm{P}(r)$ is true
That is

$$(\rho_k)^{r-1} + \left(-\frac{1}{\rho_k}\right)^{r-1} = M_{r-2} + M_r \quad ,$$
$$(\rho_k)^r + \left(-\frac{1}{\rho_k}\right)^r = M_{r-1} + M_{r+1}$$

Show P$(r+1)$ is true
That is show

$$(\rho_k)^{r+1} + \left(-\frac{1}{\rho_k}\right)^{r+1} = M_r + M_{r+2}$$

Consider

$$\begin{aligned}
M_r + M_{r+2} &= kM_{r-1} + M_{r-2} + kM_{r+1} + M_r \\
&= M_{r-2} + M_r + kM_{r-1} + kM_{r+1} \\
&= M_{r-2} + M_r + k(M_{r-1} + M_{r+1}) \\
&= (\rho_k)^{r-1} + \left(-\frac{1}{\rho_k}\right)^{r-1} + k\left[(\rho_k)^r + \left(-\frac{1}{\rho_k}\right)^r\right] \\
&= (\rho_k)^{r-1} + \left(-\frac{1}{\rho_k}\right)^{r-1} + k(\rho_k)^r + k\left(-\frac{1}{\rho_k}\right)^r \\
&= (\rho_k)^{r-1} + k(\rho_k)^r + \left(-\frac{1}{\rho_k}\right)^{r-1} + k\left(-\frac{1}{\rho_k}\right)^r \\
&= (\rho_k)^{r+1}((\rho_k)^{-2} + k(\rho_k)^{-1}) + \left(-\frac{1}{\rho_k}\right)^{r+1}\left[\left(-\frac{1}{\rho_k}\right)^{-2} + k\left(-\frac{1}{\rho_k}\right)^{-1}\right] \\
&= (\rho_k)^{r+1}\left(\frac{1}{(\rho_k)^2} + \frac{k}{\rho_k}\right) + \left(-\frac{1}{\rho_k}\right)^{r+1}[(-\rho_k)^2 + k(-\rho_k)] \\
&= (\rho_k)^{r+1}\left(\frac{1+k\rho_k}{(\rho_k)^2}\right) + \left(-\frac{1}{\rho_k}\right)^{r+1}((\rho_k)^2 - k\rho_k)
\end{aligned}$$

Since ρ_k is the root of $m^2 - km - 1 = 0$.
We obtain $(\rho_k)^2 = 1 + k\rho_k$. and $(\rho_k)^2 - k\rho_k = 1$.

$$\begin{aligned}
M_r + M_{r+2} &= (\rho_k)^{r+1}\left(\frac{(\rho_k)^2}{(\rho_k)^2}\right) + \left(-\frac{1}{\rho_k}\right)^{r+1} \\
&= (\rho_k)^{r+1} + \left(-\frac{1}{\rho_k}\right)^{r+1}
\end{aligned}$$

$\therefore$ P$(r+1)$ is true.
So by strong mathematical induction principle $(\rho_k)^n + \left(-\frac{1}{\rho_k}\right)^n = M_{n-1} + M_{n+1}$. hold for all $n \in \mathbb{N}$. □

4 Conclusion

Introducing golden, silver and bronze ratios through general metallic ratio sequence as defined in (1) and (2) I had established three interesting theorems 3.1, 3.2 and 3.3 respectively. In section of relation with respect to the general metallic ratio, I had established the same result for general metallic ratio of order k. It is amusing to notice that all metallic ratios satisfy similar relation regarding the integral powers and alternate terms of the sequence. This result may pave way for proving other similar results regarding metallic ratios.

References

[1] Margaret Lial et al. *College Algebra and Trigonometry*. Pearson Education Limited, 2014, p. 12.

[2] Raymond A. Barnett, Michael R. Ziegler, and Karl E. Byleen. *College Algebra with Trigonometry*. McGraw–Hill, a business unit of The McGraw–Hill Companies, Inc, 2008, pp. 9–15.

[3] Roland E. Larson and Robert P. Hostetler. *College Algebra*. D.C. Heath and Company, 1995, p. 32.

[4] R. Sivaraman. "Exploring Metallic Ratios, Mathematics and Statistics, Horizon Research Publication". In: *Scopus Index Journal* 8 (2020), pp. 388–391.