6.4

Ex1)

Let's set $V_1 = X_1$. then

$$V_2 = X_2 - \frac{X_2 \cdot V_1}{V_1 \cdot V_1} V_1 \quad \begin{pmatrix} V_1 \cdot V_1 = 10 \\ X_2 \cdot V_1 = 30 \end{pmatrix}$$

$$= X_2 - 3V_1$$

$$= \begin{pmatrix} 8 \\ 5 \\ -6 \end{pmatrix} - \begin{pmatrix} 9 \\ 0 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix}$$

$$\therefore W = \left\{ \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix} \right\}$$

Ex11)

By Gram-Schmit process.

$V_1 = X_1$

$$V_2 = X_2 - \frac{X_1 \cdot V_1}{V_1 \cdot V_1} \cdot V_1 \quad \begin{pmatrix} V_1 \cdot V_1 = 5 \\ X_1 \cdot V_1 = -5 \end{pmatrix}$$

$$= X_2 + V_1$$

$$= \begin{pmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{pmatrix}$$

$$V_3 = X_3 - \frac{X_3 \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{X_3 \cdot V_2}{V_2 \cdot V_2} V_2$$

$$\begin{pmatrix} V_1 \cdot V_1 = 5 & V_2 \cdot V_2 = 36 \\ X_3 \cdot V_1 = 20 & X_3 \cdot V_2 = -12 \end{pmatrix}$$

$$V_3 = X_3 - 4V_1 + \frac{1}{3} V_2$$

$$= \begin{pmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ 4 \\ 4 \\ -4 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{pmatrix}$$

$$\therefore W = \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{pmatrix} \right\}$$

Ex13).

$QR = A$

$Q^TQR = Q^TA$

$R = Q^TA$

$$= \begin{pmatrix} 5/6 & 1/6 & -3/6 & 1/6 \\ -1/6 & 5/6 & 1/6 & 3/6 \end{pmatrix} \begin{pmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 36/6 & 72/6 \\ 0 & 36/6 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 12 \\ 0 & 6 \end{pmatrix}$$

Ex15)

Let Exercise11's matrix → A.
then the orthogonal basis for column space of matrix A is

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{pmatrix} \right\}$$

And columns of Q will be normalized vector of v_1, v_2, v_3 of orthogonal basis.

$$\therefore Q = \left\{ \begin{pmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \\ -1/\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \right\}$$

$R = Q^TA$

$$= \begin{pmatrix} 1/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\therefore R = \begin{pmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

(6.5)

Ex1)

$Ax = b$

$(A^TA)x = A^Tb$

$x = A^Tb.$

$$= \begin{pmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 11 \end{pmatrix}$$

$A^TAx = A^Tb \qquad A^TA = \begin{pmatrix} 6 & -11 \\ -11 & 22 \end{pmatrix}$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$= \begin{pmatrix} 6 & -11 \\ -11 & 22 \end{pmatrix}^{-1} \begin{pmatrix} -4 \\ 11 \end{pmatrix}$$

$$= \frac{1}{11}\begin{pmatrix} 22 & 11 \\ 11 & 6 \end{pmatrix} \begin{pmatrix} -4 \\ 11 \end{pmatrix}$$

$$= \frac{1}{11}\begin{pmatrix} 33 \\ 22 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\therefore \hat{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Ex15)

Ex15)

least square solution

$$\rightarrow R\hat{x} = Q^Tb$$

$R = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} \qquad Q^Tb = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$

Then, we have to find augmented matrix

$$(R \mid Q^Tb) = \left(\begin{array}{cc|c} 3 & 5 & 7 \\ 0 & 1 & -1 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} 3 & 0 & 12 \\ 0 & 1 & -1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right)$$

$\therefore \hat{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ is the least square solution.

Ex19)

a) If $Ax = 0$, then

$$A^T A x = A^T 0 = 0.$$

b) If $A^T A x = 0$, then

$$x^T A^T A x = x^T 0 = 0$$

therefore $(Ax)^T Ax = 0$.

it means that $\|Ax\|^2 = 0$.

so, $Ax = 0$. and this shows

that $\mathrm{Nul}\, A = \mathrm{Nul}\, A^T A$.

Ex20)

Let's suppose $Ax = 0$.

$$A^T Ax = A^T 0 = 0$$

$$\therefore A^T A x = 0$$

since $A^T A$ is invertible,

x should be 0.

Therefore, columns of A are linearly independent.

Ex21)

a) Linear independence means that $Ax=0$ has only trivial solution.
by Exercise 19, $A^TAx=0$ also has trivial solution.
And A^TA is square matrix,
A^TA is invertible matrix by Invertible Matrix Theorem.

b).
If the number of columns is bigger than rows,
there exist free variable and it means that it's not linear independent.
Therefore, A must have at least as many rows as column

c) n linearly independent columns of A form basis col A.
$\therefore$ rank $A=n$.