Chapter 2
Getting Started

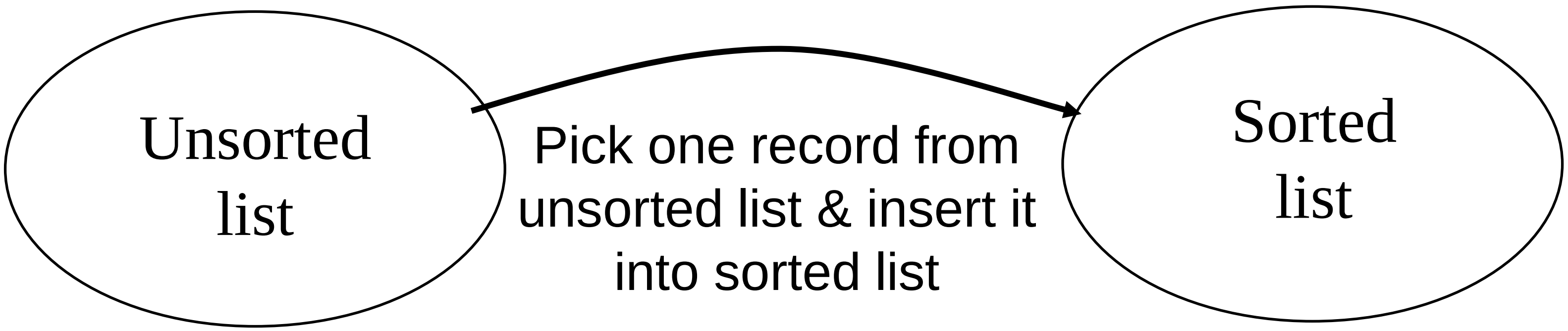
Algorithm Analysis

School of CSEE

Getting Started

- This chapter will give you an idea of the framework that will be used throughout the book.
- We will begin with the example of 'Insertion Sort' then take a look at the 'Mergesort' briefly.

An Example : Insertion Sort



Example)

27	4	21	38	1	12

4	27	21	38	1	12

4	21	27	38	1	12

4	21	27	38	1	12

1	4	21	27	38	12

1	4	12	21	27	38

An Example : Insertion Sort

1	2	3	4	5	6	
4	21	27	38	1	12	Key = 1, j = 5
4	21	27	38	38	12	i = 4
4	21	27	27	38	12	i = 3
4	21	21	27	38	12	i = 2
4	4	21	27	38	12	i = 1 ⇒ i = 0
1	4	21	27	38	12	

Insertion sort

Insertion-Sort (A);

for $j \leftarrow 2$ to $length(A)$

do $key \leftarrow A[j]$

► **Insert $A[j]$ into the sorted sequence $A[1..j-1]$.**

$i \leftarrow j - 1;$

while $i > 0$ and $A[i] > key$

do $A[i+1] \leftarrow A[i];$

$i \leftarrow i-1;$

$A[i+1] \leftarrow key;$

Algorithm

- What we are going to learn?
  - Designing the algorithm
  - Analyzing the algorithm

Design paradigms

- Insertion-sort uses ***incremental*** approach : having sorted the subarray A[1..j-1], we insert the single element A[j] into its proper place, yielding the sorted subarray A[1..j].
- cf) Divide-and-conquer approach : A problem is divided into a number of like problems of smaller size to yield small results that can be combined to produce a solution to the original problem.

  : 3 steps
  - Divide
  - Conquer
  - Combine

Other design paradigms

- Greedy
- Dynamic Programming
- Branch and Bound
- Backtracking
- Brute force ?

Analysis

- Correctness

  : Proving the correctness of the algorithm

- Efficiency

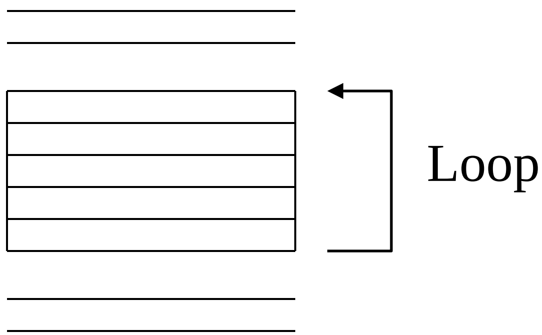
  : Obtaining the time complexity of the algorithm

Correctness

- **An algorithm is said to be *correct* if, for every input instance, it halts with the correct output.**
- **We say that a correct algorithm *solves* the given computational problem.**

Loop invariants

- Loop invariants
  - Program structure



- Definition: (Loop invariant)
  - Loop invariants are conditions and relationships that are satisfied by the variables and data structures at the end of each iteration of the loop.

Loop invariants

- Often use loop invariants to help us understand why an algorithm is correct.
- Must show three things about a loop invariants (similar to mathematical induction) :
  - **Initialization : It is true prior to the first iteration of the loop.**

    (a base case of the induction)
  - **Maintenance : If it is true before an iteration of the loop, it remains true before the next iteration.**

    (inductive step)
  - **Termination : When the loop terminates, the invariant gives us a useful property that helps show that the algorithm is correct.**

Correctness of insertion sort

- Loop invariant : At the start of each iteration of the for loop, the subarray $A[1..j\text{-}1]$ consists of the elements originally in $A[1..j\text{-}1]$ but in sorted order.
- Initialization : when j=2, $A[1..j\text{-}1]$ consists of the single element $A[1]$. Trivially sorted.
- Maintenance : Informally, the body of outer '*for'* loop works by moving $A[j\text{-}1]$, $A[j\text{-}2]$, $A[j\text{-}3]$, and so on, by one position to the right until the proper position for $A[j]$ is found.
- Termination : The outer '*for*' loop ends when $j = n$+1. Thus, $A[1..n]$ consists of the elements originally in $A[1..n]$ but in sorted order.

Efficiency

- Predicting the resources – time, storage - that the algorithm requires.

as a function of the input size *n*

  - Space requirement --- not a big deal
  - Time requirement --- in terms of the number of basic operations

Efficiency

- We need a model of the implementation technology.
- Random-access machine (RAM) model - a generic one-processor model of computation
  - **Instructions are executed one after another, with no concurrent operations.**
  - **It contains instructions commonly found in the real computers**
  - **Each instruction takes a constant time**
    - **Arithmetic (add, subtract, multiply, divide, remainder, floor, ceiling, shift left/right for mult/div by 2^k)**
    - **Data movement (load, store, copy)**
    - **Control (conditional and unconditional branch, subroutine call and return)**
  - **Data types : integers, floating point**
  - **No memory hierarchy, i.e., no cache or virtual memory**

Efficiency

- Need to specify running time for a particular input size
- Input size : depends on the problem
  – sorting *n* numbers : number of items in the input
  – multiplying two integers : total number of bits needed to represent the input in ordinary binary notation.
  – Graph algorithms : number of vertices and edges

Efficiency

- Running time of an algorithm on a particular input
  - The number of primitive operations or "steps" executed.
  - Steps are defined to be machine-independent
  - Each line of pseudocode requires a constant amount of time.
  - Each line may take different amount of time.

Time complexity Analysis

- **Worst-case:** (usually)
  - $T(n)$ = maximum time of algorithm on any input of size n.
- **Average-case:** (sometimes)
  - $T(n)$ = expected time of algorithm over all inputs of size n.
  - Need assumption of statistical distribution of inputs.
- **Best-case:** (bogus)
  - Cheat with a slow algorithm that works fast on *some* input.

Time complexity Analysis

- Usually, interested in the worst-case running time because
  - It gives an upper bound
  - For some algorithms, the worst case occurs often.
  - Average case is often as bad as the worst case.

- Average-case or ***expected*** running time – use ***probabilistic analysis***
  - Need assumption about the distribution of the input.
  - ***Randomized*** algorithm : permute the input

Analysis of insertion-sort

		Cost	times
	Inserion-Sort (*A*);		
1	for ***j*** $\leftarrow$ ***2*** to ***length(A)***	c_1	n
2	do ***key*** $\leftarrow$ **A[j]**	c_2	$n-1$
3	► Insert A[j] into the sorted sequence A[1..j-1].	0	$n-1$
4	*i* $\leftarrow$ *j* - 1;	c_4	$n-1$
5	**while** *i* > 0 and *A*[*i*] > *key*	c_5	t_j
6	**do** *A*[*i*+1] $\leftarrow$ *A*[*i*];	c_6	t_j-1
7	*i* $\leftarrow$ *i*-1;	c_7	t_j-1
8	*A*[*i*+1] $\leftarrow$ *key*;	c_8	$n-1$

For j=2,…,n, let t_j be the number of times that the while loop is executed for that value j.

Analysis of insertion-sort

- $T(n) = c_1 n + c_2(n\text{-}1) + c_4(n\text{-}1) + c_5 \Sigma\, t_j + c_6 \Sigma(t_j\text{-}1) + c_7 \Sigma(t_j\text{-}1) + c_8(n\text{-}1)$

- What can $T(n)$ be?
  - Best case -- inner loop body never executed
    - $t_j = 1$ ➔ $T(n)$ is a linear function. $T(n) = \Theta(n)$.
  - Worst case -- inner loop body executed for all previous elements
    - $t_j = i$ ➔ $T(n)$ is a quadratic function. $T(n) = \Theta(n^2)$.
  - Average case
    - ???

Merge Sort

```
MergeSort(A, left, right) {
    if (left < right) {
        mid = floor((left + right) / 2);
        MergeSort(A, left, mid);
        MergeSort(A, mid+1, right);
        Merge(A, left, mid, right);
    }
}
```

Merge( ) takes two sorted subarrays of A and merges them into a single sorted subarray of A (how long should this take?)

Merge sort

- Use divide-and-conquer paradigm
- Divide : divide the n-element sequence to be sorted into two subsequences of $n/2$ elements each.
- Conquer : sort the two subsequences recursively using merge sort.
- Combine : merge the two sorted subsequences to produce the sorted answer.

Analyzing divide-and-conquer algorithms

- Use a ***recurrence equation*** (or a ***recurrence***) to describe the running time of a divide-and-conquer algorithm.
- $T(n)$: running time on a problem of size n.
- If the problem size is small enough ($n \le c$) for some constant c, the straightforward solution takes constant time, $\Theta(1)$.
- a : number of subproblems
- n/b : input size of the subproblem
- $D(n)$: time to divide
- $C(n)$: time to combine

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le c \\ a\,T(n/b) + D(n) + C(n) & \text{otherwise.} \end{cases}$$

Analysis of merge sort

1. ***Divide:*** Trivial
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.

$$T(n) = 2\,T(n/2) + \Theta(n)$$

 # subproblems

subproblem size

Work dividing and combining

Analysis of merge sort

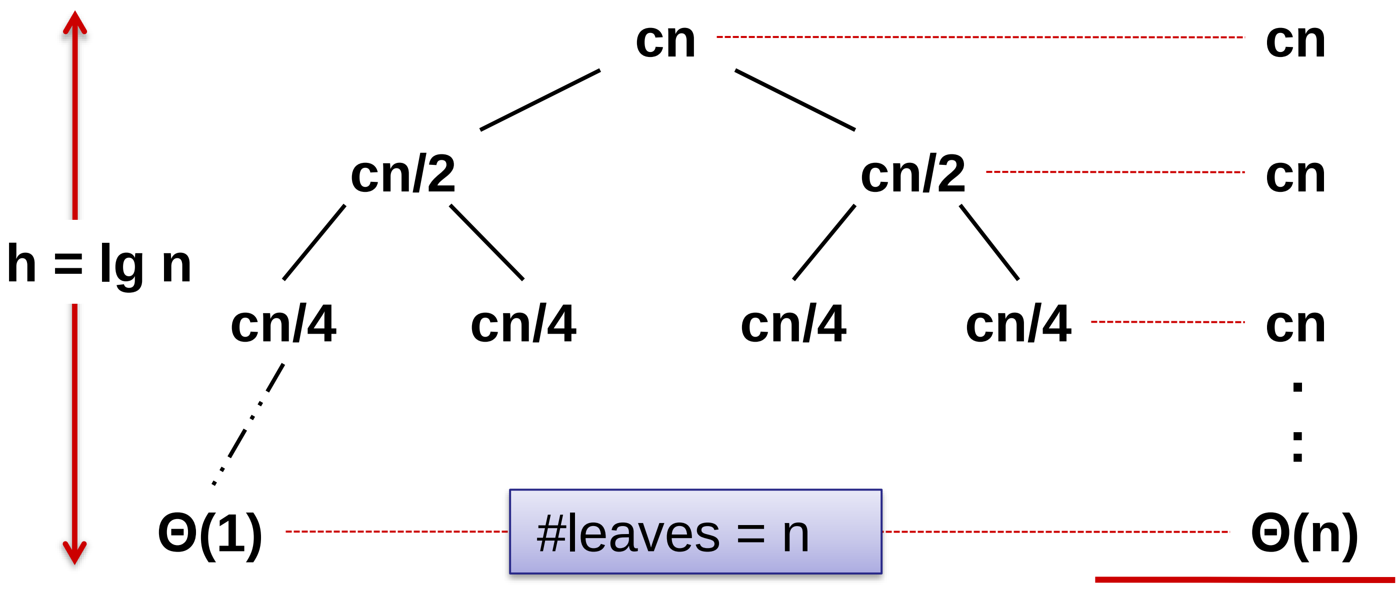
$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

By the master theorem (in Ch 4), we can show that $T(n) = \Theta(n \lg n)$.

- Compared to insertion sort ($\Theta(n^2)$ worst-case time), merge sort is faster.
- On small inputs, insertion sort may be faster. But, for large enough inputs, merge sort will always be faster, because its running time grows more slowly than insertion sort's.

Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1; \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$



Total = $\Theta(n \lg n)$

Keyword

- Designing Algorithm – design paradigms
- Analysis of Algorithm

  - Correctness : proof

  - Efficiency : time requirement

    # Worst case

    # Average case