AN APPLICATION OF OPTIMIZATION IN BUSINESS USING GRAPHICAL LINEAR PROGRAMMING

BY

BABATUNDE, KEHINDE PAUL
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DEPARTMENT OF MATHEMATICS
COLLEGE OF PHYSICAL SCIENCES
FEDERAL UNIVERSITY OF AGRICULTURE, ABEOKUTA.

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SUPERVISOR: DR. E.O. ADELEKE

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CERTIFICATION

This is to certify that this project was carried out by **BABATUNDE KEHINDE PAUL** with the matric number: **20172549** satisfying the requirements of the Department of Mathematics, College Of Physical Science, Federal University of Agricuture Abeokuta, Ogun State, Nigeria, for the partial fulfillment of the award of Bachelor of Science Degree in Mathematics.

BABATUNDE KEHINDE (Student)	Date
Dr. E.O. ADELEKE (Supervisor)	Date
Dr. E.O. ADELEKE (H.O.D)	Date

DEDICATION

I dedicate this project to God Almighty my creator, my strong pillar, my source of inspiration, wisdom, knowledge and understanding. God has been the source of my strength throughout this program and on His wings only have I soared. I also dedicate this work to my family, to my parents **Mr and Mrs Babatunde**, to my twin brother **Babatunde Taiwo**. Special thanks goes to my big Sister **Mrs Ademola Oluwaseun** for her care and love and to my big brother **Babatunde Olumide** for being a good model, his words of encouragement and advice are priceless, thank you very much Sir.

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I would like to acknowledge and convey my heartfelt gratitude to my supervisor **DR. E.O. ADELEKE** for his tremendous support and assistance in the completion of this project. My sincere appreciation also goes to all the lecturers in the department, for their patient and relentless effort in imparting knowledge into we the student. The completion of this project would not have been possible without their help and insights.

I would also like to acknowledge my dear H.O.C, the person of **ADEEKO ESTHER** for her effectiveness. I appreciate all my departmental mates, my project mates and friends in the department. May God bless you all.

ABSTRACT

This project is focused on how mathematical optimization can be applied to business processes, particularly how graphs can be used to determine optimal solution to business problems. In this project we would see three different business problem and give a step by stem solution to them.

Contents

Chapter 1

INTRODUCTION

Mathematics generally is the science and study of quality, structure, space, and change. It is a wide science that is applied to other disciplines such as physics, engineering, economics and so on. One of the branch in mathematics that is widely applicable to other fields is optimization. Optimization is derived from the Latin word "optimus" which means "best." Therefore to optimize refers to trying to bring whatever we are dealing with towards its ultimate state.

Mathematical Optimization (also known as Mathematical Programming) is the collection of mathematical principles and methods to solve quantitative problems in many disciplines including physics, biology, engineering, economics and business. In other words, optimization is a mean of identifying better solutions by utilizing scientific and mathematical approach.

This research work is based on how mathematical optimization can be applied to business, so it is worth while establishing what business really means. Business can be described as an organization or enterprising entity that engages in professional, commercial or industrial activities. Most businesses are established for profit making while some are non-profit. Some of the activities that all businesses have to undertake at some point are: accounting, sales, marketing, financing and so on. All these activity and process deals with the use of scarce resources which give raise to the need to optimize in order to get the best of result.

There are different technique and approach to mathematical optimization some of which include Integer programming, Quantitative programming, Linear programming, Functional analysis, Genetic algorithm and so on. We would be looking at how linear programming can be used to solve optimization problems in business. Linear programming (LP) is a mathematical technique for maximizing or minimizing a linear function of several variables, such as output, cost, profit and so on. The linear programming problems (LPP) can be solved using different methods, such as the graphical method, simplex method, or by using tools such as R, open solver etc. Our focus here is on the use of graphical method. We would expatiate more on this method in the next chapter.

1.1 PRELIMINARIES AND DEFINITION OF TERMS

1.1.1 Decision Variables

These are economic or physical quantities whose numerical values indicate the solution of the linear programming problem. These variables are under the control of the decision-maker and could have an impact on the solution to the problem under consideration. The relationships among these variables should be linear.

1.1.2 Objective Function

It is a linear function of the decision variables expressing the objective of the decision-maker. Objective function is prominently used to represent and solve the optimization problems of linear programming. The objective function is of the form $Z = ax_1+bx_2$, where x_1, x_2 are the decision variables and a, b are real numbers. The function $Z = ax_1 + bx_2$ is to be maximized or minimized to find the optimal solution.

1.1.3 Constraints

These are linear equations arising out of practical limitations, they define the allowable values for the variables. As an example, In manufacturing, the amount of a resource consumed cannot exceed the available amount. The mathematical forms of the constraints are: $f(x) \geq b$ or $f(x) \leq b$ or $f(x) = b$

1.1.4 Non-negativity Restrictions

In most practical problems the variables are required to be non-negative; $x_j \geq 0$, for $j = 1, 2, ...n$. This constraint is called a **non-negativity restriction**. Sometimes variables are required to be non-positive or, in fact, may be unrestricted.

1.1.5 Feasible Solution

Any non-negative solution which satisfies all the constraints is known as a **feasible solution**.

1.1.6 Feasible region

The region comprising all feasible solutions is referred to as **feasible region**.

1.1.7 Optimal Solution

The solution where the objective function is maximized or minimized is known as **optimal solution**.

1.2 Literature Review

From the mathematical standpoint, optimization or mathematical programming rests on several legs: analysis, topology, algebra, discrete mathematics, and so on. It build the foundation of the theory and applied mathematics subjects such as numerical analysis and mathematical parts of computer science build the bridge to the algorithmic side of the subject. On the other side, then with optimization we solve problems in a huge variety of areas in the technical, natural, life, engineering, sciences, business and in economics. It is worth noting that the term "program" as used here has nothing to do with "computer program", it is understood to be a "decision program", that is a strategy or decision rule. A "mathematical program" therefore is a mathematical problem designed to produce a decision program.

The history of optimization is also very long, many very often geometrical or mechanical problems that Archimedes, Euclid, Heron, and other masters from antiquity formulated and also solved are optimization problems. The masters of two millennia later, like Bernoulli, Lagrange, Euler, and Weierstrass developed variational calculus, studying problems in applied physics such as how to find the best trajectory for a flying object. The notion of optimality and especially how to characterize an optimal solution began to be developed at the same time. In the 1940s at the height of World War 2 (WW2), the US and British military commanders hired scientists from several disciplines in order to try to solve complex problems regarding the best way to construct convoys in order to avoid or protect the cargo ships from the German submarines, how to best cover the British isles with radar equipment given the scarce availability of radar systems, and so on.

Among the scientists that took part in the WW2 effort in the US and Great Britain, some were the great pioneers in placing optimization on the map after WW2. Among them are several researchers in mathematics, physics and economics who contributed greatly to the foundations of the field. To mention just a few, George W. Dantzig invented the simplex method for solving linear optimization problems during his WW2 efforts at Pentagon as well as the whole machinery of modelling such problems. Dantzig was originally a statistician and famously as a young Ph.D. student provided solutions to some then unsolved problems in mathematical statistics that he found on the black- board when he arrived late to a lecture, believing they were home work assignments in the course. Building on the knowledge of duality in the theory of two-person zero-sum games which was developed by the world-famous mathematician John von Neumann in the 1920s, Dantzig was very much involved in developing the theory of duality in linear programming together with the various characterizations of an optimal solution that is brought out from that theory. A large part of the duality theory was developed in collaboration with the mathematician Albert W. Tucker.

Speaking of linear programming, it is one of the many ways by which optimization can be done. It was used implicitly by Fourier in the early 1800s, but it was first formalized and applied to problems in economics in the 1930s by Leonid Kantorovich. Kantorivich's work was hidden behind the Iron Curtain (where it was largely ignored) and therefore unknown in the West. Linear programming was rediscovered and applied to shipping problems in the early 1940s by Tjalling Koopmans. The first complete algorithm to solve linear programming problems, called the simplex method, was published by George Dantzig in 1947. Koopmans first proposed the name "linear programming" in a discussion with Dantzig in 1948. Kantorovich and Koopmans shared the 1975 Nobel Prize in Economics "for their contributions to the theory of optimum allocation of resources". Dantzig did not; his work was apparently too pure. Koopmans wrote to Kantorovich suggesting that they refuse the prize in protest of Dantzig's exclusion, but Kantorovich saw the prize as a vindication of his use of mathematics in economics, which his Soviet colleagues had written off as "a means for apologists of capitalism".

Linear programming is a powerful algorithmic tool that allows us to express a number of optimization problems in a simple framework. Linear programming problem can be solved using different methods, such as the graphical method, simplex method, or by using tools such as R, open solver etc. In the next chapter we would discussing the graphical method and how it can use to solve Linear Programming Problems.

1.3 Motivation

Diverse work has been executed on optimization and how it can be practically applied to different fields. Since we do not study mathematics just for the sake of increasing knowledge rather to be applied to every areas of life, thus the desire to carry out this research to investigate and review how we can use graphs to solve linear programming problem in business for the sake of optimization.

1.4 Problem Statement

Quite often businesses are faced with decision making challenges such as what to produce, in what quantity to produce, number of staffs to employ and so on, decisions like this goes a long way to affect the profitability of businesses. Mathematics brings solution to these problems.

With the aid of mathematical computation we can deduce the precise quantity of products to produce, the number of labours to employ and how long they are to work.

1.5 Objectives

1. One major purpose of this project work is to show and explain the importance of mathematics in business and how it can be applied to business processes.

2. To explain how we can use equations and graphs to attain best possible results in business.

Chapter 2

DISCUSSION

In Mathematics diverse discovery has been made and applied to make tough and difficult process easy. One of the widely applied branch in mathematics is Mathematical Optimization, it deals with the collection of mathematical principles and methods for solving quantitative problems. Mathematical Optimization can be applied to different disciplines including physics, biology, engineering, economics, and business. An important class of optimization is known as linear programming.

2.1 Linear Programming

A linear programming is a mathematical technique which deals with the problem of optimizing a linear objective function subject to linear equality and inequality constraints on the decision variables. Linear indicates that no variables are raised to higher powers, such as squares. Every linear programming problem consists of a mathematical statement called an objective function. This function is to be either maximized or minimize, depending on the nature of the problem. For example, profit would be maximized but cost would be minimized. Also involved is a set of constrains equations which can either be in form of inequalities or equalities of the difference resources available and the proportion of each resource necessary to make a unit of the item of interest such as manufactured parts, personal policies, inventories etc.

Linear Programming is a versatile technique which can be applied to a variety of problems of management such as production, refinery operation, advertising, transportation, distribution and investment analysis. Over the years linear Programming has been found useful not only in business and industry but also in non-profit organizations such as government, hospitals, libraries and education. The technique is applicable in problems characterized by the presence of a number of decision variables, each of which can assume values within a certain range and affect their decision variables. The variables represent some physical or economic quantities which are of interest to the decision maker and whose domains are governed by a number of practical limitations or constraints. These may be due to availability of resources like men, material or money or may be due to a quality constraint or may arise from a variety of other reasons. The problem has a well defined objective. The common most objectives are maximization of profit/contribution or minimization of cost. Linear Programming indicates the right combination of the various decision variables which can be best employed to achieve the objective taking full account of the practical limitations within which the problem must be solved.

2.1.1 Representation of Linear Programs

A linear program can take many different forms. First, we have a minimization or a maximization problem depending on whether the objective function is to be minimized or maximized. The constraints can either be inequalities ($\leq$ or $\geq$) or equalities. Some variables might be unrestricted in sign (i.e. they can take positive or negative values) while others might be restricted to be nonnegative. A general linear program in the decision variables $x_1, x_2, ..., x_n$ is therefore of the following form:

$$\begin{array}{rlr} \textbf{Maximize or Minimize} & Z = \Sigma_{i=1}^{n} c_i X_i & \\ \textbf{subject to:} & \Sigma_{i=1}^{n} a_{j,i} X_i \leq b_j \text{ (or } \geq) & j = 1, 2, ..., m \\ & x_i \geq 0 & i = 1, 2, ..., n \end{array}$$

Where X_i = the i^{th} decision variable

c_i = the objective function coefficient corresponding to the i^{th} variable

$a_{j,i}$ = the coefficient on X_i in constraint j

b_j = the right-hand-side coefficient on constraint j.

2.1.2 Formulation Of Linear Programming Models

The usefulness of linear programming as tool for optimal decision making and resource allocation is based on its applicability to many diversified decision problems as determining the most profitable product mix, scheduling inventory, planning manpower management etc. it has been used for pollution control, personal allocation, capital budgeting and financial personnel selection.

The effective use and application require, as a first step the formulation of the model when the problem is presented. The three basic steps in formulating a linear programming are as follows:

Step 1
Identify the decision variables to be determined and express them in terms of algebraic equations.

Step 2
Identify all the limitations or constraints in the given problem and then express them as linear inequalities or equalities, in terms of above identified decision variables.

Step 3
Identify the objective (criterion) which is to be optimized (maximized or minimized) and express it as a linear function of the above defined decision variables.

Below are some illustrations on the formulation of linear programming models in various situations drawn from different area of management.

2.1.3 Example 1: Production Planning Problem

A tailor has the following materials available. 16 square meters of cotton, 11 square meters of silk, and 15 square meters of wool. He can make out two products from these three materials, namely dress and suite. A dress requires the following: 2 square meters of cotton, 1 square meter of silk and 1 square meter of wool. A suite requires 1 square meter of cotton, 2 square meter of silk and 3 square meter of wool. If the gross profit realized from a dress and as suite is respective ₦30 and ₦50. Formulate the above as a linear programming model.

Solution:

The information needed to formulate the above problem is summarized in the table below

Materials	**Dress**	**Suite**	**Available**
Cotton	2	1	16
Silk	1	2	11
Wool	1	3	15
Profit	₦30	₦50	

Table 1. Information for the Dress and Suite materials

Let x_1 = number of dresses to be made
x_2 = number of suites to be made

The following are constraints or limitations of the problem:

(a) Only 16 sq. meter available for cotton to be used hence we have $2x_1 + x_2 \leq 16$
(b) Limitation on silk would imply that $x_1 + 2x_2 \leq 11$
(c) That on wool also means $x_1 + 3x_2 \leq 15$
(d) And finally, at worst the tailor would make no garment implies that $x_1, x_2 \leq 0$ (Non-negative restriction).

Then the total profit $Z = 30x_1 + 50x_2$ which is the objective function of the problem.

Rewriting all together we have their linear programming models to be as follows:

$$\begin{array}{lcc} \text{Maximize} & Z = 30x_1 + 50x_2 & \text{(objective function)} \\ \text{s.t.} & 2x_1 + x_2 \leq 16 & \text{(cotton constraint)} \\ & x_1 + 2x_2 \leq 111 & \text{(silk constraint)} \\ & x_1 + 3x2 \leq 15 & \text{(wool constraints)} \\ & x_1,\, x2 \leq 0 & \text{(Non-negative constraints)} \end{array}$$

2.1.4 Example 2: Cost Minimization Problem

A manufacturer is to market a new fertilizer which is to be mixture of two ingredients A and B. The properties of the two ingredients are as follows: ingredient A contains 20% bone meal, 30% nitrogen, 40% lime and

10% phosphate and it cost ₦2.40 per kilogram. Ingredient B contains 40% bone meal, 10% nitrogen, 45% lime and 5% phosphate, and it costs ₦1.60 per kilogram. Furthermore it is decided that:

(i) The fertilizer will be sold in bags containing a minimum of 50 kilograms
(ii) It must contain at least 12% nitrogen
(iii) It must contain at least 6% phosphate
(iv) It must contain at least 20% bone meal

Solution:

Let x_1 = number of kilogram of ingredient A
x_2 = number of kilogram of ingredient B

The following are constraints or limitations:

(a) Total weight constraints: $x_1 + x_2 \geq 50$
(b) Bone meal constraints: $0.2x_1 + 0.4x_2 \geq 0.20$
(c) Nitrogen constraints: $0.3x_1 + 0.1x_2 \geq 0.1$
(d) Phosphate constraints: $0.1x_1 + 0.05x_2 \geq 0.06$
(e) Non negative constraints: $x_1,\ x_2 \geq 0$

The objective function is

$$\text{Minimize } Z = 2.4x_1 + 1.6x_2$$

2.2 The Graphical Method

In solving Linear Programming Problems, one part of the work is formulate the mathematical model of the problem while the other part is solving the problem. There are two most common technique in solving Linear Programming Problems: the simplex method and the graphical method, this research work is based on the graphical method. With the aid of graphs, any optimization programming problems consisting of only two variables can easily be solved. These variables can be referred as x_1 and x_2 and with the help of these variables, most of the analysis can be done on a two-dimensional graph. Although we can not generalize a large number of variables using a graphical approach, the basic concepts of Linear Programming in a two-variable context can be easily demonstrated.

We can always turn to two-variable problems if the problems seem to be complicated and we find ourselves in a pool of questions. And we can always search for answers in a two-variable case using graphs, that is solving Linear Programming problems graphically.

The graphical approach wraps itself with another advantage and that is it's visual nature. It provides us with a picture to get along with the algebra of Linear Programming. This picture can quench our thirst for understanding the basic definitions and possibilities. These reasons are proof that the graphical approach works smoothly with Linear Programming concepts.

2.2.1 The Extreme Point Theorem

The theorem state that the optimal solution to a linear programming problem will occur at one of the extreme point (also called vertex or corner) of the feasible region.

2.2.2 Steps in solving a Linear programming Problem graphically

Step 1. Construct a graph and plot the constraint lines
Constraint lines represent the limitations on available resources. Usually, constraint lines are drawn by connecting the horizontal and vertical intercepts found from each constraint equation.

Step 2. Determine the valid side of each constraint line
The simplest way to start is to plug in the coordinates of the origin (0,0) and see whether this point satisfies the constraint. If it does, then all points on the origin side of the line are feasible (valid), and all points on the other side of the line are infeasible (invalid). If (0,0) does not satisfy the constraint, then all points on the other side and away from the origin are feasible (valid), and all points on the origin side of the constraint line are infeasible (invalid).

Step 3. Identify the feasible solution region
The feasible solution region represents the area on the graph that is valid for all constraints. Choosing any point in this area will result in a valid

solution.

Step 4. Plot two objective function lines to determine the direction of improvement
Improvement is in the direction of greater value when the objective is to maximize the objective function, and is in the direction of lesser value when the objective is to minimize the objective function. The objective function lines do not have to include any of the feasible region to determine the desirable direction to move.

Another way of doing this is by applying the extreme point theorem, we might as well go ahead calculating the coordinates of the feasible region in the objective function to detect our optimum point.

Step 5. Find the most attractive corner
Optimal solutions always occur at corners. The most attractive corner is the last point in the feasible solution region touched by a line that is parallel to the two objective function lines drawn in step 5 above. When more than one corner corresponds to an optimal solution, each corner and all points along the line connecting the corners correspond to optimal solutions.

Step 6. Determine the optimal solution by algebraically calculating coordinates of the most attractive corner.

Step 7. Determine the value of the objective function for the optimal solution.

To illustrate these steps we would see different problem companies face and how it can be solved with linear programming using the graphical method.

2.2.3 Problem 1

A furniture manufacturing company looks forward to maximizing their profit. The company uses wood and labor to produce tables and chairs. the unit profit for tables is \$6, and unit profit for chairs is \$8. There are 300 board feet(bf) of wood available, and 110 hours of labor available. It takes 30 bf and 5 hours to make a table, and 20 bf and 10 hours to make a chair.

Resource	**Table**	**Chair**	**Available**
Wood(bf)	30	20	300
Labour(hr)	5	10	110
Unit profit	\$6	\$8	

Table 2. Information for the wooden tables and chairs linear programming problem.

How many units of tables and chairs should the company produce to maximize profit?

Solution:

The table 2 above will help us formulate the problem. The bottom row is used to formulate the objective function. Objective functions are developed in such a way as to be either maximized or minimized. In this case, the company's management wishes to maximize unit profit. The wood and labor rows are used to formulate the constraint set.

$$\begin{aligned} \text{Let} \quad x_1 &= \text{number of tables} \\ x_2 &= \text{number of chairs} \end{aligned}$$

$$\begin{array}{rlr} \text{Maximize:} & Z = 6x_1 + 8x_2 & \text{(objective function)} \\ \text{Subject to:} & 30x_1 + 20x_2 \leq 300 & \text{(wooden contraint: 300 bf available)} \\ & 5x_1 + 10x_2 \leq 110 & \text{(labour constraint: 110 hours available)} \\ & x_1, x_2 \leq 0 & \text{(non-negativity conditions)} \end{array}$$

Since only two variables (wood and labor) exist in this problem, it can be solved graphically. If there were more than two variables, the graph would have to be more than two dimensions.

Graphical Solution

Step 1. Construct the graph and plot constraint lines.
We would draw the graph with the x axis representing the number of tables and the y axis representing the number of chairs. We plot the two

constraint lines by finding the x and y intercepts for the two constraint equations in the following manner. First, we rewrite the constraint inequalities as equalities and solve to obtain the intercepts:

Wood	**Labour**
$30x_1 + 20x_2 = 300$	$5x_1 + 10x_2 = 100$
Set $x_2 = 0$ and solve for x_1	Set $x_2 = 0$ and solve for x_1
$30x_1 = 300$	$5x_1 = 110$
$x_1 = 10$ tables	$x_1 = 22$ tables
(All the wood is used to make tables.)	(All the labour is used to make tables.)
Next:	
Set $x_1 = 0$ and solve for x_2	Set $x_1 = 0$ and solve for x_2
$20x_2 = 300$	$10x_2 = 110$
$x_2 = 15$ chairs	$x_2 = 11$ chairs
(All the wood is used to make chairs.)	(All the labour is used to make chairs.)

Now we can plot the wood constraint line using the intercepts $x_1 = 10$ and $x_2 = 15$, and then plot the labour constraint line using the intercepts $x_1 = 22$ and $x_2 = 11$. See Figure 1.

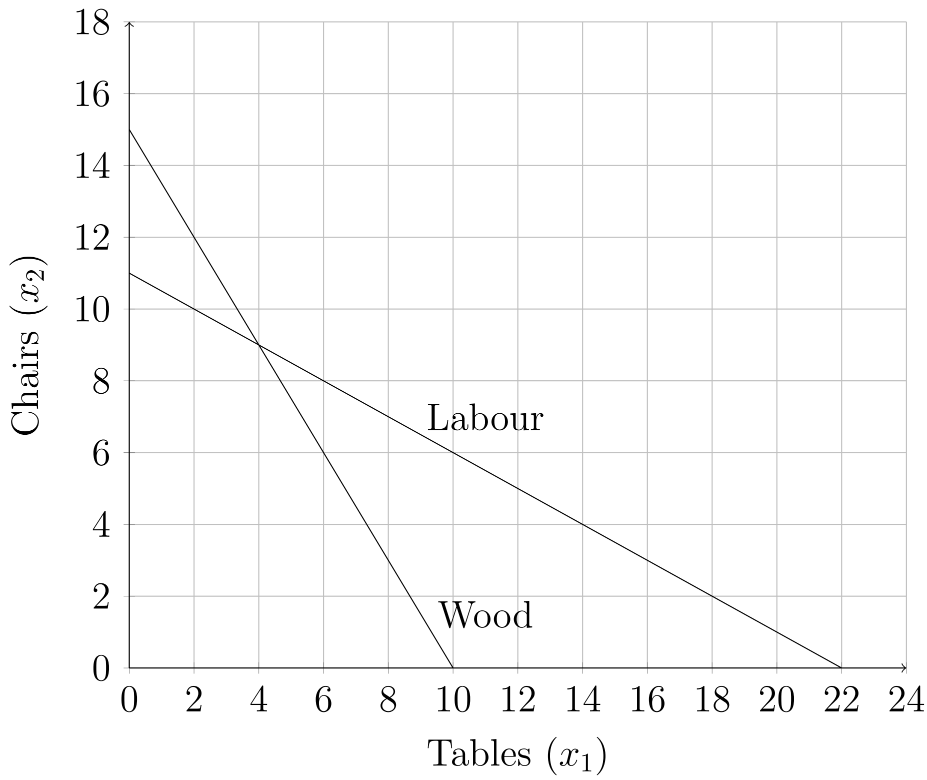

Figure 1 - Wood and labour constraint line

Step 2. Determine the valid side of each constraint line
We will use the origin (0,0) to check the valid side for both constraint lines. $30(0)+20(0) < 300$ is valid, so we know the side toward the origin (0,0) is the valid side of the wood constraint line. $5(0)+10(0) < 110$ also is valid, so we know the side toward the origin (0,0) is the valid side of the labor constraint line. We can now plot a graph to indicate the feasible region. See Figure 2.
We could have chosen any point to test for the valid side of the line. For example, setting $x_1 = 20$ and $x_2 = 10$ (clearly on the other side, away from the origin) for the wood constraint line, we get $30(20) + 20(10) > 300$, which is not valid. In other words, there simply isn't enough wood to make 20 tables and 10 chairs.

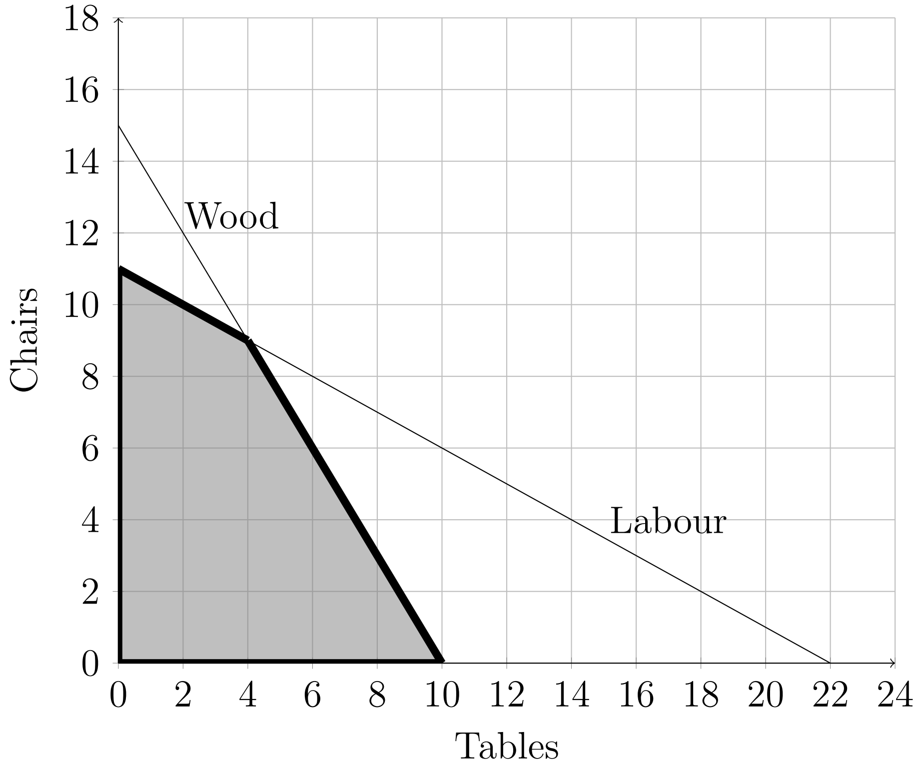

Figure 2 - Identification of the feasible region.

Step 3. Identify the feasible region
The feasible region is the area on the valid side of both constraint lines. Any point located on the invalid side of a constraint line is infeasible. Because of the non-negativity conditions, the feasible region is restricted to the positive quadrant. See Figure 2.

Step 4. Plot two objective function lines to determine the direction of improvement
First, we'll arbitrarily set profit $Z = 48$ and then set profit $Z = 72$. We'll find the x and y intercepts when $Z = 48$ and when $Z = 72$ and plot the two lines.

Set $Z = 48$	Set $Z = 72$
Set $x_2 = 0$ and solve for x_1	Set $x_2 = 0$ and solve for x_1
$48 = 6(x_1)$	$72 = 6(x_1)$
$x_1 = 8$	$x_1 = 12$
$Next:$	
Set $x_1 = 0$ and solve for x_2	Set $x_1 = 0$ and solve for x_2
$48 = 8(x_2)$	$72 = 8(x_2)$
$x_2 = 6$	$x_2 = 9$

Now we plot the objective function lines when $Z = 48$ and $Z = 72$, see Figure 3.

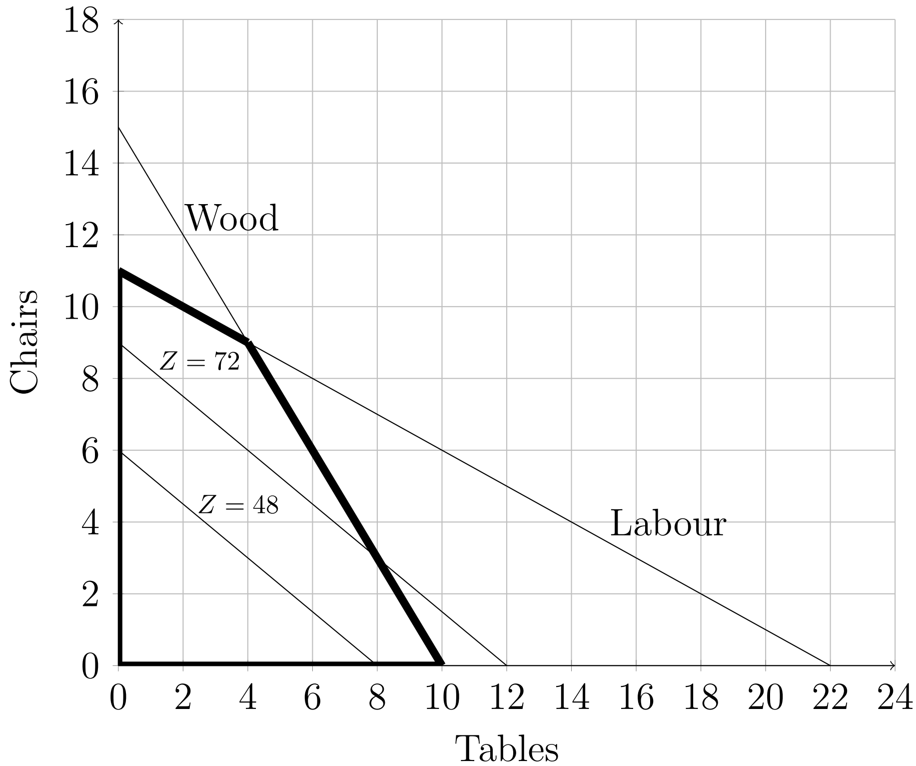

Figure 3 - Determining direction of increasing

We can see from the two objective function lines that as we move away from the origin (0,0) Z increases.

Step 5. Find the most attractive corner

Since we want to maximize Z, we can draw a line parallel to the objective function lines that touches the last point in the feasible region while moving away from the origin. This identifies the most attractive corner, which gives us the amounts of wood and labor that will result in the maximum profit (maximize Z). Thus, it represents the optimal solution to the problem (Figure 4).

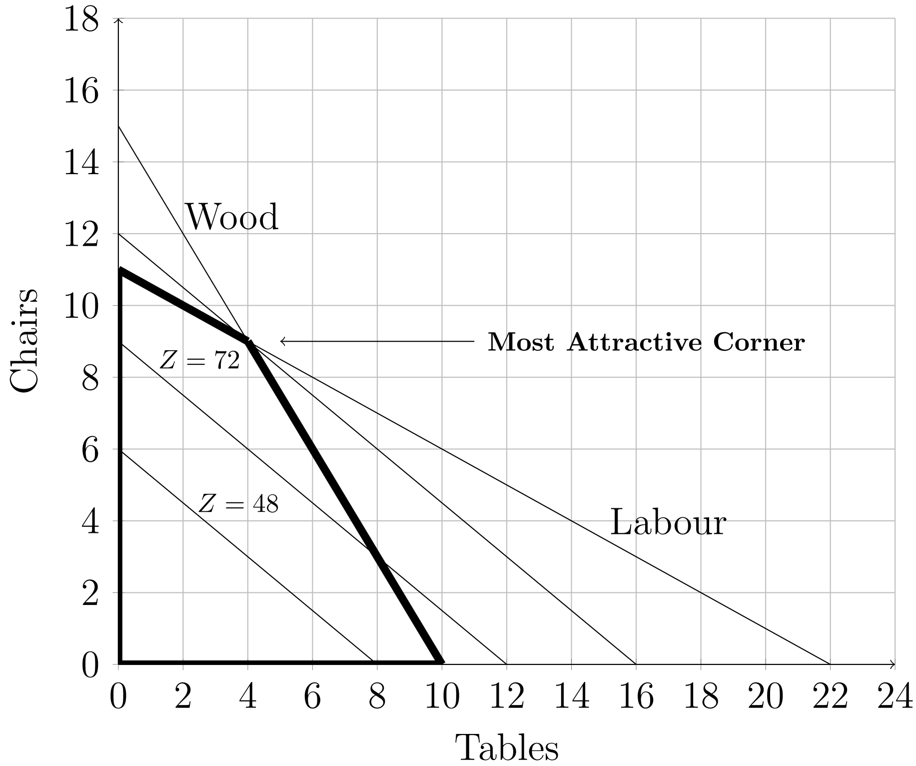

Figure 4 - Locating the most attractive corner

Step 6. Determine the optimal solution by algebraically calculating coordinates of the most attractive corner

The most attractive corner lies at the intersection of the wood and labor constraint lines. Therefore, coordinates for the most attractive corner can be found by simultaneously solving the constraint equations (wood and labor):

$$\begin{aligned} 30x_1 + 20x_2 &= 300 \text{ (wood)} \\ 5x_1 + 10x_2 &= 110 \text{ (labor)} \end{aligned}$$

To do so, we multiply the labour equation by -2 and add it to the wood equation so the x_2 variable becomes zero and we can then solve for x_1.

$$
\begin{array}{rcl}
30x_1 + 20x_2 & = 300 & \text{(wood)} \\
-2(5x_1 + 10x_2 & = 110) & \text{(labour)} \\
\hline
20x_1 + 0 & = 80 & \\
x_1 & = 4 \text{ tables} &
\end{array}
$$

Next, we substitute the number of tables calculated above into either of the constraint equations to find the number of chairs. We would substitute into both equations to illustrate that the same value is found.

Wood constraint

$$
\begin{aligned}
30(4) + 20x_2 &= 300 \\
120 + 20x_2 &= 300 \\
20x_2 &= 300 - 120 \\
20x_2 &= 180 \\
x_2 &= 9 \text{ chairs}
\end{aligned}
$$

Labour constraint

$$
\begin{aligned}
5(4) + 10x_2 &= 110 \\
20 + 10x_2 &= 110 \\
10x_2 &= 110 - 20 \\
10x_2 &= 90 \\
x_2 &= 9 \text{ chairs}
\end{aligned}
$$

Thus, the company's optimal solution is to make four tables and nine chairs. In this case, we could read this solution off the graph by finding the values on the x and y axes corresponding to the most attractive corner where the wood and labor constraint intersect. However, when the most attractive corner corresponds to an optimal solution with fractions, it is not possible to read directly from the graph. For example, the optimal solution to this problem might have been 3.8 tables and 9.2 chairs, which we probably would not be able to read accurately from the graph.

Step 7. Determine the value of the objective function for the optimal solution

We plug in the number of tables and chairs and solve for Z:

$$Z = \$6(4) + \$8(9) = \$96$$

Thus, we find that maximum profit of \$96 can be obtained by producing four tables and nine chairs.

2.2.4 Problem 2

A factory manufactures chairs and tables, each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 40 hours; the second at most 42 hours; and the third at most

25 hours. A chair requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; a table needs 2 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is 20 per unit for a chair and 30 for a table, how many units of each should be manufactured to maximize profit?

Solution:

This is another problem similar to problem one, we are to find a way to maximize the factory's profit. Since the steps to the solution has been established in problem one we would be skipping over some of them. We begin by defining the variables

Let x_1 = number of chairs made

x_2 = number of tables made

The profit would be $Z = 20x_1 + 30x_2$

For cutting, x_1 chairs will require x_1 hours and x_2 tables will require $2x_2$ hours. We can use at most 40 hours, so $x_1 + 2x_2 \leq 40$

For assembly, x_1 chairs will require $2x_1$ hours and x_2 tables will require x_2 hours. We can use at most 42 hours, so $2x_1 + x_2 \leq 42$

For finishing, x_1 chairs will require x_1 hours and x_2 tables will require x_2 hours. We can use at most 25 hours, so $x_1 + x_2 \leq 25$

Since we can't produce negative items, $x_1,\ x_2 \geq 0$

Thus we have the problem as follows:

$$
\begin{aligned}
\textbf{Maximize } & Z = 20x_1 + 30x_2 \\
\textbf{subject to: } & x_1 + 2x_2 \leq 40 \\
& 2x_1 + x_2 \leq 42 \\
& x_1 + x_2 \leq 25 \\
& x_1,\ x_2 \geq 0
\end{aligned}
$$

Now that we have the constraint, we can plot the graph by first setting the inequalities as equalities and then solve to obtain the intercepts.

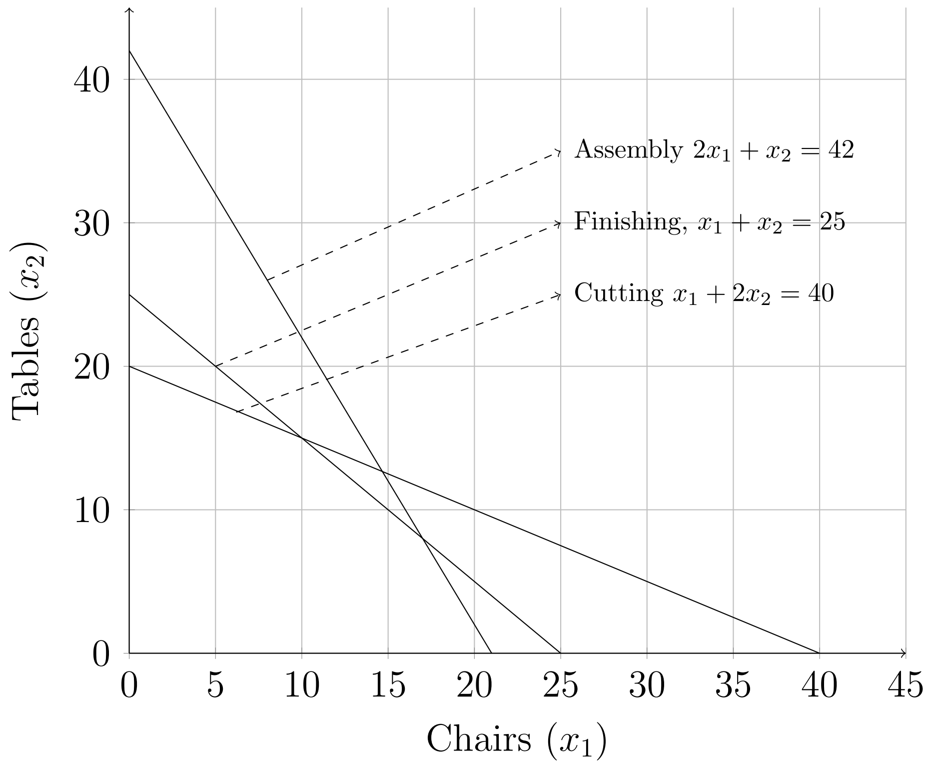

Figure 3 - Cutting, Assembly and Finishing constraint line.

From the figure 3 above it is easy to determine the feasible region:

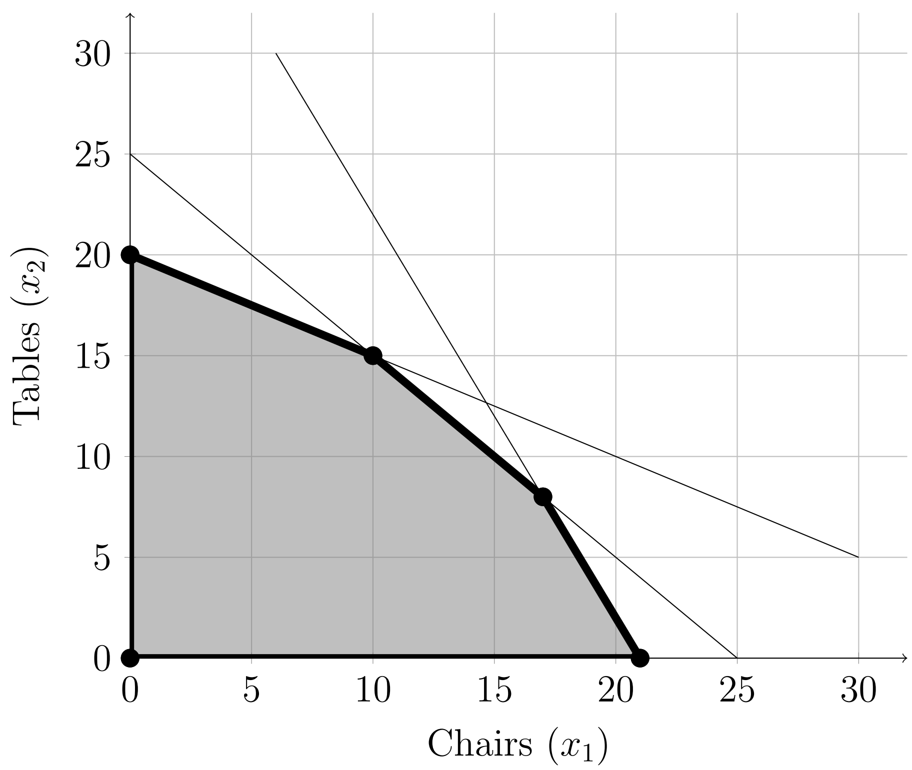

Figure 4 - Feasible region

There are five corner points on the feasible region:

Point 1:
In the lower left, where $x_1 = 0$ crosses $x_2 = 0$.
Point: (0,0)

Point 2:
In the upper left, where $x_1 = 0$ crosses $x_1 + 2x_2 = 40$. Using substitution, $2x_2 = 40$, so $x_2 = 20$.
Point: (0,20)

Point 3:
In the lower right, where $x_2 = 0$ crosses $2x_1 + x_2 = 42$. Using substitution, $2x_1 = 42$, so $x_1 = 21$.
Point: (21,0)

Point 4:
The solution to the system:

$$\begin{aligned} x_1 + 2x_2 &= 40 \\ x_1 + x_2 &= 25 \end{aligned}$$

We multiply the second equation by -1 and then add:

$$\begin{array}{cl} x_1 + 2x_2 & = 40 \\ -x_1 - x_2 & = -25 \\ \hline x_2 & = 15 \end{array}$$

Substitute to find x_1: $x_1 + 15 = 25$ so $x_1 = 10$.
Point: (10,15)

Point 5:
The solution to the system:

$$\begin{aligned} 2x_1 + x_2 &= 42 \\ x_1 + x_2 &= 25 \end{aligned}$$

We multiply the second equation by -1 and add:

$$\begin{array}{cl} 2x_1 + x_2 & = 42 \\ -x_1 - x_2 & = -25 \\ \hline x_1 & = 17 \end{array}$$

Substitute to find x_2: $17 + x_2 = 25$ so $x_2 = 8$
Point: (17,8)

To determine the optimal solution we would test the objective function at each of the corner points:

Point	Profit $Z = 20x_1 + 30x_2$
$(0, 0)$	$Z = 20(0) + 30(0) = \$0$
$(0, 20)$	$Z = 20(0) + 30(20) = \$600$
$(21, 0)$	$Z = 20(21) + 30(0) = \$420$
$(10, 15)$	$Z = 20(10) + 30(15) = \$650$
$(17, 8)$	$Z = 20(17) + 30(8) = \$580$

Thus the factory can make a maximum profit of \$650 by producing 10 chairs and 15 tables.

2.2.5 Problem 3

At a university, Professor Symons wishes to employ two people, John and Mary, to grade papers for his classes. John is a graduate student and can grade 20 papers per hour; John earns \$15 per hour for grading papers. Mary is an post-doctoral associate and can grade 30 papers per hour; Mary earns \$25 per hour for grading papers. Each must be employed at least one hour a week to justify their employment. If Prof. Symons has at least 110 papers to be graded each week, how many hours per week should he employ each person to minimize the cost?

Solution:

The first thing we do as usual is to represent our variables:

Let x_1 = number of hours per week John is employed
x_2 = number of hours per week Mary is employed

The objective function is $Z = 15x_1 + 25x_2$

Since each must work at least one hour each week results in the following two constraints: $x_1 \geq 1$, $x_2 \geq 1$

The fact that John can grade 20 papers per hour and Mary 30 papers per hour, and there are at least 110 papers to be graded per week, this would give us the constraint: $20x_1 + 30x_2 \geq 110$

x_1 and x_2 are non-negative, that is: $x_1,\ x_2 \geq 0$

The problem has been formulated as follows:

$$\begin{aligned} \textbf{Minimize } & Z = 15x_1 + 25x_2 \\ \textbf{subject to: } & x_1 \geq 1 \\ & x_2 \geq 1 \\ & 20x_1 + 30x_2 \geq 110 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Hence we graph the constraints as follows:

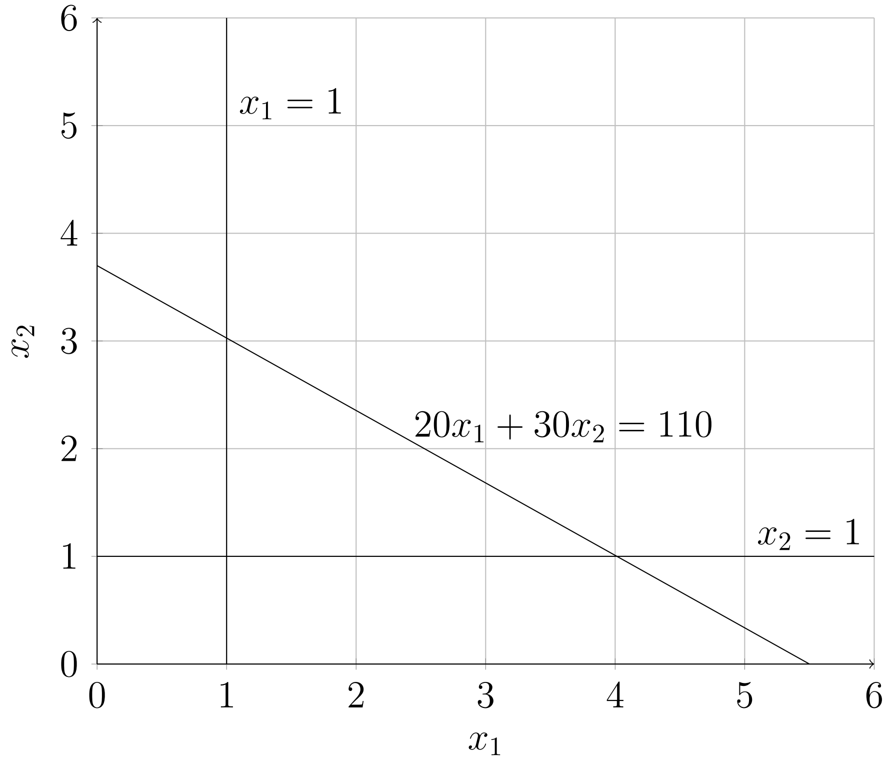

Figure 5 - John and Mary Constraint line.

Now to identify the feasible region, if we used test point (0,0) that does not lie on any of the constraints, we observe that (0,0) does not satisfy any of the constraints $x_1 \geq 1$, $x_2 \geq 1$, $20x_1 + 30x_2 \geq 110$. Thus the feasible region lies on the opposite side of the constraint lines from the point (0,0).

Alternatively we could use test point (4,6), which also does not lie on any of the constraint lines. We'd find that (4,6) does satisfy all of the inequality constraints: $4 \geq 1$, $6 \geq 1$ and $20(4) + 30(6) = 260 \geq 110$. Consequently the feasible region lies on the same side of the constraint lines as the point (4,6). Thus we have:

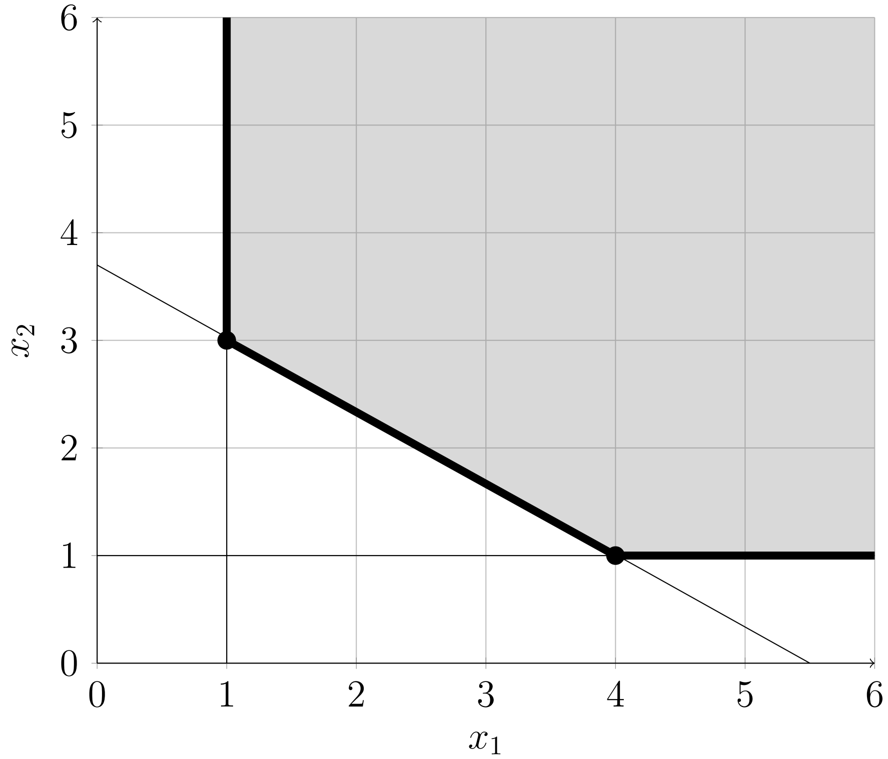

Figure 6 - Feasible region

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, thus we compute the coordinates:

Point 1:
At the intersection of $x_1 = 1$ and $20x_1 + 30x_2 = 110$

$$\begin{aligned}
20(1) + 30x_2 &= 110 \\
20 + 30x_2 &= 110 \\
30x_2 &= 110 - 20 \\
30x_2 &= 90 \\
x_2 &= 3
\end{aligned}$$

Point (1,3)

Point 2:
At the intersection of $x_2 = 1$ and $20x_1 + 30x_2 = 110$

$$\begin{aligned} 20x_1 + 30(1) &= 110 \\ 20x_1 + 30 &= 110 \\ 20x_1 &= 110 - 30 \\ 20x_1 &= 80 \\ x_1 &= 4 \end{aligned}$$

Point (4,1)

To minimize cost, we will substitute these points in the objective function to see which point gives us the minimum cost each week

Point	Cost $Z = 15x_1 + 25x_2$
$(1, 3)$	$Z = 15(1) + 25(3) = \$90$
$(4, 1)$	$Z = 15(4) + 25(1) = \$85$

The point (4, 1) gives the least cost, and that cost is \$85. Therefore, we conclude that in order to minimize grading costs, Professor Symons should employ John 4 hours a week, and Mary 1 hour a week at a cost of \$85 per week.

2.3 Limitations Of The Graphical Method

As we saw in the first problem, there is no guarantee of an integer (whole number) solution. The furniture company can't sell 0.444 of a table. An optimal solution for another problem might be to buy 6.75 trucks. Obviously, we can buy either 6 or 7 trucks, but not 0.75 of a truck. In this case, rounding up (and for many non-integer answers, rounding either up or down) offers a practical solution.

The graphical method is one of the easiest way to solve a small Linear Programming Problem but it can only be used when the decision variables are not more than two. It is not possible to plot the solution on a two-dimensional graph when there are more than two variables, we must turn to more complex methods.

Another limitation of graphical method is that, an incorrect or inconsistent graph will produce inaccurate answers, so one need to be very careful while drawing and plotting the graph.

Chapter 3

CONCLUSION AND RECOMMENDATION

3.1 Conclusion

Optimization is a broad field in mathematics that can be applied to our everyday activities and other branch of studies; medicine, statistics, engineering, economics and to businesses. The main aid of an enterprise or a business is the make profit so optimization can be used to this effect, not just to make profit but to maximize profit and minimize or cut losses.

3.2 Recommendation

I recommend that the study of Mathematics should focus more on it's applications rather than the theoretical aspect, this would encourage students to learn and be able to solve real life problems.

Chapter 4

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