Turbulence and Financial Markets?

A Comparison Between Financial Markets Dynamics and Turbulent Fluid Flow

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There are interesting similarities between turbulent fluid behavior and financial markets. In both cases, one can identify large scale disturbances being transferred to successive smaller scales (see Stanley and Mantegna). In the case of a liquid, by stirring it, one can observe the energy inputted in the system being transferred to smaller and smaller scales. In financial markets, instead of energy, information is "injected" on a large scale, and one observes the transmission of reactions to smaller scales, which in this case are individual investors. Both are extremely challenging to model due to several "types of interactions [that exist either] between their parts or between [them] and their environment."

Here, it will be shown, based on a detailed statistical analysis by Stanley and Mantegna, that though financial markets and turbulent fluids have qualitative similarities, quantitatively, the correspondence is limited.

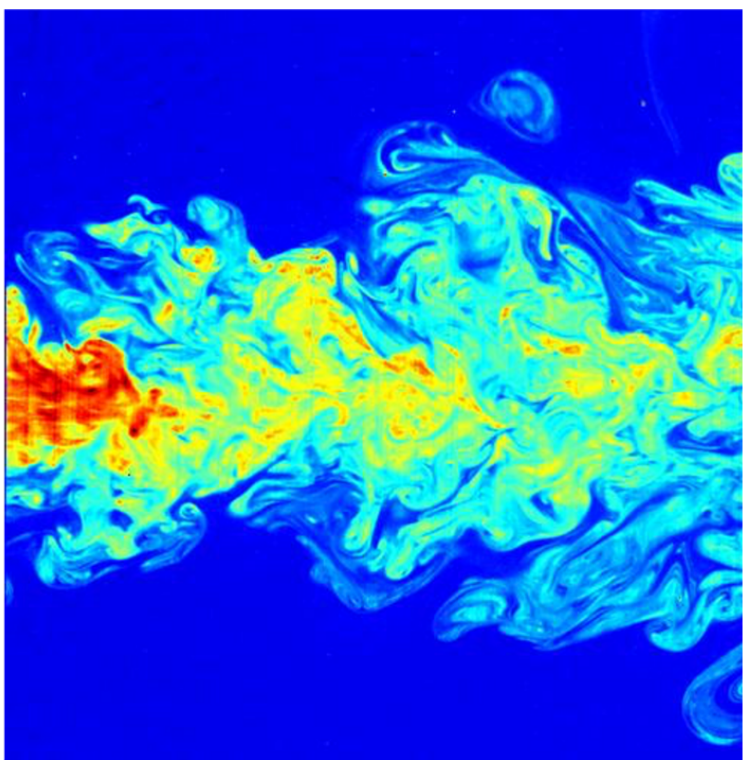


Figure 1: The flow of a turbulent jet exhibiting a broad range of length scales which is typical of turbulent behavior (source).

What is Turbulence?

Let us consider a fluid flowing in a pipe, with the following parameters:

- Kinematic viscosity ν: also known as momentum diffusivity, ν is given by the ratio of the viscosity μ (see Fig.1) to the density of the fluid ρ
- Velocity V
- The pipe has a diameter L

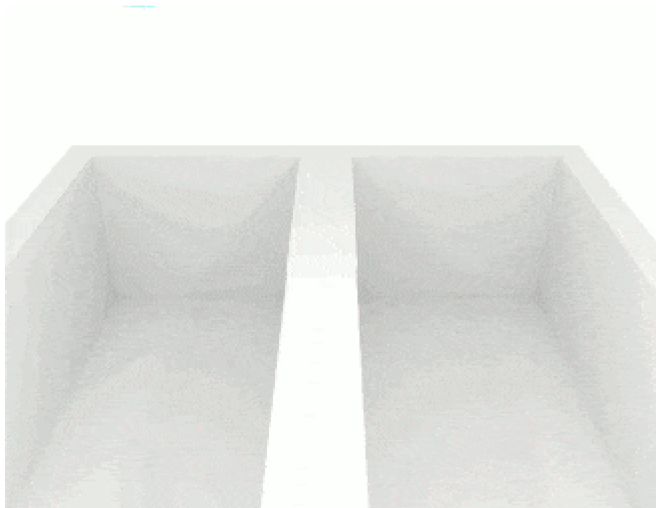


Figure 2: The viscosity of the liquid on the right is higher (source).

In fluid mechanics, the so-called Reynolds number or Re is a (dimensionless) quantity that helps to predict flow patterns. The Re of the flowing fluid with the parameters chosen above is given by:

$$\mathrm{Re} \equiv \frac{LV}{\nu}$$

Equation 1: The Reynolds number of a fluid with kinematic viscosity v, velocity V, in a pipe with diameter L. The value of Re indicates the level of complexity of the fluid.

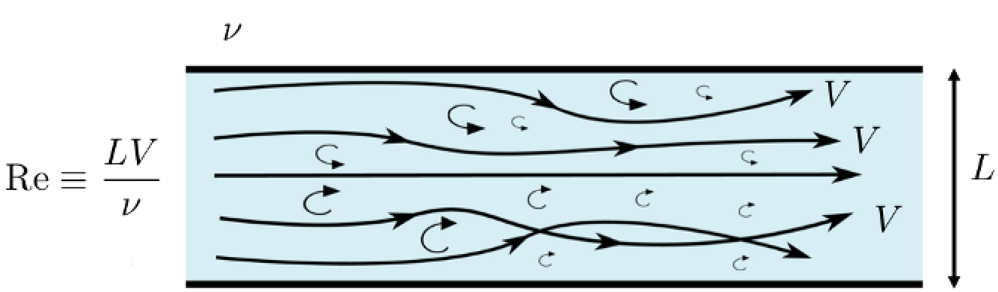


Figure 3: Turbulent fluid with kinematic viscosity v, velocity V, in a pipe with diameter L. The Reynolds number Re indicates the level of complexity of the fluid (a modified version of source).

The Reynolds number Re is a measure of the complexity of a fluid. Depending on the value of Re, the fluid is either turbulent (highly complex) or laminar (low complexity). The process of a laminar flow turning into turbulent flow is called laminar-turbulent transition.

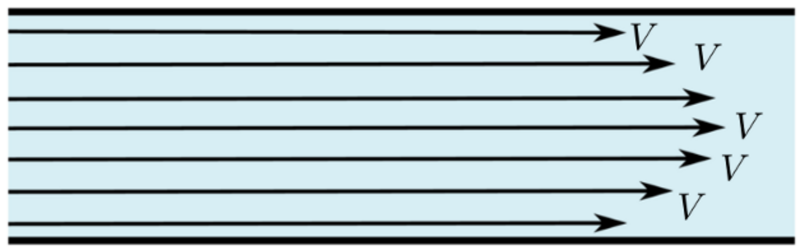


Figure 4: Laminar flow. The velocity profile "looks like a deck of cards". The fluid acts in layers sliding over each other (source).

The famous Navier-Stokes equations which describe the dynamics of incompressible fluids are given by:

$$\underbrace{\frac{\partial \mathbf{V}(\mathbf{r},t)}{\partial t}}_{\text{local acceleration}} + \underbrace{(\mathbf{V}(\mathbf{r},t)\cdot\nabla)\mathbf{V}(\mathbf{r},t)}_{\text{convective term}} = \overbrace{-\nabla P}^{\text{pressure gradient}} + \underbrace{\nu\nabla^2\mathbf{V}(\mathbf{r},t)}_{\text{viscous or diffusion term}},$$

Continuity equation

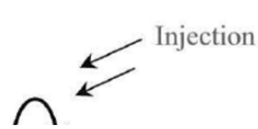
$$\nabla \cdot \mathbf{V}(\mathbf{r},t) = 0$$

Equation 2: The Navier-Stokes equations where **V**(**r**,t) is the velocity vector field and P is the pressure.

where $\mathbf{V}(\mathbf{r},t)$ is the velocity vector at $\mathbf{r}$ and time t and P is the pressure. Eq. 2 describes turbulent regimes with very high Re (fully developed turbulence).

An Aside on Energy Cascades

In a system with nonlinear dynamics, such as a fluid with fully developed turbulence, a direct (inverse) energy cascade involves the transfer of energy from large (small) scales of motion to the small (large) scales. If there are intermediate scales, this intermediate range is called inertial range or inertial subrange.



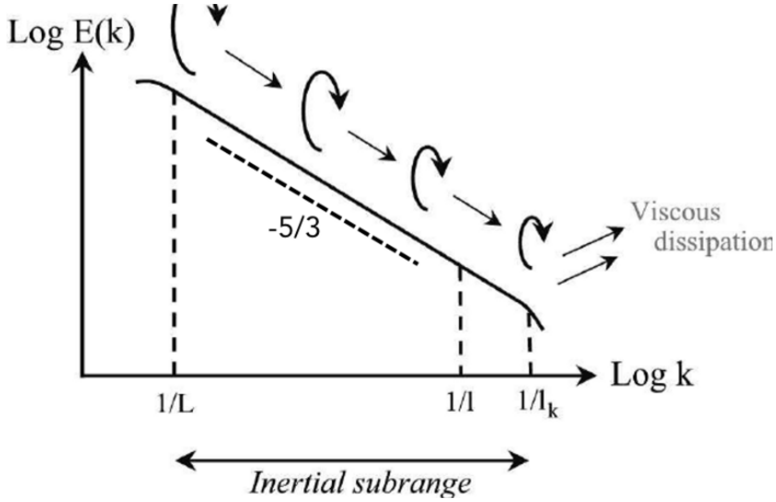


Figure 5: The plot shows the production, energy cascade and dissipation in the energy spectrum of turbulence (source).

The energy spectrum in the inertial range involves the transfer of energy from low to high wavenumbers (the energy cascade) and has the following power-law form:

$$E(k) \propto k^{-5/3}$$

Equation 3: The energy spectrum in the inertial range has this power-law form.

Kolmogorov's 1941 Theory

The renowned Soviet mathematician Andrey Kolmogorov showed, in two papers (both from 1941), that for fluids with completely developed turbulence (fluids in the limit of Re → ∞) the following behavior occurs

$$\langle [\mathbf{V}(r+l) - \mathbf{V}(r)]^2 \rangle \sim l^{2/3}$$

Equation 4: The behavior, for Re → ∞, of the mean square velocity increment, discovered by the Soviet mathematician Andrey Kolmogorov in 1941.

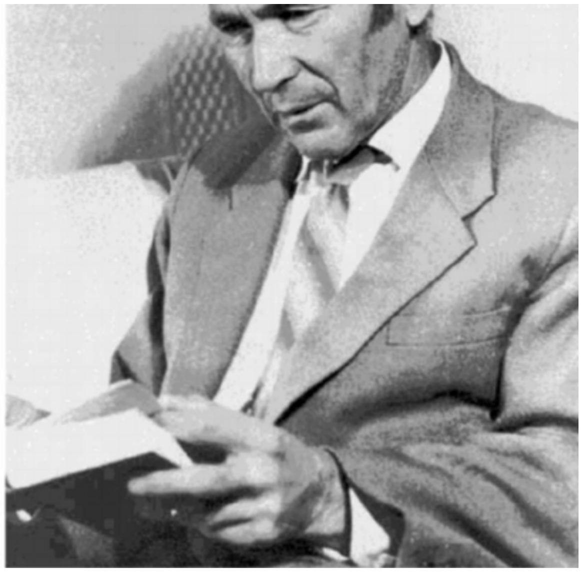


Figure 6: The Soviet mathematician Andrey Kolmogorov (source).

in the inertial range. In Eq.4, l corresponds to distances (see Stanley and Mantegna) smaller than the dimension within which the turbulent behavior occurs and larger than the length below which kinetic energy gets dissipated into heat.

However, Kolmogorov's theory fails to explain the intermittent behavior (the occurrence of sudden changes of activity in the time evolution) of the velocity changes $U(t)=\Delta V(t)$ and the consequent leptokurtic shape of the probability distribution of $U(t)$,

Comparing Financial Markets and Turbulent Dynamics

To compare the temporal behavior of financial markets and turbulent fluids, Stanley and Mantegna analyze two quantities, namely:

- The dynamics of the S&P 500 index in the period of 1984–1989
- The velocity $\mathbf{V}(t)$ of a three-dimensional fully turbulent fluid with a very high Re (more specifically they consider "the wind velocity in the atmospheric surface layer about 6 m above the canopy in Connecticut Agricultural Research Station").

For short times, both processes are nonstationary, non-Gaussian, and intermittent. However, at long times both processes are asymptotically stationary.

Stanley and Mantegna made four comparisons between the S&P 500 index and the fluid velocity, namely:

- Between their time evolutions $Y(t)$ and $V(t)$
- Between their variations $Z(t)= \Delta Y(t)$ and $U(t)=\Delta V(t)$
- Between the standard deviations $\sigma(\Delta t)$ as functions of Δt of $Z(t)$ and $U(t)$
- Between the power spectra $S(f)$ of $Y(t)$ and $V(t)$

Time Evolution of the S&P 500 Index and the Fluid Velocity

Fig. 7 below compares the time evolution of the S&P 500 index sampled in intervals Δt $=1$ hour and the atmospheric wind velocity in fully developed turbulence (at very high Reynolds number Re).

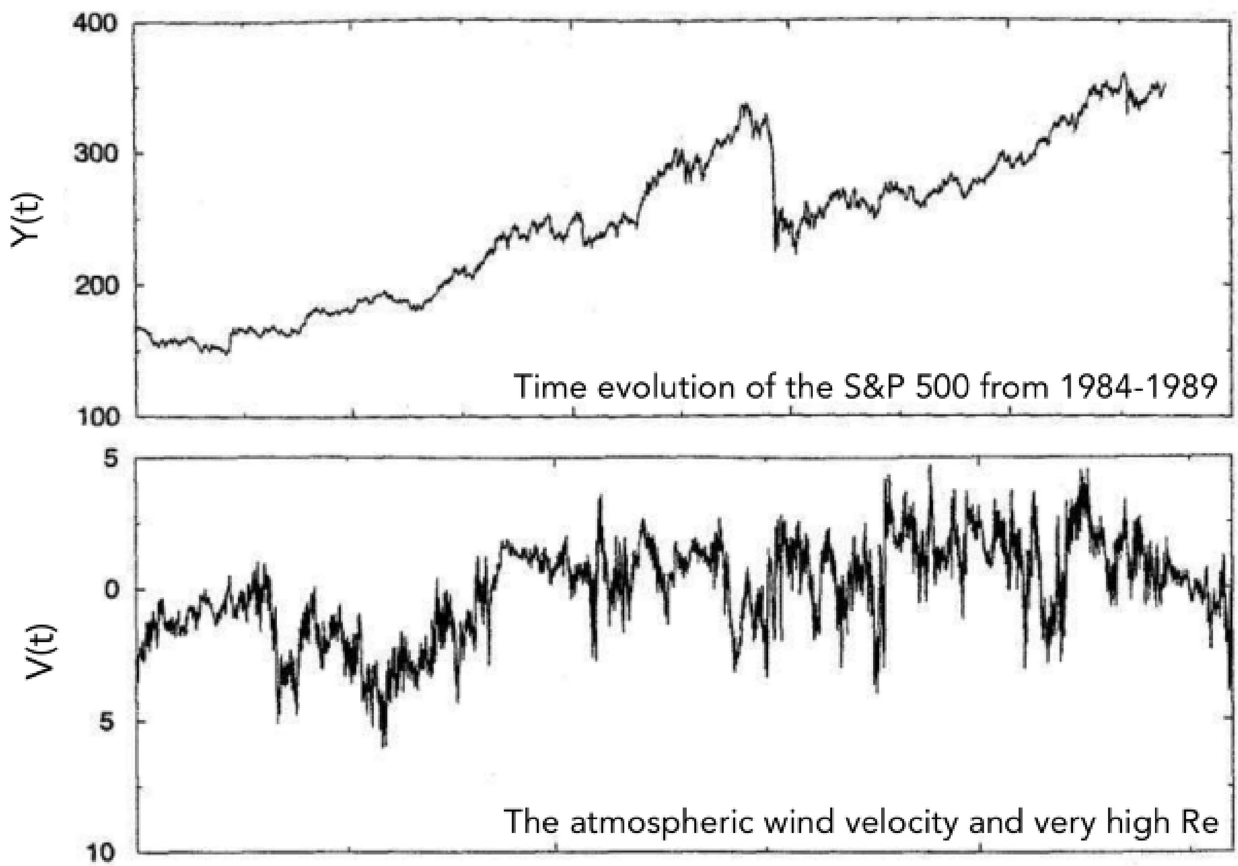


Figure 7: The S&P 500 sampled in intervals of one hour (top). The atmospheric wind velocity in fully developed turbulence (with very high Re) (bottom) (source).

Changes in the S&P 500 Index and the Fluid Velocity

Fig. 8 displays changes with intervals of $\Delta t=1$ hour of the S&P 500 index on the top and variations in fluid velocity (at a higher sampling rate) at the bottom. We see that for the turbulent fluid, there is symmetry about the x-axis, which does not occur for the financial data. This difference will be confirmed below by the behavior of the standard deviation of the increments of the index $Z(t) = \Delta Y(t)$ and the fluid velocity increments $U(t) = \Delta V(t)$ (see Fig. 9 and the discussion above it).

Figure 8: Changes in intervals of Δt=1 hour of the S&P 500 index (top). Variations in fluid velocity (at a higher sampling rate) (bottom) (source).

Standard Deviations of Increments in the S&P 500 Index and in the Fluid Velocity

Stanley and Mantegna also investigate the volatility $\sigma(\Delta t)$ as a function of Δt for processes $Z(t)$ and $U(t)$, as shown in Fig. 8. In both cases, a power law

Equation 5: Both the volatility of the probability distribution of the increments Z(t) and the standard deviation of the velocity increments V(t+Δt)-V(t) as a function of Δt display power-law behavior, with exponents v=0.53 and v=0.33 respectively.

Though $\sigma(\Delta t)$ exhibit power-law behavior in both processes, the time correlations between them differ substantially. More specifically we have:

- The volatility $\sigma(\Delta t)$ of the probability distribution $P(Z(t))$ has an initial interval of superdiffusive behavior followed by diffusive behavior, typical of random processes with uncorrelated increments (figure on top).
- The standard deviation of the probability distribution $P(U)$ where U are velocity increments $U(t) = V(t+\Delta t)\text{-}V(t)$ as a function of Δt for the turbulent fluid has initially a superdiffusive behavior but it is followed by a subdiffusive behavior with exponent 0.33, which is very close to the theoretical value of 1/3 (figure on the bottom).

Figure 9: The volatility σ(Δt) of the probability distribution P(Z) of the increments Z(t) as a function of Δt for the S&P 500 index time series (top). The standard deviation of the probability distribution P(U) where U are velocity increments U(t) = V(t+Δt)-V(t) as a function of Δt for the turbulent fluid (bottom) (source).

Spectral Densities

The power spectrum of a stationary random process is the Fourier transform of its autocorrelation function:

Equation 6: Power spectrum of a stationary random process.

The $S(f)$ of a random walk has the following functional form:

Equation 7: Spectral density of a random walk.

One can compare both processes using their power spectra. Both obey the functional form:

Equation 8: The power spectra of both processes have this form but with very different η. The S&P 500 index has η=1.98 while the velocity time series has η=5/3 and η=2 in the inertial and dissipative range respectively.

Both processes have $S(f)$ of the form Eq. 8 but the exponents differ substantially. For the S&P 500 index, one gets $\eta=1.98$ which is very close to the exponents associated with a random walk. The velocity time series has $\eta=5/3$ and $\eta=2$ at low and high frequencies respectively.

Similarities and Differences

The comparative analysis of the dynamics of highly turbulent fluids and of the S&P 500 index reveals that the same methods can be applied to examine different systems with known but (analytically) unsolvable equations of motion.
The similarities include the presence of intermittent behavior, non-Gaussian probability distributions with gradual convergence to a Gaussian attractor. The differences include

the shapes of the probability distributions in both systems and the fact that velocity fluctuations are anticorrelated in contrast to S&P 500 index fluctuations, which are uncorrelated.

This article was based on the discussion about relations between the dynamics of financial markets and turbulent behavior found in this Nature Magazine article and on the textbook by H.E. Stanley and R.N. Mantegna.

My Github and personal website www.marcotavora.me have some other interesting material both about finance and other topics such as mathematics, data science, and physics. Check them out!

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