United World College Changshu China

Optimization of a Pendulum Tuned Mass Damper

How does the damping effect of a pendulum on an oscillating mass change with the length of the pendulum rod?

Zhihan Yang
Physics
Supervised by Professor Arunananda Mukherjee
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I. Introduction

Being in school Architecture Club, I inspect buildings. Unlike model buildings, high-rises are prone to oscillations and need stabilization. After watching (curtis gulick) about the pendulum tuned mass damper (**PTMD**) in Taipei 101 (Fig 1) (Plessis), I decided to test how a PMTD works and, hopefully, optimize its damping effect.

An oscillator is damped if its amplitude of oscillation decreases over time (Douglas C. Giancoli 298). All physical oscillators experience damping from mechanical and internal friction (air friction negligible), as they eventually come to rest. However, the first few oscillations can be destructive if the building is merely damped by friction from itself (Pavlou and Constantinou).

A PTMD is a damped pendulum. As a PTMD, the pendulum has its pivot attached to the building at, for example, the bottom of a floor. When oscillation is induced in the building, the pivot also oscillates and applies a force on the top of the pendulum rod. Since the force is not applied to the center of mass of the pendulum, there will be a delay between the movement of the pivot and of the pendulum mass, resulting in large forces with horizontal components (e.g. tension) that oppose the oscillation of the building.

The pendulum and the building both experience damping from friction, hence their independent angular frequencies (not to be confused with angular velocity) are called *natural damped frequencies*, ω_{nd}. ("*Natural damped frequency*" is the damped angular frequency of a physical oscillator when it is *not* driven externally (David Halliday et al. 402).) The natural damped frequency of a building is determined by material and structure. However, the natural damped frequency of a pendulum can be changed by varying the length of its rod. It is reasonable to assume that pendulums with different ω_{nd} have different damping effects on the building. Since ω_{nd} of pendulum can be most practically varied by changing the length of the pendulum rod, it is relevant and significant to discuss the **<u>Research Question:</u> How does the**

damping effect of a pendulum on an oscillating mass change with the length of the pendulum rod?

To approach this question, I constructed a physical model (Fig 2). Before modelling this combined structure of the pendulum and the building mathematically, certain constants were needed. For the pendulum, the angular velocity and frictional torque at pivot is related by a constant. The first section of the research was dedicated to this constant. For the top of the building (*abbreviated as* **top**), spring constant and damping constant were needed. The second section was dedicated to these 2 constants. Then, the equations of motion (**EOM**) of the combined structure of the pendulum attached to the building were derived. Finally, the experimental data was collected and compared with numerical solutions (using the 3 constants) of the EOMs for different rod lengths. The accuracy of the model was evaluated finally.

This research demonstrates the feasibility of optimizing a PTMD. In my experiment, the best damping effect was achieved by the pendulum which had a natural damped frequency close to the that of the building. (Since the first few oscillations of the building can bring the most damage, "best damping effect" refers to the best ability to reduce amplitudes of the first few oscillations.) Indeed, for applications, the PTMDs are tuned to a natural damped frequency the same as that of the building (R. Lourenco et al.).

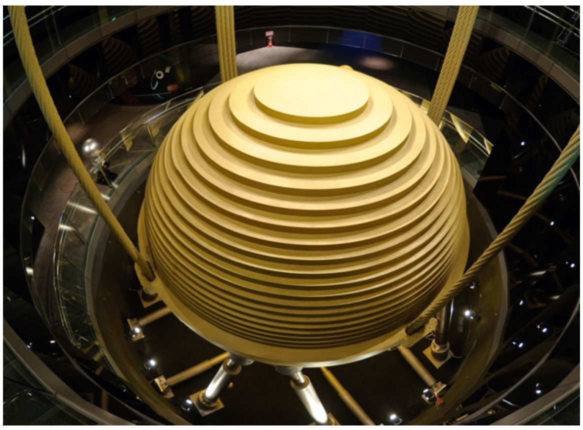

Figure 1 Armand du Plessis. "Tuned Mass Damper On Display In Taipei 101."

Figure 2 My Wooden Building with Pendulum Attached

Experimental Setup

In my experiment, a pendulum was attached to a simple wooden structure. Wood is easy to cut, and its flexibility also allowed me to induce oscillations.

The top of the building (Fig 3) was made from a square sheet of wood with 4 holes drilled. The base of the structure was made similar to the top. 4 wooden sticks with the same lengths were inserted into the 4 holes of the base, and then the top was installed.

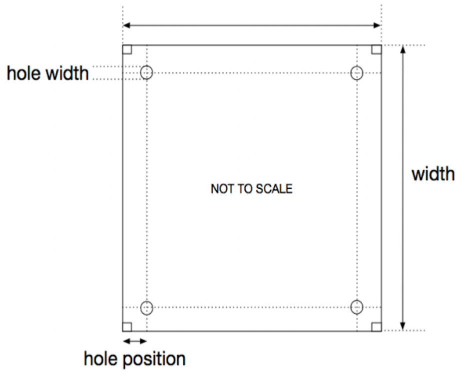

Figure 3 Blue Print of the Top of the Building

The pendulum was made from Lego® pieces and a mass. It was attached to the bottom of the top of the structure. A mpu6050 electronic gyroscope was then mounted onto the axis of rotation using glue (Fig 4). The wires from the gyroscope were then connected to the power supply and the analogue input pins on the Arduino (Atmel-Atmega328p *microcontroller* based), ready to deliver angular displacement (vertical position is $\theta = 0$) data to my computer. Thin wires were used to minimize impact on the pendulum motion.

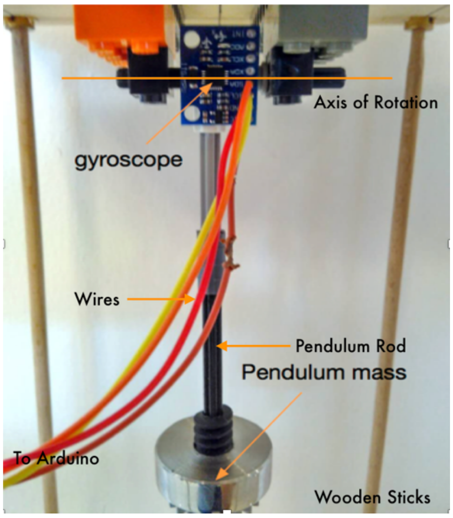

Figure 4 Gyroscope Mounted on the Pendulum

After the physical model was constructed, a heavy brick was put onto its base to make sure that the base does not oscillate. The finished physical model had a top that can only oscillate *noticeably* along the x-direction (y-axis motion negligible), and a pendulum that can only swing in the plane created by the vertical projection of x-axis onto the ground (Fig 5).

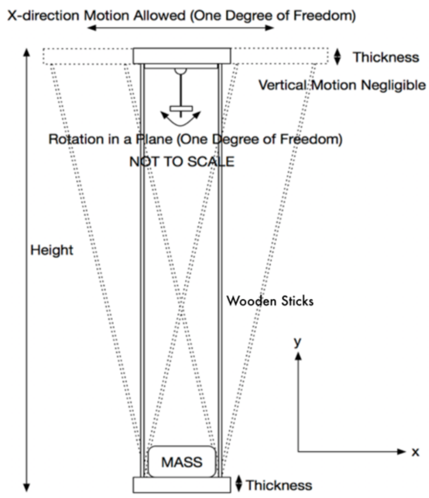

Figure 5 2 Degrees of Freedom of pendulum and the Wooden Building

Technical Data of Experimental Setup

A plastic ruler was used to measure the width, the hole width, the hole position (Fig 3) and the thickness of the top and bottom (Fig 5). (The length of pendulum rod is the independent variable in the final experiment, hence not included here.) A meter ruler was used to measure the height of the model (Fig 5). An electronic balance was used to measure the mass of the pendulum mass and the mass of the top (Table 1).

Table 1 Technical Data of the Physical Model (Refer back to Figure 3 & 5 on page 3)

Parameter Measured	Measurement and Absolute Uncertainty
Width	(0.150±0.005) m
Hole Width	(0.005±0.001) m
Hole Position	(0.015±0.001) m
Top or Bottom Thickness	(0.017±0.001) m
Height of Building	(1.02±0.01) m
Top Mass	(0.198±0.001) kg
Pendulum Mass	(0.050±0.001) kg

II. Investigating Constant b for Pendulum (Top Not Allowed to Move, Pivot Fixed)

Structure of This Section

Constants must be calculated before modelling the combined mathematical model of the pendulum attached to the building (*abbreviated as* **combined model**). Constants indicate linear relationships. Such relationships are often first *hypothesized by theoretical analysis*, and later confirmed and computed *using experimental data*. This approach was taken here.

Mathematical Model

The mechanical friction at a physical-pendulum pivot has to be considered. In linear motion,

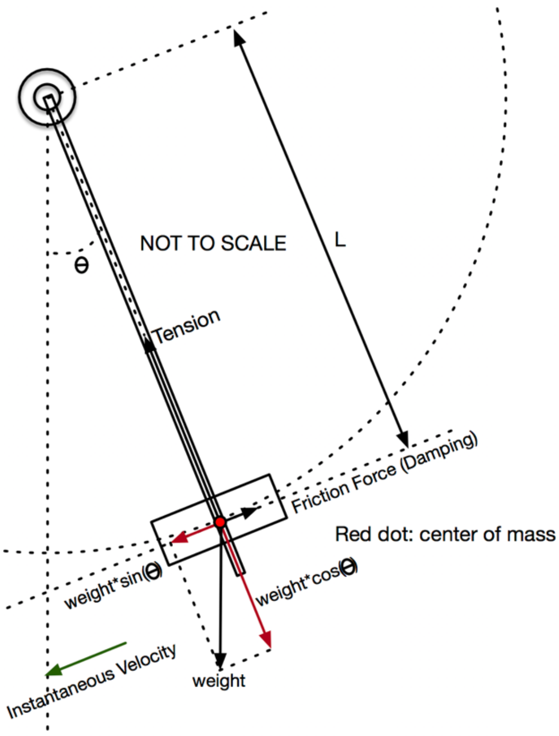

Figure 6 Pendulum (Without Pivot Moving) Free-body-force Diagram

the dynamic frictional force is proportional to objects' relative velocity; similarly, at a pivot, when the inner bearing rotates clockwise, it experiences an anticlockwise *twisting action* proportional to its *angular velocity*. (The twisting action is called a *torque* (David Halliday et al. 259).) Therefore, the *frictional torque* can be written as (Arora et al.):

$$\tau_{friction} = -b\frac{d\theta}{dt}$$

where b is the *pendulum damping torque constant* (**PDTC**), the constant to be calculated in this section.

Newton's Second Law (**NSL**) is a convenient way to mathematically model mechanical systems. To use the law on the pendulum mass for free-body-force analysis, the torque must be defined in terms of a force. By definition, torque has this magnitude if the level arm and the force are perpendicular (David Halliday et al. 282):

$$\tau = rF$$

where r is the length of the level arm, F is the force. For the pendulum, r was taken as L, the length from the center of mass (**COM**) of the pendulum mass to its pivot. The frictional force can be treated as a force acting on the COM of the pendulum mass in the *tangential direction along its circular trajectory* (Fig 6). In this way, frictional force is:

$$f_{friction} = \frac{\tau_{friction}}{L} = -\frac{b}{L}\frac{d\theta}{dt}$$

which varies according to L.

Apply NSL on the COM of the pendulum (Fig 6):

$$\sum F_{pen} = m_{pen} a_{pen}$$

$$m_{pen} a = -m_{pen} g sin(\theta) - \frac{b}{L}\frac{d\theta}{dt}$$

(Subscript "pen" refers to "pendulum".)

Instantaneous linear acceleration is equivalent to the second derivative of linear displacement with respect to time:

$$m_{pen} \frac{d^2x}{dt^2} = -m_{pen} g sin(\theta) - \frac{b}{L}\frac{d\theta}{dt}$$

To make the equation in terms of θ and t, $x = L\theta$, $\frac{d^2x}{dt^2} = \frac{d^2\theta}{dt^2}L$:

$$m_{pen} \frac{d^2\theta}{dt^2} L = -m_{pen} g sin(\theta) - \frac{b}{L}\frac{d\theta}{dt}$$

$$\frac{d^2\theta}{dt^2} + \frac{b}{m_{pen}L^2}\frac{d\theta}{dt} + \frac{g}{L} sin(\theta) = 0$$

For ±0.52 radians (about 30 degrees), $\sin(\theta) \approx \theta$ (within 5% of error). This EOM of pendulum is a derived completely from *practically-measurable variables* (**PMV**s):

$$\frac{d^2\theta}{dt^2} + \frac{b}{m_{pen}L^2}\frac{d\theta}{dt} + \frac{g}{L}\theta = 0 \quad \mathbf{(1)}$$

Constant b cannot be calculated from equation (1) alone. Therefore, I decided to compare equation (1) with the general differential equation of motion (GDEOM) of damped harmonic oscillators, which is (Haynes R. Miller and MIT 60):

$$\frac{d^2\theta}{dt^2} + 2\omega_{npen}\zeta_{pen}\frac{d\theta}{dt} + {\omega_{npen}}^2\theta = 0 \quad \textbf{(2)}$$

where ω_{npen} is the *natural un-damped frequency* of the pendulum, ζ_{pen} is the pendulum damping coefficient. ("Natural un-damped frequency" is the un-damped angular frequency of a physical oscillator when it is not driven externally (David Halliday et al. 402).) In this way, the coefficient of derivative terms in equation (1) and (2) were matched and derivative terms were cancelled out. This would result in two equations. The two equations were then merged into one using their common ω_{npen} term. In that one equation, only b needed to be calculated, since m_{pen}, g, L and ζ_{pen} (calculated from experimental data) were all known.

However, before matching up the coefficients of terms in equation (1) and (2), we must *verify* that the pendulum indeed behaves according to the GDEOM of harmonic oscillators. Fixing the top, displacing and releasing the pendulum, the angular displacement (measured by gyroscope programmed by codes (Zhaiky)) vs. time graph of the pendulum with L = (0.10021±0.00002) m was fitted well by a damped harmonic model in Logger Pro (Fig 7), a video analysis software. My pendulum was indeed a harmonic oscillator.

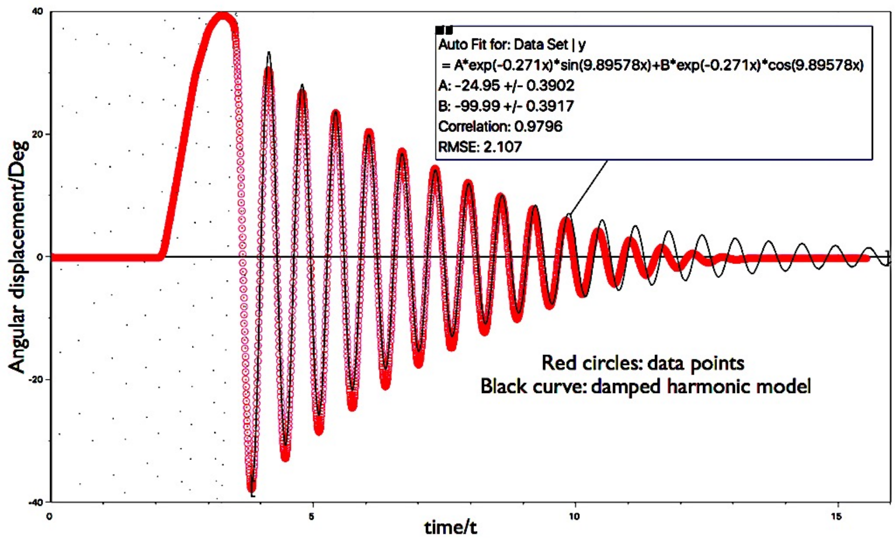

Figure 7 The damped harmonic model fitted the oscillation data of the pendulum very well.

This confirmed assumption allowed comparison between equation (1) and equation (2):

$$\frac{d^2\theta}{dt^2} + 2\omega_{npen}\zeta_{pen}\frac{d\theta}{dt} + {\omega_{npen}}^2\theta = \frac{d^2\theta}{dt^2} + \frac{b}{m_{pen}L^2}\frac{d\theta}{dt} + \frac{g}{L}\theta \quad \mathbf{(3)}$$

Matching coefficients:

$$2\omega_{npen}\zeta_{pen} = b\frac{1}{m_{pen}L^2}$$

$${\omega_{npen}}^2 = \frac{g}{L}$$

Merging two equations, cancelling ω_{npen} which is not directly computable:

$$2\sqrt{\frac{g}{L}}\zeta_{pen} = b\frac{1}{m_{pen}L^2} \quad \mathbf{(4)}$$

ζ_{pen}, the pendulum damping coefficient, can be calculated using logarithmic decrement, δ (Daniel J. Inman 43–48), which is the natural log of the ratio of the amplitudes of two successive peaks:

$$\delta = \frac{1}{n}\ln\left(\frac{\theta(t)}{\theta(t+nT)}\right)$$

$$\zeta_{pen} = \frac{1}{\sqrt{1+\left(\frac{2\pi}{\delta}\right)^2}} \quad \mathbf{(5)}$$

where n is the number of periods between the two successive peaks, and T is the natural damped period of oscillation of the pendulum.

Combining equation (4) and (5):

$$2\sqrt{\frac{g}{L}}\frac{1}{\sqrt{1+\left(\frac{2\pi}{\frac{1}{n}\ln\left(\frac{\theta(t)}{\theta(t+nT)}\right)}\right)^2}} = b\frac{1}{m_{pen}L^2} \quad \mathbf{(6)}$$

For better spacing, let the notation $y^*=2\sqrt{\frac{g}{L}}\frac{1}{\sqrt{1+\left(\frac{2\pi}{\frac{1}{n}\ln\left(\frac{\theta(t)}{\theta(t+nT)}\right)}\right)^2}}$ and $x^*=\frac{1}{m_{pen}L^2}$ be continued.

Computing Constant b Using Experimental Results

Hypothetically, the linear relationship between y^* and x^* should be a straight line through the origin with a positive slope b. Initial parameters include m_{pen} = (0.050±0.001)kg and $g = 9.81ms^{-2}$. The raw data (Table 2) includes pendulum rod lengths and corresponding two successive crests that were 5 periods apart ($n = 5$). The error of L was obtained from technical instruction paper; the error in angular displacement was obtained from my experience in using the sensor. The gyroscope was programmed in Arduino language (Zhaiky).

Table 2 Raw Data Table

$L \pm 0.00002/m$	θ (t) $\pm 0.5/Deg$	θ (t+5T) $\pm 0.5/Deg$
0.10177	27.1	11.5
0.08917	29.6	11.3
0.07914	28.0	8.3
0.07111	26.5	6.4
0.06289	23.9	5.2

From the raw data, y^* and x^* can be calculated. In table 3, the number of significant figures of x^* was limited by the number of significant figures of the mass of the pendulum mass.

Table 3 Processed Data Table

L $\pm 0.00002/m$	θ (t) $\pm 0.5/Deg$	θ (t+5T) $\pm 0.5/Deg$	y^*/s^{-1}	error of y^*/s^{-1}	x^* $/kg^{-1}m^{-2}$	error x^* $/kg^{-1}m^{-2}$
0.10177	27.1	11.5	0.535	0.037	1900	<2%
0.08917	29.6	11.3	0.643	0.044	2500	
0.07914	28.0	8.3	0.86	0.07	3200	
0.07111	26.5	6.4	1.1	0.1	4000	
0.06289	23.9	5.2	1.2	0.1	5100	

The errors of y^* and x^* were tedious to compute and display, thus one python program (Zhaiky) was used for each of them to compute their values and absolute errors.

Using Table 3, Figure 8 was produced by plotting y^* against x^* with the error bars in Logger Pro. The black line represented the best linear fit for the data set, and the red and the blue line each represented the maximum and minimum slope line.

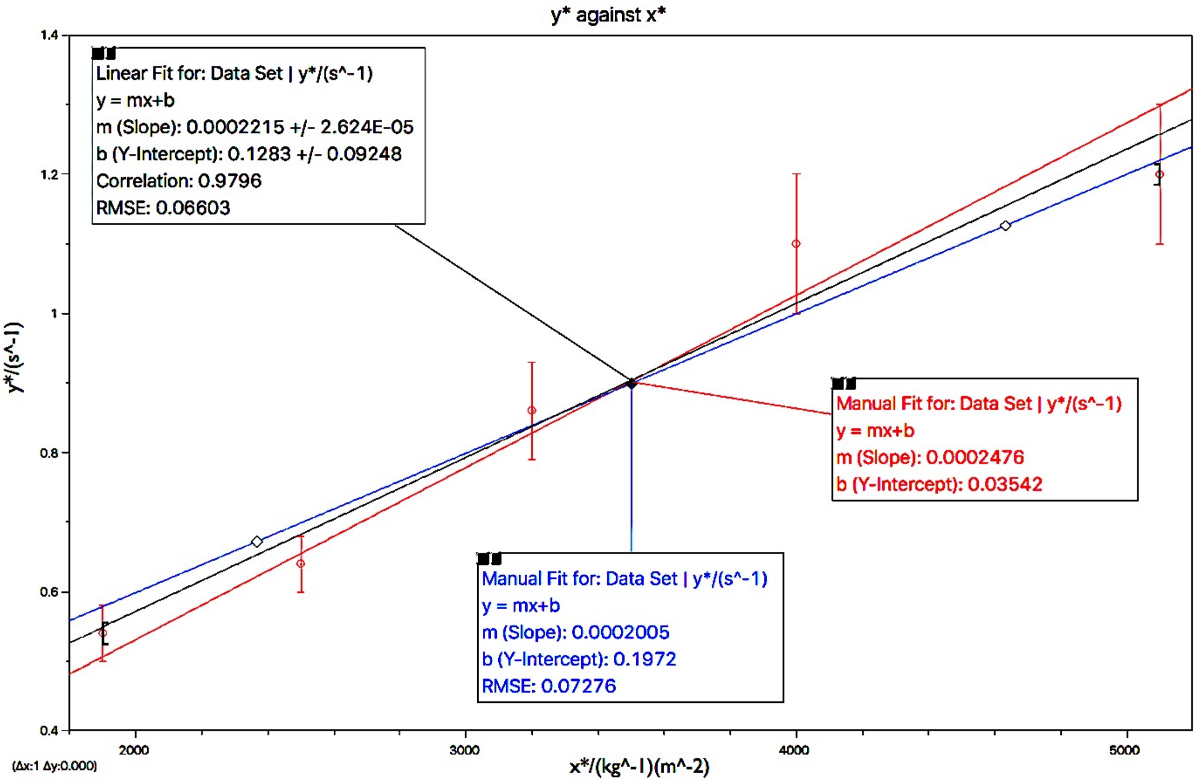

Figure 8 Plotting y against x**

The pendulum damping torque constant, b, is the slope of the graph. From the graph, we can see that the best linear fit line has a slope of 0.00022 (2s.f.), and that the maximum and the minimum slope line each has a slope of 0.00025 (2s.f.) and 0.00020 (2s.f.). By convention, we can calculate the absolute uncertainty of b in this way:

$$absolute\ uncertainty = \frac{max - min}{2} = \frac{0.00025 - 0.00020}{2}$$

$$= 0.00003\ kgm^2s^{-1}\ (1s.f.)$$

The percentage uncertainty of the slope can then be calculated:

$$percentage\ uncertainty\ in\ slope = \frac{absolute\ uncertainty\ in\ slope}{slope\ of\ best\ fit\ line} = \frac{0.00003}{0.00022}$$

$$\approx 10\%(1s.f.)$$

which is acceptable for a cheap gyroscope.

The constant b, together with its percentage uncertainty, was finally obtained. Note that the unit of b was calculated by dividing the unit of y^* by x^*.

b/ kgm^2s^{-1}	Percentage Uncertainty
0.00022 (2s.f.)	10%

III. Investigating Constants c and k for Wooden Building (Pendulum Removed)

After finding the PDTC, constants for the building must also be found before modelling the combined model.

Mathematical Model

The top of the building was not attached to any springs or dampers, but we can assume the mechanical friction of the building as a damper and the elasticity of the wooden sticks as a spring.

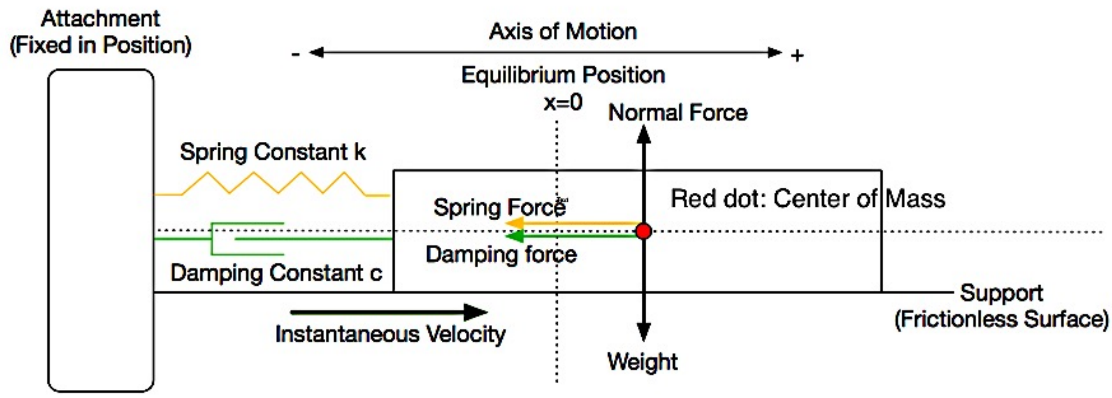

Figure 9 Free-body-force Diagram of the Top (Pendulum Removed)

The damping force on the top depends on its velocity (*Damped Oscillation*):

$$f_{damp} = -c\frac{dx}{dt}$$

where c is top damping constant.

Using NSL on the COM of the top (Fig 9):

$$\sum F_{top} = m_{top}a_{top}$$

$$-kx - c\frac{dx}{dt} = m_{top}\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{c}{m_{top}}\frac{dx}{dt} + \frac{k}{m_{top}}x = 0 \quad \textbf{(7)}$$

Just as the EOM of the pendulum derived from PMVs, constants c and k cannot be calculated from equation (7) alone. Taking a similar approach as in Section II, we can *assume* the top a damped harmonic oscillator and compare equation (7) and the GDEOM of damped harmonic oscillator. To verify the validity of this assumption, I did video analysis (Fig 10) on the displacement of the top with respect to time with Logger Pro, and the resulting data points (Fig 11) was fitted well by a damped harmonic oscillator model. The assumption was valid. (Data table offered in Appendix Part 2)

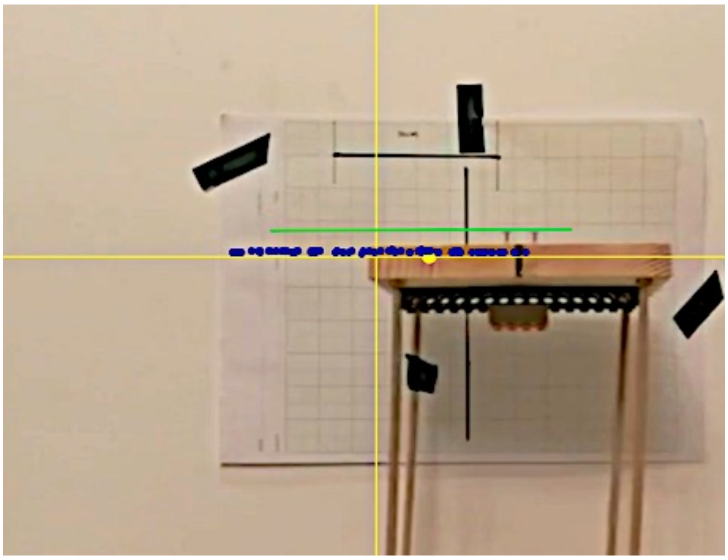

Figure 10 *Video Analysis on Oscillation of the Top*

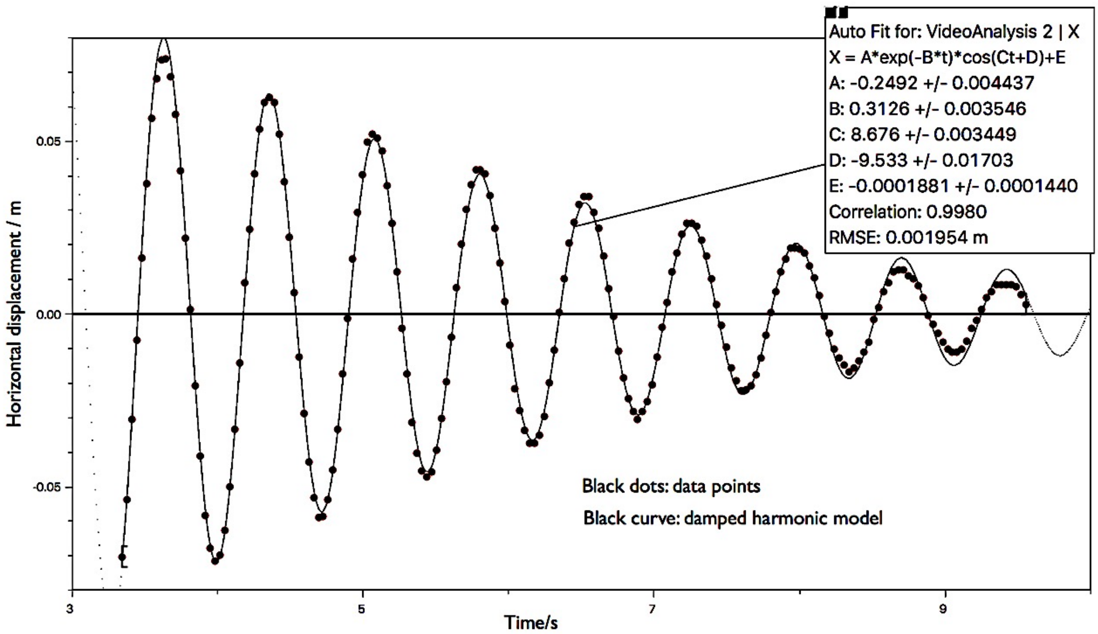

Figure 11 *Oscillation Data of the Top (Pendulum Removed) Fitted by A Damped Harmonic Model*

Comparing equation (7) with the GDEOM (Haynes R. Miller and MIT 60) :

$$\frac{d^2x}{dt^2}+\frac{c}{m_{top}}\frac{dx}{dt}+\frac{k}{m_{top}}x=\frac{d^2x}{dt^2}+2\omega_{ntop}\zeta_{top}\frac{dx}{dt}+{\omega_{ntop}}^2x \quad \mathbf{(8)}$$

where ω_{ntop} is the *natural un-damped frequency* of the top, and ζ_{top} is the top damping coefficient.

Matching coefficients:

$$\omega_{ntop}\zeta_{top}=\frac{c}{2m_{top}} \quad \mathbf{(9)}$$

$${\omega_{ntop}}^2=\frac{k}{m_{top}} \quad \mathbf{(10)}$$

Unlike for the pendulum, merging the two equations through ω_{ntop} is useless, since we would end up with one equation with two unknowns. A different approach was needed: If we can compute ω_{ntop}, the unknown constants c and k can be computed easily.

From graphs, the value of ω_{ntop} is not directly attainable because the graph (horizontal displacement plotted against time) only shows *natural damped frequency* ($2\pi/period$), ω_{ndtop}, but equation (9) and (10) only include *natural un-damped frequency*, ω_{ntop}. Without ω_{ntop}, c and k cannot be calculated, therefore, we must solve the GDEOM of damped harmonic oscillator (Haynes R. Miller and MIT 60):

$$\frac{d^2x}{dt^2}+2\omega_{ntop}\zeta_{top}\frac{dx}{dt}+{\omega_{ntop}}^2x=0 \quad \mathbf{(11)}$$

The solution (in Appendix Part 1) of equation (11) is (Haynes R. Miller and MIT 60):

$$x(t)=e^{-\omega_{ntop}\zeta_{top}t}A\cos\left(\omega_{ntop}\sqrt{1-{\zeta_{top}}^2}t+\phi\right)= e^{-\omega_{ntop}\zeta_{top}t}A\cos(\omega_{ndtop}t+\phi) \quad \mathbf{(12)}$$

where ϕ represents the initial state of the top (when $t=0$, $\cos\left(\omega_{ntop}\sqrt{1-{\zeta_{top}}^2}t+\phi\right)$ becomes $\cos(\phi)$), A represents maximum displacement.

Computing Constants c and k Using Experimental Results

Using Logger Pro, the best-fit damped harmonic model of the top's motion is (Fig 11):

$$x(t) = -0.2492e^{-0.3126}\cos(8.676t - 9.533) \quad \textbf{(13)}$$

Comparing equation (12) and equation (13), we get:

$$-0.2492e^{-0.3126t}\cos(8.676t - 9.533) = Ae^{-\omega_{ntop}\zeta_{top}t}\cos\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t + \phi\right) \quad \textbf{(14)}$$

Matching up coefficients (0.3126 and 8.676 both have the unit of s^{-1}, since ω_{ntop} has the unit of s^{-1}):

$$0.3126 = \omega_{ntop}\zeta_{top} \quad \textbf{(15)}$$

$$8.676 = \omega_{ntop}\sqrt{1-\zeta_{top}{}^2} \quad \textbf{(16)}$$

Combining equation (9) with equation (15):

$$0.3126 = \frac{c}{2m_{top}} \quad \textbf{(17)}$$

$m_{top} = 0.198\ kg$, c can be solved:

$$0.3126 = \frac{c}{2m_{top}}$$

$$c = 0.3126 * 2 * m_{top} = 0.3126 * 2 * 0.198 = 0.124\ kgs^{-1}\ (3s.f.)$$

Algebraically modifying equation (16),

$$8.676^2 = \omega_{ntop}^2\left(1-\zeta_{top}{}^2\right) \quad \textbf{(18)}$$

Combining equation (10) with equation (18):

$$8.676^2 = \frac{k}{m_{top}}\left(1-\zeta_{top}{}^2\right) \quad \textbf{(19)}$$

For solving k, the top damping coefficient ζ_{top} needed to be found from video analysis using logarithmic decrement (Daniel J. Inman 43–48):

$$\delta = \frac{1}{n}\ln\left(\frac{x(t)}{x(t+nT)}\right)$$

$$\zeta_{top} = \frac{1}{\sqrt{1+\left(\frac{2\pi}{\delta}\right)^2}}$$

From Video Analysis		Computed Using Logarithmic Decrement
$x(t)$/m	$x(t+5T)$/m	ζ_{top}
0.0896 (3s.f)	0.0303 (3s.f.)	0.0345 (3s.f.)

Substituting values into equation (19):

$$8.676^2 = \frac{k}{m_{top}}\left(1-{\zeta_{top}}^2\right)$$

$$k = \frac{8.676^2 * m_{top}}{\left(1-{\zeta_{top}}^2\right)} = \frac{8.676^2 * 0.198}{(1-0.0345^2)} = 15.0\ Nm^{-1}\ (3s.f.)$$

Value of 2 constants:

c/kgs^{-1}	k/Nm^{-1}
0.124 (3s.f.)	15.0 (3s.f.)

IV. Combined Mathematical Model of the Pendulum Attached to the Building

After obtaining the essential constants, the combined model can finally be considered through *Newton's Second Law* (**NSL**), which relates forces to acceleration.

Position, Velocity and Acceleration of the Top and the Pendulum

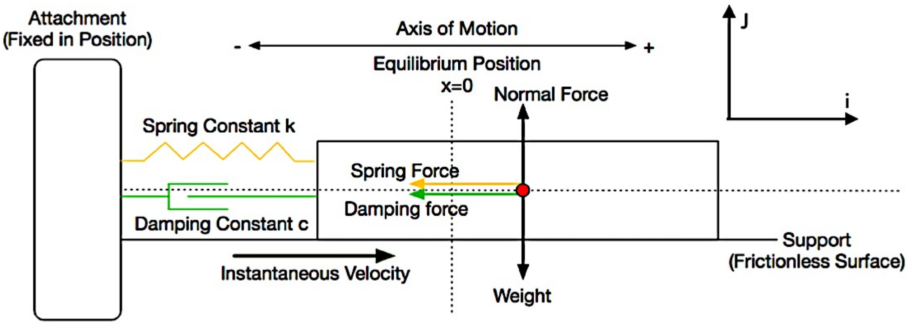

Figure 12 Diagram of the Top of the Building

From Figure 12, we define the position, velocity and acceleration of the COM of the top:

$$position_{top} = x\hat{\imath}$$

$$velocity_{top} = \frac{dx}{dt}\hat{\imath}$$

$$acceleration_{top} = \frac{d^2x}{dt^2}\hat{\imath}$$

where $\hat{\imath}$ is the *position vector* pointing in the positive x-direction.

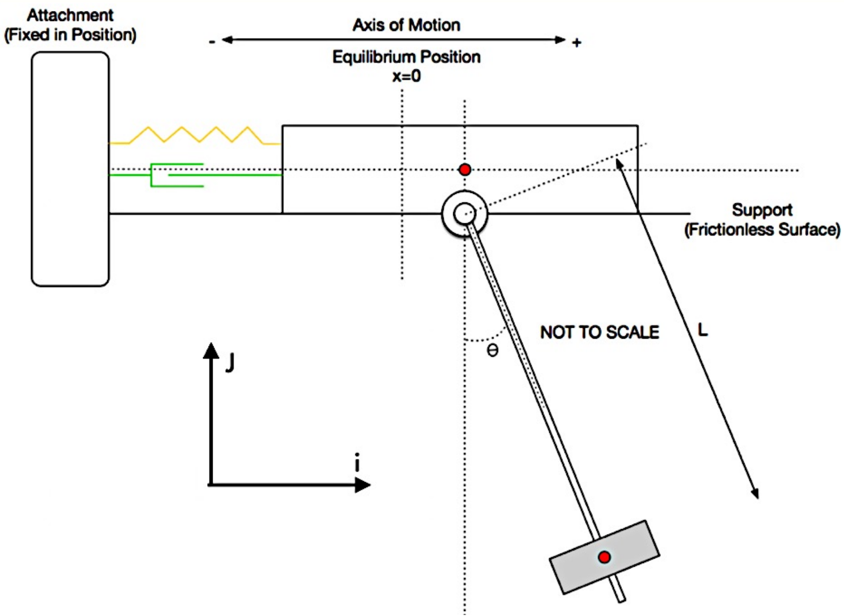

Figure 13 Diagram of the Pendulum Attached to the Top of the Building

From Figure 13, we define the position, velocity and acceleration of the COM of the pendulum mass (Erik Neumann):

$$position_{pen} = x\hat{\imath} + Lsin(\theta)\,\hat{\imath} - Lcos(\theta)\hat{\jmath}$$

Taking the derivative of position with respect to time:

$$velocity_{pen} = \frac{dx}{dt}\hat{\imath} + L\frac{d\theta}{dt}\cos(\theta)\,\hat{\imath} + L\frac{d\theta}{dt}\sin(\theta)\,\hat{\jmath}$$

Taking the derivative of velocity with respect to time using *chain rule*:

$$acceleration_{pen} = \frac{d^2x}{dt^2}\hat{\imath} + L\frac{d^2\theta}{dt^2}\cos(\theta)\,\hat{\imath} - L\left(\frac{d\theta}{dt}\right)^2\sin(\theta)\,\hat{\imath} + L\frac{d^2\theta}{dt^2}\sin(\theta)\,\hat{\jmath} + L\left(\frac{d\theta}{dt}\right)^2\cos(\theta)\hat{\jmath}$$

where $\hat{\imath}$ and $\hat{\jmath}$ are *position vectors* pointing in the positive x-direction and y-direction respectively (Fig 13).

Deriving Equations of Motion of the Combined Model

Since my research is to examine the building's oscillation in response to different pendulums, let us first focus on the forces acting on the COM of the top. The situation is *complicated* as a movable pendulum is now attached. It is worth noticing that, since the pendulum experiences a damping force of $-\frac{b}{L}\frac{d\theta}{dt}$, from *Newton's third law*, the pivot (*connected to the top*) also experiences a force ("Force from Pendulum Damping" on Fig 14) in the opposite direction of the value $\frac{b}{L}\frac{d\theta}{dt}$.

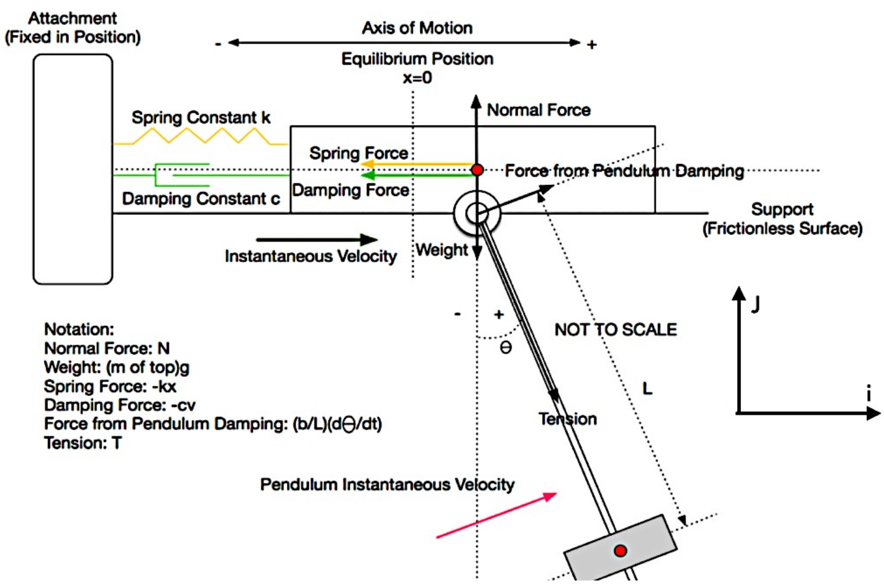

Figure 14 Free-Body-Force Diagram of the Top of the Building with Movable Pendulum Attached

We can apply NSL on the COM of the top (Fig 14):

$$\sum F = m_{top} * acceleration_{top}$$

$$N\hat{j} - m_{top}g\hat{j} + Tsin(\theta)\hat{\imath} - Tcos(\theta)\hat{j} + \frac{b}{L}\frac{d\theta}{dt}\cos(\theta)\,\hat{\imath} + \frac{b}{L}\frac{d\theta}{dt}\sin(\theta)\,\hat{j} - kx\hat{\imath} - c\frac{dx}{dt}\hat{\imath} = m_{top}\frac{d^2x}{dt^2}\hat{\imath} \quad \mathbf{(20)}$$

Now let us focus on the forces acting on the COM of the pendulum mass.

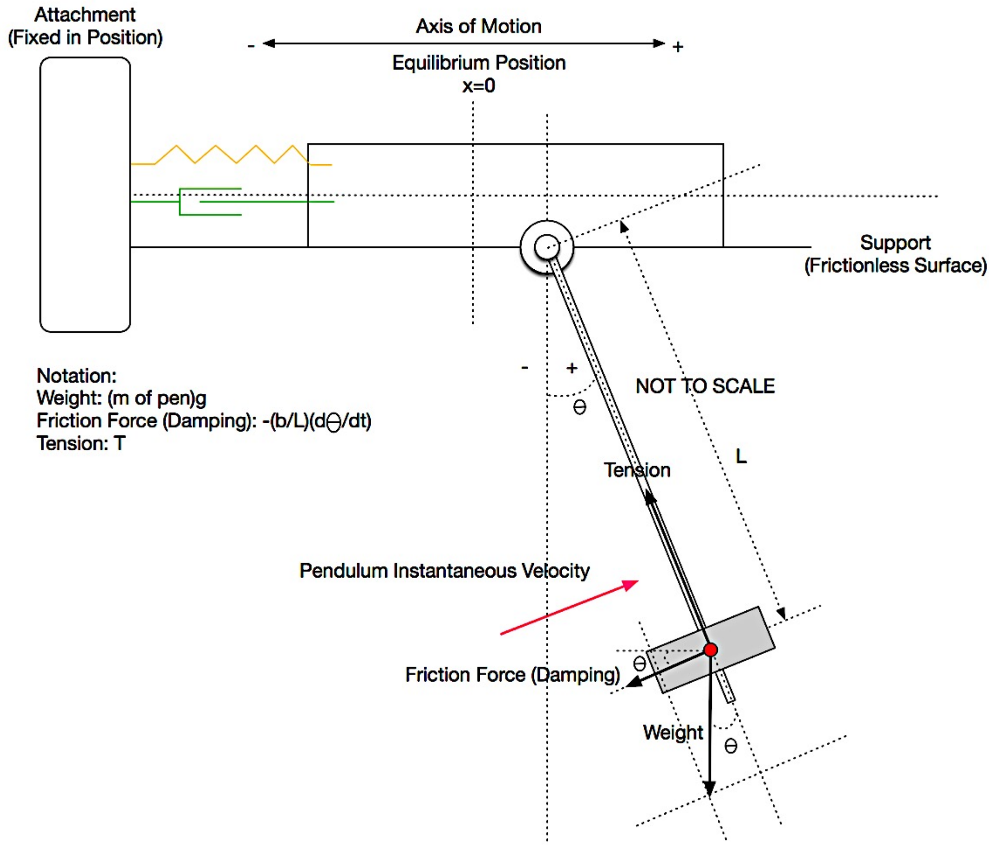

Figure 15 Free-Body-Diagram of the Pendulum (Attached to the Movable Building)

Using NSL on the COM of the pendulum mass (Fig 15):

$$\sum F = m_{pen} * acceleration_{pen}$$

$$Tcos(\theta)\hat{j} - Tsin(\theta)\hat{\imath} - m_{pen}g\hat{j} - \frac{b}{L}\frac{d\theta}{dt}\cos(\theta)\,\hat{\imath} - \frac{b}{L}\frac{d\theta}{dt}\sin(\theta)\,\hat{j}$$

$$= m_{pen}\left(\frac{d^2x}{dt^2}\hat{\imath} + L\frac{d^2\theta}{dt^2}\cos(\theta)\,\hat{\imath} - L\left(\frac{d\theta}{dt}\right)^2\sin(\theta)\,\hat{\imath} + L\frac{d^2\theta}{dt^2}\sin(\theta)\,\hat{j} + L\left(\frac{d\theta}{dt}\right)^2\cos(\theta)\,\hat{j}\right) \quad \mathbf{(21)}$$

We can take the vector components out of equation (20) and (21) as 4 equations (Erik Neumann):

$$Tsin(\theta) + \frac{b}{L}\frac{d\theta}{dt}\cos(\theta) - kx - c\frac{dx}{dt} = m_{top}\frac{d^2x}{dt^2} \quad \mathbf{(22)}$$

$$N - m_{top}g - Tcos(\theta) + \frac{b}{L}\frac{d\theta}{dt}\sin(\theta) = 0 \quad \mathbf{(23)}$$

$$-Tsin(\theta) - \frac{b}{L}\frac{d\theta}{dt}\cos(\theta) = m_{pen}\left(\frac{d^2x}{dt^2} + L\frac{d^2\theta}{dt^2}\cos(\theta) - L\left(\frac{d\theta}{dt}\right)^2\sin(\theta)\right) \quad \mathbf{(24)}$$

$$Tcos(\theta) - m_{pen}g - \frac{b}{L}\frac{d\theta}{dt}\sin(\theta) = m_{pen}\left(L\frac{d^2\theta}{dt^2}\sin(\theta) + L\left(\frac{d\theta}{dt}\right)^2\cos(\theta)\right) \quad \mathbf{(25)}$$

Since we do not need to and cannot compute the normal force, we disregard equation (23). The tension is also redundant, so we cancel it out by summing equation (22) and equation (24):

$$\left(Tsin(\theta) + \frac{b}{L}\frac{d\theta}{dt}\cos(\theta) - kx - c\frac{dx}{dt}\right) - Tsin(\theta) - \frac{b}{L}\frac{d\theta}{dt}\cos(\theta)$$

$$= \left(m_{top}\frac{d^2x}{dt^2}\right) + m_{pen}\left(\frac{d^2x}{dt^2} + L\frac{d^2\theta}{dt^2}\cos(\theta) - L\left(\frac{d\theta}{dt}\right)^2\sin(\theta)\right) \quad \mathbf{(26)}$$

Grouping similar terms:

$$(m_{top} + m_{pen})\frac{d^2x}{dt^2} = m_{pen}L\left(\frac{d\theta}{dt}\right)^2\sin(\theta) - m_{pen}L\frac{d^2\theta}{dt^2}\cos(\theta) - kx - c\frac{dx}{dt} \quad \mathbf{(27)}$$

To solve for both $x(t)$ and $\theta(t)$, we need another differential equation that is *non-equivalent*[1] to equation (27). We can multiply equation (25) by sin (θ):

$$Tcos(\theta)\sin(\theta) - m_{pen}gsin(\theta) - \frac{b}{L}\frac{d\theta}{dt}\sin(\theta)\sin(\theta)$$

$$= m_{pen}\left(L\frac{d^2\theta}{dt^2}\sin(\theta) + L\left(\frac{d\theta}{dt}\right)^2\cos(\theta)\right)\sin(\theta) \quad \mathbf{(28)}$$

and use equation (24) to substitute $Tsin(\theta)$ in equation (28).

After algebraic manipulation (steps shown in Appendix Part 3):

$$\frac{b}{L}\frac{d\theta}{dt} + m_{pen}\frac{d^2x}{dt^2}\cos(\theta) + m_{pen}L\frac{d^2\theta}{dt^2} + m_{pen}gsin(\theta) = 0 \quad \mathbf{(29)}$$

[1] Non-equivalent means that the second differential equation cannot be derived from exact same two equations as the first one.

Solving the Differential Equations of Motion

$$\left(m_{top} + m_{pen}\right)\frac{d^2x}{dt^2} = m_{pen}L\left(\frac{d\theta}{dt}\right)^2 \sin(\theta) - m_{pen}L\frac{d^2\theta}{dt^2}\cos(\theta) - kx - c\frac{dx}{dt} \quad \textbf{(30)}$$

$$\frac{b}{L}\frac{d\theta}{dt} + m_{pen}\frac{d^2x}{dt^2}\cos(\theta) + m_{pen}L\frac{d^2\theta}{dt^2} + m_{pen}gsin(\theta) = 0 \quad \textbf{(31)}$$

The complicatedness of equation (30) and equation (31) does not allow us to obtain explicit solutions to the two differential equations. Therefore, I used Runge-Kutta (Prof. Mark Owkes - Montana State University) method to obtain *numerical solutions*. This method requires four first-order differential equations, and for me to do that, two more differential equations had to be added (Erik Neumann):

$$\frac{dx}{dt} = v \quad \textbf{(32)}$$

$$\frac{d\theta}{dt} = \omega \quad \textbf{(33)}$$

$$\left(m_{top} + m_{pen}\right)\frac{dv}{dt} = m_{pen}L(\omega)^2 \sin(\theta) - m_{pen}L\frac{d\omega}{dt}\cos(\theta) - kx - cv \quad \textbf{(34)}$$

$$\frac{b}{L}\omega + m_{pen}\frac{dv}{dt}\cos(\theta) + m_{pen}L\frac{d\omega}{dt} + m_{pen}gsin(\theta) = 0 \quad \textbf{(35)}$$

where $m_{pen} = 0.050\ kg\ (2.s.f), b = 0.00022\ kgm^2s^{-1}\ (2s.f), m_{top} = 0.198\ kg$, $k = 15.0\ Nm^{-1}\ (3s.f.)$ and $c = 0.124\ kgs^{-1}(3.s.f)$. ω represents angular velocity.

A Matlab program (Zhaiky) was used to calculate the numerical solution to this system of 4 differential equations. Note that initial conditions of horizontal displacement, x, and pendulum rod length, L, were needed as input. The solutions were demonstrated later as graphs plotting consecutive x (the amplitude of oscillation of the COM of the top) against t (Fig 16 to 20).

V. Analysis – Comparing Combined Model with Experimental Data

In this section, five different pendulum lengths were experimented. Since the period is dependent on the rod length, the corresponding natural damped frequencies must also be different. It seems reasonable to assume that there exists some particular rod length that would allow the pendulum to have the best damping effect. According to an article (R. Lourenco et al.) and a response from an Internet user (Freeball), the best damping effect occurs when when the natural damped frequency of the pendulum, ω_{ndpen}, is equal to that of the top, ω_{ndtop}. This section aims to verify this statement.

Finding the Optimal Length of the Pendulum Rod in Theory

Equation (13) showed the best-fit damped harmonic model on top's oscillation data:

$$x(t) = -0.2492e^{-0.3126t}\cos(\,8.676t - 9.533)$$

Comparing equation (13) with the solution of the GDEOM of damped harmonic oscillators:

$$e^{-\omega_{ntop}\zeta_{top}t}A\cos\left(\omega_{ndtop}t + \phi\right) = -0.2492e^{-0.3126t}\cos(\,8.676t - 9.533) \quad \textbf{(36)}$$

where ω_{ndtop} is the natural damped frequency of the top.

$\omega_{ndtop} = 8.676s^{-1}$ in equation (36). Therefore, to actually see whether the pendulum with such a natural damped frequency would have the best damping effect, we need to compute the length of pendulum rod so that its natural damped frequency, ω_{ndpen}, is equal to $8.676s^{-1}$.

Note the ω_{ndpen} is related to the natural un-damped frequency, ω_{npen}, in the following way according to equation (12):

$$\omega_{ndpen} = \omega_{npen}\sqrt{1 - {\zeta_{pen}}^2}$$

Moving ω_{npen} into the square root:

$$\omega_{ndpen} = \sqrt{{\omega_{npen}}^2 - {\omega_{npen}}^2{\zeta_{pen}}^2} \quad \textbf{(37)}$$

From equation (3):

$$\frac{d^2\theta}{dt^2} + 2\omega_{npen}\zeta_{pen}\frac{d\theta}{dt} + \omega_{npen}{}^2\theta = \frac{d^2\theta}{dt^2} + \frac{b}{m_{pen}L^2}\frac{d\theta}{dt} + \frac{g}{L}\theta$$

Matching coefficients:

$$2\omega_{npen}\zeta_{pen} = \frac{b}{m_{pen}L^2} \quad (\mathbf{38})$$

$$\omega_{npen}{}^2 = \frac{g}{L} \quad (\mathbf{39})$$

Substituting equation (38) and equation (39) into equation (37):

$$\omega_{ndpen} = \sqrt{\frac{g}{L} - \left(\frac{b}{2m_{pen}L^2}\right)^2}$$

Squaring both sides:

$$\omega_{ndpen}{}^2 = \frac{g}{L} - \frac{b^2}{4m_{pen}{}^2L^4}$$

Substituting $m = 0.050\ kg$, $\omega_{ndpen} = 8.676\ s^{-1}$, $g = 9.81\ ms^{-1}$, $b = 0.00022\ kgm^2s^{-1}$:

$$8.676^2 = \frac{9.81}{L} - \frac{0.00022^2}{4 * 0.050^2L^4}$$

Solving the equation by Wolfram Alpha gives us a L of $0.13\ m\ (2s.f.)$. Due to limitation on apparatus, I was only able to measure the mass of the pendulum mass and to calculate the pendulum damping constant to 2 significant figures, and hence the optimal length is only precise to 2 significant figures.

Finding the Optimal Length of Pendulum Rod by Experiment

In theory, the optimal length of the pendulum rod for damping my building is about $0.13\ m\ (2s.f.)$, which will produce a natural damped frequency, ω_{ndpen}, that is almost identical to the top's natural damped frequency, ω_{ndtop}.

5 lengths of pendulum rod were tested. For each length, I first displaced the top to the positive x-direction by about $0.085\ m\ (2s.f.)$ (measured roughly by ruler), and then waited for the pendulum to settle to its equilibrium position, $\theta = 0$, and finally released the top and its oscillations were recorded by a phone camera. There were 5 videos produced after all the experiments were done. They were all analyzed using Logger Pro to find out the displacement of the top over time. The data was then transferred to Matlab, where I plotted the experimental data, together with the corresponding numerical solution to the mathematical model (the set of equations including equation (37), (38), (39) and (40)). Initial conditions including the initial displacement ($x = 0.085\ m$), the length of the pendulum rod, and the starting time (since the part of the data collected before I set the top in motion was useless) were essential for the Matlab program (Zhaiky) to produce the numerical solutions.

The following 5 lengths taken were (using electronic caliper):

Length$\pm$0.00002/m				
0.07130	0.10959	0.12940	0.14294	0.15762

Comparing Theory and Experimental Data

In all following graphs, the blue dots represent the data points, and the blue curves represent the numerical solutions to the system of four differential equations.

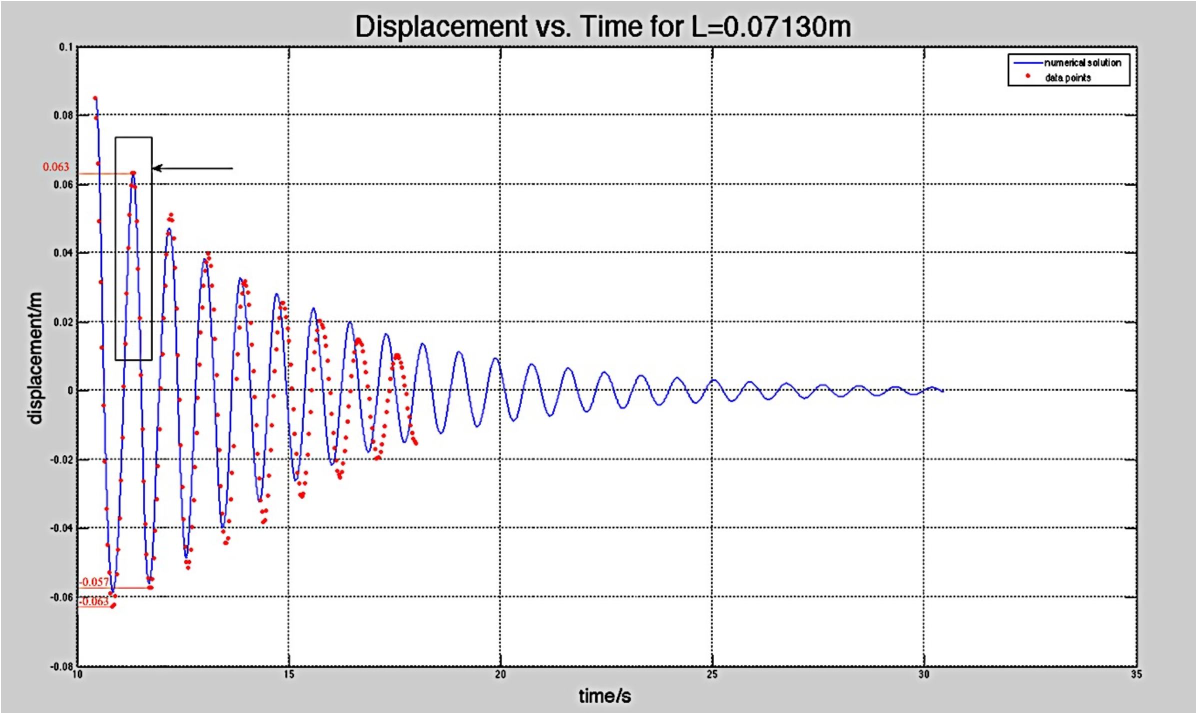

Figure 16 *Displacement vs. Time Graph for L=0.07130 m*

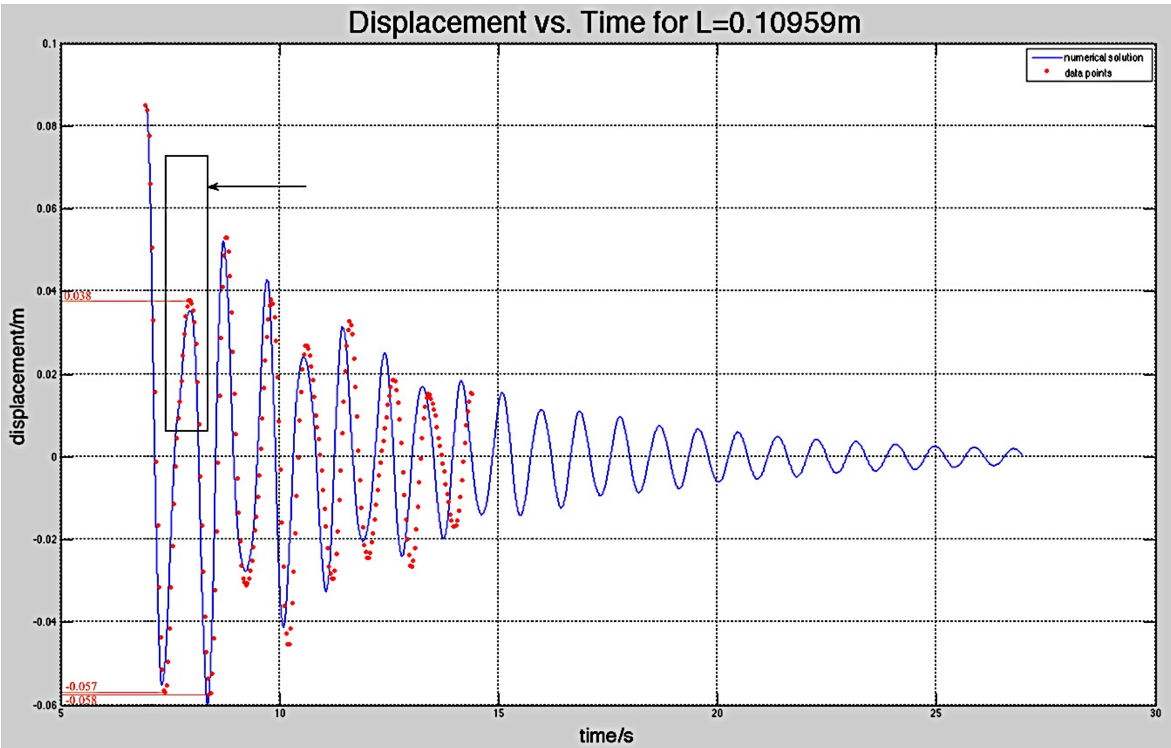

Figure 17 *Displacement vs. Time Graph for L=0.10959 m*

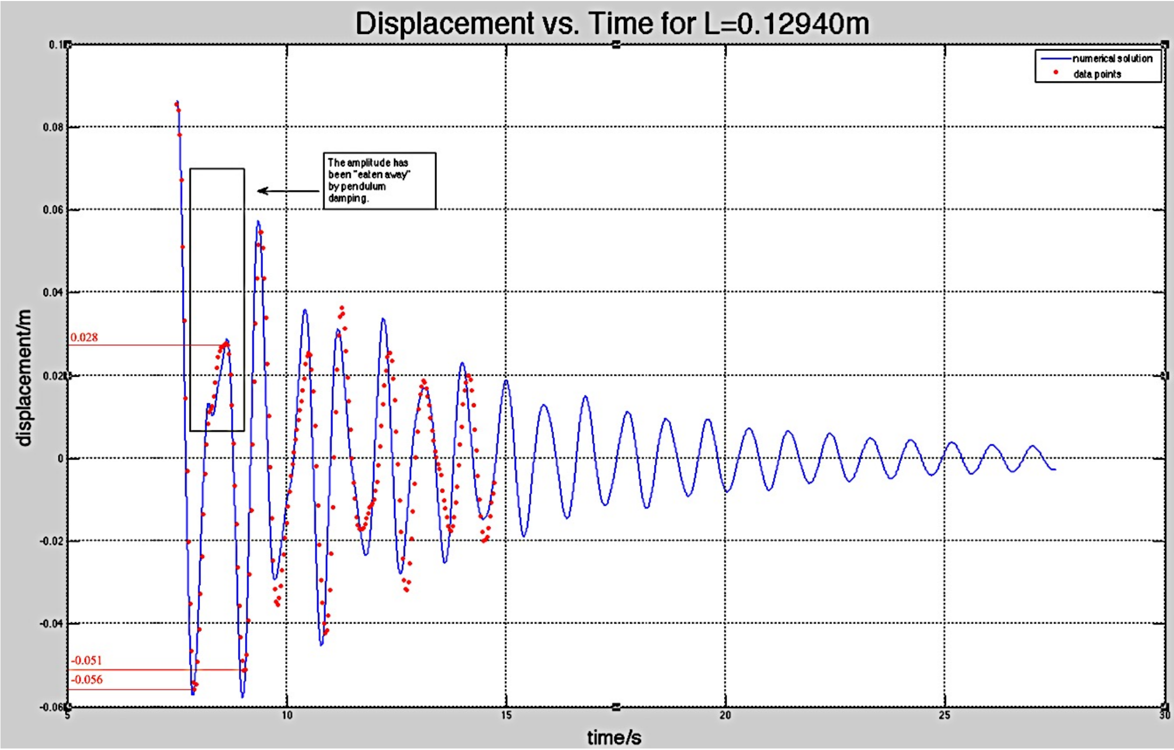

Figure 18 *Displacement vs. Time Graph for L=0.12940 m*

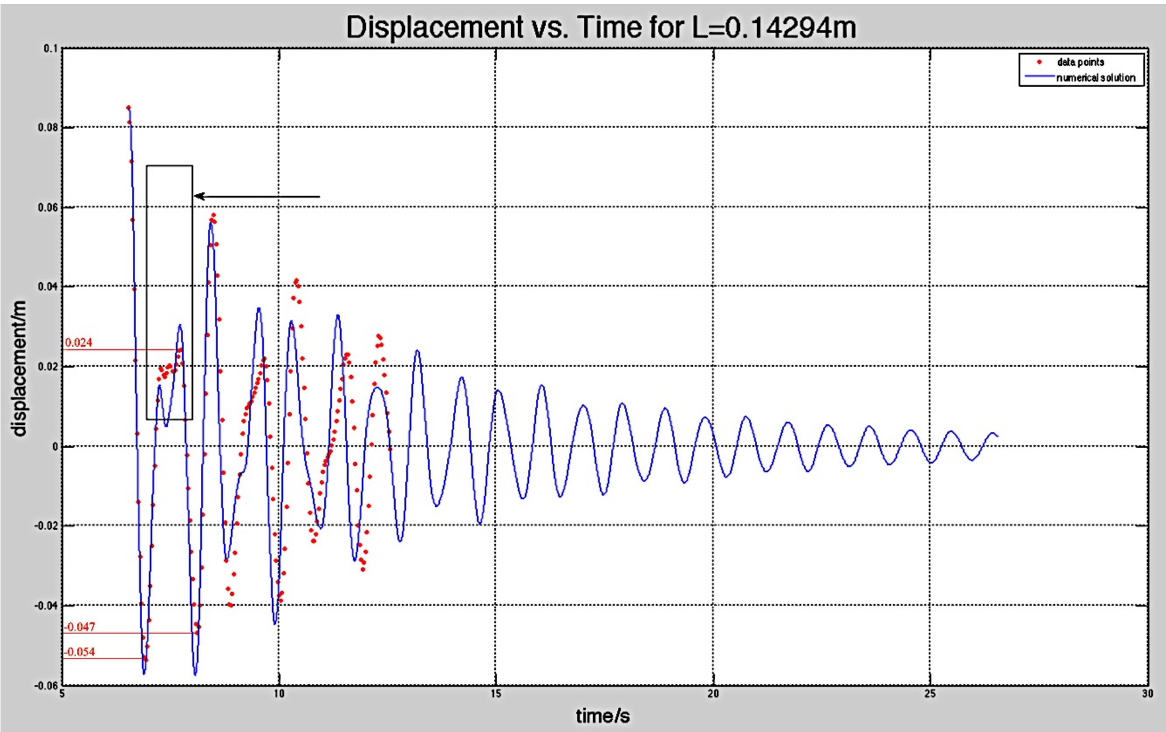

Figure 19 *Displacement vs. Time Graph for L=0.14294 m*

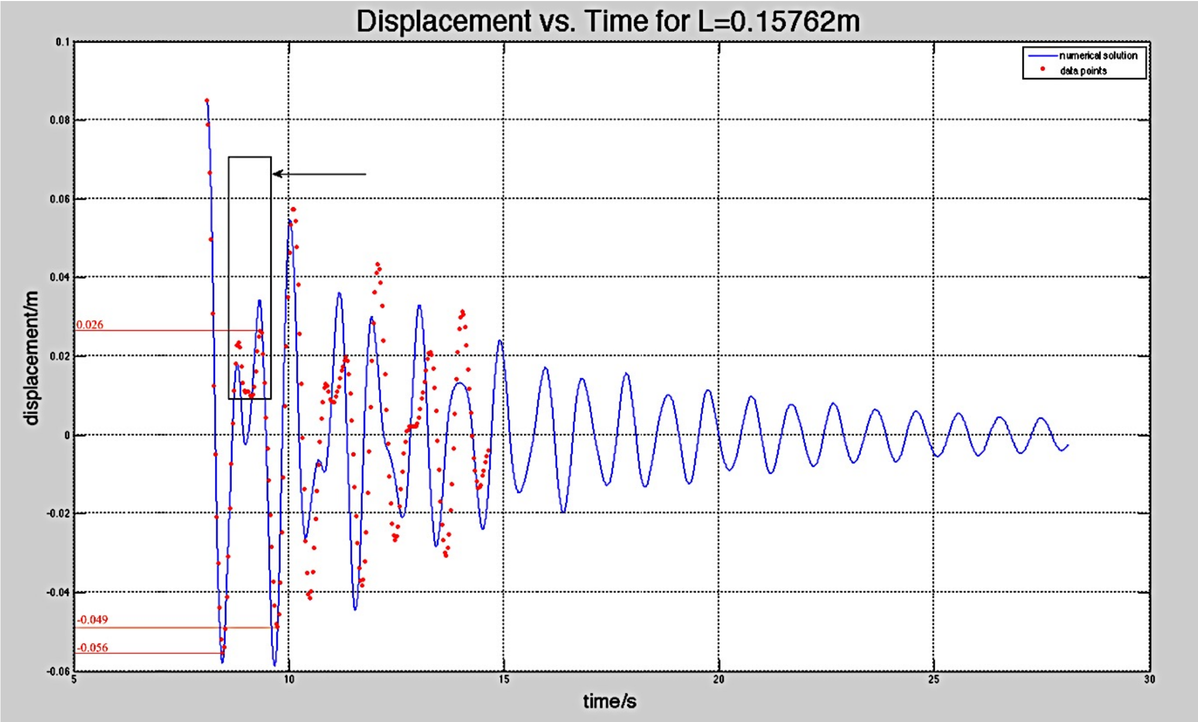

Figure 20 Displacement vs. Time Graph for L=0.15762 m

From Figure 16 to 20, the numerical solutions (represented by the blue curves) fitted the data points well in the first two oscillatory periods. However, the data points start to drift away from my model as time proceeds, especially in Figure 19 and 20, in which the length of the pendulum rod was longer then the theoretical optimum, 0.13 m $(2s.f.)$.

For the first few large-amplitude oscillations on each graph, the region where I enclosed by a black box seem to vary significantly from one figure to another. Therefore, I selected this region to represent the pendulum damping effect. For numerical solutions of my model, the theoretical optimum length “performed” the best.

Within the five rod lengths, 0.12940 m is the closest to the theoretical optimum, 0.13 m $(2s.f.)$. When the rod length increases from 0.07130 m to 0.10959 m to 0.12940 m, the damping effect of the pendulum on the top improved significantly. For length 0.07130 m, the top had a maximum displacement of 0.063 m (Fig 16) in the black box region (**BBR**); for length 0.10959 m, the top had a maximum displacement of 0.038 m (Fig 17); and for length 0.12940 m, the top had a maximum displacement of 0.028 m (Fig 18). For these 3 lengths, the model and the experimental data matched up very well, especially in the first few oscillatory periods.

According to my model, for length 0.14294 m and 0.15762 m, the top would have a maximum displacement in the BBR larger than the one produced by the top damped by the pendulum with length 0.12940 m. However, according to my experimental data, the pendulum with the 0.14294 m rod had the best damping effect on the top, and the top had a maximum displacement of only 0.024 m in the BBR. It was also worth noticing that for length 0.14294 m and 0.15762 m, the deviation between the theory and the experimental data was obvious, especially as time proceeds.

To sum up, a rod length close (if not closest) the theoretical optimum performed the best.

VI. Conclusion

My research aimed to discover the relationship between the damping effect of the pendulum on an oscillating mass and the length of the pendulum rod. I approached my research question in a seemingly indirect way, first computed the constants, and constructed the mathematical model of the combined structure at late stages. However, this approach was inevitable since constants are essential.

After verifying my mathematical model with the experimental data, it is clear that the pendulum would have a good damping effect when its natural damped frequency is close (if not closest) to that of the building. The numerical solution of the mathematical model of the combined structure fitted the experimental data reasonably well, however, the experimental data gradually drifted away from the model. This possibly indicates some factors that I failed to consider and eventually their effects accumulated over time. However, due to the inadequacy of my knowledge, this remains unsolved.

The *flaws of my model* could come from measurements. Some measurements were indeed imprecise due to limited access to high-precision instruments, which resulted relatively large uncertainties in constants. I made the wooden building using tools such as hand-drills that were lack of precision, and this could result in some redundant movement of the building not in the x direction, which might cause the gradual drifting of experimental data away from my model. Nevertheless, since the first few oscillations have the largest amplitude and hence more destructive, the ability to model them relatively well should be regarded as a success.

For real applications, the dimensions of the building put constraints on the rod length of the pendulum, and some other factors must be tuned to achieve the best damping effect. The buildings also oscillate in more than one direction. How a pendulum can compensate these factors is open to study.

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Appendix

1. Solving Equation of Motion of Damped Harmonic Oscillator (Physics This Week)

A conventional way to solve differential equations is to assume that $x = e^{\lambda t}$ is a solution, and thus we can substitute $x = e^{\lambda t}$ in equation (11):

$$\lambda^2 e^{\lambda t} + 2\omega_{ntop}\zeta_{top}\lambda e^{\lambda t} + {\omega_{ntop}}^2 e^{\lambda t} = 0$$

We could then do factorization:

$$e^{\lambda t}\left(\lambda^2 + 2\omega_{ntop}\zeta_{top}\lambda + {\omega_{ntop}}^2\right) = 0$$

Since $e^{\lambda t} > 0$:

$$\left(\lambda^2 + 2\omega_{ntop}\zeta_{top}\lambda + {\omega_{ntop}}^2\right) = 0$$

Using the quadratic equation to obtain two solutions:

$$\lambda = \frac{-2\omega_{ntop}\zeta_{top} \pm \sqrt{4{\omega_{ntop}}^2{\zeta_{top}}^2 - 4{\omega_{ntop}}^2}}{2}$$

Algebraically rearranging the terms:

$$\lambda = \frac{-2\omega_{ntop}\zeta_{top} \pm 2\omega_{ntop}\sqrt{{\zeta_{top}}^2 - 1}}{2}$$

where $1 > \zeta_{top} > 0$, since we can see from the Fig 10 that the top oscillate back and forth about its equilibrium position but the amplitude gradually decrease. The top is under-damped. (For critically damped systems, $\zeta = 1$, the oscillating object returns to equilibrium position without any oscillation (Haynes R. Miller and MIT 60). For over-damped systems, ζ >1, the oscillating object does not return to equilibrium position and no oscillation could be observed.)

Since $1 > \zeta_{top} > 0$, ${\zeta_{top}}^2 - 1 < 0$:

$$\lambda = \frac{-2\omega_{ntop}\zeta_{top} \pm 2i\omega_{ntop}\sqrt{1 - {\zeta_{top}}^2}}{2} = -\omega_{ntop}\zeta_{top} \pm i\omega_{ntop}\sqrt{1 - {\zeta_{top}}^2}$$

where $i = \sqrt{-1}$.

Substituting λ in $x = e^{\lambda t}$:

$$x = e^{\lambda t} = e^{(-\omega_{ntop}\zeta_{top} \pm i\omega_{ntop}\sqrt{1-\zeta_{top}{}^2})t}$$

$$x = e^{-\omega_{ntop}\zeta_{top}t} e^{\pm\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t}$$

If we take the positive sign for $\pm$,

$$x = e^{-\omega_{ntop}\zeta_{top}t} e^{i\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t}$$

If we take the negative sign of $\pm$

$$x = e^{-\omega_{ntop}\zeta_{top}t} e^{-i\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t}$$

In fact, the general solution of x can be any linear combination of the solution having positive sign and the equation having the negative sign:

$$x = ae^{-\omega_{ntop}\zeta_{top}t} e^{i\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t} + be^{-\omega_{ntop}\zeta_{top}t} e^{-i\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t}$$

where a and b are constants.

Equation can then be factorized,

$$x = e^{-\omega_{ntop}\zeta_{top}t}(ae^{i\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t} + be^{-i\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t})$$

$$x = e^{-\omega_{ntop}\zeta_{top}t}(\text{acos}\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t\right) + aisin\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t\right) +$$

$$+bcos\left(-\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t\right) + bisin\left(-i\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t\right))$$

Since $\cos(\theta) = \cos(-\theta)$ and $\sin(\theta) = -\sin(-\theta)$:

$$x = e^{-\omega_{ntop}\zeta_{top}t}(\text{acos}\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t\right) + aisin\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t\right) +$$

$$+bcos\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t\right) - bisin\left(i\omega_{ntop}\sqrt{1-\zeta_{top}{}^2}t\right))$$

$$x = e^{-\omega_{ntop}\zeta_{top}t}((\mathrm{a}+\mathrm{b})\cos\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^{2}}t\right) + (a-b)isin\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^{2}}t\right))$$

Let $A = a + b$ and $B = a - b$:

$$x = e^{-\omega_{ntop}\zeta_{top}t}\left(A\cos\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^{2}}t\right) + Bisin\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^{2}}t\right)\right)$$

Since we are only dealing with a one-dimensional real axis, we can simply ignore the imaginary part of equation, which represents motion alone the imaginary axis:

$$x = e^{-\omega_{ntop}\zeta_{top}t}A\cos\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^{2}}t\right)$$

By considering the initial starting position, equation (18) becomes:

$$x = e^{-\omega_{ntop}\zeta_{top}t}A\cos\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^{2}}t + \phi\right)$$

where ϕ is the argument of the imaginary number:

$$A\cos\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^{2}}t\right) + Bisin\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^{2}}t\right)$$

when $t = 0$, $\cos\left(\omega_{ntop}\sqrt{1-\zeta_{top}{}^{2}}t + \phi\right) = \cos(\phi)$, which is the projection of the imaginary position of the center of mass of the top onto the x-axis.

ϕ can also be understood as the initial condition of the center of mass of the top on the x-axis.

Equation (19) gives us a chance to actually compute the value of ω_{ntop}, c and k.

2. Oscillation Data of the Building for Calculating Constants c and k

Time/s	Displacement from Equilibrium/x
3.343	-0.075
3.377	-0.057
3.411	-0.031
3.444	-0.006
3.478	0.020
3.512	0.043
3.546	0.064
3.579	0.077
3.613	0.082
3.647	0.083
3.681	0.077
3.714	0.065
3.748	0.047
3.782	0.026
3.816	0.004
3.850	-0.021
3.883	-0.043
3.917	-0.062
3.951	-0.072
3.985	-0.076
4.018	-0.074
4.052	-0.067
4.086	-0.053
4.120	-0.034
4.153	-0.014
4.187	0.012
4.221	0.029
4.255	0.046
4.289	0.060
4.322	0.069
4.356	0.071
4.390	0.069
4.424	0.059
4.457	0.044
4.491	0.026
4.525	0.009
4.559	-0.012
4.592	-0.029
4.626	-0.045

4.660	-0.056
4.694	-0.062
4.727	-0.062
4.761	-0.057
4.795	-0.047
4.829	-0.034
4.863	-0.017
4.896	0.001
4.930	0.020
4.964	0.034
4.998	0.046
5.031	0.056
5.065	0.059
5.099	0.058
5.133	0.054
5.166	0.043
5.200	0.031
5.234	0.015
5.268	-0.002
5.302	-0.017
5.335	-0.032
5.369	-0.042
5.403	-0.048
5.437	-0.050
5.470	-0.048
5.504	-0.041
5.538	-0.031
5.572	-0.019
5.605	-0.005
5.639	0.010
5.673	0.024
5.707	0.035
5.741	0.043
5.774	0.048
5.808	0.048
5.842	0.046
5.876	0.040
5.909	0.029
5.943	0.018
5.977	0.006
6.011	-0.008
6.044	-0.022

6.078	-0.029
6.112	-0.035
6.146	-0.039
6.180	-0.039
6.213	-0.036
6.247	-0.030
6.281	-0.020
6.315	-0.009
6.348	0.002
6.382	0.013
6.416	0.025
6.450	0.031
6.483	0.037
6.517	0.039
6.551	0.039
6.585	0.034
6.618	0.029
6.652	0.020
6.686	0.010
6.720	0.001
6.754	-0.010
6.787	-0.018
6.821	-0.025
6.855	-0.029
6.889	-0.031
6.922	-0.029
6.956	-0.026
6.990	-0.020
7.024	-0.012
7.057	-0.002
7.091	0.006
7.125	0.015
7.159	0.021
7.193	0.027
7.226	0.031
7.260	0.031
7.294	0.030
7.328	0.025
7.361	0.020
7.395	0.013
7.429	0.005
7.463	-0.002

7.496	-0.009
7.530	-0.015
7.564	-0.019
7.598	-0.022
7.632	-0.022
7.665	-0.021
7.699	-0.017
7.733	-0.012
7.767	-0.005
7.800	0.003
7.834	0.009
7.868	0.015
7.902	0.019
7.935	0.023
7.969	0.023
8.003	0.023
8.037	0.021
8.071	0.017
8.104	0.014
8.138	0.008
8.172	0.001
8.206	-0.004
8.239	-0.009
8.273	-0.012
8.307	-0.014
8.341	-0.016
8.374	-0.015
8.408	-0.013
8.442	-0.010
8.476	-0.007
8.509	0.000
8.543	0.004
8.577	0.009
8.611	0.012
8.645	0.015
8.678	0.016
8.712	0.016
8.746	0.014
8.780	0.013
8.813	0.011
8.847	0.007
8.881	0.002

8.915	-0.001
8.948	-0.004
8.982	-0.007
9.016	-0.009
9.050	-0.010
9.084	-0.010
9.117	-0.009
9.151	-0.007
9.185	-0.002
9.219	0.000
9.252	0.003
9.286	0.007
9.320	0.009
9.354	0.011
9.387	0.011
9.421	0.011
9.455	0.011
9.489	0.011
9.523	0.008
9.556	0.005

3. Algebraic Manipulation

Using equation (24) to substitute $Tsin(\theta)$ in equation (28):

$$\left(-m_{pen}\left(\frac{d^2x}{dt^2}+L\frac{d^2\theta}{dt^2}\cos(\theta)-L\left(\frac{d\theta}{dt}\right)^2\sin(\theta)\right)\cos(\theta)-\frac{b}{L}\frac{d\theta}{dt}\cos^2(\theta)\right)-m_{pen}gsin(\theta)$$

$$-\frac{b}{L}\frac{d\theta}{dt}\sin^2(\theta)=m_{pen}\left(L\frac{d^2\theta}{dt^2}\sin(\theta)+L\left(\frac{d\theta}{dt}\right)^2\cos(\theta)\right)\sin(\theta)$$

Multiplying out the brackets:

$$-m_{pen}\frac{d^2x}{dt^2}\cos(\theta)-m_{pen}L\frac{d^2\theta}{dt^2}\cos^2(\theta)+m_{pen}L\left(\frac{d\theta}{dt}\right)^2\sin(\theta)\cos(\theta)-\frac{b}{L}\frac{d\theta}{dt}\cos^2(\theta)$$

$$-m_{pen}gsin(\theta)-\frac{b}{L}\frac{d\theta}{dt}\sin^2(\theta)$$

$$=m_{pen}L\frac{d^2\theta}{dt^2}\sin^2(\theta)+m_{pen}L\left(\frac{d\theta}{dt}\right)^2\sin(\theta)\cos(\theta)$$

Cancelling the red terms, and move all terms to the right-hand-side of the equation:

$$-m_{pen}\frac{d^2x}{dt^2}\cos(\theta)-m_{pen}L\frac{d^2\theta}{dt^2}\cos^2(\theta)-\frac{b}{L}\frac{d\theta}{dt}\cos^2(\theta)-m_{pen}gsin(\theta)-\frac{b}{L}\frac{d\theta}{dt}\sin^2(\theta)$$

$$-m_{pen}L\frac{d^2\theta}{dt^2}\sin^2(\theta)=0$$

Since $\sin^2(\theta)+\cos^2(\theta)=1$, the equation becomes

$$\frac{b}{L}\frac{d\theta}{dt}+m_{pen}\frac{d^2x}{dt^2}\cos(\theta)+m_{pen}L\frac{d^2\theta}{dt^2}+m_{pen}gsin(\theta)=0$$