M4P33 Algebraic Geometry

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Contents

0 Introduction

Lecture 1 Friday 11/01/19

I will not follow a particular book, but everything I am going to say will be contained in one of the following books.

- I Shafarevich, Basic algebraic geometry, 1974
- R Hartshorne, Algebraic geometry, 1977
- J Harris, Algebraic geometry: a first course, 1922

1 Affine varieties

Notation 1.1.

- R is a commutative ring with unity.
- K is a field.
- $K[x_1, \dots, x_n]$ is the ring of polynomials in n variables.
- $\mathbb{A}^n$ is K^n as a set.

Definition 1.2. Let $S \subseteq K[x_1, \dots, x_n]$ then

$$Z(S) = \{x \in \mathbb{A}^n \mid \forall f \in S,\ f(x) = 0\}$$

is called the **zero locus** of S. Subsets of $\mathbb{A}^n$ that are of this form are called **affine varieties**.

Remark 1.3. Some authors call **algebraic set** the object $Z(S)$. We will not follow this notation.

Example 1.4.

- Single points $p = (p_1, \dots, p_n)$. $p = Z(S)$ where
$$S = \{x_1 - p_1, \dots, x_n - p_n\}.$$
- $\mathbb{A}^n = Z(0)$.
- $\emptyset = Z(1)$.
- Subspaces of $\mathbb{A}^n = K^n$.
- If $X = Z(f_1, \dots, f_n) \subseteq \mathbb{A}^n$ and $Y = Z(g_1, \dots, g_m) \subseteq \mathbb{A}^n$ are affine varieties then
$$X \times Y = Z(f_1, \dots, f_n, g_1, \dots, g_m) \subseteq \mathbb{A}^{n+m}$$
is a variety.

Remark 1.5. If $S \subseteq K[x_1, \dots, x_n]$ and $I = \langle S \rangle$ then $Z(S) = Z(I)$.

Theorem 1.6 (Hilbert's basis theorem)**.** *If R is Noetherian then $R[x]$ is Noetherian.*

Corollary 1.7. *Every ideal in $K[x_1, \dots, x_n]$ is finitely generated.*

Definition 1.8. Let $X \subseteq \mathbb{A}^n$ then

$$I(X) = \{f \in K[x_1, \dots, x_n] \mid \forall x \in X,\ f(x) = 0\}.$$

Example 1.9. $I(p) = I((p_1, \dots, p_n)) = \langle x_1 - p_1, \dots, x_n - p_n \rangle$.

Goal is

$$\begin{array}{rcl} \{\text{affine varieties in } \mathbb{A}^n\} & \leftrightarrow & \{\text{ideals of } K[x_1, \dots, x_n]\} \\ X & \mapsto & I(X) \\ Z(J) & \leftarrow\!\shortmid & J \end{array}.$$

$Z(I(X)) = X$ but $I(Z(J)) \supseteq J$.

Example 1.10. $J = \langle x^2 + 1 \rangle \subseteq \mathbb{R}[x] = I(\emptyset) = I(Z(x^2 + 1))$.

Proposition 1.11.

- *If $X \subseteq Y$ then $I(Y) \subseteq I(X)$. If $I \subseteq J$ then $Z(J) \subseteq Z(I)$.*
- *$X \subseteq Z(I(X))$ and $S \subseteq I(Z(S))$.*

- *If X is affine then $Z(J(X)) = X$. If $X = Z(S)$ then take Z of $S \subseteq I(Z(S))$.*

Example 1.12. Let $J \subseteq \mathbb{C}[x]$. $J = \langle f \rangle$, where $f = (x - x_1)^{k_1} \dots (x - x_n)^{k_n}$.

Definition 1.13. Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal.

$$I \subseteq \sqrt{I} = \{f \in K[x_1, \dots, x_n] \mid \exists n \in \mathbb{N},\ f^n \in I\}.$$

If $\sqrt{I} = I$, we say I is a **radical ideal**.

(Exercise: $\sqrt{I}$ is an ideal, $I \subseteq \sqrt{I}$, and $\sqrt{I} = \bigcap_{p \text{ prime}} p$)

Theorem 1.14 (Hilbert's Nullstellensatz)**.** *$I(Z(J)) = \sqrt{J}$. If $\sqrt{J} = J$ then*

$$\begin{array}{rcl} \{\textit{affine varieties}\} & \leftrightarrow & \{\textit{radical ideals}\} \\ X & \mapsto & I(X) \\ Z(J) & \leftarrow\!\shortmid & J \end{array}.$$

Lecture 2 Monday 14/01/19

Proposition 1.15.

1. *$Z(S) \cup Z(T) = Z(ST)$.*
2. *$\bigcap_i Z(S_i) = Z(\bigcup_i S_i)$.*
3. *$Z(0) = \mathbb{A}^n$ and $Z(1) = \emptyset$.*

Proof.

1. If $p \in Z(S) \cup Z(T)$, then $f(p) = 0$ for $f \in S$ or $f \in T$, so $f(x) = 0$ for $f \in ST$, where

$$ST = \left\{ \sum_{i \in I,\ I \text{ finite}} s_i t_i \right\} \subseteq S \cap T,$$

with equality if $S + T = R$. If $p \in Z(ST)$, there exists f such that $f(p) = 0$ for $f \in S$ or $f(p) = 0$ for $f \in T$, so $p \in Z(S) \cup Z(T)$.

□

Definition 1.16. The **Zariski topology** on $\mathbb{A}^n$ is the topology generated by closed sets of the form $Z(S)$. By the above proposition this is a topology.

Example 1.17. $\mathbb{A}^1$ is not Hausdorff.

Definition 1.18. A topological space X is **irreducible** if it cannot be expressed as a union $X = A \cup B$, where A and B are proper and closed subsets. $\emptyset$ is not considered irreducible.

Example 1.19. $\mathbb{A}^1$.

Example 1.20. Any non-empty open set of irreducible X is dense and irreducible. Suppose A is open then $X = A^c \cup \overline{A}$. Since X is irreducible then $A^c = X$, a contradiction, or $\overline{A} = X$. Suppose A is reducible. Let $A = (A \cap B) \cup (A \cap C)$, where B and C are closed. Then $X = A^c \cup (B \cup C)$. $A^c = X$ or $B \cup C = X$, which are contradictions.

Example 1.21. If A is irreducible then $\overline{A}$ is also irreducible. Suppose $\overline{A}$ is not irreducible. $\overline{A} = (\overline{A} \cap B) \cup (\overline{A} \cap C)$. Take $\bigcap A$, $A = (A \cap B) \cup (A \cap C)$, a contradiction.

Definition 1.22. An affine variety is **irreducible** if it is irreducible as a topological space.

Remark 1.23. A **quasi-affine variety** is an open set of an affine variety.

Proposition 1.24.

1. $I(X \cup Y) = I(X) \cap I(Y)$.
2. $Z(I(X)) = \overline{X}$ *for any* $X \subseteq \mathbb{A}^n$.

Proof.

1. If $f \in I(X \cup Y)$ then $f(p) = 0$ for all $p \in X \cup Y$, so $f \in I(X)$ and $f \in I(Y)$.
2. We know that $X \subseteq Z(I(X))$ hence $\overline{X} \subseteq Z(I(X))$. Now, let Y be a closed set containing X, that is $X \subseteq Y$. Then
$$I(Y) \subset I(X) \qquad \implies \qquad Z(I(X)) \subset Z(I(X)) = Y,$$
so any closed set containing Y contains $Z(I(X))$.

□

Proposition 1.25. *X is irreducible if and only if $I(X)$ is prime.*

Proof.

$\implies$ Let $f, g \in I(X)$.
$$X \subseteq Z(fg) = Z(f) \cup Z(g) \qquad \implies \qquad X = (X \cap Z(f)) \cup (X \cap Z(g)).$$
$Z(f) \subseteq X$, so $f \in I(X)$, or $Z(g) \subseteq X$, so $g \in I(X)$.

$\impliedby$ Exercise.

□

Example 1.26. $\mathbb{A}^n$.

Definition 1.27. If $X \subseteq \mathbb{A}^n$, the **coordinate ring** of X is
$$A(X) = \frac{K[x_1, \dots, x_n]}{I(X)}.$$

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Example 1.28. Let $f \in K[x_1, \dots, x_n]$ be irreducible. If $n = 3$, $Z(f)$ is a surface. If $n = 2$, $Z(f)$ is a curve.

Example 1.29. Let $y - x^2 \in K[x, y]$. Then
$$\begin{array}{rcl} A(X) = \dfrac{K[x,y]}{\langle y - x^2 \rangle} \cong K\left[x, x^2\right] & \to & K[x] \\ \sum_{i,j} a_{ij} x^i x^{2j} = \sum_{i,j} a_{ij} x^{2j+i} & \mapsto & \sum_n b_n x^n \end{array}.$$

Example 1.30. Let $xy - 1 \in K[x, y]$. Then
$$A(X) = \frac{K[x,y]}{\langle xy - 1 \rangle} \cong K\left[x, \frac{1}{x}\right].$$
$A(X)$ cannot be $K[x]$.

Definition 1.31. A **Noetherian** topological space X is a topological space such that if
$$C_1 \supseteq C_2 \supseteq \dots$$
is a decreasing chain of closed sets then there is a k such that $C_k = C_{k+1} = \dots$.

Example 1.32. $\mathbb{A}^n$. Recall that if $A \subset B$ then $I(B) \subset I(A)$. So using the definition above,
$$I(C_1) \subseteq I(C_2) \subseteq \dots.$$
Since $K[x_1, \dots, x_n]$ is Noetherian then $I(C_i)$ stabilises. So $I(C_k) = I(C_{k+1}) = \dots$, but taking Z, we recover C_k so C_k stabilises as well.

Theorem 1.33. *If X is Noetherian then any non-empty closed subset can be expressed as a finite union of irreducible closed sets $X = Y_1 \cup \dots \cup Y_n$. Moreover, if we require that $Y_i \subseteq Y_j$ then this expression is unique.*

Proof. Let C be the collection of closed sets that do not satisfy that property. Let Y be a minimum closed inside C, in particular Y is reducible, so $Y = Y' \cup Y''$, for Y', Y'' closed. Hence $Y', Y'' \not\subset C$, so they can be expressed as a finite union of irreducibles, a contradiction. If $Y_i \not\subset Y_j$, then suppose

$$Y_1 \cup \dots \cup Y_n = X_1 \cup \dots \cup X_n.$$

Then $Y_1 \subset X_1 \cup X_n$, in particular $Y_1 = \bigcup_j (Y_1 \cap X_j)$, so there is a j such that $Y_1 \cap X_j = Y_1$, so $Y_1 \subset X_j$. We can assume $j = 1$ and repeat the same argument to find that $Y_1 = X_1$, so consider $\overline{Y \setminus Y_1} = Y_2 \cup \dots \cup Y_n$. But

$$Y_2 \cup \dots \cup Y_n = X_2 \cup \dots \cup X_n,$$

and the result follows by induction. $\square$

Corollary 1.34. *Any affine variety in $\mathbb{A}^n$ can be expressed equally as a union of irreducible algebraic varieties.*

Definition 1.35. The **dimension** of a topological space is the supremum of n where

$$Y_0 \subset \dots \subset Y_n$$

is a sequence of irreducible closed sets.

Example 1.36. Dimension of $\mathbb{A}^1$ is one.

Definition 1.37. Let A be a ring and $\mathfrak{p}$ be a prime ideal, then the **height** of $\mathfrak{p}$ is the supremum of n where

$$\mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_n \subset \mathfrak{p},$$

where $\mathfrak{p}_i$ are prime. The **Krull dimension** of A is

$$\sup_{\mathfrak{p} \text{ prime}} \textit{height}(\mathfrak{p}).$$

Proposition 1.38. *If Y is affine then $\dim(Y) = \dim(A(Y))$.*

Proof. Let C be a closed and irreducible set $C \subset Y$, then $I(C) \supset I(Y)$, then $I(C)$ is prime. $\square$

Proposition 1.39. *Let K be a field and B be an integral domain which is a finitely generated algebra, then*

- $\dim(B)$ *is the transcendence degree of $K(B)$ over K, and*
- *if $\mathfrak{p} \subseteq B$ is prime, then*

$$\textit{height}(\mathfrak{p}) + \dim\left(\frac{B}{\mathfrak{p}}\right) = \dim(B).$$

Proof. Atiyah Macdonald chapter 11. $\square$

Proposition 1.40 (Krull Hauptidealsatz)**.** *Let A be a Noetherian ring and $f \in A$ not a zero divisor and not a unit. Then every prime ideal containing f has height one.*

Proof. Atiyah Macdonald page 122. $\square$

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Proposition 1.41. *A Noetherian integral domain A is a UFD if and only if every prime ideal I of height one is principal.*

Theorem 1.42. *An irreducible variety $Y \subseteq \mathbb{A}^n$ has dimension $n-1$ if and only if $Y = Z(f)$ where f is an irreducible polynomial in $K[x_1, \dots, x_n]$.*

Proof.

$\implies$ If Y has dimension $n-1$ then $I(Y)$ has height one, by the above proposition $I(Y) = \langle f \rangle$, so $Y = Z(f)$.

$\impliedby$ Let $I = I(Y)$ then I is prime, by the Krull Hauptidealsatz we have that I has height one, so $\dim(Y) = n-1$.

$\square$

2 Projective varieties

Definition 2.1. The **projective space** $\mathbb{P}^n$ is defined as

$$\mathbb{P}^n = \frac{\mathbb{A}^{n+1} \setminus \{0\}}{\{x \sim \lambda x \mid \lambda \in K^*\}}.$$

A point in $\mathbb{P}^n$ is written as $[a_0 : \cdots : a_n] = \overline{(a_0, \ldots, a_n)}$.

Definition 2.2. A **graded ring** R is a ring together with a decomposition

$$R = \bigoplus_{d>0} R_d,$$

where R_d are abelian groups and $R_k \cdot R_t \subseteq R_{k+t}$.

Example 2.3. $K[x_0, \ldots, x_n]$ is a graded ring, where R_d are monomials of degree d.

Notation 2.4. Let A be $K[x_0, \ldots, x_n]$ without the grading and S be $K[x_0, \ldots, x_n]$ as a graded ring.

Definition 2.5. An ideal $I \subseteq S$ is **homogeneous** if

$$I = \bigoplus_{d \geq 0} (I \cap S_d).$$

If $f = f_0 + \cdots + f_d$, then $f_i \in I$.

Remark 2.6. I is homogeneous if and only if $I = \langle f_0, \ldots, f_n \rangle$, where f_i are homogeneous.

Lemma 2.7. *If I, J are homogeneous then*

1. *$I + J$ is homogeneous,*
2. *IJ is homogeneous,*
3. *$I \cap J$ is homogeneous, and*
4. *$\sqrt{I}$ is homogeneous.*

Proof.

4. Let $f = f_0 + \cdots + f_d \in \sqrt{I}$ then

$$f^n = (f_0 + \cdots + f_d)^n = f_d^n + \cdots \in I \qquad \implies \qquad f_d^n \in I \qquad \implies \qquad f_d \in \sqrt{I},$$

so $f - f_d \in \sqrt{I}$, by induction $f_i \in \sqrt{I}$.

□

Definition 2.8. If f is homogeneous of degree k then $f(\lambda \cdot x) = \lambda^k \cdot f(x)$, in particular $f(x) = 0$ if and only if $f(\lambda \cdot x) = 0$, so it makes sense to define

$$Z(f) = \{x \in \mathbb{P}^n \mid f(x) = 0\}.$$

More generally, if $I \subseteq S$ is a homogeneous ideal then

$$Z(I) = \{x \in \mathbb{P}^n \mid f \in I \text{ homogeneous}, \ f(x) = 0\}.$$

Definition 2.9. A subset $X \subseteq \mathbb{P}^n$ is called a **projective variety** if $X = Z(T)$ for some homogeneous ideal T.

Proposition 2.10.

- $Z(S) \cup Z(T) = Z(ST)$.
- $\bigcap_\alpha Z(S_\alpha) = Z\left(\bigcup_\alpha S_\alpha\right)$.
- $Z(0) = \mathbb{P}^n$ *and* $Z(1) = \emptyset$.

Definition 2.11. We define the **Zariski topology** on $\mathbb{P}^n$ by taking closed sets to be $Z(T)$ for some T.

Definition 2.12.

- A projective variety is **irreducible** if it is an irreducible topological space.
- An open subset of a projective variety is called a **quasi-projective variety**.
- The **dimension** of a projective variety is its dimension as a topological space.
- If $T \subseteq S$ then
$$I(T) = \langle f \in S \mid f \text{ homogeneous}, \ \forall p \in T, \ f(p) = 0 \rangle.$$

Definition 2.13. If X is a projective variety the **homogeneous coordinate ring** is
$$S(X) = \frac{S}{I(X)}.$$

Definition 2.14. If $f \in S$ is linear and homogeneous, we call $Z(f)$ a **hyperplane**.

Lecture 5 Monday 21/01/19

Proposition 2.15.
$$\begin{array}{rccc} \phi_i : & U_i = \dfrac{\mathbb{P}^n}{Z(x_i)} & \to & \mathbb{A}^n \\ & [x_0 : \dots : x_n] & \mapsto & \left(\dfrac{x_0}{x_i}, \dots, \dfrac{x_n}{x_i}\right) \end{array}$$
is a homeomorphism in the Zariski topology.

Proof. Let $\phi = \phi_0$ and $U = U_0$, let $C \subseteq \mathbb{A}^n$ be a closed set then we claim that $\phi^{-1}(C)$ is closed. Indeed, let $C = Z(S)$, then $\phi^{-1}(C) = Z(S') \cup U$ where
$$S' = \left\{ x_0^d \cdot f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \;\middle|\; f \in S \right\}.$$
Similarly, let $A \subseteq U$ is closed, we claim that $\phi(A)$ is closed. Let $\overline{A}$ be its closure in $\mathbb{P}^n$, then $\overline{A} = Z(B)$, so $\phi(A) = Z(B')$ where
$$B' = \{f(1, x_1, \dots, x_n) \mid f \in B\}.$$
So we conclude that ϕ is a homeomorphism. $\square$

Note that $\langle 1 \rangle = S$ and $\langle x_0, \dots, x_n \rangle \subsetneq S$ map to $\emptyset$ under Z. So in order to have a one-to-one correspondence we need the following.

- $Z(I) = \emptyset$ if and only if $\sqrt{I} \supseteq \langle x_0, \dots, x_n \rangle$. If we consider $Z(I)$ in $\mathbb{A}^{n+1}$, note that $x \in Z(I)$ if and only if $\lambda x \in Z(I)$. So $Z(I) = \emptyset$ if and only if $Z(I) \subseteq \{0\}$. So $\sqrt{I} \supseteq \langle x_0, \dots, x_n \rangle$.
- $I(Z(J)) = \sqrt{J}$ if $Z(J) \neq \emptyset$, since $I(Z(J)) = I(Z_a(J)) = \sqrt{J}$.

Corollary 2.16.
$$\{ \text{ projective varieties } \} \quad \leftrightsquigarrow \quad \{ \text{ homogeneous radical ideals not } \langle x_0, \dots, x_n \rangle \},$$
$$\{ \text{ irreducible projective varieties } \} \quad \leftrightsquigarrow \quad \{ \text{ homogeneous radical prime ideals } \}.$$

Example 2.17. $\mathbb{P}^n$ is irreducible.

Proposition 2.18.

- $\mathbb{P}^n$ *is Noetherian, that is satisfies the descending chain condition.*
- *Every projective variety can be written as a unique union of irreducible projective varieties. We call* ***irreducible components*** *the irreducible varieties in that decomposition.*

Theorem 2.19. *Let $Y \subseteq \mathbb{P}^n$ be an irreducible projective variety. Then*

$$\dim(S(Y)) = \dim(Y) + 1.$$

Proof. Let

$$\begin{array}{cccc} \phi_i : & Z(x_i) & \to & \mathbb{A}^n \\ & [x_0 : \cdots : x_n] & \mapsto & \left(\dfrac{x_0}{x_i}, \ldots, \dfrac{x_n}{x_i}\right) \end{array},$$

and $Y_i = \phi(Y \cap U_i)$. Let

$$\begin{array}{ccc} K[x_1, \ldots, x_n] & \to & (S_{x_i})_0 \\ f(x_1, \ldots, x_n) & \mapsto & \dfrac{x_i^{\partial f} f\left(\dfrac{x_1}{x_i}, \ldots, \dfrac{x_n}{x_i}\right)}{x_i^{\partial f}} \end{array},$$

then

$$A(Y_i) = \frac{K[x_1, \ldots, x_n]}{I(Y_i)} \cong (S_{x_i})_0,$$

moreover $S(Y)_{x_i} \cong A(Y_i)\left[x_i, x_i^{-1}\right]$. So

$$\dim(S(Y)) = \dim\left(S(Y)_{x_i}\right) = \dim\left(A(Y_i)\left[x_i, x_i^{-1}\right]\right) = tra(K(Y_i)(x_i)) = \dim(Y_i) + 1.$$

Therefore if $Y_i \neq \emptyset$, $\dim(Y_i) = \dim(S(Y)) - 1$ for all i, but since U_i cover Y we have $\dim(Y) = \max\{\dim(Y_i)\}$. (Exercise: if $\{U_n\}_n$ is a finite cover of a topological space Y then $\dim(Y) = \max\{\dim(Y_i)\}$) Since $\dim(Y_i)$ are the same if $Y_i \neq \emptyset$, we conclude that $\dim(Y) = \dim(Y_d)$ for some d. $\square$