M4P55 Commutative Algebra

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Autumn 2018

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0.1   Outline

Why study commutative algebra? Number theory and algebraic geometry use this language.
Structure of the course:

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5. Modules, Nakayama's lemma

6. Noetherian and Artinian rings

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0.2   References

1. M Reid, Undergraduate commutative algebra, 1995

2. M Atiyah and I G Macdonald, Introduction to commutative algebra, 1969

1   Rings and ideals

Definition 1.1. A commutative **ring** with 1 is a set A with two operations $+$ and $\cdot$, and two elements 0 and 1 such that the following holds.

1. $(A, +)$ is a group with zero 0.

2. Multiplication is

    (a) associative $((xy) z = x (yz)$ for all $x, y, z \in A)$,
    (b) commutative $(xy = yx$ for all $x, y \in A)$, and
    (c) distributive over addition $(x (y + z) = xy + xz$ for all $x, y, z \in A)$.

3. $x \cdot 1 = 1 \cdot x = x$ for all $x \in A$.

Example. $\mathbb{Z}$ is a ring. The set of even integers $2\mathbb{Z}$ is not a ring because it does not contain 1.

Remark 1.2. Can it happen that $0 = 1$? $x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$ gives $x \cdot 0 = 0$. But $x \cdot 1 = x$. Then $x = 0$ for all $x \in A$, so $A = \{0\}$.

Let A be a commutative ring with 1.

Definition 1.3. A **ring homomorphism** $f : A \to B$ is a homomorphism of abelian groups such that $f (xy) = f (x) f (y)$ for any $x, y \in A$ and $f (1) = 1$.

Proposition 1.4. A composition of homomorphisms is a homomorphism.

An **isomorphism** is a bijective homomorphism. If $f : A \to B$ is an isomorphism, we write $A \cong B$.

Definition 1.5. A subset $I \subset A$ is called an **ideal** if I is a subgroup of $(A, +)$ and $AI = I$. Equivalently, for any $a \in A$ and any $x \in I$ we have $ax \in I$. The **quotient ring** A/I is the quotient group $\{a + I \mid a \in A\}$, which is actually a ring by $(a + I)(b + I) = ab + I$. $1 + I$ is the 1 in A/I. $f : A \to A/I$ such that $f(a) = a + I$ is a surjective ring homomorphism. An ideal $I \subset A$ is **principal** if there is $r \in A$ such that $I = rA$.

Proposition 1.6. There is a natural bijection between the ideals of A that contain a fixed ideal I and the ideals of A/I.

Proof. Suppose $J \subset A$ is an ideal containing I. Then associate to J its image $f(J) \subset A/I$. To check this, note that since $f : A \to A/I$ is surjective, for any $x \in A/I$ there is a $y \in A$ such that $f(y) = x$. Hence $xf(J) = f(y)f(J) = f(yJ) \subset f(J)$. Conversely, take an ideal $M \subset A/I$ and associate to it $f^{-1}(M) \subset A$. This is an ideal in A. To check that for all $a \in A$ we have $af^{-1}(M) \subset f^{-1}(M)$, we note that this is equivalent to $f(a) M \subset M$, which is true. These maps are inverses to each other. $\qquad\square$

Definition 1.7. Let $g : A \to B$ be a homomorphism of rings. The **image** is the subset $Im(g) = \{x \in B \mid \exists y \in A, \ g(y) = x\}$. The **kernel** is the subset $Ker(g) = \{y \in A \mid g(y) = 0\}$.

The image is a subring of $(B, +)$ but not necessarily an ideal, but the kernel is.

Example. Let $g : \mathbb{Z} \hookrightarrow \mathbb{Q}$. $2\mathbb{Z}$ is an ideal in $\mathbb{Z}$, but not in $\mathbb{Q}$.

An isomorphism theorem states that $A/Ker(g) \cong Im(g) = g(A)$ by $a \mapsto a + Ker(g)$.

2   Polynomial rings

Let R be a ring. Define $R[X]$ as the ring of polynomials $\sum_{i=0}^{n} a_i X^i$ with coefficients $a_i \in R$ and

$$\left(\sum_{i=0}^{k} a_i X^i\right)\left(\sum_{j=0}^{m} b_j X^i\right) = \sum_{k=0}^{n+m}\left(\sum_{k=i+j} a_i b_j\right) X^k.$$

Define $R[X_1, X_2]$ to be the ring $R[X_1][X_2]$. In general, $R[X_1, \ldots, X_n] = R[X_1] \ldots [X_2]$.

3   Zero-divisors, nilpotents, units

Definition 3.1. A **zero-divisor** in A is an element $x \in A$ such that there exists $y \in A$, $y \neq 0$, with the property that $xy = 0$. A ring with no non-zero zero-divisors is called an **integral domain**. A **nilpotent** is an element $x \in A$ such that $x^n = 0$ for some $n \geq 1$. A **unit** $a \in A$ is an element such that there exists $b \in A$ with the property that $ab = 1$. Such elements are also called **invertible**. b is denoted by a^{-1}. The units form a group under multiplication, denoted by A^*.

Example. In $A = \mathbb{Z}$, $\mathbb{Z}^* = \{1, -1\}$ and $\mathbb{Z}$ is an integral domain. In $A = \mathbb{Z}/4 = \{4\mathbb{Z}, 1 + 4\mathbb{Z}, 2 + 4\mathbb{Z}, 3 + 4\mathbb{Z}\}$, $2 + 4\mathbb{Z}$ is a zero-divisor in $\mathbb{Z}/4$ that is also nilpotent.

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Definition 3.2. A **field** is a ring in which $0 \neq 1$ and every non-zero element is a unit. So if k is a field, then $k \setminus \{0\} = k^*$.

Proposition 3.3. Let A be a non-zero ring. Then the following are equivalent.

1. A is a field.

2. The only ideals in A are $(0) = \{0\}$ and $(1) = A$.

3. Every homomorphism $A \to B$, where $B \neq 0$, is injective.

Proof.

$1 \implies 2$ Let $I \subset A$ be a non-zero ideal. Then there exists $x \in I$, $x \neq 0$. Then x is a unit, i.e. there exists $y \in A$ such that $xy = 1$. For all $a \in A$, $a = a.1 = a.y.x \in (x)$. Thus $I = A$.

$2 \implies 3$ Let $f : A \to B$. $Ker(f)$ is an ideal of A. If $Ker(f) \neq \{0\}$, then $Ker(f) = A$. But then $1 \in Ker(f)$ and $f(1) = 0$ but $f(1) = 1$ so in B we have that $0 = 1$. Then $B = \{0\}$, which is a contradiction.

$3 \implies 1$ Let $x \in A$, $x \neq 0$. If $1 \in (x) = xA$, then x is a unit. If $1 \notin (x)$, then x is not a unit. If $1 \notin (x)$, then consider the map $A \to A/(x)$ sending $a \mapsto a + (x)$. Since $1 \notin (x)$, $1 + (x)$ is not zero in $A/(x)$. So this is a non-injective homomorphism to a non-zero ring. This contradicts 3.

$\square$

4 Prime ideals and maximal ideals

Definition 4.1. An ideal $P \subset A$ is a **prime ideal** if for any $x, y \in A$, $xy \in P$ implies $x \in P$ or $y \in P$. An ideal $M \subset A$ is called **maximal** if there does not exist an ideal I in A such that $M \subsetneq I \subsetneq A$.

Lemma 4.2. An ideal $P \subset A$ is prime if and only if A/P is an integral domain. An ideal $M \subset A$ is maximal if and only if A/M is a field.

Proof. Let $x, y \in A$ such that $xy \in P$. Then $(x + P)(y + P) = xy + P = P$. If $x \notin P$ and $y \notin P$, then $x + P \neq P$ and $y + P \neq P$. These are zero divisors in A/P. Conversely, if A/P is not an integral domain, then it has zero divisors. So there exists $x, y \in A$ such that $(x + P)(y + P) = P$. This implies $xy \in P$. Since P is prime, $x \in P$ or $y \in P$. So one of $x + P$ and $y + P$ is zero in A/P. Recall that there is a bijection between the ideals in A containing M with the ideals in A/M. Thus $M \subset A$ is maximal if and only if the only ideals in A/M are (0) and (1), if and only if A/M is a field. $\square$

Remark 4.3. Every field is an integral domain, hence every maximal ideal is prime. The converse is false. Take any integral domain which is not a field, such as $\mathbb{Z}$. Then $(0) \in \mathbb{Z}$ is a prime ideal which is not a maximal ideal.

Proposition 4.4. If $f : A \to B$ is a homomorphism of rings, and $P \subset B$ is a prime ideal, then $f^{-1}(P)$ is a prime ideal in A.

Proof. Assume that for some $x, y \in A$ we have $xy \in f^{-1}(P)$. Then $f(xy) = f(x)f(y) \in P$. Then $f(x) \in P$ or $f(y) \in P$. Then $x \in f^{-1}(P)$ or $y \in f^{-1}(P)$. $\square$

Remark 4.5. This does not hold for maximal ideals. Let $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$. $f^{-1}((0)) = (0)$, but (0) is maximal in $\mathbb{Q}$ and not maximal in $\mathbb{Z}$. But if $f : A \to B$ is a surjective homomorphism of rings, then f^{-1} sends maximal ideals of B to maximal ideals of A. (Exercise)

Theorem 4.6. Every non-zero ring contains at least one maximal ideal.

We need Zorn's lemma, which belongs to set theory. A **partially ordered set** or **poset** is a set S equipped with a **partial order**. By definition it is a reflexive, transitive, antisymmetric binary relation $\leq$,

$$x \leq x, \qquad x \leq y, y \leq z \implies x \leq z, \qquad x \leq y, y \leq x \implies x = y.$$

We don't require that for arbitrary x and y in S, we have either $x \leq y$ or $y \leq x$. A subset $T \subset S$ is called a **chain** if for any $x \in T$, $y \in T$ we have $x \leq y$ or $y \leq x$. An **upper bound** for a subset $T \subset S$ is an element $x \in S$ such that for any $t \in T$ we have $t \leq s$. A **maximal element** in S is an element $x \in S$ such that if $y \in S$ and $y \geq x$, then $y = x$.

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Theorem 4.7 (Zorn's lemma)**.** If S is a non-empty partially ordered set such that every chain in S has an upper bound in S, then S contains a maximal element.

Proof of Theorem 4.6. Let A be a non-zero ring. To apply Zorn's lemma it is enough to show that every growing chain of ideals $I_1 \subset I_2 \subset \ldots$, such that $1 \in I_i$ for all i, has an upper bound which is an ideal not equal to A, so not containing 1. Then Zorn's lemma applied to the set of ideals of A not containing 1 and ordered by inclusion, implies the existence of a maximal ideal. So we have a chain I_j, where j is an element of a set J. Consider $I = \cup_{j \in J} I_j$. Claim that I is an ideal in A and $1 \notin I$.

1. $1 \notin I$ is clear. Because otherwise $1 \in I$ gives $1 \in I_j$ for $j \in J$, but it is a contradiction.

2. For any $a \in A$ we have $aI \subset I$, so for all $x \in I$, $ax \in I$. But then $x \in I_j$ for some j. Then $ax \in I_j \subset I$.

3. Suppose $x, y \in I$. Must show $x + y \in I$. There exists $j_1 \in J$ such that $x \in I_{j_1}$. Similarly, there exists $j_2 \in J$ such that $y \in I_{j_2}$. Recall that I_j for $j \in J$ is a chain. Hence either $j_1 \leq j_2$ or $j_2 \leq j_1$. This means that either $I_{j_1} \subset I_{j_2}$ or $I_{j_2} \subset I_{j_1}$. Without loss of generality assume that $I_{j_1} \subset I_{j_2}$. Then $x, y \in I_{j_2}$. Hence $x + y \in I_{j_2}$, hence $x + y \in I$. This proves that I is an ideal not containing 1.

$\square$

Definition 4.8. A ring with a unique maximal ideal is called a **local ring**.

Corollary 4.9. Let I be an ideal of A and $I \neq A$. Then I is contained in a maximal ideal of A.

Proof. There is a bijection between the ideals of A containing I and the ideals in A/I. If $I \subset J \subset A$, then $J \mapsto J/I$. J/I is an ideal in A/I. By Theorem 4.6, A/I contains a maximal ideal, say $M \subset A/I$. Let $f : A \to A/I$ be the map sending $x \mapsto x + I$. Consider $f^{-1}(M) \subset A$. This is an ideal in A. In general, if $I \subset J \subset A$ are ideals, then f induces an isomorphism of rings $A/J \to (A/I)(J/I)$. For additive groups, this is one of the standard isomorphisms theorems, but this respects multiplication, so is an isomorphism of rings. Now, we know that M maximal in A/I implies that (A/I) is a field. This ring is isomorphic to $A/f^{-1}(M)$. Hence $A/f^{-1}(M)$ is also a field. Therefore, $f^{-1}(M)$ is maximal in A. $\square$

Corollary 4.10. Every non-unit is contained in a maximal ideal.

Proof. If $x \in A$ is a non-unit, consider (x). $1 \notin (x)$, otherwise x is a unit. By Corollary 4.9 (x) is contained in a maximal ideal of A. $\square$

Example.

1. Every field is a local ring. In this case (0) is a maximal ideal.

2. Let k be a field. Consider the ring of formal power series $k[[t]] = \{a_0 + a_1 t + \cdots \mid a_i \in k\}$, such that

$$\left(\sum_{i=0}^{\infty} a_i t^i \right) \left(\sum_{j=0}^{\infty} b_j t^j \right) = a_0 b_0 + (a_0 b_1 + a_1 b_0) t + \ldots.$$

Then the principal ideal (t) is a maximal ideal. Indeed, $k[[t]]/(t) \cong k$ is a field. (TODO Exercise: $k[[t]] \setminus (t) = k[[t]]^*$)

3. $\mathbb{Z}_{(p)} = \{a/b \mid a, b \in \mathbb{Z}, \ b \neq 0, \ p \nmid b\}$. (TODO Exercise: (p) is a maximal ideal. There are no other maximal ideals)

If A is a local ring with maximal ideal M, then A/M is called the **residue field** of A.

Lemma 4.11 (Prime avoidance)**.** Let A be a ring and $P \subset A$ be a prime ideal. Suppose that $I_1, \ldots, I_n$ are ideals in A such that $\cap_{i=1}^n I_i \subset P$. Then there exists j, $1 \leq j \leq n$, such that $I_j \subset P$. If $\cap_{i=1}^n I_i = P$, then there exists j, $1 \leq j \leq n$, such that $I_j = P$.

Proof. Suppose our claim is false. Then there exists $a_j \in I_j$ such that $a_j \notin P$ for $j = 1, \ldots, n$. Then $a_1 \ldots a_n \in \cap_{j=1}^n I_i \subset P$. $(a_1 \ldots a_{n-1}) a_n \in P$ gives $a_1 \ldots a_{n-1} \in P$ or $a_n \in P$. But $a_n \notin P$, so $a_1 \ldots a_{n-2} \in P$, a contradiction. The second statement follows. We know that $I_k \subset P$ for some k, $1 \leq k \leq n$, but $P = \cap_{j=1}^n I_j \subset I_k$. Hence $P = I_k$. $\square$

5   Nilradical and the Jacobson radical

Proposition 5.1. Let A be a ring. The set $N(A)$ of all nilpotent elements of A is an ideal in A. It is called the **nilradical** of A. The quotient ring $A/N(A)$ has no non-zero nilpotents.

Proof. Clearly, if $x^n = 0$ and $y^n = 0$, then $(xy)^n = 0$, if $n \geq m$. $(x+y)^{n+m}$ is the sum with coefficients of of monomials in which either the power of x is $\geq n$ or the power of y is $\geq m$. So this is zero. Let $a \in A$. Then $(ax)^n = 0$. Therefore, $N(A)$ is an ideal. Now let $t + N(A)$ for $t \in A$ be a nilpotent element in $A/N(A)$. For some k we have $t^k + N(A)$ is the trivial coset. That is, $t^k \in N(A)$. Thus $(t^k)^l = 0$ for some $l > 0$. Hence $t \in N(A)$, so $t + N(A)$ is the zero element of $A/N(A)$. $\qquad\square$

Proposition 5.2. The nilradical $N(A)$ is the intersection of all prime ideals of A.

Proof.

1. $N(A) \subset \cap_{P \subset A} P$, where P is a prime ideal of A. Take $x \in A$, $x^n = 0$. Take a prime ideal $P \subset A$. We have that $P \ni x^n = x \ldots x$ gives $x \in P$.

2. Now let $f \in A$ be a non-nilpotent element, that is $0 \notin \{f^i \mid i \geq 1\}$. Let Σ be the set of ideals of A that do not intersect $\{f^i \mid i \geq 1\}$. Σ contains the zero ideal (0), so $\Sigma \neq \emptyset$. Order the elements of Σ by inclusion. Every chain in Σ has an upper bound. If I_j for $j \in J$ is a chain, then $\cup_{j \in J} I_j$ is an ideal of A. Moreover, if $f^k \in \cup_{j \in J} I_j$, then $f^k \in I_{j_0}$ for some $j_0 \in J$, but this is impossible. By Zorn's Lemma, we know that Σ has a maximal element. Call it P. Claim that P is a prime ideal. To prove this, assume that $x, y \in A$ such that $x, y \notin P$. We must show that $xy \notin P$. Consider $P + (x)$, all elements of the form $\alpha + rx$, where $\alpha \in P$ and $r \in A$. $x \notin P$ gives $P \neq P + (x)$. By construction, P is maximal in Σ, hence $P + \sigma$ is not in Σ, that is, there exists $n \geq 1$ such that $f^n \in P + (x)$. Similarly, there exists m such that $f^m \in P + (y)$. Therefore, f^{n+m} belongs to $P + (xy)$. If $xy \in P$, then $P + (xy) = P$ but then $f^{n+m} \in P$, which is absurd because $P \in \Sigma$. Thus $xy \notin P$. This shows that P is a prime ideal and $f \notin P$.

$\qquad\square$

What happens if we consider the intersection of all maximal ideals of A. This intersection is called the **Jacobson radical** of A. It is denoted by $J(A)$.

Proposition 5.3. $x \in J(A)$ if and only if $1 - xy$ is a unit in A for all $y \in A$.

Proof. Suppose that $x \in J(A)$, that is x is contained in every maximal ideal of A, but $1 - xy$ is not a unit for some $y \in A$. By Corollary 4.10 every non-unit is contained in some maximal ideal, so there exists a maximal ideal $M \subset A$ such that $1 - xy \in M$. Since $x \in M$ we conclude that $1 \in M$, which is impossible. Conversely, suppose $x \notin J(A)$, that is, $x \notin M$ for some maximal ideal $M \subset A$. Consider the sum of two ideals $M + (x)$. This is an ideal in A, such that $M \subsetneq M + (x)$. Since M is maximal, we have $M + (x) = A$. Therefore $1 = m + xy$, where $m \in A$ and $y \in A$. Now $1 - xy = m \in M$ cannot be a unit. $\qquad\square$

Let $I \subset A$ be an ideal. The **radical** $rad(I)$ or $r(I)$ or $\sqrt{I}$ is defined as $\{x \in A \mid \exists n \geq 1, \ x^n \in I\}$.

Proposition 5.4. $r(I)$ is the intersection of all prime ideals of A that contain I.

Proof. Use the bijection between ideals containing I and the ideals in A/I. $\qquad\square$

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Definition 5.5. Let J be an index set. Suppose we have a ring R_j for $j \in J$. $\prod_{j \in J} R_j$ has a natural structure of a ring. 0 in $\prod_{j \in J} R_j$ is $(0, \ldots, 0)$ and 1 in $\prod_{j \in J}$ is defined as $(1, \ldots, 1)$, and

$$(r_j)_{j \in J} + (r'_j)_{j \in J} = (r_j + r'_j)_{j \in J}, \qquad (r_j)_{j \in J} \cdot (r'_j)_{j \in J} = (r_j \cdot r'_j)_{j \in J}.$$

$\prod_{j \in J} R_j$ is called the **product of rings** R_j for $j \in J$. If R is a ring equipped with homomorphisms $f_j : R \to R_j$ for each $j \in J$, then $(f_j) : R \to \prod_{j \in J} R_j$ is a homomorphism of rings.

Recall that $N(R) = \cap_{P \subset R} P$, where P are prime ideals of R. Consider the product ring $\prod_{P \subset R} R/P$. Putting together the canonical surjective maps $R \to R/P$ by $x \mapsto x + P$ for all $P \subset R$ we obtain a homomorphism $f : R \to \prod_{P \subset R} R/P$. $Ker(f) = \cup_{P \subset R} Ker[R \to R/P] = \cap_{P \subset R} = N(R)$. Hence we get an injective homomorphism $R/N(R) \to \prod_{P \subset R} R/P$. Similarly, we get an injective homomorphism $R/J(R) \to \prod_{M \subset R} R/M$, where M are maximal ideals of R and $J(R)$ is the Jacobson radical of R.

6   Localisation of rings

Localisation refers to introducing denominators.

Example. From $R = \mathbb{Z}$ to $\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z}, \ b \neq 0\}$.

Definition 6.1. A subset $S \subset A$ is called a **multiplicative set** if $1 \in S$, $0 \notin S$, and if $a, b \in S$, then $ab \in S$, that is S is closed under multiplication.

Example.

1. Take any $a \in A$ which is not nilpotent, that is $a^n = 0$ for $n \geq 1$. Then $\{1, a, a^2, \ldots\}$ is a multiplicative set.

2. Let $P \subset A$ be a prime ideal. Then $A \setminus P$ is a multiplicative set. Indeed, $x, y \notin P$ gives $xy \notin P$.

3. Let $P_j \subset A$, for $j \in J$, be a family of prime ideals of A. Then $A \setminus \cup_{j \in J} P_j = \cap_{j \in J}(A \setminus P_j)$ is a multiplicative set.

4. A^* is a multiplicative set in A.

5. The set of all non-zero-divisors of A is a multiplicative set.

6. Let $I \subset A$ be an ideal. Then $1 + I = \{1 + x \mid x \in I\}$ is a multiplicative set.

Definition 6.2. Let A be a ring with a multiplicative set S. Consider $A \times S$, that is the set of pairs of elements (a, s), where $a \in A$ and $s \in S$. Define an equivalence relation $\sim$ as follows. $(a, s) \sim (b, t)$ if and only if there exists $u \in S$ such that $u(at - bs) = 0$. Define $S^{-1}A$ to be the set of equivalence classes of $\sim$. Write the equivalence class of (a, s) as a/s. Define multiplication on $S^{-1}A$ as

$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

Define addition on $S^{-1}A$ as

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}.$$

Define 0 in $S^{-1}A$ as $0/1$ and we define 1 in $S^{-1}A$ as $1/1$.

(TODO Exercise: check that if $(a, s) \sim (a', s')$ and $(b, t) \sim (b', t')$, then $(ab, st) \sim (a'b', s't')$) (TODO Exercise: check that if $(a, s) \sim (a', s')$ and $(b, t) \sim (b', t')$, then $(at + bs, st) \sim (a't' + b's', s't')$) (TODO Exercise: with this definition $S^{-1}A$ is a ring)

Remark 6.3. $\sim$ is indeed an equivalence relation. $(a, s) \sim (a, s)$, $(a, s) \sim (b, t)$ gives $(b, t) \sim (a, s)$. Let us check that if $(a, s) \sim (b, t)$ and $(b, t) \sim (c, r)$, then $(a, s) \sim (c, r)$. There exists $u, v \in S$ such that $u(at - bs) = 0$ and $v(br - ct) = 0$. Then $uv(atr - bsr) = 0$ and $uv(brs - cts) = 0$, so $uvt(ar - bs) = 0$.

Lemma 6.4. Let A be a ring with a multiplicative set S. Then $f : A \to S^{-1}A$ defined by $f(x) = x/1$ is a homomorphism of rings. $Ker(f) = 0$ if and only if S contains no zero-divisors.

Proof.

$$f(x + y) = \frac{x + y}{1} = \frac{x}{1} + \frac{y}{1}, \qquad f(xy) = \frac{xy}{1} = \frac{x}{1} \cdot \frac{y}{1}.$$

$Ker(f) = \{x \mid \exists u \in S, \ ux = 0\}$ since $x/1 = 0/1$ if and only if there exists $u \in S$ such that $u(x \cdot 1 - 0 \cdot 1) = 0$. $\qquad\square$

Example. Let k be a field. Explore what happens when $A = k[x, y]/(xy)$ and $S = \{1, x, \ldots\}$. Determine $S^{-1}A$ and $Ker(f)$.

Lecture 7 is a problem class.

Lemma 6.5 (Universal property of localisation)**.** Let A be a ring with a multiplicative set $S \subset A$. Suppose $g : A \to B$ is a homomorphism such that $g(S) \subset B^*$, that is for all $s \in S$, $g(s)$ is a unit in B. Then there exists a unique homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$, where $f : A \to S^{-1}A$ is the canonical map.

Proof. Define $h(a/s) = g(a)g(s)^{-1}$ since g invertible. Check that h is well-defined, that is if $a/s = a'/s'$, then $u(as' - a's) = 0$ for $u \in S$. Apply g and get $g(u)(g(a)g(s') - g(a')g(s)) = 0$. $g(u) \in B^*$ and $g(a)g(s') = g(a')g(s)$. Hence $g(a)g(s)^{-1} = g(a')g(s')^{-1}$. Take any $a \in A$. Then $f(a) = a/1$, hence $(h \circ f)(a) = g(a)$. Finally, let us show there is only one homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$. Suppose $h' : S^{-1}A \to B$ is such that $g = h' \circ f$, so that for any $a \in A$ we have $g(a) = h'(a)$. For any $s \in S$, s^{-1} is an element of $S^{-1}A$, and so is s. $1 = s^{-1}s$ gives $1 = h'(1) = h'(s^{-1})h'(s)$. Thus $h'(s^{-1}) = h'(s)^{-1} = g(s)^{-1}$ because h' on the image of A in $S^{-1}A$ is the same as g. Comparing this with the definition of h we see that $h' = h$. $\qquad\square$

Let $I \subset A$ be an ideal. Define $S^{-1}I = \{x/s \mid x \in I, \ s \in S\}$. This is an ideal in $S^{-1}I$. It is the ideal generated by $f(I) \subset S^{-1}A$.

Proposition 6.6. Let A be a ring with a multiplicative set S. Let $I_1, \ldots, I_n$ be ideals in A. Then

1. $S^{-1}(I_1 + \cdots + I_n) = S^{-1}I_1 + \cdots + S^{-1}I_n$,

2. $S^{-1}(I_1 \ldots I_n) = S^{-1}I_1 \ldots S^{-1}I_n$,

3. $S^{-1}\left(\cap_{j=1}^n I_j\right) = \cap_{j=1}^n S^{-1}I_j$, and

4. $r\left(S^{-1}I\right) = S^{-1}r(I)$, where $r(I)$ is the radical of I.

Proposition 6.7. Every ideal of $S^{-1}A$ is of the form $S^{-1}I$ for some ideal $I \subset A$.

Proof. Start with an ideal $J \subset S^{-1}A$. Consider $f^{-1}(J) \subset A$. This is an ideal. Call it I. Claim that $J = S^{-1}I$. Pick any element $a/s \in J$. Then $a \in J$. Since $f(a) = a/1 \in J$ we have that $a \in I$. Therefore, $a/s \in S^{-1}I$. This proves $J \subset S^{-1}I$. But it is clear that $S^{-1}I \subset J$. Indeed, $x \in I$ then $x/1 \in J$. But J is an ideal, hence $x/s \in J$. $\qquad\square$

Theorem 6.8. The prime ideals in $S^{-1}A$ are the ideals $S^{-1}P$, where P is a prime ideal of A such that $P \cap S \neq \emptyset$. Thus we have a bijection between the set of prime ideals in $S^{-1}A$ and the set of prime ideals in A that do not intersect S.

Proof. Suppose that P is a prime ideal in A, $P \cap S \neq \emptyset$. Claim that $S^{-1}P$ is a prime ideal in $S^{-1}A$. If $(a/s)(b/t) \in S^{-1}P$, then $(a/s)(b/t) = c/u$, where $c \in P$, $u \in S$. This is equivalent to $v(abu - cst) = 0$ for some $v \in S$. $(ab)(vu) = c \in P$ such that $v \in P$. $vu \in S$ and $S \cap P = \emptyset$, so $vu \notin P$. But $P \subset A$ is a prime ideal, hence $ab \in P$. Thus $a \in P$ gives $a/s \in S^{-1}P$ or $b \in P$ gives $b/t \in S^{-1}P$. This proves $S^{-1}P \subset S^{-1}A$ is prime. For any ideal $J \subset S^{-1}A$, we know that $f^{-1}J$ is an ideal in S. Moreover, if J is prime, then $f^{-1}J \subset A$ is prime. Let us show that $f^{-1}J \cap S = \emptyset$. Otherwise, take $s \in S \cap f^{-1}J$, so $s/1 \in J$. But $1/s \in J^{-1}A$, hence $1 = (1/s)s \in J$, so $J = S^{-1}A$. But J is a prime ideal, so $J \neq S^{-1}A$. To show that $P \mapsto S^{-1}P$ and $J \mapsto f^{-1}J$ are the identity maps, we need to check that $P = f^{-1}\left(S^{-1}P\right)$ and $J = S^{-1}f^{-1}(J)$. $S^{-1}P = \{x/s \mid x \in P, \ s \in S\}$. If $y \in f^{-1}\left(S^{-1}P\right) \subset A$ is such that $f(y) = x/s$, then $y/1 = x/s$. Hence $ys = x \in P$. Since $P \cap S = \emptyset$, $s \notin P$. Therefore, $y \in P$. Hence $P = f^{-1}\left(S^{-1}A\right)$. Now let us prove that $J = S^{-1}f^{-1}(J)$. But in Proposition 6.7 we showed that there is an ideal $I \subset A$ such that $J = S^{-1}I$. In the proof of Proposition 6.7 we have taken $I = f^{-1}(J)$. So we are done. $\qquad\square$

7 Determinants

Lemma 7.1. Let $f(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$. If f as a function $\mathbb{Z}^n \to \mathbb{Z}$ is zero, that is f only takes zero values on arbitrary elements of $\mathbb{Z}^n$, then f is the zero polynomial.

Proof. Induction in n. If $n = 1$, then $f(x)$ is a polynomial with infinitely many roots. So $f(x)$ is the zero polynomial, so cannot have more than $\deg(f)$ roots. Assume we know the lemma for $n-1$ variables. Write $f(x_1, \ldots, x_n) = \sum_{i=0}^{N} f_i(x_1, \ldots, x_{n-1}) x_n^i$ for $f_j(x_1, \ldots, x_{n-1}) \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$. Fix $x_1, \ldots, x_{n-1}$. We get a polynomial in one variable x_n, so this polynomial has zero coefficients. This implies that each $f_i(x_1, \ldots, x_n)$ takes only zero values. By the induction assumption, each f_i is the zero polynomial. $\square$

Remark 7.2. This means that if a polynomial formula with coefficients in $\mathbb{Z}$ is true in $\mathbb{Z}$, this is true in an arbitrary commutative ring.

Example. $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$ is true in any ring.

The underlying fact is the existence of a canonical map $\mathbb{Z} \to R$ by $1 \mapsto 1$.

Definition 7.3. Let R be a commutative ring. Let $A = (a_{ij})$ be a square matrix for $1 \leq i \leq n$ and $1 \leq j \leq n$, with entries in R. Then $\det(A)$ is defined as $(-1)^{i+1} a_{i1} M_{i1} + \ldots (-1)^{i+n} a_{in} M_{in}$ for i fixed. Here M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by removing the i-th row and the j-th column.

Proposition 7.4. $\det(A) = (-1)^{i+1} a_{i1} M_{i1} + \ldots (-1)^{i+n} a_{in} M_{in}$.

Proof. This is known for matrices with entries in $\mathbb{C}$, so by Remark 7.2 this holds in any commutative ring. $\square$

Remark 7.5. The official definition is

$$\det(A) = \sum_{\pi \in S_n} sgn(\pi) \, a_{1\pi(1)} \ldots a_{n\pi(n)},$$

where $sgn : S_n \to \{\pm 1\}$.

Proposition 7.6. For $i \neq j$,

$$(-1)^{j+1} a_{i1} M_{j1} + \cdots + (-1)^{j+n} a_{in} M_{jn} = 0,$$

$$(-1)^{j+1} a_{1i} M_{1j} + \cdots + (-1)^{j+n} a_{ni} M_{nj} = 0.$$

Define the **adjacent** matrix as an $n \times n$ matrix $A_{ij}^v = (-1)^{i+j} M_{ji}$. Putting together all the previous identities we get the following.

Theorem 7.7. $A \cdot A^v = A^v \cdot A = \det(A) I_n$.

8 Modules

Definition 8.1. Let A be a ring. A **module** M over A is an abelian group $(M, 0, +)$ with an action $\cdot$ of A on M, that is $A \times M \to M$ by $a \cdot m = am$, such that the following axioms hold.

1. $1 \cdot m = m$ for all $m \in M$ and $a \in A$.

2. $\mu \cdot (\lambda \cdot m) = (\mu\lambda) \cdot m \; \lambda, \mu \in A$.

3. $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in A$ and $x, y \in M$.

4. $(\mu + \lambda) x = \mu x + \lambda x$ for all $\mu, \lambda \in A$ and $x \in M$.

Example.

1. $M = A$. More generally, consider an ideal $I \subset A$. A acts on I by $A \times I \to I$ by $a \cdot x = ax$.

2. If A is a field, then an A-module is the same as a vector space over this field.

3. Take M to be any abelian group. Take $A = \mathbb{Z}$. Define an action of $\mathbb{Z}$ as follows. $1 \cdot m = m$ and $n \cdot m = (1 + \cdots + 1) \cdot m = m + \cdots + m = nm$. $0 = n + (-n) \in \mathbb{Z}$, then $0 = (n + (-n)) \cdot m = nm + (-n) m$. Hence $(-n) \cdot m = -(n \cdot m) = -(m + \cdots + m)$. So, there is exactly one way to equip any abelian group with the structure of a $\mathbb{Z}$-module.

4. Let k be a field and let $A = k[x]$. A $k[x]$-module is a vector space over k with extra structure $x \times M \to M$. This is a linear transformation of M. It can be arbitrary. Thus a $k[x]$-module is a pair (M, f), where M is a k-vector space and $f : M \to M$ is linear transformation of M.

Definition 8.2. Let M and N be A-modules. A map $f : M \to N$ is called a **homomorphism of A-modules** if f is a homomorphism of abelian groups and $f(a, m) = af(m)$ for any $a \in A$ and $m \in M$. If $f : M \to N$ and $g : M \to N$ are homomorphisms of A-modules, then so is $f + g$, so we get $Hom_A(M, N)$, a group of such homomorphisms. This is also an A-module via the action $(a, f(a)) \mapsto a \cdot f(a)$.

Definition 8.3. A **submodule** $N \subset M$ is a subgroup, stable under the action of A. Then M/N is naturally an A-module with A-action inherited from M. Define $(N : M) = \{a \in A \mid raM \subset rN \subset N\}$. This is an ideal in A. In particular, can do this when $N = 0$. Note $Ann(M) = (0 : M) = \{a \in A \mid aM = 0\}$. This is called the **annihilator** of M.

Definition 8.4. If $f : M \to N$ is a homomorphism of A-modules, then $Ker(f)$ is an A-module and $Im(f) \cong M/Ker(f)$ is as isomorphism of A-modules.

Definition 8.5. An A-module M is **finitely generated** if there exist $m_1, \ldots, m_n$ in M such that $M = \{a_1 m_1 + \cdots + a_n m_n \mid a_i \in A\}$.

Example. A **free** A-module of rank n is the set $A^n = \{(a_1, \ldots, a_n) \mid a_i \in A\}$ with coordinate-wise addition. $a \in A$ acts on $(a_1, \ldots, a_n)$ by sending it to $(aa_1, \ldots, aa_n)$. If $f(1, 0, \ldots, 0) = m_1$, $f : A^m \to M$ is an example of an A-module homomorphism.

Lemma 8.6. Let A be a ring. Let M be a finitely generated A-module and let $A \subset A$ be an ideal such that $JM = M$, that is sums of xm, where $x \in J$ and $m \in M$, give all of M. Then there exists $a \in J$ such that $(1 - a) M = 0$.

Proof. Let $m_1, \ldots, m_n$ be a set of generators of M. $m_i \in M = JM$, so $m_i = x_{i1} m_1 + \cdots + x_{in} m_n$, where $x_{ij} \in J$. Let $X = (x_{ij})_{1 \leq i, j \leq n}$, so

$$(I_n - X) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

Let $(I_n - X)^v$ be the adjunct matrix of $I_n - X$. Then $(I_n - X)^v (I_n - X) = \det(I_n - X) I_n$. Hence

$$\det(I_n - X) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

$\det(I_n - X) = \prod_{i=1}^n (1 - x_{ii}) + J \equiv 1 \mod J$. So $\det(I_n - X) = 1 - a$, where $a \in J$. $(1 - a) m_i = 0$ for all i gives $(1 - a) M = 0$. $\square$

Corollary 8.7 (Nakayama's lemma)**.** Let A be a ring and let M be an A-module, which is finitely generated. Let $I \subset A$ be an ideal contained in the Jacobson radical $J(A)$. Then $IM = M$ implies $M = 0$.

Proof. Lemma 8.6 gives an $a \in I$ such that $(1 - a) M$. But $a \in J(A)$. By Proposition 5.3 $1 - a \in A^*$ so that there exists $u \in A^*$ such that $u(1 - a) = 1$, so $M = 1 \cdot M = u(1 - a) \cdot M = 0$. $\square$

Another proof considers $M = (m_1, \ldots, m_n)$. Let us call a generating set minimal, if no proper set is a generating set. Assume that $m_1, \ldots, m_n$ is a minimal generating set. $IM = M$ implies that $m_1 = a_1 m_1 + \cdots + a_n m_n$, where $a_i \in I$. $(1 - a_1) m_1 = a_2 m_2 + \cdots + a_n m_n$. Proposition 5.3 says that $1 - a_1 \in A^*$. Hence $m_1 = (1 - a_1)^{-1} a_2 m_2 + \cdots + (1 - a_1)^{-1} a_n m_n$. This is a contradiction, because $m_2, \ldots, m_n$ is a generating set.

9   Localisation of modules

Definition 9.1. Let A be a ring with a multiplicative set S, and let M be an A-module. Define $\sim$ on $M \times S$ by $(m, s) \sim (n, t)$ if and only if there exists $u \in S$ such that $u (tm - sn) = 0$. This is an equivalence relation. Denote the equivalence class of (m, s) by m/s. Then the set of these equivalence classes form a module denoted by $S^{-1}M$ over $S^{-1}A$. The action of $S^{-1}A$ on $S^{-1}M$ is $(a/s)(m/t) = (am/st)$. $m/s + n/t = (mt + ns)/st$. The zero in $S^{-1}M$ is $0/1$.

Definition 9.2. Let A be a ring and let $P \subset A$ be a prime ideal. Then $S = A \setminus P$ is a multiplicative set. The ring $S^{-1}A$ is denoted A_P. It is called the localisation of A at P. Recall that by Theorem 6.8 the prime ideals of A_P are of the form $S^{-1}I$, where $I \subset A$ is a prime ideal such that $I \cap (A \setminus P) = \emptyset$, if and only if $I \subset P$.

Theorem 9.3. Let A be a ring with a prime ideal P. Then $a \in A_P$ is a unit if and only if $a \notin PA_P = S^{-1}P = (A \setminus P)^{-1} P$. The ideal PA_P is the unique maximal ideal of A_P. So A_P is a local ring.

Proof. Suppose $a/s \in A_P$ is a unit. Then for some $b/t \in A_P$ we have $(a/s)(b/t) = 1$. $ab/st - 1/1 = 0$ if and only if there exists $u \in S$ such that $u(ab - st) = 0$. $uab = ust \in S = A \setminus P$. Hence $a \notin P$, so that $a/s \notin PA_P$. Conversely, if $a/s \notin PA_P$, then $a \notin P$ and $s \in S$ gives $a \in S = A \setminus P$. So a/s is a unit whose inverse is s/a. PA_P is a maximal ideal, because joining any new element will be the whole ring, as this element must be a unit. $\square$

Example. $\mathbb{Z}_{(p)} = \{a/b \mid a, b \in \mathbb{Z},\ (p, b) = 1\}$ and

$$p\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \ \middle|\ a, b \in \mathbb{Z},\ p \mid a,\ (p, b) = 1 \right\}, \qquad \mathbb{Z}_{(p)}^* = \left\{ \frac{a}{b} \ \middle|\ a, b \in \mathbb{Z},\ p \nmid a,\ (p, b) = 1 \right\}.$$

Do the same for $A = k[x]$ and $P = (f(x))$, where $f(x)$ is irreducible.

Proposition 9.4. Let M be an A-module. Then $M = 0$ if and only if $M_P = 0$ for all maximal ideals $P \subset A$.

Proof. Suppose $M \neq 0$. Choose $x \in M$, $x \neq 0$. Define $I = Ann(x) = \{a \in A \mid ax = 0\}$. This is an ideal in A, and $I \neq A$ because $1 \cdot x = x$, so $1 \notin I$. Let P be a maximal ideal such that $I \subset P$. Claim that $M_P \neq 0$. Consider $x/1 \in M_P$. If $M_P = 0$, then $x/0 = 0/1$, so $ux = 0$ for some $u \in A \setminus P$. $u \in I = Ann(x)$ but $u \notin P$. This is a contradiction because $I \subset P$. $\square$

10   Chain conditions: Noetherian and Artinian rings

Lemma 10.1. Let Σ be a partially ordered set. Then the following properties are equivalent.

1. Every non-empty subset of Σ has a maximal element.

2. Every ascending chain $x_1 \leq x_2 \leq \ldots$ is stationary, that is there exists n such that for any $m \geq 0$ we have $x_{n+m} = x_n$.

Proof.

$1 \implies 2$ Any ascending chain has a maximal element, say x_n. Hence $x_{m+n} = x_n$, for all $m \geq 0$.

$2 \implies 1$ Suppose $S \subset \Sigma$ does not have a maximal element. Choose $x_1 \in S$. There exists $x_2 \in S$ such that $x_2 > x_1$. If $x_1 < \cdots < x_2$ are chosen, then since x_n is not a maximal element, we can choose $x_{n+1} > x_n$. This constructs an ascending chain that is not stationary.

$\square$

Definition 10.2. A ring A is called **Noetherian** if every ascending chain of ideals in A is stationary. An A-module M is Noetherian if every chain of submodules of M is stationary. In particular, a ring A is Noetherian if it is a Noetherian module over A. A ring A is called **Artinian** if every descending chain of ideals is stationary. An A-module M is Artinian if every descending chain of submodules is stationary.

Example. Let $\mathbb{Z} \supset (n)$ is Noetherian. $(a) \subset \langle b \rangle$ if and only if b divides a. $(15) \subsetneq (5) \subsetneq (1) = \mathbb{Z}$. But $(2) \supsetneq (4) \supsetneq \cdots \supsetneq (2^n) \supsetneq \ldots$ is an infinite descending chain of ideals so $\mathbb{Z}$ is not Artinian. If A is a finite ring, then it is trivially both Noetherian and Artinian.

Proposition 10.3. Let A be a ring and let M be an A-module. Then M is Noetherian if and only if every submodule of M is finitely generated.

Proof. Suppose M is Noetherian, but $N \subset M$ is a submodule that is not finitely generated. Then take $x_1 \in N$. Since $N \neq (x_1)$, the submodule generated by x_1, we can find $x_2 \in N \backslash (x_1)$. This gives $(x_1) \subsetneq (x_1, x_2)$ and so on. This produces an ascending chain which is not stationary, a contradiction. Now suppose that every submodule of M is $f \cdot g$. Consider any ascending chain $M_1 \subset M_2 \subset \ldots$. Let $N = \cup_{i \geq 1} M_i$. This is a submodule of M. By assumption $N = (x_1, \ldots, x_n)$ for some $x_i \in N$. For each x_i there is an M_j in our chain such that $x_i \in M_j$. So there will be some M_l that contains $x_1, \ldots, x_n$. Then $N = M_l$. And clearly for any $m \geq 0$ we have $M_l \subset M_{l+m} \subset N = M_l$, so $M_{l+m} = M_l$. So M is Noetherian. $\square$

Remark 10.4. Applying this to the A-module A we see that A is Noetherian if and only if every ideal is finitely generated. Hence every principal ideal domain is Noetherian.

Example. $\mathbb{Z}$, $k[x]$, $k[x_1, \ldots, x_n]$. Hilbert's basis theorem says that if R is Noetherian, then $R[x]$ is also Noetherian.

Proposition 10.5. Let A be a ring. Let M be an A-module and $N \subset M$ a submodule. Then M is Noetherian if and only if N and M/N are both Noetherian A-modules.

Proof. Suppose M is Noetherian. Then clearly N is Noetherian. M/N is Noetherian too. Indeed, let L be a submodule of M/N. Let T be the inverse image of L in M. Then we have a surjective homomorphism of A-modules $T \to L$. Since T is finitely generated, so that $T = (x_1, \ldots, x_n)$ for some $x_i \in T$. Then the images of $x_1, \ldots, x_n$ generate L. Now assume N and M/N are Noetherian. This can also be proved using ascending chains. Take any ascending chain $M_1 \subset M_2 \subset \ldots$. Then $N \cap M_1 \subset N \cap M_2 \subset \ldots$ is an ascending chain of submodules of N. Let $n_1 \in \mathbb{N}$ be such that for all $i \geq 0$, $N \cap M_{n+i} = N \cap M_{n_1}$. Consider $(M_i + N)/N \subset M/N$. This is just the set of cosets $x + N$, where $x \in M_i$. In fact $(M_i + N)/N \cong M_i/M \cap N$. We obtain an ascending chain $(M_1 + N)/N \subset (M_2 + N)/N \subset \cdots \subset (M_{n_2} + N)/N = (M_{n_1} + N)/N = \ldots$. Take $n = \max(n_1, n_2)$. It works, that is $M_n = M_{n+1} = \ldots$. Indeed, take any $x \in M_{n+i}$ for $i \geq 0$. Then there exists $y \in M_n$ such that $x + N = y + N$. Thus $x - y \in N \cap M_{n+i}$. But this is $N \cap M_n$. So there exists $z \in N \cap M_n$ such that $x - y = z$. Hence $x = y + z \in M_n$. $\square$

Lecture 13 is a problem class.
(TODO Exercise: Do the same in the Artinian case)

Corollary 10.6. Let A be a Noetherian or Artinian ring. Let M be a finitely generated A-module. Then M is Noetherian or Artinian.

Proof. Let $M = (m_1, \ldots, m_n)$ for $m_i \in M$. $M = \{a_1 m_1 + \cdots + a_n m_n \mid a_i \in A\}$. Let $A^{\oplus n} = \{(a_1, \ldots, a_n) \mid a_i \in A\}$ be a free A-module of rank n. There is a homomorphism of A-modules $A^{\oplus n} \to M$ sending $(a_1, \ldots, a_n)$ to $a_1 m_1 + \cdots + a_n m_n$. It is surjective. By Proposition 10.5 it is enough to show that $A^{\oplus n}$ is Noetherian. Prove by induction in n. Clearly, A is Noetherian. $A^{\oplus(n-1)} \subset A^{\oplus n}$. The quotient $A^{\oplus n}/A^{\oplus(n-1)} \cong A$ by $(a_1, \ldots, a_n) \mapsto a_n$. By Proposition 10.5 $A^{\oplus(n-1)}$ and A Noetherian implies that $A^{\oplus n}$ is Noetherian too. $\square$

Corollary 10.7. Let A be a ring and let M be an A-module. Suppose that we have $0 = M_0 \subset \ldots M_n = M$ are A-submodules of M. Then M is Noetherian or Artinian if and only if each quotient M_{i+1}/M_i is Noetherian or Artinian.

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Proof. Use Proposition 10.5. □

Lemma 10.8. Let A be a Noetherian ring. Let $S \subset A$ be a multiplicative set. Then $S^{-1}A$ is Noetherian.

Proof. Consider a non-empty set Σ of ideals of $S^{-1}A$. There is a canonical homomorphism of rings $f : A \to S^{-1}A$ by $f(a) = a/1$. If I is an ideal of $S^{-1}A$, then $f^{-1}(I)$ is an ideal in A. Then $I = S^{-1}f^{-1}(I)$. Now Σ gives a non-empty set of ideals of A under $I \to f^{-1}(I)$. Let J be a maximal element of this set. Then $S^{-1}J$ is a maximal element of Σ. Hence $S^{-1}A$ is Noetherian. □

11 Primary decomposition

Definition 11.1. An ideal Q in a ring R not equal to R, that is a proper ideal, is called **primary** if all $x, y \in R$ such that $xy \in Q$ we have $x \in Q$ or $y^n \in Q$ for some n.

Example. Let p be a prime number. Then (p^m) for $m \geq 1$ is a primary ideal in $\mathbb{Z}$. $ab \in (p^m)$ if and only if $p^m \mid ab$. Consider a. If $p \nmid a$, then $p^m \mid b$, hence $b \in (p^m)$. Otherwise $p \mid a$, then $p^m \mid a^m$, so $a^m \in (p^m)$.