

M3P21 Geometry II: Algebraic Topology

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$$\begin{array}{ccccccc}
\begin{array}{ccccccc}
\vdots & & 0 & & \vdots & & 0 \\
\vdots & \xrightarrow{\partial} & \downarrow & \xrightarrow{\alpha} & \downarrow & \xrightarrow{\alpha} & \downarrow \\
\vdots & \xrightarrow{\partial} & A_{n+1} & \xrightarrow{\partial} & A'_{n+1} & \xrightarrow{\partial} & A'_n \\
\vdots & \xrightarrow{\partial} & \downarrow i & \xrightarrow{\beta} & \downarrow i & \xrightarrow{\beta} & \downarrow i \\
\vdots & \xrightarrow{\partial} & B_{n+1} & \xrightarrow{\partial} & B'_{n+1} & \xrightarrow{\partial} & B'_n \\
\vdots & \xrightarrow{\partial} & \downarrow j & \xrightarrow{\gamma} & \downarrow j & \xrightarrow{\gamma} & \downarrow j \\
\vdots & \xrightarrow{\partial} & C_{n+1} & \xrightarrow{\partial} & C'_{n+1} & \xrightarrow{\partial} & C'_n \\
\vdots & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & 0 & & 0 & & 0
\end{array} & \Downarrow & \begin{array}{ccccccc}
\vdots & \xrightarrow{\partial} & H_{n+1}(A) & \xrightarrow{i_*} & H_{n+1}(B) & \xrightarrow{j_*} & H_{n+1}(C) \\
\vdots & \xrightarrow{\partial} & \downarrow \alpha_* & \xrightarrow{i_*} & \downarrow \beta_* & \xrightarrow{j_*} & \downarrow \gamma_* \\
\vdots & \xrightarrow{\partial} & H_{n+1}(A') & \xrightarrow{i_*} & H_{n+1}(B') & \xrightarrow{j_*} & H_{n+1}(C') \\
\vdots & & \downarrow \alpha_* & \xrightarrow{i_*} & \downarrow \beta_* & \xrightarrow{j_*} & \downarrow \gamma_* \\
\vdots & & H_n(A) & \xrightarrow{i_*} & H_n(B) & \xrightarrow{j_*} & H_n(C) \\
\vdots & & \downarrow \alpha_* & \xrightarrow{i_*} & \downarrow \beta_* & \xrightarrow{j_*} & \downarrow \gamma_* \\
\vdots & & H_n(A') & \xrightarrow{i_*} & H_n(B') & \xrightarrow{j_*} & H_n(C') \\
\vdots & & \downarrow \alpha_* & \xrightarrow{i_*} & \downarrow \beta_* & \xrightarrow{j_*} & \downarrow \gamma_* \\
\vdots & & H_{n-1}(A) & \xrightarrow{i_*} & H_{n-1}(B) & \xrightarrow{j_*} & H_{n-1}(C) \xrightarrow{\partial} \dots \\
\vdots & & \downarrow \alpha_* & \xrightarrow{i_*} & \downarrow \beta_* & \xrightarrow{j_*} & \downarrow \gamma_* \\
\vdots & & H_{n-1}(A') & \xrightarrow{i_*} & H_{n-1}(B') & \xrightarrow{j_*} & H_{n-1}(C') \xrightarrow{\partial} \dots
\end{array}
\end{array}$$

Syllabus

Homotopy and homotopy type. Cell complexes. Basic constructions of the fundamental group. Seifert-van Kampen theorem. Covering spaces. Δ -complexes. Simplicial homology. Singular homology. Homotopy invariance. Exact sequences and excision. Mayer-Vietoris sequences. Degree.

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0 Introduction

Lecture 1
Friday
11/01/19

0.1 Introduction

Combines topological spaces with algebraic objects, which are groups.

- How to show that a torus is not homeomorphic to a sphere?
- How to show that $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n \neq m$?

We will follow chapter one and two from

- A Hatcher, Algebraic topology, 2002

The following are prerequisites.

- Point set topology. Topological spaces, continuous maps, product and quotient topologies, Hausdorff spaces, etc.
- Basic group theory. Normal subgroups and quotients, isomorphism theorems, free groups, presentation of groups, etc.

0.2 Some underlying geometric notions

0.2.1 Homotopy and homotopy type

Let X, Y be topological spaces and $I = [0, 1]$.

Definition. A **homotopy** is a continuous map $F : X \times I \rightarrow Y$. For every $t \in I$ we obtain a continuous map

$$\begin{aligned} f_t &: X \longrightarrow Y \\ x &\longmapsto f_t(x) = F(x, t) \end{aligned}$$

Definition. Two continuous maps $f_0, f_1 : X \rightarrow Y$ are **homotopic** if there exists a homotopy $F : X \times I \rightarrow Y$ such that

$$f_0(x) = F(x, 0), \quad f_1(x) = F(x, 1), \quad x \in X.$$

We write $f_0 \cong f_1$. (Exercise: this is an equivalence relation)

Definition. Let $A \subseteq X$ be a subspace. A **retraction** of X onto A is a continuous map $r : X \rightarrow A$ such that

- $r(X) = A$, and
- $r|_A = id_A$.

Example. If $X \neq \emptyset$, $p \in X$, then X retracts to p by the constant map $X \rightarrow \{p\}$.

Definition. A **deformation retraction** of X onto $A \subseteq X$ is a retraction that is homotopic to the identity. That is, there is a continuous map

$$\begin{aligned} F &: X \times I \longrightarrow A \\ (x, t) &\longmapsto f_t(x) \end{aligned}$$

such that $f_0 = id_X$ and $f_1 : X \rightarrow A$ is the deformation retraction.

Example. The closed n -dimensional n -disc $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ deformation retracts to $(0, \dots, 0) \in \mathbb{R}^n$. Let $f_t(x) = t \cdot x$. $t = 1$ implies that $f_1 = id_{D^n}$ and $t = 0$ implies that $f_0 : D^n \rightarrow (0, \dots, 0)$.

Example. Let S^n be the n -sphere, $\partial D^{n+1} = S^n = \{x \in \mathbb{R}^n \mid |x| = 1\}$. The cylinder $S^n \times I$ deformation retracts to $S^n \times \{0\}$, by defining $f_t(x, r) = (x, t \cdot r)$.

An observation is that if X is a topological space, and $f : X \rightarrow \{p\}$ for $p \in X$ is a deformation retraction of X to p , then X is path-connected. Indeed, if $F : X \times I \rightarrow X$ is a homotopy from id_X to f and $x \in X$ is a point, then this gives a path

$$\begin{aligned} I &\longrightarrow X \\ t &\longmapsto F(x, t) \end{aligned}$$

that connects x to p . This implies that not all retractions are deformation retractions.

Example. A retraction that is not a deformation retraction. Take a space that is not path-connected and retract it to a point. Let $X = \{0, 1\}$ with discrete topology. $x \mapsto 0$ is a retraction, but not a deformation retraction because X is not path-connected.

Definition. A continuous map $f : X \rightarrow Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \rightarrow X$ such that $fg \cong id_Y$ and $gf \cong id_X$. If there exists a homotopy equivalence between X and Y , X and Y are **homotopy equivalent** or they have the same **homotopy type**.

Lemma 0.1. A deformation retraction $f : X \rightarrow A$ is a homotopy equivalence.

Proof. Let $i : A \hookrightarrow X$ be the inclusion map. Then $fi = id_A$ and $if = f \cong id_X$ by definition. □

Example. The disc with two holes is equivalent to $O \cdot O$.

Example. \mathbb{R}^n deformation retracts to a point, by $f_t(x) = t \cdot x$.

Definition.

- X is **contractible** if it is homotopy equivalent to a point.
- A continuous map is **nullhomotopic** if it is homotopy equivalent to a constant map.

0.2.2 Cell complexes

Example. The torus $S^1 \times S^1$ is the union of a point, two open intervals, and the open disc $\text{Int}(D^2)$.

These are called **cells**. Can think of discs D^n glued together.

Definition. A **CW-complex**, or **cell complex**, is a topological space X such that there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X^n,$$

where the X^n are constructed inductively in the following way.

- X^n is a discrete set.
- For each $n \geq 0$ there is an collection of closed n -discs $\{D_\alpha^n\}$ together with continuous maps $\phi_\alpha : \partial D_\alpha^n \rightarrow X^{n-1}$, such that

$$X^n = \frac{X^{n-1} \sqcup \bigsqcup_\alpha D_\alpha^n}{\sim},$$

where $x \sim \phi_\alpha(x)$ for all $x \in \partial D_\alpha^n$ for all α .

- A subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n .

Remark.

- As a set,

$$X^n = X^{n-1} \sqcup \bigsqcup_\alpha e_\alpha^n,$$

where each e_α^n is homeomorphic to an open n -disc. These e_α^n are called the **n -cells** of X .

- If $X = X^m$ for some m , then X is called **finite dimensional**. The minimal m such that $X = X^m$ is the **dimension** of X .

Example. The following are CW-complexes.

$$[0, 1], \quad \mathbb{R}, \quad S^1, \quad \text{a graph}, \quad S^n = D^n / \partial D^n.$$

Can also decompose CW-complexes.

- The sphere S^2 is one 0-cell, one 1-cell, and two 2-cells.
- The torus $S^1 \times S^1$ is one 0-cell, two 1-cells, and one 2-cell.
- The Möbius strip is two 0-cells, three 1-cells, and one 2-cell.
- The Klein bottle is one 0-cell, two 1-cells, and one 2-cell.

Definition. If X is a CW-complex with finitely many cells the **Euler characteristic** $\chi(X)$ of X is the number of even cells minus the number of odd cells.

Fact. $\chi(X)$ does not depend of the choice of cells decomposition.

Example.

- $\chi(S^n) = 0$ if n is odd and $\chi(S^n) = 2$ if n is even.
- $\chi(S^1 \times S^1) = 0$.

This is the generalisation of the following observation by Leonhard Euler. Let P be a convex polyhedron, where V is the number of vertices of P , E is the number of edges of P , and F is the number of faces of P . Then $V - E + F = 2$.

Example. A topological space that is not a CW-complex. $X = \{0, 1\}$ with trivial topology does not contain any closed points.

Fact. CW-complexes are always Hausdorff.

Lecture 2
Tuesday
15/01/19

1 The fundamental group

1.1 Basic constructions

1.1.1 Paths and homotopy

Let X be a topological space. A **path** is a continuous map $f : I \rightarrow X$, where $I = [0, 1]$.

Definition. Two paths f_0, f_1 are **homotopic** if there exists a homotopy between f_0 and f_1 preserving the endpoints, that is a continuous map

$$\begin{aligned} F & : I \times I \longrightarrow X \\ (s, t) & \longmapsto f_t(s) \end{aligned} ,$$

such that

$$\begin{aligned} f_t(0) &= f_0(0), & f_t(1) &= f_0(1), & t &\in I, \\ F(s, 0) &= f_0(s), & F(s, 1) &= f_1(s), & s &\in I. \end{aligned}$$

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set. Then all the paths in X are homotopic if they have the same endpoints. Let $f_0, f_1 : I \rightarrow X$ be two paths such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$. Define

$$f_t(s) = (1 - t)f_0(s) + tf_1(s).$$

Lemma 1.1. *Being homotopic is an equivalence relation on the set of paths with fixed endpoints. We will write $f_0 \cong f_1$ for two homotopic paths f_0 and f_1 .*

Proof.

- f is homotopic to f .
- If f_0 is homotopic to f_1 by a homotopy f_t , then f_1 is homotopic to f_0 by the homotopy f_{1-t} .
- If f_0 is homotopic to f_1 by a homotopy f_t and $f_1 = g_0$ is homotopic to g_1 by a homotopy g_t , then f_0 is homotopic to g_1 by the homotopy

$$h_t = \begin{cases} f_{2t} & 0 \leq t \leq \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Then

$$\begin{aligned} H & : I \times I \longrightarrow X \\ (s, t) & \longmapsto h_t(s) \end{aligned}$$

is continuous because its restriction to the closed subsets $I \times [0, 1/2]$ and $I \times [1/2, 1]$ is continuous, since if the restriction to two closed subsets is continuous then the restriction to the union of these subsets is continuous.

□

Let X be a topological space and $I = [0, 1]$. If $f : I \rightarrow X$ is a path, $[f]$ is the class of all paths on X homotopic to f .

Definition. Let $f, g : I \rightarrow X$ be two paths such that $f(1) = g(0)$. The **product path** $f \cdot g$ is the path

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}.$$

A convention is that whenever we write $f \cdot g$ we implicitly assume $f(1) = g(0)$.

Lemma 1.2. Let f_0, f_1, g_0, g_1 be paths on X such that $f_1 \cong f_0$ and $g_0 \cong g_1$. Then $f_0 \cdot g_0 \cong f_1 \cdot g_1$.

Proof.

$$\begin{aligned} I \times I &\longrightarrow X \\ (s, t) &\longmapsto (f_t \cdot g_t)(s) \end{aligned}$$

is a homotopy between $f_0 \cdot g_0$ and $f_1 \cdot g_1$. □

Remark. Let $\phi : [0, 1] \rightarrow [0, 1]$ be continuous such that $\phi(0) = 0$ and $\phi(1) = 1$. If $f : I \rightarrow X$ is a path, then $f \circ \phi \cong f$. This is a **reparametrisation**. Define $\phi_t(s) = (1-t)\phi(s) + ts$, then $f \circ \phi_t$ is a homotopy between $f \circ \phi$ and f .

For $x \in X$, let the **constant path** at x be

$$\begin{aligned} c_x &: I \longrightarrow X \\ s &\longmapsto x \end{aligned}.$$

For a path $f : I \rightarrow X$, define

$$\begin{aligned} f^{-1} &: I \longrightarrow X \\ s &\longmapsto f(1-s) \end{aligned}.$$

Lemma 1.3. Let $f, g, h : I \rightarrow X$ be paths. Then

1. $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$,
2. $f \cdot c_{f(1)} \cong f$ and $c_{f(0)} \cdot f \cong f$, and
3. $f \cdot f^{-1} \cong c_{f(0)}$ and $f^{-1} \cdot f \cong c_{f(1)}$.

Proof.

1. $((f \cdot g) \cdot h) \phi = f \cdot (g \cdot h)$, where

$$\phi(s) = \begin{cases} \frac{s}{2} & s \in [0, \frac{1}{2}] \\ s - \frac{1}{4} & s \in [\frac{1}{2}, \frac{3}{4}] \\ 2s - 1 & s \in [\frac{3}{4}, 1] \end{cases},$$

so $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$ by reparametrisation.

2. Again reparametrisation, by

$$\psi(s) = \begin{cases} 2s & s \in [0, \frac{1}{2}] \\ 1 & s \in [\frac{1}{2}, 1] \end{cases}, \quad \chi(s) = \begin{cases} 0 & s \in [0, \frac{1}{2}] \\ 2s - 1 & s \in [\frac{1}{2}, 1] \end{cases}.$$

3. Define

$$H(s, t) = \begin{cases} f(\max\{1-2s, t\}) & s \in [0, \frac{1}{2}] \\ f(\max\{2s-1, t\}) & s \in [\frac{1}{2}, 1] \end{cases}.$$

H is continuous, and

$$H(s, 0) = f^{-1} \cdot f, \quad H(s, 1) = c_{f(1)}.$$

The inverse is similar. □

Definition. A **loop** with **basepoint** $x_0 \in X$ is a path $f : I \rightarrow X$ such that $f(0) = f(1) = x_0$.

Definition. Denote by $\pi_1(X, x_0)$ the set of **homotopy classes** $[f]$ of loops $f : I \rightarrow X$ with basepoint x_0 .

Proposition 1.4. $\pi_1(X, x_0)$ is a group with product $[f][g] = [f \cdot g]$ and neutral element $c_{x_0} : I \rightarrow X$, the constant path at x_0 .

Proof. Follows directly from Lemma 1.2 and Lemma 1.3. \square

Definition. $\pi_1(X, x_0)$ is the **fundamental group** of X at x_0 .

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set and $x_0 \in X$. Then $\pi_1(X, x_0) = 0$, since X is convex, so all loops are homotopic to each other.

Example.

- The fundamental group of a space X with the trivial topology is trivial, since X is simply-connected, because all maps $f : I \rightarrow X$ are continuous, so path-connected and all paths are homotopic.
- The fundamental group of a space X with the discrete topology is trivial, since $f : I \rightarrow X$ is continuous implies that f is constant.

Assume $x_0, x_1 \in X$ such that x_0 and x_1 are in the same path-component of X . Let $h : I \rightarrow X$ be a path such that $h(0) = x_0$ and $h(1) = x_1$. Define

$$\begin{aligned} \beta_h : \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_0) \\ [f] &\longmapsto [h \cdot f \cdot h^{-1}] \end{aligned}$$

This is well-defined by Lemma 1.2.

Proposition 1.5. $\beta_h : \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$ is an isomorphism.

Proof. It is a homomorphism.

$$\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot h^{-1}] = [h \cdot f \cdot h^{-1}] [h \cdot g \cdot h^{-1}] = \beta_h[f] \cdot \beta_h[g],$$

and $\beta_h[c_{x_1}] = [c_{x_1}]$. It is bijective with $(\beta_h)^{-1} = \beta_{h^{-1}}$. \square

If X is path-connected, we often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Definition. X is **simply-connected** if it is path-connected and $\pi_1(X) = 0$.

Proposition 1.6. X is simply-connected if and only if there exists a unique homotopy class of paths between any two points of X .

Proof.

\implies There exists a path between any two points. Let f, g be two paths from x_0 to x_1 for $x_0, x_1 \in X$. Then $f \cdot g^{-1} \cong g \cdot g^{-1}$, so

$$f \cong f \cdot g^{-1} \cdot g \cong g \cdot g^{-1} \cdot g \cong g.$$

\impliedby X is path-connected. Then $x_1 = x_0$, so all loops at x_0 are homotopic to each other, so $\pi_1(X) = 0$. \square

1.1.2 The fundamental group of the circle

The goal is to show that $\pi_1(S^1) \cong \mathbb{Z}$.

Lecture 4
Friday
18/01/19

Definition. A **covering space** of a space X is a space \tilde{X} and a continuous map $p : \tilde{X} \rightarrow X$ such that for each $x \in X$ there is an open $U \subseteq X$ such that

- $p^{-1}(U) = \bigcup_{j \in J} \tilde{U}_j$, where $\tilde{U}_j \subseteq \tilde{X}$ is open,
- $\tilde{U}_i \cap \tilde{U}_j = \emptyset$ if $i \neq j$, and
- $p|_{\tilde{U}_j} : \tilde{U}_j \rightarrow U$ is a homeomorphism for all $j \in J$.

Such a U is called **evenly covered**. The \tilde{U}_j are called **sheets**.

Example.

$$\begin{array}{ccc} p & : & \mathbb{R} \longrightarrow S^1 \\ s & \longmapsto & (\cos(2\pi s), \sin(2\pi s)) \end{array} .$$

Definition. Let $p : \tilde{X} \rightarrow X$ be a covering space. A **lift** of a continuous map $f : Y \rightarrow X$ is a continuous map $\tilde{f} : Y \rightarrow \tilde{X}$ such that $p\tilde{f} = f$, so

$$\begin{array}{ccc} & & \tilde{X} \\ & \nearrow \tilde{f} & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

Proposition 1.7 (Unique lifting property). *Let $p : \tilde{X} \rightarrow X$ be a covering space and $f : Y \rightarrow X$ be a continuous map. If there are two lifts $\tilde{f}_1, \tilde{f}_2 : Y \rightarrow \tilde{X}$ of f such that $\tilde{f}_1(y) = \tilde{f}_2(y)$ for some $y \in Y$ and if Y is connected, then $\tilde{f}_1 = \tilde{f}_2$.*

Proof. Let $y \in Y$ and $U \subseteq X$ be an evenly covered neighbourhood of $f(y)$. Then

$$p^{-1}(U) = \bigcup_j \tilde{U}_j.$$

Let \tilde{U}_1 be the sheet such that $\tilde{f}_1(y) \in \tilde{U}_1$, and let \tilde{U}_2 be the sheet such that $\tilde{f}_2(y) \in \tilde{U}_2$. Let $N \subseteq Y$ be open and $y \in N$ such that $\tilde{f}_1(N) \subseteq \tilde{U}_1$ and $\tilde{f}_2(N) \subseteq \tilde{U}_2$. We have $p\tilde{f}_1 = p\tilde{f}_2$. Then $\tilde{f}_1(y) = \tilde{f}_2(y)$ if and only if $\tilde{U}_1 = \tilde{U}_2$, if and only if $\tilde{f}_1|_N = \tilde{f}_2|_N$. Let

$$A = \{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\},$$

so A is open and $Y \setminus A$ is open. Thus $A \neq \emptyset$ implies that $A = Y$. □

Proposition 1.8 (Homotopy lifting property). *Let $p : \tilde{X} \rightarrow X$ be a covering space and $F : Y \times I \rightarrow X$ be a continuous map such that there exists a lift $\tilde{f}_0 : Y \times \{0\} \rightarrow \tilde{X}$ of $F|_{Y \times \{0\}}$. Then there is a unique lift $\tilde{F} : Y \times I \rightarrow \tilde{X}$ of F such that $\tilde{F}|_{Y \times \{0\}} = \tilde{f}_0$.*

Proof. Let $y_0 \in Y$ and $t \in I$. There are open $y_0 \in N_t \subseteq Y$ and $t \in (a_t, b_t) \subseteq I$ such that $F(N_t \times (a_t, b_t)) \subseteq U \subseteq X$, where $U \subseteq X$ is open and evenly covered. Compactness of I implies that there exist

$$0 = t_0 < \dots < t_m = 1,$$

and there exists $y_0 \in N \subseteq Y$ open such that $F(N \times [t_i, t_{i+1}]) \subseteq U_i \subseteq X$, where $U_i \subseteq X$ is open and evenly covered. We inductively construct a lift $\tilde{F}|_{N \times I}$ of $F|_{N \times I}$.

- $\tilde{F}|_{N \times [0,0]} = \tilde{f}_0|_{N \times [0,0]}$ exists.
- Assume the lift has been constructed on $N \times [0, t_i]$. Let $\tilde{U}_i \subseteq \tilde{X}$ be such that $p|_{\tilde{U}_i} : \tilde{U}_i \rightarrow U_i$ such that $\tilde{F}(y_0, t_i) \subseteq \tilde{U}_i$. After shrinking N , may assume $\tilde{F}(N \times \{t_i\}) \subseteq \tilde{U}_i$. Define \tilde{F} on $N \times [t_i, t_{i+1}]$ to be composition of F with the homeomorphism $p^{-1} : U_i \rightarrow \tilde{U}_i$.

After finitely many steps we obtain a lift $\tilde{F} : N \times I \rightarrow \tilde{X}$, where $y_0 \in N \subseteq Y$ is open, so for each $y \in Y$ there is a neighbourhood $N_y \subseteq Y$ such that $F|_{N_y \times I} : N_y \times I \rightarrow X$ lifts. For all $y \in Y$, $\{y\} \times I$ is connected and can be lifted, so Proposition 1.7 implies that the lift of $N \times I$ is unique. Thus there is a unique lift $\tilde{F} : Y \times I \rightarrow \tilde{X}$. \square

Example. Let X be a topological space and A be discrete. Then $p : X \times A \rightarrow X$ is a covering space. This is the **trivial covering**. (Exercise: show the unique lifting property and the homotopy lifting property for the trivial covering)

Corollary 1.9. Let $f : I \rightarrow X$ be a path, $f(0) = x_0$, and $p : \tilde{X} \rightarrow X$ be a covering space. For each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \rightarrow \tilde{X}$ such that $\tilde{f}(0) = \tilde{x}_0$.

Proof. Proposition 1.8 for Y a point. \square

Theorem 1.10. Let $x_0 = (1, 0) \in S^1$. $\pi_1(S^1, x_0)$ is the infinite cyclic group generated by the homotopy class of the loop

$$\begin{aligned} \omega & : I \longrightarrow S^1 \\ s & \longmapsto (\cos(2\pi s), \sin(2\pi s)) \end{aligned} .$$

Remark.

- $[\omega]^n = [\omega_n]$, where

$$\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns)) .$$

-

$$\begin{aligned} p & : \mathbb{R} \longrightarrow S^1 \\ s & \longmapsto (\cos(2\pi s), \sin(2\pi s)) \end{aligned}$$

is a covering space.

- ω_n lifts to

$$\begin{aligned} \tilde{\omega}_n & : I \longrightarrow \mathbb{R} \\ s & \longmapsto ns \end{aligned} ,$$

such that $\tilde{\omega}_n(0) = 0$ and $\tilde{\omega}_n(1) = n$.

Proof of Theorem 1.10.

- If $f : I \rightarrow S^1$ be a loop at x_0 , then the homotopy lifting property implies that there exists a lift $\tilde{f} : I \rightarrow \mathbb{R}$ such that $\tilde{f}(0) = 0$. Since $p(\tilde{f}(1)) = f(1) = x_0$, then $\tilde{f}(1) = n$ for some $n \in \mathbb{Z}$. $\tilde{\omega}_n : I \rightarrow \mathbb{R}$ is another path such that $\tilde{\omega}_n(0) = 0$ and $\tilde{\omega}_n(1) = n$, so $\tilde{f} \cong \tilde{\omega}_n$. Let $F : I \times I \rightarrow \mathbb{R}$ be a homotopy equivalence between \tilde{f} and $\tilde{\omega}_n$. Then $pF : I \times I \rightarrow S^1$ gives a homotopy between $p\tilde{f} = f$ and $p\tilde{\omega}_n = \omega_n$.
- Let $m, n \in \mathbb{Z}$ and assume $\omega_m \cong \omega_n$. Let $F : I \times I \rightarrow S^1$ be a homotopy. Then

$$F(0, t) = \omega_m(t), \quad F(1, t) = \omega_n(t), \quad F(s, 0) = F(s, 1) = x_0, \quad s, t \in I.$$

The unique lifting property implies that $\tilde{\omega}_n, \tilde{\omega}_m : I \rightarrow \mathbb{R}$ are unique lifts such that $\tilde{\omega}_n(0) = 0 = \tilde{\omega}_m(0)$. The homotopy lifting property implies that F lifts uniquely to a homotopy $\tilde{F} : I \times I \rightarrow \mathbb{R}$ between $\tilde{\omega}_n$ and $\tilde{\omega}_m$, and $\tilde{F}(s, 1) \in \mathbb{Z}$ for all $s \in I$. Thus $\tilde{F}(s, 1) = n = m$, so $\omega_m \cong \omega_n$ if and only if $n = m$. \square

Lecture 5 is a problem class.

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Theorem 1.11. *Every non-constant polynomial $p \in \mathbb{C}[z]$ has a root in \mathbb{C} .*

Proof. May assume

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n.$$

Assume p has no roots in \mathbb{C} . For each $r \in \mathbb{R}_{\geq 0}$ we obtain a loop

$$\begin{aligned} f_r &: I \longrightarrow \mathbb{C} \\ s &\longmapsto \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}, \end{aligned}$$

so $|f_r(s)| = 1$. $f_r(0) = 1$ and $f_r(1) = 1$, so f_r is a loop based at 1. f_0 is the constant loop at 1. $f_r(s)$ depends continuously on r , so $f_r \cong f_0$ for all $r \in \mathbb{R}_{\geq 0}$ and $[f_r] = [f_0] = 0 \in \pi_1(S^1)$. Fix $r \in \mathbb{R}_{\geq 0}$ such that $r > 1$ and $r > |a_1| + \cdots + |a_n|$. For $|z| = r$ we have

$$|z^n| > (|a_1| + \cdots + |a_n|)|z^{n-1}| \geq |a_1 z^{n-1}| + \cdots + |a_n| \geq |a_1 z^{n-1} + \cdots + a_n|.$$

Hence, for $0 \leq t \leq 1$ the polynomial

$$p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$$

has no root z with $|z| = r$. Define

$$F_r(t, s) = \frac{p_t(re^{2\pi i s})/p_t(r)}{|p_t(re^{2\pi i s})/p_t(r)|}.$$

$F_r(0, s) = \omega_n(s)$ and $F_r(1, s) = f_r(s)$, so $[\omega_n] = [f_r] = 0 \in \pi_1(S^1)$. Theorem 1.10 implies that $n = 0$, so p is constant. \square

See Hatcher Theorem 1.9 and Theorem 1.10 for more applications.

Proposition 1.12. *Let X, Y be topological spaces, $x_0 \in X$, and $y_0 \in Y$. Then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. A map

$$\begin{aligned} f &: Z \longrightarrow X \times Y \\ z &\longmapsto (g(z), h(z)) \end{aligned}$$

is continuous if and only if $g: Z \rightarrow X$ and $h: Z \rightarrow Y$ are continuous. For $Z = I$,

$$\{ \text{loops in } X \times Y \text{ based } (x_0, y_0) \} \quad \longleftrightarrow \quad \{ \text{loops in } X \text{ based } x_0 \} \times \{ \text{loops in } Y \text{ based } y_0 \}.$$

Two loops

$$\begin{aligned} f_1 &: I \longrightarrow X \times Y & f_2 &: I \longrightarrow X \times Y \\ s &\longmapsto (g_1(s), h_1(s)) & s &\longmapsto (g_2(s), h_2(s)) \end{aligned}$$

are homotopic if and only if $g_1 \cong g_2$ and $h_1 \cong h_2$, so there is a bijection

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

$f_1 \cdot f_2 = (g_1 \cdot g_2, h_1 \cdot h_2)$ and the constant loop is mapped to the constant loop, so this is also a group isomorphism. \square

Example. The torus $S^1 \times S^1$ has

$$\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z}^2.$$

1.1.3 Induced homomorphisms

Let X, Y be topological spaces, $x_0 \in X$, and $\phi : X \rightarrow Y$. An observation is that ϕ induces a homomorphism

$$\begin{aligned} \phi_* : \pi_1(X, x_0) &\longrightarrow \pi_1(Y, \phi(x_0)) \\ [f] &\longmapsto [\phi f] \end{aligned} .$$

ϕ_* is well-defined, since if f_t is a homotopy between the loops f_0 and f_1 based at x_0 , then ϕf_t is a homotopy of loops between ϕf_0 and ϕf_1 . Moreover, $\phi(f \cdot g) = (\phi f) \cdot (\phi g)$ and ϕ maps the constant path at x_0 to the constant path at $\phi(x_0)$, so ϕ is a homomorphism.

Proposition 1.13.

1. Let $\psi : X \rightarrow Y$ and $\phi : Y \rightarrow Z$ be continuous maps between topological spaces, $x_0 \in X$, and

$$\begin{aligned} \psi_* : \pi_1(X, x_0) &\rightarrow \pi_1(Y, \psi(x_0)), & \phi_* : \pi_1(Y, \psi(x_0)) &\rightarrow \pi_1(Z, \phi\psi(x_0)), \\ (\phi\psi)_* : \pi_1(X, x_0) &\rightarrow \pi_1(Z, \phi\psi(x_0)). \end{aligned}$$

Then $(\phi\psi)_* = \phi_*\psi_*$.

2. Let $id_X : X \rightarrow X$ be the identity then

$$(id_X)_* : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$$

is the identity.

Proof.

1. Let $f : I \rightarrow X$ be a loop at x_0 , then

$$(\phi\psi)_*([f]) = [(\phi\psi)f] = [\phi(\psi f)] = \phi_*([\psi f]) = \phi_*\psi_*([f]).$$

2. $(id_X)_*([f]) = [id_X f] = [f]$.

□

These two observations yield in particular that if $\phi : X \rightarrow Y$ is a homeomorphism with inverse $\psi : Y \rightarrow X$, then

$$\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \phi(x_0))$$

is an isomorphism with inverse ψ_* .

Proposition 1.14. Let $\phi : X \rightarrow Y$ be a homotopy equivalence. Then

$$\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \phi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Recall that if $x_0, x_1 \in X$ and $h : I \rightarrow X$ is a path such that $h(0) = x_0$ and $h(1) = x_1$, then we obtain an isomorphism

$$\begin{aligned} \beta_h : \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_0) \\ [f] &\longmapsto [h \cdot f \cdot h^{-1}] \end{aligned} .$$

Lemma 1.15. Let $\phi_t : X \rightarrow Y$ be a homotopy and $x_0 \in X$. Define the path

$$\begin{aligned} h : I &\longrightarrow Y \\ s &\longmapsto \phi_s(x_0) \end{aligned} , \quad h(0) = \phi_0(x_0), \quad h(1) = \phi_1(x_0) .$$

Then $\phi_{0*} = \beta_h \phi_{1*}$, that is the following diagram commutes.

$$\begin{array}{ccc} & \pi_1(Y, \phi_1(x_0)) & \\ \phi_{1*} \nearrow & \downarrow \sim \beta_h & \\ \pi_1(X, x_0) & & \\ \phi_{0*} \searrow & \downarrow & \\ & \pi_1(Y, \phi_0(x_0)) & \end{array} .$$

Proof. For $t \in I$, define the path

$$\begin{array}{ccc} h_t & : & I \longrightarrow X \\ & s & \longmapsto h(ts) \end{array}, \quad h_t(0) = \phi_0(x_0), \quad h_t(1) = h(t) = \phi_t(x_0).$$

Let f be a loop at x_0 . Define

$$F_t = h_t \cdot (\phi_t f) \cdot h_t^{-1}.$$

Then F_t is a loop at $\phi_0(x_0)$, which is continuous in t . So F_t is a homotopy of loops between

$$F_0 = h_0 \cdot (\phi_0 f) \cdot h_0^{-1} \cong \phi_0 f, \quad F_1 = h_1 \cdot (\phi_1 f) \cdot h_1^{-1} = h \cdot (\phi_1 f) \cdot h^{-1}.$$

Hence

$$\phi_{0*}([f]) = [\phi_0 f] = [h \cdot (\phi_1 f) \cdot h^{-1}] = \beta_h([\phi_1 f]) = \beta_h \phi_{1*}([f]).$$

□

Lemma 1.15 implies in particular the following.

Corollary 1.16. *If $\psi : X \rightarrow X$ is continuous and $\psi \cong id_X$, then*

$$\psi_* : \pi_1(X, x_0) \rightarrow \pi_1(X, \psi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Proof. By Lemma 1.15 there is a path h from $\psi(x_0)$ to x_0 such that

$$\begin{array}{ccc} & & \pi_1(X, x_0) \\ & \nearrow (id_X)_* & \downarrow \sim \beta_h \\ \pi_1(X, x_0) & & \pi_1(X, \psi(x_0)) \\ & \searrow \psi_* & \end{array},$$

so $\psi_* = \beta_h$ hence an isomorphism. □

Proof of Proposition 1.14. Let $\phi : X \rightarrow Y$ be a homotopy equivalence. Let $\psi : Y \rightarrow X$ be a homotopy inverse of ϕ , that is $\phi\psi \cong id_Y$ and $\psi\phi \cong id_X$.

$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\phi(x_0)) \xrightarrow{\phi_*} \pi_1(Y, \psi\phi\psi(x_0)).$$

Have to show that ϕ_* is bijective. The observation above implies that $(\psi\phi)_* = \psi_*\phi_*$ is an isomorphism, so ϕ_* is injective and ψ_* is surjective. Similarly $(\phi\psi)_* = \phi_*\psi_*$ is an isomorphism, so ψ_* is injective and ϕ_* is surjective. □

Lemma 1.17. *Let X be a topological space and $x_0 \in X$. Assume*

$$X = \bigcup_{\alpha \in \Lambda} A_\alpha,$$

such that

- *the A_α are all open and path-connected,*
- *$x_0 \in A_\alpha$ for all $\alpha \in \Lambda$, and*
- *all the intersections $A_\alpha \cap A_\beta$ are path-connected for all $\alpha, \beta \in \Lambda$.*

If f is a loop in X at x_0 , then we can write

$$[f] = [h_1] \dots [h_m],$$

such that the h_i are loops at x_0 , and each contained in a single A_{α_i} .

Proof. f is continuous, so for all $s \in I$ there is an open neighbourhood V_s such that $f(V_s)$ such that $f(V_s) \subseteq A_\alpha$ for some α . We can choose V_s to be an interval (a_s, b_s) such that $f([a_s, b_s]) \subseteq A_\alpha$. I is compact, so a finite number of such intervals cover I , so there is a partition

$$0 = s_0 < \dots < s_m = 1,$$

such that $f([s_{i-1}, s_i]) \subseteq A_{\alpha_i}$ for some α_i . Let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$, and rescaling. $f \cong f_1 \cdots f_m$ for $f_i \subseteq A_{\alpha_i}$ and $A_{\alpha_i} \cap A_{\alpha_j}$ is path-connected. Let g_i be a path from x_0 to $f(s_i)$ in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$. Let g_0, g_m be the constant loops at x_0 . $h_i = g_{i-1} \cdot f_i \cdot g_i^{-1}$ is a loop based at x_0 and $h_i \subseteq A_{\alpha_i}$. Thus

$$f \cong (g_0 \cdot f_1 \cdot g_1^{-1}) \cdots (g_{m-1} \cdot f_m \cdot g_m^{-1}),$$

so

$$[f] = [h_1] \dots [h_m].$$

□

Example. Möbius strip M deformation retracts to S^1 . Thus $\phi : M \rightarrow S^1$ is a homotopy equivalence, so $\pi_1(M) \cong \pi_1(S^1) \cong \mathbb{Z}$.

Example. There is no deformation retraction of S^1 to a point $p \in S^1$ because $\pi_1(S^1) \not\cong \pi_1(p)$.

Example. There is no retraction of the disc D^2 to its boundary $S^1 \subseteq D^2$. Assume there is a retraction $r : D^2 \rightarrow S^1$, consider the embedding $i : S^1 \hookrightarrow D^2$. Then $ri = id_{S^1}$. Thus

$$\begin{array}{ccccc} \pi_1(S^1) & \xrightarrow{i_*} & \pi_1(D^2) & \xrightarrow{r_*} & \pi_1(S^1) \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\ & & 0 & & \end{array},$$

so $r_*i_*(\pi_1(S^1)) = 0$ but $r_*i_* = (ri)_* = id_{\pi_1(S^1)}$, a contradiction.

Theorem 1.18 (Brouwer fixed point theorem). *Let $h : D^2 \rightarrow D^2$ be a continuous map. Then h has a fixed point, that is there exists $x \in D^2$ such that $h(x) = x$.*

Proof. Assume $h(x) \neq x$ for all $x \in D^2$. Define $r : D^2 \rightarrow S^1$ by defining $r(x)$ to be the intersection of the ray starting at $h(x)$ towards x with S^1 . r is continuous, and if $x \in S^1$, then $r(x) = x$, so r is a retraction, a contradiction. □

Lemma 1.17 implies that if $U_1, U_2 \subseteq X$ are open and path-connected such that $U_1 \cup U_2 = X$ and $U_1 \cap U_2$ is path-connected and $x_0 \in U_1 \cap U_2$, then every $[f] \in \pi_1(X, x_0)$ can be factorised as

$$[f] = [g_1][h_1] \dots [g_n][h_n],$$

such that the g_i are loops at x_0 contained in U_1 and the h_i are loops at x_0 contained in U_2 . In other words, $i_1 : U_1 \hookrightarrow X$ and $i_2 : U_2 \hookrightarrow X$, so

$$i_{1*} : \pi_1(U_1, x_0) \rightarrow \pi_1(X, x_0), \quad i_{2*} : \pi_1(U_2, x_0) \rightarrow \pi_1(X, x_0).$$

Lemma 1.17 implies that $i_{1*}(\pi_1(U_1, x_0)) \cup i_{2*}(\pi_1(U_2, x_0))$ generate $\pi_1(X, x_0)$.

Proposition 1.19. $\pi_1(S^n) = 0$ if $n \geq 2$.

Proof. Let

$$U_1 = S^n \setminus \{(1, 0, \dots, 0)\}, \quad U_2 = S^n \setminus \{(-1, 0, \dots, 0)\}.$$

Then $U_1 \cong \mathbb{R}^n$ and $U_2 \cong \mathbb{R}^n$, by stereographic projection. $U_1 \cup U_2 = S^n$ and $U_1 \cap U_2$ is path-connected. Let $x_0 \in U_1 \cap U_2$. $\pi_1(U_1, x_0) = 0$ and $\pi_1(U_2, x_0) = 0$, so Lemma 1.17 implies that $\pi_1(S^n, x_0) = 0$. □

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1.2 Seifert-van Kampen theorem

1.2.1 Free products with amalgamation

Definition. If S is a set, then F_S is the **free group** on S . We can write any group G as a quotient of some free group F_S , $G = F / \langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ is the **normal closure** of $R \subseteq F_S$, the smallest normal subgroup of F_S containing R . We write $G = \langle S \mid R \rangle$. This is called a **presentation** of G .

Let G_0, G_1, G_2 be groups, and $f_1 : G_0 \rightarrow G_1$ and $f_2 : G_0 \rightarrow G_2$ be homomorphisms.

Definition. A group H together with homomorphisms $h_1 : G_1 \rightarrow H$ and $h_2 : G_2 \rightarrow H$ such that $h_1 f_1 = h_2 f_2$ is an **amalgamated product** of G_1 and G_2 over G_0 if it satisfies the following universal property. For every group G and all homomorphisms $h'_1 : G_1 \rightarrow G$ and $h'_2 : G_2 \rightarrow G$ such that $h'_1 f_1 = h'_2 f_2$, there exists a unique homomorphism $\alpha : H \rightarrow G$ such that $h'_1 = \alpha h_1$ and $h'_2 = \alpha h_2$.

$$\begin{array}{ccccc}
 G_0 & \xrightarrow{f_1} & G_1 & & \\
 f_2 \downarrow & & \downarrow h_1 & \searrow h'_1 & \\
 G_2 & \xrightarrow{h_2} & H & \xrightarrow{\exists! \alpha} & G \\
 & \searrow h'_2 & & & \uparrow \\
 & & & & G
 \end{array}$$

Theorem 1.20. Given $f_1 : G_0 \rightarrow G_1$ and $f_2 : G_0 \rightarrow G_2$. Then there exists an amalgamated product, unique up to isomorphism. We denote it by $G_1 *_{G_0} G_2$.

Proof. Worksheet 2. □

$G_0 = \{id\}$ is the **free product**. We write $G_1 * G_2$ instead of $G_1 *_{\{id\}} G_2$. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Then $G_1 * G_2 = \langle S_1 \sqcup S_2 \mid R_1 \cup R_2 \rangle$, with injections $G_i \hookrightarrow G_1 * G_2$ for $i = 1, 2$. More generally,

$$G_1 * G_2 \cong \frac{G_1 *_{G_0} G_2}{N}$$

where N is the normal closure of the set

$$\left\{ f_1(g) f_2(g)^{-1} \mid g \in G_0 \right\} \subseteq G_1 * G_2.$$

1.2.2 The Seifert-van Kampen theorem

Theorem 1.21 (Seifert-van Kampen). Let X be a topological space and $U_1, U_2 \subseteq X$ be open and path-connected such that $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is path-connected and let $x_0 \in U_1 \cap U_2$. Then

$$\pi_1(X, x_0) \cong \pi_1(U_1, x_0) *_{\pi_1(U_1 \cap U_2, x_0)} \pi_2(U_2, x_0) \cong \frac{\pi_1(U_1, x_0) * \pi_1(U_2, x_0)}{N},$$

where N is the normal closure of the set

$$\left\{ j_{1*}(\omega) j_{2*}(\omega)^{-1} \mid \omega \in \pi_1(U_1 \cap U_2, x_0) \right\},$$

and $j_i : U_i \hookrightarrow X$.

$$\begin{array}{ccc}
 U_1 \cap U_2 & \xrightarrow{i_1} & U_1 \\
 i_2 \downarrow & & \downarrow j_1 \\
 U_2 & \xrightarrow{j_2} & X
 \end{array}
 \implies
 \begin{array}{ccc}
 \pi_1(U_1 \cap U_2, x_0) & \xrightarrow{i_{1*}} & \pi_1(U_1, x_0) \\
 i_{2*} \downarrow & & \downarrow j_{1*} \\
 \pi_1(U_2, x_0) & \xrightarrow{j_{2*}} & \pi_1(U_1, x_0) *_{\pi_1(U_1 \cap U_2, x_0)} \pi_1(U_2, x_0)
 \end{array}$$

Proof of Theorem 1.21. Appendix A.1. □

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Theorem 1.22 (Seifert-van Kampen, strong version). *Let X be a path-connected topological space such that*

- $X = \bigcup_{\alpha} A_{\alpha}$,
- $A_{\alpha}, A_{\alpha} \cap A_{\beta}, A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$ are open and path-connected for all α, β, γ , and
- $x_0 \in \bigcap_{\alpha} A_{\alpha}$.

Then

$$\pi_1(X, x_0) \cong \frac{{}^*\pi_1(A_{\alpha}, x_0)}{N},$$

where $N \subseteq {}^*\pi_1(A_{\alpha}, x_0)$ is the normal closure of the set

$$\left\{ (i_{\alpha\beta})_*(\omega) (i_{\beta\alpha})_*(\omega)^{-1} \mid \omega \in \pi_1(A_{\alpha} \cap A_{\beta}) \right\},$$

and $i_{\alpha\beta} : A_{\alpha} \cap A_{\beta} \hookrightarrow A_{\alpha}$ is the inclusion.

Example. Let $S^1 \vee S^1$ be the wedge product. Fix $x \in S^1$ and $y \in S^1$. Then

$$S^1 \vee S^1 = \frac{S^1 \sqcup S^1}{x \sim y} = \overset{b}{\mathbf{O}} \cdot \overset{a}{\mathbf{O}}.$$

Let

$$A = \mathbf{O} \cdot (, \quad B =) \cdot \mathbf{O}, \quad A \cap B =) \cdot (.$$

$\pi_1(A) \cong \langle b \rangle \cong \mathbb{Z}$, $\pi_1(B) \cong \langle a \rangle \cong \mathbb{Z}$, and $\pi_1(A \cap B) = \{id\}$. $A, B, A \cap B$ are open and path-connected. Van Kampen implies that

$$\pi_1(S^1 \vee S^1) \cong \pi_1(A) * \pi_1(B) \cong \mathbb{Z} * \mathbb{Z} \cong F_{\{a, b\}}.$$

More generally, let $X = S_{a_1}^1 \vee \cdots \vee S_{a_n}^1$. Induction implies that

$$\pi_1(X) = \mathbb{Z} * \cdots * \mathbb{Z} \cong F_{\{a_1, \dots, a_n\}}.$$

Similarly, let $X = \bigvee_{\alpha \in \Lambda} S_{\alpha}^1$. Strong version of van Kampen implies that

$$\pi_1(X) = \bigstar_{\alpha \in \Lambda} \mathbb{Z} = F_{\Lambda}.$$

Example. Let T be a torus and $x_0 \in T$. Let

$$A = T \setminus \{\text{small closed disc } D\}, \quad B = \{\text{open set that contains } D \text{ and } x_0\}.$$

- A is homotopy equivalent to $S^1 \vee S^1$, so $\pi_1(A) \cong F_{\{a, b\}}$.
- B is homeomorphic to D^2 , so $\pi_1(B) = \{id\}$.
- $A \cap B$ is homotopy equivalent to S^1 , so $\pi_1(A \cap B) \cong \mathbb{Z}$.

$A, B, A \cap B$ are open and path-connected. Van Kampen implies that

$$\pi_1(T) \cong \frac{\pi_1(A)}{\langle\langle i_*(\pi_1(A \cap B)) \rangle\rangle},$$

where $i : A \cap B \hookrightarrow A$. Then

$$i_* : \begin{array}{ccc} \pi_1(A \cap B) = \langle \omega \rangle & \longrightarrow & \pi_1(A) \\ \omega & \longmapsto & aba^{-1}b^{-1} \end{array},$$

so

$$\pi_1(T) \cong \frac{F_{\{a, b\}}}{\langle\langle aba^{-1}b^{-1} \rangle\rangle} = \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2.$$

1.2.3 Applications to CW-complexes

Let X be a path-connected topological space. Let Y be the space obtained by attaching 2-cells $\{e_\alpha^2\}$ to X along maps $\phi_\alpha : \partial D^2 = S^1 \rightarrow X$. Consider the loops

$$\begin{aligned} \phi'_\alpha &: I \longrightarrow X \\ s &\longmapsto \phi_\alpha(\cos(2\pi s), \sin(2\pi s)) \end{aligned}$$

based at $\phi'_\alpha(0)$. Let γ_α be a path from x_0 to $\phi'_\alpha(0)$ for each α . Then $\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}$ is a loop at x_0 . After attaching e_α^2 , the loop $\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}$ is homotopic to the constant loop at x_0 . Let $N \subseteq \pi_1(X, x_0)$ be the normal closure of all the elements of the form $[\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}]$. The inclusion $i : X \hookrightarrow Y$ yields

$$i_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, x_0),$$

and $N \subseteq \text{Ker}(i_*)$.

Proposition 1.23. *This inclusion $i : X \hookrightarrow Y$ induces a surjection*

$$i_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, x_0),$$

and $\text{Ker}(i_*) = N$, so $\pi_1(Y, x_0) \cong \pi_1(X, x_0) / N$.

Proof. Construct a space Z from Y by attaching a strip $I \times I$ to Y by identifying the lower edge $I \times \{0\}$ with the path γ_α and the right edge $\{1\} \times I$ with an arch on e_α^2 . Attach all the left edges of the strips with each other. Z deformation retracts to Y . Choose a point $y_\alpha \in e_\alpha^2$ for each α , such that y_α is not contained in X or in the attached strip. Let

$$A = Z \setminus \bigcup_\alpha \{y_\alpha\}, \quad B = Z \setminus X.$$

- A deformation retracts to X .
- B is homotopy equivalent to a point.
- $A \cap B$ is homotopy equivalent to

$$\{\text{paths } \gamma_\alpha \text{ from } x_0 \text{ to loops } \phi'_\alpha\} = \mathbf{O} \xrightarrow{\phi'_\alpha} \gamma_\alpha \cdot x_0 \xrightarrow{\gamma_\alpha} \mathbf{O}.$$

$A, B, A \cap B$ are open and path-connected. Van Kampen implies that

$$\pi_1(Y) \cong \pi_1(Z) = \frac{\pi_1(A)}{\langle\langle j_*(\pi_1(A \cap B)) \rangle\rangle},$$

where $j : A \cap B \hookrightarrow A$ is the inclusion. So $\langle\langle j_*(\pi_1(A \cap B)) \rangle\rangle$ is exactly N . Thus $\pi_1(A) = \pi_1(X)$. \square

Corollary 1.24. *For every group G there exists a two-dimensional CW-complex X_G such that $\pi_1(X_G) = G$.*

Proof. Let $G = \langle\{g_\alpha\} \mid \{r_\beta\}\rangle$ be a presentation of G , that is

$$G = \frac{F_{\{g_\alpha\}}}{\langle\langle\{r_\beta\}\rangle\rangle}.$$

Seen last time that $\pi_1\left(\bigvee_{g_\alpha} S_{g_\alpha}^1\right) = F_{\{g_\alpha\}}$. Each word r_β defines a loop in $\bigvee_{g_\alpha} S_{g_\alpha}^1$. Attach 2-cells to $\bigvee_{g_\alpha} S_{g_\alpha}^1$ along the loops defined by the relations $\{r_\beta\}$. Call this new CW-complex Y . Proposition 1.23 implies that

$$\pi_1(Y, x_0) \cong \frac{\pi_1(X, x_0)}{\langle\langle\{r_\beta\}\rangle\rangle} \cong \frac{F_{\{g_\alpha\}}}{\langle\langle\{r_\beta\}\rangle\rangle} \cong G.$$

\square

Remark. Let $X = \bigcup_n X^n$ be a CW-complex, path-connected. Proposition 1.23 can be used to show the following two facts.

- The inclusion $X^1 \hookrightarrow X$ induces a surjective homomorphism $\pi_1(X^1) \rightarrow \pi_1(X)$.
- The inclusion $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2) \rightarrow \pi_1(X)$.

1.3 Covering spaces

1.3.1 Lifting properties

Let X be a topological space. Recall that a **covering space** is $p : \tilde{X} \rightarrow X$ such that each $x \in X$ has an open neighbourhood U such that

$$p^{-1}(U) = \bigcup_{\alpha} \tilde{U}_{\alpha},$$

where U_{α} are pairwise disjoint and $p|_{\tilde{U}_{\alpha}} : \tilde{U}_{\alpha} \rightarrow U$ is a homeomorphism for all α .

Example.

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & S^1 \\ s & \longmapsto & (\cos(2\pi s), \sin(2\pi s)) \end{array}, \quad \begin{array}{ccc} S^1 & \longrightarrow & S^1 \\ z & \longmapsto & z^n \end{array}, \quad \text{O} \cdot \text{O} \cdot \text{O} \rightarrow S^1 \vee S^1 = \text{O} \cdot \text{O}.$$

Let $f : Y \rightarrow X$ be a continuous map. A **lift** of f is a continuous map $\tilde{f} : Y \rightarrow \tilde{X}$ such that $p\tilde{f} = f$, where $p : \tilde{X} \rightarrow X$ is a covering space. Let Y be connected.

- **Unique lifting property** states that if two lifts \tilde{f}_1 and \tilde{f}_2 of f coincide at one point, then they coincide on all of Y .
- **Homotopy lifting property** states that if $f_t : Y \rightarrow X$ is a homotopy and $\tilde{f}_0 : Y \rightarrow \tilde{X}$ is a lift of f_0 then there exists a unique homotopy $\tilde{f}_t : Y \rightarrow \tilde{X}$ of \tilde{f}_0 that lifts f_t .

Remark.

- If Y is a point, this is called the **path lifting property**. Let $f : I \rightarrow X$ be a path with $f(0) = x_0$. If $\tilde{x}_0 \in p^{-1}(x_0)$, then there is a unique path $\tilde{f} : I \rightarrow \tilde{X}$ lifting f and starting at \tilde{x}_0 .
- In particular, the lift of a constant path is constant.
- This implies in particular that the lift of a homotopy of paths is again a homotopy of paths. The endpoints $\tilde{f}_t(0)$ and $\tilde{f}_t(1)$ define constant paths as t varies.

Fix $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$, so

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0).$$

To every element in $\pi_1(X, x_0)$ we can associate a homotopy class of paths in \tilde{X} starting at \tilde{x}_0 .

Proposition 1.25.

1. $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ is injective.
2. $p_* \left(\pi_1(\tilde{X}, \tilde{x}_0) \right) \subseteq \pi_1(X, x_0)$ consists of the homotopy classes of loops at x_0 whose lifts to \tilde{X} starting at \tilde{x}_0 are loops.

Proof.

1. Let $\tilde{f}_0 : I \rightarrow \tilde{X}$ be a loop at \tilde{x}_0 such that $[\tilde{f}_0] \in \text{Ker}(p_*)$, so $p\tilde{f}_0 = f_0$ is homotopic to the constant loop at x_0 . Let $f_t : I \rightarrow X$ be a homotopy between f_0 and the constant loop. Homotopy lifting property and remark implies that f_t lifts to a homotopy \tilde{f}_t of paths between \tilde{f}_0 and the constant loop, so $[\tilde{f}_0] = \text{id} \in \pi_1(\tilde{X}, \tilde{x}_0)$ and p_* is injective.
2. Let $f : I \rightarrow X$ be a loop at x_0 that lifts to a loop \tilde{f} at \tilde{x}_0 . Then $p\tilde{f} = f$, so $p_*([\tilde{f}]) = [f]$. On the other hand, if $f : I \rightarrow X$ is a loop at x_0 such that there exists a loop $\tilde{f} : I \rightarrow \tilde{X}$ at \tilde{x}_0 with $p_*([\tilde{f}]) = [f]$, then f is homotopic to $p\tilde{f}$. Homotopy lifting property implies that there exists a loop $\tilde{f}' : I \rightarrow \tilde{X}$ at \tilde{x}_0 such that $p\tilde{f}' = f$.

□

Let $p : \tilde{X} \rightarrow X$ be a covering space. Let $U \subseteq X$ be an evenly covered neighbourhood of $x \in X$. Let

$$p^{-1}(U) = \bigsqcup_{\alpha \in \Lambda} \tilde{U}_\alpha.$$

Then the cardinality $|p^{-1}(x)|$ of $p^{-1}(x)$ is exactly the cardinality of $|\Lambda|$. The set of sheets is in bijection with $p^{-1}(x)$. So the cardinality of $p^{-1}(x)$ is locally constant. If X is connected, the cardinality of $p^{-1}(x)$ is constant.

Notation. Let X, Y be topological spaces, $x \in X$, and $y \in Y$. A continuous map

$$f : (X, x) \rightarrow (Y, y)$$

is a continuous map $f : X \rightarrow Y$ such that $f(x) = y$.

Proposition 1.26. *Let X, \tilde{X} be path-connected and*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a covering space. Then the number of sheets of p equals the index of $p_ \left(\pi_1 \left(\tilde{X}, \tilde{x}_0 \right) \right)$ in $\pi_1(X, x_0)$.*

Proof. Let g be a loop in X at x_0 and \tilde{g} be its lift to \tilde{X} starting at \tilde{x}_0 . Let $H = p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0 \right) \right)$ and let $[h] \in H$. Then $h \cdot g$ lifts to a path $\tilde{h} \cdot \tilde{g}$ in \tilde{X} starting at \tilde{x}_0 with the same endpoint as \tilde{g} , because \tilde{h} is a loop, by Proposition 1.25. Define

$$\begin{aligned} \Phi : \quad \{ \text{cosets of } H \text{ in } \pi_1(X, x_0) \} &\longrightarrow p^{-1}(x_0) \\ H[g] &\longmapsto \tilde{g}(1) \end{aligned}$$

so Φ is well-defined. Want to show that Φ is bijective.

- Φ is surjective because \tilde{X} is path-connected. Let \tilde{g} be a path in \tilde{X} from \tilde{x}_0 to any point $\tilde{x}'_0 \in p^{-1}(x_0)$, then $g = p \cdot \tilde{g}$ and $\Phi(H[g]) = \tilde{x}'_0$.
- Φ is injective, since if $\Phi(H[g_1]) = \Phi(H[g_2])$ then the lift $\tilde{g}_1 \cdot \tilde{g}_2^{-1}$ of $g_1 \cdot g_2^{-1}$ defines a loop in \tilde{X} at \tilde{x}_0 . Proposition 1.25 implies that $[g_1][g_2]^{-1} \in H$, so $H[g_1] = H[g_2]$.

□

We say that a topological space X has a certain property (P) **locally** if for each point $x \in X$ and each neighbourhood U of x there is an open neighbourhood $V \subseteq U$ having this property (P) .

Example. X is locally path-connected or X is locally simply-connected.

Proposition 1.27. *Let*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a covering space and

$$f : (Y, y_0) \rightarrow (X, x_0)$$

a continuous map, where Y is path-connected and locally path-connected. Then there is a lift

$$\tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$$

if and only if

$$f_* \left(\pi_1(Y, y_0) \right) \subseteq p_* \left(\pi_1(\tilde{X}, \tilde{x}_0) \right).$$

$$\begin{array}{ccc} & & (\tilde{X}, \tilde{x}_0) \\ & \nearrow \tilde{f} & \downarrow p \\ (Y, y_0) & \xrightarrow{f} & (X, x_0) \end{array}.$$

Proof.

\Rightarrow Clear, because $f = p\tilde{f}$ implies $f_* = p_*\tilde{f}_*$.

\Leftarrow Assume

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

For each $y \in Y$ choose a path γ from y_0 to y , so $f\gamma$ is a path in X from x_0 to $f(y)$. By path lifting, we can lift $f\gamma$ to a path $\tilde{f}\gamma$ in \tilde{X} starting at \tilde{x}_0 . Define the map

$$\begin{aligned} \tilde{f} : (Y, y_0) &\longrightarrow (\tilde{X}, \tilde{x}_0) \\ y &\longmapsto \tilde{f}\gamma(1) \end{aligned}$$

$$\begin{array}{ccc} & \tilde{x}_0 \xrightarrow[\tilde{f}\gamma']{\tilde{f}\gamma} \tilde{f}(y) & \\ & \uparrow \tilde{f} \quad \downarrow p & \\ y_0 \xrightarrow[\gamma']{\gamma} y & \xrightarrow{f} x_0 \xrightarrow[\gamma']{f\gamma} f(y) & \end{array}$$

- This map is well-defined, that is does not depend on the choice of γ . Let γ' be another path from y_0 to y . Then $h_0 = (f\gamma') \cdot (f\gamma)^{-1}$ is a loop at x_0 and

$$[h_0] \in f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Proposition 1.25 implies that can lift h_0 to a loop \tilde{h}_0 at \tilde{x}_0 . The first half of \tilde{h}_0 is $\tilde{f}\gamma'$ and the second half is $\tilde{f}\gamma^{-1}$, so $\tilde{f}\gamma(1) = \tilde{f}\gamma'(1)$. Thus \tilde{f} is well-defined.

- We have $p\tilde{f} = f$, so \tilde{f} lifts f .
- It remains to show that \tilde{f} is continuous. Let $y \in Y$ and let U be an evenly covered neighbourhood of $f(y)$. Let \tilde{U} be the sheet above U such that $\tilde{f}(y) \in \tilde{U}$, so $p|_{\tilde{U}} : \tilde{U} \rightarrow U$ is a homeomorphism. Let $V \subseteq Y$ be a path-connected neighbourhood of y such that $f(V) \subseteq U$. Fix a path γ from y_0 to y . Let $y' \in V$ be arbitrary and η be a path from y to y' , so $\gamma \cdot \eta$ is a path from y_0 to y' . Then $(f\gamma) \cdot (f\eta)$ is a path in U from x_0 to $f(y')$. $\tilde{f}\eta = (p|_{\tilde{U}})^{-1} f\eta$, so $\tilde{f}|_V = (p|_{\tilde{U}})^{-1} f$. Thus $\tilde{f}|_V : V \rightarrow \tilde{U}$ is continuous, so \tilde{f} is continuous.

□

Lecture 13
Friday
08/02/19

1.3.2 The classification of covering spaces

Definition. A covering space $p : \tilde{X} \rightarrow X$ is a **universal cover** if \tilde{X} is simply-connected.

Definition. A topological space X is **semilocally simply-connected** if each $x \in X$ has a neighbourhood U such that

$$i_* : \pi_1(U, x) \rightarrow \pi_1(X, x)$$

is trivial, where $i : U \hookrightarrow X$ is the inclusion.

Example. Let $X = \bigcup_n C_n \subseteq \mathbb{R}^2$ be the Hawaiian earrings, where $C_n \subseteq \mathbb{R}^2$ is the circle of radius $1/n$ and centre $(1/n, 0)$. Then X is not semilocally simply-connected.

Proposition 1.28. *If $p : \tilde{X} \rightarrow X$ is a universal cover, then X is semilocally simply-connected.*

Proof. Let $U \subseteq X$ be an evenly covered neighbourhood of $x_0 \in X$, $\tilde{U} \subseteq \tilde{X}$ be a sheet over U , and $\gamma \subseteq U$ be a loop at x_0 , so γ lifts to a loop $\tilde{\gamma} \subseteq \tilde{U}$ at \tilde{x}_0 . $\tilde{\gamma}$ is homotopic to the constant loop at \tilde{x}_0 . Composing this homotopy with p implies that γ is homotopic to the constant loop at x_0 in X , so

$$\pi_1(U, x_0) \rightarrow \pi_1(X, x_0)$$

is trivial. □

Theorem 1.29. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there exists a universal cover $p : \tilde{X} \rightarrow X$.*

Remark. If

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

is a universal cover, each point $\tilde{x} \in \tilde{X}$ can be joined to \tilde{x}_0 by a unique homotopy class of paths, by Proposition 1.6.

$$\{\text{points in } \tilde{X}\} \rightsquigarrow \{[\gamma] \mid \gamma \text{ is a path in } \tilde{X} \text{ starting at } \tilde{x}_0\} \rightsquigarrow \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

by the homotopy lifting property.

Proof. Let $x_0 \in X$, and

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}, \quad p : \begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ [\gamma] & \longmapsto & \gamma(1) \end{array}.$$

Have to

1. give \tilde{X} a topology,
2. show that $p : \tilde{X} \rightarrow X$ is a covering, and
3. show that \tilde{X} is simply-connected.

Recall that a **basis** for a topology on a set Y is a collection \mathcal{B} of subsets such that

- $Y = \bigcup_{U \in \mathcal{B}} U$, and
- if $U_1, U_2 \in \mathcal{B}$ and $y \in U_1 \cap U_2$ then there exists $V \in \mathcal{B}$ such that $y \in V$ and $V \subseteq U_1 \cap U_2$.

A basis defines a topology on Y , by $A \subseteq Y$ is open if and only if A is the union of elements of \mathcal{B} . A map $f : Z \rightarrow Y$ is continuous if and only if $f^{-1}(U)$ is open for all $U \in \mathcal{B}$.

1. Let \mathcal{U} be the collection of all path-connected open sets $U \subseteq X$ such that $\pi_1(U) \rightarrow \pi_1(X)$ is trivial. Then $X = \bigcup_{U \in \mathcal{U}} U$ because X is semilocally simply-connected. Let $U_1, U_2 \in \mathcal{U}$ and $y \in U_1 \cap U_2$, and let $y \in V \subseteq U_1 \cap U_2$ be path-connected and open.

$$\begin{array}{ccccc} V & \hookrightarrow & U_1 & \hookrightarrow & X \\ & & & & \uparrow \\ \pi_1(V) & \longrightarrow & \pi_1(U_1) & \xrightarrow{\text{trivial}} & \pi_1(X) \\ & & \searrow & \text{trivial} & \nearrow \end{array},$$

so $V \in \mathcal{U}$, so \mathcal{U} is a basis for the topology on X . For $U \in \mathcal{U}$ and γ a path in X from x_0 to a point in U , we define

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ a path in } U \text{ such that } \eta(0) = \gamma(1)\} \subseteq \tilde{X}.$$

$U_{[\gamma]}$ only depends on the class $[\gamma]$, so $p|_{U_{[\gamma]}} : U_{[\gamma]} \rightarrow U$ is bijective. Surjective because U is path-connected and injective because all paths η in U with the same endpoint are homotopic. Claim that $\{U_{[\gamma]}\}$ forms a basis on \tilde{X} .

- $\bigcup_{U \in \mathcal{U}, \gamma} U_{[\gamma]} = \tilde{X}$, because $\bigcup_{U \in \mathcal{U}} U = X$.
- Observe that if $[\gamma'] \in U_{[\gamma]}$ then $U_{[\gamma]} = U_{[\gamma']}$. If $\gamma' = \gamma \cdot \eta$ for η a path in U , then elements in $U_{[\gamma']}$ have the form $[\gamma \cdot \eta \cdot \mu]$, so $U_{[\gamma']} \subseteq U_{[\gamma]}$. Elements in $U_{[\gamma]}$ have the form

$$[\gamma \cdot \mu] = [\gamma \cdot \eta \cdot \eta^{-1} \cdot \mu] = [\gamma' \cdot \eta^{-1} \cdot \mu],$$

so $U_{[\gamma]} \subseteq U_{[\gamma']}$. Consider $U_{[\gamma]}$ and $V_{[\gamma']}$ and let $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, so $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. Let $W \in \mathcal{U}$ such that $W \subseteq U \cap V$ and such that $\gamma''(1) \in W$, so $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$ and $[\gamma''] \in W_{[\gamma'']}$. This proves the claim.

2. $p|_{U_{[\gamma]}} : U_{[\gamma]} \rightarrow U$ is a homeomorphism. It is bijective, let $V_{[\gamma']} \subseteq U_{[\gamma]}$ be an element of the basis, so $p(V_{[\gamma']}) = V \in \mathcal{U}$. $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$. Thus $p : \tilde{X} \rightarrow X$ is continuous. If $U \in \mathcal{U}$, then

$$p^{-1}(U) = \bigsqcup_{[\gamma]} U_{[\gamma]},$$

so $p : \tilde{X} \rightarrow X$ is a covering space.

3. Let $\tilde{x}_0 \in \tilde{X}$ be the class of the constant path at x_0 . Let $[\gamma] \in \tilde{X}$ be arbitrary. $\gamma : [0, 1] \rightarrow X$ and $\gamma(0) = x_0$. Let γ_t be the path in X defined by

$$\gamma_t(s) = \begin{cases} \gamma(s) & s \in [0, t] \\ \gamma(t) & s \in [t, 1] \end{cases}.$$

Then

$$\begin{array}{ccc} \tilde{\gamma} & : & I \longrightarrow \tilde{X} \\ & & t \longmapsto [\gamma_t] \end{array}$$

is a path in \tilde{X} from \tilde{x}_0 to $[\gamma]$, so \tilde{X} is path-connected. Recall that $p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$ consists of the classes of loops at x_0 in X that lift to loops in \tilde{X} at \tilde{x}_0 . Let $[\gamma] \in p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$. Then γ lifts to a loop at \tilde{x}_0 by $t \mapsto [\gamma_t]$. Because it is a loop we have $\tilde{x}_0 = [\gamma_1] = [\gamma]$, so γ is homotopic to the constant loop. Thus $p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right) = \{id\}$, so \tilde{X} is simply-connected. □

Lecture 14 is a problem class.

Let $p : \tilde{X} \rightarrow X$ be a covering space, so $p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right) \subseteq \pi_1(X, x_0)$.

Proposition 1.30. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \subseteq \pi_1(X, x_0)$ there is a covering space $p : X_H \rightarrow X$ such that $p_*\left(\pi_1(X_H, \tilde{x}_0)\right) = H$ for some basepoint x_0 .*

Proof. Let \tilde{X} be as constructed above. Define $X_H = \tilde{X} / \sim$, where $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot (\gamma')^{-1}] \in H$. This is an equivalence relation.

- $[\gamma] \sim [\gamma]$ because $id \in H$.
- $[\gamma] \sim [\gamma']$ implies that $[\gamma'] \sim [\gamma]$ because H contains all its inverses.
- $[\gamma] \sim [\gamma']$ and $[\gamma'] \sim [\gamma'']$ implies that $[\gamma] \sim [\gamma'']$ because H is closed under product.

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \tilde{X} \\ \downarrow & \nearrow p & \sim \\ X & & X_H \end{array}.$$

Let $U_{[\gamma]}, U_{[\gamma']}$ be basis neighbourhoods. If $[\gamma] \sim [\gamma']$ then $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$, so p is a covering space, and $p^{-1}(U) = \bigcup_{[\gamma]} U_{[\gamma]}$. Let $\tilde{x}_0 \in X_H$ be the equivalence class of the constant path c_{x_0} at x_0 . Let γ be a loop in X at x_0 such that $[\gamma] \in p_*\left(\pi_1(X_H, \tilde{x}_0)\right)$. Again $t \mapsto [\gamma_t]$ is a lift of γ at \tilde{x}_0 .

$$t \mapsto [\gamma_t] \text{ is a loop in } X_H \iff [\gamma_1] = [\gamma] = [c_{x_0}] \text{ in } X_H \iff [\gamma] \sim [c_{x_0}] \iff \gamma \in H.$$

□

Lecture 14
Tuesday
12/02/19
Lecture 15
Wednesday
13/02/19

Definition. We say that two covering spaces $p_1 : \widetilde{X}_1 \rightarrow X$ and $p_2 : \widetilde{X}_2 \rightarrow X$ are **isomorphic** if there exists a homeomorphism $f : \widetilde{X}_1 \rightarrow \widetilde{X}_2$ such that

$$\begin{array}{ccc} \widetilde{X}_1 & \xrightarrow{f} & \widetilde{X}_2 \\ & \searrow p_1 \quad \swarrow p_2 & \\ & X & \end{array} .$$

Proposition 1.31. *Let X be path-connected and locally path-connected and $x_0 \in X$. Two path-connected covering spaces $p_1 : \widetilde{X}_1 \rightarrow X$ and $p_2 : \widetilde{X}_2 \rightarrow X$ are isomorphic via an isomorphism $f : \widetilde{X}_1 \rightarrow \widetilde{X}_2$ mapping a basepoint $\widetilde{x}_1 \in p_1^{-1}(x_0)$ to a basepoint $\widetilde{x}_2 \in p_2^{-1}(x_0)$ if and only if*

$$p_{1*} \left(\pi_1 \left(\widetilde{X}_1, \widetilde{x}_1 \right) \right) = p_{2*} \left(\pi_1 \left(\widetilde{X}_2, \widetilde{x}_2 \right) \right) .$$

Proof.

\Rightarrow If

$$f : \left(\widetilde{X}_1, \widetilde{x}_1 \right) \rightarrow \left(\widetilde{X}_2, \widetilde{x}_2 \right)$$

is an isomorphism, then $p_1 = p_2 f$, so

$$p_{1*} \left(\pi_1 \left(\widetilde{X}_1, \widetilde{x}_1 \right) \right) \subseteq p_{2*} \left(\pi_1 \left(\widetilde{X}_2, \widetilde{x}_2 \right) \right) ,$$

and $p_2 = p_1 f^{-1}$, so

$$p_{2*} \left(\pi_1 \left(\widetilde{X}_2, \widetilde{x}_2 \right) \right) \subseteq p_{1*} \left(\pi_1 \left(\widetilde{X}_1, \widetilde{x}_1 \right) \right) .$$

\Leftarrow Assume

$$p_{1*} \left(\pi_1 \left(\widetilde{X}_1, \widetilde{x}_1 \right) \right) = p_{2*} \left(\pi_1 \left(\widetilde{X}_2, \widetilde{x}_2 \right) \right) .$$

By lifting criterion in Proposition 1.27, we can lift p_1 to a continuous map

$$\widetilde{p}_1 : \left(\widetilde{X}_1, \widetilde{x}_1 \right) \rightarrow \left(\widetilde{X}_2, \widetilde{x}_2 \right) ,$$

and p_2 to a continuous map

$$\widetilde{p}_2 : \left(\widetilde{X}_2, \widetilde{x}_2 \right) \rightarrow \left(\widetilde{X}_1, \widetilde{x}_1 \right) ,$$

so $p_1 \widetilde{p}_2 = p_2$ and $p_2 \widetilde{p}_1 = p_1$.

$$\begin{array}{ccc} \left(\widetilde{X}_1, \widetilde{x}_1 \right) & \xrightarrow{\quad \widetilde{p}_1 \quad} & \left(\widetilde{X}_2, \widetilde{x}_2 \right) \\ & \searrow p_1 \quad \swarrow p_2 \quad \nwarrow \widetilde{p}_2 \quad \nearrow p_1 & \\ & (X, x_0) & \end{array} .$$

$\widetilde{p}_1 \widetilde{p}_2$ fixes the point $\widetilde{x}_2 \in \widetilde{X}_2$. By the unique lifting property in Proposition 1.7, $\widetilde{p}_1 \widetilde{p}_2 = id_{\widetilde{x}_2}$. Similarly, $\widetilde{p}_2 \widetilde{p}_1 = id_{\widetilde{x}_1}$, so \widetilde{p}_1 is an isomorphism.

□

Fix $x_0 \in X$, $\widetilde{x}_1 \in p_1^{-1}(x_0)$, and $\widetilde{x}_2 \in p_2^{-1}(x_0)$. A **basepoint preserving isomorphism**

$$f : (\widetilde{X}_1, \widetilde{x}_1) \rightarrow (\widetilde{X}_2, \widetilde{x}_2)$$

is an isomorphism such that $f(\widetilde{x}_1) = \widetilde{x}_2$.

Theorem 1.32 (Galois correspondence). *Let X be path-connected, locally path-connected, and semilocally simply-connected, and $x_0 \in X$. Then*

1. *there is a bijection*

$$\left\{ \begin{array}{l} \text{path-connected covering spaces } p : (\widetilde{X}, \widetilde{x}_0) \rightarrow (X, x_0) \\ \text{up to basepoint preserving isomorphisms} \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \text{subgroups} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\},$$

2. *if we ignore the basepoints, this correspondence gives a bijection*

$$\left\{ \begin{array}{l} \text{path-connected covering spaces } p : \widetilde{X} \rightarrow X \\ \text{up to isomorphisms} \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \text{conjugacy classes of subgroups} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\}.$$

Proof.

1. To a covering space

$$p : (\widetilde{X}, \widetilde{x}_0) \rightarrow (X, x_0),$$

we associate the subgroup

$$p_* \left(\pi_1(\widetilde{X}, \widetilde{x}_0) \right) \subseteq \pi_1(X, x_0).$$

Proposition 1.30 and Proposition 1.31 show that this is well-defined on the isomorphism classes and it is bijective.

2. Let $p : \widetilde{X} \rightarrow X$ be a covering space and $\widetilde{x}_1, \widetilde{x}_2 \in p^{-1}(x_0)$. Let

$$H_i = p_* \left(\pi_1(\widetilde{X}, \widetilde{x}_i) \right) \subseteq \pi_1(X, x_0), \quad i = 1, 2.$$

Let $\widetilde{\gamma}$ be a path from \widetilde{x}_1 to \widetilde{x}_2 . Let $\gamma = p\widetilde{\gamma}$ be a loop at x_0 . Let $[f] \in \pi_1(X, x_0)$. Then $[f] \in H_1$ if and only if the lift \widetilde{f} is a loop at \widetilde{x}_1 . $\widetilde{\gamma}^{-1} \cdot \widetilde{f} \cdot \widetilde{\gamma}$ is a loop at \widetilde{x}_2 , so

$$p_* \left(\widetilde{\gamma}^{-1} \cdot \widetilde{f} \cdot \widetilde{\gamma} \right) = \gamma^{-1} \cdot f \cdot \gamma,$$

so $[\gamma]^{-1} [f] [\gamma] \in H_2$. Thus $[\gamma]^{-1} H_1 [\gamma] \subseteq H_2$. Similarly, $[\gamma] H_2 [\gamma]^{-1} \subseteq H_1$. Conversely, let $H_1 \subseteq \pi_1(X, x_0)$ as above and $[\delta] \in \pi_1(X, x_0)$ be an arbitrary element. Let $\widetilde{\delta}$ be a lift of δ such that $\widetilde{\delta}(0) = \widetilde{x}_0$ and define $x_3 = \widetilde{\delta}(1)$. Then the same construction yields

$$p_* \left(\pi_1(\widetilde{X}, \widetilde{x}_3) \right) = [\delta]^{-1} H_1 [\delta].$$

□

1.3.3 Deck transformations and group actions

Definition. Let $p : \tilde{X} \rightarrow X$ be a covering space. A **deck-transformation** is an isomorphism from \tilde{X} to itself.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & \tilde{X} \\ & \searrow p & \swarrow p \\ & X & \end{array} .$$

The group of deck-transformations is denoted by $G(\tilde{X})$.

Example.

- Let

$$\begin{array}{ccc} p & : & \mathbb{R} \longrightarrow S^1 \subseteq \mathbb{C} \\ & & t \longmapsto e^{2\pi i t} \end{array} .$$

Then $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $p(f(t)) = p(t)$ if and only if $e^{2\pi i f(t)} = e^{2\pi i t}$, if and only if $f(t) = t + n$, so $G(\mathbb{R}) \cong \mathbb{Z}$.

- Let

$$\begin{array}{ccc} p & : & S^1 \longrightarrow S^1 \\ & & z \longmapsto z^n \end{array} .$$

Then $G(S^1) \cong \mathbb{Z}/n\mathbb{Z}$.

An observation is that if \tilde{X} is path-connected then $f \in G(\tilde{X})$ is uniquely determined by where it sends a single point.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f'} & \tilde{X} \\ & \xrightarrow{f} & \\ & \searrow p & \swarrow p \\ & X & \end{array} .$$

If $f(x) = f'(x)$ for a single x , by unique lifting $f = f'$. So the identity is the only deck-transformation with a fixed point.

Definition. A covering space $p : \tilde{X} \rightarrow X$ is **normal**, or **regular**, or **Galois**, if for each $x \in X$ and every pair $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there is an $f \in G(\tilde{X})$ such that $f(\tilde{x}) = \tilde{x}'$.

Example.

- $p : \mathbb{R} \rightarrow S^1$ is normal.
- $p : S^1 \rightarrow S^1$ is normal.

Proposition 1.33. Let

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a path-connected covering space, and X be path-connected and locally path-connected. Then $p : \tilde{X} \rightarrow X$ is normal if and only if

$$H = p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0 \right) \right) \subseteq \pi_1(X, x_0)$$

is a normal subgroup.

Proof. Let $\widetilde{x}_1 \in p^{-1}(x_0)$, let $\widetilde{\gamma}$ be a path from \widetilde{x}_0 to \widetilde{x}_1 and $\gamma = p(\widetilde{\gamma})$. Then $[\gamma]$ conjugates H to $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x}_1\right)\right)$ so $[\gamma]H[\gamma]^{-1} = H$, if and only if $H = p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x}_1\right)\right)$, by Proposition 1.31 if and only if $f(\widetilde{x}_0) = \widetilde{x}_1$. So $G(\widetilde{X})$ acts transitively on $p^{-1}(x_0)$ if and only if $H \subseteq \pi_1(X, x_0)$ is a normal subgroup. Let $x'_0 \in X$ be another point and h a path from x_0 to x'_0 . Let \widetilde{h} be a lift of h such that $\widetilde{h}(0) = \widetilde{x}_0$. Set $\widetilde{x}'_0 = \widetilde{h}(1)$ and $p(\widetilde{x}'_0) = x'_0$. Then

$$\begin{array}{ccc} \pi_1\left(\widetilde{X}, \widetilde{x}_0\right) & \xrightarrow{\beta_{\widetilde{h}}} & \pi_1\left(\widetilde{X}, \widetilde{x}'_0\right) \\ p_* \downarrow & & \downarrow p_* \\ \pi_1(X, x_0) & \xrightarrow{\beta_h} & \pi_1(X, x'_0) \end{array}.$$

$H \subseteq \pi_1(X, x_0)$ is normal if and only if

$$p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x}'_0\right)\right) \subseteq \pi_1(X, x'_0)$$

is normal, as before if and only if $G(\widetilde{X})$ acts transitively on $p^{-1}(x'_0)$. □

Proposition 1.34. *Let*

$$p : (\widetilde{X}, \widetilde{x}_0) \rightarrow (X, x_0)$$

be a covering space, and X, \widetilde{X} be path-connected and locally path-connected. Let $H = p_\left(\pi_1\left(\widetilde{X}, \widetilde{x}_0\right)\right)$ and $N(H) \subseteq \pi_1(X, x_0)$ be the normaliser of H . Then $G(\widetilde{X})$ is isomorphic to $N(H)/H$. In particular,*

- *if \widetilde{X} is normal, then $G(\widetilde{X}) \cong \pi_1(X, x_0)/H$, and*
- *if \widetilde{X} is the universal cover, then $G(\widetilde{X}) \cong \pi_1(X, x_0)$.*

Proof. Exercise: read the proof of this in Hatcher. □

Example. Let $X = S^1 \vee S^1$, so $\pi_1(X) = F_{\{a, b\}}$. Then the following are covering spaces.

- A normal covering space

$$\widetilde{X} = \overset{a}{\circ} \overset{\widetilde{x}_0}{\circ} \overset{b}{\circ} \overset{a}{\circ} \rightarrow \overset{a}{\circ} \overset{b}{\circ} = X, \quad p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x}_0\right)\right) = \langle a, b^2, bab^{-1} \rangle \subseteq F_{\{a, b\}}.$$

In general, a two-oriented graph is a covering space of X .

- Not a normal covering space

$$\widetilde{X} = \overset{a}{\circ} \overset{b}{\circ} \overset{\widetilde{x}_0}{\circ} \overset{a}{\circ} \rightarrow \overset{a}{\circ} \overset{b}{\circ} = X, \quad p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x}_0\right)\right) = \langle b^2, bab^{-1}, a^2, aba^{-1} \rangle.$$

- A normal covering space

$$\widetilde{X} = \dots \overset{a}{\circ} \overset{a}{\circ} \overset{a}{\circ} \dots \rightarrow \overset{a}{\circ} \overset{b}{\circ} = X, \quad p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x}_0\right)\right) = \langle b^n ab^{-n} \mid n \in \mathbb{Z} \rangle.$$

Universal cover is a tree.

Example. Let $T = S^1 \times S^1$, so $\pi_1(T) = \mathbb{Z}^2$. This is abelian, so all covering spaces are normal. Universal cover is

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow S^1 \times S^1 \\ (s, t) &\longmapsto (e^{2\pi i s}, e^{2\pi i t}) \end{aligned} ,$$

since \mathbb{R}^2 is simply connected. (Exercise: check that it is a covering space) More generally, if $p : \tilde{X} \rightarrow X$ and $q : \tilde{Y} \rightarrow Y$ are covering spaces then

$$\begin{aligned} \tilde{X} \times \tilde{Y} &\longrightarrow X \times Y \\ (x, y) &\longmapsto (p(x), q(y)) \end{aligned}$$

is again a covering space. For example,

$$\begin{aligned} S^1 \times S^1 &\longrightarrow S^1 \times S^1 \\ (z_1, z_2) &\longmapsto (z_1^n, z_2^m) \end{aligned} .$$

Example. Worksheet 3 exercise 7. Let

$$\mathbb{RP}^n = \frac{\mathbb{R}^{n+1} \setminus \{0\}}{\sim} = \frac{S^n}{\sim}$$

be the **projective n -space**, the space of all lines through the origin in \mathbb{R}^{n+1} , where $x \sim -x$. Let $p : S^n \rightarrow \mathbb{RP}^n$ be the quotient map. Claim that this is a covering space. Let $[x] \in \mathbb{RP}^n$. Then $p^{-1}([x]) = \{\pm x\}$. Let U be an open neighbourhood of x such that $U \cap (-U) = \emptyset$, so $p(U) = \{[x] \mid x \in U\}$. Then $p^{-1}(p(U)) = U \cup (-U)$ is open and disjoint. Thus $p|_U : U \rightarrow p(U)$ is a homeomorphism, so it is a covering space.

- $n \geq 2$ implies that S^n is simply-connected, so $S^n \rightarrow \mathbb{RP}^n$ is a universal cover. Then

$$\{id\} = p_*(\pi_1(S^n)) \stackrel{2}{\subseteq} \pi_1(\mathbb{RP}^n),$$

so $|\pi_1(\mathbb{RP}^n)| = 2$. Thus $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$.

- $n = 1$ implies that $\mathbb{RP}^1 = S^1$, so

$$\begin{aligned} p : S^1 &\longrightarrow S^1 \\ z &\longmapsto z^2 \end{aligned}$$

is a covering space.

2 Homology

Higher homotopy groups $\pi_n(X, x_0)$ are groups of basepoint preserving homotopies of continuous $\phi : I^n \rightarrow X$ such that $\phi(\partial I^n) = x_0$.

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Example.

$$\pi_1(S^n) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases},$$

$$\pi_2(S^n) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases},$$

$$\pi_3(S^n) = \begin{cases} \mathbb{Z} & n = 2, 3 \\ 0 & \text{otherwise} \end{cases},$$

$$\pi_i(S^2) = \begin{cases} \mathbb{Z} & i = 4, 5 \\ 2\mathbb{Z} & i = 6 \\ 12\mathbb{Z} & i = 7 \end{cases}.$$

Homology is more suitable. The following is the plan.

- Simplicial homology.
- Singular homology.
- Technical machinery to show that they coincide.
- Applications.

2.1 Δ -complexes

Definition. Let $m, n \geq 0$.

- An **n -simplex** in \mathbb{R}^m is the convex hull of a set V of $n + 1$ points in \mathbb{R}^m that are not all contained in an affine $(n - 1)$ -dimensional subspace of \mathbb{R}^m .

- The **standard n -simplex** is the convex hull of the standard basis $\{e_1, \dots, e_{n+1}\}$ in \mathbb{R}^{n+1} ,

$$\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, x_0 + \dots + x_n = 1\}.$$

- An **ordered n -simplex** is an n -simplex with an ordering on the vertices. We denote it by $[v_0, \dots, v_n]$, where v_0, \dots, v_n are the vertices in ascending order.
- The **standard ordered n -simplex** is the ordered n -simplex $[e_1, \dots, e_{n+1}]$ in \mathbb{R}^{n+1} . It is denoted by Δ^n .
- Let $[v_0, \dots, v_{n+1}]$ be an n -simplex in \mathbb{R}^m and let $L \subseteq \mathbb{R}^m$ be the affine subspace spanned by v_0, \dots, v_n . Then there exists a unique affine morphism

$$\begin{array}{ccc} L & \longrightarrow & \mathbb{R}^{n+1} \\ v_i & \longmapsto & e_{i+1} \end{array}, \quad i = 0, \dots, n.$$

This gives a homeomorphism from $[v_0, \dots, v_n]$ to Δ^n that preserves this ordering.

- For $n \geq 1$, the **faces** of an ordered n -simplex $[v_0, \dots, v_n]$ are the ordered $(n - 1)$ -simplices

$$[v_0, \dots, \widehat{v_i}, \dots, v_n].$$

$\widehat{v_i}$ means we omit the vertex v_i .

- The union of all the faces of a simplex Δ is the **boundary** $\partial\Delta$.
- The **interior** of Δ is $\mathring{\Delta} = \Delta \setminus \partial\Delta$.

Example. Let $\Delta^2 = [e_1, e_2, e_3]$. Then $\partial\Delta^2 = [e_1, e_2] \cup [e_1, e_3] \cup [e_2, e_3]$.

Definition. Let X be a topological space. A **Δ -complex structure** on X is a collection of continuous maps $\sigma_\alpha : \Delta^{n(\alpha)} \rightarrow X$ for $\alpha \in A$ and $n(\alpha) \in \mathbb{N}$ such that

1. the restriction $\sigma_\alpha|_{\mathring{\Delta}^{n(\alpha)}}$ is injective for all $\alpha \in A$ and for each $x \in X$ there is a unique $\alpha \in A$ such that $x \in \sigma_\alpha(\mathring{\Delta}^{n(\alpha)})$,
2. the restriction of σ_α to a face of $\Delta^{n(\alpha)}$ is equal to σ_β for some $\beta \in A$ and $n(\beta) = n(\alpha) - 1$, and
3. $U \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(U)$ is open in $\Delta^{n(\alpha)}$ for all $\alpha \in A$.

An observation is that

$$\sigma : \bigsqcup_{\alpha \in A} \Delta^{n(\alpha)} \rightarrow X$$

induced by the σ_α is a quotient map, since it is surjective by 1 and $U \subseteq X$ is open if and only if $\sigma^{-1}(U)$ is open by 3.

Remark. One can show that an X with a Δ -complex structure is a CW-complex.

Example.

- Torus or Klein bottle is two Δ^2 , three Δ^1 , and one Δ^0 .
- S^2 is a tetrahedron.
- Dunce hat, by identifying all the three faces of the standard 2-simplex with each other, is one Δ^2 , one Δ^1 , and one Δ^0 .

2.2 Simplicial homology

2.2.1 Simplicial homology

Let X be a Δ -complex. The group of n -chains $\Delta_n(X)$ is the free abelian group on the n -simplices $\sigma_\alpha : \Delta^{n(\alpha)} \rightarrow X$, where $n(\alpha) = n$. So an element in $\Delta_n(X)$ is of the form

$$\sum_{\alpha \in A, n(\alpha)=n} c_\alpha \cdot \sigma_\alpha, \quad c_\alpha \in \mathbb{Z},$$

where all but finitely many of the c_α are zero.

Example. Let K be a Klein bottle.

- $\Delta_0(K) = \{n \cdot v \mid n \in \mathbb{Z}\} = \mathbb{Z} \cdot v \cong \mathbb{Z}$.
- $\Delta_1(K) = \{n_1 \cdot a + n_2 \cdot b + n_3 \cdot c \mid n_1, n_2, n_3 \in \mathbb{Z}\} = \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot b \oplus \mathbb{Z} \cdot c \cong \mathbb{Z}^3$.
- $\Delta_2(K) = \{n_1 \cdot U + n_2 \cdot V \mid n_1, n_2 \in \mathbb{Z}\} = \mathbb{Z} \cdot U \oplus \mathbb{Z} \cdot V \cong \mathbb{Z}^2$.
- $\Delta_n(K) = 0$ for $n \geq 3$.

Similarly for a torus T .

Define the **boundary homomorphism** by

$$\begin{aligned} \partial_n : \Delta_n(X) &\longrightarrow \Delta_{n-1}(X) \\ \sigma_\alpha &\longmapsto \sum_{i=0}^n (-1)^i \sigma_\alpha|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]} \end{aligned}$$

Moreover, we define $\partial_0 = 0$.

Example. Let $\sigma : [v_0, v_1, v_2, v_3] \rightarrow X$. Then

$$\partial_3(\sigma) = \sigma|_{[v_1, v_2, v_3]} - \sigma|_{[v_0, v_2, v_3]} + \sigma|_{[v_0, v_1, v_3]} - \sigma|_{[v_0, v_1, v_2]}.$$

Lemma 2.1. *The composition*

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

is the zero map.

Proof. Let $\sigma : [v_0, \dots, v_n] \rightarrow X$ be an n -simplex. Then

$$\partial_n(\sigma) = \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]},$$

so

$$(\partial_{n-1} \circ \partial_n)(\sigma) = \sum_{j < i} (-1)^i (-1)^j \sigma|_{[v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_n]} + \sum_{j > i} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_n]} = 0.$$

If $n = 1$, clear. □

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2.2.2 Algebraic situation

A **chain complex** of abelian groups is a diagram $(C., \partial)$ of the form

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0,$$

where the C_i are abelian groups and the ∂_n are group homomorphisms such that $\partial_n \circ \partial_{n-1} = 0$ for all n . ∂_n are **boundary homomorphisms**. Elements in C_n are **n -chains**.

$$Z_n = \text{Ker}(\partial_n) \subseteq C_n, \quad B_n = \text{Im}(\partial_{n+1}) \subseteq C_n.$$

Elements in Z_n are **cycles** and elements in B_n are **boundaries**. Since $\partial_{n+1} \circ \partial_n = 0$, we have that $B_n \subseteq Z_n$. The **n -th homology group** of this chain complex is defined by

$$H_n(C., \partial) = \frac{Z_n}{B_n}.$$

So, by Lemma 2.1

$$\dots \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0$$

is a chain complex. The **n -th simplicial homology group** is

$$H_n^\Delta(X) = \frac{\text{Ker}(\partial_n)}{\text{Im}(\partial_{n+1})}.$$

Example. Let $X = S^1$.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_3} & \Delta_2(X) & \xrightarrow{\partial_2} & \Delta_1(X) & \xrightarrow{\partial_1} & \Delta_0(X) \xrightarrow{\partial_0} 0 \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ & & 0 & & \mathbb{Z} & & \mathbb{Z} \end{array}.$$

- $\text{Ker}(\partial_0) = \mathbb{Z}$ and $\text{Im}(\partial_1) = 0$, so $H_0^\Delta(X) \cong \mathbb{Z}$.
- $\text{Ker}(\partial_1) = \Delta_1(X)$ and $\text{Im}(\partial_2) = 0$, so $H_1^\Delta(X) \cong \mathbb{Z}$.
- $H_n^\Delta(X) = 0$ if $n \geq 2$.

Example. Let T be a torus.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_4} & \Delta_3(T) & \xrightarrow{\partial_3} & \Delta_2(T) & \xrightarrow{\partial_2} & \Delta_1(T) \xrightarrow{\partial_1} \Delta_0(T) \xrightarrow{\partial_0} 0 \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ & & 0 & & \mathbb{Z} \cdot U \oplus \mathbb{Z} \cdot V & & \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot b \oplus \mathbb{Z} \cdot c \\ & & & & & & \mathbb{Z} \cdot v \end{array}.$$

- $\text{Ker}(\partial_0) = \mathbb{Z}$ and $\text{Im}(\partial_1) = 0$, so $H_0^\Delta(T) \cong \mathbb{Z}$.
- $\partial_2(U) = a + b - c$ and $\partial_2(V) = a + b - c$, and $\{a, b, a + b - c\}$ is a basis for $\Delta_1(T)$.

$$\text{Ker}(\partial_1) = \Delta_1(T), \quad \text{Im}(\partial_2) = \mathbb{Z} \cdot (a + b - c),$$

$$\text{so } H_1^\Delta(T) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

- $H_2^\Delta(T) \cong \mathbb{Z}$. (Exercise)

Lecture 20 is a problem class.

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2.3 Singular homology

2.3.1 Singular homology

A **singular n -simplex** in a topological space X is a continuous map $\sigma : \Delta^n \rightarrow X$. Let $C_n(X)$ be the free abelian group on the set of all singular simplices in X , that is elements in $C_n(X)$ are finite formal sums

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$$\sum_i n_i \sigma_i, \quad n_i \in \mathbb{Z},$$

where $\sigma_i : \Delta^n \rightarrow X$ are singular n -simplices. Elements in $C_n(X)$ are called **singular n -chains**. Define a **boundary map**

$$\begin{aligned} \partial_n : C_n(X) &\longrightarrow C_{n-1}(X) \\ \sigma &\longmapsto \sum_{i=0}^n (-1)^i \sigma|_{[v_1, \dots, \tilde{v}_i, \dots, v_n]} \end{aligned},$$

for a singular n -simplex σ . Extend it linearly to $C_n(X)$.

Lemma 2.2. $\partial_n \circ \partial_{n+1} = 0$.

Proof. Same proof as for Lemma 2.1. □

We obtain a chain complex

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

Remark. Often we write ∂ instead of ∂_n .

We define the **n -th singular homology group** by

$$H_n(X) = \frac{\text{Ker}(\partial_n)}{\text{Im}(\partial_{n+1})}.$$

An observation is that if X and Y are homeomorphic then $H_n(X) \cong H_n(Y)$.

Proposition 2.3. Let X be a topological space and $X = \bigcup_{\alpha} X_{\alpha}$ be the decomposition into its path-components. Then

$$H_n(X) \cong \bigoplus_{\alpha} H_n(X_{\alpha}).$$

Proof. A singular n -simplex $\sigma : \Delta^n \rightarrow X$ has a path-connected image. So

$$C_n(X) = \bigoplus_{\alpha} C_n(X_{\alpha}).$$

The boundary maps ∂_n preserve this decomposition, so $\partial_n(C_n(X_{\alpha})) \subseteq C_{n-1}(X_{\alpha})$ implies that $\text{Ker}(\partial_n)$ and $\text{Im}(\partial_{n+1})$ split as well as direct sums, so

$$H_n(X) = \frac{\text{Ker}(\partial_n)}{\text{Im}(\partial_{n+1})} \cong \bigoplus_{\alpha} H_n(X_{\alpha}).$$

□

Proposition 2.4. *If X is a path-connected, and as always $X \neq \emptyset$, topological space, then $H_0(X) \cong \mathbb{Z}$. Hence for X arbitrary $H_0(X)$ is a direct sum of \mathbb{Z} 's, one for each path-component.*

Proof. $\partial_0 = 0$, so $H_0(X) = C_0(X) / \text{Im}(\partial_1)$. Define

$$\begin{aligned} \epsilon : C_0(X) &\longrightarrow \mathbb{Z} \\ \sum_i n_i \sigma_i &\longmapsto \sum_i n_i \end{aligned}$$

ϵ is surjective. Enough to show that $\text{Ker}(\epsilon) = \text{Im}(\partial_1)$. This implies by the isomorphism theorem $H_0(X) \cong \mathbb{Z}$. Let $\sigma : \Delta^1 \rightarrow X$ be a 1-simplex. Then

$$\partial_1(\sigma) = \sigma|_{[v_1]} - \sigma|_{[v_0]},$$

so $\epsilon(\partial_1(\sigma)) = 0$, so $\text{Im}(\partial_1) \subseteq \text{Ker}(\epsilon)$. On the other hand, $\epsilon(\sum_i n_i \sigma_i) = 0$ implies that $\sum_i n_i = 0$. The σ_i correspond to points $\sigma_i([v])$ in X . Choose a basepoint $x_0 \in X$ and let

$$\begin{aligned} \sigma_0 : \Delta^0 &\longrightarrow X \\ \Delta^0 &\longmapsto x_0 \end{aligned}$$

be the singular 0-simplex. Let τ_i be a path from x_0 to $\sigma_i([v])$. Consider τ_i as a singular 1-simplex $\tau_i : [v_0, v_1] \rightarrow X$. We have $\partial_1 \circ \tau_i = \sigma_i - \sigma_0$, so

$$\partial_1 \left(\sum_i n_i \tau_i \right) = \sum_i n_i (\sigma_i - \sigma_0) = \sum_i n_i \sigma_i - \sum_i n_i \sigma_0 = \sum_i n_i \sigma_i.$$

Thus $\text{Ker}(\epsilon) \subseteq \text{Im}(\partial_1)$. □

Proposition 2.5. *If X is a point, then*

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}.$$

Proof. For each n there exists a unique singular n -simplex $\partial_n : \Delta^n \rightarrow X$, so $C_n(X) \cong \mathbb{Z}$ for all n .

$$\partial_n(\sigma_n) = \sum_{i=0}^n (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases},$$

so $\partial_n = 0$ if n is odd and ∂_n is an isomorphism if n is even.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_2} & C_1(X) & \xrightarrow{\partial_1} & C_0(X) & \xrightarrow{\partial_0} & 0 \\ & & \parallel & & \parallel & & \\ \dots & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} & \xrightarrow{0} & 0 \end{array},$$

so $H_n = \text{Ker}(\partial_n) / \text{Im}(\partial_{n+1}) = 0$ if $n \geq 1$ and $H_0(X) \cong \mathbb{Z}$. □

2.3.2 Reduced homology groups

The **reduced homology groups** $\widetilde{H}_n(X)$ are the homology groups of the **augmented chain complex**

$$\dots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} \mathbb{Z} \xrightarrow{\epsilon} 0,$$

where ϵ is as in proof of Proposition 2.4.

$$H_n(X) \cong \widetilde{H}_n(X), \quad n \geq 1.$$

Seen in the proof of Proposition 2.4 that ϵ is surjective and $\epsilon \circ \partial_1 = 0$, so $\text{Im}(\partial_1) \subseteq \text{Ker}(\epsilon)$, so ϵ induces a surjective homomorphism

$$\phi_\epsilon : H_0(X) = \frac{C_0(X)}{\text{Im}(\partial_1)} \rightarrow \mathbb{Z}.$$

Then $\text{Ker}(\phi_\epsilon) = \text{Ker}(\epsilon) / \text{Im}(\partial_1) = \widetilde{H}_0(X)$, so $H_0(X) / \widetilde{H}_0(X) \cong \mathbb{Z}$, so

$$H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}.$$

2.4 Homotopy invariance

Let (A, ∂) and (B, ∂) be two chain complexes. A **chain map** $f : (A, \partial) \rightarrow (B, \partial)$ is a collection of homomorphisms $f_n : A_n \rightarrow B_n$ such that $\partial \circ f_n = f_{n+1} \circ \partial$, that is the following diagram commutes.

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$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial} & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \xrightarrow{\partial} & \dots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ \dots & \xrightarrow{\partial} & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \xrightarrow{\partial} & \dots \end{array}.$$

If X and Y are topological spaces and $f : X \rightarrow Y$ is a continuous map define the homomorphisms

$$\begin{aligned} f_{\#} & : C_n(X) \longrightarrow C_n(Y) \\ \sigma : \Delta^n \rightarrow X & \longmapsto f \circ \sigma : \Delta^n \rightarrow Y \end{aligned},$$

and extend it linearly to $C_n(X)$.

$$(f_{\#} \circ \partial)(\sigma) = f_{\#} \left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]} \right) = \sum_{i=0}^n (f \circ \sigma)|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]} = (\partial \circ f_{\#})(\sigma),$$

so $f_{\#} \circ \partial = \partial \circ f_{\#}$, so $f_{\#}$ defines a chain map

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial} & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \dots \\ & & \downarrow f_{\#} & & \downarrow f_{\#} & & \downarrow f_{\#} & & \\ \dots & \xrightarrow{\partial} & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \xrightarrow{\partial} & \dots \end{array}.$$

$f_{\#}$ maps cycles to cycles, since $\alpha \in C_n(X)$ such that $\partial \circ \alpha = 0$, so

$$(\partial \circ f_{\#})(\alpha) = (f_{\#} \circ \partial)(\alpha) = 0.$$

$f_{\#}$ maps boundaries to boundaries, since

$$f_{\#} \circ (\partial \circ \beta) = \partial \circ (f_{\#} \circ \beta).$$

$f_{\#}(Ker(\partial_n)) \subseteq Ker(\partial_n)$ and $f_{\#}(Im(\partial_{n+1})) \subseteq Im(\partial_{n+1})$, so $f_{\#}$ induces a homomorphism

$$f_* : H_n(X) \rightarrow H_n(Y).$$

The following are observations.

- $X \xrightarrow{g} Y \xrightarrow{f} Z$, so $(f \circ g)_{\#} = f_{\#} \circ g_{\#}$, since

$$\Delta^n \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z,$$

so $f \circ (g \circ \sigma) = (f \circ g) \circ \sigma$, so $(f \circ g)_* = f_* \circ g_*$.

- $(id_X)_* = id_{H_n(X)}$.

Theorem 2.6. If two continuous maps $f, g : X \rightarrow Y$ are homotopic, then $f_* = g_* : H_n(X) \rightarrow H_n(Y)$.

Corollary 2.7. If $f : X \rightarrow Y$ is a homotopy equivalence, then $f_* : H_n(X) \rightarrow H_n(Y)$ is an isomorphism.

Proof. Let $g : Y \rightarrow X$ be a continuous map such that $f \circ g \cong id_Y$ and $g \circ f = id_X$. Then $f_* \circ g_* = (f \circ g)_* = (id_Y)_* = id$. Similarly $g_* \circ f_* = id$, so f_* is an isomorphism. \square

Example.

$$H_n(\mathbb{R}^k) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}, \quad \widetilde{H}_n(\mathbb{R}^k) = 0.$$

Proof of Theorem 2.6. Let $F : X \times I \rightarrow Y$ be a homotopy from f to g and $\sigma : \Delta_n \rightarrow X$ be a singular n -simplex. Consider the map

$$\Delta^n \times I \xrightarrow{\sigma \times I} X \times I \xrightarrow{F} Y.$$

$\Delta^n \times I$ is not a simplex. But we can subdivide $\Delta^n \times I$ into $(n+1)$ simplices. In general, we can decompose $\Delta^n \times I$ into $n+1$ $(n+1)$ -simplices

$$[v_0, \dots, v_i, w_i, \dots, w_n], \quad i = 0, \dots, n.$$

Define **prism-operators**

$$\begin{aligned} P &: C_n(X) \longrightarrow C_{n+1}(Y) \\ \sigma &\longmapsto \sum_{i=0}^n (-1)^i F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, w_n]}, \end{aligned}$$

for $\sigma : \Delta^n \rightarrow X$ a singular n -simplex.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial} & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) \xrightarrow{\partial} \dots \\ & & \swarrow P & & \downarrow g_\# & & \swarrow f_\# \\ \dots & \xrightarrow{\partial} & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) \xrightarrow{\partial} \dots \end{array}$$

Claim that

$$\partial \circ P = g_\# - f_\# - P \circ \partial,$$

if and only if $g_\# - f_\# = \partial \circ P + P \circ \partial$. The claim implies the theorem, since if $\alpha \in C_n(X)$ is a cycle, then

$$g_\#(\alpha) - f_\#(\alpha) = (\partial \circ P)(\alpha) + (P \circ \partial)(\alpha) = (\partial \circ P)(\alpha),$$

so $g_\#(\alpha) - f_\#(\alpha)$ is a boundary. Thus $g_\#(\alpha)$ and $f_\#(\alpha)$ are in the same homology class, so $g_*([\alpha]) = f_*([\alpha])$, where $[\alpha]$ is the homology class of α . Let $\sigma : \Delta^n \rightarrow X$ be a singular n -simplex.

$$\begin{aligned} (\partial \circ P)(\sigma) &= \partial \left(\sum_{i=0}^n (-1)^i F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, w_n]} \right) \\ &= \sum_{j \leq i} (-1)^i (-1)^j F \circ (\sigma \times id)|_{[v_0, \dots, \widehat{v_j}, \dots, v_i, w_i, \dots, w_n]} \\ &\quad + \sum_{j \geq i} (-1)^i (-1)^{j+1} F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, \widehat{w_j}, \dots, w_n]}. \end{aligned}$$

If $i = j$ the two sums cancel except for

$$F \circ (\sigma \times id)|_{[\widehat{v_0}, w_0, \dots, w_n]} = g \circ \sigma = g_\#(\sigma), \quad -F \circ (\sigma \times id)|_{[v_0, \dots, v_n, \widehat{w_n}]} = -f \circ \sigma = -f_\#(\sigma).$$

The terms with $i \neq j$ sum up to $(P \circ \partial)(\sigma)$, since we have

$$\begin{aligned} (P \circ \partial)(\sigma) &= \sum_{j < i} (-1)^i (-1)^j F \circ (\sigma \times id)|_{[v_0, \dots, \widehat{v_j}, \dots, v_i, w_i, \dots, w_n]} \\ &\quad + \sum_{j > i} (-1)^i (-1)^{j+1} F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, \widehat{w_j}, \dots, w_n]}. \end{aligned}$$

□

Remark. One can show that there are also induced homomorphisms $f_* : \widetilde{H}_n(X) \rightarrow \widetilde{H}_n(Y)$ invariant under homotopy. (Exercise)

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2.5 Exact sequences and excision

2.5.1 Exact sequences

Let $A \subseteq X$ be a subspace. What is the relationship between $H_n(A)$, $H_n(X)$, $H_n(X/A)$?

Definition. A sequence of group homomorphisms of abelian groups

$$\dots \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} \dots$$

is **exact at** A_n if $\text{Ker}(\alpha_n) = \text{Im}(\alpha_{n+1})$. The sequence is **exact** if it is exact at A_n for all n .

An observation is if the sequence is exact, then

- $\alpha_n \alpha_{n+1} = 0$, so exact sequences are chain complexes, and
- the homology groups of this chain complex are all trivial.

Example.

- $0 \rightarrow A \xrightarrow{\alpha} B$ is exact if and only if $\text{Ker}(\alpha) = 0$, if and only if α is injective.
- $A \xrightarrow{\alpha} B \rightarrow 0$ is exact if and only if $\text{Im}(\alpha) = B$, if and only if α is surjective.
- $0 \rightarrow A \xrightarrow{\alpha} B \rightarrow 0$ is exact if and only if α is an isomorphism.
- $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ is exact if and only if α is injective, β is surjective, and $\text{Ker}(\beta) = \text{Im}(\alpha)$, hence β induces an isomorphism

$$C \cong \frac{B}{\text{Im}(\alpha)} = \frac{B}{A}.$$

This is called a **short exact sequence**.

Definition. Let X be a topological space and $A \subseteq X$. Then A is a **strong deformation retract** of X if there exists a retraction $r : X \rightarrow A$ such that r is homotopic to the identity, and $F : I \times X \rightarrow X$ continuous such that

$$F(0, x) = x, \quad F(1, x) = r(x), \quad F(t, a) = a, \quad x \in X, \quad a \in A, \quad t \in I.$$

Let X be a topological space and $A \subseteq X$ a non-empty closed subspace. Then (X, A) is called a **good pair** if A has a neighbourhood in X that strongly deformation retracts to A .

Example.

- (D^n, S^{n-1}) is a good pair, since S^{n-1} is a deformation retract of $D^n \setminus \{0\}$.
- Let $A = \{1/n \mid n \in \mathbb{N}\} \cup \{0\} \subseteq [0, 1]$ then $([0, 1], A)$ is not a good pair.

Theorem 2.8. Let (X, A) be a good pair, then there is an exact sequence

$$\dots \rightarrow \widetilde{H}_1(A) \xrightarrow{i_*} \widetilde{H}_1(X) \xrightarrow{j_*} \widetilde{H}_1\left(\frac{X}{A}\right) \xrightarrow{\partial} \widetilde{H}_0(A) \xrightarrow{i_*} \widetilde{H}_0(X) \xrightarrow{j_*} \widetilde{H}_0\left(\frac{X}{A}\right) \rightarrow 0,$$

where $i : A \hookrightarrow X$ is the inclusion and $j : X \rightarrow X/A$ is the quotient.

Corollary 2.9.

$$\widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}.$$

Proof. (D^n, S^{n-1}) is a good pair. Let $n > 0$. Recall that $D^n/S^{n-1} \cong S^n$, so

$$\begin{array}{ccccccccccc} \dots & \rightarrow & \widetilde{H}_i(S^{n-1}) & \xrightarrow{i_*} & \widetilde{H}_i(D^n) & \xrightarrow{j_*} & \widetilde{H}_i(S^n) & \xrightarrow{\partial} & \widetilde{H}_{i-1}(S^{n-1}) & \xrightarrow{i_*} & \widetilde{H}_{i-1}(D^n) & \xrightarrow{j_*} & \widetilde{H}_{i-1}(S^n) & \rightarrow & \dots \\ & & \text{\scriptsize \mathbb{R}} & & & & & & \text{\scriptsize \mathbb{R}} & & & & & & \\ & & 0 & & & & & & 0 & & & & & & \end{array}$$

Then $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1})$ for $i > 0$, so

$$\begin{array}{ccccccccccc} \dots & \rightarrow & \widetilde{H}_1(S^{n-1}) & \xrightarrow{i_*} & \widetilde{H}_1(D^n) & \xrightarrow{j_*} & \widetilde{H}_1(S^n) & \xrightarrow{\partial} & \widetilde{H}_0(S^{n-1}) & \xrightarrow{i_*} & \widetilde{H}_0(D^n) & \xrightarrow{j_*} & \widetilde{H}_0(S^n) & \rightarrow & 0 \\ & & \text{\scriptsize \mathbb{R}} & & & & & & \text{\scriptsize \mathbb{R}} & & & & & & \\ & & 0 & & & & & & 0 & & & & & & \end{array}$$

$n > 0$ and $i > 0$, so $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1})$, and $\widetilde{H}_0(S^n) = 0$. We know that $\widetilde{H}_0(S^0) \cong \mathbb{Z}$ and $\widetilde{H}_n(S^0) = 0$, by Proposition 2.3 and Proposition 2.5. Doing induction on n ,

$$\widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}.$$

□

Corollary 2.10. *There exists no retraction $r : D^n \rightarrow \partial D^n$.*

Proof. Assume there exists such an $r : D^n \rightarrow \partial D^n$. Let $i : \partial D^n \rightarrow D^n$. Then $ri = id_{\partial D^n}$, so $r_*i_* = (ri)_* = id_*$, so

$$\begin{array}{ccccc} \widetilde{H}_{n-1}(\partial D^n) & \xrightarrow{i_*} & \widetilde{H}_{n-1}(D^n) & \xrightarrow{r_*} & \widetilde{H}_{n-1}(\partial D^n) \\ \text{\scriptsize \mathbb{R}} & & \text{\scriptsize \mathbb{R}} & & \text{\scriptsize \mathbb{R}} \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array}$$

Thus $i_* = 0$ and $r_* = 0$, a contradiction. □

Theorem 2.11 (Brouwer fixed point theorem). *Every continuous map $f : D^n \rightarrow D^n$ has a fixed point.*

Proof. Assume there exists a fixed point then construct as in dimension two a retraction $D^n \rightarrow \partial D^n$, a contradiction to Corollary 2.10. □

2.5.2 Relative homology groups

Let X be a topological space and $A \subseteq X$ be a subspace. Define

$$C_n(X, A) = \frac{C_n(X)}{C_n(A)}.$$

Let $\partial : C_n(X) \rightarrow C_{n-1}(X)$ be the boundary map then $\partial(\sigma : \Delta^n \rightarrow A) \in \partial(C_n(A)) \subseteq C_{n-1}(A)$. So ∂ induces a homomorphism

$$\partial : C_n(X, A) \rightarrow C_{n-1}(X, A),$$

such that $\partial \circ \partial = 0$. This gives a chain complex

$$\dots \rightarrow C_{n+1}(X, A) \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \rightarrow \dots$$

- The homology groups $H_n(X, A)$ of this complex are the **relative homology groups**.
- The **relative n -chains** are $C_n(X, A)$.
- The **relative n -cycles** are $\text{Ker}(\partial) \subseteq C_n(X, A)$, of the form $[\alpha]$, for $\alpha \in C_n(X)$ such that $\partial(\alpha) \in C_{n-1}(A)$.
- The **relative n -boundaries** are $\text{Im}(\partial) \subseteq C_n(X, A)$, of the form $[\alpha]$, for $\alpha \in C_n(X)$ such that $\alpha = \partial\beta + \gamma$ for $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

A **short exact sequence of chain complexes** is

$$0 \rightarrow (A, \partial) \xrightarrow{i} (B, \partial) \xrightarrow{j} (C, \partial) \rightarrow 0,$$

for i, j chain maps, where

$$0 \rightarrow A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \rightarrow 0$$

is a short exact sequence for all n , so

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{\partial} & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} \xrightarrow{\partial} \dots \\ & & \downarrow i & & \downarrow i & & \downarrow i \\ \dots & \xrightarrow{\partial} & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} \xrightarrow{\partial} \dots \\ & & \downarrow j & & \downarrow j & & \downarrow j \\ \dots & \xrightarrow{\partial} & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & B_{n-1} \xrightarrow{\partial} \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

A short exact sequence of chain complexes always yields a **long exact sequence** of homology groups

$$\dots \rightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \rightarrow \dots$$

This is the **zig-zag lemma**. First we construct the **connecting map** $\partial : H_n(C) \rightarrow H_{n-1}(A)$. Let $c \in C_n$ be a cycle.

- j is surjective, so $c = j(b)$ for some $b \in B_n$.
- $j(\partial(b)) = \partial(j(b)) = \partial c = 0$, so $\partial b \in \text{Ker}(j) \subseteq B_{n-1}$, so $\partial(b) = i(a)$ for some $a \in A_{n-1}$, by exactness.
- $\partial(a) = 0$, since $i(\partial(a)) = \partial(i(a)) = \partial(\partial(b)) = 0$ and i is injective, so $\partial(a) = 0$.

$$\begin{array}{c} a \in A_{n-1} \\ \downarrow i \\ b \in B_n \xrightarrow{\partial} \partial(b) \in B_{n-1} \\ \downarrow j \\ c \in C_n \end{array}$$

Define

$$\begin{array}{ccc} \partial & : & H_n(C) \longrightarrow H_{n-1}(A) \\ & & [c] \longmapsto [a] \end{array}$$

This is well-defined.

- a is uniquely determined by $\partial(b)$ because i is injective.
- If we choose b' instead of b , then $j(b') = j(b)$, so $j(b' - b) = j(b') - j(b) = 0$, so $b' - b \in \text{Ker}(j) = \text{Im}(i)$, hence $b' - b = i(a')$ for some $a' \in A_n$, so $b' = b + i(a')$. If we replace b by $b' = b + i(a')$ this corresponds to replacing a by $a + \partial(a')$, because

$$i(a + \partial(a')) = i(a) + i(\partial(a')) = \partial(b) + \partial(i(a')) = \partial(b + i(a')),$$

and $[a] = [a + \partial(a')]$.

- A different choice of c in its homology class has the form $c + \partial(c')$ for some $c' \in C_{n+1}$. Let $b' \in B_{n+1}$ such that $j(b') = c'$. Then

$$c + \partial(c') = c + \partial(j(b')) = j(b) + j(\partial(b')) = j(b + \partial(b')),$$

so b is replaced by $b + \partial(b')$ but $\partial(b) = \partial(b + \partial(b'))$, so $\partial(b)$ is unchanged and hence a is unchanged.

The map $\partial : H_n(C) \rightarrow H_{n-1}(A)$ is a homomorphism, since if $\partial([c_1]) = [a_1]$ and $\partial([c_2]) = [a_2]$ via elements b_1 and b_2 in B_n , then

$$j(b_1 + b_2) = j(b_1) + j(b_2) = c_1 + c_2, \quad i(a_1 + a_2) = i(a_1) + i(a_2) = \partial(b_1) + \partial(b_2) = \partial(b_1 + b_2),$$

so $\partial([c_1] + [c_2]) = [a_1] + [a_2]$.

Theorem 2.12. *The sequence*

$$\cdots \rightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \rightarrow \cdots$$

is exact.

Proof. Diagram chase, see Hatcher. □

Let i be the inclusion and j be the quotient.

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ \cdots & \xrightarrow{\partial} & C_n(A) & \xrightarrow{\partial} & C_{n-1}(A) & \xrightarrow{\partial} & \cdots \\ & & \downarrow i & & \downarrow i & & \\ \cdots & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \cdots \\ & & \downarrow j & & \downarrow j & & \\ \cdots & \xrightarrow{\partial} & C_n(X, A) & \xrightarrow{\partial} & C_{n-1}(X, A) & \xrightarrow{\partial} & \cdots \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

This diagram commutes, so this is a short exact sequence of chain complexes. Zig-zag gives a long exact sequence of homology groups

$$\cdots \rightarrow H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \rightarrow 0.$$

What is $\partial : H_n(X, A) \rightarrow H_{n-1}(A)$? If $[a] \in H_n(X, A)$ is represented by a cycle $\alpha \in C_n(X)$, then $\partial([a])$ is the class of the cycle $\partial(\alpha)$, so $\partial([a]) = [\partial(\alpha)]$. We also obtain a short exact sequence of the augmented chain complex

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & C_1(A) & \longrightarrow & C_0(A) & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & C_1(X) & \longrightarrow & C_0(X) & \longrightarrow & \mathbb{Z} \longrightarrow 0, \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & C_1(X, A) & \longrightarrow & C_0(X, A) & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

so if $A \neq \emptyset$, then $\widetilde{H}_n(X, A) = H_n(X, A)$ for all n . We also have a long exact sequence

$$\cdots \rightarrow \widetilde{H}_n(A) \rightarrow \widetilde{H}_n(X) \rightarrow \widetilde{H}_n(X, A) \rightarrow \widetilde{H}_{n-1}(A) \rightarrow \widetilde{H}_{n-1}(X) \rightarrow \widetilde{H}_{n-1}(X, A) \rightarrow \cdots$$

An observation is if $x_0 \in X$ then $H_n(X, x_0) \cong \widetilde{H}_n(X)$ for all n . Another observation is that a continuous map $f : X \rightarrow Y$ such that $f(A) \subseteq B$ induces a chain map

$$f_{\#} : C_n(X, A) \rightarrow C_n(Y, B),$$

since $f_{\#} : C_n(X) \rightarrow C_n(Y)$ maps $C_n(A)$ to $C_n(B)$ so it is well-defined on the quotient, and hence homomorphisms

$$f_* : H_n(X, A) \rightarrow H_n(Y, B).$$

This is functorial, so $(f \circ g)_* = f_* \circ g_*$.

Definition. A **homotopy** between two maps

$$f, g : (X, A) \rightarrow (Y, B)$$

is a continuous map $F : I \times X \rightarrow Y$ such that

$$F(0, x) = f(x), \quad F(1, x) = g(x), \quad F(s, a) \in B, \quad x \in X, \quad s \in I, \quad a \in A.$$

Proposition 2.13. *If*

$$f, g : (X, A) \rightarrow (Y, B)$$

are homotopic, then

$$f_* = g_* : H_n(X, A) \rightarrow H_n(Y, B).$$

Proof. Analogous to proof of Theorem 2.6. Prism operator $P : C_n(X) \rightarrow C_{n+1}(Y)$ maps $C_n(A)$ to $C_n(B)$ so it induces a map

$$P' : \frac{C_n(X)}{C_n(A)} \rightarrow \frac{C_{n+1}(Y)}{C_{n+1}(B)},$$

and $\partial P' + P' \partial = g_{\#} - f_{\#}$, so $f_* = g_*$. □

Let (X, A, B) be a triple, for X a topological space and $B \subset A \subset X$, so

$$(A, B) \rightarrow (X, B) \rightarrow (X, A).$$

There is a short exact sequence of chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_n(A, B) & \longrightarrow & C_n(X, B) & \longrightarrow & C_n(X, A) \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & \frac{C_n(A)}{C_n(B)} & & \frac{C_n(X)}{C_n(B)} & & \frac{C_n(X)}{C_n(A)} \end{array},$$

so there is a long exact sequence

$$\cdots \rightarrow H_n(A, B) \rightarrow H_n(X, B) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A, B) \rightarrow H_{n-1}(X, B) \rightarrow H_{n-1}(X, A) \rightarrow \cdots$$

2.5.3 Excision

Theorem 2.14 (Excision). *Let X be a topological space and $Z \subset A \subset X$ be subspaces such that the closure \bar{Z} of Z is contained in the interior \mathring{A} of A . Then the inclusion*

$$(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$$

induces isomorphisms

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_n(X, A),$$

for all n . Equivalently, let $A, B \subseteq X$ such that $\mathring{A} \cup \mathring{B} = X$. Then the inclusion

$$(B, A \cap B) \rightarrow (X, A)$$

induces isomorphisms

$$H_n(B, A \cap B) \xrightarrow{\sim} H_n(X, A),$$

for all n .

Why equivalent? Set $B = X \setminus Z$ and $Z = X \setminus B$. Then $A \cap B = A \setminus Z$ and $\bar{Z} = X \setminus \mathring{B}$. Then $\bar{Z} \subseteq \mathring{A}$ if and only if $X = \mathring{A} \cup \mathring{B}$.

Proof. Hatcher page 119 to 124. □

Proposition 2.15. *Let (X, A) be a good pair. Then the quotient map*

$$q : (X, A) \rightarrow \left(\frac{X}{A}, \frac{A}{A} \right)$$

induces isomorphisms

$$q_* : H_n(X, A) \xrightarrow{\sim} H_n\left(\frac{X}{A}, \frac{A}{A}\right) \cong \widetilde{H}_n\left(\frac{X}{A}\right),$$

for all n .

Proof. Let $V \subseteq X$ be a neighbourhood of A that strongly deformation retracts to A . Then (V, A) is homotopy equivalent to (A, A) , so

$$H_n(V, A) \cong H_n(A, A) = 0.$$

The triple (X, V, A) where $A \subset V \subset X$ induces a long exact sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_n(V, A) & \longrightarrow & H_n(X, A) & \longrightarrow & H_n(X, V) \longrightarrow H_{n-1}(V, A) \longrightarrow \dots \\ & & \downarrow \text{id} & & & & \downarrow \text{id} \\ & & 0 & & & & 0 \end{array},$$

so

$$H_n(X, A) \cong H_n(X, V).$$

Same with the triple $(X/A, V/A, A/A)$, so again

$$H_n\left(\frac{V}{A}, \frac{A}{A}\right) \cong H_n\left(\frac{A}{A}, \frac{A}{A}\right) = 0.$$

This gives a long exact sequence

$$H_n\left(\frac{X}{A}, \frac{A}{A}\right) \cong H_n\left(\frac{X}{A}, \frac{V}{A}\right).$$

Consider the diagram

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{\sim} & H_n(X, V) & \xleftarrow{\sim \alpha} & H_n(X \setminus A, V \setminus A) \\ \downarrow q_* & & \downarrow q_* & & \sim \downarrow j \\ H_n\left(\frac{X}{A}, \frac{A}{A}\right) & \xrightarrow{\sim} & H_n\left(\frac{X}{A}, \frac{V}{A}\right) & \xleftarrow{\sim \beta} & H_n\left(\frac{X}{A} \setminus \frac{A}{A}, \frac{V}{A} \setminus \frac{A}{A}\right) \end{array}.$$

- This diagram commutes.
- $q : X \rightarrow X/A$ induces a homeomorphism $X \setminus A \rightarrow X/A \setminus A/A$, so j is an isomorphism.
- α and β are isomorphisms by the excision theorem.

Thus

$$q_* : H_n(X, A) \rightarrow H_n\left(\frac{X}{A}, \frac{A}{A}\right)$$

is an isomorphism. □

Proof of Theorem 2.8. Long exact sequence of pair (X, A) with reduced homology

$$\dots \rightarrow \widetilde{H}_n(A) \rightarrow \widetilde{H}_n(X) \rightarrow \widetilde{H}_n(X, A) \rightarrow \widetilde{H}_{n-1}(A) \rightarrow \widetilde{H}_{n-1}(X) \rightarrow \widetilde{H}_{n-1}(X, A) \rightarrow \dots,$$

so

$$\widetilde{H}_n(X, A) = H_n(X, A) \cong \widetilde{H}_n\left(\frac{X}{A}\right),$$

by last time. □

Corollary 2.16. *Let $\{X_\alpha\}$, for $\alpha \in A$, be a collection of topological spaces and $x_\alpha \in X_\alpha$ such that (X_α, x_α) is a good pair, for all $\alpha \in A$. Let $\bigvee_\alpha X_\alpha$ be the wedge sum with respect to the points x_α . Then there is an isomorphism*

$$\widetilde{H}_n \left(\bigcup_\alpha X_\alpha \right) \cong \bigoplus_\alpha \widetilde{H}_n(X_\alpha) \xrightarrow{\sim} \widetilde{H}_n \left(\bigvee_\alpha X_\alpha \right).$$

Proof. $(X, A) = (\bigcup_\alpha X_\alpha, \bigcup_\alpha \{x_\alpha\})$ is a good pair, so Proposition 2.15 implies that

$$H_n(X, A) \cong H_n \left(\bigvee_\alpha X_\alpha, \bigcup_\alpha \{x_\alpha\} \right) \cong \widetilde{H}_n \left(\bigvee_\alpha X_\alpha \right),$$

and

$$H_n(X, A) \cong \bigoplus_\alpha H_n(X_\alpha, x_\alpha) \cong \bigoplus_\alpha \widetilde{H}_n(X_\alpha).$$

□

Example.

$$\widetilde{H}_n(S^1 \vee S^1) \cong \widetilde{H}_n(S^1) \oplus \widetilde{H}_n(S^1) \cong \begin{cases} 0 & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & n \geq 2 \end{cases}.$$

$$\widetilde{H}_n(S^1 \vee S^1 \vee S^2) \cong \widetilde{H}_n(S^1) \oplus \widetilde{H}_n(S^1) \oplus \widetilde{H}_n(S^2) \cong \begin{cases} 0 & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}.$$

Recall that

$$H_n^\Delta(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}.$$

We will see that singular and simplicial homology coincide in Appendix A.2, so $S^1 \vee S^1 \vee S^2$ and $S^1 \times S^1$ have isomorphic homology groups, but they are not homotopy equivalent.

Theorem 2.17 (Invariance of dimension). *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open, non-empty. If U and V are homeomorphic, then $m = n$.*

Proof. For $x \in U$ set $A = \mathbb{R}^m \setminus \{x\}$ and $B = U$. Excision implies that

$$H_k(U, U \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}).$$

Long exact sequence of a pair implies that

$$\begin{array}{ccccccc} \dots & \longrightarrow & \widetilde{H}_k(\mathbb{R}^m) & \longrightarrow & \widetilde{H}_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) & \longrightarrow & \widetilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\}) \longrightarrow \widetilde{H}_{k-1}(\mathbb{R}^m) \longrightarrow \dots \\ & & \downarrow \text{\scriptsize \mathbb{R}} & & & & \downarrow \text{\scriptsize \mathbb{R}} \\ & & 0 & & & & 0 \end{array}$$

so $H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\})$. $\mathbb{R}^m \setminus \{x\}$ deformation retracts to S^{m-1} , so

$$H_k(U, U \setminus \{x\}) = \begin{cases} \mathbb{Z} & k = m \\ 0 & \text{otherwise} \end{cases}.$$

Similarly

$$H_k(V, V \setminus \{x\}) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise} \end{cases}.$$

Let $h : U \rightarrow V$ be a homeomorphism then this induces isomorphisms

$$h_* : H_k(U, U \setminus \{x\}) \rightarrow H_k(V, V \setminus \{h(x)\}),$$

for all k , so $m = n$. □

2.5.4 Naturality

Proposition 2.18 (Naturality of connecting homomorphisms). *Let*

$$(A, \partial), (B, \partial), (C, \partial), (A', \partial), (B', \partial), (C', \partial)$$

be chain complexes. Consider a commutative diagram of chain maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' \longrightarrow 0 \end{array},$$

where the rows are short exact sequences. Then the induced diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H_n(A) & \xrightarrow{i_*} & H_n(B) & \xrightarrow{j_*} & H_n(C) & \xrightarrow{\partial} & H_{n-1}(A) & \xrightarrow{i_*} & H_{n-1}(B) & \xrightarrow{j_*} & H_{n-1}(C) \longrightarrow \dots \\ & & \downarrow \alpha_* & & \downarrow \beta_* & & \downarrow \gamma_* & & \downarrow \alpha_* & & \downarrow \beta_* & & \downarrow \gamma_* \\ \dots & \longrightarrow & H_n(A') & \xrightarrow{i'_*} & H_n(B') & \xrightarrow{j'_*} & H_n(C') & \xrightarrow{\partial} & H_{n-1}(A') & \xrightarrow{i'_*} & H_{n-1}(B') & \xrightarrow{j'_*} & H_{n-1}(C') \longrightarrow \dots \end{array}$$

is commutative.

Proof. The first two squares commute by functoriality.

$$\begin{array}{ccc} \partial & : & H_n(C) \longrightarrow H_{n-1}(A) \\ & & [c] \longmapsto [a] \end{array},$$

so

$$\begin{array}{c} a \in A_{n-1} \\ \downarrow i \\ b \in B_n \xrightarrow{\partial} \in \partial(b) \in B_{n-1} \\ \downarrow j \\ c \in C_n \end{array}$$

$\gamma(c) = \gamma(j(b)) = j'(\beta(b))$ and $i'(\alpha(a)) = \beta(i(a)) = \beta(\partial(b)) = \partial(\beta(b))$, so

$$\begin{array}{c} \alpha(a) \in A'_{n-1} \\ \downarrow i' \\ \beta(b) \in B'_n \xrightarrow{\partial} \in \partial(\beta(b)) \in B'_{n-1} \\ \downarrow j' \\ \gamma(c) \in C'_n \end{array}$$

so $\partial[\gamma(c)] = [\alpha(a)]$ and hence $\partial(\gamma_*[c]) = \alpha_*[a] = \alpha_*(\partial[c])$. □

2.6 Mayer-Vietoris sequences

2.6.1 The Mayer-Vietoris sequence

The main ingredient of the proof of the excision theorem is **barycentric subdivision**. Let X be a topological space and $\mathcal{U} = \{U_i\}$ be a collection of subspaces whose interiors form an open cover of X . Define $C_n^{\mathcal{U}} \subseteq C_n(X)$ as the subgroup of all chains of the form $\sum_i n_i \sigma_i$ such that the image of σ_i is contained in some $U_j \in \mathcal{U}$. Then $\partial : C_n(X) \rightarrow C_{n-1}(X)$ satisfies $\partial(C_n^{\mathcal{U}}(X)) \subseteq C_{n-1}^{\mathcal{U}}(X)$ so the $C_n^{\mathcal{U}}(X)$ define a chain complex. Let $H_n^{\mathcal{U}}(X)$ be the homology groups with respect to this chain complex.

Proposition 2.19. *The inclusion $i : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ induces isomorphisms $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all n .*
Proof. Hatcher page 119. \square

Notation. If $\mathcal{U} = \{A, B\}$ we write $C_n(A + B)$ instead of $C_n^{\mathcal{U}}(X)$.

Theorem 2.20 (Mayer-Vietoris sequence). *Let X be a topological space, $A, B \subseteq X$ such that $\mathring{A} \cup \mathring{B} = X$, and*

$$i_1 : A \cap B \hookrightarrow A, \quad i_2 : A \cap B \hookrightarrow B, \quad j_1 : A \hookrightarrow X, \quad j_2 : B \hookrightarrow X$$

be inclusions. Then there is an exact sequence

$$\cdots \rightarrow H_1(A \cap B) \xrightarrow{\Phi} H_1(A) \oplus H_1(B) \xrightarrow{\Psi} H_1(X) \xrightarrow{\partial} H_0(A \cap B) \xrightarrow{\Phi} H_0(A) \oplus H_0(B) \xrightarrow{\Psi} H_0(X) \rightarrow 0,$$

where $\Phi(x) = (i_{1*}(x), -i_{2*}(x))$, $\Psi(x, y) = j_{1*}(x) + j_{2*}(y)$, and ∂ is the connecting homomorphism.

Proof. Let a sequence of chain complexes be

$$0 \rightarrow C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \rightarrow 0,$$

where $\phi(x) = (x, -x)$ and $\psi(x, y) = x + y$.

- ϕ is injective.
- $\text{Im}(\phi) \subseteq \text{Ker}(\psi)$.
- If $(x, y) \in \text{Ker}(\psi)$, then $y = -x$, and $x \in C_n(A)$ and $y \in C_n(B)$, so $x \in C_n(A \cap B)$, so $\text{Ker}(\psi) \subseteq \text{Im}(\phi)$.
- ψ is surjective by the definition of $C_n(A + B)$.

So this is a short exact sequence of chain complexes. This induces a long exact sequence of homology groups

$$\begin{array}{ccccccc} \cdots \rightarrow H_1(A \cap B) & \xrightarrow{\Phi} & H_1(A) \oplus H_1(B) & \xrightarrow{\Psi} & H_1^{A+B}(X) & \xrightarrow{\partial} & H_0(A \cap B) \xrightarrow{\Phi} H_0(A) \oplus H_0(B) \xrightarrow{\Psi} H_0^{A+B}(X) \rightarrow 0 \\ & & & & \text{IR} & & \text{IR} \\ & & & & H_1(X) & & H_0(X) \end{array},$$

by barycentric division. \square

If $A \cap B \neq \emptyset$ we can augment these chain complexes and obtain a short exact sequence between these augmented chain complexes

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & C_0(A \cap B) & \xrightarrow{\phi} & C_0(A) \oplus C_0(B) & \xrightarrow{\psi} & C_0(A + B) \longrightarrow 0 \\ & & \downarrow \epsilon & & \downarrow \epsilon & & \downarrow \epsilon \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \longrightarrow 0 \end{array}$$

This induces a long exact sequence of homology groups

$$\cdots \rightarrow \widetilde{H}_1(A \cap B) \xrightarrow{\Phi} \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \xrightarrow{\Psi} \widetilde{H}_1(X) \xrightarrow{\partial} \widetilde{H}_0(A \cap B) \xrightarrow{\Phi} \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \xrightarrow{\Psi} \widetilde{H}_0(X) \rightarrow 0.$$

This is the Mayer-Vietoris sequence for reduced homology groups.

Note. This is the same as in the non-reduced case, but we need to assume that $A \cap B \neq \emptyset$.

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An observation is that if $A \cap B$ is path-connected, then $\widetilde{H}_0(A \cap B) = 0$, so we have an exact sequence

$$\dots \longrightarrow \widetilde{H}_1(A \cap B) \xrightarrow{\Phi} \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \xrightarrow{\Psi} \widetilde{H}_1(X) \xrightarrow{\partial} \widetilde{H}_0(A \cap B) \longrightarrow \dots$$

$\begin{array}{c} \mathbb{R} \\ 0 \end{array}$

Thus

$$H_1(X) \cong \frac{H_1(A) \oplus H_1(B)}{\Phi(H_1(A \cap B))}.$$

This is the abelianised version of the theorem of Seifert-van Kampen.

Example. Let $X = S^n \subseteq \mathbb{R}^{n+1}$ and let $x \in S^n$. Define $A = S^n \setminus \{x\}$ and $B = S^n \setminus \{-x\}$. A and B are contractible, so $\widetilde{H}_n(A) = \widetilde{H}_n(B) = 0$ for all n . $A \cap B$ deformation retracts to S^{n-1} . Mayer-Vietoris implies that

$$\dots \longrightarrow \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \longrightarrow \widetilde{H}_i(X) \longrightarrow \widetilde{H}_{i-1}(A \cap B) \longrightarrow \widetilde{H}_{i-1}(A) \oplus \widetilde{H}_{i-1}(B) \longrightarrow \dots$$

$\begin{array}{c} \mathbb{R} \\ 0 \end{array}$
 $\begin{array}{c} \mathbb{R} \\ \widetilde{H}_{i-1}(S^{n-1}) \end{array}$
 $\begin{array}{c} \mathbb{R} \\ 0 \end{array}$

so $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1})$ for $n \geq 1$. We know $\widetilde{H}_0(S^0) \cong \mathbb{Z}$ and $\widetilde{H}_0(S^n) = 0$ for $n \geq 1$, so induction and knowledge on $H_n(S^0)$ implies that

$$\widetilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise} \end{cases}.$$

Example. Let $U, V \subseteq \mathbb{R}^n$ be two path-connected open subsets such that $U \cup V = \mathbb{R}^n$. Then $U \cap V$ is path-connected as well. Enough to show that $H_0(U \cap V) \cong \mathbb{Z}$, if and only if $\widetilde{H}_0(U \cap V) = 0$. $U \cap V \neq \emptyset$ because \mathbb{R}^n is connected. U and V are open, so $\mathring{U} = U$ and $\mathring{V} = V$, so $\mathring{U} \cup \mathring{V} = \mathbb{R}^n$. Mayer-Vietoris long exact sequence for reduced homology groups implies that

$$\dots \longrightarrow \widetilde{H}_1(\mathbb{R}^n) \longrightarrow \widetilde{H}_0(U \cap V) \longrightarrow \widetilde{H}_0(U) \oplus \widetilde{H}_0(V) \longrightarrow \widetilde{H}_0(\mathbb{R}^n) \longrightarrow 0$$

$\begin{array}{c} \mathbb{R} \\ 0 \end{array}$
 $\begin{array}{c} \mathbb{R} \\ 0 \end{array}$
 $\begin{array}{c} \mathbb{R} \\ 0 \end{array}$

since \mathbb{R}^n is contractible, so $\widetilde{H}_k(\mathbb{R}^n) = 0$ for all k , and $\widetilde{H}_0(U) = \widetilde{H}_0(V) = 0$, because U and V are path-connected. Thus $\widetilde{H}_0(U \cap V) = 0$.

2.6.2 Classical applications

Definition. Let X and Y be topological spaces. A continuous map $\phi : X \rightarrow Y$ is an **embedding** if it is a homeomorphism to its image.

Example. If X is compact and Y is Hausdorff, and $\phi : X \rightarrow Y$ is a continuous and injective map, then ϕ is an embedding, since $\phi : X \rightarrow \phi(X)$ is continuous and bijective and $\phi(X)$ is Hausdorff, so worksheet 1 implies that ϕ is a homeomorphism $X \rightarrow \phi(X)$.

Proposition 2.21.

1. Let $h : D^k \rightarrow S^n$ be an embedding, then $\widetilde{H}_i(S^n \setminus h(D^k)) = 0$ for all i .
2. Let $h : S^k \rightarrow S^n$ be an embedding, with $k < n$, then

$$\widetilde{H}_i(S^n \setminus h(S^k)) = \begin{cases} \mathbb{Z} & i = n - k - 1 \\ 0 & \text{otherwise} \end{cases}.$$

Corollary 2.22. Let $h : S^1 \rightarrow S^2$ be an embedding. Then $S^2 \setminus h(S^1)$ consists of exactly two path-components.

Proof. $\widetilde{H}_0(S^2 \setminus h(S^1)) \cong \mathbb{Z}$ by Proposition 2.21. □

Corollary 2.23 (Jordan curve theorem). *Let $h : S^1 \rightarrow \mathbb{R}^2$ be an embedding. Then $\mathbb{R}^2 \setminus h(S^1)$ consists of exactly two path-components.*

Proof. \mathbb{R}^2 is homeomorphic to $S^2 \setminus \{x\}$, by stereographic projection. □

Similarly, $\mathbb{R}^n \setminus h(S^{n-1})$ consists of exactly two path-components.

Proof of Proposition 2.21.

1. Induction on k .

$k = 0$. $S^n \setminus h(D^0) \cong \mathbb{R}^n$, so $\widetilde{H}_i(S^n \setminus h(D^n)) = 0$ for all n .

$k - 1 \mapsto k$. Let $h : D^k \rightarrow S^n$ be an embedding. Replace D^k by I^k . For a contradiction, assume there is a cycle α in $S^n \setminus h(I^k)$ that is not a boundary in $S^n \setminus h(I^k)$. Claim that there is a nested sequence of intervals

$$[0, 1] = I_0 \supseteq I_1 \supseteq \dots,$$

such that I_i is of length $1/2^i$ and such that α is a cycle in $S^n \setminus h(I^{k-1} \times I_i)$ but not a boundary in $S^n \setminus h(I^{k-1} \times I_i)$. Let $A = S^n \setminus h(I^{k-1} \times [0, 1/2])$ and $B = S^n \setminus h(I^{k-1} \times [1/2, 1])$, so $A \cap B = S^n \setminus h(I^k)$ and $A \cup B = S^n \setminus h(I^{k-1} \times \{1/2\})$. Induction hypothesis implies that $\widetilde{H}_j(A \cup B) = 0$ for all j . Mayer-Vietoris implies that

$$\begin{array}{ccccccc} \dots & \rightarrow & \widetilde{H}_{j+1}(A \cup B) & \rightarrow & \widetilde{H}_j(A \cap B) & \xrightarrow{\sim} & \widetilde{H}_j(A) \oplus \widetilde{H}_j(B) \rightarrow \widetilde{H}_j(A \cup B) \rightarrow \dots \\ & & \downarrow \cong & & & & \downarrow \cong \\ & & 0 & & & & 0 \end{array},$$

so

$$\widetilde{H}_j(S^n \setminus h(I^k)) \cong \widetilde{H}_j(S^n \setminus h(I^{k-1} \times [0, \frac{1}{2}])) \oplus \widetilde{H}_j(S^n \setminus h(I^{k-1} \times [\frac{1}{2}, 1])).$$

Hence α is a cycle but not a boundary in $S^n \setminus h(I^{k-1} \times [0, 1/2])$ or $S^n \setminus h(I^{k-1} \times [1/2, 1])$. This gives us I_1 . Iterating, this proves the claim. By induction, α is a boundary of some cycle β in $S^n \setminus h(I^{k-1} \times \{x\})$ for any $x \in I$, so in particular, for $\{x\} = \bigcap_i I_i$. $\beta = \sum_i n_i \sigma_i$ is a sum of finitely many singular simplices. The images of the σ_i are compact. But $S^n \setminus h(I^{k-1} \times I_i)$ form an open cover of $S^n \setminus h(I^{k-1} \times \{x\})$. So, by compactness, β is a chain in $S^n \setminus h(I^{k-1} \times I_i)$ for some i . Thus α is a boundary in $S^n \setminus h(I^{k-1} \times I_i)$, a contradiction.

2. Induction on k .

$k = 0$. $S^n \setminus h(S^0) \cong S^{n-1} \times \mathbb{R}$, so

$$\widetilde{H}_i(S^n \setminus h(S^0)) \cong \begin{cases} \mathbb{Z} & i = n - k - 1 \\ 0 & \text{otherwise} \end{cases}.$$

$k - 1 \mapsto k$. Let $h : S^k \rightarrow S^n$ be an embedding and $S^k = D_+^k \cup D_-^k$. Let $A = S^n \setminus h(D_+^k)$ and $B = S^n \setminus h(D_-^k)$, so 1 implies that $\widetilde{H}_i(A) = 0$ and $\widetilde{H}_i(B) = 0$ for all i , and $A \cap B = S^n \setminus h(S^k)$ and $A \cup B = S^n \setminus h(S^{k-1})$. Mayer-Vietoris implies that

$$\begin{array}{ccccccc} \dots & \rightarrow & \widetilde{H}_{i+1}(A) \oplus \widetilde{H}_{i+1}(B) & \rightarrow & \widetilde{H}_i(A \cup B) & \xrightarrow{\sim} & \widetilde{H}_i(A \cap B) \rightarrow \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \rightarrow \dots \\ & & \downarrow \cong & & & & \downarrow \cong \\ & & 0 & & & & 0 \end{array},$$

by 1, so

$$\widetilde{H}_{i+1}(S^n \setminus h(S^{k-1})) \cong \widetilde{H}_i(S^n \setminus h(S^k)) \cong \begin{cases} \mathbb{Z} & i + 1 = n - (k - 1) - 1 \\ 0 & \text{otherwise} \end{cases},$$

by induction.

□

Lecture 29 is a problem class.

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2.7 Degree

Let $n \geq 1$. We have seen that $H_n(S^n) \cong \langle a \rangle \cong \mathbb{Z}$. Let $f : S^n \rightarrow S^n$ be a continuous map, so $f_* : H_n(S^n) \rightarrow H_n(S^n)$ is a homomorphism. Then f_* is given by $f_*(\alpha) = d\alpha$ for some $d \in \mathbb{Z}$ depending only on f . This integer is the **degree** of f .

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Proposition 2.24. *The following are observations.*

1. $\deg(id_{S^n}) = 1$.
2. If f is not surjective, then $\deg(f) = 0$.
3. If $f \cong g$, then $f_* = g_*$, so $\deg(f) = \deg(g)$.
4. $\deg(fg) = \deg(f)\deg(g)$. In particular, if f is a homotopy equivalence, then $\deg(f) = \pm 1$.

5. Let

$$R_i : \begin{array}{ccc} S^n & \longrightarrow & S^n \\ (x_1, \dots, x_i, \dots, x_{n+1}) & \longmapsto & (x_1, \dots, -x_i, \dots, x_{n+1}) \end{array}$$

be the reflection map. Then $\deg(R_i) = -1$.

6. The antipodal map

$$\begin{array}{ccc} -id_{S^n} & : & S^n \longrightarrow S^n \\ x & \longmapsto & -x \end{array}$$

has degree $(-1)^{n+1}$.

7. If $f : S^n \rightarrow S^n$ has no fixed points, then $\deg(f) = (-1)^{n+1}$.

Hopf implies that if $\deg(f) = \deg(g)$ then $f \cong g$.

Proof. 1 and 3 are clear.

2. Let $x_0 \in S^n \setminus f(S^n)$. So f factors as $f = i \circ f'$, where

$$S^n \xrightarrow{f'} S^n \setminus \{x_0\} \xrightarrow{i} S^n.$$

$H_n(S^n \setminus \{x_0\}) = 0$ since $S^n \setminus \{x_0\}$ is contractible, so $f_* = i_* \circ f'_* = 0$.

4. $(fg)_* = f_*g_*$, and there exists $g : S^n \rightarrow S^n$ such that $fg \cong id_{S^n}$, so

$$\deg(f)\deg(g) = \deg(fg) = \deg(id_{S^n}) = 1.$$

5. Enough to show it for $i = 1$. Induction on n .

$n = 1$. $R_1(x_1, x_2) = (-x_1, x_2)$. Then $\omega : t \mapsto (\cos(2\pi t), \sin(2\pi t))$ implies that $R_1([\omega]) = -[\omega]$, so $\deg(R_1) = -1$.

$n - 1 \mapsto n$. Claim that there is an isomorphism $\phi : H_n(S^n) \xrightarrow{\sim} H_{n-1}(S^{n-1})$ such that

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{\phi} & H_{n-1}(S^{n-1}) \\ \downarrow R_{1*} & & \downarrow R_{1*} \\ H_n(S^n) & \xrightarrow{\phi} & H_{n-1}(S^{n-1}) \end{array}$$

commutes. Let

$$N = (0, \dots, 0, 1), \quad S = (0, \dots, 0, -1), \quad U = S^n \setminus \{N\}, \quad V = S^n \setminus \{S\},$$

so $R_1(U) = U$ and $R_1(V) = V$. There is a commutative diagram of chain maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & C.(U \cap V) & \longrightarrow & C.(U) \oplus C.(V) & \longrightarrow & C.(U + V) \longrightarrow 0 \\ & & \downarrow R_{1\#} & & \downarrow R_{1\#} \oplus R_{1\#} & & \downarrow R_{1\#} \\ 0 & \longrightarrow & C.(U \cap V) & \longrightarrow & C.(U) \oplus C.(V) & \longrightarrow & C.(U + V) \longrightarrow 0 \end{array}.$$

This induces a commutative diagram

$$\begin{array}{ccccc} H_n(S^n) & \xrightarrow{\partial} & H_{n-1}(U \cap V) & \xleftarrow{i_*} & H_{n-1}(S^{n-1}) \\ \downarrow R_{1*} & & \downarrow R_{1*} & & \downarrow R_{1*} \\ H_n(S^n) & \xrightarrow{\partial} & H_{n-1}(U \cap V) & \xleftarrow{i_*} & H_{n-1}(S^{n-1}) \end{array},$$

where

$$\begin{aligned} i : S^{n-1} &\longrightarrow U \cap V \\ (x_1, \dots, x_n) &\longmapsto (x_1, \dots, x_n, 0) \end{aligned}$$

is a homotopy equivalence. i_* is an isomorphism because i is a homotopy equivalence and ∂ is an isomorphism as seen last week. The first square commutes by naturality and the second square commutes by functoriality.

6. $-id_{S^n} = R_1 \dots R_{n+1}$, so

$$\deg(-id_{S^n}) = \deg(R_1) \dots \deg(R_{n+1}) = (-1)^{n+1}.$$

7. If $f(x) \neq x$ for all $x \in S^n$, then the line segment from $f(x)$ to $-x$ defined by

$$t \mapsto (1-t)f(x) - tx$$

does not pass through the origin. Define

$$f_t(x) = \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|},$$

so f_t is a homotopy from f to $-id_{S^n}$. Thus

$$\deg(f) = \deg(-id_{S^n}) = (-1)^{n+1}.$$

□

Proposition 2.25. *If n is even, then $\mathbb{Z}/2\mathbb{Z}$ is the only non-trivial group that can act freely by homeomorphisms on S^n .*

Proof. Let G be a group acting freely by homeomorphisms on S^n , so $G \subseteq \text{Homeo}(S^n)$. So for $f \in G$, $\deg(f) = \pm 1$ by 4, and $\deg(fg) = \deg(f)\deg(g)$ for all $f, g \in G$ by 3, so the degree defines a homeomorphism $d : G \rightarrow \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$. The action is free, so if $g \in G \setminus \{id\}$, then g has no fixed points, so 7 and n even implies that $\deg(g) = (-1)^{n+1} = -1$. Then $\text{Ker}(d) = \{id\}$, so d is injective, so $G = \{id\}$ or $G \cong \mathbb{Z}/2\mathbb{Z}$. □

Definition. A **vector field** on S^n is a continuous map $v : S^n \rightarrow \mathbb{R}^{n+1}$ such that for each $x \in S^n$, $v(x)$ is **tangent** to S^n at x , that is $v(x)$ and x are orthogonal.

Theorem 2.26 (Hairy ball theorem). *S^n admits a continuous vector field $v : S^n \rightarrow \mathbb{R}^{n+1}$ that is nowhere zero if and only if n is odd.*

Proof. If $v(x) \neq 0$ for all $x \in S^n$, let

$$\begin{aligned} v' : S^n &\longrightarrow \mathbb{R}^{n+1} \\ x &\longmapsto \frac{v(x)}{|v(x)|}. \end{aligned}$$

Define

$$f_t(x) = \cos(t\pi)x + \sin(t\pi)v'(x).$$

Then $f_t(x) \in S^n$ for all $x \in S^n$ and for all $t \in I$, so f_t is a homotopy from id_{S^n} to $-id_{S^n}$, so

$$1 = \deg(id_{S^n}) = \deg(-id_{S^n}) = (-1)^{n+1}.$$

Thus n is odd. Conversely, if $n = 2k - 1$,

$$v(x_1, \dots, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$$

is a vector field on S^n . □

A Proofs

A.1 The Seifert-van Kampen theorem

Proof of Theorem 1.21. Consider the natural homomorphism

$$\Phi : \pi_1(U_1, x_0) * \pi_1(U_2, x_0) \rightarrow \pi_1(X, x_0).$$

Φ is surjective by Lemma 1.17. $N \subseteq \text{Ker}(\Phi)$. Want to show that $N = \text{Ker}(\Phi)$. A **factorisation** of an element $[f] \in \pi_1(X, x_0)$ is a formal product $[f_1] \dots [f_k]$ such that

- each f_i is a loop at x_0 in one of the U_i and $[f_i] \in \pi_1(U_i, x_0)$ is its homotopy class, and
- the loop $f_1 \cdot \dots \cdot f_k$ is homotopic to f in X .

A factorisation of $[f]$ is a word in $\pi_1(U_1, x_0) * \pi_1(U_2, x_0)$ that is mapped to $[f]$ by Φ . Two factorisations of $[f]$ are **equivalent** if they are related by finitely many of the following two moves.

- If $[f_i]$ and $[f_{i+1}]$ lie in the same group $\pi_1(U_i, x_0)$, exchange $[f_i][f_{i+1}]$ with $[f_i \cdot f_{i+1}]$. These are the relations in $\pi_1(U_i, x_0) * \pi_1(U_i, x_0)$.
- If f_i is a loop in $U_1 \cap U_2$, consider $[f_i]$ as an element in $\pi_1(U_1, x_0)$ instead of $\pi_1(U_2, x_0)$, and vice versa. These are the relations in $\pi_1(U_1, x_0) * \pi_1(U_2, x_0) / N$.

Given $[f] \in \pi_1(X, x_0)$, we want to show that any two factorisations of $[f]$ are equivalent. Let $[f_1] \dots [f_k]$ and $[f'_1] \dots [f'_l]$ be two factorisations of $[f]$, so the two loops $f_1 \cdot \dots \cdot f_k$ and $f'_1 \cdot \dots \cdot f'_l$ are homotopic. Let $F : I \times I \rightarrow X$ be a homotopy. By compactness, there exist

$$0 = s_0 < \dots < s_m = 1, \quad 0 = t_0 < \dots < t_n = 1,$$

such that $R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ and $F(R_{i,j}) \subseteq U_1$ or $F(R_{i,j}) \subseteq U_2$. May assume $0 = s_0 < \dots < s_m = 1$ subdivides the products $f_1 \cdot \dots \cdot f_k$ and $f'_1 \cdot \dots \cdot f'_l$. Relabel the $R_{i,j}$ to R_1, \dots, R_{mn} .

$mn - m + 1$	\dots	mn
\vdots	\ddots	\vdots
1	\dots	m

A path γ in $I \times I$ from left to right gives a loop $F|_\gamma$ in X at x_0 . Let γ_r be the path separating the first r rectangles from the others, so

$$F|_{\gamma_0} \cong f_1 \cdot \dots \cdot f_k, \quad F|_{\gamma_{mn}} = f'_1 \cdot \dots \cdot f'_l.$$

Let v be a grid point. Choose a path g_v in X from x_0 to $F(v)$, such that g_v is contained in $U_1 \cap U_2$ if $F(v) \in U_1 \cap U_2$ and in a single U_i otherwise. This gives us a factorisation of $[F|_{\gamma_r}]$ into loops only contained in U_1 or U_2 . The factorisations associated to γ_r and γ_{r+1} are equivalent, because the homotopy between $F|_{\gamma_r}$ and $F|_{\gamma_{r+1}}$ by pushing γ_r through R_r takes place within a single U_i . \square

A.2 The equivalence of simplicial and singular homology

Lemma A.1 (Five lemma). *Consider the following diagram of abelian groups*

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ A & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}.$$

If the rows are exact and $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is an isomorphism.

Proof. Enough to show

- if β and δ are surjective and ϵ is injective, then γ is surjective, and
- if β and δ are injective and α is surjective, then γ is injective.

□

Let $n \geq 1$. Then

$$H_n(\Delta^n, \partial\Delta^n) \cong \widetilde{H}_n\left(\frac{\Delta^n}{\partial\Delta^n}\right) \cong \widetilde{H}_n(S^n) \cong \mathbb{Z},$$

and $H_0(\Delta^0, \partial\Delta^0) \cong \mathbb{Z}$.

Lemma A.2. $H_n(\Delta^n, \partial\Delta^n)$ is generated by the class of the cycle $i_n : \Delta^n \rightarrow \Delta^n$.

Proof. i_n is a cycle. Induction on n .

$n = 0$. $H_0(\Delta^0, \emptyset)$ is generated by $[i_0]$.

$n - 1 \mapsto n$. Let $\Lambda \subseteq \partial\Delta^n$ be the union of all but one of the $(n - 1)$ -dimensional faces of Δ^n . Δ^n strongly deformation retracts to Λ , so

$$H_i(\Delta^n, \Lambda) = H_i(\Lambda, \Lambda) = 0.$$

Long exact sequence of the triple $\Lambda \subseteq \partial\Delta^n \subseteq \Delta^n$ implies that

$$\begin{array}{ccccccc} \dots & \rightarrow & H_n(\Delta^n, \Lambda) & \rightarrow & H_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\sim} & H_{n-1}(\partial\Delta^n, \Lambda) \rightarrow H_{n-1}(\Delta^n, \Lambda) \rightarrow \dots \\ & & \parallel & & & & \parallel \\ & & 0 & & & & 0 \end{array}.$$

Note that $\partial\Delta^n/\Lambda$ is homeomorphic to $\Delta^{n-1}/\partial\Delta^{n-1}$, which are good pairs, so

$$H_n(\Delta^n, \partial\Delta^n) \cong H_{n-1}(\partial\Delta^n, \Lambda) \cong \widetilde{H}_{n-1}\left(\frac{\partial\Delta^n}{\Lambda}\right) \cong \widetilde{H}_{n-1}\left(\frac{\Delta^{n-1}}{\partial\Delta^{n-1}}\right) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$

One can check that $[i_n]$ maps to $[\pm i_{n-1}]$ along these isomorphisms, so induction implies that $H_n(\Delta^n, \partial\Delta^n)$ is generated by $[i_n]$.

□

Let X be a topological space with a Δ -complex structure, so there is a simplicial chain complex

$$\cdots \rightarrow \Delta_{n+1}(X) \rightarrow \Delta_n(X) \rightarrow \Delta_{n-1}(X) \rightarrow \cdots$$

Every simplicial chain complex can be viewed as a singular n -chain, so we obtain an inclusion of chain complexes $\Delta_*(X) \rightarrow C_*(X)$.

Theorem A.3. *This inclusion of chain complexes induces an isomorphism $H_n^\Delta(X) \xrightarrow{\sim} H_n(X)$ for all n .*

Proof. We only consider the case, where the Δ -complex structure on X is finite dimensional, that is $\Delta_m(X) = 0$ for all $m > k$, and the maximal such k is $\dim(X)$. Induction on k , the dimension.

$k = 0$. $H_n^\Delta(X) \cong H_n(X)$ for X points.

$k - 1 \mapsto k$. Let X^l be the l -skeleton of X consisting of all simplices of dimension at most l . Then $H_n^\Delta(X^k, X^{k-1})$ are the homology groups of the chain complex

$$\cdots \rightarrow \frac{\Delta_{k+1}(X^k)}{\Delta_{k+1}(X^{k-1})} \rightarrow \frac{\Delta_k(X^k)}{\Delta_k(X^{k-1})} \rightarrow \frac{\Delta_{k-1}(X^k)}{\Delta_{k-1}(X^{k-1})} \rightarrow \cdots,$$

so

$$H_n^\Delta(X^k, X^{k-1}) = \begin{cases} 0 & n \neq k \\ \text{free abelian group with basis the } k\text{-simplices of } X & n = k \end{cases}.$$

The short exact sequence of chain complexes

$$0 \rightarrow \Delta_n(X^{k-1}) \rightarrow \Delta_n(X^k) \rightarrow \frac{\Delta_n(X^k)}{\Delta_n(X^{k-1})} \rightarrow 0$$

gives a long exact sequence

$$\begin{array}{ccccccccc} \rightarrow H_{n+1}^\Delta(X^k, X^{k-1}) & \rightarrow & H_n^\Delta(X^{k-1}) & \rightarrow & H_n^\Delta(X^k) & \rightarrow & H_n^\Delta(X^k, X^{k-1}) & \rightarrow & H_{n-1}^\Delta(X^{k-1}) \rightarrow \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ \rightarrow H_{n+1}(X^k, X^{k-1}) & \rightarrow & H_n(X^{k-1}) & \rightarrow & H_n(X^k) & \rightarrow & H_n(X^k, X^{k-1}) & \rightarrow & H_{n-1}(X^{k-1}) \rightarrow \end{array},$$

which commutes by naturality. β and ϵ are isomorphisms by induction. Consider the continuous map

$$\Phi : \bigsqcup_{\alpha} (\Delta_{\alpha}^k, \partial \Delta_{\alpha}^k) \rightarrow (X^k, X^{k-1}).$$

This induces an isomorphism

$$H_n(X^k, X^{k-1}) \cong H_n\left(\bigsqcup_{\alpha} \Delta_{\alpha}^k, \bigsqcup_{\alpha} \partial \Delta_{\alpha}^k\right) = \bigoplus_{\alpha} H_n(\Delta_{\alpha}^k, \partial \Delta_{\alpha}^k),$$

which is the free abelian group on $i_{n\alpha} : \Delta_{\alpha}^n \rightarrow \Delta_{\alpha}^n$ by Lemma A.2, so α and δ are isomorphisms. Thus five lemma implies that γ is an isomorphism. □