

M4P54 Differential Topology

Lectured by Prof Paolo Cascini
Typed by David Kurniadi Angdinata

Spring 2020

Syllabus

Contents

0	Introduction	3
1	Differential forms on manifolds	4
1.1	Alternating p -forms on a vector space	4
1.2	Differential forms on manifolds	5
1.3	Local description of p -forms	6
1.4	Integrations on manifolds	7
1.5	Orientation	8
1.6	Partitions of unity	9
1.7	Manifolds with boundary	11
1.8	Stokes' theorem	11
1.9	Applications of Stokes' theorem	13
2	De Rham cohomology	15
2.1	De Rham cohomology	15
2.2	Homotopy invariance	16
2.3	Some homological algebra	20
2.4	The Mayer-Vietoris sequence	22
2.5	Compactly supported de Rham cohomology	25
2.6	Poincaré duality	27
2.7	Degree of a morphism	32
3	Morse theory	34
3.1	Introduction	34
3.2	Cell decomposition	34
3.3	CW-complexes	36
3.4	The CW-complex structure associated to a Morse function	37

0 Introduction

Differential topology is the study of the topology of a manifold using analysis. The topics are

- a review of differential forms,
- de Rham cohomology,
- Morse theory, and
- singular homology.

The following are references.

- J M Lee, Introduction to smooth manifolds, 2000
- L W Tu, Introduction to smooth manifolds, 2008
- J Milnor, Morse theory, 1960
- A Banyaga and D Hurtubise, Lectures on Morse homology, 2004

Lecture 1
Thursday
09/01/20

1 Differential forms on manifolds

1.1 Alternating p -forms on a vector space

Let V be a vector space over \mathbb{R} , and let $p \geq 0$. Then $V^p = V \times \cdots \times V$.

Definition 1.1. A multilinear map $\omega : V^p \rightarrow \mathbb{R}$ is called an **alternating p -form** if we have

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(p)}) = \epsilon(\sigma) \omega(v_1, \dots, v_p), \quad v_1, \dots, v_p \in V \quad \sigma \in \mathcal{S}_p,$$

where \mathcal{S}_p is the group of permutations of p elements and $\epsilon(\sigma)$ is the signature of σ .

Recall that if m is the number of transpositions in a decomposition of σ , then $\epsilon(\sigma) = (-1)^m$, where a **transposition** is $(a_i a_j)$ for $a_i \neq a_j$.

Notation 1.2. $\Lambda^p V^* = \{\text{alternating } p\text{-forms } \omega \text{ on } V\}$ is called the **p -th exterior power** of V .

Check that it is a vector space.¹

Example 1.3.

- $\Lambda^0 V^* = \mathbb{R}$.
- $\Lambda^1 V^* = V^* = \text{Hom}(V, \mathbb{R})$, the **dual** of V .

Definition 1.4. Let $\omega_1 \in \Lambda^p V^*$ and $\omega_2 \in \Lambda^q V^*$. We define the **exterior product** $\omega_1 \wedge \omega_2 \in \Lambda^{p+q} V^*$ of ω_1 and ω_2 by

$$\omega_1 \wedge \omega_2(v_1, \dots, v_{p+q}) = \sum_{\sigma \in \mathcal{S}_{p,q}} \epsilon(\sigma) \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \omega_2(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}), \quad v_1, \dots, v_{p+q} \in V,$$

where

$$\mathcal{S}_{p,q} = \{\sigma \in \mathcal{S}_{p+q} \mid \sigma(1) < \cdots < \sigma(p), \sigma(p+1) < \cdots < \sigma(p+q)\}.$$

Example 1.5.

- Assume $\omega_1, \omega_2 \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \omega_2(v_1, v_2) = \omega_1(v_1) \omega_2(v_2) - \omega_1(v_2) \omega_2(v_1), \quad v_1, v_2 \in V.$$

- Assume $\omega_1, \dots, \omega_p \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \cdots \wedge \omega_p(v_1, \dots, v_p) = \det(\omega_i(v_j))_{i,j=1,\dots,p}, \quad v_1, \dots, v_p \in V.$$

Proposition 1.6. Let $\omega_i \in \Lambda^{p_i} V^*$ for $i = 1, 2, 3$.

- *Associativity* $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$.
- *Distributivity* $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$, assuming $p_2 = p_3$.
- *Supercommutativity* $\omega_1 \wedge \omega_2 = (-1)^{p_1 p_2} \omega_2 \wedge \omega_1$.

Definition 1.7. Let $\Phi : V \rightarrow W$ be a linear map between vector spaces over \mathbb{R} . Let $\omega \in \Lambda^p W^*$. Then the **pull-back** $\Phi^* \omega \in \Lambda^p V^*$ of ω is an alternating p -form on V defined by

$$\Phi^* \omega(v_1, \dots, v_p) = \omega(\Phi(v_1), \dots, \Phi(v_p)), \quad v_1, \dots, v_p \in V.$$

¹Exercise

Proposition 1.8. *Given $\Phi : V \rightarrow W$ a linear map,*

- *the pull-back*

$$\begin{aligned} \Phi^* &: \Lambda^p W^* \longrightarrow \Lambda^p V^* \\ \omega &\longmapsto \Phi^* \omega \end{aligned}$$

is a linear map that preserves exterior products, that is

$$\Phi^* (\omega_1 \wedge \omega_2) = \Phi^* \omega_1 \wedge \Phi^* \omega_2, \quad \omega_1 \in \Lambda^p W^*, \quad \omega_2 \in \Lambda^q W^*,$$

- *if $\Psi : W \rightarrow Z$ is linear then*

$$(\Psi \circ \Phi)^* \omega = \Phi^* \Psi^* \omega, \quad \omega \in \Lambda^p Z^*,$$

- *assuming $V = W$ and $p = \dim V$, then*

$$\Phi^* \omega = (\det \Phi) \omega, \quad \omega \in \Lambda^p V^*.$$

1.2 Differential forms on manifolds

Let M be a smooth manifold of dimension n , and let $x \in M$. Then the tangent space $T_x M$ of M at x is a vector space of dimension n .

Notation 1.9. Let

$$\Lambda^p T_x^* M = \Lambda^p (T_x M)^*.$$

Consider the set

$$\Lambda^p T^* M = \bigsqcup_{x \in M} \Lambda^p T_x^* M,$$

the **p -th exterior bundle** on M . There exists a morphism $\pi : \Lambda^p T^* M \rightarrow M$ such that for all $x \in M$, $\pi^{-1}(x) = \Lambda^p T_x^* M$, so $\Lambda^p T^* M$ is a vector bundle and it is a smooth manifold, and π is a smooth morphism.

Example 1.10.

- $\Lambda^0 T^* M = M \times \mathbb{R}$.
- $\Lambda^1 T^* M$ is the **cotangent bundle**, the dual of the tangent bundle.

Definition 1.11. A **differential p -form** ω on M is a smooth section of π , that is it is a smooth morphism $\omega : M \rightarrow \Lambda^p T^* M$ such that $\pi \circ \omega = \text{id}_M$.

Thus, $\omega(x) \in \Lambda^p T_x^* M$.

Notation 1.12.

$$\Omega^p(M) = \{\text{differential } p\text{-forms } \omega \text{ on } M\}, \quad \Omega^\bullet(M) = \bigoplus_p \Omega^p(M).$$

Example 1.13.

$$\Omega^0(M) \cong \{f : M \rightarrow \mathbb{R} \text{ } C^\infty\text{-function}\}.$$

Exercise. If $n = \dim M$, then $\Omega^{n+1}(M) = 0$.

The algebra is the same as last week.

Definition 1.14. Let $\omega_1 \in \Omega^p(M)$ and $\omega_2 \in \Omega^q(M)$. Then $\omega_1 \wedge \omega_2 \in \Omega^{p+q}(M)$ is defined by

$$\omega_1 \wedge \omega_2(x) = \omega_1(x) \wedge \omega_2(x) \in \Lambda^{p+q} T_x^* M, \quad x \in M.$$

By Proposition 1.6, associativity, distributivity, and supercommutativity hold for $\Omega^p(M)$. Let $F : M \rightarrow N$ be a smooth morphism between manifolds. Then for all $x \in M$, the differential of F at x is the linear map

$$DF_x : T_x M \rightarrow T_{F(x)} N.$$

Lecture 2
Monday
13/01/20

Thus, for all $p \geq 0$, we have a natural map, called the **pull-back**,

$$\begin{aligned} F_x^* : \Lambda^p T_{F(x)}^* N &\longrightarrow \Lambda^p T_x^* M \\ \omega(v_1, \dots, v_p) &\longmapsto \omega(DF_x(v_1), \dots, DF_x(v_p)) \end{aligned} \quad , \quad \omega \in \Lambda^p T_{F(x)}^* N, \quad v_1, \dots, v_p \in T_x^* M.$$

Thus, we can define

$$\begin{aligned} F^* : \Omega^p(N) &\longrightarrow \Omega^p(M) \\ \omega(x) &\longmapsto F^* \omega(F(x)) \end{aligned} \quad , \quad \omega \in \Omega^p(N).$$

By Proposition 1.8, the pull-back preserves the exterior product, so

$$F^*(\omega_1 \wedge \omega_2) = F^* \omega_1 \wedge F^* \omega_2.$$

If $G : N \rightarrow P$,

$$(G \circ F)^* \omega = F^* G^* \omega.$$

1.3 Local description of p -forms

Let M be a manifold of dimension n , let $x_0 \in M$, let (U, ϕ) be a local chart around x_0 , and let (x_1, \dots, x_n) be local coordinates around x_0 . A basis of $T_{x_0} M$ is given by

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}.$$

A basis of $T_{x_0}^* M$ is given by

$$\{dx_1, \dots, dx_n\}, \quad dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij}.$$

A basis of $\Lambda^p T_{x_0}^* M$ is

$$dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad i_1 < \dots < i_p.$$

Thus, $\omega \in \Omega^p(M)$ is locally given by

$$\omega(x) = \sum_{|I|=p} f_I(x) dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad I = (i_1, \dots, i_p), \quad i_1 < \dots < i_p,$$

where f_I is a C^∞ -function on U for all I .

Example 1.15. Let $F : M \rightarrow N$ be a smooth morphism between manifolds of dimension n , and let $\omega \in \Omega^n(N)$. Locally,

$$\omega(y) = f(y) dy_1 \wedge \dots \wedge dy_n, \quad y \in N,$$

for some $f \in C^\infty$. By Proposition 1.8,

$$F^* \omega(x) = (f \circ F)(x) \det DF_x dx_1 \wedge \dots \wedge dx_n, \quad x \in M,$$

where $y_i = p_i \circ F$ and $p_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is the i -th projection.

Let $f : M \rightarrow \mathbb{R}$ be a smooth function, so $f \in \Omega^0(M)$. Locally, the **differential** is

$$\begin{aligned} d : \Omega^0(M) &\longrightarrow \Omega^1(M) \\ f &\longmapsto \sum_{i=1}^n \frac{\partial}{\partial x_i} f dx_i \end{aligned}.$$

Check that $df \in \Omega^1(M)$, so df is a 1-form on M . Alternatively, $df = f^* dx$ for dx a 1-form on \mathbb{R} , or $df(X) = X(f)$ for any vector field X on M . More in general, let $\omega \in \Omega^p(M)$. Locally,

$$\omega = \sum_{|I|=p} f_I dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad f_I \in C^\infty,$$

so $d\omega \in \Omega^{p+1}(M)$. Then the **de Rham differential** is

$$\begin{aligned} d : \Omega^p(M) &\longrightarrow \Omega^{p+1}(M) \\ \omega &\longmapsto \sum_{|I|=p} df_I \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p} \end{aligned}.$$

Proposition 1.16.

- The Leibnitz rule

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2, \quad \omega_1 \in \Omega^p(M), \quad \omega_2 \in \Omega^q(M).$$

- $d^2 = 0$, that is

$$d(d\omega) = 0, \quad \omega \in \Omega^p(M).$$

- Let $F : M \rightarrow N$ be a smooth morphism between manifolds. Then

$$F^*d\omega = d(F^*\omega), \quad \omega \in \Omega^p(M),$$

so

$$\begin{array}{ccc} \Omega^p(M) & \xrightarrow{d} & \Omega^{p+1}(M) \\ F^* \uparrow & & \uparrow F^* \\ \Omega^p(N) & \xrightarrow{d} & \Omega^{p+1}(N) \end{array}.$$

Definition 1.17.

- $\omega \in \Omega^p(M)$ is **closed** if $d\omega = 0$.
- $\omega \in \Omega^p(M)$ is **exact** if there exists $\omega' \in \Omega^{p-1}(M)$ such that $d\omega' = \omega$.

ω is exact implies that ω is closed, since if $\omega = d\omega'$ then $d\omega = d^2\omega' = 0$.

1.4 Integrations on manifolds

Let M be a manifold of dimension n , let $F : M \rightarrow M$ be a smooth morphism, and let $\omega \in \Omega^n(M)$. Then

$$F^*\omega(x) = \det DF_x \omega(F(x)).$$

Locally, assume $\omega = f dy_1 \wedge \cdots \wedge dy_n$ for some coordinates (y_1, \dots, y_n) and $f \in C^\infty$. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of M , where $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n$. Then

$$h_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \rightarrow \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n,$$

such that

$$h_{\alpha\beta}^* \omega(x) = (f \circ h_{\alpha\beta})(x) \det(Dh_{\alpha\beta})_x dx_1 \wedge \cdots \wedge dx_n.$$

Let $D \subset \mathbb{R}^n$ be compact such that ∂D has zero measure, so D is a domain of integration, let $f : U \rightarrow \mathbb{R}$ be a C^∞ -function where $U \subset \mathbb{R}^n$ is open such that $D \subset U$, and let $h : U \rightarrow h(U)$ be a diffeomorphism. Then

$$\int_{h^{-1}(D)} f(y) dy_1 \cdots dy_n = \int_{h^{-1}(D)} f(y) dy_1 \wedge \cdots \wedge dy_n = \int_D (f \circ h)(x) |\det Dh_x| dx_1 \wedge \cdots \wedge dx_n.$$

Let us assume that $\omega = f(y) dy_1 \wedge \cdots \wedge dy_n$ on U . We define

$$\int_D \omega = \int_D f(y) dy_1 \wedge \cdots \wedge dy_n, \quad D \subset U.$$

Definition 1.18. Let $U \subset \mathbb{R}^n$ be an open set. We define the **support** of ω as

$$\text{supp } \omega = \overline{\{x \in U \mid \omega(x) \neq 0\}}, \quad \omega(x) \in \Lambda^p T_x^* U.$$

Then ω has **compact support**, if $\text{supp } \omega$ is compact.

Fact. Under this assumption, we can define

$$\int_U \omega = \int_D \omega \in \mathbb{R},$$

which is well-defined. Under the same assumption, if $\phi : V \rightarrow U$ is a diffeomorphism, provided that $\det D\phi_x > 0$, since $\det D\phi_x \neq 0$ for all x , then

$$\int_U \omega = \int_V \phi^* \omega.$$

Lecture 3
Tuesday
14/01/20

1.5 Orientation

Let V be a vector space over \mathbb{R} of dimension n , and let $B = (b_1, \dots, b_n) \subset V$ and $B' = (b'_1, \dots, b'_n) \subset V$ be ordered bases of V . Then B and B' have the **same orientation** if $\det T > 0$ where

$$\begin{array}{ccc} T & : & V \longrightarrow V \\ & & b_i \longmapsto b'_i \end{array}$$

is a linear map. Let $\omega \in \Lambda^n V^*$ for $\omega \neq 0$. Then B and B' have the same orientation if and only if $\omega(b_1, \dots, b_n)$ has the same sign as $\omega(b'_1, \dots, b'_n)$, by Proposition 1.8. An **orientation** Λ of V is a set of all the ordered basis of V with the same orientation. Let $\phi : V \rightarrow W$ be an isomorphism of vector spaces with fixed orientations Λ_v and Λ_w respectively. We say that ϕ is **orientation preserving** if an ordered basis of V induces an ordered basis of W , so Λ_v induces Λ_w . Let $V = \mathbb{R}^n$, and let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. Then e_1, \dots, e_n defines an orientation of V called **positive**. Let M be a manifold. The idea is to find an orientation Λ_x of $T_x M$ for all $x \in M$.

Special case. Let $M = U \subset \mathbb{R}^n$ be open. There exists a natural isomorphism $\phi_x : T_x U \rightarrow \mathbb{R}^n$. Let Λ_x^+ be an orientation on $T_x U$ such that ϕ_x is orientation preserving with respect to the positive orientation on \mathbb{R}^n . Let $\Lambda^+ = \{\Lambda_x^+\}$.

General case. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M . On U_α , we define the orientation so that $(D\phi_\alpha)_x : T_x U_\alpha \rightarrow T_{\phi_\alpha(x)} \phi_\alpha(U) \subset \mathbb{R}^n$ is orientation preserving. This is called the positive orientation on the chart (U_α, ϕ_α) . We define Λ on M , which is a collection of Λ^+ on $T_x M$ for all $x \in M$. Then M is **orientable** if there exists an atlas with positive orientation charts. This coincides in assuming that $\det D(\phi_\beta^{-1} \circ \phi_\alpha) > 0$ for all α and β .

Notation 1.19. For all $p \geq 0$,

$$\Omega_c^p(M) = \{\omega \in \Omega^p(M) \mid \text{supp } \omega \text{ is compact}\}.$$

If M is compact $\Omega_c^p(M) = \Omega^p(M)$. Let $\omega \in \Omega_c^p(M)$. Assume $\text{supp } \omega \subset U$ where (U, ϕ) is a chart of M , and $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$. Assume also that (U, ϕ) is positively oriented. Let $\phi^{-1} : \phi(U) \rightarrow U$ such that $(\phi^{-1})^* \omega \in \Omega_c^p(\phi(U))$, that is $\text{supp } (\phi^{-1})^* \omega \subset \phi(U)$. We define

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega. \quad (1)$$

We need to show that, under the assumptions above, $\int_M \omega$ does not depend on (U, ϕ) . Let $(\bar{U}, \bar{\phi})$ be also a positively oriented chart such that $\text{supp } \omega \subset \bar{U}$. We want to show that

$$\int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\bar{\phi}(\bar{U})} (\bar{\phi}^{-1})^* \omega.$$

Let $\bar{\phi} \circ \phi^{-1} : \phi(U \cap \bar{U}) \rightarrow \bar{\phi}(U \cap \bar{U})$, so

$$\begin{array}{ccc} & U \cap \bar{U} & \\ \phi \swarrow & & \searrow \bar{\phi} \\ \mathbb{R}^n \supset \phi(U \cap \bar{U}) & \xrightarrow{\bar{\phi} \circ \phi^{-1}} & \bar{\phi}(U \cap \bar{U}) \subset \mathbb{R}^n \end{array}.$$

Since both charts are positively oriented the determinant of the differential $D(\bar{\phi} \circ \phi^{-1})$ is positive, so

$$\begin{aligned} \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi}^{-1})^* \omega &= \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi}^{-1})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi} \circ \phi^{-1})^* (\bar{\phi}^{-1})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* \bar{\phi}^* (\bar{\phi}^{-1})^* \omega \\ &= \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* (\bar{\phi}^{-1} \circ \bar{\phi})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* \omega = \int_{\phi(U)} (\phi^{-1})^* \omega, \end{aligned}$$

by a property of the pull-back and since $(\bar{\phi}^{-1})^* \omega = 0$ outside $\bar{\phi}(U \cap \bar{U})$.

Lecture 4
Thursday
16/01/20

1.6 Partitions of unity

Definition 1.20. Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering. A **partition of unity** with respect to U is a collection of smooth functions $f_\alpha : M \rightarrow [0, 1]$ such that

1. $\text{supp } f_\alpha = \overline{\{x \in M \mid f_\alpha(x) > 0\}} \subset U_\alpha$ for all α ,
2. $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$, and
3. for all $x \in M$, there exists $U \ni x$ open such that $\text{supp } f_\alpha \cap U \neq \emptyset$ for only finitely many α .

Remark. 3 implies that 2 is a finite sum.

Example 1.21. Let

$$M = S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}, \quad U_1 = S^1 \setminus \{(1, 0)\}, \quad U_2 = S^1 \setminus \{(-1, 0)\},$$

so $\{U_i\}$ is a cover. Let

$$f_1(\cos \theta, \sin \theta) = \frac{1}{2} - \frac{1}{2} \cos \theta, \quad f_2(\cos \theta, \sin \theta) = \frac{1}{2} + \frac{1}{2} \cos \theta.$$

Then f_i is a partition of unity.

Proposition 1.22. Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering of M . Then there exists a partition of unity f_α with respect to U .

Proof. We omit the proof. □

Proposition 1.23. Let M be a manifold, and let $n = \dim M$. Then M is orientable if and only if there exists $\omega \in \Omega^n(M)$ which is never vanishing on M , so $\omega(x) \neq 0$ for all $x \in M$.

ω is called a **volume form** on M .

Proof.

\Leftarrow Assume $\omega \in \Omega^n(M)$ is a volume form. We want to construct an orientation Λ on M , that is Λ_x on $T_x M$ for all $x \in M$. Given an oriented basis v_1, \dots, v_n of $T_x M$ we say that it is **positively oriented** if $\omega(x)(v_1, \dots, v_n) > 0$. For all $x \in M$, we define the orientation Λ_x on $T_x M$ by considering the class of positively oriented ordered basis of $T_x M$ which is compatible with the choice of an atlas on M . Take any atlas $\{(U_\alpha, \phi_\alpha)\}$, where $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$. On U_α ,

$$\omega = g_\alpha \phi_\alpha^* dx_1 \wedge \dots \wedge dx_n.$$

Since $\omega \neq 0$, $g_\alpha > 0$ or $g_\alpha < 0$. If $g_\alpha < 0$ then switch x_1 with x_2 , so $g_\alpha > 0$. After this change of coordinates, (U_α, ϕ_α) is positively oriented, so M is orientable.

\Rightarrow Assume that M is orientable, that is there exists an atlas $\{(U_\alpha, \phi_\alpha)\}$ of positively oriented charts. On U_α , we consider

$$\omega_\alpha = \phi_\alpha^* dx_1 \wedge \dots \wedge dx_n.$$

Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Let $\widetilde{\omega}_\alpha = f_\alpha \omega_\alpha \in \Omega^n(U_\alpha)$. We may assume that $\widetilde{\omega}_\alpha \in \Omega^n(M)$ by extending equal to zero outside U_α . We define $\omega = \sum_\alpha \widetilde{\omega}_\alpha \in \Omega^n(M)$. For all α , since $\sum_\alpha f_\alpha = 1$ there exists α such that $\widetilde{\omega}_\alpha \neq 0$, so $\omega \neq 0$. □

Let M be an orientable manifold of dimension n , and let $\omega \in \Omega_c^n(M)$. We want to define $\int_M \omega$. So far we defined for ω such that $\text{supp } \omega \subset U_\alpha$ where (U_α, ϕ_α) is a chart.

Definition 1.24. Let $\{(U_\alpha, \phi_\alpha)\}$ be a positively oriented atlas on M , and let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $\text{supp } f_\alpha \omega \subset U_\alpha$, so let

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega.$$

Remark 1.25. Note that for each α , we have that the support of $f_\alpha \omega$ is contained in U_α and therefore each term of the sum is well-defined as in (1). Indeed, we have

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* f_\alpha.$$

Lemma 1.26. $\int_M \omega$ does not depend on $\{(U_\alpha, \phi_\alpha)\}$ and f_α .

Proof. Under the assumption that $\text{supp } \omega \subset U_\alpha$ then we showed $\int_{U_\alpha} \omega$ does not depend on (U_α, ϕ_α) . Let $\{(U_\alpha, \phi_\alpha)\}$ and $\{(\overline{U}_\alpha, \overline{\phi}_\alpha)\}$ be two atlases with positively oriented charts, and let f_α and \overline{f}_α be two partitions of unity with respect to $\{U_\alpha\}$ and $\{\overline{U}_\alpha\}$ respectively. Then $\sum_\alpha f_\alpha = \sum_\alpha \overline{f}_\alpha = 1$, so $\int_M f_\alpha \omega = \sum_\beta \int_M \overline{f}_\beta f_\alpha \omega$. Thus

$$\int_M \omega = \sum_\alpha \int_M f_\alpha \omega = \sum_{\alpha, \beta} \int_M \overline{f}_\beta f_\alpha \omega = \sum_\beta \int_M \sum_\alpha f_\alpha \overline{f}_\beta \omega = \sum_\beta \int_M \overline{f}_\beta \omega.$$

□

Proposition 1.27. Let M and N be orientable manifolds of dimension n , and let $\omega, \eta \in \Omega_c^n(M)$.

1. *Linearity*

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

2. *Orientation reversal.* Let \overline{M} be the manifold M with opposite orientation $\Lambda^- = \{\Lambda_x^- \mid x \in M\}$, which is the orientation opposite than the one induced by M with orientation Λ . Then

$$\int_M \omega = - \int_{\overline{M}} \omega.$$

3. *Positivity.* Let ω be the volume form on M . Then

$$\int_M \omega > 0.$$

4. *Diffeomorphism invariance.* Let $F : N \rightarrow M$ be an orientation preserving diffeomorphism. Then

$$\int_M \omega = \int_N F^* \omega.$$

Proof.

1. Exercise. ²

2. Exercise. ³

3. Choose a positively oriented chart (U_α, ϕ_α) on U_α , so

$$\omega = g_\alpha \phi_\alpha^* dx_1 \wedge \cdots \wedge dx_n, \quad g_\alpha > 0.$$

Then $\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega$ where f_α is a partition of unity. For all $x \in M$ there exists α such that $x \in U_\alpha$ and $\int_{U_\alpha} f_\alpha \omega > 0$, so $\int_M \omega > 0$.

4. Let (U_α, ϕ_α) be a positively oriented atlas on M . Then $(F^{-1}(U_\alpha), \phi_\alpha \circ F)$ is an atlas on N which is positively oriented. Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $f_\alpha \circ F$ is a partition of the unity with respect to $\{F^{-1}(U_\alpha)\}$, so

$$\int_N F^* \omega = \sum_\alpha \int_N (f_\alpha \circ F) F^* \omega = \sum_\alpha \int_N F^* (f_\alpha \omega) = \sum_\alpha \int_M f_\alpha \omega = \int_M \omega.$$

□

²Exercise

³Exercise

1.7 Manifolds with boundary

Denote

$$\mathbb{R}_{\geq 0}^n = (\mathbb{R}_{\geq 0})^n, \quad \mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Let $U \subset \mathbb{R}_+^n$ be open, and let $F : U \rightarrow \mathbb{R}^m$ be a function. Then F is C^∞ if it can be extended to a C^∞ -function $\tilde{F} : \tilde{U} \rightarrow \mathbb{R}^m$ where $\tilde{U} \supset U$ and \tilde{U} is open.

Definition 1.28. A **manifold with boundary** of dimension n is a Hausdorff topological space M such that there exists an open covering $\{U_\alpha\}$, and for all α , there exists a homeomorphism $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ such that for all α and β ,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n \rightarrow \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n$$

is a diffeomorphism, so

$$\begin{array}{ccc} & U_\alpha \cap U_\beta & \\ \phi_\alpha \swarrow & & \searrow \phi_\beta \\ \mathbb{R}_+^n \supset \phi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow{\phi_\alpha \circ \phi_\beta^{-1}} & \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n \end{array}.$$

The **boundary** of M is

$$\partial M = \{x \in M \mid \exists \alpha, \phi_\alpha(x) \in \partial \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}\}.$$

Then (U_α, ϕ_α) is called a **chart** and $\{(U_\alpha, \phi_\alpha)\}$ is called an **atlas**.

Remark 1.29.

- ∂M is closed in M .
- $\mathring{M} = M \setminus \partial M$ is a manifold of dimension n .

Example 1.30.

- $M = [0, 1]$ is a manifold with boundary $\partial M = \{0, 1\}$.
- The closed disc $D = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is a manifold with boundary $\partial D = S^{n-1}$.
- $M = [0, 1] \times S^1$ is a manifold with boundary $\partial M = S^1 \sqcup S^1$.

Remark 1.31.

- We can define tangent spaces and differential forms exactly in the same way as usual manifolds.
- The definition of orientability is the same. If M is orientable, then ∂M is also orientable. As a convention, the positive orientation on the boundary of $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}$ is given by $(-1)^n dx_1 \wedge \dots \wedge dx_{n-1}$. This induces a positive orientation on ∂M .
- Also partitions of unity for any open cover U_α of M is defined the same way. If M is orientable, for any manifold with boundary, for all open covering $U = \{U_\alpha\}$, there exists a partition of unity f_α . This implies that if $\omega \in \Omega_c^n(M)$, then $\int_M \omega$ is defined the same way for manifolds.

1.8 Stokes' theorem

Theorem 1.32 (Stokes). *For any manifold with boundary M of dimension n , and for any $\omega \in \Omega_c^{n-1}(M)$ we have*

$$\int_M d\omega = \int_{\partial M} \omega \in \Omega_c^n(M).$$

Lecture 6
Tuesday
21/01/20

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas, and let $f_\alpha : M \rightarrow \mathbb{R}$ be a partition of unity with respect to this cover. Then $\sum_\alpha f_\alpha = 1$ on M , so

$$\int_M d\omega = \int_M d\left(\sum_\alpha f_\alpha \omega\right) = \sum_\alpha \int_M d(f_\alpha \omega) = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* d(f_\alpha \omega).$$

By Proposition 1.16,

$$(\phi_\alpha^{-1})^* d(f_\alpha \omega) = d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right).$$

Then $(\phi_\alpha^{-1})^* (f_\alpha \omega)$ is an $(n-1)$ -form on $\phi_\alpha(U_\alpha)$. In coordinates,

$$(\phi_\alpha^{-1})^* (f_\alpha \omega) = \sum_{j=1}^n \widetilde{f_\alpha \omega_j} dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

where ω_j is a smooth function on $\phi_\alpha(U_\alpha)$ and

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\widetilde{\phi_\alpha}} & \phi_\alpha(U_\alpha) \\ f_\alpha \downarrow & \swarrow \widetilde{f_\alpha} & \\ [0, 1] & & \end{array}.$$

Then

$$\begin{aligned} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right) &= d\left(\sum_{j=1}^n \widetilde{f_\alpha \omega_j} dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n\right) \\ &= \sum_{j=1}^n \sum_{k=1}^n \frac{\partial}{\partial x_k} \left(\widetilde{f_\alpha \omega_j}\right) dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha \omega_j}\right) dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha \omega_j}\right) dx_1 \wedge \cdots \wedge dx_n, \end{aligned}$$

so

$$\sum_\alpha \int_{\phi_\alpha(U_\alpha)} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right) = \sum_\alpha \int_{\mathbb{R}_+^n} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right),$$

because $\widetilde{f_\alpha} = 0$ outside $\phi_\alpha(U_\alpha)$. Thus

$$\begin{aligned} \int_M d\omega &= \sum_\alpha \int_{\mathbb{R}_+^n} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha \omega_j}\right) dx_1 \wedge \cdots \wedge dx_n \\ &= \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha \omega_j}\right) dx_n dx_{n-1} \cdots dx_1 \\ &= \sum_\alpha \sum_{j=1}^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} (-1)^{j-1} \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha \omega_j}\right) \Big|_{x_n=0} dx_n dx_{n-1} \cdots \widehat{dx_j} \cdots dx_1 \\ &= \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^{n-1} \left(\widetilde{f_\alpha \omega_j}\right) \Big|_{x_n=0} dx_{n-1} \cdots dx_1, \end{aligned}$$

since $(f_\alpha \omega_j)|_{x_n=0} = 0$ for $j = 1, \dots, n-1$, so

$$\int_M d\omega = \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^{n-1} \left(\widetilde{f_\alpha \omega_j}\right) \Big|_{x_n=0} dx_{n-1} \cdots dx_1 = \sum_\alpha \int_{\partial U_\alpha} f_\alpha|_{\partial U_\alpha} \omega = \int_{\partial M} \omega,$$

where $\partial U_\alpha = U_\alpha \cap \partial M$. □

1.9 Applications of Stokes' theorem

Theorem 1.33 (Integration by parts). *Let M be an orientable n -dimensional manifold with boundary, let $\omega \in \Omega_c^p(M)$, let $\eta \in \Omega_c^{n-p-1}(M)$, and let $p \in \{0, \dots, n-1\}$. Then*

$$\int_{\partial M} \omega \wedge \eta = \int_M d\omega \wedge \eta + (-1)^p \int_M \omega \wedge d\eta.$$

Proof.

$$\int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta) = \int_M (d\omega \wedge \eta + (-1)^p \omega \wedge d\eta),$$

by Stokes and the Leibnitz rule. □

Theorem 1.34 (Brouwer's fixed point theorem). *Let*

$$D = \{x \in \mathbb{R}^n \mid |x| \leq 1\},$$

so

$$\partial D = S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\},$$

and let $f : D \rightarrow D$ be a smooth morphism. Then f admits a fixed point, that is there exists $x \in D$ such that $f(x) = x$.

Proof. Assume that $f(x) \neq x$ for all $x \in D$. For any $x \in D$, consider the ray starting from $f(x)$ and passing through x . Let $g(x)$ be the point where this ray intersects ∂D away from $f(x)$. Note that if $x \in \partial D$ then $g(x) = x$. Then $g : D \rightarrow \partial D$. It is easy to check that g is smooth. Since $\partial D = S^{n-1}$ is orientable by Proposition 1.23 there exists a volume form $\omega \in \Omega^{n-1}(\partial D)$, so $\omega(x) \neq 0$. Since $\omega \in \Omega^{n-1}(\partial D)$, $d\omega \in \Omega^n(\partial D)$, which is an n -dimensional manifold, so $d\omega = 0$. Thus

$$0 < \int_{\partial D} \omega = \int_{\partial D} g^* \omega = \int_D dg^* \omega = \int_D g^* d\omega = 0,$$

by Stokes, a contradiction. □

Example 1.35. Recall any exact form is closed, since $d^2 = 0$. But the opposite is not always true. Let $M = \mathbb{R}^2 \setminus \{0\}$, and let

$$\omega = \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx \in \Omega^1(M).$$

Then ω is closed, since

$$d\omega = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) dx \wedge dy - \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) dy \wedge dx = 0.$$

We want to show that ω is not exact. Assume that

$$\omega = df, \quad f \in \Omega^0(M) = \{C^\infty\text{-function}\}.$$

In particular $\omega = df$ on $S^1 \subset M$. Let

$$\begin{aligned} \gamma &: [0, 2\pi] \longrightarrow S^1 \\ \theta &\longmapsto (\cos \theta, \sin \theta) \end{aligned}$$

Then

$$\int_{S^1} \omega = \int_0^{2\pi} \gamma^* \omega = \int_0^{2\pi} \left(\left(\frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} \right) \cos \theta d\theta - \left(\frac{\sin \theta}{\cos^2 \theta + \sin^2 \theta} \right) (-\sin \theta) d\theta \right) = \int_0^{2\pi} d\theta = 2\pi,$$

and

$$\int_{S^1} \omega = \int_{S^1} df = \int_{\partial S^1} f = \int_{\emptyset} f = 0,$$

so ω is not exact.

Proposition 1.36. *Let M be an orientable manifold of dimension n without boundary, and let $\omega \in \Omega_c^n(M)$. Assume ω is exact. Then*

$$\int_M \omega = 0.$$

Proof. Easy from Stokes. □

Proposition 1.37. *Let M be an orientable manifold of dimension n with boundary, and let $\omega \in \Omega_c^{n-1}(M)$ be a closed form. Then*

$$\int_{\partial M} \omega = 0.$$

Proof. Easy from Stokes. □

Let M be an orientable manifold of dimension n , let $\omega \in \Omega_c^k(M)$, and let $N \subset M$ be a submanifold of dimension k . We can define

$$\int_M \omega = \int_N i^* \omega,$$

where $i : N \hookrightarrow M$ is the inclusion. We will denote

$$\omega|_N = i^* \omega \in \Omega_c^k(N).$$

Proposition 1.38. *Let M be an oriented manifold of dimension n , let $\omega \in \Omega_c^k(M)$, and let $S \subset M$ be a compact orientable submanifold of dimension k such that $\partial S = \emptyset$ and $\int_S \omega \neq 0$. Then*

- ω is not exact,
- $\omega|_S$ is not exact, and
- S is not the boundary of an orientable manifold $N \subset M$ of dimension $k + 1$.

Proof. Exercise. ⁴ □

⁴Exercise

2 De Rham cohomology

2.1 De Rham cohomology

Definition 2.1. Let M be a manifold of dimension n , and let $p \geq 0$. Then $\omega_1, \omega_2 \in \Omega^p(M)$ are said to be **cohomologous** if $\omega_1 - \omega_2 = d\eta$ where $\eta \in \Omega^{p-1}(M)$. In particular $\omega \in \Omega^p(M)$ is cohomologous to zero if it is exact. Let

$$\mathcal{Z}^p(M) = \text{Ker}(d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)) = \{\omega \in \Omega^p(M) \mid \omega \text{ is closed}\} \subset \Omega^p(M),$$

and let

$$\mathcal{B}^p(M) = \text{Im}(d : \Omega^{p-1}(M) \rightarrow \Omega^p(M)) = \{\omega \in \Omega^p(M) \mid \omega \text{ is exact}\} \subset \Omega^p(M).$$

Then $\mathcal{B}^p(M) \subset \mathcal{Z}^p(M)$ for all $p \geq 0$.

Notation. If $p = 0$, then $\mathcal{B}^0(M) = 0$.

Note. If $\omega_1, \omega_2 \in \mathcal{Z}^p(M)$ then $\omega_1 - \omega_2 \in \mathcal{B}^p(M)$ if and only if ω_1 and ω_2 are cohomologous.

Definition 2.2. Denote the p -th de Rham cohomology group as

$$H^p(M) = \mathcal{Z}^p(M) / \mathcal{B}^p(M) = \{[\omega] \mid \omega \in \mathcal{Z}^p(M)\}, \quad p \geq 0.$$

where

$$[\omega] = \{\omega' \in \Omega^p(M) \text{ cohomologous to } \omega\}$$

is the **de Rham class** of ω .

Remark. $H^p(M)$ is a vector space over \mathbb{R} .

Definition 2.3. $b_p(M) = \dim H^p(M) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is the p -th **Betti number** of M .

Proposition 2.4. If M is connected then

$$H^0(M) = \mathbb{R},$$

that is $b_0(M) = 1$. More in general, $b_0(M)$ is the number of connected components of M .

Proof. Assume M is connected. Then $\mathcal{B}^0(M) = 0$, so

$$\begin{aligned} H^0(M) &= \mathcal{Z}^0(M) = \{f \in \Omega^0(M) \text{ closed}\} \\ &= \left\{ f \in \Omega^0(M) \mid \text{locally } \forall x \in M, \frac{\partial}{\partial x_i} f(x) = 0 \right\} \\ &= \{f \in \Omega^0(M) \text{ locally constant}\} = \mathbb{R}. \end{aligned}$$

□

Example. Let $M = S^1$. Then $H^0(M) = \mathbb{R}$.

Proposition 2.5. Let M be a manifold of dimension n . Then

$$H^p(M) = 0, \quad p \geq n+1.$$

Proof. Recall $\Omega^p(M) = 0$ if $p \geq n+1$ because all alternating p -forms for $p \geq n+1$ on an n -dimensional vector space are zero, so $\mathcal{Z}^p(M) = 0$. Thus $H^p(M) = 0$. □

Proposition 2.6. Let M be a compact orientable manifold of dimension n without boundary. Then

$$H^n(M) \neq 0.$$

Proof. M is orientable, so there exists a volume form $\omega \in \Omega^n(M) = \Omega_c^n(M)$, by Proposition 1.23. Then ω is closed, because $d\omega$ is an $(n+1)$ -form on M , so $\omega \in \mathcal{Z}^n(M)$. We want to show that $[\omega] \neq 0$ in $H^n(M)$. Assume $[\omega] = 0$, so ω is exact. Thus $\omega = d\eta$ where η is an $(n-1)$ -form on M , so

$$0 < \int_M \omega = \int_M d\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0,$$

by Stokes, a contradiction. □

Proposition 2.7. *Let $G : M \rightarrow N$ be a smooth morphism between manifolds. Then*

$$G^* : \Omega^p(N) \rightarrow \Omega^p(M), \quad p \geq 0$$

takes closed forms of N to closed forms on M and exact forms of N to exact forms on M .

Proof. By Proposition 1.16, $G^*d = dG^*$. If ω is closed then $dG^*\omega = G^*d\omega = G^*0 = 0$, so $G^*\omega$ is closed. If $\omega = d\eta$ is exact then $G^*\omega = dG^*\eta$ is also exact. \square

Thus $G^* : \mathcal{Z}^p(N) \rightarrow \mathcal{Z}^p(M)$ and $G^* : \mathcal{B}^p(N) \rightarrow \mathcal{B}^p(M)$, so there exists a linear map

$$\begin{array}{ccc} G^* & : & \mathcal{H}^p(N) \longrightarrow \mathcal{H}^p(M) \\ & & [\omega] \longmapsto [G^*\omega] \end{array}.$$

Corollary 2.8. *Let M and N be diffeomorphic manifolds. Then*

$$\mathcal{H}^p(M) \cong \mathcal{H}^p(N), \quad p \geq 0,$$

that is $\mathcal{H}^p(M)$ is a diffeomorphic invariant.

Proof. By Proposition 2.7 there exists $F^* : \mathcal{H}^p(N) \rightarrow \mathcal{H}^p(M)$ and $(F^{-1})^* : \mathcal{H}^p(M) \rightarrow \mathcal{H}^p(N)$. By Proposition 1.8,

$$(F^{-1})^* F^* \omega = (F \circ F^{-1})^* \omega = \text{id}_N^* \omega = \omega, \quad \omega \in \mathcal{H}^p(N)$$

so $(F^{-1})^* \circ F^* = \text{id}_{\mathcal{H}^p(N)}$. Similarly $F^* \circ (F^{-1})^* = \text{id}_{\mathcal{H}^p(M)}$, so F^* is an isomorphism. \square

2.2 Homotopy invariance

Definition 2.9. Let M_0 and M_1 be manifolds, and let $f_0, f_1 : M_0 \rightarrow M_1$ be smooth morphisms. Then f_0 and f_1 are **smoothly homotopic equivalent** if there exists a smooth morphism $H : M_0 \times [0, 1] \rightarrow M_1$ such that $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$ for all $x \in M_0$. A **homotopy** is a smooth morphism $H : M_0 \times [0, 1] \rightarrow M_1$ where M_0 and M_1 are smooth manifolds.

Notation 2.10. Let $f_t(x) = H(x, t)$, so $f_t : M_0 \rightarrow M_1$ is a smooth morphism. Then f_0 and f_1 are said to be homotopic equivalent, denoted by $f_0 \sim f_1$. Then \sim is an equivalence. ⁵

Definition 2.11. M_0 and M_1 are **homotopy equivalent** if there exist smooth morphisms $f : M_0 \rightarrow M_1$ and $g : M_1 \rightarrow M_0$ such that $f \circ g \sim \text{id}_{M_1}$ and $g \circ f \sim \text{id}_{M_0}$.

Example 2.12.

- Let $M_0 = \mathbb{R}^n$ and $M_1 = \{0\}$. Then M_0 and M_1 are homotopy equivalent. Let

$$\begin{array}{ccc} f & : & M_0 \longrightarrow M_1 \\ x & \longmapsto & 0 \end{array}, \quad \begin{array}{ccc} g & : & M_1 \longrightarrow M_0 \\ 0 & \longmapsto & 0 \end{array}.$$

Then

$$\begin{array}{ccc} f \circ g & : & M_1 \longrightarrow M_1 \\ 0 & \longmapsto & 0 \end{array},$$

so $f \circ g = \text{id}_{M_1}$, and

$$\begin{array}{ccc} g \circ f & : & M_0 \longrightarrow M_0 \\ x & \longmapsto & 0 \end{array}.$$

We want to show that $g \circ f \sim \text{id}_{M_0}$. Define a smooth morphism

$$\begin{array}{ccc} H & : & M_0 \times [0, 1] \longrightarrow M_0 \\ (x, t) & \longmapsto & tx \end{array}.$$

Then $H(x, 0) = 0 = (g \circ f)(x)$ for all x , and $H(x, 1) = x = \text{id}_{M_0}(x)$ for all x , so $g \circ f \sim \text{id}_{M_0}$. More in general $M \subset \mathbb{R}^n$ is called **convex** if for all $x, y \in M$ the segment joining x to y is contained inside M . If M is convex then M is homotopy equivalent to $M \times \{0\}$.

⁵Exercise

- Let $M_0 = \mathbb{R}^2 \setminus \{0\}$ and $M_1 = S^1$. Then M_0 and M_1 are homotopy equivalent. Let

$$\begin{aligned} f &: M_0 \longrightarrow M_1 \\ x &\longmapsto \frac{x}{|x|}, \end{aligned} \quad \begin{aligned} g &: M_1 \longrightarrow M_0 \\ x &\longmapsto x \end{aligned}.$$

Then

$$\begin{aligned} f \circ g &: M_1 \longrightarrow M_1 \\ x &\longmapsto x \end{aligned},$$

so $f \circ g = \text{id}_{M_1}$, and

$$\begin{aligned} g \circ f &: M_0 \longrightarrow M_0 \\ x &\longmapsto \frac{x}{|x|}. \end{aligned}$$

Let

$$\begin{aligned} H &: M_0 \times [0, 1] \longrightarrow M_0 \\ (x, t) &\longmapsto tx + (1-t) \frac{x}{|x|} \end{aligned}$$

be smooth. Then $H(x, 0) = x/|x| = (g \circ f)(x)$ and $H(x, 1) = x = \text{id}_{M_0}(x)$, so $g \circ f \sim \text{id}_{M_0}$.

Proposition 2.13. *Let M and N be manifolds, and let $H : M \times [0, 1] \rightarrow N$ be smooth. Denote*

$$\begin{aligned} f_t &: M \longrightarrow N \\ x &\longmapsto H(x, t), \end{aligned} \quad t \in [0, 1].$$

Then $f_t^* : H^p(N) \rightarrow H^p(M)$ does not depend on t for all $p \geq 0$.

Proof. Let $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$. The goal is $f_{t_1}^*[\eta] = f_{t_2}^*[\eta]$ for all $[\eta] \in H^p(N)$. Let

$$\begin{aligned} i_k &: M \longrightarrow M \times [0, 1] \\ x &\longmapsto (x, t_k) \end{aligned}, \quad k = 1, 2.$$

Claim that for all p there exists a linear map $h : \Omega^p(M \times [t_1, t_2]) \rightarrow \Omega^{p-1}(M)$ such that

$$d(h(\omega)) + h(d\omega) = i_2^*\omega - i_1^*\omega \in \Omega^p(M), \quad \omega \in \Omega^p(M \times [0, 1]). \quad (2)$$

Step 1. The claim implies the proposition. Let $\eta \in \Omega^p(N)$ be closed, so $d\eta = 0$. Then $H^*\eta$ is also closed, so let $\omega = H^*\eta \in \Omega^p(M \times [t_1, t_2])$. Apply h . Then $d\omega = 0$, so $d(h(\omega)) = i_2^*\omega - i_1^*\omega$ is exact. Thus

$$f_{t_1}^*[\eta] = [f_{t_1}^*\eta] = [i_1^*H^*\eta] = [i_1^*\omega] = [i_2^*\omega] = [i_2^*H^*\eta] = [f_{t_2}^*\eta] = f_{t_2}^*[\eta],$$

so the proposition follows.

Step 2. The proof of the claim. Let $\omega \in \Omega^p(M \times [t_1, t_2])$. Then for all $(x, t) \in M \times [t_1, t_2]$, $\omega(x, t)$ is an alternating p -form on $T_{(x,t)}(M \times [t_1, t_2])$. We want an alternating $(p-1)$ -form $h(\omega)(x)$ on $T_x M$. Let $v_1, \dots, v_{p-1} \in T_x M$. Then

$$h(\omega)(x)(v_1, \dots, v_{p-1}) = \int_{t_1}^{t_2} \omega(x, t) \left(\frac{\partial}{\partial t}, v_1, \dots, v_{p-1} \right) dt$$

is a $(p-1)$ -form on M , and $\frac{\partial}{\partial t}$ is a global vector field. Check h is linear.⁶ It is enough to prove (2) locally. Remark that exactness is not a local property. Fix local coordinates (x_1, \dots, x_n, t) around a point of $M \times [0, 1]$. Then

$$\omega = \sum_{|I|=p} \omega_I + \sum_{|J|=p-1} \omega_J, \quad \omega_I = g_I dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad \omega_J = g_J dx_{j_1} \wedge \dots \wedge dx_{j_{p-1}} \wedge dt,$$

where g_I and g_J are smooth functions. Any term of (2) is linear. So we just need to check (2) holds for ω_I and ω_J .

⁶Exercise

ω_I . Let $\omega = g(x, t) dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Then

$$d \left(h \left(\omega(x, t) \left(\frac{\partial}{\partial t}, v_1, \dots, v_{p-1} \right) \right) \right) = d(h(0)) = 0,$$

and

$$\begin{aligned} h(d\omega) &= h \left(\frac{\partial}{\partial t} g(x, t) dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} + \sum_{j=1}^n \frac{\partial}{\partial x_j} g(x, t) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \right) \\ &= \left(\int_{t_1}^{t_2} \frac{\partial}{\partial t} g(x, t) dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_p} + 0 \\ &= (g(x, t_2) - g(x, t_1)) dx_{i_1} \wedge \cdots \wedge dx_{i_p} = i_2^* \omega - i_1^* \omega, \end{aligned}$$

so (2) holds.

ω_J . Let $\omega = g(x, t) dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \wedge dt$. Then

$$\begin{aligned} d(h(\omega)) &= (-1)^{p-1} d \left(\left(\int_{t_1}^{t_2} g(x, t) dt \right) dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \right) \\ &= (-1)^{p-1} \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\int_{t_1}^{t_2} g(x, t) dt \right) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \\ &= (-1)^{p-1} \sum_{j=1}^n \left(\int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x, t) dt \right) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}}, \end{aligned}$$

and

$$\begin{aligned} h(d\omega) &= h \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} g(x, t) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \wedge dt + 0 \right) \\ &= (-1)^{p-1} \sum_{j=1}^n \left(\int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x, t) dt \right) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} = -d(h(\omega)), \end{aligned}$$

and $i_2^* \omega = i_1^* \omega = 0$, so (2) holds. □

Corollary 2.14. *Assume M and N are homotopy equivalent. Then there exist isomorphisms*

$$H^p(N) \rightarrow H^p(M), \quad p \geq 0.$$

Proof. There exist $f : M \rightarrow N$ and $g : N \rightarrow M$ such that $g \circ f \sim \text{id}_M$ and $f \circ g \sim \text{id}_N$. By Proposition 2.13 $(g \circ f)^* : H^p(M) \rightarrow H^p(M)$ coincides with $\text{id}_M^* = \text{id}_{H^p(M)}$. Then $f^* \circ g^* = (g \circ f)^* = \text{id}_{H^p(M)}$. Similarly $g^* \circ f^* = \text{id}_{H^p(N)}$, so g^* and f^* are isomorphisms. □

Definition 2.15. Let M be a manifold. Then M is **smoothly contractible** if M is homotopy equivalent to a point.

Example. \mathbb{R}^n is contractible, by Example 2.12. If $M \subset \mathbb{R}^n$ is convex then M is contractible.

Theorem 2.16 (Poincaré lemma). *If M is a contractible manifold then*

$$H^p(M) = 0, \quad p \geq 1.$$

Proof. By previous Corollary 2.14, there exists an isomorphism $H^p(M) \rightarrow H^p(\{\text{point}\})$. Then $\{\text{point}\}$ is a zero-dimensional manifold, so by Proposition 2.5, $H^p(\{\text{point}\}) = 0$ for all $p > 0$. □

Thus $H^p(\mathbb{R}^n) = 0$ for all $p > 0$, so \mathbb{R}^n is not diffeomorphic to any compact orientable manifold.

Proposition 2.17. *Let M be a manifold, and let $\omega \in \Omega^p(M)$ be a closed p -form for $p > 0$. Then for all $x \in X$, there exists a neighbourhood $U \ni x$ such that ω is exact on U , that is there exists $\eta \in \Omega^{p-1}(U)$ such that $\omega = d\eta$ on U .*

Proof. Let (U, ϕ) be a chart around x . I may assume that $V = \phi(U)$ is a ball in \mathbb{R}^n . Then U is diffeomorphic to $B = \{z \mid |z - z_0| < r\}$ for some $z_0 \in \mathbb{R}^n$ and $r > 0$, so $H^p(U) \cong H^p(B)$ for all $p \geq 0$. Since B is contractible, $H^p(B) = 0$ for all $p > 0$. The restriction of ω on U gives a class $[\omega] \in H^p(U) = 0$, so ω is cohomologous to zero on U . Thus ω is exact on U . \square

Definition 2.18. Let M be a manifold, let $\gamma : [0, 1] \rightarrow M$ be a continuous or smooth path, and let $x = \gamma(0)$ and $y = \gamma(1)$. A **homotopy of paths** from x to y is a map

$$\begin{aligned} F : [0, 1] \times [0, 1] &\longrightarrow M \\ (0, t) &\longmapsto x \\ (1, t) &\longmapsto y \end{aligned}$$

Proposition 2.19. *Let γ_0 and γ_1 be homotopic paths on a manifold M , and let $\omega \in \Omega^1(M)$ be closed. Then*

$$\int_0^1 \gamma_0^* \omega = \int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

Proof. Lee's introduction to smooth manifolds. The idea is

$$\int_{\gamma_0} \omega - \int_{\gamma_1} \omega = \int_{\gamma_0 \cup \gamma_1} \omega = \int_{\text{Im } F} \omega = 0,$$

by Stokes' theorem. \square

Recall that M is **simply connected**, so $\pi_1(M) = 0$, if any path γ from x to x is homotopic equivalent to a point.

Proposition 2.20. *Let M be a simply connected orientable manifold. Then*

$$H^1(M) = 0.$$

Proof. Let $\omega \in \Omega^1(M)$ be a closed form. Then claim that ω is exact if and only if $\int_\gamma \omega = 0$ for all loops γ , that is paths from x to x .

- The proof of the claim. Assume that $\omega = df$ is exact for $f \in \Omega^0(M)$. By Proposition 2.19,

$$\int_\gamma \omega = \int_{\text{trivial loop}} \omega = 0.$$

Assume that $\int_\gamma \omega = 0$ for all loops γ . Fix x . Let

$$f(y) = \int_x^y \omega.$$

Since $\int_{\gamma_1 \cup \gamma_2} \omega = 0$, f is well-defined, that is it does not depend on the choice of the path. Then $df = \omega$. This can be checked locally, that is in an open set of \mathbb{R}^n . Here it follows from the fundamental theorem of calculus.

- The claim implies the proposition. Being simply connected, any loop inside M is homotopic equivalent to the trivial loop. For all loops γ and for all closed ω , $\int_\gamma \omega = 0$ by Proposition 2.19, so ω is exact. Thus $[\omega] = 0$ in $H^1(M)$. \square

2.3 Some homological algebra

Let C^\bullet be a sequence of vector spaces, that is C^k is a vector space for $k \in \mathbb{Z}$.

Definition 2.21. (C^\bullet, d^\bullet) is a **cochain complex** if C^\bullet is a sequence of vector spaces and d^\bullet is a sequence of linear maps $d^k : C^k \rightarrow C^{k+1}$ such that the composition $d^{k+1} \circ d^k : C^k \rightarrow C^{k+1} \rightarrow C^{k+2}$ is zero for all k . Then d^\bullet is the **differential**.

Definition 2.22. The elements of

$$\mathcal{Z}^k(C^\bullet, d^\bullet) = \text{Ker}(d^k : C^k \rightarrow C^{k+1}) \subset C^k$$

are called **cocycles**. The elements of

$$\mathcal{B}^k(C^\bullet, d^\bullet) = \text{Im}(d^k : C^{k-1} \rightarrow C^k) \subset C^k$$

are called **coboundaries**. Then $d^{k-1} \circ d^k = 0$, so $\mathcal{B}^k \subset \mathcal{Z}^k$. The quotients

$$H^k(C^\bullet, d^\bullet) = \mathcal{Z}^k(C^\bullet, d^\bullet) / \mathcal{B}^k(C^\bullet, d^\bullet)$$

are the **k -th cohomology groups** of (C^\bullet, d^\bullet) .

Definition 2.23. Let (C^\bullet, d^\bullet) and (D^\bullet, d^\bullet) be two cochain complexes. A map $f : (C^\bullet, d^\bullet) \rightarrow (D^\bullet, d^\bullet)$ is a sequence of linear maps $f^k : C^k \rightarrow D^k$ such that $f^{k+1} \circ d^k = d^k \circ f^k$ for all k , so

$$\begin{array}{ccccccc} \dots & \longrightarrow & C^k & \xrightarrow{d^k} & C^{k+1} & \xrightarrow{d^{k+1}} & C^{k+2} \longrightarrow \dots \\ & & \downarrow f^k & & \downarrow f^{k+1} & & \downarrow f^{k+2} \\ \dots & \longrightarrow & D^k & \xrightarrow{d^k} & D^{k+1} & \xrightarrow{d^{k+1}} & D^{k+2} \longrightarrow \dots \end{array}.$$

Proposition 2.24. Let $f : (C^\bullet, d^\bullet) \rightarrow (D^\bullet, d^\bullet)$ be a map between cochain complexes. Then there exists a natural induced map

$$f^k : H^k(C^\bullet, d^\bullet) \rightarrow H^k(D^\bullet, d^\bullet).$$

Proof. Let $[\omega] \in H^k(C^\bullet, d^\bullet) = \mathcal{Z}^k(C^\bullet, d^\bullet) / \mathcal{B}^k(C^\bullet, d^\bullet)$ for $\omega \in \mathcal{Z}^k(C^\bullet, d^\bullet)$, that is $d^k(\omega) = 0$. I want to check that $f^k(\omega) \in \mathcal{Z}^k(D^\bullet, d^\bullet)$. By definition of maps, $d^k(f^k(\omega)) = f^{k+1}(d^k(\omega)) = 0$, so there is a map

$$\mathcal{Z}^k(C^\bullet, d^\bullet) \rightarrow \mathcal{Z}^k(D^\bullet, d^\bullet).$$

Now I need to check that if $\omega \in \mathcal{B}^k(C^\bullet, d^\bullet)$ then $f^k(\omega) \in \mathcal{B}^k(D^\bullet, d^\bullet)$. ⁷

□

Definition 2.25. A sequence of linear maps

$$C^1 \xrightarrow{f^1} C^2 \rightarrow \dots \rightarrow C^{k-1} \xrightarrow{f^{k-1}} C^k$$

between vector spaces is said to be **exact** if for all i , $\text{Ker } f^i = \text{Im } f^{i-1}$.

Example 2.26.

- A sequence

$$0 \rightarrow C^1 \xrightarrow{f^1} C^2$$

is exact if and only if f^1 is injective.

- A sequence

$$C^1 \xrightarrow{f^1} C^2 \rightarrow 0$$

is exact if and only if f^1 is surjective.

- An exact sequence

$$0 \rightarrow C^1 \xrightarrow{f^1} C^2 \xrightarrow{f^2} C^3 \rightarrow 0$$

is called a **short exact sequence**. In particular f^1 is injective and f^2 is surjective.

⁷Exercise

- Any long exact sequence

$$C^1 \xrightarrow{f^1} C^2 \rightarrow \dots \rightarrow C^{q-1} \xrightarrow{f^{q-1}} C^q$$

can be split into short exact sequences

$$\begin{array}{ccccccc} \dots \rightarrow C^{k-1} & \xrightarrow{f^{k-1}} & C^k & \xrightarrow{f^k} & C^{k+1} & \rightarrow & \dots \\ & \searrow & \nearrow & \searrow & \nearrow & & \\ & \text{Im } f^{k-1} = \text{Ker } f^k & & \text{Im } f^k = \text{Ker } f^{k+1} & & & \\ & \nearrow & \searrow & \nearrow & \searrow & & \\ 0 & & 0 & & 0 & & \end{array}, \quad k = 2, \dots, q-1.$$

Lemma 2.27 (Snake lemma). *Consider the commutative diagram*

$$\begin{array}{ccccccc} C^1 & \xrightarrow{f^1} & C^2 & \xrightarrow{f^2} & C^3 & \longrightarrow & 0 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \\ 0 & \longrightarrow & D^1 & \xrightarrow{g^1} & D^2 & \xrightarrow{g^2} & D^3 \end{array},$$

such that both the horizontal lines are exact sequences. Then there exists a long exact sequence

$$\text{Ker } \alpha_1 \rightarrow \text{Ker } \alpha_2 \rightarrow \text{Ker } \alpha_3 \xrightarrow{\delta} \text{Coker } \alpha_1 \rightarrow \text{Coker } \alpha_2 \rightarrow \text{Coker } \alpha_3.$$

If

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^1 & \xrightarrow{f^1} & C^2 & \xrightarrow{f^2} & C^3 \longrightarrow 0 \\ & & \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 \\ 0 & \longrightarrow & D^1 & \xrightarrow{g^1} & D^2 & \xrightarrow{g^2} & D^3 \longrightarrow 0 \end{array},$$

then

$$0 \rightarrow \text{Ker } \alpha_1 \rightarrow \text{Ker } \alpha_2 \rightarrow \text{Ker } \alpha_3 \xrightarrow{\delta} \text{Coker } \alpha_1 \rightarrow \text{Coker } \alpha_2 \rightarrow \text{Coker } \alpha_3 \rightarrow 0.$$

Proof. We are going to construct $\delta : \text{Ker } \alpha_3 \rightarrow \text{Coker } \alpha_1$. Let $x \in \text{Ker } \alpha_3$. There exists $y \in C^2$ such that $f^2(y) = x$ because f^2 is surjective. Let $z = \alpha_2(y)$ then

$$g^2(z) = g^2(\alpha_2(y)) = \alpha_3(f^2(y)) = \alpha_3(x) = 0,$$

since $x \in \text{Ker } \alpha_3$. Then $z \in \text{Ker } g^2 = \text{Im } g^1$, so there exists $w \in D^1$ such that $z = g^1(w)$. The idea is

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{Ker } \alpha_1 & \longrightarrow & \text{Ker } \alpha_2 & \longrightarrow & \text{Ker } \alpha_3 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ C^1 & \xrightarrow{f^1} & y \in C^2 & \xrightarrow{f^2} & x \in C^3 & \longrightarrow & 0 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \\ 0 & \longrightarrow & w \in D^1 & \xrightarrow{g^1} & z \in D^2 & \xrightarrow{g^2} & D^3 \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{Coker } \alpha_1 & \longrightarrow & \text{Coker } \alpha_2 & \longrightarrow & \text{Coker } \alpha_3 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}.$$

Define $\delta(x) = [w] \in \text{Coker } \alpha_1 = D^1 / \text{Im } \alpha_1$. Need to check that δ is well-defined, so $[w]$ does not depend on our choice of w and y . The rest is an exercise. ⁸ \square

⁸Exercise

2.4 The Mayer-Vietoris sequence

The idea is given a manifold M , we may write $M = U \cup V$ with open U and V so that $H^i(U)$, $H^i(V)$, and $H^i(U \cap V)$ are easy to compute, so this will give us $H^i(M)$. Let M be a manifold, and let U and V be open such that $M = U \cup V$. Assume $U \cap V \neq \emptyset$. Let

$$i_U : U \rightarrow M, \quad i_V : V \rightarrow M, \quad j_U : U \cap V \rightarrow U, \quad j_V : U \cap V \rightarrow V$$

be inclusions, and let $i_U^*, i_V^*, j_U^*, j_V^*$ be pull-backs.

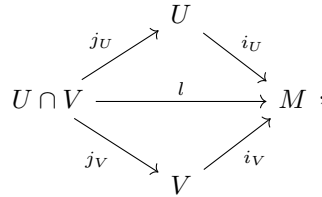
Proposition 2.28. *For all p there exist short exact sequences*

$$0 \rightarrow \Omega^p(M) \xrightarrow{f} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{g} \Omega^p(U \cap V) \rightarrow 0,$$

where $f = (i_U^*, i_V^*)$ and $g = j_V^* - j_U^*$. More precisely, if $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^p(V)$ then $g(\omega_1, \omega_2) = j_V^* \omega_2 - j_U^* \omega_1$.

Proof.

- f is injective. Assume $\omega \in \Omega^p(M)$ such that $f(\omega) = 0$, so $i_U^* \omega = i_V^* \omega = 0$. Since $M = U \cup V$ then $\omega = 0$ on M , so f is injective.
- $\text{Im } f = \text{Ker } g$. Let $f(\omega) \in \text{Im } f$, so $f(\omega) = (i_U^* \omega, i_V^* \omega)$. Then $g(f(\omega)) = j_V^* i_V^* \omega - j_U^* i_U^* \omega = l^* \omega - l^* \omega = 0$, where



so $\text{Im } f \subset \text{Ker } g$. Now let $(\omega_1, \omega_2) \in \text{Ker } g$, so $j_V^* \omega_2 = j_U^* \omega_1$ for $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^p(V)$. The restriction of ω_2 on $U \cap V$ coincides with the restriction of ω_1 on $U \cap V$. Then define

$$\omega = \begin{cases} \omega_1 & \text{on } U \\ \omega_2 & \text{on } V \end{cases}.$$

Then $f(\omega) = (\omega_1, \omega_2)$, so $\text{Ker } g \subset \text{Im } f$.

- g is surjective. Let $\eta \in \Omega^p(U \cap V)$, and let $\{f_U, f_V\}$ be a partition of unity with respect to $\{U, V\}$. Then $\text{supp } f_U \subset U$ and $f_U + f_V = 1$. Let $\eta_1 \in \Omega^p(U)$ be defined by

$$\eta_1 = \begin{cases} f_V \cdot \eta & \text{on } U \cap V \\ 0 & \text{outside } \text{supp } f_V \end{cases},$$

and let $\eta_2 \in \Omega^p(V)$ be defined by

$$\eta_2 = \begin{cases} f_U \cdot \eta & \text{on } U \cap V \\ 0 & \text{outside } \text{supp } f_U \end{cases}.$$

Then $g(-\eta_2, \eta_1) = \eta_1|_{U \cap V} + \eta_2|_{U \cap V} = (f_U + f_V) \cdot \eta = \eta$, so $\eta \in \text{Im } g$.

☐

Theorem 2.29 (Mayer-Vietoris). *Let M be a manifold, and let U and V be open in M such that $M = U \cup V$ and $U \cap V \neq \emptyset$. Then for all $p \geq 0$ there exists a linear $\delta : H^p(U \cap V) \rightarrow H^{p+1}(M)$ such that*

$$\begin{array}{c} \dots \longrightarrow \mathbf{H}^p(M) \xrightarrow{(i_U^*, i_V^*)} \mathbf{H}^p(U) \oplus \mathbf{H}^p(V) \xrightarrow{j_V^* - j_U^*} \mathbf{H}^p(U \cap V) \longrightarrow \\ \hspace{10em} \delta \\ \longleftarrow \mathbf{H}^{p+1}(M) \xleftarrow{(i_U^*, i_V^*)} \mathbf{H}^{p+1}(U) \oplus \mathbf{H}^{p+1}(V) \xleftarrow{j_V^* - j_U^*} \mathbf{H}^{p+1}(U \cap V) \longrightarrow \dots \end{array}$$

is exact.

Example 2.30. Let $M = S^1$, let $N = (0, 1)$ and $S = (0, -1)$, and let $U = M \setminus \{N\}$ and $V = M \setminus \{S\}$, so $M = U \cup V$ and $U \cap V = M \setminus \{N, S\}$. Then

$$H^p(U) \cong H^p(V) \cong H^p((0, 1)) \cong \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \quad (0, 1) \subset \mathbb{R},$$

and

$$H^p(U \cap V) = H^p(U \setminus \{S\}) = H^p\left(\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)\right) = \begin{cases} \mathbb{R}^2 & p = 0 \\ 0 & p > 0 \end{cases}, \quad \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right) \subset \mathbb{R},$$

so

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(M) & \rightarrow & H^0(U) \oplus H^0(V) & \xrightarrow{\phi} & H^0(U \cap V) \xrightarrow{\delta} H^1(M) \rightarrow H^1(U) \oplus H^1(V) \rightarrow H^1(U \cap V) \rightarrow \dots \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R}^2 & & \mathbb{R} & & \mathbb{R} \oplus 0 & & 0 \end{array}.$$

Then $\text{Im } \phi = \mathbb{R} \subset H^0(U \cap V) = \mathbb{R}^2$. Thus

$$H^1(M) = \text{Coker } \phi = \mathbb{R}^2 / \text{Im } \phi \cong \mathbb{R}.$$

Remark 2.31. Let

$$0 \rightarrow C^1 \rightarrow \dots \rightarrow C^k \rightarrow 0$$

be an exact sequence. Then

$$\sum_k (-1)^k \dim C^k = 0.^9$$

In our $M = S^1$ case $1 - 2 + 2 - \dim H^1(M) = 0$, so $\dim H^1(M) = 1$. Thus $H^1(M) \cong \mathbb{R}$.

Example 2.32. Let $M = S^n \subset \mathbb{R}^{n+1}$ be the n -dimensional sphere. Then

$$H^p(M) = \begin{cases} \mathbb{R} & p = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

By induction on n .

$n = 1$. Ok.

$n > 1$. Let $U = M \setminus \{N\}$ and $V = M \setminus \{S\}$, so $U \cap V \neq \emptyset$ and $U \cup V = M$. Then

$$U \cong V \cong \mathbb{R}^n, \quad U \cap V = V \setminus \{N\} \cong \mathbb{R}^n \setminus \{0\} \cong S^{n-1},$$

so

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(M) & \rightarrow & H^0(U) \oplus H^0(V) & \rightarrow & H^0(U \cap V) \xrightarrow{\delta} H^1(M) \rightarrow H^1(U) \oplus H^1(V) \rightarrow \dots \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R} & & \mathbb{R} \oplus 0 \end{array}.$$

Then $1 - 2 + 1 - \dim H^1(M) = 0$, so $\dim H^1(M) = 0$. Thus $H^1(M) = 0$. Then for $p > 0$

$$\begin{array}{ccccccc} \dots & \rightarrow & H^p(U) \oplus H^p(V) & \rightarrow & H^p(U \cap V) & \xrightarrow{\delta} & H^{p+1}(M) \rightarrow H^{p+1}(U) \oplus H^{p+1}(V) \rightarrow \dots \\ & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \oplus 0 \end{array}$$

is exact, so $H^p(U \cap V) \cong H^{p+1}(M)$. By induction

$$H^p(U \cap V) = H^{p+1}(M) = \begin{cases} \mathbb{R} & p = n - 1 \\ 0 & \text{otherwise} \end{cases}.$$

⁹Exercise

Proof of Theorem 2.29. By Proposition 2.28 for all p

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^p(M) & \longrightarrow & \Omega^p(U) \oplus \Omega^p(V) & \longrightarrow & \Omega^p(U \cap V) \longrightarrow 0 \\ & & \downarrow d_M^p & & \downarrow (d_U^p, d_V^p) & & \downarrow d_{U \cap V}^p \\ 0 & \longrightarrow & \Omega^{p+1}(M) & \longrightarrow & \Omega^{p+1}(U) \oplus \Omega^{p+1}(V) & \longrightarrow & \Omega^{p+1}(U \cap V) \longrightarrow 0 \end{array}$$

are exact. Recall d commutes with the pull-back. By the strong snake lemma,

$$\begin{array}{ccccccc} \text{Coker } d_M^{p-1} & \longrightarrow & \text{Coker } (d_U^{p-1}, d_V^{p-1}) & \longrightarrow & \text{Coker } d_{U \cap V}^{p-1} & \longrightarrow & 0 \\ & & \downarrow \partial_M^p = d_M^p & & \downarrow (\partial_U^p, \partial_V^p) = (d_U^p, d_V^p) & & \downarrow \partial_{U \cap V}^p = d_{U \cap V}^p \\ 0 & \longrightarrow & \text{Ker } d_M^{p+1} & \longrightarrow & \text{Ker } (d_U^{p+1}, d_V^{p+1}) & \longrightarrow & \text{Ker } d_{U \cap V}^{p+1} \end{array},$$

which is well-defined, since $d^{p+1} \circ d^p = 0$. By the weak snake lemma again,

$$\text{Ker } \partial_M^p \rightarrow \text{Ker } (\partial_U^p, \partial_V^p) \rightarrow \text{Ker } \partial_{U \cap V}^p \xrightarrow{\delta} \text{Coker } \partial_M^p \rightarrow \text{Coker } (\partial_U^p, \partial_V^p) \rightarrow \text{Coker } \partial_{U \cap V}^p.$$

Then $\text{Coker } d_M^{p-1} = \Omega^p(M) / \text{Im } d_M^{p-1}$. There exists

$$H^p(M) = \text{Ker } d_M^p / \text{Im } d_M^{p-1} \xrightarrow{\sim} \text{Ker } (\Omega^p(M) / \text{Im } d_M^{p-1} \rightarrow \text{Ker } d_M^{p+1}) = \text{Ker } \partial_M^p.$$

Similarly, $\text{Ker } (\partial_U^p, \partial_V^p) \cong H^p(U) \oplus H^p(V)$ and $\text{Ker } \partial_{U \cap V}^p \cong H^p(U \cap V)$. There exists

$$H^{p+1}(M) = \text{Ker } d_M^{p+1} / \text{Im } d_M^p \xrightarrow{\sim} \text{Coker } (\Omega^p(M) / \text{Im } d_M^{p-1} \rightarrow \text{Ker } d_M^{p+1}) = \text{Coker } \partial_M^p.$$

Similarly, $\text{Coker } (\partial_U^p, \partial_V^p) \cong H^{p+1}(U) \oplus H^{p+1}(V)$ and $\text{Coker } \partial_{U \cap V}^p \cong H^{p+1}(U \cap V)$. □

Example 2.33. Let $\mathbb{T}^2 = S^1 \times S^1$ be the torus. Then

$$H^p(\mathbb{T}^2) = \begin{cases} \mathbb{R} & p = 0, 2 \\ \mathbb{R} \oplus \mathbb{R} & p = 1 \end{cases}.$$

We leave the proof as an exercise. ¹⁰

Definition 2.34. Let M be a manifold, and let $U = \{U_i\}$ be an open cover of M . Then U is said to be **good** if for all $I = (i_1, \dots, i_p)$, $U_{i_1} \cap \dots \cap U_{i_p}$ is either \emptyset or contractible.

Lemma 2.35. Let M be a connected manifold which admits a finite good cover. Then for all $p \geq 0$, $H^p(M)$ is a finite dimensional vector space.

Exercise. Find a counterexample without assuming there exists a finite good cover.

Proof. Let U be a finite good cover. Define $k = \#U$. By induction on k .

$k = 1$. $M = U_1$ is contractible, so

$$H^p(M) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & \text{otherwise} \end{cases}.$$

$k > 1$. Assume ok for covers with at most $k-1$ elements. Let $U = \bigcup_{i=1}^{k-1} U_i$ and $V = U_k$. Then $U \cup V = M$ and $U \cap V \neq \emptyset$, so Mayer-Vietoris holds. By induction $H^p(U)$ and $H^p(V)$ are finite dimensional, since $H^p(U)$ is covered by $k-1$ of U_i and $H^p(V)$ is contractible. Then $U \cap V = \bigcup_{i=1}^{k-1} (U_i \cap U_k)$, and $\{U_i \cap U_k\}$ is a good cover of $U \cap V$ with $k-1$ elements. ¹¹ By induction $H^p(U \cap V)$ is finite dimensional. Thus $H^p(M)$ is also finite dimensional. □

¹⁰Exercise

¹¹Exercise

Fact. Any manifold admits a good cover.

Theorem 2.36. Let M be a compact connected manifold. Then $H^p(M)$ is finite dimensional.

Proof. Follows from the fact and Lemma 2.35. □

2.5 Compactly supported de Rham cohomology

Let M be a manifold, and let $\omega \in \Omega_c^p(M)$. Then $d\omega \in \Omega_c^{p+1}(M)$ and $d^2 = 0$, so

$$\Omega_c^p(M) \xrightarrow{d} \Omega_c^{p+1}(M) \xrightarrow{d} \dots$$

Definition 2.37. The p -th compactly supported de Rham cohomology group is

$$H_c^p(M) = \mathcal{Z}_c^p(M) / \mathcal{B}_c^p(M) = \text{Ker}(d : \Omega_c^p(M) \rightarrow \Omega_c^{p+1}(M)) / \text{Im}(d : \Omega_c^{p-1}(M) \rightarrow \Omega_c^p(M)).$$

Example. If M is compact, then

$$H_c^p(M) = H^p(M), \quad p \geq 0.$$

Proposition 2.38. Let M be a non-compact connected manifold. Then

$$H_c^0(M) = 0.$$

Recall if M is connected $H^0(M) = \mathbb{R}$, since $H^0(M) = \{f \text{ constant on } M\}$.

Proof. $H_c^0(M) = \{f \text{ constant on } M \text{ and with compact support}\}$. Since M is non-compact, if $f \in \Omega_c^0(M)$, then $\text{supp } f \subsetneq M$. Thus there exists $x \in M$ such that $f(x) = 0$, so $f \equiv 0$, since f is constant. □

Remark 2.39. Let $f : M \rightarrow N$ be a smooth morphism between manifolds, and let $\omega \in \Omega_c^p(N) \subset \Omega^p(N)$. Then $f^*\omega \in \Omega^p(M)$, and $\text{supp } f^*\omega \subset f^{-1}(\text{supp } \omega)$, which is not compact in general, so $f^*\omega \notin \Omega_c^p(M)$ in general. If f is **proper**, that is $f^{-1}(K)$ is compact for all compact subsets $K \subset N$, then $f^* : \Omega_c^p(N) \rightarrow \Omega_c^p(M)$ is well-defined.

Exercise. If f is a diffeomorphism then f^* induces an isomorphism $H_c^p(N) \rightarrow H_c^p(M)$.

Definition 2.40. Let M_0 and M_1 be manifolds without boundary, and let $f_i : M_0 \rightarrow M_1$ be smooth morphisms for $i = 0, 1$. Then f_0 and f_1 are **properly smoothly homotopic** if there exists a smooth $H : M_0 \times [0, 1] \rightarrow M_1$ such that $H(\cdot, i) = f_i(\cdot)$ for $i = 0, 1$ and H is proper. Then M_0 and M_1 are **properly smoothly homotopically equivalent** if there exist smooth morphisms $f : M_0 \rightarrow M_1$ and $g : M_1 \rightarrow M_0$ such that $f \circ g \sim \text{id}_{M_1}$ and $g \circ f \sim \text{id}_{M_0}$, where the equivalences are properly homotopic.

Notation. $f_t(\cdot) = H(\cdot, t) : M_0 \rightarrow M_1$.

Remark 2.41. To say that H is proper is not the same as saying f_t is proper for all t .

Exercise. Find H such that f_t is proper but H is not. A hint is to let $M_0 = M_1 = \mathbb{R}$ and $H : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ such that $f_t^{-1}(0)$ is bounded for all t but $H^{-1}(0)$ is not.

Proposition 2.42. If M_0 and M_1 are properly homotopically equivalent then

$$H_c^p(M_0) \cong H_c^p(M_1).$$

Let M be a manifold, and let $i : U \hookrightarrow M$ be an open set. Then there exist linear **push-forwards**

$$i_* : \Omega_c^p(U) \rightarrow \Omega_c^p(M), \quad p \geq 0.$$

Let $\omega \in \Omega_c^p(U)$. Then $\omega = 0$ outside U . We can define

$$i_*\omega = \begin{cases} \omega & \text{on } U \\ 0 & \text{on } M \setminus U \end{cases}.$$

If $j : V \hookrightarrow U$ and $i : U \hookrightarrow M$, then $(i \circ j)_* = i_* \circ j_*$.

Lecture 15
Tuesday
11/02/20

Lemma 2.43. *Let M be a manifold, and let $i : U \hookrightarrow M$ be an immersion such that U is open. Then for all $p \geq 0$, $i_* : \Omega_c^p(U) \rightarrow \Omega_c^p(M)$ commutes with d , that is*

$$d(i_*\omega) = i_*d\omega, \quad \omega \in \Omega_c^p(U).$$

In particular if ω is closed then i_ω is closed, and if ω is exact then $i_*\omega$ is exact.*

Proof.

$$d(i_*\omega) = \begin{cases} d\omega & \text{on } U \\ 0 & \text{on } M \setminus U \end{cases} = i_*d\omega.$$

Let ω be closed, so $d\omega = 0$. Then $d(i_*\omega) = i_*d\omega = 0$, so $i_*\omega$ is closed. Similarly for exactness. \square

Let $U \hookrightarrow M$ be as before. Then there exist

$$i_* : H_c^p(U) \rightarrow H_c^p(M), \quad p \geq 0.$$

Proposition 2.44 (Punctured manifolds). *Let M be a manifold of dimension n , let $x \in M$, and let $i : M \setminus \{x\} \hookrightarrow M$. Then*

- for all $p \geq 2$, $i_* : H_c^p(M \setminus \{x\}) \rightarrow H_c^p(M)$ is an isomorphism.
- for all $p \geq 1$, if M is compact $i_* : H_c^p(M \setminus \{x\}) \rightarrow H_c^p(M) = H^p(M)$ is an isomorphism.

Proof.

- Injectivity.

$p \geq 2$. Let $\omega \in \Omega_c^p(M \setminus \{x\})$ be closed such that $i_*[\omega] = 0$, so $[i_*\omega] = 0$ in $H_c^p(M)$. The goal is $[\omega] = 0$. There exists $\eta \in \Omega_c^{p-1}(M)$ such that $i_*\omega = d\eta$. By Poincaré lemma there exists $U \subset M$ containing x such that $H^q(U) = 0$ for all $q \geq 1$. Then $i_*\omega = 0$ in a neighbourhood of x because $\text{supp } \omega \subset M \setminus \{x\}$, so $d\eta = 0$ in a neighbourhood of x . By taking U smaller we can assume η is closed. Since $p \geq 2$, $[\eta] \in H^{p-1}(U) = 0$, so η is exact. Then there exists $\sigma \in \Omega^{p-2}(U)$ such that $\eta = d\sigma$ on U . Let $(U, M \setminus \{x\})$ be an open cover of M , let $(f_U, f_{M \setminus \{x\}})$ be a partition of unity, and let $\eta' = \eta - d(i_*(f_U\sigma))$. On a neighbourhood of x , $\eta' = 0$ because $i_*(f_U\sigma) = \sigma$, so $\text{supp } \eta' \subset M \setminus \{x\}$. Thus $\eta' \in \Omega_c^{p-1}(M \setminus \{x\})$ and $\omega = d\eta'$, so $[\omega] = 0$.

$p = 1$. The same proof. Let $\omega \in \Omega_c^1(M \setminus \{x\})$ be closed such that $[i_*\omega] = 0$. There exists $\eta \in \Omega_c^0(M)$ such that $i_*\omega = d\eta$. By taking an open set $U \subset M$ such that $x \in U$, we may assume $d\eta = 0$, so $\eta = c$ is constant on U . Let $\eta' = \eta - c$. Then $\eta' = 0$ on U . If M is compact then $\eta' \in \Omega_c^0(M \setminus \{x\})$. Thus $\omega = d\eta'$, so $[\omega] = 0$.

- Surjectivity.

$p \geq 1$. Let $[\omega] \in H_c^p(M)$ such that ω is closed. By Poincaré lemma there exists an open $U \ni x$ such that ω is exact, so there exists $\sigma \in \Omega^{p-1}(U)$ such that $\omega = d\sigma$. Let $(f_U, f_{M \setminus \{x\}})$ be a partition of unity as before, and let $\omega' = \omega - d(i_*(f_U\sigma))$. Then $\omega' = 0$ in a neighbourhood of x and $[\omega'] = [\omega]$, and $\omega'|_{M \setminus \{x\}} \in \Omega_c^p(M \setminus \{x\})$. Thus $[i_*\omega']_{M \setminus \{x\}} = [\omega'] = [\omega]$.

\square

Exercise. Compute $H_c^1(\mathbb{R}^2 \setminus \{0\})$ by hands.

Example 2.45.

$$H_c^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}.$$

Recall $\mathbb{R}^n \cong S^n \setminus \{x\}$ for $x \in S^n$. By Proposition 2.44, by $M = S^n$,

$$H_c^p(\mathbb{R}^n) = H_c^p(S^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}, \quad p \geq 1,$$

and $H_c^0(\mathbb{R}^n) = 0$.

Let M be a manifold such that $M = U \cup V$ for open U and V such that $U \cap V \neq \emptyset$, and let

$$\begin{array}{ccc} & U & \\ j_U \nearrow & & \searrow i_U \\ U \cap V & & M \\ j_V \searrow & & \nearrow i_V \\ & V & \end{array}, \quad \begin{array}{ccc} & \Omega^p(U) & \\ j_{U*} \nearrow & & \searrow i_{U*} \\ \Omega^p(U \cap V) & & \Omega^p(M) \\ j_{V*} \searrow & & \nearrow i_{V*} \\ & \Omega^p(V) & \end{array}, \quad p \geq 0.$$

Proposition 2.46. *We have a short exact sequence*

$$0 \leftarrow \Omega^p(M) \xleftarrow{i} \Omega^p(U) \oplus \Omega^p(V) \xleftarrow{j} \Omega^p(U \cap V) \leftarrow 0,$$

where $i = i_{U*} + i_{V*}$ and $j = (-j_{U*}, j_{V*})$.

Proof.

- j is injective. Let $\omega \in \Omega^p(U \cap V)$ such that $j(\omega) = 0$, so $j_{U*}\omega = j_{V*}\omega = 0$. Then $\omega = 0$, so j is injective.
- $\text{Ker } i = \text{Im } j$. Let $\omega \in \Omega^p(U \cap V)$. Then $i(j(\omega)) = i(-j_{U*}\omega, j_{V*}\omega) = -i_{U*}j_{U*}\omega + i_{V*}j_{V*}\omega = 0$, so $\text{Ker } i \supset \text{Im } j$. Let $(\omega_1, \omega_2) \in \text{Ker } i$. Then $i_{U*}\omega_1 + i_{V*}\omega_2 = 0$, so $i_{V*}\omega_1 = -i_{V*}\omega_2$, so $\text{supp } \omega_1 \subset U \cap V$ and $\text{supp } \omega_2 \subset U \cap V$, so there exists $\eta \in \Omega^p(U \cap V)$ such that $j_{U*}\eta = -\omega_1$ and $j_{V*}\eta = \omega_2$, so $(\omega_1, \omega_2) = j(\eta)$, so $\text{Ker } i \subset \text{Im } j$.
- i is surjective. Let $\omega \in \Omega_c^p(M)$, and let $\{f_U, f_V\}$ be a partition of unity with respect to $\{U, V\}$. Define $\omega_U = f_U \cdot \omega|_U \in \Omega_c^p(U)$ and $\omega_V = f_V \cdot \omega|_V \in \Omega_c^p(V)$. Then $i(\omega_U, \omega_V) = i_{U*}\omega_U + i_{V*}\omega_V = f_U \cdot \omega + f_V \cdot \omega = (f_U + f_V) \cdot \omega = \omega$.

□

Thus for all p we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_c^p(U \cap V) & \longrightarrow & \Omega_c^p(U) \oplus \Omega_c^p(V) & \longrightarrow & \Omega_c^p(M) \longrightarrow 0 \\ & & \downarrow d & & \downarrow (d, d) & & \downarrow d \\ 0 & \longrightarrow & \Omega_c^{p+1}(U \cap V) & \longrightarrow & \Omega_c^{p+1}(U) \oplus \Omega_c^{p+1}(V) & \longrightarrow & \Omega_c^{p+1}(M) \longrightarrow 0 \end{array}.$$

Theorem 2.47. *There exists $\delta : H_c^p(M) \rightarrow H_c^{p+1}(U \cap V)$ such that*

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_c^p(U \cap V) & \longrightarrow & H_c^p(U) \oplus H_c^p(V) & \longrightarrow & H_c^p(M) \longrightarrow \\ & & & & \delta & & \\ & \longleftarrow & H_c^{p+1}(U \cap V) & \longrightarrow & H_c^{p+1}(U) \oplus H_c^{p+1}(V) & \longrightarrow & H_c^{p+1}(M) \longrightarrow \dots \end{array}$$

Proof. Same proof as Mayer-Vietoris for $H^p(M)$.

□

2.6 Poincaré duality

Let M be an orientable manifold. Then $H^p(M) \cong H_c^{n-p}(M)^*$, the dual of $H_c^{n-p}(M)$.

Proposition 2.48. *Let M be a manifold. Then the bilinear map*

$$\begin{aligned} \cup : H^p(M) \times H^q(M) &\longrightarrow H^{p+q}(M) \\ ([\omega], [\eta]) &\longmapsto [\omega \wedge \eta] \end{aligned}$$

is well-defined, and

$$[\omega] \cup [\eta] = (-1)^{p \cdot q} [\eta] \cup [\omega].$$

Proof. Follows from the Leibnitz rule and Proposition 1.6.

□

Lemma 2.49. *Let M be oriented without boundary of dimension n . Then there exists a linear map*

$$\begin{aligned} I_M &: H_c^n(M) \longrightarrow \mathbb{R} \\ [\omega] &\longmapsto \int_M \omega \end{aligned}$$

and I_M is surjective.

I_M is called **integration**.

Proof. Let $\omega \in \Omega_c^n(M)$ such that $[\omega] = 0$, so ω is exact. By Stokes $\int_M \omega = 0$, so I_M is well-defined and it is linear. It is enough to show there exists closed $\omega \in \Omega_c^n(M)$ such that $\int_M \omega \neq 0$. Take a volume form ω_0 , which exists because M is oriented. Take $f \in C^\infty(M)$ for $f \geq 0$ and with compact support. Let $\omega = f \cdot \omega_0 \in \Omega_c^n(M)$. Then ω is closed because $\Omega_c^{n+1}(M) = 0$ and $\int_M \omega = \int_M (f \cdot \omega_0) > 0$, by definition of volume forms. \square

Example 2.50. Let $M = S^n$, and let $\omega \in \Omega_c^n(M)$ such that $\int_M \omega = 0$. We want to show that ω is exact. Since M is compact, $H_c^n(M) = H^n(M) = \mathbb{R}$. By Lemma 2.49 $I_M : H_c^n(M) \rightarrow \mathbb{R}$ is surjective, and $H_c^n(M) = \mathbb{R}$, so I_M is injective. Since $\int_M \omega = 0$, $I_M([\omega]) = 0$, so $[\omega] = 0$. Thus ω is exact.

Let M be a connected manifold of dimension n . If $\omega_2 \in H_c^q(M)$ then $[\omega_1 \wedge \omega_2] \in H_c^{p+q}(M)$. Then

$$\cup : H^p(M) \times H_c^q(M) \rightarrow H_c^{p+q}(M).$$

Let M be an oriented manifold without boundary of dimension n . Then

$$\begin{aligned} I_M &: H_c^n(M) \longrightarrow \mathbb{R} \\ [\omega] &\longmapsto \int_M \omega \end{aligned}$$

Choose $q = n - p$. Then

$$I_M \circ \cup : H^p(M) \times H_c^{n-p}(M) \rightarrow H_c^n(M) \rightarrow \mathbb{R}.$$

Recall that if $\phi : V \times W \rightarrow \mathbb{R}$ is bilinear, then there exists

$$\begin{aligned} V &\longrightarrow W^* = \text{Hom}(W, \mathbb{R}) & \phi_v &: W \longrightarrow \mathbb{R} \\ v &\longmapsto \phi_v & w &\longmapsto \phi(v, w) \end{aligned}$$

Thus, we get

$$H^p(M) \rightarrow H_c^{n-p}(M)^*.$$

Poincaré duality says that this is an isomorphism.

Example. Assume M is compact and oriented. Then $H^p(M) \xrightarrow{\sim} H^{n-p}(M)$, so

$$b^p(M) = b^{n-p}(M).$$

Example 2.51. Let $U \subset \mathbb{R}^n$ be an open subset diffeomorphic to \mathbb{R}^n . Then

$$H^p(U) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \quad H_c^p(U) = \begin{cases} 0 & p < n \\ \mathbb{R} & p = n \end{cases}.$$

We want to show that Poincaré duality holds. We just need to check that Poincaré duality holds for $p = 0$. It is enough to show that $\phi : H^0(U) \rightarrow H_c^n(U)^*$ is injective, that is there exists ω such that $\phi(\omega) \neq 0$. Given $\omega \in H^0(U)$,

$$\begin{aligned} \phi(\omega) &: H_c^n(U) \longrightarrow \mathbb{R} \\ \eta &\longmapsto \int_U \eta \wedge \omega \end{aligned}$$

Then $\omega = c$ is a constant function on U , so

$$\begin{aligned} \phi(\omega) &: H_c^n(U) \longrightarrow \mathbb{R} \\ \eta &\longmapsto \int_U c\omega \end{aligned}$$

If $c \neq 0$ there exists η such that this map is not zero, so $\phi(\omega) \neq 0$. Thus ϕ is an isomorphism.

Lecture 17
Monday
17/02/20

We will prove the following.

Theorem 2.52 (Poincaré duality). *Assume that M is an oriented manifold, without boundary, such that there exists a finite open cover $U = \{U_i\}$ such that $U_{i_1} \cap \cdots \cap U_{i_q}$ is \emptyset or diffeomorphic to \mathbb{R}^n . Then*

$$\mu_M : H^p(M) \xrightarrow{\sim} H_c^{n-p}(M)^*, \quad p \geq 0, \quad n = \dim M$$

is an isomorphism.

Any compact manifold M admits such a cover.

Lemma 2.53. *Let*

$$C^1 \xrightarrow{f^1} C^2 \xrightarrow{f^2} C^3$$

be exact, where C^i are vector spaces of finite dimension. Then there exists

$$(C^3)^* \xrightarrow{(f^2)^*} (C^2)^* \xrightarrow{(f^1)^*} (C^1)^*,$$

which is also exact, where $(f^1)^* \phi = \phi \circ f^1$ and $(f^2)^* \phi = \phi \circ f^2$.

Proof. By assumption $\text{Ker } f^2 = \text{Im } f^1$. We want to prove $\text{Ker } (f^1)^* = \text{Im } (f^2)^*$.

- Let $\phi \in \text{Im } (f^2)^*$. Then there exists $\psi \in (C^3)^*$ such that $(f^2)^* \psi = \phi$, so $\psi \circ f^2 = \phi$, so $0 = \psi \circ f^2 \circ f^1 = \phi \circ f^1 = (f^1)^* \phi$, so $\phi \in \text{Ker } (f^1)^*$.
- Let $\phi \in \text{Ker } (f^1)^*$. Then $\phi \circ f^1 = 0$, so $\text{Ker } f^2 = \text{Im } f^1 \subset \text{Ker } \phi$, so there exists $\bar{\phi} : C^2 / \text{Ker } f^2 \rightarrow \mathbb{R}$, so there exists $\psi : C^3 \rightarrow \mathbb{R}$ extending $\bar{\phi}$ such that $\psi \circ f^2 = \phi$, so $(f^2)^* \psi = \phi$, so $\phi \in \text{Im } (f^2)^*$.

□

Lemma 2.54 (Five lemma). *Let*

$$\begin{array}{ccccccccc} C^1 & \xrightarrow{f^1} & C^2 & \xrightarrow{f^2} & C^3 & \xrightarrow{f^3} & C^4 & \xrightarrow{f^4} & C^5 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\ D^1 & \xrightarrow{g^1} & D^2 & \xrightarrow{g^2} & D^3 & \xrightarrow{g^3} & D^4 & \xrightarrow{g^4} & D^5 \end{array},$$

such that the horizontal lines are exact. Suppose

- α_1 is surjective,
- α_5 is injective, and
- α_2 and α_4 are isomorphisms.

Then α_3 is an isomorphism.

Proof. Let $x \in C^3$ such that $\alpha_3(x) = 0$, so if $y = f^3(x)$ then $\alpha_4(y) = 0$. Since α_4 is an isomorphism, $y = 0$. Then $x \in \text{Ker } f^3 = \text{Im } f^2$, so there exists $z \in C^2$ such that $f^2(z) = x$. Let $w = \alpha_2(z)$ then $g^2(w) = 0$, so $w \in \text{Ker } g^2 = \text{Im } g^1$. Then there exists $t \in D^1$ such that $g^1(t) = w$. Since α_1 is surjective there exists $s \in C^1$ such that $\alpha_1(s) = t$, so

$$\begin{array}{ccccccccc} s \in C^1 & \xrightarrow{f^1} & z \in C^2 & \xrightarrow{f^2} & x \in C^3 & \xrightarrow{f^3} & y \in C^4 & \xrightarrow{f^4} & C^5 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\ t \in D^1 & \xrightarrow{g^1} & w \in D^2 & \xrightarrow{g^2} & 0 \in D^3 & \xrightarrow{g^3} & 0 \in D^4 & \xrightarrow{g^4} & D^5 \end{array}.$$

We want to show that $f^1(s) = z$, and $\alpha_2(f^1(s)) = g^1(\alpha_1(s)) = g^1(t) = w = \alpha_2(z)$, so $f^1(s) = z$, since α_2 is injective. Thus $x = f^2(z) = f^2(f^1(s)) = 0$, so α_3 is injective. Show that α_3 is surjective. ¹² □

¹²Exercise

Proof of Theorem 2.52. Let $N = \#U$. We proceed by induction on N . Then $N = 1$ is ok, so let $N > 1$. Let $U = \bigcup_{i=1}^{N-1} U_i$ and $V = U_N$, so $M = U \cup V$. Both U and V , and $U \cap V$, satisfy Poincaré duality by induction. The idea is to use classical Mayer-Vietoris and compact support Mayer-Vietoris, and the five lemma. By Mayer-Vietoris,

$$H^{p-1}(U) \oplus H^{p-1}(V) \xrightarrow{g} H^{p-1}(U \cap V) \xrightarrow{\delta} H^p(M) \xrightarrow{f} H^p(U) \oplus H^p(V) \rightarrow \dots,$$

where $f = (i_U^*, i_V^*)$ and $g = j_V^* - j_U^*$. By compact support Mayer-Vietoris,

$$\dots \rightarrow H_c^{n-p}(U) \oplus H_c^{n-p}(V) \xrightarrow{i} H_c^{n-p}(M) \xrightarrow{\delta_c} H_c^{n-(p-1)}(M) \xrightarrow{j} H_c^{n-(p-1)}(U) \oplus H_c^{n-(p-1)}(V),$$

where $j = (-j_{U*}, j_{V*})$ and $i = i_{U*} + i_{V*}$. Taking the dual, by Lemma 2.53,

$$H_c^{n-(p-1)}(U)^* \oplus H_c^{n-(p-1)}(V)^* \xrightarrow{j^*} H_c^{n-(p-1)}(U \cap V)^* \xrightarrow{\delta_c^*} H_c^{n-p}(M)^* \xrightarrow{i^*} H_c^{n-p}(U)^* \oplus H_c^{n-p}(V)^* \rightarrow \dots$$

We get a diagram

$$\begin{array}{ccccccc} H^{p-1}(U) \oplus H^{p-1}(V) & \xrightarrow{g} & H^{p-1}(U \cap V) & \xrightarrow{\delta} & H^p(M) & \xrightarrow{f} & H^p(U) \oplus H^p(V) \longrightarrow \dots \\ \downarrow n_{p-1} \cdot \mu_U \oplus \mu_V & & \downarrow n_{p-1} \cdot \mu_{U \cap V} & & \downarrow n_p \cdot \mu_M & & \downarrow n_p \cdot \mu_U \oplus \mu_V \\ H_c^{n-(p-1)}(U)^* \oplus H_c^{n-(p-1)}(V)^* & \xrightarrow{j^*} & H_c^{n-(p-1)}(U \cap V)^* & \xrightarrow{\delta_c^*} & H_c^{n-p}(M)^* & \xrightarrow{i^*} & H_c^{n-p}(U)^* \oplus H_c^{n-p}(V)^* \rightarrow \dots \end{array},$$

where $n_0 = 1$ and $n_p = (-1)^{p-1} n_{p-1}$. The goal is to show that μ_M is an isomorphism. The idea is by the five lemma, it is enough to show that

1. all the other vertical arrows are isomorphisms, and
2. the diagram is commutative.

We know 1 is ok by induction on N . We need to show 2.

- The first square. We want to show that

$$\mu_{U \cap V} \circ g = j^* \circ (\mu_U \oplus \mu_V).$$

Let $\omega_U \in \Omega^{p-1}(U)$ and $\omega_V \in \Omega^{p-1}(V)$ be closed forms. We want to show

$$\mu_{U \cap V}(g([\omega_U], [\omega_V])) = j^*(\mu_U([\omega_U]), \mu_V([\omega_V])),$$

in $H_c^{n-(p-1)}(U \cap V)^*$, that is we want to show that on any element of $H_c^{n-(p-1)}(U \cap V)$ they coincide.

Let $\eta \in \Omega_c^{n-(p-1)}(U \cap V)$. Recall $g = j_V^* - j_U^*$. Then

$$\int_{U \cap V} g(\omega_U, \omega_V) \wedge \eta = - \int_U \omega_U \wedge j_{U*} \eta + \int_V \omega_V \wedge j_{V*} \eta,$$

since $g(\omega_U, \omega_V) = j_V^* \omega_V - j_U^* \omega_U$.

- The second square. We want an explicit construction of δ and δ_c . Let $\omega \in \Omega^p(M)$ be a closed form, and let $\{f_U, f_V\}$ be a partition of the unity with respect to $\{U, V\}$. Define

$$\omega_U = f_U \cdot \omega|_U \in \Omega_c^p(U), \quad \omega_V = f_V \cdot \omega|_V \in \Omega_c^p(V),$$

so $(\omega_U, \omega_V) \in \Omega_c^p(U) \oplus \Omega_c^p(V)$. Recall $i = i_{U*} + i_{V*}$. Then

$$i(\omega_U, \omega_V) = i_{U*} \omega_U + i_{V*} \omega_V = \omega_U + \omega_V = f_U \cdot \omega + f_V \cdot \omega = \omega.$$

If ω is closed, then $i(d\omega_U, d\omega_V) = d(i_{U*}\omega_U) + d(i_{V*}\omega_V) = 0$, so $(d\omega_U, d\omega_V) \in \text{Ker } i = \text{Im } j \subset \Omega_c^{p+1}(U) \oplus \Omega_c^{p+1}(V)$. Since j is injective there exists a unique $\delta_c(\omega) \in \Omega_c^{p+1}(U \cap V)$ such that $j(\delta_c(\omega)) = (d\omega_U, d\omega_V)$. Since $f_U + f_V = 1$, $df_U + df_V = 0$, so $df_U = -df_V$. Then

$$j(\delta_c(\omega)) = (d\omega_U, d\omega_V) = (df_U \wedge \omega|_U, df_V \wedge \omega|_V) = (-df_V \wedge \omega|_U, df_V \wedge \omega|_V) = j(df_V \wedge \omega|_{U \cap V}).$$

Since j is injective, $\delta_c(\omega) = df_V \wedge \omega|_{U \cap V}$, so $\delta_c : \Omega_c^p(M) \rightarrow \Omega_c^{p+1}$. Let η be a form on M . Since $\delta_c(d\eta) = df_V \wedge d\eta|_{U \cap V} = -d\delta_c(\eta)$, δ_c maps closed forms to closed forms and exact forms to exact forms, so

$$\begin{aligned} \delta_c : \Omega_c^p(M) &\longrightarrow \Omega_c^{p+1}(U \cap V) \\ \omega &\longmapsto df_V \wedge \omega|_{U \cap V}. \end{aligned}$$

By construction, it makes the long exact sequence exact. Similarly

$$\begin{aligned} \delta : \Omega^p(U \cap V) &\longrightarrow \Omega^{p+1}(M) \\ \omega &\longmapsto \begin{cases} df_V \wedge \omega & \text{on } U \cap V \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Now we check that the second square is commutative, that is

$$n_{p-1} \cdot \mu_M(\delta([\omega_1])) = n_p \cdot \delta_c^*(\mu_{U \cap V}([\omega_1])), \quad \omega_1 \in \Omega^{p-1}(U \cap V).$$

That is,

$$n_{p-1} \int_M \delta(\omega_1) \wedge \omega_2 = n_p \int_{U \cap V} \omega_1 \wedge \delta_c(\omega_2), \quad \omega_2 \in \Omega_c^{n-p}(M).$$

For all $\omega_2 \in \Omega_c^{n-p}(M)$,

$$n_{p-1} \int_M \delta(\omega_1) \wedge \omega_2 = n_{p-1} \int_{U \cap V} df_V \wedge \omega_1 \wedge \omega_2 = n_p \int_{U \cap V} \omega_1 \wedge df_V \wedge \omega_2 = n_p \int_{U \cap V} \omega_1 \wedge \delta_c(\omega_2).$$

- The third square. To check

$$(\mu_U \oplus \mu_V) \circ f = i^* \circ \mu_M,$$

so

$$(\mu_U \oplus \mu_V)(f([\omega])) = i^*(\mu_M([\omega])), \quad \omega \in \Omega^p(M),$$

in $\Omega_c^{n-p}(U)^* \oplus \Omega_c^{n-p}(V)^*$. Let $\eta_U \in \Omega_c^{n-p}(U)$ and $\eta_V \in \Omega_c^{n-p}(V)$. Then

$$\int_U \omega|_U \wedge \eta_U + \int_V \omega|_V \wedge \eta_V = \int_M \omega \wedge i(\eta_U, \eta_V).$$

□

The following is an easy consequence.

Corollary 2.55. *Let M be an oriented compact connected manifold of dimension n . Then*

$$H^n(M) = \mathbb{R},$$

and

$$H_c^p(M \setminus \{x\}) = H^p(M), \quad x \in M, \quad 1 \leq p < n.$$

Definition 2.56. The **Euler characteristic** of M is

$$\chi(M) = \sum_{p=0}^n (-1)^p \dim H^p(M).$$

Corollary 2.57. *If M is a compact oriented manifold of odd dimension then $\chi(M) = 0$.*

Proof. By Poincaré duality, $\dim H^i(M) = \dim H^{n-i}(M)$. □

2.7 Degree of a morphism

Let M and N be connected oriented manifolds of dimension n , and let $f : M \rightarrow N$ be a proper smooth morphism. Then $f^* : H_c^n(N) \cong \mathbb{R} \rightarrow H_c^n(M) \cong \mathbb{R}$ by Poincaré duality and connectedness, so $f(x) = c \cdot x$ for some $\deg f = c \in \mathbb{R}$. For any $\omega \in \Omega_c^n(M)$,

$$\int_M f^* \omega = \deg f \cdot \int_M \omega.$$

Proposition 2.58. *Let M, N, P be connected oriented manifolds of dimension n .*

- *If $f : M \rightarrow N$ and $g : N \rightarrow P$ are smooth and proper morphisms then*

$$\deg(g \circ f) = \deg f \cdot \deg g$$

- *If f is a diffeomorphism then*

$$\deg f = \begin{cases} 1 & f \text{ is orientation preserving} \\ -1 & \text{otherwise} \end{cases}.$$

- *If $f, g : M \rightarrow N$ are smooth proper and properly homotopic equivalent then*

$$\deg f = \deg g.$$

Theorem 2.59 (Mapping degree theorem). *Let $f : M \rightarrow N$ be a proper smooth morphism between connected oriented manifolds of dimension n . Then $\deg f \in \mathbb{Z}$.*

Definition 2.60. Let $f : M \rightarrow N$ be a smooth morphism. Then $y \in N$ is **regular** if for all $x \in f^{-1}(y)$, Df_x has maximal rank.

Theorem 2.61 (Preimage theorem). *Let $f : M \rightarrow N$ be a smooth morphism, and let $y \in N$ be a regular value. Then $f^{-1}(y)$ is a manifold of dimension $\dim M - \text{rk } Df_x$ where $x \in f^{-1}(y)$.*

Theorem 2.62 (Implicit function theorem). *Let $f : M \rightarrow N$ be a smooth morphism, and let $x \in M$ be such that Df_x is an isomorphism. Then there exists an open $x \in U \subset M$ such that $f|_U : U \rightarrow f(U)$ is an isomorphism.*

Theorem 2.63 (Sard's theorem). *Let $f : M \rightarrow N$ be smooth. Then if $Z \subset N$ is the set of regular values of f then $Z \cap f(M)$ is dense in $f(M)$.*

Proof of Theorem 2.59. Recall $\dim M = \dim N$, and if $\omega \in \Omega_c^n(N)$ and if Df_x is of rank less than n for all x , then $\deg f = 0$. We are done. In particular we may assume there exists x such that Df_x has rank equal to n . Let $y = f(x)$. By Sard's theorem, we may assume that for all $x \in f^{-1}(y)$, Df_x has rank n . By the preimage theorem $f^{-1}(y)$ is a manifold of dimension zero, such as $\mathbb{Z} \subset \mathbb{R}$, so

$$f^{-1}(y) = \{x_1, \dots, x_k\}$$

is a finite set, because f is proper. By the implicit function theorem, for all i there exists an open set $U_i \ni x_i$ such that $f|_{U_i}$ is a diffeomorphism and $f(U_i) = U$. Let $\omega \in \Omega_c^n(N)$ be such that $\int_U \omega = 1$ and $\text{supp } \omega \subset U$. Since $f|_{U_i}$ is a diffeomorphism

$$\int_{U_i} f|_{U_i}^* \omega = \text{sgn}(\det Df_{x_i}) \int_U \omega,$$

and $f|_{U_i}^* \omega$ has support in U_i . Since $\text{supp } f^* \omega \subset \bigcup_i U_i$,

$$\int_M f^* \omega = \sum_{i=1}^k \int_{U_i} f^* \omega = \sum_{i=1}^k \text{sgn}(\det Df_{x_i}) \int_U \omega = \sum_{i=1}^k \text{sgn}(\det Df_{x_i}) \int_M \omega,$$

so $\deg f = \sum_{i=1}^k \text{sgn}(\det Df_{x_i}) \in \mathbb{Z}$, which does not depend on y , if y is a regular point. \square

Lecture 20
Monday
24/02/20

Exercise. Suppose that $f : M \rightarrow N$ is a proper morphism between oriented connected manifolds. If $\deg f \neq 0$, then f is surjective.

Example 2.64. Let $M = S^n = N$, and let

$$\begin{aligned} f &: M \longrightarrow N \\ x &\longmapsto -x \end{aligned}$$

be the antipodal map. Claim that $\deg f = (-1)^{n+1}$. Let $i : S^n \hookrightarrow \mathbb{R}^{n+1}$, let

$$\tilde{\omega} = x_1 dx_2 \wedge \cdots \wedge dx_{n+1} \in \Omega^n(\mathbb{R}^{n+1}),$$

and let $\omega = i_* \tilde{\omega} \in \Omega^n(S^n)$. By Stokes and $S^n = \partial D^{n+1}$,

$$\int_{S^n} \omega = \int_{S^n} i^* \tilde{\omega} = \int_{D^{n+1}} d\tilde{\omega} = \int_{D^{n+1}} dx_1 \wedge \cdots \wedge dx_{n+1} \neq 0,$$

so f can be extended to

$$\begin{aligned} \tilde{f} &: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1} \\ x &\longmapsto -x \end{aligned}.$$

Then $\tilde{f} \circ i = i \circ f$ and $\tilde{f}^* \tilde{\omega} = (-1)^{n+1} \tilde{\omega}$, so

$$f^* \omega = f^* i^* \tilde{\omega} = (i \circ f)^* \tilde{\omega} = (\tilde{f} \circ i)^* \tilde{\omega} = i^* \tilde{f}^* \tilde{\omega} = (-1)^{n+1} i^* \tilde{\omega} = (-1)^{n+1} \omega.$$

Thus

$$(-1)^{n+1} \int_{S^n} \omega = \int_{S^n} f^* \omega = \deg f \int_{S^n} \omega,$$

so $\deg f = (-1)^{n+1}$.

3 Morse theory

3.1 Introduction

Definition 3.1. Let M be a manifold of dimension n , and let $f : M \rightarrow \mathbb{R}$ be smooth. A **critical point** of f is a point $x \in M$ such that $Df_x = 0$, that is if x_1, \dots, x_n are local coordinates at x , then

$$\frac{\partial}{\partial x_i} f(x) = 0, \quad i = 1, \dots, n.$$

For such x , we define the Hessian of f to be

$$H_f(x) = \left(\frac{\partial^2}{\partial x_i \partial x_j} f(x) \right).$$

Then x is called **non-degenerate** if $\det H_f(x) \neq 0$. A function f such that every critical point of f is non-degenerate is called a **Morse function**.

Fact. By Sard's theorem most of the functions satisfy this property.

3.2 Cell decomposition

Notation. Let $D_n = \{x \mid |x| \leq 1\} \subset \mathbb{R}^n$ be the unit ball, and let $S^{n-1} = \partial D_n$.

Definition 3.2. An n -cell is a topological space which is homeomorphic to the open ball $D_n \setminus \partial D_n$. A **cell decomposition** of a topological space M is a family F of pairwise disjoint subspaces of M which are n -cells and such that $M = \bigsqcup_{e_i \in F} e_i$. If F is finite, then this is called a **finite cell decomposition**. For all $m \geq 0$,

$$SK_m M = \bigsqcup_{\dim e_i \leq m} e_i.$$

Example 3.3. $S^1 = (S^1 \setminus \{p\}) \sqcup \{p\}$, where $S^1 \setminus \{p\}$ is a 1-cell and $\{p\}$ is a 0-cell.

Notation 3.4. Let M be a topological space, and let $f_\partial : S^{n-1} \rightarrow M$ be continuous. We construct a new topological space

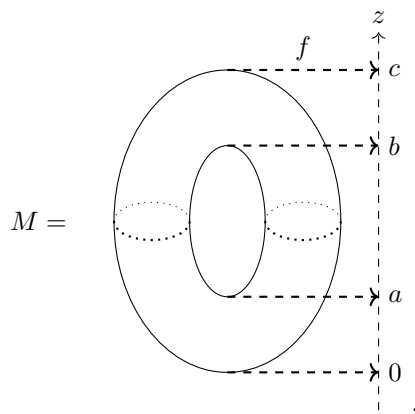
$$M \cup_{f_\partial} D_n = M \sqcup D_n / \sim,$$

where $M \ni x \sim y \in S^{n-1} \subset D_n$ if $f_\partial(y) = x$. Then $M \cup_{f_\partial} D_n$ is said to be obtained by **attaching** an n -cell to M via f_∂ .

Example. Let $M = \{p\}$, and let $f_\partial : S^0 = \partial D_1 = \partial[0, 1] \rightarrow \{p\}$. Then $S^1 = M \cup_{f_\partial} D_1$.

Exercise. If M admits a cell decomposition then also $M \cup_{f_\partial} D_n$ does.

Example 3.5. Let $M = S^1 \times S^1 \subset \mathbb{R}^3$ be the torus



where

$$f : \begin{array}{ccc} M & \longrightarrow & \mathbb{R} \\ (x, y, z) & \longmapsto & z \end{array}.$$

Lecture 21
Tuesday
25/02/20

For all h define

$$S_h = \{p \in M \mid f(p) \leq h\} = f^{-1}((-\infty, h]).$$

- If $h < 0$ then $S_h = \emptyset$.
- If $0 < h < a$ then S_h is homotopically equivalent to a 0-cell, where

$$S_h = \text{[shaded bowl shape]} \sim \bullet$$

- If $a < h < b$ then S_h is homotopically equivalent to a 1-cell attached to the previous S_h , where

$$S_h = \text{[U-shaped bowl with two shaded top rims]} \sim \text{[shaded bowl with two points connected by a vertical line segment]}$$

- If $b < h < c$ then S_h is homotopically equivalent to a 1-cell attached to the previous S_h , where

$$S_h = \text{[torus with shaded top disk]} \sim \text{[U-shaped bowl with two shaded top rims connected by a horizontal line segment]}$$

- If $h > c$ then S_h is homotopically equivalent to a 2-cell attached to the previous S_h , where

$$S_h = \text{[torus with shaded top disk]} \sim \text{[torus with shaded top disk and a dashed line indicating the boundary of the shaded region]}$$

Thus

$$M = 0\text{-cell} \sqcup \text{two } 1\text{-cells} \sqcup 2\text{-cell}.$$

Given a Morse function $f : M \rightarrow \mathbb{R}$, the goal is to study the level sets of f , $S_h = f^{-1}((-\infty, h])$.

Definition 3.6. Let M be a manifold, let $f : M \rightarrow \mathbb{R}$ be a Morse function, and let $x \in M$ be critical. Denote

$$\text{Eig}^- H_f(x) = \{\text{eigenvectors of } H_f \text{ with negative eigenvalues}\}.$$

Recall that H_f is a symmetric matrix. The **index** of f at x is the dimension of $\text{Eig}^- H_f(x)$.

Lemma 3.7 (Morse). *Let M be a manifold of dimension n , let $f : M \rightarrow \mathbb{R}$ be a Morse function, and let $x_0 \in M$ be a critical point. Then there exist local coordinates (x_1, \dots, x_n) around x_0 such that $x_0 = (0, \dots, 0)$ and*

$$f = f(x_0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^n x_i^2,$$

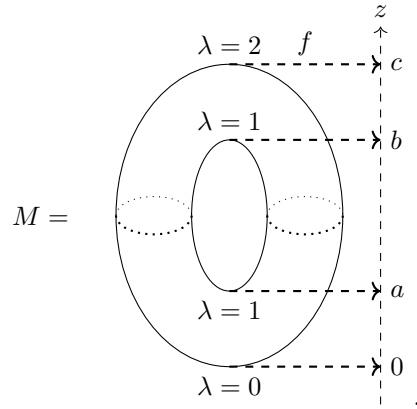
where λ is the index of f at x .

Thus the set of critical points of f is discrete, since locally at critical x_0 ,

$$f = f(x_0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^n x_i^2$$

has no more critical points.

Example 3.8. Let $f : M \rightarrow \mathbb{R}$ be as in the previous example. Then



3.3 CW-complexes

Definition 3.9. A topological space M admits a **CW-structure** if there exists a sequence of topological subspaces

$$M^{(0)} \subset \dots \subset M^{(n)},$$

such that

1. $M^{(0)}$ is a discrete subset of M ,
2. $M^{(k)}$ is obtained by attaching k -cells to $M^{(k-1)}$, and
3. $V \subset M$ is closed if and only if $V \cap M^{(k)}$ is closed for all k .

Such M is called a **CW-complex**. Then M is a **finite CW-complex** if it is obtained by attaching finitely many cells. In this case 3 is not needed. A **subcomplex** of a CW-complex M is a closed subspace of M which is a union of cells of M . A **closed cell** is the image of D_n in a cell. An **open cell** is the image of $D_n \setminus \partial D_n$ in a cell. Open cells are not open in M in general.

Example 3.10.

- $S^n = \{p\} \cup D_n = M^{(0)} \cup M^{(1)}$.
- If $M = \mathbb{R}^n$, and Λ is the integral points in \mathbb{R}^n , then Λ gives a decomposition of \mathbb{R}^n into n -cubes, which are n -cells, where 0-cells are points of Λ , 1-cells are edges of Λ , etc.
- If $n \neq 4$ and M is a manifold of dimension n , then M is a CW-complex. If $n = 4$, then it is open.

Proposition 3.11. *Let M be a CW-complex. Then*

1. *if $K \subset M$ is a compact subset, then K is contained in a finite union of open cells, and*
2. *the closure of every cell of M is contained in a finite subcomplex of M .*

Proof.

1. We first prove 1. Let $K \subset M$ be a compact subset. We want to show that K only intersects finitely many cells of M . Assume by contradiction that there is an infinite sequence of points $S = \{x_j\} \subset K$ all lying in distinct cells. We claim that $S \cap M^{(n)}$ is closed and discrete for all $n \geq 0$. We proceed by induction on n . For $n = 0$, this follows from the fact that $M^{(0)}$ is closed and discrete. Assume now that $S \cap M^{(n)}$ is closed and discrete. Then, if $\{e_i\}_I$ are the $(n+1)$ -cells, then the open cell corresponding to e_i contains at most one $x_j \in S$. Thus $S \cap (\bigcup_i e_i)$ is closed and discrete. It follows that $S \cap M^{(n+1)}$ is closed and discrete, as claimed. Since $S \subset K$, it follows that S is finite, a contradiction.
2. We now prove 2. To this end, we proceed by induction on the dimension n of the cell. For $n = 0$, the result is clear. Assume now that the result is true for any m -cell with $m < n$ and let e_n be an n -cell. In particular, the border K of e_n is the image of S^{n-1} and it is compact. Hence, it is contained in a finite union of open cells of dimension smaller than n by 1. By induction, each of these cells is contained in a finite subcomplex. The union of these subcomplexes is a finite subcomplex containing K . Hence attaching e_n results in a finite subcomplex containing e_n .

□

Corollary 3.12. *Let M be a CW-complex. Then any compact subset of M is contained in a finite subcomplex.*

Proof. Since a finite union of finite subcomplex is again a finite subcomplex, the result follows immediately from Proposition 3.11. □

3.4 The CW-complex structure associated to a Morse function

Definition 3.13. Let M be a manifold, then a **flow** or a **group of diffeomorphisms** of M is the collection of diffeomorphisms $\phi_t : M \rightarrow M$ for $t \in \mathbb{R}$ such that there exists $\phi : \mathbb{R} \times M \rightarrow M$ with $\phi_t = \phi(t, \cdot)$ and

- $\phi_0 = \text{id}_M$, and
- $\phi_{t+s} = \phi_t \circ \phi_s$.

For all $x \in M$, the **flow line** or the **integral curve** for the flow is

$$\begin{array}{rcl} \gamma_x & : & \mathbb{R} \longrightarrow M \\ & & t \longmapsto \phi(t, x) \end{array},$$

where $\gamma_x(0) = x$.

Since $\frac{d}{dt} \gamma_x|_{t=0} \in T_x M$, the flow defines a vector field on M , that is a smooth section of TM . On the other hand, given a vector field, we can find a flow. Let X be a section of TM . There exists ϕ_t , and γ_x , as above such that $\frac{d}{dt} \gamma_x|_{t=0} \in T_x M$, locally around $t = 0$.

Lemma 3.14. *Let M be a manifold, and let X be a smooth compactly supported vector field on M . Then X generates a unique one-parameter group of diffeomorphisms $\phi_t : M \rightarrow M$ such that, for all $x \in M$, we have*

$$(X \circ \gamma_x)(t) = \frac{d}{dt} \gamma_x(t).$$

It can be shown that two distinct flow lines are disjoint. Thus, the manifold M decomposes into a disjoint union of flow lines.

Example. Let $M = \mathbb{R}$, and let $X = x^2 \partial_x$. Find the flow. ¹³

¹³Exercise

Definition 3.15. Let M be a manifold. A **Riemannian metric** g on M is a collection of inner products for $T_x M$ for $x \in M$ given by

$$g_x : T_x M \times T_x M \rightarrow \mathbb{R},$$

such that for all vector fields X and Y we have that $g_x(X(x), Y(x)) : M \rightarrow \mathbb{R}$ is smooth. A **Riemannian manifold** (M, g) is a manifold M and a Riemannian metric g on M .

Then g gives $TM \xrightarrow{\sim} \Omega^1(M) = (TM)^*$.

Definition 3.16. Let (M, g) be a Riemannian manifold, and let $f : M \rightarrow \mathbb{R}$ be a smooth function. The **gradient vector field** ∇f of f is the unique vector field on M such that for all vector fields X ,

$$g(\nabla f, X) = X(f) = Df(X),$$

the derivative of f with respect to X .

Thus,

$$\|\nabla f\|^2 = g(\nabla f, \nabla f) = Df(\nabla f).$$

In particular $\nabla f(x) = 0$ if and only if x is a critical point of f , and ∇f is orthogonal to any vector tangent to $f^{-1}(c)$ for all regular values $c \in \mathbb{R}$. In particular, for all smooth functions $f : M \rightarrow \mathbb{R}$, we can take the flow ϕ associated to $-\nabla f$ such that if $\gamma_x(t) = \phi(t, x)$ then

$$\frac{d}{dt} \gamma_x(0) = -\nabla f(\gamma_x(0)), \quad \gamma_x(0) = x.$$

This is called the **gradient flow** of f . The integral curve of this flow are called **gradient flow lines**.

Lemma 3.17. f decreases along the gradient lines, that is $f(\gamma_x(t))$ is a decreasing function with respect to t .

Proof.

$$0 \leq -\|\nabla f(\gamma_x(t))\|^2 = Df_{\gamma_x(t)}(-\nabla f(\gamma_x(t))) = Df_{\gamma_x(t)}\left(\frac{d}{dt} \gamma_x(t)\right) = \frac{d}{dt} f(\gamma_x(t)).$$

□

Proposition 3.18. Let M be a compact manifold, and let $f : M \rightarrow \mathbb{R}$ be a Morse function. Then every gradient flow line begins and ends with a critical point of f , that is $\lim_{t \rightarrow \pm\infty} \gamma_x(t)$ exist and are critical points of f .

Proof. First we prove that if the limit exists then it is a critical point of f . For all x , $f(\gamma_x(\cdot)) : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded function. Then

$$0 = \lim_{t \rightarrow \pm\infty} \frac{d}{dt} f(\gamma_x(t)) = \lim_{t \rightarrow \pm\infty} -\|\nabla f(\gamma_x(t))\|^2,$$

so the limit is critical. The goal is to show that limits exist. Since M is compact and f is a Morse function, the set of critical points is finite. Fix $\epsilon > 0$. Let U be the union of open balls of radius $\epsilon > 0$ around each critical point, so U is open in M . Then $M \setminus U$ is compact, so $\|\nabla f(\cdot)\|^2$ admits a minimum inside $M \setminus U$, but it cannot be zero, but $\lim_{t \rightarrow \pm\infty} \|\nabla f(\gamma_x(t))\|^2 = 0$. If $\pm t$ is very large then $\gamma_x(t) \notin M \setminus U$, so if ϵ is sufficiently small, $\gamma_x(t)$ is in a ball around a single critical point for $t \rightarrow \pm\infty$. This implies that the limit is the critical point. □

Definition 3.19. Let X be a topological space, and let $S \subset X$. Then S is called a **deformation retract** of X if there exists $F : X \times [0, 1] \rightarrow X$ such that $F(x, 0) = x$ and $F(x, 1) \in S$ for all $x \in X$, and $F(s, 1) = s$ for all $s \in S$.

This implies that S and X are homotopy equivalent.

Theorem 3.20 (First fundamental theorem of Morse theory). Let M be a manifold, let $f : M \rightarrow \mathbb{R}$, let $a < b \in \mathbb{R}$ such that $f^{-1}([a, b])$ does not contain any critical point, and let

$$S_t = f^{-1}((-\infty, t]), \quad t \in \mathbb{R}.$$

Then S_a is a deformation retract of S_b .

The idea is that we will use a perturbation of the gradient flow.

Lecture 23
Monday
02/03/20

Proof. There exists $\epsilon > 0$ such that $f^{-1}([a - \epsilon, b + \epsilon])$ does not contain any critical point. Fix a metric g on M . Define ρ smooth on M such that $\rho(x) \geq 0$ and

$$\rho(x) = \begin{cases} \frac{1}{\|\nabla f\|^2} & x \in f^{-1}([a, b]) \\ 0 & x \in M \setminus f^{-1}([a - \epsilon, b + \epsilon]) \end{cases}.$$

Define

$$X(x) = -\rho(x) \nabla f(x).$$

There exists a flow $\phi(t, x)$ induced by X , that is if $\gamma_x(t) = \phi(t, x)$ then $\frac{d}{dt} \gamma_x(0) = X(x)$. By definition of ∇f ,

$$\frac{d}{dt} f(\gamma_x(t)) = Df_{\gamma_x(t)} \left(\frac{d}{dt} \gamma_x(t) \right) = g \left(\nabla f, \frac{d}{dt} \gamma_x(t) \right) = g(\nabla f, -\rho(\gamma_x(t)) \nabla f) = -\rho(\gamma_x(t)) \|\nabla f\|^2 \leq 0,$$

so $f(\gamma_x(t))$ is decreasing. Moreover for all t such that $f(\gamma_x(t)) \in [a, b]$ we have that

$$\frac{d}{dt} f(\gamma_x(t)) = -1,$$

by definition of ρ . If $\gamma_x(s) \in f^{-1}([a, b])$ for all $s \in [0, t]$, by the fundamental theorem of calculus

$$f(\gamma_x(t)) - f(\gamma_x(0)) = \int_0^t \frac{d}{ds} f(\gamma_x(s)) ds = -t,$$

so

$$f(\gamma_x(t)) = f(x) - t.$$

Then

1. if $f(x) \leq b$ then $f(\phi_{b-a}(x)) = f(\gamma_x(b-a)) \leq a$, and
2. if $f(x) > b$ then $f(\phi_{b-a}(x)) > a$.

1 implies that $\phi_{b-a}(S_b) \subset S_a$ and 2 implies that $\phi_{a-b}(S_a) \subset S_b$. Recall that $\phi_{a-b} = \phi_{b-a}^{-1}$, so S_a and S_b are diffeomorphic. Now we define

$$\begin{aligned} F : S_b \times [0, 1] &\longrightarrow S_b \\ (x, t) &\longmapsto \begin{cases} x & f(x) \leq a \\ \phi_{t(f(x)-a)}(x) & a \leq f(x) \leq b \end{cases}. \end{aligned}$$

Then $F(x, 0) = x$, since $\phi_0(x) = x$, and

$$F(x, 1) = \begin{cases} x & f(x) \leq a \\ \phi_{f(x)-a}(x) = \gamma_x(f(x)-a) & a \leq f(x) \leq b \end{cases}.$$

In particular if $x \in S_a$, then $F(x, 1) = x$ and for all $x \in S_b$, $F(x, 1) \in S_a$. □

Theorem 3.21 (Reeb's theorem). *Let M be a compact manifold, and let $f : M \rightarrow \mathbb{R}$ be a Morse function. Assume that f admits exactly two critical points. Then M is homeomorphic to a sphere S^n .*

Proof. There exists a unique x_{\min} such that $h_{\min} = f(x_{\min})$ is the minimum and there exists a unique x_{\max} such that $h_{\max} = f(x_{\max})$ is the maximum. Both x_{\min} and x_{\max} are critical points. Then $\lambda(x_{\min}) = n$ and $\lambda(x_{\max}) = 0$. By the Morse lemma, around x_{\min} , we can write locally

$$f(x) = h_{\min} + \sum_{i=1}^n x_i^2,$$

for some local coordinates x_1, \dots, x_n such that $x_{\min} = (0, \dots, 0)$. Let $a > h_{\min}$ be sufficiently close to h_{\min} . Then

$$S_a = \left\{ h_{\min} + \sum_{i=1}^n x_i^2 \leq a \right\} = D_n.$$

Similarly there exists $b < h_{\max}$ sufficiently close such that $M \setminus S_b \cong D_n$. By Theorem 3.20, since there does not exist critical points in $f^{-1}([a, b])$ we know that $S_b \cong S_a \cong D_n$. We proved that there exist

$$\phi_+ : D_n^+ \xrightarrow{\sim} H_+, \quad \phi_- : D_n^- \xrightarrow{\sim} H_-,$$

where $H_- = S_b$ and $H_+ = \overline{M \setminus S_b}$, such that $M = H_+ \cup H_-$ and $\phi_+ (\partial D_n^+) = \phi_- (\partial D_n^-) = H_+ \cap H_-$, so

$$\begin{array}{ccccc} S^{n-1} = \partial D_n^+ & \subset & D_n^+ & \xrightarrow{\phi_+} & H_+ \\ & \searrow \phi_+ & & \subset & \\ & & H_+ \cap H_- & & \\ & \nearrow \phi_- & & \subset & \\ S^{n-1} = \partial D_n^- & \subset & D_n^- & \xrightarrow{\phi_-} & H_- \end{array} \quad \begin{array}{c} \\ \\ \\ \\ \end{array} \quad \begin{array}{c} \\ \\ H_+ \cup H_- = M \\ \\ \end{array}$$

The problem is that in general $\phi_+|_{\partial D_n^+} \neq \phi_-|_{\partial D_n^-}$. Let

$$f = (\phi_+^{-1} \circ \phi_-)|_{S^{n-1}} : \partial D_n^- \rightarrow \partial D_n^+.$$

I want a homeomorphism $F : D_n^+ \rightarrow D_n^+$ such that $F|_{\partial D_n^-} = f$. By taking $\phi_+ \circ F$ we obtain that M is obtained by attaching D_n^+ with D_n^- so that they coincide at the boundary with the identity on S^{n-1} , so M is homeomorphic to S^n . The goal is given a homeomorphism $f : S^{n-1} \rightarrow S^{n-1}$, there exists a homeomorphism $F : D_n \rightarrow D_n$ such that $F|_{S^{n-1}} = f$. Indeed, if $v \in \mathbb{R}^n$ such that $|v| = 1$, then let

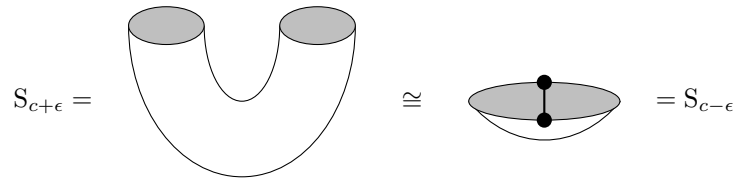
$$F(tv) = tf(v).$$

We do the same with the inverse. □

Lecture 24 is a problems class.

Theorem 3.22 (Second fundamental theorem of Morse theory). *Let M be a manifold of dimension n , let $f : M \rightarrow \mathbb{R}$ be a Morse function, let $x_0 \in M$ be a critical point for f such that if $c = f(x_0)$ then there exists $\epsilon > 0$ such that $f^{-1}([c - \epsilon, c + \epsilon])$ contains only one critical point, and let λ be the index of f at x_0 . Then if ϵ is sufficiently small, $S_{c+\epsilon}$ is homotopy equivalent to $S_{c-\epsilon}$ attached to a λ -dimensional cell.*

Example.



Proof.

$\lambda = 0$. x_0 is a local minimum for d . Around x_0 , $S_{c-\epsilon}$ is empty and $S_{c+\epsilon}$ is homotopy equivalent to a point. Indeed it is a ball. There exists $U \ni x_0$ such that outside U , $f|_{M \setminus \overline{U}} : M \setminus \overline{U} \rightarrow \mathbb{R}$ does not contain any critical points. By the first fundamental theorem $S_{c-\epsilon} \setminus \overline{U} \cong S_{c+\epsilon} \setminus \overline{U}$.

$\lambda = n$. x_0 is a local maximum. Let $U \ni x_0$ be a ball D_n^+ , by the Morse lemma. Like in the proof of Reeb's theorem, $S_{c+\epsilon}$ is homotopy equivalent to $S_{c-\epsilon}$ attached to D_n^+ , so ok.

$1 \leq \lambda \leq n - 1$. We apply the Morse lemma. There exists $U \ni x_0$ such that on U , there exist coordinates x_1, \dots, x_n such that $x_0 = (0, \dots, 0)$ and

$$f(x) = f(x_0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^n x_i^2.$$

We just need to study $S_{c-\epsilon} \cap U$ and $S_{c+\epsilon} \cup U$. Define

$$B_{\sqrt{2\epsilon}} = \left\{ (x_1, \dots, x_n) \mid \sum_{i=1}^n x_i^2 \leq 2\epsilon \right\},$$

so $B_{\sqrt{2\epsilon}} \subset U$ if ϵ is sufficiently small, and

$$e_\lambda = \left\{ (x_1, \dots, x_n) \mid \sum_{i=1}^\lambda x_i^2 \leq \epsilon, x_{\lambda+1} = \dots = x_n = 0 \right\} \cong D_\lambda.$$

An easy case is $n = 2$ and $\lambda = 1$. Then

$$U \cap f^{-1}(c) = \{x_1^2 - x_2^2 = 0\}, \quad e_1 \cap U = \{x_1^2 \leq \epsilon, x_2 = 0\},$$

$$U \cap S_{c-\epsilon} = \{x_1^2 - x_2^2 \geq \epsilon\}, \quad U \cap S_{c+\epsilon} = \{x_1^2 - x_2^2 \leq -\epsilon\},$$

so $S_{c+\epsilon} \cong S_{c-\epsilon} \cup e_1$. We want to perturb f , and obtain $g \leq f$ with the same critical value, and $g = f$ outside U . We define

$$\begin{aligned} \mu : \mathbb{R}_{\geq 0} &\longrightarrow \mathbb{R}_{\geq 0} \\ t &\longmapsto \begin{cases} \epsilon & t = 0 \\ 0 & t \geq 2\epsilon \end{cases}, \end{aligned}$$

such that $-1 < \mu'(t) \leq 0$. We define

$$g = \begin{cases} f & \text{outside } U \\ f - \mu(\xi + 2\eta) & \text{in } U \end{cases}, \quad \xi = x_1^2 + \dots + x_\lambda^2, \quad \eta = x_{\lambda+1}^2 + \dots + x_n^2.$$

Then $g \leq f$, since $f(x) = f(x_0) - \xi + \eta$, so $g(x) = f(x_0) - \xi + \eta - \mu(\xi + 2\eta)$ inside U , and $g = f$ in the ellipsoid

$$E = \{\xi + 2\eta \leq 2\epsilon\} \subset B_{\sqrt{2\epsilon}} \subset U.$$

In E , $\eta - \xi \leq \frac{1}{2}(\xi + 2\eta) \leq \epsilon$, so $E \subset S_{c+\epsilon} = \{\eta - \xi \leq \epsilon\}$.

□