M3P20 Geometry I: Algebraic Curves

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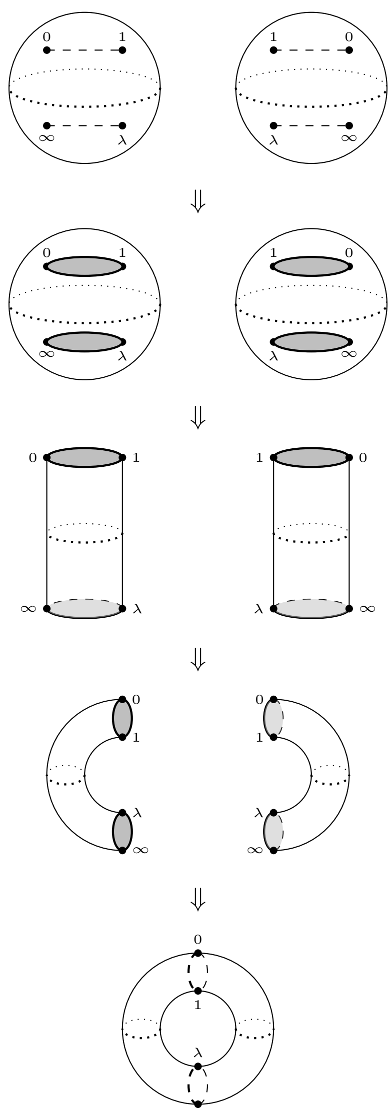

Syllabus

Affine plane algebraic curves. Projective space. Plane projective curves. Projectivisation. Points at infinity. Singularities. Smoothness. Intersections of plane curves. Resultants. Multiplicities. Bézout's theorem. Conics. Cubic curves. Riemann surfaces. Genus. Ramification. The Riemann-Hurwitz formula. The degree-genus formula.

Contents

1 Introduction

Lecture 1 Monday 08/10/18

This course is intended as a first course in algebraic geometry. It will focus on one-dimensional algebraic varieties. The following are the reference books for the course.

- F Kirwan, Complex algebraic curves, 1992
- W Fulton, Algebraic curves, an introduction to algebraic geometry, 1969

Note. The official notes are integrated in these unofficial notes.

Geometry is the study of shapes in suitable spaces, such as sets of points on the real line $\mathbb{R}$, lines and circles in $\mathbb{R}^2$, spheres in higher dimensional Euclidean spaces $\mathbb{R}^n$, etc. One way to think about shapes is to see them as the locus of zeroes defined by

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid f(x_1, \dots, x_n) = 0\} \subseteq \mathbb{R}^n,$$

for some suitable function f.

Example 1.1.

- Circles with centre at $(0,0)$ in $\mathbb{R}^2$ are
$$\left\{f_1(x,y) = x^2 + y^2 - R^2 = 0\right\}, \qquad R \in \mathbb{R}.$$
- The unit square with vertices at $\{(\pm 1, 0), (0, \pm 1)\}$ in $\mathbb{R}^2$ is
$$\{f_2(x,y) = |x| + |y| - 1 = 0\}.$$
- Spheres in $\mathbb{R}^n$ are
$$\left\{f_3(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2 - R^2 = 0\right\}, \qquad R \in \mathbb{R}.$$

Remark 1.2. Note that every subset $S \subseteq \mathbb{R}^n$ is the zero set of some function, by just defining

$$\begin{array}{cccc} \chi_S & : & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & \begin{cases} 0 & x \in S \\ 1 & x \notin S \end{cases} \end{array}.$$

The class of functions used to define our shapes has great consequences on their geometry. In Example 1.1, f_1 is a polynomial so that it is differentiable and also C^∞, while f_2 is continuous but not differentiable at $\{(0, \pm 1), (\pm 1, 0)\}$, the vertices of the square. The function χ_S is not even continuous, unless S is empty, or the whole $\mathbb{R}^n$. As these examples illustrate, an underlying principle is the equivalence between the regularity properties of f and the regularity properties of $\{f = 0\}$. **Algebraic geometry** is the area of mathematics that studies the shapes in spaces defined by polynomial equations using algebra. Such shapes are called **algebraic varieties**. Their geometric properties are intimately related to the algebraic properties of the defining polynomial equations.

Example 1.3.

- Let $f(x)$ be a polynomial. Then the zero set $\{f(x) = 0\} \subseteq \mathbb{R}$ is a finite set of points in $\mathbb{R}$, and every finite set of points arises in this manner.
- The circle $\left\{x^2 + y^2 - 1 = 0\right\} \subseteq \mathbb{R}^2$ is an algebraic variety.
- Spheres in higher dimensions are algebraic varieties, defined by the equation $\left\{x_1^2 + \dots + x_n^2 = r^2\right\} \subseteq \mathbb{R}^n$, where $r \in \mathbb{R}_{\geq 0}$ is the radius.

Exercise 1.

- Is $\mathbb{Z} \subseteq \mathbb{R}$ an algebraic variety?
- Is the unit square an algebraic variety?

Definition 1.4. Let K be a field, such as $K = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ a multi-index, denote a **monomial** by

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \qquad |\alpha| = \sum_{i=1}^n \alpha_i.$$

A **polynomial** of degree d in n variables with coefficients in K is a finite sum

$$P(x_1, \dots, x_n) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha, \qquad a_\alpha \in K,$$

where $a_\alpha = 0$ for all $|\alpha| > d$ and $a_\alpha \neq 0$ for some α with $|\alpha| = d$. The set of polynomials of arbitrary degree in n variables with coefficients in K is denoted $K[x_1, \dots, x_n]$.

Example. Let $n = 3$. Then

$$P(x_1, x_2, x_3) = 3 + x_1^2 x_2 + x_3^{10}, \qquad \alpha = (0,0,0), (2,1,0), (0,0,10)$$

has degree ten.

Exercise 2.

- Show that $K[x_1, \dots, x_n]$ is a ring, and that if P and Q are polynomials of degrees p and q respectively, then the degree of $\lambda P + \mu Q$ for $\lambda, \mu \in K$ is at most $\max\{p, q\}$. Give an example of polynomials $P, Q \in K[x]$ such that
$$\deg(P + Q) < \max\{\deg(P), \deg(Q)\}.$$
- Show that $(P \cdot Q)(x_1, \dots, x_n) = P(x_1, \dots, x_n) Q(x_1, \dots, x_n)$ is a polynomial $P \cdot Q \in K[x_1, \dots, x_n]$ with $\deg(P \cdot Q) = \deg(P) + \deg(Q)$. What if $P = 0$? What is $\deg(0)$?

Definition 1.5. An **affine plane curve** defined over K is

$$C = \left\{(x, y) \in K^2 \mid P(x, y) = 0\right\} \subseteq K^2,$$

where $P \in K[x, y]$ is non-constant. More generally, an **algebraic variety** $V \subseteq K^n$ is a subset of K^n defined as the locus

$$\{f_1 = \dots = f_k = 0\} \subseteq K^n,$$

where $f_1, \dots, f_k \in K[x_1, \dots, x_n]$ are polynomials in n variables with coefficients in K.

Example 1.6.

- If we allow constant polynomials, then for $P = 0$ we get $C = K^2$, and if P is a non-zero constant, then C is the empty set. Neither of those really look like curves.
- Let $a, b, c \in \mathbb{R}$ with $(a, b) \neq (0, 0)$. The curve
$$\left\{(x, y) \in \mathbb{R}^2 \mid ax + by + c = 0\right\}$$
is a line.
- Let $a, b \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$. The curve
$$\left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\right\}$$
is an ellipse.
- Let $a, b \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$. The curve
$$\left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0\right\}$$
is a hyperbola.
- Spheres, and quadrics such as ellipsoids, paraboloids, and hyperboloids in $\mathbb{R}^3$ are all defined via a single polynomial equation of degree two. A line in $\mathbb{R}^3$ can be defined by two equations in degree one.

The following is the first property of algebraic curves.

Lemma 1.7. *The union of two affine plane curves is again an affine plane curve.*

Proof. Let $f_1, f_2 \in K[x, y]$ and let

$$C_1 = \left\{(x, y) \in K^2 \mid f_1(x, y) = 0\right\}, \qquad C_2 = \left\{(x, y) \in K^2 \mid f_2(x, y) = 0\right\}.$$

Then $f_1 \cdot f_2 \in K[x, y]$ is a polynomial and

$$C_1 \cup C_2 = \left\{(x, y) \in K^2 \mid (f_1 \cdot f_2)(x, y) = 0\right\},$$

so that $C_1 \cup C_2$ is an affine plane curve. $\square$

Exercise 3. Write down an equation for the plane curve that is the union of the lines through any two vertices of the unit square.

Recall the following.

Definition 1.8. A polynomial $P \in K[x_1, \dots, x_n]$ is **reducible** over K if there are non-constant polynomials $Q, R \in K[x_1, \dots, x_n]$ such that $P = Q \cdot R$. A polynomial P is **irreducible** if it is not reducible. Recall that a polynomial P is called non-constant if $\deg(P) > 0$.

Example. $x_1 x_2$ is reducible, and $x_1 + x_2$ is irreducible.

Remark 1.9. Recall also that every polynomial $P \in K[x_1, \dots, x_n]$ can be written as a product of irreducible factors

$$P = f_1 \dots f_k,$$

in an essentially unique way up to multiplication by constants. We have

$$\{P = 0\} = \{f_1 = 0\} \cup \dots \cup \{f_k = 0\} \subseteq K^n,$$

so in particular, for $n = 2$, every algebraic curve is a union of algebraic curves defined by irreducible polynomials.

In the course, we will consider questions such as the following.

- When do polynomials $f, g \in K[x, y]$ define the same affine plane curve?
- What can be said about the intersection $\{f = 0\} \cap \{g = 0\} \subseteq K^2$?

Very different questions can be approached through algebraic curves. For example, we can study integer solutions to some Diophantine equations.

Example 1.10. The unit circle is the curve

$$C = \left\{x^2 + y^2 = 1\right\} \subseteq \mathbb{R}^2.$$

Several parametrisations are known, such as

$$\begin{array}{rcl} [0, 2\pi) & \longrightarrow & \mathbb{R}^2 \\ t & \longmapsto & (\cos(t), \sin(t)) \end{array}.$$

We can write down another parametrisation of C by considering lines through the point $P = (-1, 0)$, using a stereographic projection. A line through P with slope $t \in \mathbb{R}$ has equation

$$L_t = \{y = t(x + 1)\} \subseteq \mathbb{R}^2$$

and meets C in two points, P and $P_t = (x(t), y(t))$. We can determine the coordinate of P_t by solving the system

$$L_t \cap C = \begin{cases} y = t(x + 1) \\ x^2 + y^2 = 1 \end{cases}.$$

Replacing the value of y given by the first equation into the second yields two solutions for $x(t)$. The first one is $x = -1$ and corresponds to the point $P = (-1, 0)$. The second is $(x(t), y(t))$, where

$$x(t) = \frac{1-t^2}{1+t^2}, \qquad y(t) = \frac{2t}{1+t^2}. \tag{1}$$

Note that when $t \to \infty$, $(x(t), y(t)) \to (-1, 0)$, so that $t \mapsto (x(t), y(t))$ is a parametrisation of C that identifies it with $\mathbb{R} \cup \{\infty\}$. The advantage of this parametrisation is that it is given by rational functions, that is $x(t)$ and $y(t)$ are of the form $t \mapsto p(t)/q(t)$, where p and q are polynomials. One can use this parametrisation to get the general solution of the equation

$$x^2 + y^2 = z^2 \tag{2}$$

for $x, y, z \in \mathbb{Z}$ coprime. If $t = p/q \in \mathbb{Q}$, where $p, q \in \mathbb{Z}$ are coprime, then $x(t), y(t) \in \mathbb{Q}$ in (1) becomes

$$x(t) = \frac{p^2-q^2}{p^2+q^2}, \qquad y(t) = \frac{2pq}{p+q^2}.$$

If

$$x = p^2 - q^2, \qquad y = 2pq, \qquad z = p^2 + q^2,$$

$x, y, z \in \mathbb{Z}$ satisfy (2). They are coprime precisely when p and q are coprime and not both odd. When p and q are coprime and both odd, then

$$x = \frac{p^2-q^2}{2}, \qquad y = pq, \qquad \frac{p^2+q^2}{2}$$

satisfy (2). Conversely, this is the general form of solutions in (2). Indeed, given $x, y, z \in \mathbb{Z}$ coprime that satisfy (2), $z \neq 0$ and

$$\frac{x^2}{z^2} + \frac{y^2}{z^2} = 1,$$

so that $(x/z, y/z) \in \mathbb{C}$ and if $(x, y, z) \neq (-1, 0, 1)$, there is $t \in \mathbb{R}$ such that

$$\left(\frac{x}{z}, \frac{y}{z}\right) = (x(t), y(t)).$$

But then since $x/z, y/z \in \mathbb{Q}$, we can take $t \in \mathbb{Q}$ and x, y, z have the form above.

Lecture 2 Thursday 11/10/18

Definition 1.11. Let $f \in \mathbb{R}[x, y]$ and let $C = \{f = 0\}$. A **rational point** of C is a point $(x, y) \in C$, that is $f(x, y) = 0$, such that $x, y \in \mathbb{Q}$.

Example 1.12. There are infinitely many rational points on the circle

$$\left\{x^2 + y^2 = 1\right\} \subseteq \mathbb{R}^2,$$

which can be described explicitly, and can be used to solve

$$a^2 + b^2 = c^2, \qquad a, b, c \in \mathbb{Z},$$

a problem in number theory. Take $n \geq 3$ and consider

$$C = \{x^n + y^n - 1 = 0\}.$$

What are the rational points of C? Write

$$x = \frac{a}{c}, \qquad y = \frac{b}{c}, \qquad a, b, c \in \mathbb{Z}, \qquad c \neq 0.$$

Then

$$(x, y) \in C \qquad \Longleftrightarrow \qquad a^n + b^n = c^n.$$

Fermat's last theorem by Wiles then states that there exists no solution with $a, b \neq 0$.

2 Complex plane curves

Let $P \in \mathbb{R}[x, y]$ be a polynomial with coefficients in $\mathbb{R}$. A priori, it is natural to study the real plane curve

$$C_{\mathbb{R}} = \left\{ (x, y) \in \mathbb{R}^2 \mid P(x, y) = 0 \right\}.$$

However, P can also been seen as a polynomial with coefficients in $\mathbb{C}$, and it will often be simpler to study the complex plane curve

$$C_{\mathbb{C}} = \left\{ (x, y) \in \mathbb{C}^2 \mid P(x, y) = 0 \right\}.$$

We first explain some of the properties of algebraic curves that we would like to hold and explain why these properties do not necessarily hold for real plane curves, and some unpleasant things happen.

Fact. Many real curves are so degenerate that they do not even have points, that is $C_{\mathbb{R}} = \emptyset$. If $C_{\mathbb{R}} \neq \emptyset$, the dimension of $C_{\mathbb{R}}$, that is whether it is a union of points or a genuine curve, is difficult to determine.

Example 2.1. Let $t \in \mathbb{R}$ and consider

$$f_t(x, y) = x^2 + y^2 - t,$$

and the real plane curve

$$C_t = \{ f_t(x, y) = 0 \} \subseteq \mathbb{R}^2.$$

- If $t > 0$, C_t is a circle with radius $\sqrt{t}$.
- If $t = 0$, $C_0 = \{(0, 0)\}$.
- If $t < 0$, $C_t = \emptyset$.

Fact. In general, it is not clear when two polynomials $f, g \in \mathbb{R}[x, y]$ define the same real plane curve, that is when

$$\left\{ (x, y) \in \mathbb{R}^2 \mid f(x, y) = 0 \right\} = \left\{ (x, y) \in \mathbb{R}^2 \mid g(x, y) = 0 \right\}.$$

Example 2.2. Let f and g denote the polynomials

$$f(x, y) = x^2 y + y^2 + x^3 + x, \qquad g(x, y) = x^2 + 2xy + y^2.$$

Then, since $f(x, y) = (x + y) \cdot \left(x^2 + 1\right)$ and $g(x, y) = (x + y)^2$,

$$\left\{ (x, y) \in \mathbb{R}^2 \mid f(x, y) = 0 \right\} = \left\{ (x, y) \in \mathbb{R}^2 \mid g(x, y) = 0 \right\}.$$

Fact. In general, it is hard to predict when a curve intersects a fixed line, or more generally when two real curves intersect.

Example 2.3. In the notation of Example 2.1, let

$$C = C_1 = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 1 \right\} \subseteq \mathbb{R}^2$$

be the unit circle. Consider the line

$$L = \{ ax + by + c = 0 \}, \qquad (a, b) \neq (0, 0).$$

Then, depending on $(a, b, c) \in \mathbb{R}^3$, $L \cap C$ consists of two points, one point, or is empty.

Most of these difficulties disappear when working with curves $C_{\mathbb{C}} \subseteq \mathbb{C}^2$, essentially because $\mathbb{C}$ is algebraically closed, in other words the following theorem holds.

Theorem 2.4 (Fundamental theorem of algebra)**.** *Let $P \in \mathbb{C}[x]$ be a non-constant polynomial. Then P has at least one complex root, that is there exists $\alpha \in \mathbb{C}$ such that $P(\alpha) = 0$.*

This, first of all, has the following consequence.

Proposition 2.5. *Let $P \in \mathbb{C}[x,y]$ be a non-constant polynomial. Then the algebraic curve*

$$C = \left\{(x,y) \in \mathbb{C}^2 \mid P(x,y) = 0\right\}$$

contains infinitely many points.

Proof. Because P is not constant, one of x and y will show up in a monomial of P. Assume that it is x. Otherwise, the same argument swapping x and y. By grouping together monomials with the same degree in x, we can write $P(x,y)$ as

$$P(x,y) = f_0(y) + \dots + x^d \cdot f_d(y), \qquad d \geq 1,$$

thanks to the assumption above, and $f_0(y), \dots, f_d(y)$ are polynomials in y alone, with $f_d(y) \neq 0$. Now note that since $P(x,y) \neq 0$, there will be infinitely many values $y_0 \in \mathbb{C}$ of y for which $P(x,y_0) \neq 0$. Indeed, the equation $P(x,y) = 0$ has finitely many solutions, and $\mathbb{C}$ has infinitely many elements. For any one of those values $y_0 \in C$, the polynomial of x alone given by $P(x,y_0) \in \mathbb{C}[x]$ is non-constant and has at least a root, by the fundamental theorem of algebra. Call this root $x_0 = \alpha(y_0) \in \mathbb{C}$. Then the points (x_0,y_0) are infinitely many points, all belonging to the curve $C = \{P = 0\}$. They are all distinct, because the second coordinate is always different. ☐

Example 2.6. Let $a, b, c \in \mathbb{C}$ with $(a,b) \neq (0,0)$, and let

$$f(x,y) = ax + by + c.$$

If $a \neq 0$, for each $y \in \mathbb{C}$, there is precisely one solution of $f(x,y) = 0$, namely

$$x = -\frac{b}{a}y - \frac{c}{a}.$$

Thus there is a one-to-one correspondence

$$C = \{f = 0\} \subseteq \mathbb{C}^2 \qquad \leftrightsquigarrow \qquad \mathbb{C},$$

that is a plane when seen as an $\mathbb{R}$-vector space $\mathbb{C} \cong \mathbb{R}^2$. We will call C a **complex line**.

Remark 2.7. It is difficult to draw complex curves. Our intuition is for real vector spaces, and this makes complex curves hard to visualise. They are objects of real dimension two in $\mathbb{C}^2 \cong \mathbb{R}^4$, a four-dimensional real vector space.

Example 2.8. Let

$$f(x,y) = x^2 + y^2 = (x + iy) \cdot (x - iy).$$

Then, as in Lemma 1.7, $C = \{f = 0\} \subseteq \mathbb{C}^2$ is the union of the two complex lines

$$\{x + iy = 0\}, \qquad \{x - iy = 0\}.$$

When seen as $\mathbb{R}$-vector spaces, these two planes meet at exactly one point corresponding to $(0,0) \in \mathbb{R}^2 \subseteq \mathbb{C}^2$, the only real point of C. It is difficult to imagine two planes meeting in one point, because our intuition relies on the three-dimensional space $\mathbb{R}^3$, while $\mathbb{C}^2 \cong \mathbb{R}^4$.

Describing intersections is also easier.

Example 2.9. Consider

$$C = \left\{x^2 + y^2 - 1 = 0\right\} \subseteq \mathbb{C}^2, \qquad L = \{ax + by + c = 0\} \subseteq \mathbb{C}^2.$$

If $b \neq 0$, we determine the intersection $C \cap L$ by solving the linear system

$$\begin{cases} x^2 + y^2 = 1, \\ y = -\frac{a}{b}x - \frac{c}{b} \end{cases}.$$

Unless $a^2 = -b^2$ and $c = 0$, there are one or two solutions. Again, it is hard to imagine a two-dimensional real surface which meets a real plane in two points.

We now turn to the question of recognising when two polynomials define the same plane curve. Here again, working in $\mathbb{C}$ is a simplification.

Theorem 2.10 (Consequence of Hilbert's Nullstellensatz)**.** *Let $f, g \in \mathbb{C}[x,y]$ be two polynomials. Then*

$$\left\{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\right\} = \left\{(x,y) \in \mathbb{C}^2 \mid g(x,y) = 0\right\}$$

if and only if there exist

$$P_1, \dots, P_k \in \mathbb{C}[x,y], \qquad a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{Z}_{>0}, \qquad \lambda_1, \lambda_2 \in \mathbb{C}^*,$$

such that

$$\begin{cases} f(x,y) = \lambda_1 P_1^{a_1} \dots P_k^{a_k} \\ g(x,y) = \lambda_2 P_1^{b_1} \dots P_k^{b_k} \end{cases}. \tag{3}$$

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Proof. Assume that (3) holds. Then by the proof of Lemma 1.7,

$$\{f = 0\} = \{P_1^{a_1} = 0\} \cup \dots \cup \{P_k^{a_k} = 0\} = \{P_1 = 0\} \cup \dots \cup \{P_k = 0\},$$

because if $\alpha \in \mathbb{C}$ is such that $\alpha^n = 0$, then $\alpha = 0$. The same holds for $\{g = 0\}$. Therefore $\{f = 0\} = \{g = 0\}$. The second half of the proof needs tools of commutative algebra, and is omitted. ☐

Thus, the relation between the geometric shape $C = \{f = 0\}$ in $\mathbb{C}^2$ and the polynomial $f \in \mathbb{C}[x,y]$ is more transparent than in $\mathbb{R}^2$.

Remark 2.11. In fact, the statement of Theorem 2.10 is not true over $\mathbb{R}$, as Example 2.2 shows. Even better, we can just take

$$f(x,y) = x^2 + 1, \qquad g(x,y) = 1$$

as polynomials in $\mathbb{R}[x,y]$. Their zero locus is

$$\{f = 0\} = \{g = 0\} = \emptyset$$

in both cases, but f and g cannot be written in the form guaranteed by (3).

We will always work in $\mathbb{C}$. Let us introduce some important notions for the study of polynomials.

Definition 2.12. A polynomial $f \in K[x,y]$ has **no repeated factors** over K if it cannot be written as a product of the form

$$f(x,y) = g(x,y)^2 \cdot h(x,y), \qquad g, h \in K[x,y]$$

where g is non-constant. Equivalently,

$$f = P_1 \cdot \dots \cdot P_k,$$

where $P_1, \dots, P_k$ are distinct irreducible polynomials.

Exercise 4. Prove the equivalence of the two different definitions.

Corollary 2.13. *Let $f, g \in \mathbb{C}[x,y]$ be polynomials with non-repeated factors. Then f and g define the same complex plane curve $\{f = 0\} = \{g = 0\}$ if and only if there is a non-zero constant $\lambda \in \mathbb{C}^*$ such that $f = \lambda g$.*

Proof. Easy consequence of Theorem 2.10. ☐

Remark 2.14. Note that we do not lose anything by only working with polynomials with no repeated factors. Indeed, if

$$f = P_1^{a_1} \dots P_k^{a_k}, \qquad a_i \in \mathbb{N}$$

is a factorisation of f in distinct irreducible polynomials P_i for all i, and we set

$$g = P_1 \cdot \dots \cdot P_k,$$

then we have $\{f = 0\} = \{g = 0\}$, and g has no repeated factors.

Let $C \subseteq \mathbb{C}^2$ be a complex plane curve. We have proved that, up to multiplication by $\lambda \in \mathbb{C}^*$, there is a unique non-constant polynomial $f \in \mathbb{C}[x, y]$ with no repeated factors such that $C = \{f = 0\}$. It makes sense to define the following.

Definition 2.15. The **degree** of an affine curve $C \subseteq \mathbb{C}^2$ is the degree of any polynomial with no repeated factors f which defines C, such that $C = \{f = 0\}$, that is $\deg(C) = \deg(f)$.

Example 2.16.

- A complex line has always degree one, since they are defined by a linear polynomial.
- A conic, a curve defined by a polynomial $f(x, y)$ of degree two, has degree two, unless it is a double line, that is
$$f(x, y) = L(x, y)^2,$$
for some linear polynomial $L(x, y)$. In that case, it has degree equal to $\deg(L) = 1$.
- If $P \in \mathbb{C}[x, y]$ is an irreducible polynomial of degree two and $L \in \mathbb{C}[x, y]$ is a polynomial of degree one, then the curve $\{P \cdot L = 0\}$ has degree three. For example,
$$\left\{x^2 y + y^2 + x + 1 = 0\right\}$$
has degree three, assuming it has no repeated factors.

Unless mentioned otherwise, in the first few weeks, we will assume that polynomials have no repeated factors.

Definition 2.17. Let $f_1, f_2 \in \mathbb{C}[x, y]$ be polynomials with no repeated factors and let
$$C_1 = \{f = 0\}, \qquad C_2 = \{g = 0\}$$
be the associated complex curves. The curves C_1 and C_2 have **no common component** if there is no non-constant polynomial P that divides both f and g.

We can read off whether C_1 and C_2 have common components from the factorisation of f_1 and f_2 in irreducible polynomials. If
$$f = P_1^{a_1} \dots P_k^{a_k}, \qquad g = Q_1^{b_1} \dots Q_k^{b_k},$$
where all P_i and Q_i are irreducible, and $P_i \neq P_j$ and $Q_i \neq Q_j$ for $i \neq j$, then C_1 and C_2 have no common component if and only if $\lambda P_i \neq Q_j$ for all i and j and $\lambda \in \mathbb{C}^*$.

Remark 2.18. The terminology comes from the fact that if $f = P_1 \dots P_k$ as above, the algebraic curve $\{P_i = 0\}$ is said to be **irreducible**, and it is called an **irreducible component** of the algebraic curve $\{f = 0\}$, which is the union of all its components.

Exercise 5. Show that if C_1 and C_2 have no common component, then
$$\deg(C_1 \cup C_2) = \deg(C_1) + \deg(C_2).$$

Exercise 6. Let L and L' be the lines
$$L = \{ax + by + c = 0\} \subseteq \mathbb{C}^2, \qquad L' = \{a'x + b'y + c' = 0\} \subseteq \mathbb{C}^2.$$

- Show that L and L' meet at exactly one point if and only if $ab' - a'b \neq 0$.
- Show that $L = L'$ if and only if there exists $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$ and
$$a' = \lambda a, \qquad b' = \lambda b, \qquad c' = \lambda c.$$

Remark 2.19 (First aid topology).

- A **topological space** X is a set, equipped with a collection of **open subsets** $\{U_i \subseteq X\}$, such that
  - $\emptyset$ and X are open,
  - any union
  $$\bigcup_{i \in I} U_i$$
  of open sets U_i is open, and
  - any finite intersection
  $$\bigcap_{i=1}^{k} U_i$$
  of open sets U_i is open.
- A **metric space** X, such as $(\mathbb{C}^n, \|.\|)$, is a topological space. The open sets are given by arbitrary unions and finite intersections of the familiar **open balls**
$$B(x, \epsilon) = \{z \in X \mid \|z - x\|\} < \epsilon.$$
- A subset $X \subseteq Y$ of a topological space Y inherits a topology from Y. The open sets of X are the sets $X \cap U$, where $U \subseteq Y$ is an open set of Y.
- X is **compact** if for all open covering
$$X = \bigcup_{i \in I} U_i,$$
where U_i are open, there exists a finite subcovering
$$X = \bigcup_{i_1, \dots, i_k} U_{i_j}, \qquad \{i_1, \dots, i_k\} \subseteq I.$$
- The **Heine-Borel theorem** states that a subset X of $\mathbb{R}^n$ or of $\mathbb{C}^m$ is compact if and only if X is **closed**, that is its complement is open, and **bounded**, for the usual norm.
- A closed subset of a compact space is compact.
- A map $f : X \to Y$ between topological spaces is **continuous** if and only if $f^{-1}(U)$ is open, in X, whenever $U \subseteq Y$ is open. It follows that $f^{-1}(F)$ is closed whenever $F \subseteq Y$ is closed. In particular, if $f \in \mathbb{C}[x_1, \dots, x_n]$ is a polynomial, f defines a map $f : \mathbb{C}^n \to \mathbb{C}$ that is continuous, and
$$f^{-1}(\{0\}) = \{f = 0\} \subseteq \mathbb{C}^n$$
is closed because $\{0\}$ is a closed subset of $\mathbb{C}$.

In particular, $\mathbb{C}^2$ is a topological space with the Euclidean distance in $\mathbb{R}^4$, and if
$$C = \{f = 0\} \subseteq \mathbb{C}^2$$
is an affine plane curve, then C is a topological space that inherits a topology as a subset of $\mathbb{C}^2 \cong \mathbb{R}^4$. The open sets of C are $U \cap C$ where $U \subseteq \mathbb{C}^2$ is open. So algebraic curves have a natural topology.

Lemma 2.20. *Let $C \subseteq \mathbb{C}^2$ be an affine plane curve, then C is not compact.*

Proof. Since f is a continuous function $\mathbb{C}^2 \to \mathbb{C}$, $C = \{f = 0\} = f^{-1}(\{0\})$, and $\{0\}$ is closed in $\mathbb{C}$, C is closed in $\mathbb{C}^2$. We show that $C \subseteq \mathbb{C}^2$ is not bounded. Assume that it is, then there is a constant $M > 0$ such that $C \subseteq B(0, M)$, where the open ball is
$$B(0, M) = \left\{(x, y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 < M\right\}.$$
Want to show that some points in $\mathbb{C}$ are outside this open ball. Let $x_0 \in \mathbb{C}$ be such that $|x_0| > M$ and assume we can arrange for $g = f(x_0, y)$ to be a non-constant polynomial of y. (Exercise: what if $f(x, y)$ happens to be a polynomial of x alone, so that this cannot be arranged?) By the fundamental theorem of algebra, g has a root $y_0 \in \mathbb{C}$ and the point $(x_0, y_0) \in C$. This is a contradiction, as $(x_0, y_0) \notin B(0, M)$. $\square$

3 Projective space

Lecture 4 Monday 15/10/18

Recall that it is difficult to determine when two affine plane curves $C, C' \in \mathbb{C}^2$ intersect, and some curves do not in fact intersect, even over $\mathbb{C}$. We want to fix that, and the key is adding points at infinity.

Example 3.1.

- Consider two distinct lines
$$L_1 = \{ax + by + c = 0\}, \qquad L_2 = \{a'x + b'y + c' = 0\}.$$
Then L_1 and L_2 meet at exactly one point if and only if
$$\det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \neq 0.$$
But we can pretend that parallel lines meet at a point at infinity corresponding to the direction vector.
- Consider the asymptotic curve and line
$$C = \{xy - 1 = 0\}, \qquad L = \{x = 0\}.$$
Then C and L do not meet, but again we can pretend that they meet at a point at infinity.

Informally, a heuristic trick is to introduce a variable z.

1. Replace $(x, y) \mapsto (x/z, y/z)$.
2. Solve $z = 0$.

Example 3.2.

- Consider the lines
$$L_1 = \{ax + by + c = 0\}, \qquad L_2 = \{a'x + b'y + c' = 0\}.$$
Clearly L_1 and L_2 do not meet. Let us apply the trick. By 1 and 2 we get
$$\begin{cases} \dfrac{x}{z} + \dfrac{y}{z} + 1 = 0 \\ \dfrac{x}{z} + \dfrac{y}{z} - 1 = 0 \end{cases} \qquad \implies \qquad \begin{cases} x + y + z = 0 \\ x + y - z = 0 \end{cases} \qquad \implies \qquad \begin{cases} x + y = 0 \\ x + y = 0 \end{cases}.$$
We get that the point $(1, -1, 0)$ is a common solution. This will be called the point at infinity.
- Consider the asymptotic curve and line
$$C = \{xy - 1 = 0\}, \qquad L = \{x = 0\}.$$
Apply 1 and 2 to get
$$\begin{cases} xy - z^2 = 0 \\ \dfrac{x}{z} = 0 \end{cases} \qquad \implies \qquad \begin{cases} xy = 0 \\ x = 0 \end{cases}.$$
We get that $(0, 1, 0)$ is a common solution. Again, this will be called the point at infinity.

To make this formal, we introduce the projective plane $\mathbb{P}^2$. We will add points at infinity to $\mathbb{C}^2$, in such a way that asymptotic curves meet at infinity. We will then compactify an affine plane curve C so that the two compactifications are compatible, that is
$$\left(C \subseteq \mathbb{C}^2\right) \hookrightarrow \left(\overline{C} \subseteq \mathbb{P}^2\right).$$

Notation 3.3. Fix $n \geq 0$ and $\mathbb{C}^{n+1}$. Let $\underline{0} = (0, \ldots, 0) \in \mathbb{C}^{n+1}$ be the origin of the $(n+1)$-dimensional complex Euclidean space. We will denote

$$W = \mathbb{C}^{n+1} \setminus \{\underline{0}\},$$

that is a point $x \in W$ is given by $x = (x_0, \ldots, x_n)$ where $x_0, \ldots, x_n \in \mathbb{C}$ are not all zero. We define the equivalence relation on W, for any $x, y \in W$ by

$$x \sim y \qquad \iff \qquad \exists \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \ x = \lambda y.$$

Exercise 7. Show that $\sim$ is an equivalence relation on W.

Notation 3.4. Given $x \in W$, we denote

$$[x] = \{y \in W \mid x \sim y\}.$$

For simplicity, if $x = (x_0, \ldots, x_n)$ we will denote

$$[x] = [x_0, \ldots, x_n],$$

instead of $x = [(x_0, \ldots, x_n)]$.

Exercise 8. Show that $[x] = [y]$ if and only if $x \sim y$. Show that if $y \notin [x]$ then $[x] \cap [y] = \emptyset$.

Definition 3.5. The n**-dimensional complex projective space** $\mathbb{P}^n_{\mathbb{C}}$, or $\mathbb{P}^n(\mathbb{C})$, or simply $\mathbb{P}^n$, is defined as the quotient of W by $\sim$, that is

$$\mathbb{P}^n_{\mathbb{C}} = \frac{W}{\sim} = \left\{[x] \mid x \in W = \mathbb{C}^{n+1} \setminus \{\underline{0}\}\right\}.$$

The coordinates of $\mathbb{P}^n$ are $[x] \in \mathbb{P}^n$ except $[0, \ldots, 0]$, and

$$[\lambda x_0, \ldots, \lambda x_n] = [x_0, \ldots, x_n].$$

In other words, in $\mathbb{P}^n$, two points $[x_0, \ldots, x_n]$ and $[y_0, \ldots, y_n]$ are the same point if and only if there exists a non-zero constant λ such that

$$x_0 = \lambda y_0, \qquad \ldots, \qquad x_n = \lambda y_n.$$

Example 3.6. The point $[1, 2, i]$ is the same as the point $[i, 2i, -1]$.

Exercise 9. Show that there exists a bijection

$$\mathbb{P}^n \qquad \leftrightsquigarrow \qquad \left\{\text{one-dimensional subspaces of } \mathbb{C}^{n+1}\right\}.$$

In fact, if V is a finite-dimensional vector space over $\mathbb{C}$ without the choice of a basis, we can define the associated projective space $\mathbb{P}(V)$ as the set of one-dimensional linear subspaces of V.

Example 3.7. For any non-zero $x \in \mathbb{C}$ we have $[x] = [1]$. So

$$\mathbb{P}^0 = \frac{\mathbb{C}^1 \setminus \{0\}}{\sim} = \{[1]\}$$

is a point.

Notation 3.8. For any $i = 0, \ldots, n$, denote the **affine chart**

$$U_i = \{[x] = [x_1, \ldots, x_n] \in \mathbb{P}^n \mid x_i \neq 0\} \subseteq \mathbb{P}^n.$$

Lemma 3.9.

$$\mathbb{P}^n = U_0 \cup \cdots \cup U_n.$$

Proof. Take $[x] = [x_0, \ldots, x_n] \in \mathbb{P}^n$ then $x \in W$ and in particular $x = (x_0, \ldots, x_n)$ where at least one of the coefficients is non-zero, say $x_i \neq 0$. Then $[x] \in U_i$. Thus any $[x] \in \mathbb{P}^n$ is contained in the union of $U_0, \ldots, U_n$. $\square$

Lemma 3.10. *Pick $i = 0, \dots, n$. Define*

$$\begin{array}{rccc} \phi_i & : & \mathbb{C}^n & \longrightarrow & U_i \\ & & (y_1, \dots, y_n) & \longmapsto & [y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n] \end{array}.$$

Then ϕ_i is a bijection and its inverse is given by

$$\begin{array}{rccc} \psi_i & : & U_i & \longrightarrow & \mathbb{C}^n \\ & & [x_0, \dots, x_n] & \longmapsto & \left(\dfrac{x_0}{x_i}, \dots, \dfrac{x_{i-1}}{x_i}, \dfrac{x_{i+1}}{x_i}, \dots, \dfrac{x_n}{x_i}\right) \end{array}.$$

Proof. First note that both ϕ_i and ψ_i is well-defined, indeed, if $(y_1, \dots, y_n) \in \mathbb{C}^n$ then

$$(y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n) \in W$$

and therefore

$$[y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n] \in \mathbb{P}^n.$$

Similarly, if $[x_0, \dots, x_n] = [x_0', \dots, x_n']$ then it follows that

$$\psi_i [x_0, \dots, x_n] = \psi_i [x_0', \dots, x_n'].$$

Thus, it is enough to show that both $\phi_i \circ \psi_i$ and $\psi_i \circ \phi_i$ coincide with the identity. We have

$$\psi_i (\phi_i (y_1, \dots, y_n)) = \psi_i [y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n] = \left(\frac{y_1}{1}, \dots, \frac{y_i}{1}, \frac{y_{i+1}}{1}, \dots, \frac{y_n}{1}\right) = (y_1, \dots, y_n).$$

Similarly,

$$\phi_i (\psi_i [x_0, \dots, x_n]) = \phi_i \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right) = \left[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right] = [x_0, \dots, x_n].$$

Thus they are inverses. $\square$

Example. Let $n = 2$ and $[x_0, x_1, x_2] \in \mathbb{P}^2$. Then

$$\begin{array}{rccc} \phi_1 & : & \mathbb{C}^2 & \longrightarrow & U_1 \\ & & (0,0) & \longmapsto & [0,1,0] \end{array}, \qquad \begin{array}{rccc} \phi_2 & : & \mathbb{C}^2 & \longrightarrow & U_2 \\ & & (0,0) & \longmapsto & [0,0,1] \end{array}.$$

Lemma 3.9 and Lemma 3.10 can be used to define a topology on $\mathbb{P}^n$. Let $U \subseteq \mathbb{P}^n$, then U is open if and only if $\phi_i^{-1} (U \cap U_i) \subseteq \mathbb{C}^n$ is open, in $U_i \cong \mathbb{C}^n$, for any $i = 0, \dots, n$.

Exercise 10. Show that $U_i \subseteq \mathbb{P}^n$ is open in $\mathbb{P}^n$ for all $i = 0, \dots, n$.

Exercise 11. We can define another topology on $\mathbb{P}^n$ as the quotient topology induced by the map

$$\begin{array}{rccc} \pi & : & W & \longrightarrow & \mathbb{P}^n \\ & & (x_0, \dots, x_n) & \longmapsto & [x_0, \dots, x_n] \end{array},$$

so a subset $U \subseteq \mathbb{P}^n$ is open if and only if its preimage $\pi^{-1} (U) \subseteq W$ is open in W. Show that this indeed defines a topology, and that this topology coincides with the one defined above using the maps ϕ_i. Check that with this topology on $\mathbb{P}^n$, π is continuous.

Exercise 12. Prove that $\mathbb{P}^n$ is compact. A hint is to restrict the projection π of Exercise 11 to the $(n+1)$-dimensional sphere $S^{n+1} \subseteq W$, and check that this restriction is surjective and continuous. Since S^{n+1} is compact, it follows that $\mathbb{P}^n$ is compact as well.

Example 3.11.

- $\mathbb{P}^1 = \left\{[x_0, x_1] \mid (x_0, x_1) \in \mathbb{C}^2 \setminus \{\underline{0}\}\right\}$ is the union of two copies U_0 and U_1 of $\mathbb{C}^1$. The intersection of U_0 and U_1 is
$$U_0 \cap U_1 = \{[x_0, x_1] \mid x_0 \neq 0,\ x_1 \neq 0\},$$
which can be identified with $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ via the map
$$\begin{array}{ccc} U_0 \cap U_1 & \longrightarrow & \mathbb{C}^* \\ [x_0, x_1] & \longmapsto & \dfrac{x_1}{x_0} \end{array}.$$
Using this identification and the maps ψ_i, the inclusions $U_0 \cap U_1 \subseteq U_0$ and $U_0 \cap U_1 \subseteq U_1$ are the maps
$$\begin{array}{ccc} \mathbb{C}^* & \longrightarrow & \mathbb{C}^1 \\ z & \longmapsto & z \end{array}, \qquad \begin{array}{ccc} \mathbb{C}^* & \longrightarrow & \mathbb{C}^1 \\ z & \longmapsto & \dfrac{1}{z} \end{array}.$$
Then $\mathbb{P}^1$ is glued together from two copies of $\mathbb{C}$ along $U_0 \cap U_1$ by these inclusions, so $\mathbb{P}^1 = \mathbb{C}^1 \cup \{\infty\}$. Over the real numbers, $\mathbb{P}^1(\mathbb{R})$ is built up in the same way from two copies of $\mathbb{R}^1$, and can be identified with the circle S^1.
- $\mathbb{P}^2 = \left\{[x_0, x_1, x_2] \mid (x_0, x_1, x_2) \in \mathbb{C}^3 \setminus \{\underline{0}\}\right\}$ is the union of three copies of $U_0 \cong U_1 \cong U_2 \cong \mathbb{C}^2$, and the intersection can be described similarly. At infinity we will have a line.
- More generally, $\mathbb{P}^n$ can be described similarly.

In practice, we will view the affine complex plane $\mathbb{C}^2$ as being embedded in $\mathbb{P}^2$ as one of the open sets U_i.

Example. We identify $(x, y) \in \mathbb{C}^2$ and $[x, y, 1] \in \mathbb{P}^2$.

What is the complement of this embedding, that is the points at infinity?

Notation 3.12. For any $i = 0, \dots, n$, denote
$$\mathcal{P}_i = \{[x_0, \dots, x_n] \in \mathbb{P}^n \mid x_i = 0\} \subseteq \mathbb{P}^n.$$

Lemma 3.13. *For any $i = 0, \dots, n$, we have $\mathbb{P}^n = U_i \sqcup \mathcal{P}_i$. Moreover if we define*
$$\begin{array}{cccc} f_i & : & \mathbb{P}^{n-1} & \longrightarrow & \mathcal{P}_i \\ & & [z_0, \dots, z_{n-1}] & \longmapsto & [z_0, \dots, z_{i-1}, 0, z_i, \dots, z_{n-1}] \end{array},$$
then f_i is a bijection.

Proof. Exercise: both statements are easy to check. □

In conclusion, we have that
$$\mathbb{P}^n = U_i \sqcup \mathcal{P}_i, \qquad U_i \cong \mathbb{C}^n, \qquad \mathcal{P}_i \cong \mathbb{P}^{n-1},$$
so $\mathbb{P}^n \cong \mathbb{C}^n \sqcup \dots \sqcup \mathbb{C}^0$. $\mathcal{P}_i$ is called the **hyperplane at infinity**.

Example 3.14.

- We have already seen that $\mathbb{P}^0$ is a point.
- $\mathbb{P}^1 \cong \mathbb{C}^1 \cup \mathbb{P}^0$. In other words, $\mathbb{P}^1$ is obtained by adding a point at infinity to the complex line $\mathbb{C}$. It is a way to compactify the real plane. If the $\mathbb{C}^1$ above is U_0, this point at infinity is the origin of the other open subset $U_1 \cong \mathbb{C}^1$. One can show that there is an bijection, which is also a homeomorphism
$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & S^2 \subseteq \mathbb{R}^3 \\ r & \longmapsto & \dfrac{1}{r} \end{array}.$$
$\mathbb{P}^1$ is called the **projective line**, by problem sheet 1.
- $\mathbb{P}^2 \cong \mathbb{C}^2 \cup \mathbb{P}^1$. Thus $\mathbb{P}^2$ is obtained by adding a projective **line at infinity** to the complex plane $\mathbb{C}^2$. $\mathbb{P}^2$ is called the **projective plane**.

Note. The point at infinity or the line at infinity is not unique. It depends on the choice of the coordinate, and in our settings, it depends on i. In the future, we will often fix i, and this will give us a unique choice of the point or line at infinity.

4 Projective curves

Lecture 5
Thursday
18/10/18

Recall that an algebraic curve is given by

$$C = \{f(x,y) = 0\} \subseteq \mathbb{C}^2, \qquad f \in \mathbb{C}[x,y].$$

In this section, we want to define projective curves similarly in $\mathbb{P}^2$ and then compactify these affine curves. If we try to define a plane projective curve in the same way as an affine curve, that is as

$$\{f = 0\} \subseteq \mathbb{P}^2, \qquad f \in \mathbb{C}[x_0,x_1,x_2],$$

the first hurdle we encounter is that f does not define a function on $\mathbb{P}^2$.

Example 4.1.

- Let
$$f(x,y,z) = x^2 + y^2 - z^2.$$
Then $f(1,1,1) \neq f(2,2,2)$, so that f does not define a function on $\mathbb{P}^2$. It makes no sense to talk about $f(1,1,1)$, because in $\mathbb{P}^2$, $[1,1,1] = [2,2,2]$, so the value would not be well-defined. However, the locus where f vanishes in this case is well-defined. If $f(x,y,z) = 0$, then $f(\lambda x, \lambda y, \lambda z) = 0$ for all $\lambda \in \mathbb{C}^*$, so that the subset
$$\left\{[x,y,z] \in \mathbb{P}^2 \mid f(x,y,z) = 0\right\}$$
is well-defined.

- Let
$$g(x_0,x_1,x_2) = x_0^2 + x_1,$$
then $g(i,1,0) = 0$, but $g(2i,2,0) \neq 0$, so that in this case, the vanishing locus
$$\left\{[x_0,x_1,x_2] \in \mathbb{P}^2 \mid g(x_0,x_1,x_2) = 0\right\}$$
is not even well-defined.

The example shows that if we want to define projective curves as

$$\left\{[x_0,x_1,x_2] \in \mathbb{P}^2 \mid f(x_0,x_1,x_2) = 0\right\},$$

then $f \in \mathbb{C}[x_0,x_1,x_2]$ has to satisfy some additional properties.

Definition 4.2. A polynomial $f \in K[x_0,\dots,x_n]$ is **homogeneous** if all its monomials have the same degree $d \in \mathbb{N}$, that is

$$f(x_0,\dots,x_n) = \sum_{\alpha,\ |\alpha|=d} a_\alpha x^\alpha, \qquad \alpha \in \mathbb{N}^{n+1}.$$

Example 4.3.

- The polynomial
$$f(x,y,z) = x^2 + y^2 - z^2$$
is homogeneous of degree two.

- The polynomial
$$g(x_0,x_1,x_2) = x_0^2 + x_1$$
is not homogeneous.

- The polynomial
$$f(x,y) = Ax^3 + Bx^2y + Cxy^2 + Dy^3$$
is homogeneous of degree three, and all the homogeneous polynomials of degree three in x and y can be written in this form.

- The polynomial
$$g(x,y,z) = x^4 - 2x^2yz + yz^3$$
is homogeneous of degree four.

Lemma 4.4. *Let $P \in \mathbb{C}[x_0, \dots, x_n]$ be a polynomial. If P is homogeneous of degree d, then*

$$P(\lambda x_0, \dots, \lambda x_n) = \lambda^d P(x_0, \dots, x_n), \qquad \lambda \in \mathbb{C}, \qquad (x_0, \dots, x_n) \in \mathbb{C}^{n+1}.$$

Proof. Let P be a homogeneous polynomial of degree d. Then

$$P = M_1 + \dots + M_k,$$

where each M_i is a monomial of degree d. For each monomial

$$M = a_\alpha x^\alpha = a_\alpha x_0^{\alpha_0} \dots x_n^{\alpha_n}, \qquad \alpha \in \mathbb{N}^{n+1}, \qquad a_\alpha \in \mathbb{C}$$

of degree d, we have

$$M(\lambda x_0, \dots, \lambda x_n) = a_\alpha \left(\lambda^{\alpha_0} x_0^{\alpha_0}\right) \dots \left(\lambda^{\alpha_n} x_n^{\alpha_n}\right) = \lambda^{\sum_{i=0}^n \alpha_i} a_\alpha x^\alpha = \lambda^d a_\alpha x^\alpha = \lambda^d M(x_0, \dots, x_n),$$

because $\sum_{i=0}^n \alpha_i = |\alpha| = d$. For the arbitrary homogeneous polynomial P, write $P = \sum_{i=1}^k M_i$. Thus

$$P(\lambda x_0, \dots, \lambda x_n) = \sum_{i=1}^k M_i(\lambda x_0, \dots, \lambda x_n) = \lambda^d \sum_{i=1}^k M_i(x_0, \dots, x_n) = \lambda^d P(x_0, \dots, x_n).$$

$\square$

Exercise 13. Prove the converse implication of Lemma 4.4, that is if $P \in \mathbb{C}[x_0, \dots, x_n]$ and

$$P(\lambda x_0, \dots, \lambda x_n) = \lambda^d P(x_0, \dots, x_n), \qquad \lambda \in \mathbb{C}, \qquad (x_0, \dots, x_n) \in \mathbb{C}^{n+1},$$

then P is homogeneous of degree d.

Proposition 4.5. *Let P be a homogeneous polynomial. Then*

$$\{[x_0, \dots, x_n] \in \mathbb{P}^n \mid P(x_0, \dots, x_n) = 0\}$$

is well-defined.

Proof. We have to check that if $(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$, $P(x_0, \dots, x_n) = 0$ if and only if $P(y_0, \dots, y_n) = 0$. This follows immediately from Lemma 4.4, because by definition of $\sim$,

$$(y_0, \dots, y_n) = (\lambda x_0, \dots, \lambda x_n), \qquad \lambda \in \mathbb{C}^*,$$

so $P(x_0, \dots, x_n) = 0$ if and only if $P(\lambda x_0, \dots, \lambda x_n) = \lambda^d P(x_0, \dots, x_n) = 0$. $\square$

Notation 4.6. Unless mentioned otherwise, arbitrary polynomials will be denoted $f, g, h, \dots$, while homogeneous polynomials will be denoted $P, Q, R, \dots$.

Let $P_1, \dots, P_k \in \mathbb{C}[x_0, \dots, x_n]$ be homogeneous polynomials. Then the vanishing locus

$$\{P_1 = \dots = P_k = 0\} \subseteq \mathbb{P}^n$$

is a **projective variety**. These are the main object of study of algebraic geometry. In this class, we will focus on plane projective curves.

Definition 4.7. A **complex plane projective algebraic curve** is

$$C = \left\{[x_0, x_1, x_2] \in \mathbb{P}^2 \mid P(x_0, x_1, x_2) = 0\right\} \subseteq \mathbb{P}^2,$$

where P is a non-constant homogeneous polynomial $P \in \mathbb{C}[x_0, x_1, x_2]$.

Example 4.8. The hyperplanes at infinity

$$\mathcal{P}_i = \{x_i = 0\} \subseteq \mathbb{P}^2, \qquad i = 0, 1, 2$$

are projective curves, of degree one.

Definition 4.9. A **projective line** is a projective curve defined by a homogeneous polynomial $ax_0 + bx_1 + cx_2 = 0$ of degree one.

We have seen that in the case of affine plane curves $\{f = 0\} \subseteq \mathbb{C}^2$, the irreducible factors of the polynomial f were important when studying curves. The next lemma ensures that the same type of results hold for projective curves.

Lemma 4.10. *Let $P \in K[x_0, \dots, x_n]$ be a non-zero homogeneous polynomial. Assume that*

$$P = Q \cdot R, \qquad Q, R \in K[x_0, \dots, x_n].$$

Then the polynomials Q and R are homogeneous.

Proof. Exercise. □

Remark 4.11. As in the case of affine plane curves, we can therefore make sense of **irreducible** projective plane curves, $C = \{P = 0\} \subseteq \mathbb{P}^2$ with P irreducible. When P is reducible as

$$P = P_1^{a_1} \dots P_k^{a_k}, \qquad a_i \in \mathbb{N}^*,$$

with P_i distinct irreducible polynomials, the projective curves $C_i = \{P_i = 0\}$ are called the **irreducible components** of $C = \{P = 0\}$.

Remark 4.12. Hilbert's Nullstellensatz, and Theorem 2.10, still holds. In particular, if $P, Q \in \mathbb{C}[x, y, z]$ are homogeneous polynomials with no repeated factors,

$$\{P = 0\} = \{Q = 0\} \subseteq \mathbb{P}^2$$

if and only if $P = \lambda Q$ for $\lambda \in \mathbb{C}^*$.

Definition 4.13. Let $C \subseteq \mathbb{P}^2$ be a projective plane curve and P any homogeneous polynomial with no repeated factor such that $C = \{P = 0\} \subseteq \mathbb{P}^2$. The **degree** of C is the degree of P.

Exercise 14. Show that the union of two projective curves $C_1, C_2 \subseteq \mathbb{P}^2$ is a projective curve. More generally, show that if $X_1, X_2 \subseteq \mathbb{P}^2$ are projective varieties, then $X_1 \cup X_2 \subseteq \mathbb{P}^n$ is again a projective variety.

Exercise 15. Show that if $X = \{P = 0\} \subseteq \mathbb{P}^n$ for a non-zero homogeneous polynomial $P \in \mathbb{C}[x_0, \dots, x_n]$, then

$$X = \{x_0 P = \dots = x_n P = 0\}.$$

Lemma 4.14. *Let $C \subseteq \mathbb{P}^2$ be a projective curve. Then C is compact.*

Recall that affine curves are never compact.

Proof. Recall that a closed subset of a compact set is compact itself. Since $\mathbb{P}^2$ is compact, by Exercise 12, we only need to prove that C is closed, or equivalently, that $\pi^{-1}(C) \subseteq W = \mathbb{C}^3 \setminus \{\underline{0}\}$ is closed, where

$$\begin{array}{cccc} \pi & : & W & \longrightarrow & \frac{W}{\sim} = \mathbb{P}^2 \\ & & (x_0, x_1, x_2) & \longmapsto & [x_0, x_1, x_2] \end{array},$$

by Remark 2.19. Assume that $C = \{P = 0\}$ for a homogeneous polynomial P. Then, $P : W \to \mathbb{C}$ is a continuous map, so that

$$\pi^{-1}(C) = P^{-1}(\{0\}) \cap W = \{P(x_0, x_1, x_2) = 0\} \cap W$$

is a closed subset of W. □

Remark 4.15. You may have heard of the **Zariski topology** for algebraic varieties. The Zariski topology is defined on affine spaces $\mathbb{C}^n$ or on projective spaces $\mathbb{P}^n$ as the topology whose closed sets are of the form

$$V(S) = \{\underline{x} \in \mathbb{C}^n \mid f(\underline{x}) = 0\},$$

where S is a set of polynomials, or homogeneous polynomials, $f \in \mathbb{C}[x_0, \dots, x_n]$. In this course, we do not work in the Zariski topology, but in the classical topology, metric or Euclidean, on $\mathbb{C}^{n+1}$ and in the topology this induces on $\mathbb{P}^n$ as outlined above. One reason for that is that the Zariski topology is not Hausdorff, or separated.

Lecture 6 is a problem class.

Lecture 6 Friday 19/10/18

5 Affine vs projective plane curves

We now show how to projectivise affine curves, and study the relationship between affine and projective curves.

Notation 5.1. We will typically use coordinates x and y on $\mathbb{C}^2$ and x_0, x_1, x_2 on $\mathbb{P}^2$.

Let $C = \{f = 0\} \subseteq \mathbb{C}^2$ for $f \in \mathbb{C}[x, y] \setminus \{0\}$ be an affine curve, and want to construct a projective curve $\overline{C} \subseteq \mathbb{P}^2$, a projectivisation of C. Recall that we have identified $\mathbb{C}^2$ with the open subset $U_2 = \{x_2 \neq 0\} \subseteq \mathbb{P}^2$, via the inverse functions

$$\begin{array}{cccc} \phi_2 & : & U_2 & \longrightarrow & \mathbb{C}^2 \\ & & [x_0, x_1, x_2] & \longmapsto & \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) \end{array}, \qquad \begin{array}{cccc} \psi_2 & : & \mathbb{C}^2 & \longrightarrow & U_2 \\ & & (x, y) & \longmapsto & [x, y, 1] \end{array}.$$

We now want to identify $U_2 \cap \overline{C} \subseteq \mathbb{C}^2$ with the original curve C for a suitable projective plane curve $\overline{C} = \{P = 0\} \subseteq \mathbb{P}^2$ for P homogeneous, picking $\mathbb{C}^2 \cong U_2 \subseteq \mathbb{P}^2$ to be the points not at infinity.

Example. The idea is to let $x = x_0/x_2$ and $y = x_1/x_2$. For example,

$$f(x, y) = x^3 + y + 1 = \left(\frac{x_0}{x_2}\right)^3 + \left(\frac{x_1}{x_2}\right) + 1.$$

Clear denominators to get

$$P(x_0, x_1, x_2) = x_0^3 + x_1 x_2^2 + x_2^3,$$

homogeneous of degree three. Restricting P to $U_2 \cong \mathbb{C}^2$ I get back

$$P(x, y, 1) = x^3 + y + 1 = f(x, y).$$

Thus $\{P = 0\} = \overline{C}$ is the projectivisation of $\{f = 0\} = C$.

Theorem 5.2. *There is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{projective plane curves } \overline{C} = \{P = 0\} \subseteq \mathbb{P}^2 \text{ that} \\ \text{do not contain the line at infinity } \mathcal{P}_2 = \{x_2 = 0\} \end{array} \right\} \quad \leftrightsquigarrow \quad \left\{ \text{ affine curves } C = \{f = 0\} \subseteq \mathbb{C}^2 \right\}.$$

The bijection is obtained by

$$\begin{array}{ccc} C & \longleftrightarrow & \overline{C} \\ f & \longmapsto & \left[P : (x_0, x_1, x_2) \mapsto x_2^d \cdot f\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right)\right] \\ [f : (x, y) \mapsto P(x, y, 1)] & \longleftarrow & P \end{array},$$

where $d = \deg(f)$. The affine curve $\{f = 0\} \subseteq \mathbb{C}^2$ is $\phi_2\left(U_2 \cap \overline{C}\right)$, where $\overline{C} = \{P = 0\}$.

Notation 5.3. In general, P is called the **homogenisation** of the polynomial f, and $\overline{C}$ is the **projectivisation** of C.

Proof. Let $\overline{C} = \{P = 0\} \subseteq \mathbb{P}^2$ be a projective curve that does not contain $\{x_2 = 0\}$, as in the statement. Then P contains at least one monomial without x_2, so that $f : (x, y) \mapsto P(x, y, 1)$ is a polynomial in x and y. If not,

$$P = x_2 \cdot Q, \qquad \{P = 0\} = \{x_2 = 0\} \cup \{Q = 0\}.$$

So $f(x, y)$ has degree equal to $d = \deg(P)$. Under the identification $\phi_2 : U_2 \to \mathbb{C}^2$ defined in the previous section,

$$\phi_2\left(U_2 \cap \overline{C}\right) = \left\{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\right\}.$$

Conversely, if $C = \{f = 0\}$, then C is the image by ϕ_2 of the intersection of U_2 and $\overline{C}$, where $\overline{C} = \{P = 0\}$, where the polynomial P is defined by

$$P(x_0, x_1, x_2) = x_2^d \cdot f\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right).$$

Check that P is a well-defined homogeneous polynomial of degree d, and that the two constructions are the inverses of each other. (Exercise) $\square$

Example. Look at an example where $\overline{C}$ does contain $\{x_2 = 0\}$. Let

$$P(x_1, x_1, x_2) = x_0 x_2^2 + x_1^2 x_2 = x_2 \left(x_0 x_2 + x_1^2\right).$$

Then

$$f(x, y) = P(x, y, 1) = x + y^2,$$

so

$$P(x_0, x_1, x_2) = \left(\frac{x_0}{x_2} + \left(\frac{x_1}{x_2}\right)^2\right) x_2^2 = x_0 x_2 + x_1^2.$$

Remark. To intersect $\{P = 0\}$ and $\{Q = 0\}$ in $\mathbb{P}^2$, solve $P = 0$ and $Q = 0$ in x_0, x_1, x_2 to get homogeneous coordinates of points of intersection, where $\underline{0}$ is not a valid solution and $(\lambda x_0, \lambda x_1, \lambda x_2) = (x_0, x_1, x_2)$.

Example 5.4.

- Let $\overline{C}$ be the projective curve

$$\overline{C} = \left\{P(x_0, x_1, x_2) = x_0^2 + x_1^2 + x_2^2 = 0\right\}.$$

  Then $C = \phi_2\left(\overline{C} \cap U_2\right)$ is $\{f = 0\} \subseteq \mathbb{C}^2$, where

$$f(x, y) = x^2 + y^2 + 1.$$

- Let C be the affine curve

$$C = \left\{x^2 y + y - 1 = 0\right\}.$$

  Then $\overline{C}$ is defined by

$$x_2^3 \left(\left(\frac{x_0}{x_2}\right)^2 \left(\frac{x_1}{x_2}\right) + \left(\frac{x_1}{x_2}\right) - 1\right) = x_0^2 x_1 + x_1 x_2^2 - x_2^3.$$

Exercise 16. Let $i = 0$ or $i = 1$. Recall that ϕ_i denotes the homeomorphism $\phi_i : U_i \to \mathbb{C}^2$. Show that if $\overline{C} \subseteq \mathbb{P}^2$ is a projective curve that does not contain $\mathcal{P}_i = \{x_i = 0\}$, then $C_i = \phi_i\left(\overline{C} \cap U_i\right) \subseteq \mathbb{C}^2$ is an affine curve.

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Example 5.5. Intersect projectivisations to find the points of intersection at infinity.

- Consider the parallel lines

$$L_1 = \{x + y + 1 = 0\}, \qquad L_2 = \{x + y - 1 = 0\}.$$

  The corresponding projective lines are given by

$$\overline{L_1} = \{x_0 + x_1 + x_2 = 0\}, \qquad \overline{L_2} = \{x_0 + x_1 - x_2 = 0\},$$

  then solve to give

$$\begin{cases} x_0 + x_1 = -x_2 \\ x_0 + x_1 = x_2 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x_0 + x_1 = 0 \\ x_2 = 0 \end{cases}.$$

  Inside $\mathbb{P}^2$, the intersection $L_1 \cap L_2$ consists of exactly one point $p = [1, -1, 0]$. Thus, the two projective lines meet at one point $p \in \mathcal{P}_0$, that is p is a point at infinity.

- Similarly, let

$$C = \{xy - 1 = 0\}, \qquad L = \{x = 0\},$$

  then the projectivisations are

$$\overline{C} = \left\{x_0 \cdot x_1 - x_2^2 = 0\right\}, \qquad \overline{L} = \{x_0 = 0\},$$

  then intersect to give

$$\begin{cases} x_0 = 0 \\ x_0 x_1 - x_2^2 = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x_0 = 0 \\ x_2 = 0 \end{cases}.$$

  As above, in $\mathbb{P}^2$, $\overline{C} \cap \overline{L}$ consists of a single point $\{[0, 1, 0]\}$ lying on $\mathcal{P}_2 = \{x_2 = 0\}$, the hyperplane at infinity.

Example 5.6. Projective conics over $\mathbb{R}$ are defined by degree two equations. In $\mathbb{R}^2$, smooth conics are three kinds, ellipses, hyperbolas, and parabolas. Passing to $\mathbb{P}^2(\mathbb{R})$, these three kinds of curves become the same. Ellipses have no new points at infinity, hyperbolas have two new points at infinity, and parabolas have one new point at infinity.

- Consider the hyperbola C with equation
$$f(x,y) = x^2 - y^2 + 1.$$
Projectivising, we obtain the curve $\overline{C}$ with equation
$$P(x_0, x_1, x_2) = x_0^2 - x_1^2 + x_2^2.$$
Restrict to $U_1 = \{x_1 \neq 0\} \cong \mathbb{R}^2$. In $\overline{C} \cap U_1$, the equation of the unit circle
$$g(x,y) = x^2 - 1 + y^2 = x^2 + y^2 - 1$$
is obtained by setting $x_1 = 1$.

- Consider the parabola
$$C = \left\{y = x^2\right\} \subseteq \mathbb{R}^2$$
Projectivising, this becomes the curve
$$\overline{C} = \left\{x_1 x_2 = x_0^2\right\} \subseteq \mathbb{P}^2(\mathbb{R})$$
The intersection with the line at infinity $x_2 = 0$ gives $x_0^2 = 0$, so the only point is $[0,1,0]$. The square suggests some kind of tangency, and indeed $\overline{C}$ is tangent to $x_2 = 0$. In the chart $x_0 \neq 0$, the equation becomes
$$xy = 1,$$
and we see a hyperbola. In the chart $x_1 \neq 0$ we have again a parabola,
$$y = x^2.$$

- Consider the unit circle
$$C = \left\{x^2 + y^2 - 1 = 0\right\} \subseteq \mathbb{R}^2.$$
Projectivising we obtain
$$\overline{C} = \left\{x_0^2 + x_1^2 = x_2^2\right\} \subseteq \mathbb{P}^2(\mathbb{R}).$$
The intersection with the line at infinity $x_2 = 0$ is empty, as we could expect. In the chart $x_0 \neq 0$ we obtain the curve
$$1 + x^2 - y^2 = 0,$$
which is a hyperbola, and similarly in $x_1 \neq 0$.

Exercise 17. Let $a, b, c, d, e, f \in \mathbb{C}$, with $(a,b,c) \neq (0,0,0)$, and define
$$C = \left\{ax^2 + bxy + cy^2 + dx + ey + f = 0\right\}.$$
Define the projectivisation $\overline{C}$ of C and determine its points on the line at infinity.

Exercise 18.

- Show that there exists a unique projective line $L \subseteq \mathbb{P}^2$ through two distinct points $P, Q \in \mathbb{P}^2$.
- Show that two distinct projective lines in $\mathbb{P}^2$ meet in exactly one point.

Exercise 19. Let P_1, P_2, P_3 be three points of $\mathbb{P}^2$, and denote
$$P_i = [b_{i1}, b_{i2}, b_{i3}], \qquad i = 1, 2, 3,$$
their coordinates. Let $B = (b_{ij})$ be the associated 3×3 matrix. Show that P_1, P_2, P_3 lie in the same projective line if and only if $\det(B) = 0$.

6 Points at infinity

Given an affine curve $C \subseteq \mathbb{C}^2$ and its projectivisation $\overline{C} \subseteq \mathbb{P}^2$, we study the points at infinity of C, that is the points of $\overline{C}$ that do not lie on C. Let us first recall the notation in use. Denote by U_2 the open set of $\mathbb{P}^2$ defined by $\mathbb{C}^2 \cong U_2 = \{x_2 \neq 0\} \subseteq \mathbb{P}^2$, and let $\mathcal{P}_2 = \mathbb{P}^2 \setminus U_2 = \{x_2 = 0\}$ be the corresponding hyperplane or line at infinity. The homeomorphism is defined by inverses

$$\begin{array}{cccc} \phi & : & \mathbb{C}^2 & \longrightarrow & U_2 \\ & & (x, y) & \longmapsto & [x, y, 1] \end{array}, \qquad \begin{array}{cccc} \psi & : & U_2 & \longrightarrow & \mathbb{C}^2 \\ & & [x_0, x_1, x_2] & \longmapsto & \left[\dfrac{x_0}{x_2}, \dfrac{x_1}{x_2}\right] \end{array}.$$

Let C be the affine curve

$$C = \left\{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\right\},$$

where $f \in \mathbb{C}[x, y]$ is a polynomial of degree d with no repeated factors. The projectivisation of C is then the curve $\overline{C}$ such that $\overline{C} \cap U_2 = \phi(C)$, or equivalently such that $\psi\left(\overline{C} \cap U_2\right) = C$. The curve $\overline{C}$ is

$$\overline{C} = \left\{[x_0, x_1, x_2] \in \mathbb{P}^2 \mid P(x_0, x_1, x_2) = 0\right\}, \qquad P(x_0, x_1, x_2) = x_2^d f\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right).$$

where P is a homogeneous polynomial. Let $f_i \in \mathbb{C}[x, y]$ be the homogeneous polynomials of degree $0 \leq i \leq d$ with

$$f(x, y) = f_0(x, y) + \dots + f_d(x, y),$$

and note that

$$P(x_0, x_1, x_2) = f_0(x_0, x_1) x_2^d + \dots + f_d(x_0, x_1). \tag{4}$$

Definition 6.1. The **points at infinity** of C are the points on $\overline{C} \setminus \phi(C)$. We will often just write $\overline{C} \setminus C$.

How many points at infinity can there be? If $\{x_2 = 0\} \subseteq \overline{C}$, if and only if x_2 divides $P(x_0, x_1, x_2)$, then there are infinitely many. Assume $\{x_2 = 0\} \not\subseteq \overline{C}$. Look at $P(x_0, x_1, 0)$, and find solutions of $P(x_0, x_1, 0) = 0$ in $\mathbb{P}^1 \cong \mathcal{P}_2$. The points at infinity of C are thus

$$\overline{C} \setminus C = \overline{C} \cap \mathcal{P}_2 = \left\{[x_0, x_1, 0] \in \mathbb{P}^2 \mid P(x_0, x_1, 0) = 0\right\} = \left\{[x_0, x_1] \in \mathbb{P}^1 \mid f_d(x_0, x_1) = 0\right\},$$

from (4), and under the identification $\mathcal{P}_2 \cong \mathbb{P}^1$, that is a projective variety in $\mathcal{P}_2 \cong \mathbb{P}^1$. It is natural to expect that such a projective variety consists of a finite number of points. This is what we will prove next.

Lemma 6.2. *Let $Q \in \mathbb{C}[x_0, x_1]$ be a non-zero homogeneous polynomial of degree $d \geq 1$. Then there are $\alpha_i, \beta_i \in \mathbb{C}$ with $(\alpha_i, \beta_i) \neq (0, 0)$ for $i = 1, \dots, d$ such that*

$$Q(x_0, x_1) = \prod_{i=1}^{d} (\alpha_i x_0 + \beta_i x_1).$$

Therefore,

$$\left\{[x_0, x_1] \in \mathbb{P}^1 \mid Q(x_0, x_1) = 0\right\} = \{P_i = [-\beta_i, \alpha_i]\} \subseteq \mathbb{P}^1.$$

Note. The P_i need not be distinct.

Proof. Let us pass to $U_1 \cong \mathbb{C}^1 \subseteq \mathbb{P}^1$, one of the two affine charts. Write

$$Q(x_0, x_1) = \sum_{r=0}^{d} a_r x_0^r x_1^{d-r} = x_1^d \cdot \sum_{r=0}^{e} a_r \left(\frac{x_0}{x_1}\right)^r, \qquad e = \max\{r \mid a_r \neq 0\}.$$

Define $f \in \mathbb{C}[x]$ as $f(x) = \sum_{r=0}^{e} a_r x^r$. By the fundamental theorem of algebra, in Theorem 2.4, there are $\lambda_1, \dots, \lambda_e \in \mathbb{C}$ such that

$$f(x) = a_e \cdot \prod_{i=1}^{e} (x - \lambda_i),$$

where λ_i are the roots of f, and hence

$$Q(x_0, x_1) = x_1^d \cdot a_e \cdot \prod_{i=1}^{e} \left(\frac{x_0}{x_1} - \lambda_i \right) = a_e \cdot x_1^{d-e} \cdot \prod_{i=1}^{e} (x_0 - \lambda_i x_1).$$

This proves Lemma 6.2, if we set

$$(\alpha_i, \beta_i) = \begin{cases} (1, -\lambda_i) & i < e \\ (a_e, -a_e \lambda_e) & i = e \\ (0, 1) & i > e \end{cases}.$$

The description that $\{Q = 0\} \subseteq \mathbb{P}^1$ is clear, once we note that $(\alpha_i, \beta_i) \neq (0, 0)$. $\square$

We have proved the following.

Theorem 6.3. *Let $\overline{C} \subseteq \mathbb{P}^2$ be a projective curve defined by a polynomial of degree d that does not contain $\mathcal{P}_2$. Then $\overline{C} \cap \mathcal{P}_2$ is a non-empty set of at most $d = \deg(Q)$ points.*

Definition 6.4. Let $C = \{f = 0\} \subseteq \mathbb{C}^2$ for $f \in \mathbb{C}[x, y]$ be an affine curve of degree d and let $\overline{C}$ be its projectivisation, where, as usual, $C = \psi\left(\overline{C} \cap U_2\right)$. Denote

$$f = f_d + \dots + f_0, \qquad \deg(f_i) = i, \qquad f_d \not\equiv 0,$$

the decomposition of f into its homogeneous polynomial parts, so that

$$P(x_0, x_1, x_2) = \sum_{i=0}^{d} x_2^{d-i} f_i(x_0, x_1) = f_d(x_0, x_1) + \dots + f_0(x_0, x_1) x_2^d.$$

Then, if $(\alpha_i, \beta_i) \in \mathbb{C}^2 \setminus \{\underline{0}\}$ for $i = 1, \dots, d$ are such that

$$f_d(x_0, x_1) = \prod_{i=1}^{d} (\alpha_i x_0 + \beta_i x_1),$$

the affine lines

$$\{\alpha_i x + \beta_i y = 0\} \subseteq \mathbb{C}^2$$

are the **asymptotes** of C. The **points at infinity** at $x_2 = 0$ of C depend on the equation $f_d(x_0, x_1) = 0$, and are

$$\overline{C} \setminus C = \{[-\beta_i, \alpha_i]\} \subseteq \mathcal{P}_2.$$

Each asymptote meets $\overline{C}$ at the point $[-\beta_i, \alpha_i] \in \mathbb{P}^1$, the line at infinity. (Exercise: check this)

Remark 6.5. It is clear from Lemma 6.2 that asymptotes are well-defined. Note that the asymptotes are lines in $\mathbb{C}^2$ because $(\alpha_i, \beta_i) \neq (0, 0)$.

Example 6.6. Consider the affine curve

$$C = \left\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 - 1 = 0\right\}.$$

The projectivisation of C is

$$\overline{C} = \left\{[x_0, x_1, x_2] \in \mathbb{P}^2 \mid x_0^2 + x_1^2 - x_2^2 = 0\right\},$$

so that the points at infinity of C are given by

$$\overline{C} \setminus C = \left\{[x_0, x_1, 0] \in \mathbb{P}^2 \mid x_0^2 + x_1^2 = 0\right\}.$$

This shows that C has two asymptotes, namely $L_\pm = \{x \pm iy = 0\}$, and two points at infinity $\{[\pm i, 1, 0]\}$.

Exercise 20. Find the asymptotes of the affine curve given by the equation

$$x^2 y + y^3 + xy + x + 1 = 0.$$

Exercise 21. Write an example of an affine curve whose asymptotes include the lines

$$x + 2y = 0, \qquad 4x - 3y = 0, \qquad 7x + 9y = 0.$$

7 Smoothness and singularities

Lecture 8 Thursday 25/10/18

Smoothness is a very important notion in algebraic geometry, and in geometry in general. The idea is that an algebraic, affine or projective, curve $C \subseteq \mathbb{C}^2, \mathbb{P}^2$ will be called smooth at a point p if C does not have an angle, a corner, or a pinched point, etc. at p. We are going to formalise this by requiring that we can make sense of the tangent line of C at p, and this will involve derivatives of a defining equation, without repeated factors, of C. When C is smooth at a point p, topologically around p it will look like a two-dimensional small disc in the complex plane $\mathbb{C}$. In the following, we always assume that polynomials defining curves do not have repeated factors.

Notation 7.1. Given a function $f(x_1, \dots, x_n)$, we will denote by f_{x_i} or $\partial_{x_i} f$ the partial derivative of f with respect to x_i.

Let $C = \{f = 0\} \subseteq \mathbb{C}^2$ for $f \in \mathbb{C}[x, y]$ and let $(a, b) \in C$. Then the **tangent line** of C at the point $(a, b) \in C$ is the line defined by the equation

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) = 0.$$

Note. This equation defines a complex line if and only if either $f_x(a, b)$ or $f_y(a, b)$, or both, is not zero.

Definition 7.2. The affine curve C is **smooth** at a point $(a, b) \in C$ if at least one of $f_x(a, b)$ and $f_y(a, b)$ is not zero, so that the above equation defines a line in $\mathbb{C}^2$. On the other hand if

$$f_x(a, b) = f_y(a, b) = 0,$$

then (a, b) is called a **singular** point of C. We will simply say that C is **smooth** if C is smooth at every point.

Example 7.3.

- Any line $C \subseteq \mathbb{C}^2$ is smooth.
- The curve
$$\left\{x^2 + y^2 - 1 = 0\right\}$$
is smooth.
- The curve
$$\left\{y^2 - x^2(x + 1) = 0\right\}$$
is singular in one point $(0, 0)$. This kind of singularity is called a **node**.
- The curve
$$\left\{y^2 - x^3 = 0\right\}$$
is singular in one point $(0, 0)$. This kind of singularity is called a **cusp**.

How about projective curves? We give a similar definition, but this time we will have three variables and three partial derivatives.

Definition 7.4. If a point $p = [z_0, z_1, z_2]$ of a projective curve $C = \{P = 0\} \subseteq \mathbb{P}^2$ for $P \in \mathbb{C}[x_0, x_1, x_2]$ homogeneous satisfies

$$P_{x_i}(z_0, z_1, z_2) = 0, \qquad i = 0, 1, 2,$$

then we will say that C is **singular** at p. C is **smooth** at p if it is not singular. C is said to be **smooth** if it does not admit any singular point, so it is smooth at every point.

Lemma 7.5. *Let $C = \{P = 0\} \subseteq \mathbb{P}^2$ be a projective curve which does not contain the line $\{x_2 = 0\}$, and let $C_0 = \{f = 0\} \subseteq \mathbb{C}^2$ be the associated affine curve, where $f(x, y) = P(x, y, 1)$. Then (a, b) is a singular point of C_0 if and only if $[a, b, 1]$ is a singular point of C.*

For the proof we need the following.

Theorem 7.6 (Euler relation)**.** *Let $P \in \mathbb{C}[x_0, x_1, x_2]$ be a homogeneous polynomial of degree d. Then the following relation holds at each point $[x_0, x_1, x_2] \in \mathbb{P}^2$.*

$$x_0 \cdot P_{x_0} + x_1 \cdot P_{x_1} + x_2 \cdot P_{x_2} = d \cdot P.$$

Example. Let

$$P(x_0, x_1, x_2) = x_0^2 + x_1^2 + x_2^2.$$

Then $P_{x_0} = 2x_0$, $P_{x_1} = 2x_1$, and $P_{x_2} = 2x_2$. Thus

$$x_0 \cdot P_{x_0} + x_1 \cdot P_{x_1} + x_2 \cdot P_{x_2} = 2\left(x_0^2 + x_1^2 + x_2^2\right) = d \cdot P.$$

Proof. By Lemma 4.4, for any $\lambda \in \mathbb{C}$ we have

$$P(\lambda x_0, \lambda x_1, \lambda x_2) = \lambda^d \cdot P(x_0, x_1, x_2).$$

We now compute the derivative with respect to λ of both sides of this equation. By the equality above we have

$$\sum_{i=0}^{2} x_i P_{x_i}(\lambda x_0, \lambda x_1, \lambda x_2) = d\lambda^{d-1} P(x_0, x_1, x_2), \qquad \lambda \in \mathbb{C}.$$

Thus, if we plug in $\lambda = 1$, we get the claim. □

Proof of Lemma 7.5. Let $(a, b) \in C_0$ be a singular point of C_0. Note that $P(x, y, 1) = f(x, y)$ by construction, so $P_{x_0}(x, y, 1) = f_x(x, y)$ and $P_{x_1}(x, y, 1) = f_y(x, y)$. Then, since $f(a, b) = 0$ we have $P(a, b, 1) = 0$. Since $f_x(a, b) = f_y(a, b) = 0$, we have $P_{x_0}(a, b, 1) = P_{x_1}(a, b, 1) = 0$. Finally by Theorem 7.6, we have

$$P_{x_2}(a, b, 1) = x_0 \cdot P_{x_0}(a, b, 1) + x_1 \cdot P_{x_1}(a, b, 1) + x_2 \cdot P_{x_2}(a, b, 1) = dP(a, b, 1) = 0,$$

since everything is zero except $P_{x_2}(a, b, 1)$. Thus, we have

$$P_{x_i}(a, b, 1) = 0, \qquad i = 0, 1, 2.$$

The converse implication is proved similarly. □

Exercise 22. Show that

- $C = \left\{x_0^2 + x_1^2 + x_2^2 = 0\right\}$ is smooth,
- $C = \left\{x_1^2 x_0 - x_2^3 = 0\right\}$ is singular at the point $[1, 0, 0]$, and
- any projective line is smooth.

Definition 7.7. Let $p = [z_0, z_1, z_2]$ be a smooth point of the projective curve $C = \{P = 0\}$. The **projective tangent line** of the curve $C = \{P = 0\}$ at the point p is given by the equation

$$\sum_{i=0}^{2} P_{x_i}(z_1, z_2, z_3) \cdot x_i = 0. \tag{5}$$

Note. The line is well-defined since the curve C is smooth at the point p.

Proposition 7.8. *Let $C \subseteq \mathbb{P}^2$ be a projective algebraic curve not containing the line $\{x_2 = 0\}$. Then the affine line in U_2 associated to the projective tangent line at $[a, b, 1] \in C$ is the tangent line at a smooth point (a, b) of the affine curve C_0 associated to C in U_2.*

Equivalently, the projectivisation of the tangent line to $C_0 \subseteq \mathbb{C}^2$ at $(a, b) \in C_0$ is the projective tangent line to the projectivisation C of C_0 at $[a, b, 1]$.

Proof. Assume $C = \{P = 0\} \subseteq \mathbb{P}^2$ does not contain the line $\{x_2 = 0\}$. Let $f(x, y) = P(x, y, 1)$ and let $C_0 = \{f = 0\}$. The affine line associated to (5) at the point $(a, b) \in C_0$ is given by the equation

$$P_{x_0}(a, b, 1) \cdot x_0 + P_{x_1}(a, b, 1) \cdot x_1 + P_{x_2}(a, b, 1) \cdot x_2 = 0.$$

Since $P(a, b, 1) = 0$, by Theorem 7.6 we have

$$P_{x_2}(a, b, 1) = -a \cdot P_{x_0}(a, b, 1) - b \cdot P_{x_1}(a, b, 1).$$

Moreover, as above we have $f_x(a, b) = P_{x_0}(a, b, 1)$ and $f_y(a, b) = P_{x_1}(a, b, 1)$. Combining everything together we get

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) = 0,$$

which is the equation of the tangent line of C_0 at (a, b). ☐

Theorem 7.9. *Let C be a smooth projective curve. Then C is irreducible.*

Assume that any two projective curves in $\mathbb{P}^2$ intersect in at least one point, so there exists $p \in C_1 \cap C_2$. Later on we will prove this.

Proof. Suppose not. Then in particular,

$$\{P = 0\} = C = C_1 \cup C_2 = \{Q = 0\} \cup \{R = 0\} = \{Q \cdot R = 0\},$$

where $C_1 = \{Q = 0\}$ and $C_2 = \{R = 0\}$ are projective curves. We want to show that $A = [a, b, c] \in C_1 \cap C_2 \neq \emptyset$ is a singular point for C, which contradicts the assumption on C. We have

$$P_{x_i}(A) = (Q \cdot R)_{x_i}(A) = Q(A) \cdot R_{x_i}(A) + R(A) \cdot Q_{x_i}(A) = 0, \qquad i = 1, 2.$$

The last equality follows from the fact that $Q(A) = R(A) = 0$. Thus C is singular at A as claimed, a contradiction. ☐

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Example. An example of a reducible curve, where the two pieces have the same tangent line where they intersect. $C_1 = \{y - x^2 = 0\}$ and $C_2 = \{y + x^2 = 0\}$ has $(0, 0)$ as the only point of intersection. The tangent line at $(0, 0)$ to both C_1 and C_2 is $y = 0$. The definition implies that

$$C = C_1 \cup C_2 = \{(y - x^2)(y + x^2) = 0\} = \{y^2 - x^4 = 0\}$$

is not smooth at $(0, 0)$. Note that the tangent line at $(0, 0) \in C$ for some affine curve C is given by the linear part of $f = 0$ where $\{f = 0\} = C$. The equation

$$f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) = 0$$

is a linear approximation of C around $(0, 0)$. Now $y^2 - x^4 = 0$ does not have a linear part. The best possible approximation is a double line $y^2 = 0$, the term of lowest degree of f. Thus $(0, 0) \in \{y^2 - x^4 = 0\}$ should not be a smooth point.

Remark 7.10. The same reasoning shows that the curve $C = \{f = 0\}$ defined by a polynomial f with a repeated factor, say $f = g^2 \cdot h$ for g non-constant, is singular at all points of the curve $\{g = 0\}$. This is why it is important that we assume that polynomials defining curves have no repeated factors in this section.

Theorem 7.11. *Let C be an irreducible projective curve of degree d. Then the number of singular points of C is at most $d \cdot (d - 1)$. In particular it is finite.*

We will prove this later on, after seeing some results on intersections of curves.

Exercise 23. For which values of $\lambda \in \mathbb{C}$ does the algebraic curve

$$y^2 - x(x - 1)(x - \lambda) = 0$$

admit at least one singular point? For each of those values of λ, what are the singular points?

8 Projective transformations

How do we move around points, or change coordinates, in $\mathbb{P}^n$? In linear algebra, that is in $\mathbb{C}^n$, we do that with linear maps $L : \mathbb{C}^n \to \mathbb{C}^n$, that is functions such that

$$L(\lambda v + \mu w) = \lambda L(v) + \mu L(w), \qquad v, w \in \mathbb{C}^n, \qquad \lambda, \mu \in \mathbb{C}.$$

These correspond to $n \times n$ matrices $A = (a_{ij})$ with coefficients in $\mathbb{C}$. Given such a matrix, we can define a linear function by

$$\begin{array}{rccc} L & : & \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ & & (x_1, \dots, x_n) & \longmapsto & A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{ij} x_j \\ \vdots \\ \sum_{j=1}^n a_{nj} x_j \end{pmatrix} \end{array},$$

and a linear function L gives a matrix by setting a_{ij} as the i-th coefficient of the vector $L(e_j) \in \mathbb{C}^n$. How about in $\mathbb{P}^n = W/\sim$? Use matrices or linear transformations in $\mathbb{C}^{n+1}$.

Lemma 8.1. *Assume that A is an $(n+1) \times (n+1)$ matrix with coefficients in $\mathbb{C}$ such that $\det(A) \neq 0$, if and only if it is invertible. Then the function defined by*

$$\begin{array}{rccc} \Phi & : & \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ & & [x_0, \dots, x_n] & \longmapsto & \left[A \cdot \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \right] \end{array}$$

is well-defined, and a bijection.

Proof. Since $\det(A) \neq 0$, we know from linear algebra that if

$$A \cdot \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = \underline{0},$$

then

$$\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = \underline{0}.$$

If A is not invertible there is no associated Φ. Thus if $x \in W = \mathbb{C}^{n+1} \setminus \{\underline{0}\}$, then $A \cdot x \in W$. Moreover, if $x, y \in W$ are such that $[x] = [y]$, then there exists a non-zero $\lambda \in \mathbb{C}$ such that $x = \lambda y$ and $A \cdot x = \lambda A \cdot y$, which implies that $[A \cdot x] = [A \cdot y]$. It follows that Φ is well-defined. It is a bijection, with inverse given by the same kind of transformation obtained from the inverse matrix A^{-1}. $\square$

Definition 8.2. Any such function like Φ in Lemma 8.1 is called a **projective transformation**.

Remark 8.3. Note that an arbitrary $(n+1) \times (n+1)$ matrix A does not define a well-defined function as above, because there might be non-zero vectors $v \in \mathbb{C}^{n+1}$ such that $A \cdot v = \underline{0}$, and thus $[A \cdot v]$ would not define a point of projective space.

Example 8.4. You probably have encountered Möbius transformations in complex analysis. They are rational functions $f : \mathbb{P}^1 \to \mathbb{P}^1$ of one complex variable z

$$f : z \mapsto \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{C}, \qquad ad - bc \neq 0.$$

In terms of geometry, these are the projective transformations of $\mathbb{P}^1$ to itself. Indeed, if $[x_0, x_1]$ are the homogeneous coordinates on $\mathbb{P}^1$, a projective transformation Φ associated to a 2×2 invertible matrix T is given by

$$\begin{array}{cccc} \Phi & : & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\ & & [x_0, x_1] & \longmapsto & [A \cdot (x_0, x_1)] = [ax_0 + bx_1, cx_0 + dx_1] \end{array}, \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where T is invertible. If, as in Lemma 3.13, we write

$$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}, \qquad \{\infty\} = \{x_1 = 0\} = \{[1, 0]\},$$

then if $x_1 \neq 0$,

$$[x_0, x_1] = \left[\frac{x_0}{x_1}, 1\right] = [z, 1], \qquad z = \frac{x_0}{x_1},$$

and

$$\Phi [z, 1] = [ax_0 + bx_1, cx_0 + dx_1] = [az + b, cz + d],$$

and if $cz + d \neq 0$, that is $[z, 1] \neq [-d, c]$, this is

$$\Phi [z, 1] = \left[\frac{az+b}{cz+d}, 1\right] = [f(z), 1].$$

Here, unlike in complex analysis, the point $\{\infty\} = \{[1, 0]\}$ plays no special role, and we see right away that

$$\Phi [1, 0] = [a, c], \qquad \Phi [-d, c] = [1, 0].$$

Theorem 8.5. *Assume that P_1, P_2, P_3 are three non-collinear points in $\mathbb{P}^2$, that is not on the same line. Then there exists a projective transformation $\Phi : \mathbb{P}^2 \to \mathbb{P}^2$ such that*

$$\Phi (P_1) = [1, 0, 0], \qquad \Phi (P_2) = [0, 1, 0], \qquad \Phi (P_3) = [0, 0, 1].$$

Remark. Every projective transformation $\Phi = [A]$ is invertible. Just take $\Phi^{-1} = \left[A^{-1}\right]$.

Proof. Let A be the transpose of the matrix defined in Exercise 19, so

$$P_i = (a_{1i}, a_{2i}, a_{3i}) \in \mathbb{C}^3, \qquad A = (a_{ij}).$$

Then, since P_1, P_2, P_3 are not collinear, it follows that $\det(A) \neq 0$ and A is invertible. Let B be the inverse matrix of A and let $\Phi = [B] : \mathbb{P}^2 \to \mathbb{P}^2$ be the projective transformation associated to the matrix B. Then $BP_i = e_i$, so

$$\Phi (P_1) = [1, 0, 0], \qquad \Phi (P_2) = [0, 1, 0], \qquad \Phi (P_3) = [0, 0, 1],$$

as desired. ☐

Exercise 24. When do two linear transformations $L, L' : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ give rise to the same projective transformation $\mathbb{P}^n \to \mathbb{P}^n$?

Exercise 25. Show that if $P_1, P_2, P_3 \in \mathbb{P}^1_{\mathbb{C}}$ are distinct points, there exists a unique projective transformation $\Psi : \mathbb{P}_1 \to \mathbb{P}_1$ such that

$$\Psi (P_1) = [1, 0], \qquad \Psi (P_2) = [0, 1], \qquad \Psi (P_3) = [1, 1].$$

Exercise 26. Assume that $P_1, P_2, P_3, P_4 \in \mathbb{P}^2_{\mathbb{C}}$ are four points such that no three of them lie in the same line. Show that there exists a unique projective transformation $\Psi : \mathbb{P}^2 \to \mathbb{P}^2$ such that

$$\Psi (P_1) = [1, 0, 0], \qquad \Psi (P_2) = [0, 1, 0], \qquad \Psi (P_3) = [0, 0, 1], \qquad \Psi (P_4) = [1, 1, 1].$$

Proof of Exercise 26. Take a Φ as in Theorem 8.5. I can scale the representatives of the P_i, so $[v_i] = P_i$. Choose v_4 such that

$$P_4 = [v_4], \qquad v_4 = \sum_{i=1}^{3} \lambda_i v_i = \lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \lambda_3 \cdot v_3, \qquad \lambda_i \in \mathbb{C}, \qquad \lambda_i^*.$$

We are in $\mathbb{C}^3$, and the v_1, v_2, v_3 are linearly independent, so they are a basis. If $\lambda_i = 0$ for some i, the remaining vectors are in the same two-dimensional subspace of $\mathbb{C}$, contradicting non-collinearity in $\mathbb{P}^2$. Take $\lambda_i \cdot v_i$ as representatives for P_i for $i = 1, 2, 3$ and v_4 for P_4, so get a matrix A as before, so $A \cdot e_i = P_i$ and

$$A \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \sum_{i=1}^{3} \lambda_i v_i = v_4,$$

so $B = A^{-1}$, $\Psi = [B]$, and

$$\Psi(P_1) = [1, 0, 0], \qquad \Psi(P_2) = [0, 1, 0], \qquad \Psi(P_3) = [0, 0, 1], \qquad \Psi(P_4) = [1, 1, 1].$$

$\square$

There is a version of the previous statement in higher dimensions.

Definition 8.6. A **hyperplane** in $\mathbb{P}^n$ is the image of a subspace of dimension n in $\mathbb{C}^{n+1}$ via the map $\pi : W \to \mathbb{P}^n$. Equivalently, it is the locus of points $[x_0, \dots, x_n]$ of $\mathbb{P}^n$ satisfying some linear equation

$$\sum_{i=1}^{n} a_i x_i = 0, \qquad (a_0, \dots, a_n) \neq \underline{0}.$$

Example 8.7. Projective lines are precisely the hyperplanes of $\mathbb{P}^2$.

Exercise 27. Let $\mathcal{S} = \{p_0, \dots, p_n, q\} \subseteq \mathbb{P}^n$ be a collection of $n+2$ distinct points such that no $n+1$ points of $\mathcal{S}$ lie in a hyperplane. Denote by e_i, for $i = 0, \dots, n$, the vectors of the standard basis of $\mathbb{C}^{n+1}$. Then there is a unique projective transformation $f : \mathbb{P}^n \to \mathbb{P}^n$ such that

$$f(p_i) = [e_i] = [0, \dots, 0, 1, 0, \dots, 0],$$

for all i and $f(q) = [1, \dots, 1]$. Sometimes such a set of points is called a **projective basis**, in analogy with bases of vector spaces.

Exercise 28. Let L be a line in $\mathbb{P}^2$. Then given $i \in \{0, 1, 2\}$, there exists a projective transformation $f : \mathbb{P}^2 \to \mathbb{P}^2$ such that $f(L) = \mathcal{P}_i = \{x_i = 0\}$.

Exercise 29. Let $C = \{P = 0\} \subseteq \mathbb{P}^2$ be a projective curve and let $\Phi = [A] : \mathbb{P}^2 \to \mathbb{P}^2$ be a projective transformation, a 3×3 matrix A. Show that

- $\Phi(C)$ is again a projective curve of degree $\deg(\Phi(C)) = \deg(C)$, in particular C is a line if and only if $\Phi(C)$ is a line,
- $\Phi(C)$ is irreducible if and only if C is, and there is a natural bijection between the sets of components of the two curves,
- C is smooth if and only if $\Phi(C)$ is smooth,
- if $p \in C$ is a smooth point and l is the tangent line of C at p then $\Phi(C)$ is smooth at the point $\Phi(p)$ and $\Phi(l)$ is the tangent line of $\Phi(C)$ at $\Phi(p)$, and
- if $D \subseteq \mathbb{P}^2$ is another projective curve, then Φ induces a bijection between $C \cap D$ and $\Phi(C) \cap \Phi(D)$.

What is the equation of $\Phi(C)$?

Remark 8.8. As Exercise 29 suggests, one should think of curves only differing from each other via a projective transformation to be the same curve, since projective transformations preserve their properties. These transformations should be thought of as a change of a projective coordinate system.

9 Resultants and weak Bézout

Lecture 10 Monday 29/10/18

Now we start studying how projective curves intersect in the projective plane, heading towards Bézout's theorem. Let $C = \{P = 0\}$ and $D = \{Q = 0\}$ be two projective curves of degree n and m defined by homogeneous polynomials $P, Q \in \mathbb{C}[x_0, x_1, x_2]$. We want to study

$$C \cap D = \{P = 0\} \cap \{Q = 0\},$$

where P and Q have no repeated factors, that is points $[x_0, x_1, x_2]$ that are solutions up to scaling of the equations for C and D,

$$P(x_0, x_1, x_2) = 0, \qquad Q(x_0, x_1, x_2) = 0, \qquad (x_0, x_1, x_2) \neq (0, 0, 0).$$

Bézout's theorem states that $C \cap D$ has $n \cdot m$ points, unless C and D have a common component, and count points with multiplicity.

Example. If you do not use multiplicity, there might be less than $n \cdot m$ points.

- Tangencies. Let $C = \left\{y = x^2\right\}$ and $D = \{y = 0\}$. In $\mathbb{P}^2$, they intersect in $1 \neq (2)(1) = \deg(C)\deg(D)$ point in this case.
- Singularities of C and D. Let C be two lines and D be a line passing through the intersection of C. Then there is $1 \neq (2)(1) = \deg(C)\deg(D)$ point of intersection.

Think of perturbing the equations.

We first have to develop some algebraic tools, the resultant. We first take a step back to the case of polynomials in one variable and study solutions of $p(x) = q(x) = 0$ for polynomials $p, q \in \mathbb{C}[x]$. Let p and q be

$$p(x) = a_0 + \dots + a_n x^n = \sum_{i=0}^{n} a_i x^i, \qquad a_n \neq 0, \qquad a_i \in \mathbb{C}, \qquad \deg(p) = n,$$

$$q(x) = b_0 + \dots + b_m x^m = \sum_{i=0}^{m} b_i x^i, \qquad b_m \neq 0, \qquad b_i \in \mathbb{C}, \qquad \deg(q) = m.$$

When does $p(x) = q(x) = 0$ have solutions? Transform this question into linear algebra. The polynomials p and q have a common root, that is there is $\lambda \in \mathbb{C}$ such that $p(\lambda) = q(\lambda) = 0$, precisely if there is a non-constant polynomial $l \in \mathbb{C}[x]$ such that $p(x) = l(x) r(x)$ and $q(x) = l(x) s(x)$, where r and s are non-zero polynomials, which thus have $\deg(r) \leq n - 1$ and $\deg(s) \leq m - 1$.

Lemma 9.1. *The polynomials p and q have a common root if and only if there are non-zero polynomials r and s with $\deg(r) \leq n-1$ and $\deg(s) \leq m-1$ with*

$$p(x) \cdot s(x) - q(x) \cdot r(x) = 0, \tag{6}$$

the zero polynomial.

Proof. By taking r and s in the previous discussion, it is clear that if p and q have a common root, then (6) holds. Conversely assume that (6) holds. There is a unique decomposition

$$p(x) = c p_1(x)^{a_1} \dots p_k(x)^{a_k}, \qquad c \in \mathbb{C}^*, \qquad a_i \in \mathbb{N}^*,$$

where $p_1, \dots, p_k$ are monic non-constant irreducible polynomials. Since

$$p_1(x)^{a_1} \dots p_k(x)^{a_k} \mid q(x) r(x), \qquad \deg(r) \leq n - 1 < \sum_{i=1}^{k} a_i \deg(p_i),$$

at least one of the irreducible polynomials $p_i(x)$ divides $q(x)$, by irreducibility. It follows that p and q have a common factor and, by the fundamental theorem of algebra, that p and q have a common root. ∎

We now show that (6) is a system of $m+n$ equations in $m+n$ variables. Now write $r(x)$ and $s(x)$ in terms of unknown coefficients. Let $\underline{v} = (v_0, \dots, v_{n+m-1}) \in \mathbb{C}^{n+m}$ be defined by

$$s(x) = v_0 + \dots + v_{m-1}x^{m-1} = \sum_{i=0}^{m-1} v_i x^i, \qquad -r(x) = v_m + \dots + v_{n+m-1}x^{n-1} = \sum_{i=0}^{n-1} v_{m+i} x^i.$$

Substitute these into $p(x)s(x) - q(x)r(x) = 0$, and get a big system of equations. Then (6) is

$$(v_0a_0 + v_mb_0) + \dots + \left(\sum_{k=0}^{i} (v_k a_{i-k} + v_{m+k} b_{i-k})\right) x^i + \dots + (v_{m-1}a_n + v_{n+m-1}b_m) x^{n+m-1} = 0.$$

Since this is a polynomial of degree $n+m-1$, we thus have a system of $n+m$ equations in $n+m$ variables of the form

$$A \cdot v = 0, \qquad (7)$$

where A is an $(n+m) \times (n+m)$ matrix

$$A = \begin{pmatrix} a_0 & & 0 & b_0 & & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ a_n & & a_0 & b_m & & b_0 \\ & \ddots & \vdots & & \ddots & \vdots \\ 0 & & a_n & 0 & & b_m \end{pmatrix},$$

where there are m columns of a_i and n columns of b_i. From linear algebra, we know that (7) has a non-trivial solution if and only if $\det(A) = 0$. This motivates the following definition.

Definition 9.2. Let $p, q \in \mathbb{C}[x]$ be as above. Then the **resultant** of p and q is the determinant of the $(n+m) \times (n+m)$ matrix A^T,

$$R_{p,q} = \det\left(A^T\right) = \det \begin{pmatrix} a_0 & \dots & a_n & & 0 \\ & \ddots & & \ddots & \\ 0 & & a_0 & \dots & a_n \\ b_0 & \dots & b_m & & 0 \\ & \ddots & & \ddots & \\ 0 & & b_0 & \dots & b_m \end{pmatrix}.$$

Thus, Lemma 9.1 can be reformulated as the following.

Theorem 9.3. *Let $p, q \in \mathbb{C}[x]$ be two non-zero polynomials. Then, p and q have a common root if and only if $R_{p,q} = 0$.*

Example 9.4.

- Let
$$p(x) = x^2 - 1, \qquad q(x) = x^2 + 2x + 1,$$
so
$$a_0 = 1, \qquad a_1 = 0, \qquad a_2 = -1, \qquad b_0 = 1, \qquad b_1 = 2, \qquad b_2 = 1.$$
Then the resultant of p and q is the determinant of a 4×4 matrix,
$$R_{p,q} = \det \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$
(Exercise: check that the determinant is $R_{p,q} = 0$) Thus, p and q have a common solution, $x = -1$.

- Let $n = 1$ and $m = 4$, so
$$p(x) = a_0 + a_1 x, \qquad q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4.$$
Then A is a 5×5 matrix
$$A = \begin{pmatrix} a_0 & 0 & 0 & 0 & b_0 \\ a_1 & a_0 & 0 & 0 & b_1 \\ 0 & a_1 & a_0 & 0 & b_2 \\ 0 & 0 & a_1 & a_0 & b_3 \\ 0 & 0 & 0 & a_1 & b_4 \end{pmatrix}.$$
- Recall that $p \in \mathbb{C}[x]$ has a repeated root if and only if p and p' have a common root, that is by what precedes if and only if $R_{p,p'} = 0$. If
$$p(x) = ax^2 + bx + c, \qquad a \neq 0,$$
then
$$R_{p,p'} = \det \begin{pmatrix} c & b & a \\ b & 2a & 0 \\ 0 & b & 2a \end{pmatrix} = a\left(4ac - b^2\right).$$
We see that $-R_{p,p'} = a \cdot \Delta$, where Δ is the discriminant of the quadratic equation $p(x) = 0$.
- Let $n = 2$ and $m = 3$, so
$$p(x) = a_0 + a_1 x + a_2 x^2, \qquad q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3.$$
Then A is a 5×5 matrix
$$A = \begin{pmatrix} a_0 & 0 & 0 & b_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 \\ 0 & a_2 & a_1 & b_3 & b_2 \\ 0 & 0 & a_2 & 0 & b_3 \end{pmatrix}.$$

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In general, the resultant is useful because it tends to be easier to compute the determinant of a matrix than to factorise polynomials in order to determine whether they have a common root. Assume now that $P, Q \in \mathbb{C}[x_0, x_1, x_2]$ are homogeneous polynomials of degrees n and m respectively. We write P and Q as polynomials in x_0, with coefficients in $\mathbb{C}[x_1, x_2]$. Then
$$P(x_0, x_1, x_2) = \sum_{i=0}^{n} a_i(x_1, x_2) x_0^i, \qquad Q(x_0, x_1, x_2) = \sum_{j=0}^{m} b_j(x_1, x_2) x_0^j,$$
where $a_i \in \mathbb{C}[x_1, x_2]$ is a homogeneous polynomial of degree $n - i$, and $b_j \in \mathbb{C}[x_1, x_2]$ is homogeneous of degree $m - j$. In particular, $a_n = P(1, 0, 0)$ and $b_m = Q(1, 0, 0)$ are constants.

Definition 9.5. Assume that $P(1, 0, 0) \neq 0$ and $Q(1, 0, 0) \neq 0$. The **resultant** of P and Q is the determinant of an $(n + m) \times (n + m)$ matrix,
$$R_{P,Q}(x_1, x_2) = \det \begin{pmatrix} a_0(x_1, x_2) & \dots & a_n(x_1, x_2) & & 0 \\ & \ddots & & \ddots & \\ 0 & & a_0(x_1, x_2) & \dots & a_n(x_1, x_2) \\ b_0(x_1, x_2) & \dots & b_m(x_1, x_2) & & 0 \\ & \ddots & & \ddots & \\ 0 & & b_0(x_1, x_2) & \dots & b_m(x_1, x_2) \end{pmatrix}.$$
The entries of the matrix above are homogeneous polynomials in $\mathbb{C}[x_1, x_2]$. If you specify $x_1 = a$ and $x_2 = b$ for $a, b \in \mathbb{C}$, then $R_{P,Q}$ is the resultant of $P(x_0, a, b)$ and $Q(x_0, a, b)$. We admit the following theorem, which can be proven using tools from commutative algebra.

Theorem 9.6. *Let $P, Q \in \mathbb{C}[x_0, x_1, x_2]$ be homogeneous polynomials of degrees n and m, and assume that $a_n, b_m \neq 0$. Then P and Q have a common factor if and only if $R_{P,Q} \equiv 0$, that is if $R_{P,Q}(x_1, x_2) = 0$ for all $x_1, x_2 \in \mathbb{C}$.*

In fact, slightly more is true.

Theorem 9.7. *Let $P, Q \in \mathbb{C}[x_0, x_1, x_2]$ be homogeneous polynomials such that $P(1,0,0), Q(1,0,0) \neq 0$. Then $R_{P,Q} \in \mathbb{C}[x_1, x_2]$ is a homogeneous polynomial of degree nm.*

Proof. Let A be the $(n+m) \times (n+m)$ matrix above, that is such that $R_{P,Q} = \det(A)$. Recall from linear algebra that

$$\det(A) = \sum_{\sigma \in S_{n+m}} sign(\sigma) \prod_{k=1}^{n+m} A_{k,\sigma(k)}.$$

The entries of A are either zero or are homogeneous polynomials. We thus only need to check that for each $\sigma \in S_{n+m}$ such that $A_{k,\sigma(k)} \neq 0$ for all k, $\prod_{k=1}^{n+m} A_{k,\sigma(k)}$ is of degree nm. By definition of A, the non-zero entries of A are

$$A_{k,l} = \begin{cases} a_{l-k}(x_1, x_2) & k \leq m, \ 0 \leq l-k \leq n \\ b_{l-k+m}(x_1, x_2) & k \geq m+1, \ 0 \leq l-k+m \leq m \end{cases}.$$

Then

$$\deg(a_{l-k}) = n - (l-k), \qquad \deg(b_{l-k+m}) = m - (l-k+m) = k - l,$$

so that when non-zero, $\prod_{k=1}^{n+m} A_{k,\sigma(k)}$ has degree

$$\deg\left(\prod_{k=1}^{n+m} A_{k,\sigma(k)}\right) = \sum_{k=1}^{m} (n - (\sigma(k) - k)) + \sum_{k=m+1}^{m+n} (k - \sigma(k)) = nm + \sum_{k=1}^{m+n} k - \sum_{k=1}^{m+n} \sigma(k).$$

The last two terms are equal because σ is a permutation, so this is equal to nm for all permutations $\sigma \in S_{m+n}$, and this proves that $R_{P,Q}$ is a homogeneous polynomial of degree nm. ☐

Recall that homogeneous polynomials in two variables always factor as a product of linear homogeneous polynomials. We can now prove a weak form of Bézout's theorem. We will refine the statement later on.

Theorem 9.8 (Weak Bézout theorem)**.** *Let $C, D \subseteq \mathbb{P}^2$ be projective curves of degrees $\deg(C) = n$ and $\deg(D) = m$ respectively. Then*

1. *C and D intersect in at least one point, that is $C \cap D$ is not empty, and*
2. *C and D have at most $\#\{C \cap D\} \leq nm$ distinct points of intersection, unless they have a common component.*

Proof. Exercise: prove that given finitely many projective curves $C_1, \dots, C_k \subseteq \mathbb{P}^2$, one can always find a point $p \in \mathbb{P}^2$ in the complement $\mathbb{P}^2 \setminus \bigcup_{i=1}^{k} C_i$.

1. We first prove 1. Let $p \in \mathbb{P}^2$ be a point that lies neither on C nor on D, so not on $C \cup D$. There is a projective transformation

$$\begin{array}{cccc} \Phi & : & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\ & & p & \longmapsto & [1,0,0] \end{array}.$$

Transform C and D using Φ, so replace them with $\Phi(C)$ and $\Phi(D)$. Note that $\Phi(C)$ and $\Phi(D)$ are projective curves of degrees $\deg(\Phi(C)) = \deg(C)$ and $\deg(\Phi(D)) = \deg(D)$, and that $\#\{C \cap D\} = \#\{\Phi(C) \cap \Phi(D)\}$, by Exercise 29. Thus, after replacing C with $\Phi(C)$ and D with $\Phi(D)$ we may assume that $[1,0,0] \notin C \cup D$. Let $P, Q \in \mathbb{C}[x_0, x_1, x_2]$ be homogeneous polynomials of degrees n and m with no repeated factors that define C and D, that is $C = \{P = 0\}$ and $D = \{Q = 0\}$. So $P(1,0,0) \neq 0$ and $Q(1,0,0) \neq 0$ if and only if $[1,0,0] \notin C \cup D$. Using the projective transformation to ensure this, $P(1,0,0) \neq 0$ and $Q(1,0,0) \neq 0$. We assume that C and D have no common component, as the result is trivial if they do. Denote by R the resultant polynomial of P and Q, $R(x_1, x_2) = R_{P,Q}(x_1, x_2)$ for $x_1, x_2 \in \mathbb{C}$, and factor it. By Theorem 9.6 and Theorem 9.7, R is a homogeneous polynomial of degree nm that is not identically zero. By Lemma 6.2, there are $(\alpha_i, \beta_i) \in \mathbb{C}^2 \setminus \{(0,0)\}$ for $i = 1, \dots, nm$ such that

$$R(x_1, x_2) = \prod_{i=1}^{nm} (\alpha_i x_1 + \beta_i x_2).$$

Fix $(a, b) \in \mathbb{C}^2 \setminus \{(0,0)\}$, and let $f, g \in \mathbb{C}[x]$ be the polynomials defined by $f(x) = P(x, a, b)$ and $g(x) = Q(x, a, b)$. Then, $R(a, b) = R_{f,g}$. In particular, if $(a, b) = (-\beta_1, \alpha_1)$, $R(-\beta_1, \alpha_1) = 0$, and by Theorem 9.3, there is $\lambda \in \mathbb{C}$ such that

$$P(\lambda, -\beta_1, \alpha_1) = f(\lambda) = g(\lambda) = Q(\lambda, -\beta_1, \alpha_1) = 0,$$

so $f(x_0)$ and $g(x_0)$ have a common root λ. Since $(-\beta_1, \alpha_1) \neq (0,0)$, $(\lambda, \beta_1, \alpha_1) \neq (0,0,0)$, so that $p = [\lambda, -\beta_1, \alpha_1]$ is a well-defined point of $\mathbb{P}^2$ and is on both C and D, so $p \in C \cap D$.

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2. We will show that if $C \cap D$ contains at least $nm+1$ distinct points, C and D have a common factor. Let $\mathcal{S} = \{p_1, \dots, p_{nm+1}\}$ be any set of $nm+1$ distinct points and denote by $L_{i,j}$ the unique line containing p_i and p_j for $1 \leq i, j \leq nm+1$. For each i, denote by $\left[x_0^i, x_1^i, x_2^i\right]$ the coordinates of p_i. The point $x = [x_0, x_1, x_2] \in \mathbb{P}^2$ belongs to $L_{i,j}$ if and only if $\det(A) = 0$, where

$$A_{i,j} = \begin{pmatrix} x_0 & x_0^i & x_0^j \\ x_1 & x_1^i & x_1^j \\ x_2 & x_2^i & x_2^j \end{pmatrix}. \tag{8}$$

Let $p \in \mathbb{P}^2$ be any point that lies neither on C nor D nor on any $L_{i,j}$, that is $p \notin C \cup D \cup \left(\bigcup_{i,j} L_{i,j}\right)$. Can apply a projective transformation

$$\begin{array}{cccc} \Phi & : & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\ & & p & \longmapsto & [1,0,0] \end{array}.$$

Taking images along Φ, the image of $\mathcal{S}$ is a set of $nm+1$ distinct points and the image of $L_{i,j}$ by Φ is the unique line through $\Phi(p_i)$ and $\Phi(p_j)$. As in the proof of 1, after replacing $C, D, \mathcal{S}$ by their images by Φ, we may assume that $[1,0,0] \notin C \cup D$ and that $[1,0,0]$ lies on no line through two points $p_i, p_j \in \mathcal{S}$. In particular, by (8),

$$\det \begin{pmatrix} 1 & x_0^i & x_0^j \\ 0 & x_1^i & x_2^j \\ 0 & x_2^i & x_2^j \end{pmatrix} \neq 0,$$

that is $\left(x_1^i, x_2^i\right) \neq (0,0)$ for all i, otherwise determinant is zero, so that $\left[x_1^i, x_2^i\right]$ is a well-defined point of $\mathbb{P}^1$, and $\left[x_1^i, x_2^i\right] \neq \left[x_1^j, x_2^j\right]$ for all $i \neq j$. Let $P, Q \in \mathbb{C}[x_0, x_1, x_2]$ be homogeneous polynomials of degrees n and m that define C and D, that is $C = \{P = 0\}$ and $D = \{Q = 0\}$. Since $[1,0,0] \notin C \cup D$, $P(1,0,0) \neq 0$ and $Q(1,0,0) \neq 0$. Denote by R the resultant $R_{P,Q}$ of P and Q. This is homogeneous of degree nm and factors into linear factors

$$R(x_1, x_2) = \prod_{i=1}^{nm} (\alpha_i x_1 + \beta_i x_2), \qquad (\alpha_i, \beta_i) \neq (0,0),$$

so $\{R = 0\} \subseteq \mathbb{P}^1$ has at most nm distinct elements. Now assume $C \cap D$ contains at least $nm+1$ points and let $\mathcal{S} \subseteq C \cap D$ be a set of $nm+1$ distinct points. Set $(\alpha_i, \beta_i) = \left(-x_2^i, x_1^i\right)$ for $i = 1, \dots, nm+1$. By construction, for all i, $f_i(x) = P\left(x_0, x_1^i, x_2^i\right)$ and $g_i(x) = Q\left(x_0, x_1^i, x_2^i\right)$ have a common root so that $R_{f_i, g_i} = 0$. But we have $R_{f_i, g_i} = R\left(x_1^i, x_2^i\right)$, so that $(\alpha_i x_1 + \beta_i x_2)$ is a factor of the polynomial $R(x_1, x_2)$. Since $\left[x_1^i, x_2^i\right] \neq \left[x_1^j, x_2^j\right]$ these factors are distinct, so $\left\{\left[x_1^i, x_2^i\right]\right\} \subseteq \mathbb{P}^1$ has at least $nm+1$ distinct points and

$$R(x_1, x_2) = \prod_{i=1}^{nm+1} (\alpha_i x_1 + \beta_i x_2) S(x_1, x_2), \tag{9}$$

for some, possibly constant, homogeneous polynomial S. If $R(x_1, x_2)$ is not identically zero, by Theorem 9.7, it has degree nm. Thus, (9) shows that R is identically zero, because the degree of the polynomial on the right hand side of (9) is at least $nm+1$ if this polynomial is not identically zero. Thus, by Theorem 9.6, P and Q have a common factor and C and D have a common component.

□

This finishes the proof of Theorem 7.9, and now we can also prove Theorem 7.11, stated before.

Proof of Theorem 7.11. Use Bézout's theorem. Let $C = \{P = 0\}$ for P a polynomial of degree d defining C. The singular points are solutions $[x_1, x_2, x_3]$ at

$$P = 0, \qquad P_{x_0} = 0, \qquad P_{x_1} = 0, \qquad P_{x_2} = 0.$$

Without loss of generality we may assume that there exists $i = 0, 1, 2$ such that $Q = P_{x_i}$ is not zero, otherwise P is constant. Thus Q is a homogeneous polynomial of degree $d - 1$. If p is a singular point of C then $P(p) = Q(p) = 0$. Clearly the opposite is not true. Let $D = \{Q = 0\}$. Then D is another projective curve of degree at most $d - 1$ and the intersection $C \cap D$ contains all the singular points of C. Moreover, since C is irreducible, it follows that C and D do not have any common component. By Theorem 9.8, it follows that the number of intersection points of C and D is at most $\deg(C)\deg(D) = d \cdot (d - 1)$ and the claim follows. $\square$

Proposition 9.9 (Pascal's mystic hexagon). *Let $C \subseteq \mathbb{P}^2$ be an irreducible conic and let $p_1, \dots, p_6 \in C$ be six distinct points that form a hexagon, that is no three of the points $p_1, \dots, p_6$ lie on a projective line. Then the points of intersection of lines passing through opposite sides are collinear.*

Proof. Assume that $p_1, \dots, p_6$ are ordered such that the lines $L_1, \dots, L_6$, with L_i the unique line through p_i and p_{i+1}, form the sides of a hexagon. Denote by D_1 and D_2 the cubic curves

$$D_1 = L_1 \cup L_3 \cup L_5, \qquad D_2 = L_2 \cup L_4 \cup L_6.$$

Both cubic curves are reducible by definition, and both contain the points $p_1, \dots, p_6$. Let

$$q_1 = L_1 \cap L_4, \qquad q_2 = L_3 \cap L_6, \qquad q_3 = L_5 \cap L_2$$

be the not necessarily distinct points of intersection of opposite sides. We want to prove that there is a projective line L through q_1, q_2, q_3. First note that if q_1, q_2, q_3 are not distinct, there is a projective line through q_1, q_2, q_3 by Exercise 18. We therefore assume that q_1, q_2, q_3 are three distinct points. The points q_1, q_2, q_3 are not in the set $\{p_1, \dots, p_6\}$ because otherwise, one of the lines $L_1, \dots, L_6$ would have to contain at least three of the points $p_1, \dots, p_6$. Since C is irreducible, it contains none of the lines $L_1, \dots, L_6$. In particular, C has no common component with D_1 or with D_2. Since

$$C \cap D_1 \supseteq \{p_1, \dots, p_6\}, \qquad C \cap D_2 \supseteq \{p_1, \dots, p_6\},$$

and since C and D_1 and C and D_2 have no common component, their intersections consist of at most $(2)(3) = 6$ points and therefore

$$C \cap D_1 = C \cap D_2 = \{p_1, \dots, p_6\}.$$

This shows that the points q_1, q_2, q_3 do not lie on C. I will construct another cubic, having seven points of intersection with C. Let $p_0 \in C$ be a point that does not belong to the set $\{p_1, \dots, p_6\}$. In particular, the point p_0 does not lie on D_1 or on D_2. Let P and Q be homogeneous polynomials that define the cubic curves D_1 and D_2, products of the equations of the lines, that is

$$D_1 = \{P = 0\}, \qquad D_2 = \{Q = 0\}.$$

Since $p_0 \notin D_1 \cup D_2$, $P(p_0) \neq 0$ and $Q(p_0) \neq 0$, so that there exists $[\lambda, \mu] \in \mathbb{P}^1$ such that $\lambda P(p_0) + \mu Q(p_0) = 0$, such as $[-Q(p_0), P(p_0)]$. Consider the cubic curve

$$D = \{\lambda P + \mu Q = 0\} \subseteq \mathbb{P}^2,$$

where $\lambda P + \mu Q$ is a homogeneous polynomial of degree three. The points $p_1, \dots, p_6$ lie on $D_1 \cap D_2$, hence they satisfy $P(p_i) = Q(p_i) = 0$ for all i, so $(\lambda P + \mu Q)(p_i) = 0$, and $p_1, \dots, p_6 \in D$. Since $p_0 \in D$, the intersection $C \cap D$ thus contains at least seven distinct points $p_0, \dots, p_6$. By Bézout's theorem, this implies that C and D have a common component, and since C is irreducible, that has to be C and $C \subseteq D$. If Q is a homogeneous polynomial of degree two that defines C, there is a homogeneous polynomial S of degree one such that

$$D = \{Q \cdot S = 0\} = C \cup \{S = 0\},$$

of which $\{S = 0\}$ has to be a line. Let L be the projective line $L = \{S = 0\}$ that I was looking for. Since the points $q_1, q_2, q_3 \in D_1 \cap D_2$, they also lie on D. We have seen that $q_1, q_2, q_3 \notin C$, thus, they lie on L. $\square$

Lecture 13 is a problem class.

Lecture 13 Monday 05/11/18

10 Conics

Lecture 14 Thursday 08/11/18

In $\mathbb{R}^2$ there are three types of conics, which are projective curves of degree two,

1. ellipse, which is compact and therefore it does not admit any point at infinity,
2. parabola, which admits a unique point at infinity, and
3. hyperbola, which admits two points at infinity.

We have seen how looking at them in projective space eliminates the difference, but we still have examples with only one point, or without any points, such as

$$x^2 + y^2 = 0, \qquad x^2 + y^2 = -1.$$

In $\mathbb{C}^2$, case 1 cannot happen because no affine curve is compact or, in other words, any affine curve has a point at infinity, and we know that in any case there are infinitely many points. Let $C = \{P = 0\}$, where P has degree two for $P \in \mathbb{C}[x_0, x_1, x_2]$. By Theorem 9.8, the number of points at infinity in the line at infinity $\{x_2 = 0\}$ of any conic C is either one or two.

Example 10.1.

- $[0, 1, 0]$ is the only point at infinity of
$$\left\{x_2 x_1 - x_0^2 = 0\right\},$$
since $x_2 = 0$, so $x_0^2 = 0$.
- $[1, \pm i, 0]$ are the two points at infinity of
$$\left\{x_0^2 + x_1^2 + x_2^2 = 0\right\}.$$

We now prove that all irreducible conics are the same up to projective equivalence.

Theorem 10.2. *Let C be an irreducible conic, if and only if P is irreducible. Then there exists a projective transformation $\Psi : \mathbb{P}^2 \to \mathbb{P}^2$ such that*

$$\Psi(C) = \left\{x_2^2 + x_1 x_0 = 0\right\}.$$

Remark 10.3. Note that this equation is the same as

$$x_0^2 + x_1^2 + x_2^2 = 0$$

up to projective transformation, via

$$x_0 \mapsto x_0 + i x_1, \qquad x_1 \mapsto x_0 - i x_1, \qquad x_2 \mapsto x_2.$$

Proof. By Theorem 7.11 there exists a smooth point $p \in C$. After taking a projective transformation we may assume that $p = [0, 1, 0]$ and that the tangent line of C at the point p has equation

$$x_0 = 0,$$

by Exercise 29. Let $C = \{P = 0\}$, where $P \in \mathbb{C}[x_0, x_1, x_2]$ is a homogeneous polynomial of degree two. We may write

$$P(x_0, x_1, x_2) = a x_0^2 + b x_1^2 + c x_2^2 + d x_0 x_1 + e x_0 x_2 + f x_1 x_2, \qquad a, b, c, d, e, f \in \mathbb{C}.$$

The tangent line of C at $[0, 1, 0]$ is given by

$$\sum_{i=0}^{2} x_i \cdot P_{x_i}(0, 1, 0) = x_0 \cdot P_{x_0}(0, 1, 0) + x_1 \cdot P_{x_1}(0, 1, 0) + x_2 \cdot P_{x_2}(0, 1, 0) = 0.$$

But we know that this line is $x_0 = 0$ and therefore

$$P_{x_1}(0,1,0) = P_{x_2}(0,1,0) = 0.$$

Compute these, and see that they are b and f, so it follows that $b = f = 0$ and

$$P(x_0, x_1, x_2) = ax_0^2 + cx_2^2 + dx_0x_1 + ex_0x_2 = cx_2^2 + x_0(ax_0 + dx_1 + ex_2).$$

Want to change coordinates to get $x_2^2 + x_1x_0 = 0$. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & d & e \\ 0 & 0 & \sqrt{c} \end{pmatrix},$$

where $\sqrt{c}$ denotes any one of the complex square roots of c. Check that A is invertible. Assume that $\det(A) = d \cdot \sqrt{c} = 0$. Then either $c = 0$ or $d = 0$. If $c = 0$ then

$$P = x_0(ax_0 + dx_1 + ex_2),$$

which is not irreducible. If $d = 0$ then

$$P = cx_2^2 + ax_0^2 + ex_0x_2,$$

which can be factored, being a homogeneous polynomial in two variables. In both cases, P is not irreducible, which contradicts the assumption. Then $\det(A) \neq 0$. Let $\Psi : \mathbb{P}^2 \to \mathbb{P}^2$ be the projective transformation associated to A, by $\Psi = [A]$. Then if $[z_0, z_1, z_2] = \Psi([x_0, x_1, x_2])$, we have

$$z_0 = x_0, \qquad z_1 = ax_0 + dx_1 + ex_2, \qquad z_2 = \sqrt{c}x_2,$$

and the equation of $\Psi(C)$ becomes

$$z_2^2 + z_1z_0 = 0,$$

as claimed. (Exercise: convince yourself that this is the same that you get by doing $P\left(A^{-1}(z_0, z_1, z_2)\right)$) $\square$

Corollary 10.4. *Let C be a conic. Then C is smooth if and only if it is irreducible.*

Proof. By Theorem 7.9, if C is reducible then it is not smooth, so if C is smooth then it is irreducible. Assume now that C is irreducible. Then by Theorem 10.2, after taking a projective transformation we may assume that C is given by the equation

$$x_2^2 + x_0x_1 = 0,$$

which defines a smooth conic, by Exercise 29. $\square$

Remark 10.5. In general it is not true that if C is irreducible then C is smooth, for higher degree curves.

Example 10.6. Let

$$C = \left\{x_2^3 - x_0x_1^2 = 0\right\} \subseteq \mathbb{P}^2.$$

Then C is singular at the point $[1,0,0]$ but C is irreducible otherwise C would contain a line. (Exercise: check that C does not contain any line)

Exercise 30.

- Show that for every reducible conic C there exists a projective transformation $\Psi : \mathbb{P}^2 \to \mathbb{P}^2$ such that

$$\Psi(C) = \left\{x_0^2 + x_1^2 = 0\right\}.$$

- Show that for any linear homogeneous polynomial $L(x_0, x_1, x_2)$, there exists a projective transformation $\Psi : \mathbb{P}^2 \to \mathbb{P}^2$ such that

$$\Psi(C) = \left\{x_0^2 = 0\right\},$$

where $C = \left\{L^2 = 0\right\}$.

In conclusion, if $C = \{f = 0\}$ is a projective curve defined by a homogeneous polynomial f of degree two, which could be reducible or the square of a linear form, then after a change of coordinates, by a projective transformation, C is of one of the following three forms.

- A double line
$$x_0^2 = 0.$$
This according to our definitions is not actually a curve of degree two, but of degree one.

- Two distinct lines intersecting at a point
$$x_0^2 + x_1^2 = 0.$$

- An irreducible and smooth conic
$$x_0^2 + x_1^2 + x_2^2 = 0.$$

These three forms are clearly distinct, that is they cannot be brought into one another via a projective transformation.

Remark 10.7. There exists a natural bijection between $\mathbb{P}^1$ and the conic
$$C = \left\{x_2^2 + x_1 x_0 = 0\right\} \subseteq \mathbb{P}^2,$$
defined by
$$\begin{array}{cccc} f & : & \mathbb{P}^1 & \longrightarrow & C \subseteq \mathbb{P}^2 \\ & & [z_0, z_1] & \longmapsto & \left[z_0^2, -z_1^2, z_0 z_1\right] \end{array},$$
which satisfies the equation of C. (Exercise: check that this is a bijection) Thus, it follows by Theorem 10.2 that any irreducible conic admits a natural bijection with $\mathbb{P}^1$ and therefore with the sphere
$$\left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\} \subseteq \mathbb{R}^3.$$
Thus, the affine curve defined by the equation $x^2 + y^2 = 1$ admits a bijection with the sphere minus two distinct points.

Remark 10.8. The bijection between a smooth conic and $\mathbb{P}^1$, of which there are actually many, can be seen geometrically as follows. Fix a point $p \in C$, and look at the set S of lines through p. On the one hand, S is in bijection with C itself, via the map that sends a line L through p to the other point of $L \cap C$, which we take to be p itself if L is the tangent line to C in p. On the other hand, S is also naturally in bijection with $\mathbb{P}^1$, in the following way. The space of lines in $\mathbb{P}^2$ can be identified itself with $\mathbb{P}^2$. Every line has an equation $ax_0 + bx_1 + cx_2 = 0$ with $(a, b, c) \neq (0, 0, 0)$, and this represents the same line as another equation $a'x_0 + b'x_1 + c'x_2 = 0$ if and only if $[a, b, c] = [a', b', c']$ as points of $\mathbb{P}^2$. Moreover, the space of lines passing through $p = [z_0, z_1, z_2]$ is given by the equation $az_0 + bz_1 + cz_2 = 0$ inside this $\mathbb{P}^2$ with coordinates $[a, b, c]$. Here, a, b, c play the role usually played by x_0, x_1, x_2, and z_0, z_1, z_2 are numbers. This identifies the space of lines through p with a copy of $\mathbb{P}^1$. The following is another way. Fix equations $L_1(x_0, x_1, x_2)$ and $L_2(x_0, x_1, x_2)$ passing through p. Then every other line passing through p has equation $\lambda L_1 + \mu L_2 = 0$ for a uniquely determined point $[\lambda, \mu] \in \mathbb{P}^1$. (Exercise: prove this) This also identifies the space of lines through p with a $\mathbb{P}^1$.

Exercise 31. Let $C \subseteq \mathbb{P}^2$ be a conic. Show that there exists a symmetric 3×3 matrix $A = (a_{ij})$ such that
$$P(z_0, z_1, z_2) = \sum_{i,j=0}^{2} a_{ij} z_i z_j$$
is the homogeneous polynomial which defines C. Show that C is irreducible if and only if $\det(A) \neq 0$.

More generally, the rank of this matrix determines if the conic is a double line, a pair of distinct lines, or a smooth conic.

11 Multiplicities and strong Bézout

In order to refine the statement of Bézout's theorem, we need to introduce the intersection multiplicity of curves $C, D \subseteq \mathbb{P}^2$ at a point $p \in \mathbb{P}^2$. This will encode both how singular p is as a point of C and D and the relative position, or tangency, of C and D at p. We start by defining a number that measures just the singularity of one of the two curves at p. If $f(x) \in \mathbb{C}[x]$ is a polynomial of one variable, the multiplicity of f in some $\alpha \in \mathbb{C}$ is the number of times that the factor $(x-\alpha)$ appears in the factorisation of f in linear factors, so $\max\left\{d \mid (x-\alpha)^d \mid f(x)\right\}$. Let

$$f(x) = a \cdot (x-\lambda_1)^{a_1} \dots (x-\lambda_k)^{a_k}.$$

If $\alpha = \lambda_i$, the multiplicity is a_i. The multiplicity is zero if α is not a root, that is $f(\alpha) \neq 0$. In order to generalise the definition to polynomials in more variables, we can observe that this multiplicity is k if and only if we have

$$f^{(0)}(\alpha) = \dots = f^{(k-1)}(\alpha) = 0, \qquad f^{(k)}(\alpha) \neq 0.$$

By convention, $f^{(0)} = f$.

Example. If multiplicity is one, then $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

Notation 11.1. Let $f \in \mathbb{C}[x_0, \dots, x_n]$ and let $\alpha \in \mathbb{N}^{n+1}$ be a multi-index. Recall that $|\alpha| = \alpha_0 + \dots + \alpha_n$. Denote

$$\partial^\alpha f = f_{x_0^{\alpha_0} \dots x_n^{\alpha_n}} = \frac{\partial^{|\alpha|} f}{\partial x_0^{\alpha_0} \dots \partial x_n^{\alpha_n}},$$

and recall that for all α, $\partial^\alpha f$ is a polynomial, and that $\partial^\alpha f \equiv 0$ whenever all its coefficients are zero. Set $\partial^{(0,\dots,0)} f = f$.

Example. Let $n = 1$. Then $\partial^{(1,1)} f = f_{x_1 x_0} = f_{x_0 x_1}$.

Definition 11.2. Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be a polynomial and $p = (a_0, \dots, a_n) \in \mathbb{C}^{n+1}$ be a point. The **multiplicity** of f at p is

$$mult_p(f) = \min\{|\alpha| \mid \partial^\alpha f(a_0, \dots, a_n) \neq 0\} = \max\{k \in \mathbb{N} \mid \forall |\alpha| < k,\ \partial^\alpha f(a_0, \dots, a_n) = 0\}.$$

Let $C = \{P = 0\} \subseteq \mathbb{P}^2$ be a projective curve defined by a homogeneous polynomial $P \in \mathbb{C}[x_0, x_1, x_2]$ with no repeated factors. The multiplicity of C at $p = [a, b, c] \in \mathbb{P}^2$ is

$$mult_p(C) = mult_{(a,b,c)}(P).$$

Want to check

$$mult_{(\lambda a, \lambda b, \lambda c)}(P) = mult_{(a,b,c)}(P).$$

This is true as $\partial^\alpha P(\lambda a, \lambda b, \lambda c) = \lambda^{\deg(P) - |\alpha|} \partial^\alpha P(a, b, c)$, and the left hand side is zero if and only if $\partial^\alpha P(a, b, c)$ is zero. In other words, this is the order of vanishing of f at p. Lecture 15 Friday 09/11/18

Let $C = \{P = 0\}$ be a projective curve and $p \in \mathbb{P}^2$. Then the following holds.

- $mult_p(C) = 0$ if and only if $P(p) = \partial^{(0,0,0)} P(p) \neq 0$, that is if and only if $p \notin C$.
- $mult_p(C) = 1$ if and only if $P(p) = \partial^{(0,0,0)} P(p) = 0$ and there is α with $|\alpha| = 1$ such that $\partial^\alpha P(p) \neq 0$. Since the only indices α with $|\alpha| = 1$ are $(1,0,0), (0,1,0), (0,0,1)$, this happens if and only if at least one of $P_{x_0}(p), P_{x_1}(p), P_{x_2}(p)$ is non-zero, if and only if p is not a singular point of C. Thus, $mult_p(C) = 1$ if and only if p is a smooth point of C.
- Similarly $mult_p(C) \geq 2$ if and only if p is a singular point of C.

Example 11.3.

- The curve with equation
$$f(x_0,x_1,x_2) = x_0^2 x_1 + x_0 x_1 x_2$$
has multiplicity three at $p = (0,0,0)$.
  - $|\alpha| = 0$ gives $\partial^{(0,0,0)} f(p) = f(p) = 0$.
  - $|\alpha| = 1$ gives
    * $\partial^{(1,0,0)} f(p) = f_{x_0}(p) = (2x_0x_1 + x_1x_2)(0,0,0) = 0$,
    * $\partial^{(0,1,0)} f(p) = f_{x_1}(p) = \left(x_0^2 + x_0x_2\right)(0,0,0) = 0$, and
    * $\partial^{(0,0,1)} f(p) = f_{x_2}(p) = (x_0x_1)(0,0,0) = 0$.
  - $|\alpha| = 2$ gives $\partial^\alpha f(p) = 0$ for all α with $|\alpha| = 2$.
  - But $\partial^{(2,1,0)} f(p) = f_{x_0^2x_1}(p) = 2$.
- The curve with equation
$$P(x_0,x_1,x_2) = x_0^2 x_2 - x_1^2(x_1 + x_2)$$
has multiplicity two at $p = [0,0,1]$. Indeed,
  - $P_{x_0} = 2x_0x_2$ vanishes at p,
  - $P_{x_1} = -3x_1^2 - 2x_1x_2$ vanishes at p,
  - $P_{x_2} = x_0^2 - x_1^2$ vanishes at p, and
  - $P_{x_0x_0} = 2x_2$ does not vanish at p.

Exercise 32. Multiplicity behaves well under projective transformations. Assume that $C = \{P = 0\}$ is a projective curve and that χ is a projective transformation. Show that
$$mult_p(C) = mult_{\chi(p)}(\chi(C)).$$
A hint is that partial derivatives are linear maps.

Exercise 33. Let $\overline{C} = \{P = 0\}$ be a projective curve that does not contain the line $\{x_2 = 0\}$. Denote by $f(x,y) = P(x,y,1)$ and let $C = \{f = 0\}$ be the associated affine curve. Let $(a,b) \in C$. Then
$$mult_{[a,b,1]}\left(\overline{C}\right) = mult_{(a,b)}(f).$$

Note. Another way to think about multiplicity $mult_p(f)$ of $f \in \mathbb{C}[x_0,\dots,x_n]$ at $p \in \mathbb{C}^{n+1}$ is exactly the lowest degree of terms with non-zero coefficients in the Taylor expansion of f around the point p, given by
$$mult_p(f) = \sum_{\alpha \in \mathbb{N}^{n+1}} \frac{\partial^\alpha f}{\prod_{i=0}^n \alpha_i!}(p) \prod_{i=0}^n (x_i - p_i)^{\alpha_i}.$$
If $p = (0,\dots,0)$, then this coincides with f.

Remark 11.4. A consequence of Exercise 32 and Exercise 33 is that $mult_p(C) \le \deg(C)$ for all $p \in C$. Indeed, if $f \in \mathbb{C}[x,y]$ is a polynomial of degree d, it is clear that $mult_{(0,0)}(f) \le d$, so that $mult_{[0,0,1]}(C) \le d$. For any $p \in \mathbb{P}^2$, let
$$\begin{array}{rccc} \Phi & : & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\ & & p & \longmapsto & [0,0,1] \end{array}.$$
Then $mult_p(C) = mult_{[0,0,1]}(\Phi(C))$, and the result follows because $\Phi(C)$ is a curve of degree $\deg(C)$. Why? Suppose $\deg(C) = n$. Pick a monomial $x_{i_1}\dots x_{i_n}$ that appears with non-zero coefficient in P. Then $P_{x_{i_1}\dots x_{i_n}}(p) \ne 0$ for $p \in \mathbb{P}^2$.

Lemma 11.5. *Let $P \in \mathbb{C}[x_0,x_1]$ be a homogeneous polynomial of degree $d > 0$, and denote $\{P = 0\} = \{p_1,\dots,p_k\} \subseteq \mathbb{P}^1$. Then,*
$$\sum_{i=1}^k mult_{p_i}(P) = d,$$
and, in particular, $k \le d$.

Example. Let $P = x_0^2$ then $\{P = 0\} = \{[0,1]\}$ then $mult_{[0,1]}\left(x_0^2\right) = 2$. If $a \ne 0$ then $mult_{[a,b]}\left(x_0^2\right) = 0$.

This is essentially the statement that a degree d polynomial in one variable has d roots when counted with multiplicity.

Proof. As in the proof of Lemma 6.2, we may write

$$P(x_0, x_1) = \lambda x_0^{d-e} \cdot \prod_{i \in I} (x_1 - a_i x_0)^{d_i}, \qquad \lambda \neq 0, \qquad d_i, e \in \mathbb{N}, \qquad \sum_{i \in I} d_i = e,$$

where $a_i \in \mathbb{C}$ are such that $a_i \neq a_j$ and $a_i \neq 0$ for all $i \neq j \in I$. Note that in the expression of P, e is not necessarily distinct from zero, in which case $I = \emptyset$ and the product is empty, or from d. Let us denote

$$\{p_1, \dots, p_k\} = \begin{cases} \{[1, a_i]\}_{i \in I} \cup \{[0, 1]\} & 0 < d < e \\ \{[0, 1]\} & e = 0 \\ \{[1, a_i]\}_{i \in I} & e = d \end{cases}.$$

It is easy to check from the definition of multiplicity that, for each i, $mult_{[1,a_i]}(P) = d_i$ and $mult_{[0,1]}(P) = d - e$, and the claim follows from $\sum_{i \in I} d_i = e$. $\square$

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Multiplicity is the measure of the singularity of C at p. Using multiplicities of curves, one can already improve Theorem 9.8. We will not prove the following, but rather prove directly the strong version of Bézout's theorem, which will have the following as a corollary.

Theorem 11.6. *Let C_1 and C_2 be two plane projective curves with no common components of degree n and m respectively. Let $C_1 \cap C_2 = \{p_1, \dots, p_k\}$. Then*

$$k \leq \sum_{i=1}^{k} mult_{p_i}(C_1) \cdot mult_{p_i}(C_2) \leq n \cdot m.$$

In particular, it follows that $k \leq n \cdot m$, as in Theorem 9.8. Using multiplicities like this is still not enough to attain equality. For example, for a line tangent to a conic at a point, the multiplicity of both curves at the intersection point is one, but the product of the degrees is two. We have to define a multiplicity of intersection of the two curves, that on top of their singularities also keeps track of how they interact. To also detect tangencies, have to look at the multiplicity of the resultant. Let us start by proving the following.

Lemma 11.7. *Let $P, Q \in \mathbb{C}[x_0, x_1, x_2]$ be homogeneous polynomials such that $P(1,0,0) \neq 0$, $Q(1,0,0) \neq 0$, $\deg(P) = n$, and $\deg(Q) = m$. Let $p = [z_0, z_1, z_2] \in \mathbb{P}^2$ be such that $p \neq [1,0,0]$ and let $r = mult_p(P)$ and $s = mult_p(Q)$. Then if $R_{P,Q}(x_1, x_2) = \det(R(x_1, x_2))$ is the resultant of P and Q, we have*

$$mult_{[z_1, z_2]}(R_{P,Q}) \geq r \cdot s.$$

Proof. After applying a projective transformation we may assume that $p = [0,0,1]$, by Exercise 32. Note that $mult_{[0,1]}(R(x_1, x_2)) = mult_0(R(y, 1))$, since R is homogeneous. We may write

$$P = \sum_{i=0}^{n} a_i(x_1, x_2) x_0^i, \qquad Q = \sum_{i=0}^{m} b_i(x_1, x_2) x_0^i,$$

where a_i and b_i are homogeneous polynomials. Let

$$f(x, y) = P(x, y, 1) = \sum_{i=0}^{n} a_i(y, 1) x^i = \sum_{i=0}^{n} \widetilde{a_i}(y) x^i, \qquad \widetilde{a_i}(y) = a_i(y, 1),$$

and similarly let

$$g(x, y) = Q(x, y, 1) = \sum_{i=0}^{m} b_i(y, 1) x^i = \sum_{i=0}^{m} \widetilde{b_i}(y) x^i, \qquad \widetilde{b_i}(y) = b_i(y, 1).$$

Then $mult_{(0,0)}(f) = r$ and $mult_{(0,0)}(g) = s$. In particular, since the minimum degree of any monomial that appears on f with non-zero coefficient is r, it follows that $mult_0(\widetilde{a_i}(y)) \geq r - i$ if $i < r$, that is $\widetilde{a_i}(y) = y^{r-i} \cdot A_i(y)$ if $i < r$, and similarly $\widetilde{b_i}(y) = y^{s-i} \cdot B_i(y)$ if $i < s$, for some polynomials A_i and B_i in y. Note that by Remark 11.4, we have $r \leq n$ and $s \leq m$. By Definition 9.5, it follows that

$$R(y,1) = \begin{pmatrix} y^r \cdot A_0(y) & \dots & y^0 \cdot A_r(y) & \dots & A_n(y) & & 0 \\ & \ddots & & \ddots & & \ddots & \\ 0 & & y^r \cdot A_0(y) & \dots & y^0 \cdot A_r(y) & \dots & A_n(y) \\ y^s \cdot B_0(y) & \dots & y^0 \cdot B_s(y) & \dots & B_m(y) & & 0 \\ & \ddots & & \ddots & & \ddots & \\ 0 & & y^s \cdot B_0(y) & \dots & y^0 \cdot B_s(y) & \dots & B_m(y) \end{pmatrix}.$$

Want to show $mult_0(\det(R(y,1))) \geq r \cdot s$. Let $R_1(y)$ be the matrix obtained by multiplying the i-th row of $R(y,1)$ by y^{s-i+1} for any $i \leq s$, and multiplying its $(m+j)$-th row by y^{r-j+1} for any $j \leq r$. Then we obtain

$$R_1(y) = \begin{pmatrix} y^{r+s} \cdot A_0(y) & \dots & y^s \cdot A_r(y) & \dots & y^s \cdot A_n(y) & & 0 \\ & \ddots & & \ddots & & \ddots & \\ 0 & & y^{r+1} \cdot A_0(y) & \dots & y^1 \cdot A_r(y) & \dots & y^1 \cdot A_n(y) \\ y^{r+s} \cdot B_0(y) & \dots & y^r \cdot B_s(y) & \dots & y^r \cdot B_m(y) & & 0 \\ & \ddots & & \ddots & & \ddots & \\ 0 & & y^{s+1} \cdot B_0(y) & \dots & y^1 \cdot B_s(y) & \dots & y^1 \cdot B_m(y) \end{pmatrix}.$$

Now let $R_2(y)$ be the matrix obtained by dividing the i-th column of $R_1(y)$ by $y^{r+s+1-i}$ for any $i \leq r+s$. Then we obtain

$$R_2(y) = \begin{pmatrix} A_0(y) & \dots & A_r(y) & \dots & y^s \cdot A_n(y) & & 0 \\ & \ddots & & \ddots & & \ddots & \\ 0 & & A_0(y) & \dots & A_r(y) & \dots & y \cdot A_n(y) \\ B_0(y) & \dots & B_s(y) & \dots & y^r \cdot B_m(y) & & 0 \\ & \ddots & & \ddots & & \ddots & \\ 0 & & B_0(y) & \dots & B_s(y) & \dots & y \cdot B_m(y) \end{pmatrix}.$$

In particular $\det(R_2)$ is a polynomial in y and it follows from linear algebra that

$$\begin{aligned} \det(R(y,1)) &= \det(R_1(y)) \cdot y^{-s} \cdot \dots \cdot y^{-1} \cdot y^{-r} \cdot \dots \cdot y^{-1} = \det(R_1(y)) \cdot y^{-\frac{s(s+1)+r(r+1)}{2}} \\ &= \det(R_2(y)) \cdot y^{r+s} \cdot \dots \cdot y^1 \cdot y^{-\frac{s(s+1)+r(r+1)}{2}} = \det(R_2(y)) \cdot y^{\frac{(r+s+1)(r+s)}{2} - \frac{s(s+1)+r(r+1)}{2}} \\ &= \det(R_2(y)) \cdot y^{rs}. \end{aligned}$$

Since $\det(R_2(y))$ is a polynomial, it follows that $mult_{[0,1]}(R_{P,Q}) = mult_0(\det(R(y,1))) \geq r \cdot s$, where for the first equality it is essential that R is homogeneous, and the claim follows. □

Inspired by Lemma 11.7 above, we can now define the intersection number of two curves at a given point.

Definition 11.8. Let $C = \{P = 0\} \subseteq \mathbb{P}^2$ and $D = \{Q = 0\} \subseteq \mathbb{P}^2$ be projective curves defined by homogeneous polynomials $P, Q \in \mathbb{C}[x_0, x_1, x_2]$ with no repeated factor, and assume that $[1,0,0] \notin C \cup D$.

- Assume that C and D have no common component, and let $\{p_1, \dots, p_k\} = C \cap D$, with $[a_i, b_i, c_i] = p_i$. Assume that $[1,0,0]$ lies on none of the lines through p_i and p_j, where $1 \leq i, j \leq k$. The **intersection multiplicity**, or **intersection number**, of C and D at $p = [a,b,c] \in \mathbb{P}^2$ is

$$I(p,C,D) = I(p,P,Q) = \begin{cases} mult_{[b,c]}(R_{P,Q}(x_1,x_2)) & p \in C \cap D \\ 0 & p \notin C \cap D \end{cases}.$$

- If E is a common component of C and D, set $I(p,C,D) = \infty$ for all $p \in E$.

Remark 11.9. As in the proof of Theorem 9.8, the assumption on $[1,0,0]$ guarantees that if $p_i = [a_i, b_i, c_i] \in C \cap D$, then $(b_i, c_i) \neq (0,0)$, so that $[b_i, c_i] \in \mathbb{P}^1$, and that p_i is the only point of $C \cap D$ with coordinates $[\lambda, b_i, c_i]$ for some $\lambda \in \mathbb{C}$. In other words, it guarantees that $\{[b_i, c_i] \mid 1 \leq i \leq k\}$ is a set of k distinct points of $\mathbb{P}^1$.

Recall that given p, C, D, we may assume that the hypotheses of Definition 11.8 are satisfied after a projective transformation $\Psi : \mathbb{P}^2 \to \mathbb{P}^2$. The following exercise shows that intersection numbers can be computed in any coordinate system for which the assumptions on the point $[1,0,0]$ are satisfied.

Exercise 34. Let $\Psi : \mathbb{P}^2 \to \mathbb{P}^2$ be a projective transformation, and let $C, D \subseteq \mathbb{P}^2$ be projective curves and $p \in \mathbb{P}^2$. Assume that $p \neq [1,0,0]$ and $\Psi(p) \neq [1,0,0]$, and that $[1,0,0] \notin C \cup D \cup \Psi(C) \cup \Psi(D)$, and also that $[1,0,0]$ does not lie on any line through two points of $C \cap D$ and of $\Psi(C) \cap \Psi(D)$. Show that

$$I(p, C, D) = I(\Psi(p), \Psi(C), \Psi(D)).$$

Note. These numbers are well-defined because of the assumptions on the two curves. This seems hard to do by hand. See Remark 12.9.

Remark 11.10. Assume that C and D have no common component, and let $\{p_i = [a_i, b_i, c_i] \mid 1 \leq i \leq k\} = C \cap D$. Then, by Theorem 9.3, $R_{P,Q}(b_i, c_i) = 0$ for all $i = 1, \dots, k$, so that $I(p_i, C, D) > 0$. This shows that $I(p, C, D) \neq 0$ if and only if $p \in C \cap D$.

Theorem 11.11 (Strong Bézout theorem)**.** *Let $C, D \subseteq \mathbb{P}^2$ be projective curves of degrees n and m that have no common component. Let $C \cap D = \{p_i = [a_i, b_i, c_i] \mid 1 \leq i \leq k\}$. Then*

$$\sum_{i=1}^{k} I(p_i, C, D) = n \cdot m.$$

Proof. After projective transformation, we may assume that $[1,0,0]$ is contained in none of the lines $L_{i,j}$ through p_i and p_j where $1 \leq i < j \leq k$. Let $R(x_1, x_2)$ be the resultant of P and Q. As is recalled in Remark 11.10, the k points $[b_i, c_i] \in \mathbb{P}^1$ are distinct and by definition, $I(p_i, C, D) = mult_{[b_i, c_i]}(R(x_1, x_2))$. By Theorem 9.6, $R(x_1, x_2)$ is a homogeneous polynomial of degree $n \cdot m$, so that the claim follows from Lemma 11.5. ☐

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Example 11.12. Consider the curves C_1 and C_2 defined by

$$C_1 = \left\{x_0 x_2 - x_1^2 = 0\right\}, \qquad C_2 = \left\{x_0 x_2 + x_1^2 = 0\right\}.$$

We want to compute $I(p, C_1, C_2)$ for $p = [0,0,1]$. As $[1,0,0] \in C_1 \cap C_2$, we first need to change variables. We consider the projective transformation $\Psi([x_0, x_1, x_2]) = [x_1, x_0, x_2]$, so that

$$\Psi(C_1) = \left\{x_1 x_2 - x_0^2 = 0\right\}, \qquad \Psi(C_2) = \left\{x_1 x_2 + x_0^2 = 0\right\}, \qquad \Psi(C_1) \cap \Psi(C_2) = \{[0,1,0], [0,0,1]\}.$$

Note that $\Psi([0,0,1]) = [0,0,1]$. The resultant of $\Psi(C_1)$ and $\Psi(C_2)$ is

$$R(x_1, x_2) = \det \begin{pmatrix} x_1 x_2 & 0 & -1 & 0 \\ 0 & x_1 x_2 & 0 & -1 \\ x_1 x_2 & 0 & 1 & 0 \\ 0 & x_1 x_2 & 0 & 1 \end{pmatrix} = 4 x_1^2 x_2^2.$$

Thus

$$I([0,0,1], C_1, C_2) = I([0,0,1], \Psi(C_1), \Psi(C_2)) = mult_{[0,1]}(R(x_1, x_2)) = 2,$$

and similarly, $I([1,0,0], C_1, C_2) = 2$. Bézout's theorem is satisfied in this case.

Now we can note that Theorem 11.6 is a corollary of the strong theorem.

Proof of Theorem 11.6. Follows directly from Theorem 11.11 and Lemma 11.7. ☐

Exercise 35. Show that an irreducible projective curve C of degree d has at most $d(d-1)/2$ singular points.

Exercise 36. Let C be an irreducible projective curve of degree d with a point $p \in C$ with multiplicity $q = mult_p(C)$. Show that there exists a line $L \subseteq \mathbb{P}^2_{\mathbb{C}}$ such that $p \in L$ and which meets C in exactly $d - q + 1$ points. In particular, for any projective curve C of degree d there exists a line which meets C in d points.

12 More about intersection multiplicities

We now look at a few more properties of intersection multiplicities. We start by giving an alternative description of the resultant of two polynomials in one variable.

Lemma 12.1. *Let $p, q \in \mathbb{C}[x]$ be monic polynomials of degrees n and m, so that there are $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $\mu_1, \dots, \mu_m \in \mathbb{C}$ such that $p(x) = \prod_{i=1}^n (x - \lambda_i)$ and $q(x) = \prod_{j=1}^m (x - \mu_j)$. Then, $R_{p,q}$, the resultant of p and q, is a homogeneous polynomial of degree nm in the variables $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_m$. More precisely,*

$$R_{p,q} = \prod_{i,j} (\lambda_i - \mu_j) = (-1)^{nm} \prod_{j=1}^m p(\mu_j) = \prod_{i=1}^n q(\lambda_i).$$

Proof. Write

$$p(x) = \sum_{i=1}^n a_i(\lambda_1, \dots, \lambda_n) \cdot x^i, \qquad q(x) = \sum_{j=1}^m b_j(\mu_1, \dots, \mu_m) \cdot x^j,$$

where a_i is a homogeneous polynomial of degree $n - i$ in the variables $\lambda_1, \dots, \lambda_n$ and b_j is a homogeneous polynomial of degree $m - j$ in the variables $\mu_1, \dots, \mu_m$. The proof that $R_{p,q}$ is a homogeneous polynomial of degree nm in $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m$ is identical to the proof of Theorem 9.7. We now see the expression for p as a homogeneous polynomial of degree n in $\mathbb{C}[x, \lambda_1, \dots, \lambda_n]$ and q as a homogeneous polynomial of degree m in $\mathbb{C}[x, \mu_1, \dots, \mu_m]$. By Theorem 9.3, for each i and j, if some $\lambda_i = \mu_j$, $R_{p,q} = 0$, so that, as polynomials in $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m$, we have $\lambda_i - \mu_j \mid R_{p,q}$, so

$$R_{p,q} = \prod_{i,j} (\lambda_i - \mu_j) S,$$

where S is a homogeneous polynomial in $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m$. S is homogeneous by Lemma 4.10. Since $\deg(R_{p,q}) = nm$, the degree of S is zero and S is a constant in $\mathbb{C}^*$ with

$$R_{p,q} = S \prod_{i,j} (\lambda_i - \mu_j).$$

The constant $S = 1$, as can be checked by computing the resultant of $p(x) = (x-1)^n$ and $q(x) = x^m$. $\square$

Remark 12.2. More generally, if the leading coefficients of p and q are $a_n \neq 0$ and $b_m \neq 0$ then

$$R_{p,q} = a_n^m b_m^n \prod_{i,j} (\lambda_i - \mu_j).$$

Remark 12.3. Let $f_1, f_2, g \in \mathbb{C}[x]$ be monic polynomials. An immediate consequence of Lemma 12.1 is that $R_{f_1 f_2, g} = R_{f_1, g} \cdot R_{f_2, g}$.

Lemma 12.4. *Let $P_1, P_2, Q \in \mathbb{C}[x_0, x_1, x_2]$ be homogeneous polynomials with $P_i(1,0,0), Q(1,0,0) \neq 0$, and let $P = P_1 \cdot P_2$. Let $R(x_1, x_2)$ be the resultant of P and Q and $R_i(x_1, x_2)$ be the resultant of P_i and Q, for $i = 1, 2$. For all $[b, c] \in \mathbb{P}^1$, we have*

$$mult_{[b,c]}(R) = mult_{[b,c]}(R_1) + mult_{[b,c]}(R_2).$$

Proof. $P_i(1,0,0) \neq 0$, so $a \cdot x_0^{\deg(P_i)}$ is a monomial of P_i for $a \neq 0$. Let $a_1, a_2, b \in \mathbb{C}^*$ be constants such that $P_1 = a_1 P_1'$, $P_2 = a_2 P_2'$, and $Q = bQ'$, where P_1', P_2', Q' are homogeneous polynomials that are monic in x_0. Then, if $m = \deg(Q)$, $P' = P_1' P_2'$, and $a = a_1 a_2 \in \mathbb{C}^*$, the resultants are scalar multiples that satisfy $R_{P_1,Q} = a_1^m b^{\deg(P_1)} R_{P_1',Q'}$, $R_{P_2,Q} = a_2^m b^{\deg(P_2)} R_{P_2',Q'}$, and $R_{P,Q} = a^m b^{\deg(P)} R_{P',Q'}$. In particular, for any $[b, c] \in \mathbb{P}^1$, the multiplicities of $R_{P_1,Q}, R_{P_2,Q}, R_{P,Q}$ and of $R_{P_1',Q'}, R_{P_2',Q'}, R_{P',Q'}$ coincide. We may therefore assume that P_1, P_2, Q are monic with respect to x_0. Fix $[b, c] \in \mathbb{P}^1$ and let $f_i(x) = P_i(x, b, c)$, $f(x) = f_1(x) \cdot f_2(x)$, and $g(x) = Q(x, b, c)$. The polynomials f_1, f_2, f, g are monic. (Exercise) By Lemma 12.1 and the consequence in Remark 12.3, $R_{f,g} = R_{f_1,g} \cdot R_{f_2,g}$, that is we have $R_{P,Q}[b, c] = R_{P_1,Q}[b, c] \cdot R_{P_2,Q}[b, c]$. As this holds for every $[b, c] \in \mathbb{P}^1$, $R_{P,Q} = R_{P_1,Q} \cdot R_{P_2,Q}$ in $\mathbb{C}[x_1, x_2]$, and

$$mult_{[b,c]}(R) = mult_{[b,c]}(R_1) + mult_{[b,c]}(R_2), \qquad [b, c] \in \mathbb{P}^1.$$

$\square$

The following proposition gathers a few important properties of intersection multiplicities.

Proposition 12.5. *Let $C \subseteq \mathbb{P}^2$ and $D \subseteq \mathbb{P}^2$ be projective curves defined by homogeneous polynomials $P, Q \in \mathbb{C}[x_0, x_1, x_2]$. Let $p \in \mathbb{P}^2$ be a point. The intersection numbers $I(p, P, Q)$ satisfy the following.*

1. *Intersection numbers are symmetric, so $I(p, C, D) = I(p, D, C)$.*
2. *$I(p, C, D) = \infty$ if p lies on a common component of C and D and $I(p, C, D) \in \mathbb{Z}$ otherwise.*
3. *$I(p, C, D) \neq 0$ if and only if $p \in C \cap D$.*
4. *If C and D are distinct lines and if $\{p\} = C \cap D$ then $I(p, C, D) = 1$.*
5. *If $P = P_1 \cdot P_2$, for homogeneous polynomials P_1 and P_2, then $I(p, P_1 \cdot P_2, Q) = I(p, P_1, Q) + I(p, P_2, Q)$.*
6. *If $P = P_1 \cdot Q + P_2$, for homogeneous polynomials P_1 and P_2, then $I(p, P, Q) = I(p, P_2, Q)$.*

Proof.

1. Since the effect of exchanging the rows of a matrix on its determinant is to multiply it by (-1), $R_{P,Q} = \pm R_{Q,P}$, and since multiplication by a constant does not affect the multiplicity of a polynomial, 1 follows.
2. 2 is by definition and is part of Definition 11.8.
3. 3 is by construction and was noted in Remark 11.10.
4. For 4, after projective transformation, we may assume that $C = \{x_0 = 0\}$ and $D = \{x_0 + x_1 = 0\}$, so that $[1, 0, 0] \notin C \cup D$ and $\{p\} = C \cap D = \{[0, 0, 1]\}$. Then, $I(p, C, D) = 1$ by an easy computation. (Exercise)
5. 5 is an application of the statement about multiplicities of resultants in Lemma 12.4.
6. We now prove 6. Let $R(x_1, x_2)$ be the resultant of $P_1 \cdot Q + P_2$ and Q and let $R'(x_1, x_2)$ be the resultant of P_2 and Q. Then, denote by
$$P_2 = \sum_{i=1}^{n} a_i(x_1, x_2) \cdot x_0^i, \qquad P_1 = \sum_{i=1}^{n} c_i(x_1, x_2) \cdot x_0^i, \qquad Q = \sum_{i=1}^{n} b_i(x_1, x_2) \cdot x_0^i.$$
Then
$$P_1 \cdot Q + P_2 = \sum_{i=1}^{n} \left(a_i + \sum_{j=0}^{i} c_j \cdot b_{i-j} \right) \cdot x_0^i,$$
so
$$R(x_1, x_2) = \det \begin{pmatrix} a_0 + b_0 c_0 & \dots & a_n + \sum_{j=0}^{n} c_j b_{i-j} & & 0 \\ & \ddots & & \ddots & \\ 0 & & a_0 + b_0 c_0 & \dots & a_n + \sum_{j=0}^{n} c_j b_{i-j} \\ b_0 & \dots & b_n & & 0 \\ & \ddots & & \ddots & \\ 0 & & b_0 & \dots & b_n \end{pmatrix},$$
so that the resultant matrix of R is obtained from the resultant matrix of R' by performing the following row operations.
   - Add c_{j-1} times the $m + j$ row to the 1-st row, for each $j = 1, \dots, n$.
   - $\dots$.
   - Add c_{j-1} times the $m + j + n - 1$ row to the n-th row, for each $j = 1, \dots, n$.

   Since performing these row operations do not affect the determinant, $R = R'$ and 6 follows.

This finishes the proof of Proposition 12.5. $\square$

Example 12.6. Consider the curves

$$C_1 = \left\{x_0 x_2 - x_1^2 = 0\right\}, \qquad C_2 = \left\{x_0 x_2 + x_1^2 = 0\right\},$$

and let $p = [0, 0, 1] \in C_1 \cap C_2$. Then

$$\begin{aligned}
I(p, C_1, C_2) &= I\left(p, x_0 x_2 - x_1^2, x_0 x_2 + x_1^2\right) && \text{by definition,} \\
&= I\left(p, 2x_0 x_2, x_0 x_2 + x_1^2\right) && \text{by 6,} \\
&= I\left(p, x_0, x_0 x_2 + x_1^2\right) + I\left(p, x_2, x_0 x_2 + x_1^2\right) && \text{by 5,} \\
&= I\left(p, x_0, x_0 x_2 + x_1^2\right) && \text{by 3,} \\
&= I\left(p, x_0, x_1^2\right) && \text{by 6,} \\
&= 2 \cdot I(p, x_0, x_1) && \text{by 5,} \\
&= 2 && \text{by 4.}
\end{aligned}$$

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Exercise 37. Let

$$C = \left\{x_0 x_2^2 - x_1 (x_1 - x_0)(x_1 + x_0) = 0\right\} \subseteq \mathbb{P}^2, \qquad L = \{a x_0 + b x_1 = 0\} \subseteq \mathbb{P}^2,$$

for some $a, b \in \mathbb{C}$ not both zero, and let $p = [0, 0, 1]$. Compute $I(p, C, L)$.

Proposition 12.7. *Let $C \subseteq \mathbb{P}^2$ be a projective curve and $p \in C$ a smooth point. If $L = T_p C$ is the tangent line to C at p, $I(p, C, T_p C) > 1$.*

Proof. Let $C = \{P = 0\}$ and $p = [a, b, c]$. We may assume that $[1, 0, 0] \notin C \cup T_p C$ so that $P_{x_0}(a, b, c) \neq 0$. The tangent to C at p is

$$T_p C = \{P_{x_0}(a, b, c) \cdot x_0 + P_{x_1}(a, b, c) \cdot x_1 + P_{x_2}(a, b, c) \cdot x_2 = 0\},$$

so

$$x_0 = -\frac{P_{x_1}(a, b, c) x_1 + P_{x_2}(a, b, c) x_2}{P_{x_0}(a, b, c)},$$

and from applying Lemma 12.1 to

$$p(x) = P(x, x_1, x_2), \qquad q(x) = x + \frac{P_{x_1}(a, b, c) x_1 + P_{x_2}(a, b, c) x_2}{P_{x_0}(a, b, c)},$$

it follows that the resultant $R_{C,L}$ is equal to a scalar multiple of the polynomial in x_1 and x_2,

$$Q(x_1, x_2) = P\left(-\frac{P_{x_1}(a, b, c) x_1 + P_{x_2}(a, b, c) x_2}{P_{x_0}(a, b, c)}, x_1, x_2\right).$$

If P is not monic in x_0, there will be a non-zero constant factor, irrelevant when taking multiplicity. The multiplicity $I(p, C, T_p C) > 1$ if and only if Q has a repeated factor $(c x_1 - b x_2)$ at p. This holds because

$$Q(b, c) = P(a, b, c) = 0, \quad Q_{x_1}(b, c) = P_{x_0}(a, b, c)\left(-\frac{P_{x_1}(a, b, c)}{P_{x_0}(a, b, c)}\right) + P_{x_1}(a, b, c) + 0 = 0, \quad Q_{x_2}(b, c) = 0,$$

by the chain rule, so that $[b, c]$ is a common root of Q and Q_{x_i}. Claim that this implies that $Q(x_1, x_2) = (c x_1 - b x_2)^2 \cdot R(x_1, x_2)$, for some polynomial R. This concludes the proof. $\square$

Exercise 38. Show that the claim holds, so $(c x_1 - b x_2)^2 \mid Q(x_1, x_2)$.

Exercise 39. Show that the converse of the above is also true. If $p \in C$ is a smooth point, and L is a line with $I(p, C, L) > 1$, then L is the tangent line to C at p.

One can actually define intersection multiplicities of projective curves $C, D \subseteq \mathbb{P}^2$ by the properties listed in Proposition 12.5. We will not prove this, see Theorem 3.18 in Kirwan's book if you are interested.

Proposition 12.8. *Properties* 1 *to* 6 *in Proposition 12.5 determine uniquely and characterise completely* $I(p, C, D)$ *for any point* $p \in \mathbb{P}^2$ *and any projective curves* $C, D \subseteq \mathbb{P}^2$*.*

Remark 12.9. Proposition 12.8 is another proof that the intersection multiplicity is defined independent of the choice of coordinates on $\mathbb{P}^2$, that is that

$$I(p, C, D) = I(\Psi(p), \Psi(C), \Psi(D)),$$

for any projective transformation $\Psi : \mathbb{P}^2 \to \mathbb{P}^2$.

Remark 12.10. Another consequence is that the intersection multiplicity $I(p, C, D)$ depends only on the components of C and D that contain the point p.

The intersection of two distinct lines at their point of intersection is one. The next question we could ask is, when do curves C and D intersect at a point $p \in C \cap D$ with multiplicity one? The following proposition, that we state without proving, gives the answer.

Proposition 12.11. *Let* $C, D \subseteq \mathbb{P}^2$ *be projective curves and* p *a point of* $\mathbb{P}^2$*. Then* $I(p, C, D) = 1$ *if and only if* $p \in C \cap D$ *is a smooth point of* C *and of* D*, and the tangent lines to* C *and* D *at* p *are distinct. Then* C *and* D *meet* ***transversely*** *at* p*.*

Remark 12.12. By Proposition 12.5.2 and Proposition 12.5.3, $I(p, C, D)$ is a non-zero finite number if and only if $p \in C \cap D$ and p does not lie on a common component of C and D, and by Lemma 11.7,

$$1 \le mult_p(C) \cdot mult_p(D) \le I(p, C, D) = 1$$

implies that $mult_p(C) = mult_p(D) = 1$, so that p is a smooth point of C and of D. We would therefore have to show that if $p \in C \cap D$ is a smooth point of C and of D, $I(p, C, D) = 1$ if and only if the tangent lines L_C and L_D of C and D at p are distinct.

Proposition 12.11 implies immediately the following corollary.

Corollary 12.13. *Let* $C, D \subseteq \mathbb{P}^2$ *be two projective curves with no common component,* $n = \deg(C)$*, and* $m = \deg(D)$*. Then for any* $p \in C \cap D$*,* C *and* D *are smooth at* p *and have distinct tangent lines at* p *if and only if*

$$\#\{C \cap D\} = n \cdot m.$$

Proposition 12.14. *Let* L *be a projective line and* $C \subseteq \mathbb{P}^2$ *a curve of degree* d*. If* $p \in C \cap L$*,* $I(p, C, L) > 1$ *if and only if either* C *is singular at* p *or* L *is the tangent line to* C *at* p*. Moreover, more generally, if* C *is singular at* p*, then* $I(p, C, L) > mult_p(C)$ *if and only if* L *is one of the higher tangent lines to* C *at* p*.*

Proof. See Exercise 6 of problem sheet 2. □

Example.

- Let
$$y^2 = x^2(x+1) = x^3 + x^2.$$
The lowest degree form is $y^2 = x^2$, so the higher tangent lines at the origin are $y = \pm x$.

- Let
$$y^2 = x^3.$$
The lowest degree form is $y^2 = 0$, so the higher tangent lines at the origin are $y^2 = 0$.

13 Cubic curves

In this section, we investigate the geometry of cubic curves. Up to projective transformation,

- a projective line is
$$L = \{x_0 = 0\},$$
- a smooth irreducible conic is
$$C = \left\{x_0^2 + x_1^2 + x_2^2 = 0\right\} \subseteq \mathbb{P}^2.$$

The equation of a smooth line or conic can be brought in a uniquely determined **standard form**. What about cubics? By contrast, we will see that the equation of a smooth cubic curve can be brought in standard form, but that standard form depends on a parameter. We prove the following.

Theorem 13.1. *Let $C \subseteq \mathbb{P}^2$ be a smooth projective cubic curve of degree three. Then, there exists a projective transformation $\Psi : \mathbb{P}^2 \to \mathbb{P}^2$ and $\lambda \in \mathbb{C} \setminus \{0, 1\}$ such that*

$$\Psi(C) = \left\{x_1^2 x_2 = x_0 (x_0 - x_2)(x_0 - \lambda x_2)\right\} \subseteq \mathbb{P}^2.$$

Dehomogenising with respect to x_2, $y^2 = x(x-1)(x-\lambda)$ is the **Legendre form** for **elliptic curves**. In order to prove Theorem 13.1, we need to define inflection points of projective curves. These will generalise the points of inflection on graphs of functions.

Definition 13.2. Let $P \in \mathbb{C}[x_0, x_1, x_2]$ be a homogeneous polynomial. The **Hessian matrix** of P is the symmetric matrix whose entries are the second order differentials of P,

$$H_P = \left(\frac{\partial^2 P}{\partial x_i \partial x_j}\right)_{0 \le i,j \le 2} = \begin{pmatrix} P_{x_0,x_0} & P_{x_0,x_1} & P_{x_0,x_2} \\ P_{x_1,x_0} & P_{x_1,x_1} & P_{x_1,x_2} \\ P_{x_2,x_0} & P_{x_2,x_1} & P_{x_2,x_2} \end{pmatrix}.$$

The **Hessian** of P is $\mathcal{H}_P(x_0, x_1, x_2) = \det(H_P)$. An **inflection point**, or **flex point**, of C is a smooth point $p = [a, b, c]$ of C such that $\mathcal{H}_P(a, b, c) = 0$.

$\mathcal{H}_P$ is a homogeneous polynomial of degree $3(d-2)$ when $d \ge 3$. The only thing there is to check is that $\mathcal{H}_P$ is a homogeneous polynomial of the specified degree. When $d \ge 3$, all non-zero entries of H_P are homogeneous polynomials of degree $d-2$, and H_P is a 3×3 matrix, so $\mathcal{H}_P$ is indeed a homogeneous polynomial of degree $3(d-2)$. (Exercise: an inflection point p is a smooth point $p \in C$ whose intersection multiplicity with the tangent line to C at p is at least three)

Lemma 13.3. *Let $P \in \mathbb{C}[x_0, x_1, x_2]$ be a homogeneous polynomial of degree $d > 1$. Then*

$$x_2^2 \cdot \mathcal{H}_P = (d-1)^2 \cdot \det \begin{pmatrix} P_{x_0,x_0} & P_{x_0,x_1} & P_{x_0} \\ P_{x_1,x_0} & P_{x_1,x_1} & P_{x_1} \\ P_{x_0} & P_{x_1} & \frac{d}{d-1} \cdot P \end{pmatrix}. \tag{10}$$

Remark 13.4. Analogous formulae to (10) can be stated for x_0 and x_1.

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Proof. For ease of notation, let us label the rows and columns of the Hessian matrix as follows.

$$H_P = \begin{pmatrix} C_0 & C_1 & C_2 \end{pmatrix} = \begin{pmatrix} R_0 \\ R_1 \\ R_2 \end{pmatrix}.$$

Each P_{x_i} is a homogeneous polynomial of degree $d-1$, so that by the Euler relation, in Theorem 7.6, we have

$$(d-1) P_{x_i} = x_0 P_{x_0,x_i} + x_1 P_{x_1,x_i} + x_2 P_{x_2,x_i},$$

and

$$x_2 \cdot \mathcal{H}_P = \det\begin{pmatrix} C_0 & C_1 & x_2 C_2 \end{pmatrix} = \det\begin{pmatrix} C_0 & C_1 & x_0 C_0 + x_1 C_1 + x_2 C_2 \end{pmatrix}$$
$$= \det\begin{pmatrix} P_{x_0,x_0} & P_{x_0,x_1} & (d-1)\,P_{x_0} \\ P_{x_1,x_0} & P_{x_1,x_1} & (d-1)\,P_{x_1} \\ P_{x_2,x_0} & P_{x_2,x_1} & (d-1)\,P_{x_2} \end{pmatrix} = (d-1)\det\begin{pmatrix} P_{x_0,x_0} & P_{x_0,x_1} & P_{x_0} \\ P_{x_1,x_0} & P_{x_1,x_1} & P_{x_1} \\ P_{x_2,x_0} & P_{x_2,x_1} & P_{x_2} \end{pmatrix} = (d-1)\det\begin{pmatrix} R_0' \\ R_1' \\ R_2' \end{pmatrix}.$$

Similarly,

$$x_2^2 \cdot \mathcal{H}_P = (d-1)\det\begin{pmatrix} R_0' \\ R_1' \\ x_2 R_2' \end{pmatrix} = (d-1)\det\begin{pmatrix} R_0' \\ R_1' \\ x_0 R_0' + x_1 R_1' + x_2 R_2' \end{pmatrix}.$$

Using the Euler relation for P_{x_0} and P_{x_1} and the Euler relation for P

$$dP = x_0 P_{x_0} + x_1 P_{x_1} + x_2 P_{x_2},$$

we find that this last matrix has the desired form. □

Note. For a line every point is an inflection point, because the Hessian matrix is zero.

Lemma 13.5. *Let C be a smooth projective curve of degree d. If $d \geq 3$, C has at least one point of inflection.*

Proof. Let C be a projective curve of degree $d \geq 2$. If $d = 2$, $\mathcal{H}_P$ is a constant polynomial, and it is non-zero, as one can check directly, using the standard form for a conic that we have seen before. Therefore, C has no point of inflection, which agrees with Lemma 13.5. We now assume that $d > 2$, so that $\mathcal{H}_P$ is a homogeneous polynomial of degree $\deg(\mathcal{H}_P) = 3\,(d-2) \geq 3$ by Definition 13.2. Then it follows from the weak version of Bézout's Theorem 9.8 that there is at least a point in $C \cap \{\mathcal{H}_P = 0\}$, which is then an inflection point, since C is smooth. □

Remark 13.6. It also true that if $d \geq 2$, then $\deg(C) \cdot \deg(\mathcal{H}_P) = 3d\,(d-2)$, so C has at most $3d\,(d-2)$ points of inflection. Note that this follows from the strong form of Bézout's theorem, once we know that C and $\{\mathcal{H}_P = 0\}$ have no common component. Unfortunately this does not immediately follow from irreducibility of C, because $3\,(d-2) > d$ as soon as $d \geq 4$, and $\{\mathcal{H}_P = 0\}$ could well be reducible, or have multiple components, for that matter. The way around this is to prove that if every smooth point of an irreducible curve is an inflection point, which would happen if C were a component of $\{\mathcal{H}_P = 0\}$, then C has to be a line, by Lemma 3.32 in Kirwan's book. (Exercise)

We now see that the presence of inflection points guarantees that the equation of every smooth cubic curve can be put in a very simple form.

Proof of Theorem 13.1. Let P be the irreducible homogeneous polynomial of degree three such that $C = \{P = 0\}$. By Lemma 13.5, C has at least one inflection point. Up to projective transformation, we may thus assume that $q = [0,1,0]$ is an inflection point of C, and that the tangent line to C at q is $T_q C = \{x_2 = 0\}$. This implies that

$$P(0,1,0) = 0, \qquad P_{x_0}(0,1,0) = 0, \qquad P_{x_1}(0,1,0) = 0, \qquad P_{x_2}(0,1,0) \neq 0, \qquad \mathcal{H}_P(0,1,0) = 0.$$

We may write the equation of P as

$$P(x_0,x_1,x_2) = Ax_1^3 + Bx_0x_1^2 + Cx_2x_1^2 + Dx_0^2x_1 + Ex_0x_2x_1 + Fx_2^2x_1 + \Phi(x_0,x_2), \qquad A,B,C,D,E,F \in \mathbb{C},$$

where Φ is a homogeneous polynomial of degree three in the variables x_0 and x_2. Since $P(0,1,0) = 0$, $A = 0$. Since $P_{x_0}(0,1,0) = 0$, $B = 0$. Since $P_{x_2}(0,1,0) \neq 0$, $C \neq 0$. Now we would like to say something about the other coefficients. This will involve some manipulation of $\mathcal{H}_P$. By the analogous result of Lemma 13.3, we have

$$x_1^2 \cdot \mathcal{H}_P = (d-1)^2 \cdot \det\begin{pmatrix} P_{x_0,x_0} & P_{x_0} & P_{x_0,x_2} \\ P_{x_0} & \frac{d}{d-1}\cdot P & P_{x_2} \\ P_{x_0,x_2} & P_{x_2} & P_{x_2,x_2} \end{pmatrix} = 4\det\begin{pmatrix} P_{x_0,x_0} & P_{x_0} & P_{x_0,x_2} \\ P_{x_0} & \frac{3}{2}\cdot P & P_{x_2} \\ P_{x_0,x_2} & P_{x_2} & P_{x_2,x_2} \end{pmatrix},$$

so that

$$0 = \mathcal{H}_P(0,1,0) = 4\det\begin{pmatrix} P_{x_0,x_0} & 0 & P_{x_0,x_2} \\ 0 & 0 & P_{x_2} \\ P_{x_0,x_2} & P_{x_2} & P_{x_2,x_2}\end{pmatrix} = -4P_{x_2}(0,1,0)^2 P_{x_0,x_0}(0,1,0),$$

so that $P_{x_0,x_0}(0,1,0) = 0$, since $P_{x_2}(0,1,0)$ is non-zero. Since $P_{x_0,x_0}(0,1,0) = 0$, $D = 0$. We may thus write

$$P(x_0,x_1,x_2) = x_1x_2(Ex_0 + Cx_1 + Fx_2) + \Phi(x_0,x_2) = C\left(x_1 + \frac{Ex_0+Fx_2}{2C}\right)^2 x_2 + \Phi'(x_0,x_2),$$

where $\Phi' \in \mathbb{C}[x_0,x_2]$ is a homogeneous polynomial of degree three. Consider the projective transformation

$$\begin{array}{cccc} \Psi_1 & : & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\ & & [x_0,x_1,x_2] & \longmapsto & \left[x_0, x_1 + \frac{Ex_0+Fx_2}{2C}, x_2\right] \end{array}.$$

Note that Ψ_1 is well-defined because $C \neq 0$. Then I get that the equation of $\Psi_1(C)$ is

$$\Psi_1(C) = \left\{x_1^2x_2 = \Phi''(x_0,x_2)\right\} \subseteq \mathbb{P}^2,$$

where $\Phi'' \in \mathbb{C}[x_0,x_2]$ is a homogeneous polynomial of degree three. By Lemma 6.2, Φ'' is a product of linear factors. Since C and hence $\Psi_1(C)$ are irreducible, x_2 does not divide Φ''. Thus

$$\Phi''(x_0,x_2) = K(x_0 - ax_2)(x_0 - bx_2)(x_0 - cx_2), \qquad K \in \mathbb{C}^*, \qquad a,b,c \in \mathbb{C}.$$

After a suitable diagonal projective transformation Ψ_2, $K = 1$, and the equation of $\Psi_2 \circ \Psi_1(C)$ is

$$\Psi_2 \circ \Psi_1(C) = \left\{x_1^2x_2 = (x_0 - ax_2)(x_0 - bx_2)(x_0 - cx_2)\right\} \subseteq \mathbb{P}^2.$$

Since C is non-singular, a,b,c are distinct. (Exercise: check) The map

$$\begin{array}{cccc} \Psi_3 & : & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\ & & [x_0,x_1,x_2] & \longmapsto & \left[\frac{x_0-ax_2}{b-a}, \eta x_1, x_2\right] \end{array}, \qquad \frac{1}{\eta^2} = (b-a)^3$$

is thus a well-defined projective transformation and

$$\Psi_3 \circ \Psi_2 \circ \Psi_1(C) = \left\{x_1^2x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2)\right\} \subseteq \mathbb{P}^2, \qquad \lambda \in \mathbb{C}.$$

Note that $\lambda \neq 0, 1$ because C is non-singular. This is exactly the form we wanted. $\square$

Lecture 20 Thursday 22/11/18

Lecture 20 is a problem class.

Corollary 13.7. *Let $C \subseteq \mathbb{P}^2$ be a smooth cubic curve. Then C has precisely nine points of inflection.*

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Proof. Let $C = \{P = 0\}$, where P is a homogeneous polynomial of degree three. Define $D = \{\mathcal{H}_P = 0\}$ to be the curve defined by the Hessian of P. Recall that $\mathcal{H}_P$ may have repeated factors. Note that C and D do not have common components. Since C is irreducible, they would have to coincide. By Theorem 13.1 we can assume that C has equation

$$x_1^2x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2),$$

and it is easy to check that $[0,0,1]$ is not an inflection point of this particular cubic. (Exercise) By Bézout's Theorem 11.11, we then have

$$9 = \deg(C) \cdot \deg(D) = \sum_{p \in C \cap D} I(p,C,D).$$

It is thus enough to prove that for each inflection point $p \in C \cap D$, $I(p,C,D) = 1$. By Proposition 12.11, this is equivalent to proving that p is a smooth point of C and D and that $T_pC \neq T_pD$. Let $p \in C \cap D$, then by Theorem 13.1, up to projective transformation, we may assume that $p = [0,1,0]$, and that the equation of C is

$$\left\{x_1^2x_2 = x_0(x_0 - x_2)(x_0 - \lambda x_2)\right\}, \qquad \lambda \in \mathbb{C} \setminus \{0,1\}.$$

Then $T_pC = \{x_2 = 0\}$, while

$$\partial_{x_0}\mathcal{H}_P(0,1,0) = 24, \qquad \partial_{x_1}\mathcal{H}_P(0,1,0) = 0, \qquad \partial_{x_2}\mathcal{H}_P(0,1,0) = 8(\lambda - 1),$$

so that p is a smooth point of D and $T_pD \neq T_pC$. This finishes the proof. $\square$

14 Linear systems

Before moving on to Riemann surfaces, we briefly turn our attention to the way curves behave in families. Here are two basic questions.

- Given two irreducible curves $C = \{P = 0\}$ and $D = \{Q = 0\}$ of the same degree, we can consider a family of curves $C_{[\lambda,\mu]} = \{\lambda P + \mu Q = 0\}$ parametrised by $[\lambda, \mu] \in \mathbb{P}^1$. The curves C and D are special members of the family. How do properties of $C_{[\lambda,\mu]}$ relate to properties of C and D?
- Let $p_1, \dots, p_k$ be points in $\mathbb{P}^2$. When can we find a curve of degree d that passes through $p_1, \dots, p_k$? When can we find a curve C of degree d with $mult_{p_i}(C) = q_i$, for a collection $q_1, \dots, q_k \in \mathbb{N}$?

Example.

- Let $d = 1$, $k = 2$, and $q_1 = q_2 = 1$. Then there is only one projective line.
- Let $d = 1$, $k = 3$, and $q_1 = q_2 = q_3 = 1$. Then it depends on whether p_1, p_2, p_3 are collinear or not.

Example 14.1. Let us consider projective lines in $\mathbb{P}^2$. Recall that the equation of every line L is given by

$$L = \{ax_0 + bx_1 + cx_2 = 0\} \subseteq \mathbb{P}^2, \qquad (a, b, c) \in \mathbb{C}^3, \qquad (a, b, c) \neq (0, 0, 0).$$

Then (a, b, c) and (a', b', c') define the same line if and only if there exists $\lambda \in \mathbb{C}^*$ such that $(a, b, c) = (\lambda a', \lambda b', \lambda c')$, that is when $[a, b, c] = [a', b', c']$ as points of $\mathbb{P}^2$. This shows that

$$\left\{\text{projective lines } L \subseteq \mathbb{P}^2\right\} \cong \mathbb{P}^2.$$

Let $p \in \mathbb{P}^2$ be a point. Then the space of lines passing through $p = [z_0, z_1, z_2]$ for $z_i \in \mathbb{C}$ is given by the points $[a, b, c]$ with $a \cdot z_0 + b \cdot z_1 + c \cdot z_2 = 0$, which are lines $\mathbb{P}^1$ defined by equations $\{ax_0 + bx_1 + cx_2 = 0\} \subseteq \mathbb{P}^2$ in $\mathbb{P}^2$. We may assume that $p = [0, 0, 1]$, after projective transformation. Then the set of lines $L \subseteq \mathbb{P}^2$ that contain p is the set of lines such that $a \cdot 0 + b \cdot 0 + c \cdot 1 = c = 0$. In other words,

$$\left\{\text{projective lines } L \subseteq \mathbb{P}^2 \mid p \in L\right\} \cong \left\{[a, b, 0] \in \mathbb{P}^2\right\} \cong \mathbb{P}^1.$$

We have seen that given two points $p \neq q \in \mathbb{P}^2$ there is a unique line $L_{p,q}$ through p and q. This shows that passage through q gives a single line

$$\left\{\text{projective lines } L \subseteq \mathbb{P}^2 \mid p, q \in L\right\} = \{L_{p,q}\} \cong \mathbb{P}^0.$$

We can parametrise projective curves of degree d in a similar way. More precisely, we have seen that any curve of degree d, $C = \{P = 0\}$ is defined by a homogeneous polynomial $P \in \mathbb{C}[x_0, x_1, x_2]$ of degree d with no repeated factors. Write

$$P(x_0, x_1, x_2) = \sum_{(i,j,k) \in \mathbb{N}^3} a_{i,j,k} x_0^i x_1^j x_2^k,$$

where the only non-zero coefficients $a_{i,j,k}$ correspond to multi-indices (i, j, k) with $i + j + k = d$. The set

$$I_d = \left\{(i, j, k) \in \mathbb{N}^3 \mid i + j + k = d\right\}$$

has precisely $(d + 1)(d + 2)/2$ elements, since the number of (i, j) such that $i + j \leq e \leq d$ is $e + 1$, and the number of (i, j) such that $i+j \leq d$ is $1+\dots+(d + 1) = (d + 1)(d + 2)/2$. We may order the triples $(i, j, k) \in I$ by lexicographic order. Recall from Remark 4.12 that if P and Q are two homogeneous polynomials with no repeated factors, $C = \{P = 0\} = \{Q = 0\}$ precisely when $P = \lambda Q$ for some $\lambda \in \mathbb{C}^*$. This shows that there is a well-defined map

$$\begin{array}{cccc} \Psi_d & : & \{C = \{P = 0\} \mid \deg(C) = d\} & \longrightarrow & \mathbb{P}^{N_d} \\ & & C = \left\{\sum_{(i,j,k) \in \mathbb{N}^3} a_{i,j,k} x_0^i x_1^j x_2^k = 0\right\} & \longmapsto & [a_{i,j,k}] = [a_{0,0,d}, \dots, a_{d,0,0}] \end{array},$$

where $N_d = (d + 1)(d + 2)/2 - 1 = d(d + 3)/2$.

Example.

- $d = 1$ gives $N_1 = (1)(4)/2 = 2$.
- $d = 2$ gives $N_2 = (2)(5)/2 = 5$.
- $d = 3$ gives $N_3 = (3)(6)/2 = 9$.

Remark 14.2. The map Ψ_d is not surjective when $d > 1$. In $\mathbb{P}^{N_d}$, there are points that correspond to P with repeated factors. Indeed, for instance, the point $[1, 0, \dots, 0] \in \mathbb{P}^{N_d}$ corresponds to $P(x_0, x_1, x_2) = x_0^d$, so that it defines a line $L = \{x_0 = 0\}$ with multiplicity d. We will include this case in our description by counting that curve component with multiplicity d. Also, we will set $mult_p(C) = d \cdot mult_p(L) = d$ for $p \in C$, and analogously for curves with more than one repeated component, and not necessarily of degree one, and for intersection numbers of two curves with repeated components, so that for example for the curves $C = \left\{x_0^d = 0\right\}$ and $D = \{x_1^e = 0\}$, we have $I(p, C, D) = de$ in the only point of intersection $p = [0, 0, 1]$. Bézout's theorem continues to hold with these more general definitions.

Definition 14.3. Let $\mathcal{L}_d$ denote the set of curves $C \subseteq \mathbb{P}^2$ defined by a homogeneous polynomial of degree d, possibly with repeated factors. Then Ψ_d defines a bijection

$$\begin{array}{rrcl} \Psi_d : & \mathcal{L}_d & \longrightarrow & \mathbb{P}^{N_d} \\ & C = \left\{\sum_{(i,j,k)\in I} a_{i,j,k} x_0^i x_1^j x_2^k = 0\right\} & \longmapsto & [a_{i,j,k}] \end{array}, \qquad N_d = \frac{d(d+3)}{2}.$$

Example 14.4.

- We have seen that $\mathcal{L}_1 \cong \mathbb{P}^2$.
- Similarly, $\mathcal{L}_2 \cong \mathbb{P}^5$, and $\mathcal{L}_2$ contains smooth conics, reducible conics with no repeated factors, and a subspace $F \subseteq \mathcal{L}_2$ of fake conics that are defined by polynomials with repeated factors, so double lines in $\mathbb{P}^2$. The subset F parametrises lines counted with multiplicity two, $F \cong \mathcal{L}_1 \cong \mathbb{P}^2$.
- Last, $\mathcal{L}_3 \cong \mathbb{P}^9$, and $\mathcal{L}_3$ contains a subspace $F_1 \cong \mathbb{P}^2$ parametrising lines counted with multiplicity three and a subspace $F_2 \cong \mathbb{P}^2 \times \mathbb{P}^2$ parametrising the union of a line counted with multiplicity two and a line counted with multiplicity one.

Remark 14.5.

- We have proved in previous lectures that up to a projective transformation, there are only three kinds of conics. Here the space of conics $\mathcal{L}_2$ on the other hand has infinitely many elements, but in this space we are also remembering the way that the conic sits inside $\mathbb{P}^2$, via its equation. What relates the two different points of view, is that the group of projective transformations $\Psi : \mathbb{P}^2 \to \mathbb{P}^2$, that is usually denoted by $PGL(3, \mathbb{C})$, acts on the space $\mathcal{L}_2$, and there are exactly three orbits for the action, described by the three conics

  $$x_0^2 = 0, \qquad x_0^2 + x_1^2 = 0, \qquad x_0^2 + x_1^2 + x_2^2 = 0,$$

  mentioned after Exercise 30.
- For cubics something similar happens. $PGL(3, \mathbb{C})$ acts on the space $\mathcal{L}_3$, but this time there are infinitely many orbits. The equation of Theorem 13.1 gives one such cubic for every $\lambda \in \mathbb{C} \setminus \{1, 0\}$, and although it is not true that these are all non-isomorphic, for every λ there is a finite number of λ', at most six, for which the two curves are isomorphic, that is one equation can be brought to the other via a projective transformation, so there is still an infinite number of non-isomorphic smooth cubics.

Lemma 14.6. *Let $d, q \in \mathbb{N}$. Let $p \in \mathbb{P}^2$, then*

$$\mathcal{S} = \{C \in \mathcal{L}_d \mid mult_p(C) \geq q\} \cong \mathbb{P}^{N_{d,q}}, \qquad N_{d,q} = \frac{d(d+3)}{2} - \frac{q(q+1)}{2}.$$

Proof. We will show that $\Psi_d(\mathcal{S}) \cong \mathbb{P}^{N_{d,q}}$, where as above, Ψ_d is the map

$$\begin{array}{cccc} \Psi_d & : & \mathcal{L}_d & \longrightarrow & \mathbb{P}^{N_d} \\ & & C = \left\{\sum_{(i,j,k)\in I} a_{i,j,k} x_0^i x_1^j x_2^k = 0\right\} & \longmapsto & [a_{i,j,k}] \end{array}.$$

We denote by $[C] = \Psi_d(C)$ for each $C \in \mathcal{L}_d$. We show that $C \in \mathcal{S}$ if and only if $[C] \in \mathbb{P}^{N_d}$ is in the subspace of solutions of a linear system of $q(q+1)/2$ independent equations, that is a linear system defined by a matrix of size $q(q+1)/2 \times N_d$ of rank $q(q+1)/2$. After projective transformation, we may assume that $p = [0,0,1]$. Let $C \in \mathcal{L}_d$, and denote by P a homogeneous polynomial of degree d with $C = \{P = 0\}$. Let

$$f(x,y) = P(x,y,1) = \sum_{(i,j,k)\in I} a_{i,j,k} x^i y^j.$$

Then $mult_p(P) = mult_{(0,0)}(f) \geq q$ if and only if $a_{i,j,k} = 0$ for all $(i,j,k) \in I$ with $i + j < q$. Denote by $J \subseteq \{0, \dots, N_d\}$ the set of indices in lexicographic order that correspond to the subset

$$\{(i,j,k) \in I \mid i + j < q\}$$

of I. In $[a_{i,j,k}]$, a bunch of these are zero. As there are $q(q+1)/2$ triples (i,j,k) with $i + j + k = d$ and $i + j < q$, (Exercise) the ones that are left are the coordinates of the $\mathbb{P}^{N_{d,q}}$,

$$\Psi_d(\mathcal{S}) \cong \mathbb{P}^{N_{d,q}} = \left\{[C] = [C_0, \dots, C_{N_d}] \in \mathbb{P}^{N_d} \mid \forall j \in J,\ C_j = 0\right\}.$$

□

Example. Let $[x_1, x_2, x_3, x_4] \in \mathbb{P}^3$. Then $x_0 = 0$ and $x_1 = 0$ implies that

$$\begin{array}{cccc} \Psi_d & : & \mathcal{L}_d & \longrightarrow & \mathbb{P}^1 \\ & & [0,0,x_2,x_3] & \longmapsto & [x_2,x_3] \end{array}.$$

Example 14.7. Let $p \in \mathbb{P}^2$, and consider the space

$$\mathcal{S} = \{C \in \mathcal{L}_2 \mid mult_p(C) \geq 2\}.$$

Then, by Lemma 14.6, $\mathcal{S} \cong \mathbb{P}^2$. Recall that an irreducible conic is always smooth, by Corollary 10.4. Therefore, the curves $C \in \mathcal{S}$ parametrise **degenerate conics** of the form $C = L \cup L'$, where L and L' are projective lines containing p. The lines L and L' need not be distinct, as the points of $\mathcal{L}_2$ also parametrise double lines. It is easy to check by direct methods (Exercise) that

$$\{(L, L') \in \mathcal{L}_1 \mid p \in L \cap L'\} \cong \mathbb{P}^2.$$

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Definition 14.8. Let $p_1, \dots, p_k$ be distinct points of $\mathbb{P}^2$, and fix $d, q_1, \dots, q_k \in \mathbb{Z}_{>0}$. Then

$$\mathcal{S} = \{C \in \mathcal{L}_d \mid \forall i = 1, \dots, k,\ mult_{p_i}(C) \geq q_i\}$$

is the **linear system of curves** of degree d going through the points p_i with multiplicity at least q_i for $i = 1, \dots, k$.

This is a copy of $\mathbb{P}^N$ inside $\mathcal{L}_d$. Every condition on p_i gives some linear equations in the coefficients of a polynomial defining C, but these equations might not be independent.

Theorem 14.9. *With the notation of Definition 14.8,*

$$\mathcal{S} \cong \mathbb{P}^N, \qquad N \geq N' = \frac{d(d+3)}{2} - \sum_{i=1}^{k} \frac{q_i(q_i+1)}{2}.$$

The number N' is the expected dimension of $\mathcal{S}$, which is what you get if the equations are independent, while N is its actual dimension.

Proof. Exercise: prove this precisely. □

Corollary 14.10. *Let $p_1, \dots, p_k$ be distinct points of $\mathbb{P}^2$ and $d, q_1, \dots, q_k \in (\mathbb{N}^*)^{k+1}$. If $N' \geq 0$, there exists a curve $C \in \mathcal{L}_d$ that passes through $p_1, \dots, p_k$ with the assigned multiplicities $q_1, \dots, q_k$.*

Proof. This is an immediate consequence of Theorem 14.9. $\square$

Example. Let $k = 1$ and $q_1 = 2$. Then $d(d+3)/2 \geq 2 \cdot 3/2$, so $d(d+3) \geq 6$. Thus $d \geq 2$.

Linear systems of dimension one have a special name.

Definition 14.11. A **pencil of curves** of degree d is a family of plane curves

$$C_{[\lambda_0,\lambda_1]} = \{\lambda_0 \cdot P_1 + \lambda_1 \cdot P_2 = 0\} \subseteq \mathbb{P}^2,$$

where P_1 and P_2 are homogeneous polynomials of degree d with no common factor of $C_1 = \{P_1 = 0\}$ and $C_2 = \{P_2 = 0\}$ in $\mathcal{S}$, and $[\lambda_0, \lambda_1] \in \mathbb{P}^1$. In other words $\left\{C_{[\lambda_0,\lambda_1]}\right\}_{[\lambda_0,\lambda_1]\in\mathbb{P}^1}$ is a family of curves of degree d of dimension one, that is parametrised by $\mathbb{P}^1$. If $\mathcal{S} \cong \mathbb{P}^1$ is a linear system of curves of degree d, it defines naturally a pencil of curves of degree d.

Exercise 40. Let $p_1, \dots, p_4 \in \mathbb{P}^2$ be four non-collinear points. Prove that

$$\mathcal{S} = \{C \in \mathcal{L}_2 \mid p_1, \dots, p_4 \in C\} \cong \mathbb{P}^1.$$

Example 14.12. By Exercise 40 above, $\mathcal{S}$ defines a pencil of conics, the pencil of conics through $p_1, \dots, p_4$. Let $C_1, C_2 \in \mathcal{S}$ be distinct conics in $\mathcal{S}$ and assume that $C_1 = \{P_1 = 0\}$ and $C_2 = \{P_2 = 0\}$ for homogeneous polynomials P_1 and P_2 of degree two. Then every other conic is $\lambda \cdot P_1 + \mu \cdot P_2 = 0$. How many reducible conics are there in $\mathcal{S}$? Since $p_1, \dots, p_4$ are non-collinear, neither C_1 nor C_2 is a line with multiplicity two, and P_1 and P_2 are polynomials with no repeated factor. Further P_1 and P_2 have no common factor. The pencil of conics through $p_1, \dots, p_4$ parametrises the family of curves

$$C_{[\lambda,\mu]} = \{\lambda \cdot P_1 + \mu \cdot P_2 = 0\} \subseteq \mathbb{P}^2.$$

Recall that $C_{[\lambda,\mu]}$ is smooth if and only if it is irreducible, that is if and only if

$$F(\lambda, \mu) = \det\left(M_{[\lambda,\mu]}\right) = \det\left(\lambda \cdot M_{P_1} + \mu \cdot M_{P_2}\right) \neq 0,$$

where $M_{[\lambda,\mu]}, M_{P_1}, M_{P_2}$ are the matrices associated to $C_{[\lambda,\mu]}, C_1, C_2$ as in Exercise 31. The polynomial $F \in \mathbb{C}[\lambda, \mu]$ is homogeneous of degree three in λ and μ, so that it is either identically zero, and one can prove that this does not happen, or it has at least one and at most three roots by Lemma 6.2. Geometrically, if three among the p_i are contained in a line L, all conics through $p_1, \dots, p_4$ have to have L as a component, by Bézout's theorem, and the other line has to pass through the fourth point, but is arbitrary otherwise. In this case the determinant is identically zero, and every conic of the linear system is reducible. Otherwise, there are exactly three reducible conics in the family, given by the three possible pairs of lines passing through $p_1, \dots, p_4$. So the pencil of conics through $p_1, \dots, p_4$ has at least one and at most three reducible elements.

Proposition 14.13. *Let $p_1, \dots, p_8 \in \mathbb{P}^2$ be eight distinct points and suppose that no four of the points lie on a line and no seven on a conic. Then*

$$\mathcal{S} = \{C \in \mathcal{L}_3 \mid p_1, \dots, p_8 \in C\} \cong \mathbb{P}^1.$$

Corollary 14.14. *Let C_1 and C_2 be two cubic curves whose intersection consists of nine distinct points $C_1 \cap C_2 = \{p_1, \dots, p_9\}$. Then any cubic curve $D \subseteq \mathbb{P}^3$ that contains $p_1, \dots, p_8$ passes through p_9.*

Proof. One proves that $p_1, \dots, p_8$ satisfy the assumptions of Theorem 14.9, so that cubics through these points form a pencil. Then if P_1 and P_2 are homogeneous polynomials of degree three with $C_1 = \{P_1 = 0\}$ and $C_2 = \{P_2 = 0\}$, P_1 and P_2 form a basis of the two-dimensional vector space of homogeneous polynomials defining a curve in $\mathcal{S}$. In other words,

$$\mathcal{S} = \left\{P_{[\lambda,\mu]} = \{\lambda \cdot P_1 + \mu \cdot P_2 = 0\} \mid [\lambda, \mu] \in \mathbb{P}^1\right\}.$$

It follows that if $p_9 \in C_1 \cap C_2$, $P_1(p_9) = P_2(p_9) = 0$, then $P_{[\lambda,\mu]}(p_9) = 0$ and $p_9 \in P$ for every $P \in \mathcal{S}$. $\square$

15 Riemann surfaces

In this part of the course, we will see that a smooth projective curve has a single topological invariant that characterises its topology, its genus, and we will introduce the formalism of Riemann surfaces to study this genus. We have seen that a projective line $\mathbb{P}^1$ is homeomorphic to the sphere $S^2 \subseteq \mathbb{R}^3$. We also saw that stereographic projection defined a homeomorphism (Exercise: check that it is in fact a diffeomorphism) between any smooth conic C and $\mathbb{P}^1$, by a bijection between C and lines through p_0, or a bijection to any line L_0 not containing p_0, so that a smooth conic is also homeomorphic to a sphere $S^2 \subseteq \mathbb{R}^3$. If $C \subseteq \mathbb{P}^2$ is a smooth projective plane curve of degree $d \geq 3$, can do a similar projection. We will see that C is a homeomorphic to a sphere with g handles, where g is the genus of C and can be determined in terms of d. In these cases, stereographic projection turns out to be a bit more complicated than for smooth conics. We now look at the case $d = 3$, where the study of stereographic projection gives us a good grasp of the topology of a smooth cubic. Stereographic projection of C with respect to $p_0 \in C$ is defined as follows. Fix a smooth point $p_0 \in C$, recall that $\mathcal{S} = \{L \in \mathcal{L}_1 \mid p_0 \in L\} \cong \mathbb{P}^1$ and we can identify $\mathcal{S}$ with any line $L_0 \subseteq \mathbb{P}^2$ that does not contain p_0 by the bijection $L_0 \in \mathcal{S} \mapsto L \cap L_0 \in L$. Then, the stereographic projection is the surjective map

$$\begin{array}{cccc} \pi & : & C & \longrightarrow & L_0 \\ & & p & \longmapsto & L_{p,p_0} \cap L_0 \end{array},$$

where L_{p,p_0} is the unique line through p and p_0 if $p \neq p_0$ or $T_{p_0}C$ if $p = p_0$, which is not one-to-one anymore. Equivalently, L_{p,p_0} is the line through p_0 and $\pi(p)$. By Proposition 12.14 and Bézout's Theorem 11.11, $\pi^{-1}(\pi(p)) \subseteq C$ consists of $d-1$ points unless L_{p,p_0} is tangent to C at some point in $\pi^{-1}(p)$. π is a $(d-1)$-to-one covering space of $\mathbb{P}^1$.

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Example.

- The covering space

$$\begin{array}{cccc} f & : & S^1 \subseteq \mathbb{C} & \longrightarrow & S^1 \subseteq \mathbb{C} \\ & & z & \longmapsto & z^2 \end{array}$$

  is two-to-one everywhere. Pretend I do not know the source S^1. Could have a disjoint union of two copies of S^1 with the identity map on each copy, which is a trivial covering space. Glue back with a cross to give the S^1 from before, which is a non-trivial covering space.

- Assume $d = 3$. Recall that up to projective transformation, a smooth cubic is defined by an equation of the form

$$C = \left\{P(x_0, x_1, x_2) = x_1^2 x_2 - x_0(x_0 - x_2)(x_0 - \lambda x_2) = 0\right\}, \qquad \lambda \neq 0, 1.$$

  The affine equation is $C_0 = \left\{y^2 = x(x-1)(x-\lambda)\right\}$. If $\pi : C \to \mathbb{P}^1$ is the stereographic projection from the inflection point $p_0 = [0,1,0]$, we see by direct calculation that π is two-to-one and the fibre $\pi^{-1}(\pi(p))$ consists of precisely two points unless $p = p_0 = [0,1,0]$, or $p_0 \in T_pC$, that is $P_{x_1}(p) = 0$, which occurs when p is one of the points $p_0 = [0,1,0]$, $p_1 = [0,0,1]$, $p_2 = [1,0,1]$, or $p_3 = [\lambda, 0, 1]$. The projection π presents C as a two sheeted cover of $\mathbb{P}^1$ ramified at the points $p_0, \ldots, p_3$. Informally, in the neighbourhood of a ramification point, the stereographic projection π behaves like

$$\begin{array}{ccc} C_0 & \longrightarrow & \mathbb{C} = \{y = 0\} \\ (x,y) & \longmapsto & x \end{array}.$$

  Projectively, $[x_0, x_1, x_2] \mapsto [x_0, x_1] \in \mathbb{P}^1$. From this, we construct a topological model of C as follows. Take two spheres $\mathbb{P}^1 \cong S^2$ and slit each sphere twice in identical ways along paths from $\pi(p_0)$ to $\pi(p_1)$ and from $\pi(p_2)$ to $\pi(p_3)$. Open up the slits to make two holes, then turn one of the spheres over and glue it to the other respecting the markings. In this way, you obtain a torus. For any value of λ, we obtain a fixed topological object, the torus $T \cong S^1 \times S^1$. So C is homeomorphic to a torus. Where did λ go? It is in the structure of the Riemann surface. The parameter λ that distinguishes different, non-isomorphic, cubics will influence the complex structure that we obtain on T, that is the structure of complex-analytic manifold.

- Similarly, if
$$C = \left\{ y^2 = \prod_{i=1}^{2g+1} (x - a_i) \right\} \subseteq \mathbb{C}^2,$$
for distinct $a_i \in \mathbb{C}$, the projectivisation of C is a hyper elliptic curve $\overline{C} \subseteq \mathbb{P}^2$. Just like the cubic above, the stereographic projection from $p_0 = [0, 1, 0]$ is a two-sheeted cover ramified at $2g + 2$ points $p_0, \dots, p_{2g+1}$, where $p_i = [a_i, 0, 1]$. The topological model of $\overline{C}$ is then a sphere with g holes.

We will now adopt a more analytic viewpoint on the study of smooth projective plane curves, and see how this complements the algebraic approach. The end goal for this section is to prove the degree-genus formula. We will see that a smooth projective curve $C \subseteq \mathbb{P}^2$ is topologically a compact orientable surface. These are classified by a single number, their Euler characteristic, or equivalently their genus $g \in \mathbb{N}$. An orientable surface of genus g is homeomorphic to a sphere with g holes or handles attached in the surface. The degree-genus formula says that a smooth projective curve $C \subseteq \mathbb{P}^2$ of degree d has genus $g = (d-1)(d-2)/2$. Let us start by defining and studying the basic properties of Riemann surfaces.

Definition 15.1. A **Riemann surface** is

- a Hausdorff topological space S, endowed with
- an **atlas** $\{(U_i, \phi_i)\}_{i \in I}$, where
  - for each $i \in I$, $U_i \subseteq S$ is an open subset,
  - for each $i \in I$, $\phi_i : U_i \to V_i$ is a homeomorphism, where $V_i \subseteq \mathbb{C}$ is an open subset,
  - for each $i \in I$, U_i form an **open cover** of S, that is $S = \bigcup_{i \in I} U_i$, and
  - if $U_i \cap U_j \neq \emptyset$, the **transition function**
  $$\phi_j \circ \phi_i^{-1} : \phi_i (U_i \cap U_j) \to \phi_j (U_i \cap U_j),$$
  where $\phi_i (U_i \cap U_j)$ and $\phi_i (U_i \cap U_j)$ are open subsets of $\mathbb{C}$, is **biholomorphic**, so holomorphic and invertible with holomorphic inverse. This is to talk about holomorphicity of functions defined on S.

For each $i \in I$, U_i is a **coordinate neighbourhood** and $\phi_i : U_i \to \mathbb{C}$ is a **holomorphic coordinate** on U_i.

Remark 15.2. Being locally homeomorphic to an open subset of $\mathbb{C} \cong \mathbb{R}^2$, a Riemann surface is really a surface in the sense of differential geometry, that is a space with two real dimensions.

Remark 15.3. Keeping the notation of Definition 15.1 above, let $f_i : V_i \to f_i (V_i) \subseteq \mathbb{C}$ be a biholomorphic function, with $f_i (V_i)$ open in $\mathbb{C}$. If S is a Riemann surface, and $\{(U_i, \phi_i)\}_{i \in I}$ is an atlas, then for each $i \in I$, $f_i \circ \phi_i$ is also a holomorphic coordinate on $U_i \subseteq S$, so that $\{(U_i, f_i \circ \phi_i)\}_{i \in I}$ is another atlas for S.

We are interested in properties of Riemann surfaces that are independent of the choice of holomorphic coordinates. In some cases, we will use holomorphic functions to change the atlas to more suitable holomorphic coordinates.

Example 15.4.

- Let $S = \mathbb{P}^1$, and consider $\{(U_0, \phi_0), (U_1, \phi_1)\}$, where
$$U_0 = \left\{ [x_0, x_1] \in \mathbb{P}^1 \mid x_0 \neq 0 \right\}, \qquad \begin{array}{cccc} \phi_0 & : & U_0 & \longrightarrow & \mathbb{C} \\ & & [x_0, x_1] & \longmapsto & \dfrac{x_1}{x_0} \end{array},$$
$$U_1 = \left\{ [x_0, x_1] \in \mathbb{P}^1 \mid x_1 \neq 0 \right\}, \qquad \begin{array}{cccc} \phi_1 & : & U_1 & \longrightarrow & \mathbb{C} \\ & & [x_0, x_1] & \longmapsto & \dfrac{x_0}{x_1} \end{array}.$$
Then $S = U_0 \cup U_1$, and ϕ_0 and ϕ_1 are homeomorphic, and on $U_0 \cap U_1 = \{x_0 \neq 0, \ x_1 \neq 0\}$,
$$\begin{array}{cccccc} \phi_0 \circ \phi_1^{-1} : & \mathbb{C} \setminus \{0\} & \to & U_1 \cap U_0 & \to & \mathbb{C} \\ & z & \mapsto & [z, 1] & \mapsto & \dfrac{1}{z} \end{array},$$
which is biholomorphic because $z \neq 0$ on $\mathbb{C}^* = \phi_1 (U_0 \cap U_1)$. So this is an atlas, and $\mathbb{P}^1$ acquires the structure of a Riemann surface.

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- Let $\omega_1, \omega_2 \in \mathbb{C}^*$ be linearly independent over $\mathbb{R}$, that is $\omega_1/\omega_2 \notin \mathbb{R}$, and let
$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \left\{n\omega_1 + m\omega_2 \mid (n,m) \in \mathbb{Z}^2\right\} \subseteq \mathbb{C}$$
be a subgroup of $\mathbb{C}$ with respect to $+$. Define an equivalence relation $\sim$ on $\mathbb{C}$ by
$$z_1 \sim z_2 \qquad \iff \qquad z_1 - z_2 \in \Lambda,$$
and we are going to consider the quotient $X = \mathbb{C}/\Lambda = \mathbb{C}/\sim$. There is a natural quotient map $\pi : \mathbb{C} \to X = \mathbb{C}/\Lambda$ and we endow X with the quotient topology, that is $U \subseteq X$ is open if and only if $\pi^{-1}(U) \subseteq \mathbb{C}$ is open. Note that for all $z \in \mathbb{C}$, z has a unique representative in the **fundamental parallelogram** $\mathcal{P}$ with vertices at $0, \omega_1, \omega_2, \omega_1 + \omega_2$, where opposite sides of $\mathcal{P}$ are identified. Topologically, $X \cong T$, where T is a complex torus. From this description, is not very hard to see that X is compact. If $z \in \mathbb{C}$, then for $\epsilon > 0$ small enough, the open disc $D(z,\epsilon) = \{\omega \in \mathbb{C} \mid |z - \omega| < \epsilon\}$ is homeomorphic to $\pi(D(z,\epsilon))$. Denote ϕ_z the resulting homeomorphism $\phi_z : \pi(D(z,\epsilon)) \to D(z,\epsilon)$. If ϵ is as above, since $\mathbb{C} = \bigcup_{z \in \mathbb{C}} D(z,\epsilon)$ and X is compact, I can choose a finite subcover, so we may choose $z_1, \dots, z_N \in X$ such that
$$X = \bigcup_{1 \le i \le N} \pi(D(z_i,\epsilon)),$$
and use these as coordinate neighbourhoods. Then, $\{(U_i = \pi(D(z_i,\epsilon)) \subseteq \mathbb{C}, \phi_{z_i})\}$ is an atlas for X. Indeed, the only thing left to check is that transition functions are biholomorphic, but this is clear, indeed note that $\phi_j \circ \phi_i^{-1} : z \mapsto z + \lambda$ are all translations by an element $\lambda \in \Lambda$ for all $z \in U_i \cap U_j$. A hint is to pick $x \in U_i \cap U_j$ then its preimages are $x_i \in D(z_i,\epsilon)$ and $x_j \in D(z_j,\epsilon)$, so we have $x_i = \lambda + x_j$ for some $\lambda \in \Lambda$, and λ cannot vary as you move around in $U_i \cap U_j$. These things are exactly the smooth cubics in $\mathbb{P}^2$. Note that $\mathbb{C}/\Lambda$ is a group. Every cubic in $\mathbb{P}^2$ also has a group structure.

Next, we consider what will be for us the fundamental example of Riemann surfaces, smooth plane projective curves.

Lemma 15.5. *Let $C = \{P(x_0,x_1,x_2) = 0\} \subseteq \mathbb{P}^2$ be a smooth plane projective curve. Then C has a natural structure of a Riemann surface.*

To prove this, we need a version of the implicit function theorem for holomorphic functions, and polynomials in particular.

Theorem 15.6 (Implicit function theorem for complex polynomials)**.** *Let $f(x,y) \in \mathbb{C}[x,y]$ be a polynomial and assume that $(a,b) \in \mathbb{C}^2$ satisfies $f(a,b) = 0$ and $\frac{\partial f}{\partial x}(a,b) \ne 0$. Let $C = \{f(x,y) = 0\}$. Then locally around (a,b), C is the graph of a holomorphic function. There are open neighbourhoods $a \in U \subseteq \mathbb{C}$ and $b \in V \subseteq \mathbb{C}$ and a holomorphic function $\lambda : V \to U$ such that $\lambda(b) = a$ and $f(\lambda(y),y) = 0$ for $y \in V$, such that $C \cap (U \times V) = \{(\lambda(y),y) \mid y \in V\}$ is the graph of λ.*

Note. In particular,

- around (a,b), the function λ is uniquely determined, because two functions having the same graph are equal, and
- the functions
$$\begin{array}{ccc} V & \longrightarrow & C \cap (U \times V) \\ y & \longmapsto & (\lambda(y),y) \end{array}, \qquad \begin{array}{ccc} C \cap (U \times V) & \longrightarrow & V \\ (x,y) & \longmapsto & y \end{array}$$
are inverse homeomorphisms.

Example.

- Let
$$x^2 + y^2 - 1 = 0.$$
Use $y = \sqrt{x^2+1}$ around $(0,1)$ and use $x = \sqrt{y^2+1}$ around $(1,0)$.
- If
$$f(x,y) = x - p(y),$$
then $\frac{\partial f}{\partial x} \equiv 1$, so $f(x,y) = 0$, so $x = p(y)$.

Proof of Lemma 15.5. We define an atlas that gives C the structure of a Riemann surface. Recall that $U_i \subseteq \mathbb{P}^2$ were the open covers $U_i = \{x_i \neq 0\}$, we denote by $U_i \cap C$ the affine curve that is the restriction of C to the open set $U_i \cong \mathbb{C}^2$. We show that each $p \in C$ lies in an open coordinate neighbourhood $p \in U_p \subseteq C$ and construct homeomorphisms $\phi_p : U_p \to \phi_p(U_p) \subseteq \mathbb{C}$ such that $\{(U_p, \phi_p)\}_{p \in C}$ form an atlas. Let us first assume that $p = [a, b, 1] \in U_2$, and write

$$C_2 = C \cap U_2 = \{f(x, y) = P(x, y, 1) = 0\}, \qquad x = \frac{x_0}{x_2}, \qquad y = \frac{x_1}{x_2}.$$

Since C is smooth, $p \in C$ is a smooth point, so $(f_x(a, b), f_y(a, b)) \neq (0, 0)$. If $f_y(a, b) \neq 0$, then, by the implicit function theorem, there are neighbourhoods $a \in V \subseteq \mathbb{C}$ and $b \in W \subseteq \mathbb{C}$ and a holomorphic function $g : V \to W$ such that $C_2 \cap (V \times W) = \{(x, g(x)) \mid x \in V\}$. We may further assume, since f_y is continuous that $f_y(x, y) \neq 0$ for all $(x, y) \in C_2 \cap (V \times W)$. Define

$$U_p^{2,y} = C_2 \cap (V \times W), \qquad \phi_p \quad : \quad \begin{array}{ccc} U_p^{2,y} & \longrightarrow & \mathbb{C} \\ [x, y, 1] & \longmapsto & x \end{array}.$$

This is a homeomorphism onto its image with inverse $x \mapsto [x, g(x), 1]$. If on the other hand $f_y(a, b) = 0$, then we have $f_x(a, b) \neq 0$, and we define in the same way using x as a function of y to get

$$U_p^{2,x} = \{(h(y), y) \mid y \in W\}, \qquad \phi_p \quad : \quad \begin{array}{ccc} U_p^{2,x} & \longrightarrow & \mathbb{C} \\ [x, y, 1] & \longmapsto & y \end{array}.$$

Moreover, if $p \in U_1 \setminus U_2$ or $p \in U_0 \setminus (U_1 \cup U_2)$, we also define $U_p^{1,y}, U_p^{1,x}, U_p^{0,y}, U_p^{0,x}$ in the exact same manner. Note that x and y mean different things in different affine charts. We have to check that transition functions between these open sets are biholomorphic. There are three cases.

- If $U_p^{2,y} \cap U_{p'}^{2,y} \neq \emptyset$, then $\phi_p = \phi_{p'}$, so that the transition function is the identity, which is holomorphic.
- If $[a, b, 1] \in U_p^{2,y} \cap U_{p'}^{2,x}$, then $f_x(a, b) \neq 0$ and $f_y(a, b) \neq 0$ and on $U_p^{2,y} \cap U_{p'}^{2,x}$, there are holomorphic functions $g : V \to W$ and $h : W \to V$ such that $y = g(x)$ and $x = h(y)$. By construction, g, h are the transition functions between the holomorphic coordinates $[x, y, 1] \mapsto x$ and $[x, y, 1] \mapsto y$.
- Now we have to look at open subsets that we get from points that live in different affine charts U_i. Assume for example that $U_p^{2,y} \cap U_{p'}^{1,\widetilde{z}} \neq \emptyset$. The other cases are analogous. Write

  $$C \cap U_1 = \{P(\widetilde{x}, 1, \widetilde{z}) = 0\}, \qquad \tilde{x} = \frac{x_0}{x_1}, \qquad \tilde{z} = \frac{x_2}{x_1},$$

  and $C \cap U_2 = \{P(x, y, 1) = 0\}$, where x, y are as above. Then in points of $U_p^{2,y} \cap U_{p'}^{1,\widetilde{z}}$ the two charts are given by

  $$\phi_p \quad : \quad \begin{array}{ccc} U_p^{2,y} & \longrightarrow & \mathbb{C} \\ [x, y, 1] & \longmapsto & x \end{array}, \qquad \phi_{p'} \quad : \quad \begin{array}{ccc} U_{p'}^{1,\widetilde{z}} & \longrightarrow & \mathbb{C} \\ [\widetilde{x}, 1, \widetilde{z}] & \longmapsto & \widetilde{x} \end{array}$$

  respectively. If $y = g(x)$ on $U_p^{2,y}$ and $\widetilde{z} = h(\widetilde{x})$ on $U_{p'}^{1,\widetilde{z}}$ are the two holomorphic functions coming from the implicit function theorem, then the transition function is given by

  $$\begin{array}{ccccc} \phi_{p'} \circ \phi_p^{-1} : \quad \phi_p\left(U_p^{2,y} \cap U_{p'}^{1,\widetilde{z}}\right) & \to & U_p^{2,y} \cap U_{p'}^{1,\widetilde{z}} & \to & \phi_{p'}\left(U_p^{2,y} \cap U_{p'}^{1,\widetilde{z}}\right) \\ x & \mapsto & [x, g(x), 1] = \left[\dfrac{x}{g(x)}, 1, \dfrac{1}{g(x)}\right] & \mapsto & \dfrac{x}{g(x)} = \widetilde{x} \end{array},$$

  since $g(x) \neq 0$, which is biholomorphic where defined. Note that we already know that this function is a homeomorphism, the new bit is the fact that it is holomorphic.

This shows that we may cover C with open sets U_p endowed with homeomorphisms ϕ_p such that the transition functions are biholomorphic. This atlas $\{(U_p, \phi_p)\}_{p \in C}$ gives C the structure of a Riemann surface. □

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Remark 15.7. The proof gives actually something more. For every projective curve C, if $Sing(C)$ denotes the set of singular points of C, then $C \setminus Sing(C) \subseteq C$ is naturally a Riemann surface.

Using atlases, we can define holomorphic functions on a Riemann surface.

Definition 15.8. Let S be a Riemann surface, and $f : S \to \mathbb{C}$ be a continuous function. Then f is **holomorphic** at $x \in S$ if for some coordinate neighbourhood with a chart (U_i, ϕ_i) around $x \in U$, the function

$$f \circ \phi_i^{-1} : \phi_i(U_i) \to \mathbb{C}$$

is holomorphic at $\phi_i(x)$. The function f is **holomorphic** if it is holomorphic at every $x \in S$.

Exercise 41. Check that Definition 15.8 above is independent of the chosen coordinate chart around x. In particular, a function $f : X \to \mathbb{C}$ is holomorphic if and only if the function $f \circ \phi_i^{-1} : \phi_i(U_i) \to \mathbb{C}$ is holomorphic for every i.

Definition 15.9. Let $f : X \to Y$ be a continuous function between Riemann surfaces. Then f is **holomorphic** if, for the atlases $\{(U_i, \phi_i)\}_{i \in I}$ of X and $\{(V_j, \psi_j)\}_{j \in J}$ of Y, the function

$$\psi_j \circ f \circ \phi_i^{-1} : \phi_i(U_i) \to \psi_j(V_j)$$

is holomorphic for all i and j.

Definition 15.10. A holomorphic function $f : X \to Y$ is an **isomorphism**, or **biholomorphism**, of Riemann surfaces if it is bijective, and its inverse f^{-1} is holomorphic.

Remark 15.11. Two different atlases $\mathcal{U} = \{(U_p, \phi_p)\}_{p \in S}$ and $\mathcal{U}' = \left\{\left(U_p', \phi_p'\right)\right\}_{p \in S}$, in the sense of Definition 15.1, on the same topological space S are said to be **compatible** if the transition functions from $\mathcal{U}$ to $\mathcal{U}'$ are holomorphic, or if both identity maps $(S,\mathcal{U}) \to (S,\mathcal{U}')$ and $(S,\mathcal{U}') \to (S,\mathcal{U})$ are holomorphic. Being compatible is an equivalence relation, and one can say that a Riemann surface is a topological space S, together with an equivalence class of compatible atlases.

Exercise 42. Let S and S' be Riemann surfaces in the sense of Remark 15.11, and let $f : X \to Y$ be a function. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two compatible atlases for X, and $\mathcal{V}_1$ and $\mathcal{V}_2$ be two compatible atlases for Y. Show that f is holomorphic with respect to $\mathcal{U}_1$ and $\mathcal{V}_1$ if and only if it is holomorphic with respect to $\mathcal{U}_2$ and $\mathcal{V}_2$.

This says that it does not matter what specific atlas we use on a Riemann surface, in the given equivalence class, to define holomorphic functions.

Exercise 43. Show that every equivalence class of atlases on a topological space S contains exactly one maximal element, with respect to inclusion. Describe this maximal atlas.

Exercise 44. Given an example of a topological space S with two non-equivalent atlases.

Exercise 45. Let S_1, S_2, S_3 be Riemann surfaces, and $f : S_1 \to S_2$ and $g : S_2 \to S_3$ be holomorphic functions. Show that $g \circ f : S_1 \to S_3$ is holomorphic.

Example 15.12. Let $C \subseteq \mathbb{P}^2$ be a smooth plane projective curve and assume that $[0, 0, 1] \notin C$. Then

$$\begin{array}{cccc} f & : & C & \longrightarrow & \mathbb{P}^1 \\ & & [x_0, x_1, x_2] & \longmapsto & [x_0, x_1] \end{array}$$

is not defined on $[0, 0, 1]$, so assume $[0, 0, 1] \notin C$. This function defines a holomorphic map (Exercise) that we will eventually use to prove the degree-genus formula.

16 Reminder about complex analysis

We recall a few results in the theory of holomorphic and meromorphic functions. Let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be a function. For ease of notation, we will always assume that $U \subseteq K$, where K is compact. The function f is **holomorphic** if it satisfies one of the equivalent conditions.

- f is **complex differentiable**. For each point $z_0 \in U$, $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.
- f admits **power series expansions**. For a disc $D(a,r) \subseteq U$, then $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ for all $z \in D(a,r)$.
- **Cauchy's integral formula** holds for f. If a closed disc $\overline{D}(a,r) \subseteq U$, then $f(z) = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(\omega)}{\omega - z} \mathrm{d}\omega$ if $|z-a| < r$, where $\gamma(a,r)$ is the circle with centre a and radius r that bounds $D(a,r)$.

If f and g are holomorphic functions, $f+g, fg, f \circ g$, where defined, are holomorphic, and $1/f$ is holomorphic at a if $f(a) \neq 0$. The **maximum modulus principle** says that if $f : U \to \mathbb{C}$ is holomorphic with U open connected, and if $|f|$ attains a maximum at $z_0 \in D(a,r)$, an interior point of a disc, then f is constant. If f is holomorphic on a **punctured disc** $D^*(a,r) = \{z \in \mathbb{C} \mid 0 < |z-a| < r\}$ about a, then f has an isolated singularity at a, and f expands in a unique way in a **Laurent series** locally at a,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k, \qquad 0 < |z-a| < r, \qquad a_k = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(z)}{(z-a)^{k+1}} \mathrm{d}z.$$

The sum $\sum_{k=-\infty}^{-1} a_k (z-a)^k$ is the **principal part** of the Laurent series. Then f has a **removable singularity** if $a_k = 0$ for all $k < 0$, a **pole of order** m if $a_{-m} \neq 0$ and $a_k = 0$ for all $k < -m$, and an **essential singularity** otherwise. The **identity theorem** says that zeroes of a holomorphic function $f : U \to \mathbb{C}$ are isolated in U unless $f \equiv 0$. If $f(a) = 0$, there exists an open $a \in U$ such that a is the only zero of f in U, so $f(z) \neq 0$ for $z \in U$ such that $z \neq a$. If $f(a) = 0$ and f is holomorphic, there exists $m \in \mathbb{N}^*$ and $r > 0$ such that for all $|z-a| < r$, the power series expansion is

$$f(z) = c_m (z-a)^m + c_{m+1} (z-a)^{m+1} + \dots, \qquad c_m \neq 0.$$

The function f has a **zero of order** m at a, and m is the **order of vanishing** of f at a, which is the multiplicity of f at a if f is a polynomial. In this case, can write $f(z) = (z-a)^m g(z)$ around a, where $g(z)$ is holomorphic and $g(a) \neq 0$. If $f : U \to \mathbb{C}$ and $K \subseteq U$ is a compact subset, then f has finitely many zeroes in K. Let $z_1, \dots, z_N \in K$ be all the zeroes of f in K and $m_1, \dots, m_N$ be their orders, then

$$f(z) = (z-z_1)^{m_1} \dots (z-z_N)^{m_N} g(z), \qquad z \in K,$$

where g is holomorphic on K and $g(z) \neq 0$.

Proposition 16.1. *Let $f : U \to \mathbb{C}$. If f is holomorphic on $D^*(a,r) = \{z \in \mathbb{C} \mid 0 < |z-a| < r\}$ and $|f(z)| \leq M|z-a|^{-m}$ for some $M > 0$ and $m \in \mathbb{N}^*$, then f has at $z = a$ a pole of order at most m.*

A function $f : U \to \mathbb{C}$ is **meromorphic** if there are finitely many points $a_1, \dots, a_N \in U$ such that f is holomorphic on $U \setminus \{a_1, \dots, a_N\}$ and has poles at $a_1, \dots, a_N$. When f is meromorphic, we write $f(a) = \infty$ when a is a pole of f. Poles are isolated by definition, and as above, if $K \subseteq U$ is compact, then f has finitely many poles in K. If $f : D^*(a,\epsilon) \to \mathbb{C}$ is meromorphic, with a pole of order m, then can write $f(z) = g(z)/(z-a)^m$, for some g holomorphic at $z = a$, and $g(a) \neq 0$.

Proposition 16.2. *Let $f : U \to \mathbb{C}$ be meromorphic and $K \subseteq U$ be a compact subset. Let $a_1, \dots, a_N$ be all the poles of f in K and $k_1, \dots, k_N$ be their orders, and let $b_1, \dots, b_M$ be the zeroes of f in K and $l_1, \dots, l_M$ be their orders. Then*

$$f(z) = \frac{(z-b_1)^{l_1} \dots (z-b_M)^{l_M}}{(z-a_1)^{k_1} \dots (z-a_N)^{k_N}} \widetilde{g}(z), \qquad z \in K,$$

for a uniquely determined holomorphic function $\widetilde{g}$ on K.

If f and g are meromorphic functions, then $f+g, fg, f \circ g$, when defined, and $1/f$ are meromorphic. Meromorphic functions form a field.

17 Holomorphic functions and ramification

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We now study holomorphic functions between Riemann surfaces and the concept of ramification. From now on, Riemann surfaces will mostly be compact and connected. Compact is exactly the ones that can be embedded in some $\mathbb{P}^n$. First, we can observe that on a compact Riemann surface, holomorphic functions to $\mathbb{C}$ are uninteresting.

Proposition 17.1. *Let X be a compact and connected Riemann surface and $f : X \to \mathbb{C}$ be a holomorphic function. Then, f is constant.*

Recall that connected means cannot be written as a disjoint union of two open subsets. If X is connected, and $Y \subseteq X$ open and closed, then $Y = \emptyset$ or $Y = X$.

Proof. By compactness of $X, |f| : X \to \mathbb{R}_{\geq 0}$ has a maximum on X, say at a point a. Around a, f has to be constant, and equal to $f(a)$. Let U be an open set with $a \in U$, ϕ_U be a holomorphic coordinate on U with $\phi(a) = 0$, and let $F = f \circ \phi_U^{-1}$ be the holomorphic function representing f on U. Then F has a maximum at an interior point, so that F is constant near a. The set $\{z \in U \mid f(z) = f(a)\}$ is open and non-empty, by the maximum modulus principle. The same argument shows that $\{x \in X \mid f(x) = f(a)\} \subseteq X$ is open in X. But it is also closed, because f is continuous, and we are looking at $f^{-1}(f(a))$ and $\{f(a)\}$ is closed in $\mathbb{C}$. By connectedness, we get $\{x \in X \mid f(x) = f(a)\} = X$, so f is constant. $\square$

Remark 17.2. Note that the situation is very different for non-compact Riemann surfaces. There are plenty of interesting holomorphic functions $f : \mathbb{C} \to \mathbb{C}$, such as polynomials, but also $z \mapsto e^z$, or $z \mapsto \sin(z)$, etc.

A **meromorphic** function f on a Riemann surface X is a map $f : X \to \mathbb{C} \cup \{\infty\}$ such that locally around every point $a \in X$ has a coordinate chart (U, ϕ_U) with $a \in U$, and for which $f \circ \phi_U^{-1} : \phi_U(U) \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function, so only has poles.

Lemma 17.3. *If X is connected, there is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{meromorphic functions} \\ f : X \to \mathbb{C} \cup \{\infty\} \end{array} \right\} \qquad \leftrightsquigarrow \qquad \left\{ \begin{array}{c} \textit{holomorphic functions } \widetilde{f} : X \to \mathbb{P}^1 \\ \textit{that are not constantly } \infty \end{array} \right\}.$$

Fact. One can prove that there always are non-constant meromorphic functions on any Riemann surface.

Proof. We cover $\mathbb{P}^1$ with the two standard charts $U_0 = \{[x_0, x_1] \mid x_0 \neq 0\}$ and $U_1 = \{[x_0, x_1] \mid x_1 \neq 0\}$, and identify $\mathbb{P}^1$ with $\mathbb{C} \cup \{\infty\}$, with $\mathbb{C} = U_1$ and $\infty = [1, 0]$. This gives a bijection

$$\left\{ \text{ functions } f : X \to \mathbb{C} \cup \{\infty\} \right\} \qquad \leftrightsquigarrow \qquad \left\{ \text{ functions } \widetilde{f} : X \to \mathbb{P}^1 \right\}.$$

We just have to prove that f is meromorphic if and only if $\widetilde{f}$ is holomorphic. Let $a \in X$ be such that $f(a) \neq \infty$. Then there is a chart (U, ϕ_U) around a, where f never takes the value ∞, so f restricts to $f : U \to \mathbb{C} \subseteq \mathbb{C} \cup \{\infty\} = U_1 \cup \{\infty\} = \mathbb{P}^1$, and the function $F = f \circ \phi_U^{-1}$ is holomorphic if and only if it is meromorphic. All that is left is to look at points with $f(a) = \infty$. Around such a point a, assume first that f is meromorphic. Then there is a chart (U, ϕ_U) around a such that the function $F = f \circ \phi_U^{-1} : \phi_U(U) \to \mathbb{C} \cup \{\infty\}$ is meromorphic. By shrinking U we may assume that a is the only pole in U and U contains no zeroes of f. To pass to the chart U_0 in $\mathbb{P}^1$, where $\infty \in \mathbb{P}^1$ becomes $0 \in \mathbb{C}$, just have to take $1/F(z)$, which is a well-defined holomorphic map on $\phi_U(U)$, and it is the function that represents $\widetilde{f}$ using the chart U_0 on the target $\mathbb{P}^1$. Therefore $\widetilde{f}$ is holomorphic around a. On the other hand, assume that $\widetilde{f}$ is holomorphic around a. Choose a chart (U, ϕ_U) around a such that a is the only point such that $f(a) = \infty$, $\phi_U(a) = 0$, and U contains no zeroes of $\widetilde{f}$. And let (V, ψ_V) be the chart of $\mathbb{P}^1$ around ∞ given by $V = U_0$, and

$$\begin{array}{cccc} \psi_V & : & U_0 & \longrightarrow & \mathbb{C} \\ & & [x_0, x_1] & \longmapsto & \dfrac{x_1}{x_0} \end{array}.$$

In this chart, $\infty \in \mathbb{P}^1$ corresponds to $0 \in \mathbb{C}$. Then by assumption the function $F = \psi_V \circ \widetilde{f} \circ \phi_U^{-1}$ is holomorphic. Write, after possibly shrinking U, $F(z) = z^n g(z)$, with $g(z) \neq 0$ and holomorphic. Then passing back to the chart $U_1 = \mathbb{C}$ in the target $\mathbb{P}^1$, this becomes $1/F(z) = h(z)/z^n$, where $h(z) = 1/g(z)$ is holomorphic around $z = 0$, and therefore in these charts, f becomes a meromorphic function around a. $\square$

Example 17.4. Recall that a rational function over $\mathbb{C}$ is of the form $f(z) = p(z)/q(z)$, where $p, q \in \mathbb{C}[z]$ are polynomials with no common factor. Then f defines a meromorphic function on $\mathbb{P}^1$, or equivalently a holomorphic function $f : \mathbb{P}^1 \to \mathbb{P}^1$. Explicitly, if $f : \mathbb{C} \to \mathbb{C}$ is rational, the associated map is defined by

$$\begin{array}{cccc} f & : & \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} & \longrightarrow & \mathbb{C} \cup \{\infty\} \\ & & z & \longmapsto & \begin{cases} \dfrac{p(z)}{q(z)} & q(z) \neq 0 \\ \infty & q(z) = 0 \\ \lim_{|z|\to\infty} \dfrac{p(z)}{q(z)} & z = \infty \end{cases} \end{array}.$$

This is holomorphic. (Exercise) Note that if $f = p/q$ is non-constant, $f : \mathbb{P}^1 \to \mathbb{P}^1$ is surjective. Indeed, if f is non-constant, $\max\{\deg(p), \deg(q)\} > 0$. Assume for example that $\deg(p) > \deg(q)$. Other cases are similar. Then if $a \in \mathbb{C}$, $f(z) = a$ if and only if $p(z)/q(z) = a$, if and only if $p(z) - aq(z) = 0$. This is a polynomial equation of degree $\deg(p)$, so has $\deg(p)$ solutions, counted with multiplicity, for $z \in \mathbb{C}$. It remains to check that there is $z \in \mathbb{C}$ for which $f(z) = \infty$, which is also true. But since $\deg(p) > \deg(q)$, $p(z)/q(z)$ behaves like a polynomial of degree $\deg(p) - \deg(q)$ when $z \to \infty$, so f has a pole of order $\deg(p) - \deg(q)$ at ∞, and the other poles of f are solutions to $q(z) = 0$ counted with multiplicity, which are the $\deg(q)$ zeroes of q, so that $f(z) = \infty$ also has $\deg(p)$ solutions counted with multiplicity. In particular, the number of solutions of $\{z \in \mathbb{P}^1 \mid f(z) = a\}$ is independent of $a \in \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$ counted with multiplicity. It is called the **degree** of the holomorphic function f and is equal to $\max\{\deg(p), \deg(q)\}$. In this case, a rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ is biholomorphic if and only if $\deg(f) = 1$, that is if and only if f is a Möbius transformation

$$z \mapsto \frac{az + b}{cz + d},$$

by Example 8.4. Strictly speaking, we have that f is biholomorphic implies that $\deg(f) = 1$, the converse is easy to check.

Exercise 46. Check that the equality $\deg(f) = \# f^{-1}(Q)$, for $Q \in \mathbb{P}^1$, counted with multiplicities, holds when $\deg(p) < \deg(q)$, and when $\deg(p) = \deg(q)$.

Exercise 47. Any holomorphic map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is defined by a rational function.

Remark 17.5. The example of

$$\left\{ \text{ holomorphic maps } \mathbb{P}^1 \to \mathbb{P}^1 \ \right\} \qquad \leftrightsquigarrow \qquad \left\{ \text{ rational functions } \right\}$$

illustrates an important point. We have combined the notions of holomorphy and compactness, both on the source and target, and have found that the result is algebraic, defined by polynomials. This is the first instance of a remarkable phenomenon connecting algebraic and analytic geometries, here Riemann surfaces and algebraic curves, called the **GAGA principle**, from the paper géométrie algébrique et géométrie analytique by J P Serre, that turns out to be very general.

Example 17.6. Recall the holomorphic map $f : C \to \mathbb{P}^1$ defined in Example 15.12. Then f is the meromorphic function

$$\begin{array}{cccc} f & : & C & \longrightarrow & \mathbb{C} \cup \{\infty\} \\ & & [x_0, x_1, x_2] & \longmapsto & \dfrac{x_0}{x_1} \end{array},$$

using the chart $U_1 = \mathbb{C}$ on the target $\mathbb{P}^1$. Note that the only point where x_0/x_1 is undefined is $[0, 0, 1]$, which we are assuming is not contained in C. To check carefully that f is meromorphic, use the charts constructed in Lemma 15.5. (Exercise)

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We could define more complicated holomorphic maps $C \to \mathbb{P}^1$ by taking any polynomial in f, or any ratio of such polynomials. Now we are going to define the concept of ramification of holomorphic maps of Riemann surfaces.

Proposition 17.7. *Let X and Y be compact connected Riemann surfaces, and $f : X \to Y$ a non-constant holomorphic map. Then for every $a \in Y$, $f^{-1}(a) \subseteq X$ is a finite set.*

In the proof we will use the following.

Exercise 48. Prove that for an f as above, we have that its restriction to any non-empty open subset of X is also non-constant. A hint is to suppose that f is constant on a non-empty open $U \subseteq X$, with value $a \in Y$. Show that the union

$$\emptyset \neq U \subseteq \bigcup_{V \subseteq X \text{ open}, \ f(V)=\{a\} \subseteq Y} V \subseteq X$$

is both open and closed in X, and non-empty. Then by connectedness it is equal to X.

Remark 17.8. The conclusion of Exercise 48 above is false for real C^∞ functions, such as bump functions.

Proof of Proposition 17.7. Using compactness of X, the conclusion follows from the fact that $f^{-1}(a)$ is closed and discrete, that is for every $x \in f^{-1}(a)$ there is an open neighbourhood $x \in U \subseteq X$ such that $f(y) \neq a$ for every $y \in U \setminus \{x\}$, so $U \cap f^{-1}(a) = \{x\}$. After I show this, X is compact and a closed discrete subset has to be finite. The discreteness follows from Exercise 48 and the identity theorem in usual complex analysis. Indeed, fix some $x \in f^{-1}(a)$ and take a coordinate neighbourhood (U, ϕ_U) around x. Then the zeroes of the function $g(z) = f(z) - a$ are isolated on U. If $f^{-1}(a)$ is not discrete at x, then there exist $x_n \to x$ of distinct points in $f^{-1}(a) \setminus \{x\}$. Then $g^{-1}(0)$ has an accumulation point in the disc, so $g(z) = 0$. Then $f(z)$ is constant, which cannot happen because of Exercise 48, therefore we can find a small neighbourhood $U' \subseteq U$ where x is the only point with $f(z) = a$ in U'. $\square$

Remark 17.9. Another basic fact about non-constant holomorphic maps between connected compact Riemann surfaces is that they are always surjective. Indeed, every such f is an open map, because of exercise 4 in the third problem sheet, but also closed, since every closed $Z \subseteq X$ is compact, therefore $f(Z) \subseteq Y$ is compact, and since Y is Hausdorff, $f(Z)$ is closed. In particular $f(X) \subseteq Y$ is both open and closed, and non-empty. By connectedness of Y, it follows that $f(X) = Y$.

Example 17.10. Let $a = [a_0, a_1] \in \mathbb{P}^1$, and consider $f : C \to \mathbb{P}^1$, the map defined in Example 15.12. The set

$$f^{-1}(a) = \{[ta_0, ta_1, z] \mid t \in \mathbb{C}^*, \ z \in \mathbb{C}, \ P(ta_0, ta_1, z) = 0\}$$

is almost the set of solutions of the homogeneous polynomial $Q(t, z) = P(ta_0, ta_1, z)$ of degree $\deg(P)$, which is finite as a subset of $\mathbb{P}^1$, by Lemma 6.2, of cardinality $\deg(Q) = \deg(P)$ if we count with multiplicity. Note that there is no solution with $t = 0$, because that would correspond to the point $[0, 0, 1]$, which is not in C.

Now let us look at points where $f' = 0$. Let us look at f in charts instead. Recall that for each $p \in X$, let $p \in U$ be a coordinate neighbourhood, ϕ_U be a holomorphic coordinate on U, and similarly, (V, ψ_V) be a coordinate neighbourhood and holomorphic coordinate with $f(p) \in V$. Then, f is described locally by the holomorphic function

$$F = \psi_V \circ f \circ \psi_U^{-1} : \phi_U\left(U \cap f^{-1}(V)\right) \to \psi_V(V).$$

By the same kind of argument used above, there is a finite number of points at which the derivative F' vanishes.

Exercise 49. Take a moment to think about the derivative f' of the holomorphic function of Riemann surfaces $X \to Y$. Does it make sense? How would you define it? What kind of object is it? A spoiler is that to define it properly, we would need to talk about tangent bundles.

Exercise 50. Let $f : X \to Y$ be a holomorphic function of Riemann surfaces and $x \in X$. Consider charts (U, ϕ_U) and $(U', \phi_{U'})$ around x and (V, ψ_V) and $(V', \psi_{V'})$ around $f(x)$. Let $F = \psi_V \circ f \circ \phi_U^{-1}$ and $G = \psi_{V'} \circ f \circ \phi_{U'}^{-1}$. Show that $F'(\phi_U(x)) = 0$ if and only if $G'(\phi_{U'}(x)) = 0$. These are derivatives of usual holomorphic functions between opens of $\mathbb{C}$.

Therefore it makes sense to say that the derivative of f vanishes at x, and to talk about the number of points $x \in X$ where $f'(x) = 0$.

Definition 17.11. Let $f : X \to Y$ be a non-constant holomorphic map of Riemann surfaces. The point $x \in X$ is a **ramification point** of f if $f'(x) = 0$, that is in local coordinates f is represented by a holomorphic function F with $F'(x) = 0$ at $x \in X$.

Example 17.12. Let $n \in \mathbb{N}$, and consider the holomorphic function $z \mapsto z^n$. Then $0 \in \mathbb{C}$ is the only ramification point of f, unless $n = 1$, in which case there is not any.

In fact, we will show that this Example 17.12 gives a local model for every holomorphic function $X \to Y$ of Riemann surfaces. Let $x \in X$, U a coordinate neighbourhood around x with holomorphic coordinate ϕ_U, and V a coordinate neighbourhood around $f(x) \in Y$ with holomorphic coordinate ψ_V. Up to composition of ϕ_U and ψ_V with suitable biholomorphic functions, we may assume that $\phi_U(x) = 0$ and $\psi_V(f(x)) = 0$. Denote by $F = \psi_V \circ f \circ \phi_U^{-1} : \phi_U\left(U \cap f^{-1}(V)\right) \to \psi_V(V)$ the holomorphic function representing f in the charts U and V. Then $F(0) = 0$, so after possibly shrinking U, $F(z) = z^n \cdot g(z)$ for all $z \in \phi_U(U)$, where $n \in \mathbb{N}$ is the order of F at zero and g is a holomorphic function with $g(0) \neq 0$. The point x is a ramification point of f if and only if $n \geq 2$. In this case we call $n = v_f(x) \in \mathbb{N}$ the **ramification index** of f at $x \in X$. This is a sort of multiplicity of the solution x for the equation $f(z) = a$, where $a = f(x) \in Y$ and z is a coordinate on X. Note that since g is a holomorphic function with $g(0) \neq 0$, I can define a holomorphic $g(z)^{1/n}$ around x, so the function $h : z \mapsto zg(z)^{1/n}$ is holomorphic, and since $h'(0) \neq 0$ (Exercise: check) h is biholomorphic on $D(0, r)$ for some $r > 0$ by the inverse function theorem. Replacing V with $\widetilde{V} = V \cap \psi_V^{-1}(h(D(0, r)))$, and ψ_V with the holomorphic coordinate $\widetilde{\psi}_V = h^{-1} \circ \psi_V$, we obtain a simpler local holomorphic representative for f, $\widetilde{F} = \widetilde{\psi}_V \circ f \circ \phi_U^{-1} : z \mapsto z^n$. We see immediately that no other point of U is a ramification point, and that for every $y \neq f(x) \in \widetilde{V}$ in a small neighbourhood of $f(x)$, $\#\left\{f^{-1}(y) \cap U\right\} = n = v_f(x)$. Let $f : X \to Y$ be a non-constant holomorphic map between connected compact Riemann surfaces.

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Lemma 17.13. *For every $y \in Y$, define the **degree** of f*

$$k(y) = \sum_{x \in f^{-1}(y)} v_f(x) \in \mathbb{N},$$

a finite sum. Then for all $y, y' \in Y$, we have $k(y) = k(y')$, so k is independent of $y \in Y$.

Proof. We have seen that $f^{-1}(y)$ is a finite set so that $k(y)$ is well-defined. As above, for all $x \in X$ with $f(x) = y$, let N_x be an open coordinate neighbourhood of $x \in X$ and V_x be an open coordinate neighbourhood of $y \in Y$ such that f looks like $z \mapsto z^{v_f(x)}$ around x and y, so for all $x' \in N_x \setminus \{x\}$, $v_f(x') = 1$, and for all $y' \in V_x \setminus \{y\}$, $f(x') = y'$ has precisely $v_f(x)$ solutions in N_x. Let $V = \bigcap_{x \in f^{-1}(y)} V_x$ and $U_x = f^{-1}(V) \cap N_x$. Possibly shrinking V, we may assume that the neighbourhoods U_x are disjoint, pairwise, so that $f^{-1}(V) = \bigsqcup_{x \in f^{-1}(y)} U_x$. For all $y' \in V \setminus \{y\}$, $f(x') = y'$ admits $v_f(x)$ solutions in U_x and none of these are ramification points, so that

$$k(y') = \sum_{x' \in f^{-1}(y')} v_f(x') = \sum_{x' \in f^{-1}(y')} 1 = \sum_{x \in f^{-1}(y)} \left(\sum_{x' \in f^{-1}(y'),\ x' \in U_x} 1 \right) = \sum_{x \in f^{-1}(y)} v_f(x) = k(y),$$

since $v_f(x') = 1$ for all such x'. The function $k : Y \to \mathbb{Z}$ is then locally constant, and Y is connected. Therefore k is constant and the degree is well-defined. ☐

Remark. For all but finitely many points of Y, we have $v_f(x) = 1$ for all $x \in f^{-1}(y)$.

Example 17.14. If $f : C \to \mathbb{P}^1$ is as in Example 15.12, for fixed $[a_0, a_1] \in \mathbb{P}^1$,

$$f^{-1}(a_0, a_1) = \{[ta_0, ta_1, z] \mid t \in \mathbb{C}^*,\ z \in \mathbb{C},\ P(ta_0, ta_1, z) = 0\},$$

where $Q(t, z) = P(ta_0, ta_1, z)$ is a homogeneous polynomial of two variables of degree $\deg(P)$. A point $[ta_0, ta_1, z]$ is a ramification point of f over $[a_0, a_1]$ if and only if $\{P(ta_0, ta_1, z) = 0\}$ has repeated roots, if and only if (t_0, z_0) is a repeated root of $Q(t, z)$, and the multiplicity of the root corresponds to the ramification index. (Exercise) We thus have $\deg(f) = \deg(P)$.

The following characterisation of the ramification points and indices of f will be used in the proof of the degree-genus formula. We will admit it without proof.

Proposition 17.15. *Let $f : C \to \mathbb{P}^1$ be the map defined in Example 15.12. The ramification points are those $p \in C$ such that $[0, 0, 1] \in T_pC$, and the ramification index at p is then $I(p, C, T_pC)$.*

18 The degree-genus formula

Let X be a compact connected Riemann surface. We temporarily disregard the complex structure on X, and just consider the underlying topological surface. Since X is a topological manifold, it is orientable. Orientable compact connected topological surfaces are classified up to homeomorphism by their Euler characteristic or equivalently, by their genus g. Each such surface is homeomorphic to a sphere $S^2 \subseteq \mathbb{R}^3$ with g handles attached. Here we collect a few definitions and facts on the topology of connected compact orientable surfaces. These are non-examinable, and you may be familiar with part of it. As a topological surface, A Riemann surface X is always orientable because of the Cauchy-Riemann equations. Informally, this means that X has two sides. For a precise definition of orientability, start from a real vector space $W \cong \mathbb{R}^k$. An **orientation** of W is an equivalence class of bases of W. Two bases are equivalent, or have the same orientation, if and only if the matrix expressing the transition between these bases has positive determinant. An abstract **surface** X, or a two-dimensional real differentiable manifold, is a given by an atlas of charts $\{(U, \phi_U)\}$, where $\phi_U : U \to \mathbb{R}^2$ are homeomorphic and define local coordinates on the coordinate patches U, and the transition functions $\phi_{U'} \circ \phi_U^{-1} : \phi_U(U \cap U') \to \phi_{U'}(U \cap U')$ are differentiable where defined. In general, a transition map $\mathbb{R}^2 \to \mathbb{R}^2$ is non-linear but, at each point, it has a good linear approximation given by its derivative. Here, we view the derivative as a 2×2 matrix of partial differentials that acts on vectors in $\mathbb{R}^2$. Then, X is **orientable** if it has an atlas $\{(U, \phi_U)\}$ such that for all transition functions $\tau = \phi_{U'} \circ \phi_U^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$, the 2×2 Jacobian matrices of partial differentials of τ preserves the orientation, that is has positive determinant at every point where it is defined. In the case of a Riemann surface, the transition functions are holomorphic, and we are looking at linearisations of holomorphic functions, that is we consider biholomorphic transition functions $z \mapsto \tau(z)$ between holomorphic coordinates. The corresponding 2×2 real matrices always have positive determinant. This is a consequence of the holomorphicity of τ. Compact connected orientable surfaces are classified up to a homeomorphism by their Euler characteristic χ, or, equivalently, by their genus. In order to define χ, we need a concept of triangulation. I only include an outline, and state some results without proof. Let X be a compact connected orientable surface. Then, informally a **triangulation** T is obtained by cutting X into a finite number of polygonal triangle regions, called faces, by smooth non-self-intersecting arcs, called edges, joined at vertices, so that a triangulated surface looks like a topological polyhedron. More precisely is the following.

- An **edge** of T is the image in X of a homeomorphic map $\phi : [0,1] \subseteq \mathbb{R} \to \phi([0,1]) \subseteq X$.
- The images of 0 and 1 are **vertices** of T.
- The complement in X of the edges of T consists of finitely many connected components. Each one is required to be homeomorphic to an open disc. The **faces** of T are the closures of these components.

In addition, one requires the following properties.

- Two faces share at most one edge, and each edge belongs to the boundaries of exactly two faces.
- Two edges meet only in one common end-point, a vertex of both, if at all.
- Any vertex has a neighbourhood homeomorphic to an open disc with edges corresponding to rays from the centre to the boundary of that disc. Any two distinct sectors cut out by these rays correspond to distinct faces of T. In particular, at least three edges meet at each vertex.

Remark 18.1. There are variations in the literature on what is allowed or not in a triangulation. These variations all lead to the same Euler characteristic, and can be ignored from our point of view.

Let $V(T), E(T), F(T)$ denote the number of vertices, edges, and faces of a triangulation T of X. A remarkable fact is that the quantity

$$\chi(X, T) = V(T) - E(T) + F(T)$$

only depends on X itself, and not on the choice of triangulation T of X. It is called the **Euler characteristic** of X, and denoted by $\chi(X)$. The **genus** $g(X)$ of X is related to the Euler characteristic by

$$\chi(X) = 2 - 2g(X).$$

As has been already mentioned, $g(X)$ can be visualised as the number of handles that one needs to attach to the sphere in order to obtain X. In particular, $g(X) \geq 0$, and so $\chi(X) \leq 2$.

Example.

- Let $\mathbb{P}^1 \cong S^2$. The surface of a tetrahedron is a triangulation, so $\chi\left(S^2\right) = \chi\left(\mathbb{P}^1\right) = 4 - 6 + 4 = 2$ and $g\left(S^2\right) = g\left(\mathbb{P}^1\right) = 0$.
- The torus $X = \mathbb{C}/\Lambda$ has $\chi(X) = 1 - 3 + 2 = 0$ and $g(X) = 1$.

Remark 18.2. The Euler characteristic $\chi(X)$ is always even. Recall that we only consider orientable surfaces here. An example of topological surface with an odd χ is the quotient $S^2/\{\pm 1\}$ of $S^2 \subseteq \mathbb{R}^3$ by the antipodal map which has $\chi = 1$, but it is not orientable, in particular, not a Riemann surface.

We shall use without proof the following topological result.

Theorem 18.3. *Every topological surface underlying a compact Riemann surface has a triangulation.*

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Therefore, it makes sense to talk about Euler characteristic and genus of a Riemann surface. We now go back to the study of holomorphic maps between compact, connected Riemann surfaces. So far, we have seen how to associate to such a map

- an algebraic quantity, the degree of the map $d = \deg(f)$, which generalises the degree of polynomials,
- analytic quantities, the ramification index $v_f(x) \in \mathbb{N}$ at points $x \in X$, that are defined via the local holomorphic descriptions of the map, where all but finitely many are one, and
- topological quantities, the Euler characteristics $\chi(X)$ and $\chi(Y)$ or genera $g(X)$ and $g(Y)$ of the source and target surfaces.

The next theorem relates these different quantities. We only give a sketch of the proof in simple cases. The proof is not examinable.

Theorem 18.4 (Riemann-Hurwitz formula)**.** *Let $f : X \to Y$ be a non-constant holomorphic map between compact connected Riemann surfaces, and denote $d = \deg(f)$ its degree. Then the Euler characteristics of X and Y satisfy*

$$\chi(X) = d \cdot \chi(Y) - \sum_{p \in X} (v_f(p) - 1),$$

or, equivalently, the genera of X and Y are related by

$$2(g(X) - 1) = 2d \cdot (g(Y) - 1) + \sum_{p \in X} (v_f(p) - 1).$$

Remark 18.5. It follows in particular that the sum $\sum_{p \in X} (v_f(p) - 1)$ is always an even number.

Example 18.6.

- It immediately follows from the Riemann-Hurwitz formula that if there exists a non-constant and holomorphic $f : X \to Y$, then we have $g(X) \geq g(Y)$.
- Consider the holomorphic map
$$f \quad : \quad \begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ z & \longmapsto & z^2 \end{array},$$
which is non-constant. This is degree two, ramified over 0 and ∞, so $v_f(0) = v_f(\infty) = 2$. Therefore, by the Riemann-Hurwitz formula we have $2\left(g\left(\mathbb{P}^1\right) - 1\right) = 2\left(2g\left(\mathbb{P}^1\right) - 2\right) + 2$, and so $g\left(\mathbb{P}^1\right) = 0$.
- Let us go back to the example of the projection
$$f \quad : \quad \begin{array}{ccc} C = \left\{y^2 = x(x-1)(x-\lambda)\right\} & \longrightarrow & \mathbb{P}^1 \\ (x,y) & \longmapsto & x \end{array}, \qquad \lambda \in \mathbb{C} \setminus \{0,1\}$$
from a point of a smooth cubic C. For this map, the degree is $\deg(f) = 2$ and there are exactly four ramification points $x = 0, 1, \lambda, \infty$. The ramification index can only be $v_f(p) = 2$ at each of these points. The Riemann-Hurwitz formula gives $2g(C) - 2 = \deg(f)\left(g\left(\mathbb{P}^1\right) - 2\right) + \sum_{p \in C} (v_f(p) - 1) = 2(-2) + 4$, so $g(C) = 1$. This proves that smooth cubics are homeomorphic to a torus.

Proof. The idea is to consider a triangulation of Y and pull it back to X.

- Suppose first that there is no branching, that is that there is no ramification point. This means that for all $y \in Y$, there is a coordinate neighbourhood $y \in V_y \subseteq Y$ such that $f^{-1}(V_y) = \bigsqcup_{f(x)=y} U_x$ is a disjoint union of d open sets U_x, each of which is mapped biholomorphically onto V_y by $f : U_x \to V_y$, which looks like $z \mapsto z$ in charts. By compactness of Y, can choose finitely many V_y that cover all of Y, $V_1, \dots, V_k$. Choose a triangulation T_Y of Y that is so fine that every face of T_Y is contained in one of the V_i. A triangulation can always be refined by adding vertices in the interior of faces and subdividing. Then, f lifts back to a triangulation T_X of X, and clearly f is d-to-one everywhere, so
$$V(T_X) = d \cdot V(T_Y), \qquad E(T_X) = d \cdot E(T_Y), \qquad F(T_X) = d \cdot F(T_Y),$$
so that $\chi(X) = d \cdot \chi(Y)$ in this case.

- Next, assume that f has a single ramification point $x \in X$, and focus around this x. Denote $v_f(x) = r \le d$ and $y = f(x)$. Everything still holds, except at $y = f(x)$. We may assume, up to subdivision of T_Y that y is a vertex of the triangulation T_Y. Let $x \in U$ and $y \in V$ be coordinate neighbourhoods of $x \in X$ and $y \in Y$. A local holomorphic representative $\psi_V \circ f \circ \phi_U^{-1}$ of f is $z \mapsto z^r$, and this is r-to-one for $z \ne 0$. Lifting the triangulation T_Y back to X will yield a triangulation T_X with d times of everything, so everything still has $\deg(f) = d$ preimages, except for the vertex point y, which has only $d - (r-1)$ preimages in X. Now when computing χ, one has $\chi(X) = d \cdot \chi(Y) - (r-1)$.

This can be turned into a proof in the general case, with some work. ☐

We now use the Riemann-Hurwitz formula to prove the degree-genus formula.

Theorem 18.7 (Degree-genus formula)**.** *Let $C \subseteq \mathbb{P}^2$ be a non-singular projective curve of degree $\deg(C) = d$. Then the genus of C is*
$$g(C) = \frac{(d-1)(d-2)}{2}.$$

Remark 18.8. For $d = 1, 2$ we get $g(C) = 0$, and for $d = 3$ we get $g(C) = 1$. Note that on the other hand $d = 4$ already gives $g(C) = 3$, so that there is no plane curve of genus two, and more generally, of any number not of the form $k(k-1)/2$. There are, though, compact Riemann surfaces of any genus g. If g is not of the form $k(k-1)/2$, it is not possible to embed these Riemann surfaces in $\mathbb{P}^2$.

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Proof. Let $C = \{P = 0\} \subseteq \mathbb{P}^2$, where P is a homogeneous polynomial of degree d. Up to projective transformation, we may assume that $[0, 0, 1] \notin C$ and we consider, as above, the holomorphic map $f : C \to \mathbb{P}^1$ defined in Example 15.12. Then as noted in Example 17.14, the degree of f is n, so that, by the Riemann-Hurwitz formula, the genus of C satisfies
$$g(C) = 1 - d + \frac{1}{2} \sum_{p \in C} (v_f(p) - 1),$$
since we have seen $\deg(f) = \deg(P) = d$. Claim that, after possibly applying a projective transformation, I can assume $v_f(p) = 2$ for all ramification points in C.

- By Proposition 17.15, the ramification points of C are those $p \in C$ such that $[0, 0, 1] \in T_pC$, and the ramification index at p is then precisely $I(p, C, T_pC)$.

- One can show that (Exercise) $I(p, C, T_pC) \ge 3$ precisely when p is an inflection point of C. For a sketch, let $P(x_0, x_1, x_2) = 0$ be the equation of C and $a_0x_0 + a_1x_1 + a_2x_2 = 0$ be the tangent line. Say $a_0 \ne 0$, so $x_0 = -(a_1/a_0)x_1 - (a_2/a_0)x_2$. Plugging this in P,
$$Q(x_1, x_2) = P\left(-\frac{a_1}{a_0}x_1 - \frac{a_2}{a_0}x_2, x_1, x_2\right).$$
Check that p has multiplicity at least three in this, if and only if $\mathcal{H}_P(p) = 0$, the determinant of the Hessian.

- As mentioned in Remark 13.6, C has finitely many points of inflection.

Consider
$$C' = C \cup \bigcup_{q \text{ inflection point of } C} T_q(C) \subseteq \mathbb{P}^2.$$
Pick p_0 in the complement of C' and send that to $[0,0,1]$, and consider the new $f : C \to \mathbb{P}^1$. We may therefore assume, up to projective transformation of $\mathbb{P}^2$, that $[0,0,1]$ does not lie on C', so that $v_f(p) = I(p, C, T_pC) = 2$ for every ramification point of C. Let N be the number of ramification points of f. We have
$$g(C) = 1 - d + \frac{1}{2} \cdot N,$$
so that in order to prove the degree-genus formula, we have to show that $N = d(d-1)$. We use Bézout's theorem to count ramification points. Since
$$\begin{aligned} p \in C \text{ is a ramification point} \qquad &\iff \qquad [0,0,1] \in T_pC \\ &\iff \qquad P_{x_0}(p) \cdot x_0 + P_{x_1}(p) \cdot x_1 + P_{x_2}(p) \cdot x_2 = 0 \\ &\iff \qquad P_{x_2}(p) = 0, \end{aligned}$$
the ramification points of f are the points of intersection of C with the curve $D = \{P_{x_2} = 0\} \subseteq \mathbb{P}^2$. Since C is irreducible and D has degree $d-1 < \deg(C)$, noting that P_{x_2} cannot be identically zero, since $[0,0,1] \notin C$ implies that x_2^d is a monomial in P, otherwise every point of C would be a ramification point, by the strong Bézout theorem,
$$\deg(C) \cdot \deg(D) = d(d-1) = \sum_{p \in C \cap D} I(p, C, D),$$
and the right hand side is equal to $N = \#(C \cap D)$ if and only if $I(p, C, D) = 1$ for all $p \in C \cap D$. By Proposition 12.11, this is the case if and only if, for all $p \in C \cap D$, p is a smooth point of C, p is a smooth point of D, and $T_pC \neq T_pD$, so C and D intersect transversally. We already know that p is a smooth point of C. Let $p = [p_0, p_1, p_2]$, and denote $P_{x_i,x_j} = \frac{\partial^2 P}{\partial x_i \partial x_j}$, recalling that $(p_0, p_1) \neq (0,0)$, since $[0,0,1] \notin C$. The point p is a smooth point of D, because otherwise $(P_{x_2,x_0}(p), P_{x_2,x_1}(p), P_{x_2,x_2}(p)) = (0,0,0)$, and then p would be an inflection point of C, by Definition 13.2. If these vanish, then
$$\mathcal{H}_P(p) = \det \begin{pmatrix} P_{x_0,x_0} & P_{x_0,x_1} & P_{x_0,x_2} \\ P_{x_0,x_1} & P_{x_1,x_1} & P_{x_1,x_2} \\ P_{x_0,x_2} & P_{x_1,x_2} & P_{x_2,x_2} \end{pmatrix} = 0.$$
So p is a smooth point of D. Finally let us prove that $T_pC \neq T_pD$. Otherwise, we would have $[0,0,1] \in T_pD$, because $[0,0,1] \in T_pC$, and therefore $P_{x_2}(p) = P_{x_2,x_2}(p) = 0$. Recall that, by Lemma 13.3,
$$x_1^2 \cdot \mathcal{H}_P = (d-1)^2 \cdot \det \begin{pmatrix} P_{x_0,x_0} & P_{x_0} & P_{x_0,x_2} \\ P_{x_0} & \frac{d}{d-1} \cdot P & P_{x_2} \\ P_{x_0,x_2} & P_{x_2} & P_{x_2,x_2} \end{pmatrix},$$
so that
$$p_1^2 \cdot \mathcal{H}_P(p) = (d-1)^2 \cdot \det \begin{pmatrix} P_{x_0,x_0} & P_{x_0} & P_{x_0,x_2} \\ P_{x_0} & 0 & 0 \\ P_{x_0,x_2} & 0 & 0 \end{pmatrix} = 0.$$
Doing the same for p_0,
$$p_0^2 \cdot \mathcal{H}_P(p) = (d-1)^2 \cdot \det \begin{pmatrix} 0 & P_{x_1} & 0 \\ P_{x_1} & P_{x_1,x_1} & P_{x_1,x_2} \\ 0 & P_{x_1,x_2} & 0 \end{pmatrix} = 0.$$
Since $(p_0, p_1) \neq (0,0)$, p would be an inflection point, a contradiction. We have thus proved that $I(p, C, D) = 1$ for all $p \in C \cap D$, and this finishes the proof of the degree-genus formula. $\square$

19 What's next?

In this course, we have only looked at complex projective plane curves, or complex-analytic manifolds of dimension one.

- Higher dimensional algebraic geometry. For example, surfaces in $\mathbb{P}^3$, threefolds, etc. The classification problem is to try to classify smooth projective varieties up to birational transformations. This is birational geometry.

- Moduli spaces. Classify smooth curves up to isomorphism. One discrete invariant is the genus. For $g = 0$, there is only $\mathbb{P}^1$. For $g = 1$, cubics, there is an infinite family

  $$y^2 = x(x-1)(x-\lambda), \qquad \lambda \in \mathbb{C} \setminus \{0, 1\}.$$

  The space of curves of genus one is isomorphic to $\mathbb{C}$. The coordinate is the j-invariant. The moduli space of isomorphism classes of genus g,

  $$M_g = \{\text{isomorphism classes of genus } g \text{ curves}\},$$

  has a natural structure of an algebraic variety, defined by polynomial equations in some $\mathbb{P}^N$.

- Arithmetic geometry. Work over other fields, such as $\mathbb{Q}$, or over rings, such as $\mathbb{Z}$,

  $$2x^2 + 3y^2 = 100.$$

  Characteristic zero talks to characteristic p.

- Computational or combinatorial algebraic geometry. Algorithms to find nice sets of equations to

  $$\begin{cases} p_1 = 0 \\ \vdots \\ p_k = 0 \end{cases}.$$