

# M4P58 Modular Forms

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Autumn 2019

**Syllabus**

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## 0 Introduction

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The following are textbooks.

- Serre, A course in arithmetic, 1973
- J Shurman and F Diamond, A first course in modular forms, 2005

Let

$$f = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 = \sum_{n=1} b_n q^n = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \dots,$$

and let  $a_n$  be the number of solutions modulo  $n$  to the elliptic curve

$$E = \{(x, y) \in \mathbb{Z} \mid y^2 + y = x^3 - x^2 - 10x - 20\}.$$

- Modulo 2, there are  $a_2 = 4$  solutions  $(0, 0), (0, 1), (1, 0), (1, 1)$ .
- Modulo 3, there are  $a_3 = 4$  solutions  $(1, 0), (1, -1), (-1, 0), (-1, -1)$ .
- Modulo 5, there are  $a_5 = 4$  solutions  $(0, 0), (0, -1), (1, 0), (-1, -1)$ .
- Modulo 7, there are  $a_7 = 9$  solutions  $(1, 3), (2, 2), (2, -3), (-1, 1), (-1, -2), (-2, 1), (-2, -2), (-3, 1), (-3, -2)$ .

If  $p \neq 11$ , then

$$a_p - p = -b_p.$$

The following are some questions.

- What is the relationship between  $E$  and  $f$ ?
- Can we find similar relationships for other  $E$ ?
- How does one prove something like this?

Let

$$\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\} \subseteq \mathbb{C}.$$

Then  $\mathbb{H}$  has an action of

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

Modular forms are complex functions on  $\mathbb{H}$  with a high degree of symmetry. These functions are symmetric under the action of large discrete subgroups of  $\mathrm{SL}_2(\mathbb{R})$ , in particular

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} \subseteq \mathrm{SL}_2(\mathbb{R}).$$

Why are these interesting to number theorists? Power series expansions often involve expressions of interest to number theorists. For example,

- Bernoulli numbers,
- divisor functions  $\sigma_k(n) = \sum_{d|n} d^k$ ,
- number of points on elliptic curves, and
- traces of Galois representations.

# 1 Modular forms of level one

## 1.1 Modular functions and forms

### 1.1.1 Modular actions

Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

Then  $\mathrm{SL}_2(\mathbb{R})$  acts on  $\mathbb{C} \cup \{\infty\}$  by

$$\gamma \cdot z = \begin{cases} \frac{az+b}{cz+d} & z \neq -\frac{d}{c} \\ \infty & z = -\frac{d}{c} \end{cases} \quad \gamma \cdot \infty = \frac{a}{c}.$$

One checks that this gives a bijection from  $\mathbb{C} \cup \{\infty\}$  to  $\mathbb{C} \cup \{\infty\}$ , where inverse is given by the inverse matrix

$$\gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and  $\gamma \cdot (\gamma' \cdot z) = \gamma\gamma' \cdot z$ . One obtains a left action of  $\mathrm{SL}_2(\mathbb{R})$  on  $\mathbb{C} \cup \{\infty\}$ . An observation is

$$\mathrm{Im} \gamma z = \mathrm{Im} \frac{az+b}{cz+d} = \mathrm{Im} \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{\mathrm{Im}(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{(ad-bc)\mathrm{Im} z}{|cz+d|^2}.$$

In particular, if  $\gamma \in \mathrm{SL}_2(\mathbb{R})$ , then

$$\mathrm{Im} \gamma z = \frac{\mathrm{Im} z}{|cz+d|^2}.$$

So  $\mathrm{SL}_2(\mathbb{R})$  preserves  $\mathbb{H} \cup \{\infty\}$ . More generally, if  $\gamma \in \mathrm{GL}_2(\mathbb{R})$ , then

$$\mathrm{Im} \gamma z = \frac{\det \gamma \mathrm{Im} z}{|cz+d|^2}.$$

So

$$\mathrm{GL}_2(\mathbb{R})_+ = \{\gamma \in \mathrm{GL}_2(\mathbb{R}) \mid \det \gamma > 0\}$$

preserves  $\mathbb{H} \cup \{\infty\}$ . Define

$$\begin{aligned} f|_{k,\gamma} : \mathbb{H} &\longrightarrow \mathbb{C} \\ z &\longmapsto \det \gamma^{k-1} f(\gamma z) (cz+d)^{-k}, \end{aligned} \quad f : \mathbb{H} \rightarrow \mathbb{C}, \quad \gamma \in \mathrm{GL}_2(\mathbb{R})_+, \quad k \in \mathbb{Z},$$

where  $\det \gamma^{k-1}$  is the fudge factor, which is one for  $\gamma \in \mathrm{SL}_2(\mathbb{R})$ , and  $(cz+d)^{-k}$  is the twisted action on functions. Check that

$$f|_{k,\mathrm{id}} = f, \quad \left(f|_{k,\gamma}\right)|_{k,\gamma'} = f|_{k,\gamma'\gamma}.$$

This gives, for each  $k$ , a left action of  $\mathrm{GL}_2(\mathbb{R})_+$  on functions  $\mathbb{H} \rightarrow \mathbb{C}$ , a **modular action of weight  $k$** . A modular form of weight  $k$  will be a sufficiently nice function  $f : \mathbb{H} \rightarrow \mathbb{C}$  such that  $f|_{k,\gamma} = f$  for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ . That is, for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  and all  $z \in \mathbb{H}$ ,

$$f(\gamma z) (cz+d)^{-k} = f(z), \quad \implies \quad f(\gamma z) = f(z) (cz+d)^k,$$

the **modular transformation law of weight  $k$** . The following are some observations.

- Let  $k = 0$ . Then constant functions satisfy  $f(\gamma z) = f(z)$ . It will turn out that all functions of weight zero are constant.
- Let  $k$  be odd, and  $\gamma = -\mathrm{id}$ . Then  $\gamma z = z$  for all  $z$  and  $cz+d = -1$ , so  $f(\gamma z) = f(z) (cz+d)^k$  gives  $f(z) = f(z) (-1)^k$ , so  $f(z) = -f(z)$ , so  $f(z) = 0$  for all  $z$ . So no functions  $f : \mathbb{H} \rightarrow \mathbb{C}$  satisfy the modular transformation law of weight  $k$ , for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , when  $k$  is odd.

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### 1.1.2 Review of complex analysis

Let  $f : U \rightarrow \mathbb{C}$ , for  $U \subseteq \mathbb{C}$  open, and let  $p \in U$ .

**Definition 1.1.1.**  $f$  is **holomorphic** at  $p$  if

$$f'(p) = \lim_{\epsilon \rightarrow 0, \epsilon \in \mathbb{C}} \frac{f(p' + \epsilon) - f(p')}{\epsilon}$$

exists for all  $p'$  in a neighbourhood of  $p$ .

**Proposition 1.1.2.**  $f$  is holomorphic at  $p$  implies that  $f$  is continuous.

**Proposition 1.1.3.**  $f$  is holomorphic at  $p$  implies that  $f$  is infinitely differentiable at  $p$ , that is  $f^{(n)}(p)$  exists for all  $n \geq 0$ . Moreover, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (z-p)^n = f(p) + f'(p)(z-p) + \frac{f''(p)}{2} (z-p)^2 + \dots,$$

for all  $z$  in a neighbourhood of  $p$ .

**Corollary 1.1.4.** If  $f$  is holomorphic and not identically zero on an open set  $U$ , then the zeroes of  $f$  are isolated on  $U$ .

More generally is the following.

**Definition 1.1.5.**  $f$  is **meromorphic** at  $p$  if there exists a neighbourhood  $U$  of  $p$  and  $g, h : U \rightarrow \mathbb{C}$  holomorphic on  $U$  such that  $f = g/h$  on  $U \setminus \{p\}$ . Such an  $f$  has a **Laurent series expansion** at  $p$ ,

$$f(z) = \sum_{i=-N}^{\infty} c_i (z-p)^i.$$

The smallest  $i$  such that  $c_i \neq 0$  is denoted by  $\text{ord}_p f$ , the **order of vanishing** of  $f$  at  $p$ .

- If  $\text{ord}_p f = -n$  for  $n > 0$ , we say  $f$  has a **pole of order  $n$** .
- If  $\text{ord}_p f = n$  for  $n > 0$ , we say  $f$  has a **zero of order  $n$** .

**Proposition 1.1.6.**

- $\text{ord}_p fg = \text{ord}_p f + \text{ord}_p g$ .
- $\text{ord}_p (f + g) \geq \min \{\text{ord}_p f, \text{ord}_p g\}$ , with equality if  $\text{ord}_p f \neq \text{ord}_p g$ .

If  $f$  is holomorphic on  $U \setminus \{p\}$  for  $U$  a neighbourhood of  $p$ , then  $f$  may or may not be meromorphic at  $p$ .

**Example.**  $f(z) = e^{-1/z^2}$  is holomorphic on  $\mathbb{C} \setminus \{0\}$ , but not meromorphic at zero.

**Theorem 1.1.7.** Let  $f$  be holomorphic on  $U \setminus \{p\}$ , and there exists  $n > 0$  such that

$$\lim_{x \rightarrow p} (x-p)^n f(x)$$

exists. Then  $f$  is meromorphic on  $U$ , and  $\text{ord}_p f \geq -n$ .

### 1.1.3 Modular functions

**Definition 1.1.8.**  $f : \mathbb{H} \rightarrow \mathbb{C}$  is a **weakly modular function of weight  $k$**  if

- $f$  is meromorphic on  $\mathbb{H}$ , and
- $f$  satisfies the modular transformation law of weight  $k$ .

Consider

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so  $\gamma z = z + 1$  and  $cz + d = 1$ . The modular transformation law gives  $f(z + 1) = f(z)$ . Let

$$D = \{q \mid |q| < 1\}.$$

Can define a function

$$\begin{aligned} g : D \setminus \{0\} &\longrightarrow \mathbb{C} \\ q &\longmapsto f\left(\frac{\log q}{2\pi i}\right), \end{aligned}$$

that is  $f(z) = g(e^{2\pi iz})$  for  $z \in \mathbb{H}$ , where  $g$  is holomorphic or meromorphic on  $\{z \mid 0 < |z| < 1\}$  if and only if  $f$  is holomorphic or meromorphic on  $\mathbb{H}$ .

**Definition 1.1.9.**  $f : \mathbb{H} \rightarrow \mathbb{C}$  is a **modular form of weight  $k$**  if

1.  $f$  satisfies the modular transformation law of weight  $k$ ,
2.  $f$  is holomorphic on  $\mathbb{H}$ , and
3.  $f$  is holomorphic at  $\infty$ , so the function  $g : D \setminus \{0\} \rightarrow \mathbb{C}$ , which is holomorphic on  $D \setminus \{0\}$  by 2, extends to a holomorphic function on  $D$ .

Then  $q \rightarrow 0$  in  $D$  if and only if  $\text{Im } z \rightarrow +\infty$ . Then 3 means  $g(q)$  is bounded as  $q \rightarrow 0$  so  $f(z)$  is bounded as  $\text{Im } z \rightarrow +\infty$ . For  $f$  satisfying 3,  $g : D \setminus \{0\} \rightarrow \mathbb{C}$  has a series expansion

$$g(q) = \sum_n a_n q^n = a_0 + a_1 q + \dots$$

in  $q = e^{2\pi iz}$ . We call this the  **$q$ -expansion** for  $f$ .

**Definition 1.1.10.**  $f : \mathbb{H} \rightarrow \mathbb{C}$  is a **meromorphic modular form of weight  $k$**  if the same conditions 1 to 3 hold, but with holomorphic weakened to meromorphic.

**Note.** If  $f$  is only meromorphic at  $\infty$  then a finite number of negative powers of  $q$  can appear.

**Example.**

- The **modular discriminant**

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

is a modular form of weight 12.

- The  **$j$ -invariant**

$$j(z) = \frac{1}{q} + 744 + 196844q + 21493760q^2 + \dots$$

is a meromorphic modular form of weight 0.

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### 1.1.4 Lattice functions

How can we construct modular forms?

**Definition 1.1.11.** A **lattice** in  $\mathbb{C}$  is an abelian subgroup of  $\mathbb{C}$  of the form  $\mathbb{Z}w_1 + \mathbb{Z}w_2$ , where  $w_1, w_2 \in \mathbb{C}$  are  $\mathbb{R}$ -linearly independent. More generally if  $V$  is an  $\mathbb{R}$ -vector space, a **lattice**  $L$  in  $V$  is a discrete abelian subgroup of  $V$  that spans  $V$  over  $\mathbb{R}$ . For  $L \subseteq \mathbb{C}$  a lattice and  $\lambda \in \mathbb{C}^\times$ , let

$$\lambda L = \{\lambda x \mid x \in L\} \subseteq \mathbb{C}.$$

We say that  $L$  and  $\lambda L$  are **homothetic**. For  $z \in \mathbb{H}$ , let

$$L_{z,1} = \mathbb{Z} + \mathbb{Z}z = \{az + b \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}.$$

A question is when is  $L_{z,1}$  homothetic to  $L_{z',1}$ , and what is a homothety factor?

- Suppose  $L_{z,1} = \lambda L_{z',1}$ . Then there exist  $a, b, c, d$  such that  $\lambda z' = az + b$  and  $\lambda = cz + d$ , so

$$\begin{pmatrix} \lambda z' \\ \lambda \end{pmatrix} = \gamma \begin{pmatrix} z \\ 1 \end{pmatrix}. \quad (1)$$

On the other hand there exist  $a', b', c', d'$  such that  $z = a'\lambda z' + b'\lambda$  and  $1 = c'\lambda z' + d'\lambda$ , so

$$\gamma' \begin{pmatrix} \lambda z' \\ \lambda \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix}. \quad (2)$$

(1) and (2) imply that

$$\gamma' \gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix},$$

so  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ . Moreover (1) implies that  $z' = (az + b) / (cz + d)$ .

- Conversely, if  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , then  $\gamma z = (az + b) / (cz + d)$ , so

$$L_{\gamma z,1} = (cz + d)^{-1} L_{az+b, cz+d}.$$

But certainly  $L_{az+b, cz+d} \subseteq L_{z,1}$ . On the other hand if  $\gamma'$  is inverse to  $\gamma$ ,

$$\begin{pmatrix} z \\ 1 \end{pmatrix} = \gamma' \gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \gamma \begin{pmatrix} az + b \\ cz + d \end{pmatrix} = \begin{pmatrix} a'(az + b) + b'(cz + d) \\ c'(az + b) + d'(cz + d) \end{pmatrix},$$

so  $z \in L_{az+b, cz+d}$  and  $1 \in L_{az+b, cz+d}$ . So  $L_{az+b, cz+d} = L_{z,1}$ , so  $L_{\gamma z,1} = (cz + d)^{-1} L_{z,1}$ .

**Definition 1.1.12.** A **lattice function of weight  $k$**  is a function  $F : \{\text{lattices in } \mathbb{C}\} \rightarrow \mathbb{C}$  such that

$$F(\lambda L) = \lambda^{-k} F(L),$$

for all lattices  $L$ . Given such an  $F$ , can define

$$\begin{aligned} f &: \mathbb{H} \longrightarrow \mathbb{C} \\ z &\longmapsto F(L_{z,1}). \end{aligned}$$

If  $F$  has weight  $k$ , then

$$f(\gamma z) = F(L_{\gamma z,1}) = F((cz + d)^{-1} L_{z,1}) = (cz + d)^k F(L_{z,1}) = (cz + d)^k f(z).$$

## 1.2 Eisenstein series

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**Definition 1.2.1.** For  $L \in \mathbb{C}$ , define the **Eisenstein series**

$$G_k(L) = \sum_{w \in L, w \neq 0} \frac{1}{w^k}, \quad g_k(z) = G_k(L_{z,1}) = \sum_{\substack{m=-\infty \\ (m,n) \neq (0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k}.$$

Then

$$G_k(\lambda L) = \sum_{w' \in \lambda L, w' \neq 0} \frac{1}{w'^k} = \sum_{w \in L, w \neq 0} \frac{1}{(\lambda w)^k} = \lambda^{-k} G_k(L).$$

**Corollary 1.2.2.**  $g_k$  satisfies the modular transformation law of weight  $k$ .

The following are some questions.

- Does  $G_k$ , or  $g_k$ , converge?
- Is  $g_k$  holomorphic or meromorphic on  $\mathbb{H}$ ?
- Is  $g_k$  holomorphic at  $\infty$ ?
- What is the  $q$ -expansion of  $g_k$ ?

### 1.2.1 Convergence and holomorphy on $\mathbb{H}$

**Definition 1.2.3.** Let  $U \subseteq \mathbb{C}$  be open. A sequence of functions  $f_n : U \rightarrow \mathbb{C}$  **converges uniformly on compact sets** to  $f$  if for all  $C \subseteq U$  compact and all  $\epsilon > 0$ , there exists  $N \in \mathbb{Z}$  such that for all  $n > N$ ,

$$|f(z) - f_n(z)| < \epsilon, \quad z \in C.$$

**Theorem 1.2.4.** A uniform limit of holomorphic functions is holomorphic. If  $f_n$  converges to  $f$  uniformly on compact sets and  $f_n$  is holomorphic on  $U$ , then  $f$  is holomorphic on  $U$ .

**Theorem 1.2.5.** Let  $k \geq 4$ . The series  $g_k(z)$  converges absolutely and uniformly on compact subsets of  $\mathbb{H}$ .

*Proof.* Let

$$P_{z,r} = \{az + b \mid a, b \in \mathbb{R}, \max(|a|, |b|) = r\} \subseteq \mathbb{C},$$

so  $P_{z,r} = rP_{z,1}$ , and there are  $8r$  points on  $P_{z,r} \cap L_{z,1}$ . Then

$$g_k(z) = \sum_{r=1}^{\infty} \sum_{w \in L_{z,1} \cap P_{z,r}} \frac{1}{w^k}.$$

The function  $z \mapsto |z|$  attains a non-zero minimum  $\delta(z)$  on  $P_{z,1}$ , so on  $P_{z,1}$ , have  $|z| > \delta(z)$ , so  $1/|z|^k < 1/\delta(z)^k$ . On  $P_{z,r}$ , have  $|z| > r\delta(z)$ , so  $1/|z|^k < 1/r^k \delta(z)^k$ . Let  $C \subseteq \mathbb{H}$  be compact. Then  $z \mapsto \delta(z)$  is a continuous function on  $C$  and attains a minimum  $\delta_C$ . For all  $z \in C$  and all  $w \in P_{z,r}$ , get  $|w| > r\delta_C$ , so

$$\frac{1}{|w|^k} < \frac{1}{r^k \delta_C^k}.$$

Thus for  $z \in C$ ,  $g_k(z)$  is dominated by

$$\sum_{r=1}^{\infty} \frac{8r}{r^k \delta_C^k} = \frac{8}{\delta_C^k} \sum_{r=1}^{\infty} \frac{1}{r^{k-1}},$$

which converges absolutely for  $k \geq 4$ . □

**Corollary 1.2.6.**  $g_k(z)$  is holomorphic on  $\mathbb{H}$ .



### 1.2.2 $q$ -expansion and holomorphy at $\infty$

The idea is to understand series of the form

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k}.$$

**Theorem 1.2.7.** *A bounded holomorphic function on all of  $\mathbb{C}$  is constant.*

**Lemma 1.2.8.**

1.

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

2.

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

*Proof.*

1. The right hand side converges absolutely and uniformly on compact subsets of  $\mathbb{C} \setminus \mathbb{Z}$ , so the right hand side is holomorphic on  $\mathbb{C} \setminus \mathbb{Z}$ . Locally around  $z = n$ , the series looks like

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \cdots + \frac{1}{(z-n+1)^2} + \frac{1}{(z-n)^2} + \frac{1}{(z-n-1)^2} + \cdots = \frac{1}{(z-n)^2} + h_1(z),$$

where  $h_1(z)$  is holomorphic in a neighbourhood of  $z = n$ . Similarly, the left hand side is meromorphic on  $\mathbb{C}$ , and the Laurent series near  $z = n$  is

$$\frac{\pi^2}{\sin^2 \pi z} = \pi \left( \frac{1}{\pi^2 (z-n)^2} + \frac{1}{3} + \frac{1}{15} \pi^2 (z-n)^2 + \cdots \right) = \frac{1}{(z-n)^2} + h_2(z),$$

where  $h_2(z)$  is a holomorphic function. So the difference

$$g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} - \frac{\pi^2}{\sin^2 \pi z}$$

is meromorphic on  $\mathbb{C}$  and holomorphic on  $\mathbb{C} \setminus \mathbb{Z}$ , and the Laurent expression around  $z = n$  is

$$g(z) = \frac{1}{(z-n)^2} + h_1(z) - \left( \frac{1}{(z-n)^2} + h_2(z) \right) = h_1(z) - h_2(z),$$

so  $g(z)$  is holomorphic at  $z = n$  for all  $n$ . Consider  $t \rightarrow \pm\infty$  for  $z = a + it$ . The right hand side is

$$R = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=a-N}^{a+N} \frac{1}{(z-n)^2} + \sum_{n=-\infty}^{a-N-1} \frac{1}{(z-n)^2} + \sum_{n=a+N+1}^{\infty} \frac{1}{(z-n)^2} = R_0 + R_- + R_+,$$

where  $R_0$  has finitely many terms that converge to less than  $\epsilon/2$  as  $t \rightarrow \pm\infty$  and  $R_- + R_+ < \epsilon/2$  for  $N \gg 0$  independent of  $t$ , so  $R < \epsilon$  converges to zero. Similarly, the left hand side is

$$\left| \frac{\pi^2}{\sin^2 \pi z} \right| = \left| \frac{2\pi^2}{e^{\pi i z} - e^{-\pi i z}} \right| \rightarrow 0,$$

so  $\lim_{t \rightarrow \infty} g(a + it) = 0$ . Moreover,  $g(z+1) = g(z)$  for all  $z$ . Then

$$S = \{z \in \mathbb{C} \mid n-1 \leq \operatorname{Re} z \leq n, -N \leq \operatorname{Im} z \leq N\}, \quad n \in \mathbb{Z}$$

is compact, so  $|g(z)|$  attains a maximum in  $S$ , so  $g(z)$  is bounded in  $S$ . Since  $g(z)$  is also bounded in  $R_- + R_+$ ,  $g(z)$  is bounded in  $\mathbb{C}$ , so  $g$  is constant. Since  $\lim_{t \rightarrow \infty} g(a + it) = 0$ ,  $g = 0$ .

2. Check that the right hand side converges absolutely and uniformly on compact subsets of  $\mathbb{C} \setminus \mathbb{Z}$ , so the right hand side is meromorphic on  $\mathbb{C} \setminus \mathbb{Z}$ . Similarly, the left hand side is also meromorphic on  $\mathbb{C} \setminus \mathbb{Z}$ . Comparing derivatives,

$$-\frac{\pi^2}{\sin^2 \pi z} = -\frac{1}{z^2} - \sum_{n=1}^{\infty} \left( \frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} \right),$$

so the difference is constant. Let  $z = \frac{1}{2}$ . The left hand side is  $\pi \cot \pi/2 = 0$  and the right hand side is

$$\frac{2}{1} + \left( -\frac{2}{1} + \frac{2}{3} \right) + \left( -\frac{2}{3} + \frac{2}{5} \right) + \cdots \rightarrow 0, \quad n \rightarrow \infty,$$

so the difference is zero. □

Thus

$$\frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z-n} + \frac{1}{z+n} \right) = \pi \cot \pi z = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = \pi i \frac{q+1}{q-1} = \pi i - \frac{2\pi i}{1-q} = \pi i - 2\pi i \sum_{n=0}^{\infty} q^n.$$

Take  $\frac{d^{k-1}}{dz^{k-1}}$ . For  $k \geq 2$  even, get

$$-(k-1)! \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = -(2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} q^n,$$

so

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n.$$

Collecting powers of  $q$ ,

$$\begin{aligned} g_k(z) &= \sum_{\substack{m=-\infty \\ (m,n) \neq (0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{n^k} + 2 \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{k-1} q^{nm} \\ &= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \end{aligned} \quad \begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} n^{-s} \\ \sigma_{k-1}(n) &= \sum_{d|n, d>0} d^{k-1}. \end{aligned}$$

**Corollary 1.2.9.**  $g_k(z)$  is holomorphic at  $\infty$ . In particular,  $g_k$  is a modular form of weight  $k$ .

### 1.2.3 Bernoulli numbers

**Definition 1.2.10.** The **Bernoulli numbers**  $b_k$  are defined by

$$\sum_{k=0}^{\infty} b_k \frac{x^k}{k!} = \frac{x}{e^x - 1},$$

a formal power series with rational coefficients.

Then

$$b_0 = 1, \quad b_1 = -\frac{1}{2}, \quad b_2 = \frac{1}{6}, \quad b_3 = 0, \quad b_4 = -\frac{1}{20}, \quad \dots, \quad b_{2k} \in \mathbb{Q}, \quad b_{2k+1} = 0, \quad \dots$$

**Proposition 1.2.11.** *For all even  $k$ ,*

$$\zeta(k) = -b_k \frac{(2\pi i)^k}{2k!}.$$

*Proof.* On one hand,

$$\pi z \cot \pi z = \pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1} = \pi i z + \sum_{k=0}^{\infty} b_k \frac{(2\pi i z)^k}{k!}.$$

On the other hand,

$$\begin{aligned} \pi \cot \pi z &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \frac{1}{z} - \frac{2z}{n^2} \sum_{n=1}^{\infty} \frac{1}{1 - z^2/n^2} \\ &= \frac{1}{z} - \sum_{n=1}^{\infty} \frac{2}{z} \sum_{k=1}^{\infty} \left(\frac{z^2}{n^2}\right)^k = \frac{1}{z} - \frac{2}{z} \sum_{k=1}^{\infty} z^{2k} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{1}{z} - \frac{2}{z} \sum_{k=1}^{\infty} \zeta(2k) z^{2k}, \end{aligned}$$

so

$$\pi i z + \sum_{k=0}^{\infty} b_k \frac{(2\pi i z)^k}{k!} = \pi z \cot \pi z = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k}.$$

Comparing,

$$b_{2k} \frac{(2\pi i)^{2k}}{(2k)!} = -2\zeta(2k),$$

get the desired formula. □

So

$$g_k(z) = \frac{-b_k (2\pi i)^k}{k!} + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Set the **normalised Eisenstein series**

$$E_k = \frac{g_k}{2\zeta(k)} = 1 - \frac{2k}{b_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

**Example.**

$$\begin{aligned} E_4 &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, & E_6 &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \\ E_8 &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, & E_{12} &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n. \end{aligned}$$

An observation is if  $f$  is modular of weight  $k$  and  $g$  is modular of weight  $k'$ , then  $fg$  is modular of weight  $k + k'$ , and if  $k = k'$ , then  $f + g$  is modular of weight  $k$ .

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**Example.** Important examples.

- The **modular discriminant**

$$\Delta(z) = \frac{E_4 - E_6^2}{1728} = q - 24q^2 + 252q^3 + \dots$$

is a modular form of weight 12.

- The **j-invariant**

$$j(z) = \frac{E_4^3}{\Delta} = \frac{1}{q} + 744 + 196844q + \dots$$

is a meromorphic modular form of weight 0.

### 1.3 Controlling modular forms

#### 1.3.1 The fundamental domain

The idea is to control the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathbb{H}$ . If  $f : \mathbb{H} \rightarrow \mathbb{C}$  satisfies  $f(\gamma z) = (cz + d)^k f(z)$  for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , and if  $D \subseteq \mathbb{H}$  such that  $D$  meets every  $\mathrm{SL}_2(\mathbb{Z})$ -orbit in  $\mathbb{H}$ , then  $f$  is determined by its values on  $D$ .

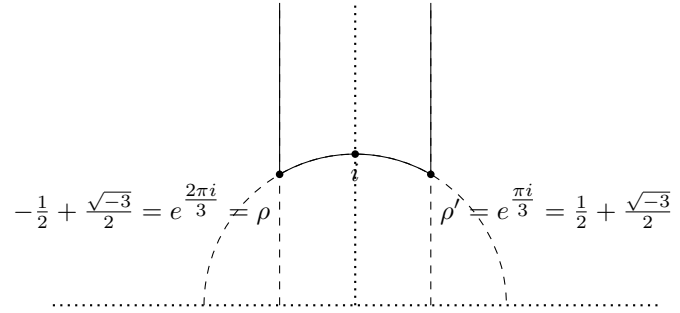
**Definition 1.3.1.** Let  $G$  be a group acting continuously on a complex analytic space  $X$ , such as  $X = \mathbb{H}$ . A subset  $D \subseteq X$  is a **fundamental domain** for the action of  $G$  if

- $D$  meets every  $G$ -orbit in  $X$ ,
- the subset  $\{x \in D \mid \exists g \in G, gx \in D, gx \neq x\}$  has measure zero, and
- $D$  is closed in  $X$ .

Define

$$\mathcal{D} = \{z \in \mathbb{H} \mid \tfrac{1}{2} \leq \operatorname{Re} z \leq \tfrac{1}{2}, |z| \geq 1\} \subseteq \mathbb{H},$$

so



Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : z \mapsto -\frac{1}{z}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : z \mapsto z + 1,$$

and let  $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$  be the subgroup generated by  $S$  and  $T$ . We will see later that  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ .

**Theorem 1.3.2.**

1. For all  $z \in \mathbb{H}$ , there exists  $\gamma \in \Gamma$  such that  $\gamma z \in \mathcal{D}$ .
2. Suppose  $z, z' \in \mathcal{D}$  and  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  with  $\gamma z = z'$ . Then either
  - $z = z'$ ,
  - $\operatorname{Re} z = \pm \frac{1}{2}$  and  $z' = z \mp 1$ , or
  - $|z| = 1$  and  $z' = -1/z$ .

In particular, if  $z \neq z'$ , then  $z$  and  $z'$  are on the boundary of  $\mathcal{D}$ .

3. For  $z \in \mathcal{D}$ , let  $I_z$  be the stabiliser of  $z$  in  $\mathrm{SL}_2(\mathbb{Z})$ , that is

$$I_z = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma z = z\}.$$

Then  $I_z = \{\pm \operatorname{id}\}$  unless

- $z = i$ , where  $I_z = \{\pm \operatorname{id}, \pm S\}$ ,
- $z = \rho$ , where  $I_z = \{\pm \operatorname{id}, \pm (ST), \pm (T^{-1}S)\}$ , or
- $z = \rho'$ , where  $I_z = \{\pm \operatorname{id}, \pm (TS), \pm (ST^{-1})\}$ .

**Corollary 1.3.3.**  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ .

*Proof.* Fix  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  and  $z \in \mathring{\mathcal{D}}$  so  $\mathrm{SL}_2(\mathbb{Z})z \cap \mathcal{D} = \{z\}$  and  $I_z = \{\pm \operatorname{id}\}$ . Consider  $\gamma z$ . There exists  $\gamma' \in \Gamma$  such that  $\gamma'\gamma z \in \mathcal{D}$ , so  $\gamma'\gamma z = z$ . So  $\gamma'\gamma = \pm \operatorname{id}$ , so  $\gamma = \pm \gamma'^{-1}$ . But  $\gamma'^{-1} \in \Gamma$  and  $-\operatorname{id} = S^2 \in \Gamma$ , so  $\gamma \in \Gamma$ .  $\square$

*Proof of Theorem 1.3.2.* Recall  $\operatorname{Im} \gamma z = \operatorname{Im} z / |cz + d|^2$  for  $\gamma \in \operatorname{SL}_2(\mathbb{Z})$ .

1. As  $c$  and  $d$  vary,  $\{cz + d\}$  forms a lattice in  $\mathbb{C}$ , so there exist only finitely many  $c$  and  $d$  such that  $|cz + d| < 1$ . So  $\operatorname{Im} \gamma z$  attains a maximum as  $\gamma$  varies over  $\Gamma$ , so there exists  $\gamma \in \Gamma$  such that  $\operatorname{Im} \gamma z$  is maximal. There exists  $n \in \mathbb{Z}$  such that  $T^n \gamma z$  has real part between  $-\frac{1}{2}$  and  $\frac{1}{2}$ . Consider  $|T^n \gamma z|$ . If this is less than one, then

$$\operatorname{Im} ST^n \gamma z = \operatorname{Im} \frac{-1}{T^n \gamma z} > \operatorname{Im} T^n \gamma z = \operatorname{Im} \gamma z.$$

Since  $ST^n \gamma \in \Gamma$ , this contradicts maximality so  $|T^n \gamma z| \geq 1$ , so  $T^n \gamma z \in \mathcal{D}$ .

- 2, 3. Let  $z, z' \in \mathcal{D}$  such that  $\gamma z = z'$ . Without loss of generality  $\operatorname{Im} z' \geq \operatorname{Im} z$ , so  $|cz + d| \leq 1$ . Note that  $|cz + d| \geq \operatorname{Im}(cz + d) \geq \frac{\sqrt{3}}{2}c$ , so  $c = -1, 0, 1$ . Note that can replace  $\gamma$  with  $-\gamma$  if convenient.

$c = 0$ . Then  $ad = 1$ , so can assume  $a = d = 1$ , so  $\gamma z = z + b$ . Since  $z, z + b \in \mathcal{D}$ ,  $b = \pm 1$  and  $\operatorname{Re} z = \mp \frac{1}{2}$ .

$c = 1$ . Have  $|z + d| \leq 1$  and  $|z| \geq 1$ , so  $d = -1, 0, 1$ .

$d = 0$ . Then  $|z| = 1$ , and  $\gamma z = (az - 1)/z = a - 1/z$ . The only possibilities are

- \*  $a = 0$  and  $\gamma = S$ ,
- \*  $a = 1$  and  $\gamma = TS$ , so  $z = \rho'$ , or
- \*  $a = -1$  and  $\gamma = T^{-1}S$ , so  $z = \rho$ .

$d = 1$ . Then  $z = \rho$ , and  $\gamma z = ((b + 1)z + b)/(z + 1) = b + 1 - 1/(z + 1)$ , so  $b = 0$  or  $b = -1$ .

$d = -1$ . Then  $z = \rho'$  is similar.

$c = -1$ . Similar.

□

### 1.3.2 Further review of complex analysis

Recall that on any compact set, a meromorphic function has only finitely many zeroes and poles. If  $f(z) = g(e^{2\pi iz})$  is meromorphic at infinity and  $g$  is meromorphic on  $D = \{q \mid |q| < 1\}$ , zeroes and poles of  $g$  are discrete with respect to  $q$ , and  $\operatorname{Im} z \gg 0$  if and only if  $|q| < \epsilon$ .

**Definition 1.3.4.** Let  $U \subseteq \mathbb{C}$  be open, and let  $f : U \rightarrow \mathbb{C}$  be meromorphic on  $U$ . If  $f$  has a pole at  $p$ , can write

$$f(z) = \sum_{n=\operatorname{ord}_p f < 0}^{\infty} a_n (z - p)^n.$$

The coefficient  $a_{-1}$  is called the **residue**  $\operatorname{Res}_p f$  of  $f$  at  $p$ .

**Theorem 1.3.5** (Residue theorem). *Let  $V$  be a region in  $\mathbb{C}$  whose boundary  $\partial V$  is a simple closed curve. Then*

$$\frac{1}{2\pi} \int_{\partial V} f(z) dz = \sum_{p \in V \text{ pole of } f} \operatorname{Res}_p f.$$

**Definition 1.3.6.** Let  $f$  be meromorphic on  $U \subseteq \mathbb{C}$  open. Then the **logarithmic derivative**  $d \log f$  is the function  $f'/f$ .

If  $f(z) = c_n (z - p)^n + c_{n+1} (z - p)^{n+1} + \dots$ , then if  $n \neq 0$ , then the leading term of  $f'$  is  $nc_n (z - p)^{n-1}$  and the leading term of  $f$  is  $c_n (z - p)^n$ , so the leading term of  $f'/f$  is  $n(z - p)^{-1}$ . If  $n = 0$ , then  $f'/f$  is holomorphic. So  $f'/f$  is meromorphic with simple poles precisely at the points where  $\operatorname{ord}_p f \neq 0$ , and  $\operatorname{Res}_p f'/f$  at such  $p$  is  $\operatorname{ord}_p f$ .

**Theorem 1.3.7** (Argument principle).

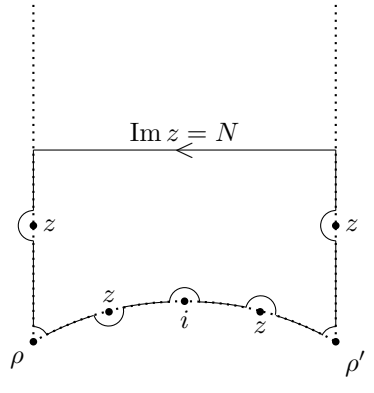
$$\frac{1}{2\pi i} \int_{\partial V} \frac{f'(z)}{f(z)} dz = \sum_{p \in V} \operatorname{ord}_p f.$$

### 1.3.3 Controlling modular forms

**Theorem 1.3.8.** *Let  $f$  be a non-zero meromorphic modular form of weight  $k$ . Then*

$$\text{ord}_\infty f + \frac{\text{ord}_\rho f}{3} + \frac{\text{ord}_i f}{2} + \sum_{p \in \mathbb{H}/\text{SL}_2(\mathbb{Z}), p \neq \{i, \rho\}} \text{ord}_p f = \frac{k}{12}.$$

*Proof.* Consider the closed curve  $C_{N,\epsilon}$ ,



where the  $z$ 's are zeroes or poles of  $f$ , and the circles are of radius  $\epsilon$ . Consider

$$\frac{1}{2\pi i} \int_{C_{N,\epsilon}} \frac{f'(z)}{f(z)} dz = \sum_{p \in \mathbb{H}/\text{SL}_2(\mathbb{Z}), p \neq \{i, \rho\}} \text{ord}_p f, \quad \epsilon \rightarrow 0.$$

So it suffices to show

$$\lim_{\epsilon \rightarrow 0, N \rightarrow \infty} \frac{1}{2\pi i} \int_{C_{N,\epsilon}} \frac{f'(z)}{f(z)} dz = -\text{ord}_\infty f - \frac{\text{ord}_\rho f}{3} - \frac{\text{ord}_i f}{2} + \frac{k}{12}.$$

The vertical parts of the boundary cancel. The integral over the circular part of  $\partial\mathcal{D}$  approaches

$$\frac{1}{2\pi i} \int_\rho^i \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i} \int_i^{\rho'} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \left( \int_\rho^i \frac{f'(z)}{f(z)} dz - \int_\rho^i \frac{f'(-1/z)}{f(-1/z)} dz \right)$$

Since  $f(-1/z) = z^k f(z)$ ,

$$d(z^k f(z)) = (kz^{k-1} f(z) + z^k f'(z)) dz,$$

so

$$\frac{1}{2\pi i} \int_\rho^i \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i} \int_i^{\rho'} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_\rho^i \frac{f'(z)}{f(z)} dz - \frac{kz^{k-1} f(z) + z^k f'(z)}{z^k f(z)} dz = -\frac{1}{2\pi i} \int_\rho^i \frac{k}{z} dz = \frac{k}{12}.$$

Since  $dq = 2\pi i q dz$ , the top part is

$$\frac{1}{2\pi i} \int_{\frac{1}{2} - iN}^{\frac{1}{2} - iN} \frac{f'(z)}{f(z)} dz = -\frac{1}{2\pi i} \int_{\text{circle of radius } \epsilon} \frac{g'(q)}{g(q)} dq = -\text{ord}_\infty f.$$

Near  $i$ ,  $f'/f = \text{ord}_i f (z - i)^{-1} + h(z)$ , where  $h(z)$  is holomorphic and  $h(z) \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Then the circle  $C_{\epsilon,i}$  of radius  $\epsilon$  centered at  $i$  is

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{C_{\epsilon,i}} \frac{f'(z)}{f(z)} dz = \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\text{arc of half circle centered at } i} \frac{\text{ord}_i f}{z - i} dz = -\frac{\text{ord}_i f}{2}.$$

Similarly, at  $\rho$  and  $\rho'$ , get that the circles  $C_{\epsilon,\rho}$  and  $C_{\epsilon,\rho'}$  of radius  $\epsilon$  centered at  $\rho$  and  $\rho'$  are

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{C_{\epsilon,\rho}} \frac{f'(z)}{f(z)} dz = \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{C_{\epsilon,\rho'}} \frac{f'(z)}{f(z)} dz = -\frac{\text{ord}_\rho f}{6},$$

which gives  $-\text{ord}_\rho f/3$ . □

### 1.3.4 Holomorphic modular forms

Let

$$M_k = \{\text{holomorphic modular forms of weight } k\},$$

and let

$$S_k = \{\text{cusp forms of weight } k\} = \{f \in M_k \mid \text{ord}_\infty f > 0\} \subseteq M_k.$$

**Corollary 1.3.9.**

- $M_k = 0$  if  $k < 0$ ,  $k = 2$ , or  $k$  odd.
- $M_0$  are constants.
- $M_4 = \mathbb{C}E_4$ , where  $\text{ord}_p E_4 = 1$  and no other zeroes.
- $M_6 = \mathbb{C}E_6$ , where  $\text{ord}_i E_6 = 1$  and no other zeroes.
- $M_8 = \mathbb{C}E_8$ , where  $\text{ord}_p E_8 = 2$  and no other zeroes.
- $M_{10} = \mathbb{C}E_{10}$ , where  $\text{ord}_p E_{10} = \text{ord}_i E_{10} = 1$  and no other zeroes.
- $M_{12} = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta$ , where  $\text{ord}_\infty \Delta = 1$  and no other zeroes.

**Corollary 1.3.10.**  $\Delta : M_k \rightarrow S_{k+12}$  is an isomorphism. On the other hand,

$$M_k \cong \mathbb{C}E_k \oplus S_k, \quad k \geq 4 \text{ even},$$

so

$$M_k \cong \mathbb{C}E_k \oplus \cdots \oplus \mathbb{C}E_{k-12r}\Delta^r, \quad k - 12r \in \{0, 4, 6, 8, 10, 14\}.$$

So for  $k \geq 4$ , the set

$$\begin{cases} E_k, \dots, E_{k-12\lfloor k/12 \rfloor} \Delta^{\lfloor k/12 \rfloor} & k \not\equiv 2 \pmod{12} \\ E_k, \dots, E_{14} \Delta^{\lfloor k/12 \rfloor - 1} & k \equiv 2 \pmod{12} \end{cases}$$

is a basis for  $M_k$ .

**Corollary 1.3.11.**  $E_4^2 = E_8$  and  $E_4 E_6 = E_{10}$ .

A variant is to write  $k = 4n + 6m$  with  $m = 0, 1$  and  $n \geq 0$ , for  $k \geq 4$ . Then  $M_k = \mathbb{C}E_4^n E_6^m \oplus S_k$  gives a basis

$$E_4^n E_6^m, \dots, E_4^{n-3\lfloor n/3 \rfloor} E_6^m \Delta^{\lfloor n/3 \rfloor}$$

for  $M_k$ . Since  $\Delta = (E_4^3 - E_6^2)/1728$ , we see every modular form of weight  $k$  is a polynomial in  $E_4$  and  $E_6$ , and

$$\Delta \in q + q^2 \mathbb{Z}[[q]], \quad E_4^n E_6^m \in 1 + q\mathbb{Z}[[q]], \quad E_4^{n-3\lfloor n/3 \rfloor} E_6^m \Delta \in q + q^2 \mathbb{Z}[[q]], \quad \dots$$

have integer coefficients. The upshot is if the  $q$ -expansion of  $f$  has integer coefficients, then  $f$  is an integer combination of

$$E_4^n E_6^m, \dots, E_4^{n-3\lfloor n/3 \rfloor} E_6^m \Delta^{\lfloor n/3 \rfloor}.$$

**Notation.**  $M_k(\mathbb{Z}) \subseteq M_k$  consists of modular forms with integer  $q$ -expansions.

**Theorem 1.3.12.**  $M_k(\mathbb{Z})$  spans  $M_k$ , and  $f \in M_k$  lies in  $M_k(\mathbb{Z})$  if and only if  $f$  is an integral polynomial in  $E_4, E_6, \Delta$ .

**Definition 1.3.13.** A **graded ring** is a ring  $R$ , together with a direct sum decomposition, as abelian groups,

$$R = \bigoplus_{i \in \mathbb{Z}} R_i,$$

such that  $R_i \cdot R_j \subseteq R_{i+j}$  for all  $i, j \in \mathbb{Z}$ .

**Example.**

- $R = \mathbb{C}[X, Y]$ , where  $R_i$  are polynomials homogeneous of degree  $i$ .
- $R = \bigoplus_{k \in \mathbb{Z}} M_k$ .

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Let  $\mathbb{C}[X, Y]$  be graded with  $\deg X = 4$  and  $\deg Y = 6$ . Have a homomorphism of graded rings

$$\begin{aligned} \mathbb{C}[X, Y] &\longrightarrow \bigoplus_{k \in \mathbb{Z}} M_k \\ (X, Y) &\longmapsto (E_4, E_6) \end{aligned}.$$

**Theorem 1.3.14.** *This is an isomorphism of graded rings.*

*Proof.* This map is surjective, since every  $f \in M_k$  is a polynomial in  $E_4$  and  $E_6$ . Remains to show this map is injective. Suppose not. There exists  $P(X, Y)$ , homogeneous of degree  $k$ , such that  $P(E_4, E_6) = 0$ . Write  $k = 4n + 6m$  with  $m = 0, 1$ . If  $P = c_0 X^n Y^m + \dots + c_r X^{n-3r} Y^{m+2r}$  where  $r = \lfloor n/3 \rfloor$ , then

$$c_0 E_4^n E_6^m + \dots + c_r E_4^{n-3r} E_6^{m+2r} = 0.$$

Dividing by  $E_4^{n-3r} E_6^{m+2r}$ , get  $Q(E_4^3/E_6^2) = 0$  where  $Q(X) = c_0 X^r + \dots + c_r$ . Since the roots of  $Q$  are discrete, and  $E_4^3/E_6^2$  is non-constant, this is impossible.  $\square$

### 1.3.5 Meromorphic modular forms

**Note.** The meromorphic modular forms of weight zero form a field. For example,  $j(z) = E_4^3/\Delta = 1728E_4^3/(E_4^3 - E_6^2)$  is a non-constant meromorphic modular form, with a pole of order one at infinity, a zero of order three at  $\rho$ , and no other zeroes or poles.

**Theorem 1.3.15.**  *$j$  gives a bijection between  $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$  and  $\mathbb{C}$ .*

*Proof.* Given  $\lambda \in \mathbb{C}$ , want  $z \in \mathbb{H}$  such that  $j(z) = \lambda$ . Consider  $g = j - \lambda$ . This is meromorphic of weight zero. There is a pole at infinity, and no other poles, and

$$\mathrm{ord}_\infty g + \frac{\mathrm{ord}_\rho g}{3} + \frac{\mathrm{ord}_i g}{2} + \sum_{p \in \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}), p \neq \{i, \rho\}} \mathrm{ord}_p g = 0.$$

The only possibilities are

- $g$  has a zero at  $\rho$  of order three, and no other zeroes,
- $g$  has a zero at  $i$  of order two, and no other zeroes, or
- $g$  has a simple zero somewhere else, and no others.

In each case, the zero of  $g$  is a unique  $\mathrm{SL}_2(\mathbb{Z})$ -orbit on which  $j(z) = \lambda$ . So  $j$  is bijective.  $\square$

**Theorem 1.3.16.** *Every meromorphic modular form of weight zero is a rational function in  $j$ . That is, the field of meromorphic modular forms is  $\mathbb{C}(j)$ .*

*Proof.* Let  $g$  be meromorphic of weight zero. Then  $g$  has finitely many  $\mathrm{SL}_2(\mathbb{Z})$ -orbits worth of poles in  $\mathbb{H}$ . Saw last time that  $j$  is holomorphic in  $\mathbb{H}$ . If  $p$  is a pole of  $g$ , then  $(j(z) - j(p))^{n_p}$  is holomorphic on  $\mathbb{H}$  and zero at  $z = p$ . Doing this for all poles, there exists  $P \in \mathbb{C}[X]$  such that  $P(j)g(z)$  is holomorphic on  $\mathbb{H}$ . Then for some  $m$ ,  $P(j)g(z)\Delta^m$  is holomorphic of weight  $12m$ . So it suffices to show if  $h$  is holomorphic of weight  $12m$ , then  $h/\Delta^m$  is a rational function in  $j$ , since if  $P(j)g(z)\Delta^m = h$  then  $P(j)g(z) \in \mathbb{C}(j)$ , so  $g(z) \in \mathbb{C}(j)$ . Then  $h$  is a sum of terms

$$h = \sum_{a,b} c_{a,b} E_4^a E_6^b, \quad c_{a,b} \in \mathbb{C}, \quad 4a + 6b = 12m.$$

Considering this equation modulo four and modulo three, find  $3 \mid a$  and  $2 \mid b$ , so

$$\frac{h}{\Delta^m} = \sum_{a,b} c_{a,b} \left( \frac{E_4^3}{\Delta} \right)^{\frac{a}{3}} \left( \frac{E_6^2}{\Delta} \right)^{\frac{b}{2}}.$$

So it suffices to show  $E_4^3/\Delta$  and  $E_6^2/\Delta$  are rational functions in  $j$ . Then  $j = E_4^3/\Delta$ , and

$$\frac{E_6^2}{\Delta} = \frac{1728E_6^2}{E_4^3 - E_6^2} = \frac{1728(E_6^2 - E_4^3) + 1728E_4^3}{E_4^3 - E_6^2} = -1728 + \frac{1728E_4^3}{E_4^3 - E_6^2} = j - 1728.$$

$\square$

Lecture 10  
Friday  
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## 1.4 Theta series

Let  $L \subseteq \mathbb{R}^n$  be a lattice. For  $x, y \in L$ ,  $x \cdot y \in \mathbb{R}$ . Suppose  $x \cdot y \in \mathbb{Z}$  for all  $x, y \in L$ . A question is for  $n \in \mathbb{Z}$ , how many  $x \in L$  have  $x \cdot x = n$ ? The rough idea is to form the series

$$\sum_{x \in L} q^{x \cdot x} = \sum_{n=0}^{\infty} a_n q^n, \quad a_n = \# \{x \in L \mid x \cdot x = n\}.$$

We will show, with some slight modifications, and extra hypotheses on  $L$ , this generating function turns out to be a modular form.

### 1.4.1 Lattices

Fix a lattice  $L \subseteq \mathbb{R}^n$ , so

$$L = \mathbb{Z} \cdot e_1 \oplus \cdots \oplus \mathbb{Z} \cdot e_n.$$

Given these  $e_i$ , form a matrix  $A$  such that  $A_{ij} = e_i \cdot e_j$ .

**Note.**  $A = B^T B$ , where  $B$  is the matrix whose columns are the  $e_i$ , and  $|\det B|$  is the volume of the parallelogram spanned by  $e_i$ , so  $\det A = (\det B)^2 > 0$ .

**Definition 1.4.1.** The **dual lattice**  $L^\vee$  is the set of  $y \in \mathbb{R}^n$  such that  $y \cdot x \in \mathbb{Z}$  for all  $x \in L$ .

Let  $f_1, \dots, f_n$  be the dual basis to  $e_1, \dots, e_n$ , that is the unique set of solutions  $f_1, \dots, f_n$  such that

$$f_i \cdot e_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then  $L^\vee$  is spanned by the  $f_i$ . Clearly  $f_i \in L^\vee$  for all  $i$ . Conversely, if  $y \in L^\vee$ , then  $y \cdot e_i = a_i \in \mathbb{Z}$ , then  $y = \sum_{i=1}^n a_i f_i$ .

**Proposition 1.4.2.** Let  $C = A^{-1}$ . Then

$$f_i = \sum_{j=1}^n C_{ij} e_j.$$

*Proof.*

$$f_i \cdot e_k = \sum_{j=1}^n C_{ij} e_j \cdot e_k = \sum_{j=1}^n C_{ij} A_{jk} = (CA)_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}.$$

□

**Definition 1.4.3.** A lattice  $L$  is **self-dual** if  $L^\vee = L$  as subsets of  $\mathbb{R}^n$ .

**Proposition 1.4.4.**  $L$  is self-dual if and only if the associated matrix  $A$  has integer entries and determinant 1.

*Proof.* Clearly if  $L = L^\vee$ , then  $e_i \cdot e_j \in \mathbb{Z}$ , so  $A$  has integer entries. Since  $L^\vee \subseteq L$ ,  $f_i$  is an integer combination of the  $e_j$ , so  $C = A^{-1}$  has integer entries. So  $\det A = \pm 1$ , but already saw  $\det A > 0$ . Conversely if  $A$  has integer entries and determinant one,  $C = A^{-1}$  has integer entries. Then  $A$  has integer entries implies that  $e_i \cdot e_j \in \mathbb{Z}$  for all  $i$  and  $j$ , so  $e_i \in L^\vee$  for all  $i$ , so  $L \subseteq L^\vee$ . Similarly,  $C$  has integer entries implies that  $L^\vee \subseteq L$ . □

If  $L$  is self-dual, get an integer-valued quadratic form

$$\begin{aligned} Q_L : \quad \mathbb{Z}^n &\longrightarrow \mathbb{Z} \\ (a_1, \dots, a_n) &\longmapsto (a_1 e_1 + \cdots + a_n e_n) \cdot (a_1 e_1 + \cdots + a_n e_n) = (a_1 \ \dots \ a_n) A \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}. \end{aligned}$$

A question is given  $m$ , how often does  $Q_L$  represent  $m$ ?

### 1.4.2 Fourier analysis

Let  $f$  be a  $C^\infty$  function on  $\mathbb{R}^n \rightarrow \mathbb{C}$ .

**Definition 1.4.5.** We will say  $f$  is **rapidly decreasing** if for all  $m$ ,

$$\|x\|^m \cdot f(x) \rightarrow 0, \quad |x| \rightarrow \infty,$$

where  $|x| = (x \cdot x)^{1/2}$ . For  $f \in C^\infty$ , rapidly decreasing, define

$$\widehat{f}(y) = \int_{\mathbb{R}^n} e^{-2\pi i(x \cdot y)} dx : \mathbb{R}^n \rightarrow \mathbb{C}.$$

**Fact 1.4.6.** If  $f$  is smooth and rapidly decreasing, so is  $\widehat{f}$ .

**Fact 1.4.7.** If  $f(x) = e^{-\pi(x \cdot x)}$ , then  $\widehat{f}(x) = f(x)$ .

**Fact 1.4.8.** If  $f$  is smooth and rapidly decreasing, and  $\mathbb{R}^n$  is a lattice with volume  $V$ , then

$$\sum_{x \in L} f(x) = \frac{1}{V} \sum_{x \in L^\vee} \widehat{f}(x).$$

### 1.4.3 Theta series

A crucial assumption is that  $L$  is self-dual. An assumption that can be removed is that  $L$  is even, so for all  $x \in L$ ,  $Q_L(x) \in 2\mathbb{Z}$ .

**Definition 1.4.9.** The **theta series**  $\Theta_L$  is defined by

$$\Theta_L(z) = \sum_{x \in L} q^{\frac{1}{2}x \cdot x} = \sum_{m=0}^{\infty} a_m q^m, \quad a_m = \# \{x \in \mathbb{Z}^n \mid Q_L(x) = 2m\}.$$

**Theorem 1.4.10.**  $\Theta_L$  is modular of weight  $n/2$ .

**Example.** Let  $\Gamma_8 \subseteq \mathbb{R}^8$  be spanned by

$$\begin{aligned} e_1 &= \left( \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right), & e_2 &= (1, 1, 0, 0, 0, 0, 0, 0), \\ e_3 &= (1, -1, 0, 0, 0, 0, 0, 0), & e_4 &= (0, 1, -1, 0, 0, 0, 0, 0), & e_5 &= (0, 0, 1, -1, 0, 0, 0, 0), \\ e_6 &= (0, 0, 0, 1, -1, 0, 0, 0), & e_7 &= (0, 0, 0, 0, 1, -1, 0, 0), & e_8 &= (0, 0, 0, 0, 0, 1, -1, 0). \end{aligned}$$

Then

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

and

$$Q_L(z_1, \dots, z_8) = 2(z_1^2 + \dots + z_8^2 - z_1 z_3 - z_2 z_4 - z_3 z_4 - z_4 z_5 - z_6 z_7 - z_7 z_8).$$

If  $L \subseteq \mathbb{R}^n$  is even and self-dual, and  $\Theta_L$  is modular of weight  $n/2$ , then dimension is  $\sim 24$ .

**Fact.**  $L \subseteq \mathbb{R}^n$  even and self-dual implies that  $8 \mid n$ .

*Proof.* Serre V.2.1 Corollary 2. □

*Proof of Theorem 1.4.10.* Know, since  $L$  is even, that  $\Theta_L(z+1) = \Theta_L(z)$ . It suffices to show

$$\Theta_L\left(-\frac{1}{z}\right) = z^{\frac{n}{2}} \Theta_L(z).$$

Both sides are holomorphic on  $\mathbb{H}$ , so it suffices to show

$$\Theta_L\left(-\frac{1}{it}\right) = (it)^{\frac{n}{2}} \Theta_L(it).$$

For  $t \in \mathbb{R}^\times$ , let

$$L_t = t^{\frac{1}{2}} \cdot L, \quad L_t^\vee = t^{-\frac{1}{2}} \cdot L = L_{t^{-1}},$$

so  $\text{vol } L_t = t^{n/2}$ . By the facts,

$$\sum_{x \in L_t} e^{-\pi(x \cdot x)} = t^{-\frac{n}{2}} \sum_{x \in L_{t^{-1}}} e^{-\pi(x \cdot x)},$$

so

$$\sum_{x \in L} e^{-\pi(x \cdot x)t} = t^{-\frac{n}{2}} \sum_{x \in L} e^{-\frac{\pi(x \cdot x)}{t}}.$$

Now return to  $\Theta_L$ . The left hand side is

$$\Theta_L\left(-\frac{1}{it}\right) = \sum_{x \in L} e^{\frac{1}{2} \cdot 2\pi i \cdot \left(-\frac{1}{it}\right) \cdot (x \cdot x)} = \sum_{x \in L} e^{-\frac{\pi(x \cdot x)}{t}},$$

and the right hand side is

$$\Theta_L(it) = \sum_{x \in L} e^{\frac{1}{2} \cdot 2\pi i \cdot (it) \cdot (x \cdot x)} = \sum_{x \in L} e^{\pi(x \cdot x)t},$$

so the result follows. □

Let  $\Theta_L = \sum_{m=1}^{\infty} a_m q^m$ , where  $a_m$  is the number of ways  $Q_L$  represents  $2m$ , so  $a_0 = 1$ . Then  $\Theta_L = E_{n/2} + g$ , where  $g$  is a cusp form. The coefficients of  $E_{n/2}$  are constants times  $\sigma_{n/2-1}(m)$ , which grows like  $m^{n/2-1}$ .

**Theorem 1.4.11** (Hasse). *If  $g = \sum b_m q^m$  is a cusp form of weight  $k$ , then  $|b_m| m^{-k/2}$  is bounded as  $n \rightarrow \infty$ .*