

# M3P21 Geometry II: Algebraic Topology

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Spring 2019

# Contents

<b>0</b>	<b>Introduction</b>	<b>3</b>
0.1	Introduction . . . . .	3
0.2	Some underlying geometric notions . . . . .	3
0.2.1	Homotopy . . . . .	3
0.2.2	Cell complexes . . . . .	4
<b>1</b>	<b>The fundamental group</b>	<b>6</b>
1.1	Basic constructions . . . . .	6
1.1.1	Paths and homotopy . . . . .	6
1.1.2	The fundamental group of the circle . . . . .	9
1.1.3	Induced homomorphisms . . . . .	12
1.2	Seifert-van Kampen theorem . . . . .	15
1.2.1	Free products with amalgamation . . . . .	15
1.2.2	The Seifert van-Kampen theorem . . . . .	15
1.2.3	Applications to CW-complexes . . . . .	17
1.3	Covering spaces . . . . .	19
1.3.1	Lifting properties . . . . .	19
1.3.2	The classification of covering spaces . . . . .	21
1.3.3	Deck transformations and group actions . . . . .	25
<b>2</b>	<b>Homology</b>	<b>28</b>
2.1	Simplicial and singular homology . . . . .	28
2.1.1	$\Delta$ -complexes . . . . .	28
2.1.2	Simplicial homology . . . . .	29

## 0 Introduction

Lecture 1  
Friday  
11/01/19

### 0.1 Introduction

Combines topological spaces with algebraic objects, which are groups.

- How to show that a torus is not homeomorphic to a sphere?
- How to show that  $\mathbb{R}^n \not\cong \mathbb{R}^m$  if  $n \neq m$ ?

Content is fundamental groups and homology. We will follow chapter one and two from

- A Hatcher, Algebraic topology, 2002

The following are prerequisites.

- Point set topology. Topological spaces, continuous maps, product and quotient topologies, Hausdorff spaces, etc.
- Basic group theory. Normal subgroups and quotients, isomorphism theorems, free groups, presentation of groups, etc.

### 0.2 Some underlying geometric notions

#### 0.2.1 Homotopy

Let  $X, Y$  be topological spaces and  $I = [0, 1]$ .

**Definition.** A **homotopy** is a continuous map  $F : X \times I \rightarrow Y$ . For every  $t \in I$  we obtain a continuous map

$$\begin{aligned} f_t : X &\rightarrow Y \\ x &\mapsto f_t(x) = F(x, t) \end{aligned} .$$

**Definition.** Two continuous maps  $f_0, f_1 : X \rightarrow Y$  are **homotopic** if there exists a homotopy  $F : X \times I \rightarrow Y$  such that

$$f_0(x) = F(x, 0), \quad f_1(x) = F(x, 1),$$

for all  $x \in X$ . We write  $f_0 \cong f_1$ . (Exercise: this is an equivalence relation)

**Definition.** Let  $A \subseteq X$  be a subspace. A **retraction** of  $X$  onto  $A$  is a continuous map  $r : X \rightarrow A$  such that

- $r(X) = A$ , and
- $r|_A = id_A$ .

**Example.** If  $X \neq \emptyset$ ,  $p \in X$ , then  $X$  retracts to  $p$  by the constant map  $X \rightarrow \{p\}$ .

**Definition.** A **deformation retraction** of  $X$  onto  $A \subseteq X$  is a retraction that is homotopic to the identity. That is, there is a continuous map

$$\begin{aligned} F : X \times I &\rightarrow A \\ (x, t) &\mapsto f_t(x) \end{aligned} ,$$

such that  $f_0 = id_X$  and  $f_1 : X \rightarrow A$  is the deformation retraction.

**Example.** The closed  $n$ -dimensional  $n$ -disc

$$D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$

deformation retracts to  $(0, \dots, 0) \in \mathbb{R}^n$ . Let  $f_t(x) = t \cdot x$ .  $t = 1$  gives  $f_1 = id_{D^n}$  and  $t = 0$  gives  $f_0 : D^n \rightarrow (0, \dots, 0)$ .

**Example.** Let  $S^n$  be the  $n$ -sphere,

$$\partial D^{n+1} = S^n = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

The cylinder  $S^n \times I$  deformation retracts to  $S^n \times \{0\}$ , by defining  $f_t(x, r) = (x, t \cdot r)$ .

An observation is if  $X$  is a topological space, and  $f : X \rightarrow \{p\}$  for  $p \in X$  is a deformation retraction of  $X$  to  $p$ , then  $X$  is path-connected. Indeed, if  $F : X \times I \rightarrow X$  is a homotopy from  $id_X$  to  $f$  and  $x \in X$  is a point, then this gives a path

$$\begin{aligned} I &\rightarrow X \\ t &\mapsto F(x, t) \end{aligned}$$

that connects  $x$  to  $p$ . This implies that not all retractions are deformation retractions.

**Example.** A retraction that is not a deformation retraction. Take a space that is not path-connected and retract it to a point. Let  $X = \{0, 1\}$  with discrete topology.  $x \mapsto 0$  is a retraction, but not a deformation retraction because  $X$  is not path-connected.

**Definition.** A continuous map  $f : X \rightarrow Y$  is a **homotopy equivalence** if there is a continuous map  $g : Y \rightarrow X$  such that  $fg \cong id_Y$  and  $gf \cong id_X$ . If there exists a homotopy equivalence between  $X$  and  $Y$ ,  $X$  and  $Y$  are **homotopy equivalent** or they have the same **homotopy type**.

**Lemma 0.1.** A deformation retraction  $f : X \rightarrow A$  is a homotopy equivalence.

*Proof.* Let  $i : A \hookrightarrow X$  be the inclusion map. Then  $fi = id_A$  and  $if = f \cong id_X$  by definition.  $\square$

**Example.** The disc with two holes is equivalent to  $O \cdot O$ .

**Example.**  $\mathbb{R}^n$  deformation retracts to a point, by  $f_t(x) = t \cdot x$ .

**Definition.**

- $X$  is **contractible** if it is homotopy equivalent to a point.
- A continuous map is **nullhomotopic** if it is homotopy equivalent to a constant map.

## 0.2.2 Cell complexes

**Example.** The torus  $S^1 \times S^1$  is the union of a point, two open intervals, and the open disc  $Int(D^2)$ .

These are called **cells**. Can think of discs  $D^n$  glued together.

**Definition.** A **CW-complex**, or **cell complex**, is a topological space  $X$  such that there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X^n,$$

where the  $X^n$  are constructed inductively in the following way.

- $X^n$  is a discrete set.
- For each  $n \geq 0$  there is an collection of closed  $n$ -discs  $\{D_\alpha^n\}$  together with continuous maps  $\phi_\alpha : \partial D_\alpha^n \rightarrow X^{n-1}$ , such that

$$X^n = \frac{X^{n-1} \sqcup \bigsqcup_\alpha D_\alpha^n}{\sim},$$

where  $x \sim \phi_\alpha(x)$  for all  $x \in \partial D_\alpha^n$  for all  $\alpha$ .

- A subset  $U \subseteq X$  is open if and only if  $U \cap X^n$  is open for all  $n$ .

*Remark.*

Lecture 2  
Tuesday  
15/01/19

- As a set,

$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha} e_{\alpha}^n,$$

where each  $e_{\alpha}^n$  is homeomorphic to an open  $n$ -disc. These  $e_{\alpha}^n$  are called the  $n$ -**cells** of  $X$ .

- If  $X = X^m$  for some  $m$ , then  $X$  is called **finite dimensional**. The minimal  $m$  such that  $X = X^m$  is the **dimension** of  $X$ .

**Example.**

- $[0, 1]$  is a CW-complex.
- $\mathbb{R}$  is a CW-complex.
- $S^1$  is a CW-complex.
- A graph is a CW-complex.
- $S^n = D^n / \partial D^n$  is a CW-complex. See worksheet 1.

Can also decompose CW-complexes.

- The sphere  $S^2$  is one 0-cell, one 1-cell, and two 2-cells.
- The torus  $S^1 \times S^1$  is one 0-cell, two 1-cells, and one 2-cell.
- The Möbius strip is two 0-cells, three 1-cells, and one 2-cell.
- The Klein bottle is one 0-cell, two 1-cells, and one 2-cell.

**Definition.** If  $X$  is a CW-complex with finitely many cells the **Euler characteristic**  $\chi(X)$  of  $X$  is the number of even cells minus the number of odd cells.

*Fact.*  $\chi(X)$  does not depend of the choice of cells decomposition.

**Example.**

- $\chi(S^n) = 0$  if  $n$  is odd and  $\chi(S^n) = 2$  if  $n$  is even.
- $\chi(S^1 \times S^1) = 0$ .

This is the generalisation of the following observation by Leonhard Euler. Let  $P$  be a convex polyhedron, where

- $V$  is the number of vertices of  $P$ ,
- $E$  is the number of edges of  $P$ , and
- $F$  is the number of faces of  $P$ .

Then  $V - E + F = 2$ .

**Example.** A topological space that is not a CW-complex.  $X = \{0, 1\}$  with trivial topology does not contain any closed points.

*Fact.* CW-complexes are always Hausdorff.

# 1 The fundamental group

## 1.1 Basic constructions

### 1.1.1 Paths and homotopy

Let  $X$  be a topological space. A **path** is a continuous map  $f : I \rightarrow X$ , where  $I = [0, 1]$ .

**Definition.** Two paths  $f_0, f_1$  are **homotopic** if there exists a homotopy between  $f_0$  and  $f_1$  preserving the endpoints, that is a continuous map

$$F : I \times I \rightarrow X \\ (s, t) \mapsto f_t(s) ,$$

such that

$$f_t(0) = f_0(0), \quad f_t(1) = f_0(1),$$

for all  $t \in I$ , and

$$F(s, 0) = f_0(s), \quad F(s, 1) = f_1(s),$$

for all  $s \in I$ .

**Example.** Let  $X \subseteq \mathbb{R}^n$  be a convex set. Then all the paths in  $X$  are homotopic if they have the same endpoints.

*Proof.* Let  $f_0, f_1 : I \rightarrow X$  be two paths such that  $f_0(0) = f_1(0)$  and  $f_0(1) = f_1(1)$ . Define

$$f_t(s) = (1 - t)f_0(s) + tf_1(s).$$

□

**Lemma 1.1.** *Being homotopic is an equivalence relation on the set of paths with fixed endpoints. We will write  $f_0 \cong f_1$  for two homotopic paths  $f_0$  and  $f_1$ .*

*Proof.*

- $f$  is homotopic to  $f$ .
- If  $f_0$  is homotopic to  $f_1$  by a homotopy  $f_t$ , then  $f_1$  is homotopic to  $f_0$  by the homotopy  $f_{1-t}$ .
- If  $f_0$  is homotopic to  $f_1$  by a homotopy  $f_t$  and  $f_1 = g_0$  is homotopic to  $g_1$  by a homotopy  $g_t$ , then  $f_0$  is homotopic to  $g_1$  by the homotopy

$$h_t = \begin{cases} f_{2t} & 0 \leq t \leq \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Then

$$H : I \times I \rightarrow X \\ (s, t) \mapsto h_t(s)$$

is continuous because its restriction to the closed subsets  $I \times [0, 1/2]$  and  $I \times [1/2, 1]$  is continuous, since if the restriction to two closed subsets is continuous then the restriction to the union of these subsets is continuous.

□

Let  $X$  be a topological space and  $I = [0, 1]$ . If  $f : I \rightarrow X$  is a path,  $[f]$  is the class of all paths on  $X$  homotopic to  $f$ .

**Definition.** Let  $f, g : I \rightarrow X$  be two paths such that  $f(1) = g(0)$ . The **product path**  $f \cdot g$  is the path

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \leq s \leq 1 \end{cases}.$$

A convention is that whenever we write  $f \cdot g$  we implicitly assume  $f(1) = g(0)$ .

**Lemma 1.2.** *Let  $f_0, f_1, g_0, g_1$  be paths on  $X$  such that  $f_1 \cong f_0$  and  $g_0 \cong g_1$ . Then  $f_0 \cdot g_0 \cong f_1 \cdot g_1$ .*

*Proof.*

$$\begin{aligned} I \times I &\rightarrow X \\ (s, t) &\mapsto (f_t \cdot g_t)(s) \end{aligned}$$

is a homotopy between  $f_0 \cdot g_0$  and  $f_1 \cdot g_1$ . □

*Remark.* Let  $\phi : [0, 1] \rightarrow [0, 1]$  be continuous such that  $\phi(0) = 0$  and  $\phi(1) = 1$ . If  $f : I \rightarrow X$  is a path, then  $f \circ \phi \cong f$ . This is a **reparametrisation**.

*Proof.* Define

$$\phi_t(s) = (1 - t)\phi(s) + ts,$$

then  $f \circ \phi_t$  is a homotopy between  $f \circ \phi$  and  $f$ . □

For  $x \in X$ , let the **constant path** at  $x$  be

$$\begin{aligned} c_x : I &\rightarrow X \\ s &\mapsto x \end{aligned}.$$

For a path  $f : I \rightarrow X$ , define

$$\begin{aligned} f^{-1} : I &\rightarrow X \\ s &\mapsto f(1 - s) \end{aligned}.$$

**Lemma 1.3.** *Let  $f, g, h : I \rightarrow X$  be paths. Then*

1.  $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$ ,
2.  $f \cdot c_{f(1)} \cong f$  and  $c_{f(0)} \cdot f \cong f$ , and
3.  $f \cdot f^{-1} \cong c_{f(0)}$  and  $f^{-1} \cdot f \cong c_{f(1)}$ .

*Proof.*

1.  $((f \cdot g) \cdot h) \phi = f \cdot (g \cdot h)$ , where

$$\phi(s) = \begin{cases} \frac{s}{2} & s \in [0, \frac{1}{2}] \\ s - \frac{1}{4} & s \in [\frac{1}{2}, \frac{3}{4}] \\ 2s - 1 & s \in [\frac{3}{4}, 1] \end{cases},$$

so  $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$  by reparametrisation.

2. Again reparametrisation, by

$$\psi(s) = \begin{cases} 2s & s \in [0, \frac{1}{2}] \\ 1 & s \in [\frac{1}{2}, 1] \end{cases}, \quad \chi(s) = \begin{cases} 0 & s \in [0, \frac{1}{2}] \\ 2s - 1 & s \in [\frac{1}{2}, 1] \end{cases}.$$

3. Define

$$H(s, t) = \begin{cases} f(\max\{1 - 2s, t\}) & s \in [0, \frac{1}{2}] \\ f(\max\{2s - 1, t\}) & s \in [\frac{1}{2}, 1] \end{cases}.$$

$H$  is continuous, and

$$H(s, 0) = f^{-1} \cdot f, \quad H(s, 1) = c_{f(1)}.$$

The inverse is similar. □

**Definition.** A **loop** with **basepoint**  $x_0 \in X$  is a path  $f : I \rightarrow X$  such that  $f(0) = f(1) = x_0$ .

**Definition.** Denote by  $\pi_1(X, x_0)$  the set of homotopy classes  $[f]$  of loops  $f : I \rightarrow X$  with basepoint  $x_0$ .

**Proposition 1.4.**  $\pi_1(X, x_0)$  is a group with product  $[f][g] = [f \cdot g]$  and neutral element  $c_{x_0} : I \rightarrow X$ , the constant path at  $x_0$ .

*Proof.* Follows directly from Lemma 1.2 and Lemma 1.3.  $\square$

**Definition.**  $\pi_1(X, x_0)$  is the **fundamental group** of  $X$  at  $x_0$ .

**Example.** Let  $X \subseteq \mathbb{R}^n$  be a convex set and  $x_0 \in X$ . Then  $\pi_1(X, x_0) = 0$ .

*Proof.*  $X$  is convex gives that all loops are homotopic to each other.  $\square$

**Example.**

- The fundamental group of a space  $X$  with the trivial topology is trivial, since  $X$  is simply-connected, because all maps  $f : I \rightarrow X$  are continuous, so path-connected and all paths are homotopic.
- The fundamental group of a space  $X$  with the discrete topology is trivial, since  $f : I \rightarrow X$  continuous gives  $f$  constant.

Assume  $x_0, x_1 \in X$  such that  $x_0$  and  $x_1$  are in the same path component of  $X$ . Let  $h : I \rightarrow X$  be a path such that  $h(0) = x_0$  and  $h(1) = x_1$ . Define

$$\begin{aligned} \beta_h : \pi_1(X, x_1) &\rightarrow \pi_1(X, x_0) \\ [f] &\mapsto [h \cdot f \cdot h^{-1}] \end{aligned} .$$

This is well-defined by Lemma 1.2.

**Proposition 1.5.**  $\beta_h : \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$  is an isomorphism.

*Proof.* It is a homomorphism.

$$\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot h^{-1}] = [h \cdot f \cdot h^{-1}] [h \cdot g \cdot h^{-1}] = \beta_h[f] \cdot \beta_h[g],$$

and  $\beta_h[c_{x_1}] = [c_{x_1}]$ . It is bijective with  $(\beta_h)^{-1} = \beta_{h^{-1}}$ .  $\square$

If  $X$  is path-connected, we often write  $\pi_1(X)$  instead of  $\pi_1(X, x_0)$ .

**Definition.**  $X$  is **simply-connected** if it is path-connected and  $\pi_1(X) = 0$ .

**Proposition 1.6.**  $X$  is simply-connected if and only if there exists a unique homotopy class of paths between any two points of  $X$ .

*Proof.*

$\implies$  There exists a path between any two points. Let  $f, g$  be two paths from  $x_0$  to  $x_1$  for  $x_0, x_1 \in X$ .  $f \cdot g^{-1} \cong g \cdot g^{-1}$  gives  $f \cong f \cdot g^{-1} \cdot g \cong g \cdot g^{-1} \cdot g \cong g$ .

$\impliedby$   $X$  is path-connected.  $x_1 = x_0$  gives that all loops at  $x_0$  are homotopic to each other, so  $\pi_1(X) = 0$ .  $\square$



### 1.1.2 The fundamental group of the circle

Goal is to show that  $\pi_1(S^1) \cong \mathbb{Z}$ .

Lecture 4  
Friday  
18/01/19

**Definition.** A **covering space** of a space  $X$  is a space  $\tilde{X}$  and a continuous map  $p : \tilde{X} \rightarrow X$  such that for each  $x \in X$  there is an open  $U \subseteq X$  such that

- $p^{-1}(U) = \bigcup_{j \in J} \tilde{U}_j$ , where  $\tilde{U}_j \subseteq \tilde{X}$  is open,
- $\tilde{U}_i \cap \tilde{U}_j = \emptyset$  if  $i \neq j$ , and
- $p|_{\tilde{U}_j} : \tilde{U}_j \rightarrow U$  is a homeomorphism for all  $j \in J$ .

Such a  $U$  is called **evenly covered**. The  $\tilde{U}_j$  are called **sheets**.

**Example.**

$$\begin{aligned} p : \mathbb{R} &\rightarrow S^1 \\ s &\mapsto (\cos(2\pi s), \sin(2\pi s)) \end{aligned}$$

**Definition.** Let  $p : \tilde{X} \rightarrow X$  be a covering space. A **lift** of a continuous map  $f : Y \rightarrow X$  is a continuous map  $\tilde{f} : Y \rightarrow \tilde{X}$  such that  $p\tilde{f} = f$ , so

$$\begin{array}{ccc} & \tilde{X} & \\ \tilde{f} \nearrow & \downarrow p & \\ Y & \xrightarrow{f} & X \end{array}$$

**Proposition 1.7** (Unique lifting property). *Let  $p : \tilde{X} \rightarrow X$  be a covering space and  $f : Y \rightarrow X$  be a continuous map. If there are two lifts  $\tilde{f}_1, \tilde{f}_2 : Y \rightarrow \tilde{X}$  of  $f$  such that  $\tilde{f}_1(y) = \tilde{f}_2(y)$  for some  $y \in Y$  and if  $Y$  is connected, then  $\tilde{f}_1 = \tilde{f}_2$ .*

*Proof.* Let  $y \in Y$  and  $U \subseteq X$  be an evenly covered neighbourhood of  $f(y)$ . Then

$$p^{-1}(U) = \bigcup_j \tilde{U}_j.$$

Let  $\tilde{U}_1$  be the sheet such that  $\tilde{f}_1(y) \in \tilde{U}_1$ , and let  $\tilde{U}_2$  be the sheet such that  $\tilde{f}_2(y) \in \tilde{U}_2$ . Let  $N \subseteq Y$  be open and  $y \in N$  such that  $\tilde{f}_1(N) \subseteq \tilde{U}_1$  and  $\tilde{f}_2(N) \subseteq \tilde{U}_2$ . We have  $p\tilde{f}_1 = p\tilde{f}_2$ .

$$\tilde{f}_1(y) = \tilde{f}_2(y) \iff \tilde{U}_1 = \tilde{U}_2 \iff \tilde{f}_1|_N = \tilde{f}_2|_N.$$

Let

$$A = \{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\},$$

so  $A$  is open and  $Y \setminus A$  is open. Thus  $A \neq \emptyset$  gives  $A = Y$ .  $\square$

**Proposition 1.8** (Homotopy lifting property). *Let  $p : \tilde{X} \rightarrow X$  be a covering space and  $F : Y \times I \rightarrow X$  be a continuous map such that there exists a lift  $\tilde{f}_0 : Y \times \{0\} \rightarrow \tilde{X}$  of  $F|_{Y \times \{0\}}$ . Then there is a unique lift  $\tilde{F} : Y \times I \rightarrow \tilde{X}$  of  $F$  such that  $\tilde{F}|_{Y \times \{0\}} = \tilde{f}_0$ .*

*Proof.* Let  $y_0 \in Y$  and  $t \in I$ . There are open  $y_0 \in N_t \subseteq Y$  and  $t \in (a_t, b_t) \subseteq I$  such that  $F(N_t \times (a_t, b_t)) \subseteq U \subseteq X$ , where  $U \subseteq X$  is open and evenly covered. Compactness of  $I$  gives that there exist

$$0 = t_0 < \dots < t_m = 1,$$

and there exists  $y_0 \in N \subseteq Y$  open such that  $F(N \times [t_i, t_{i+1}]) \subseteq U_i \subseteq X$ , where  $U_i \subseteq X$  is open and evenly covered. We inductively construct a lift  $\tilde{F}|_{N \times I}$  of  $F|_{N \times I}$ .

- $\tilde{F}|_{N \times [0,0]} = \tilde{f}_0|_{N \times [0,0]}$  exists.
- Assume the lift has been constructed on  $N \times [0, t_i]$ . Let  $\tilde{U}_i \subseteq \tilde{X}$  be such that  $p|_{\tilde{U}_i}: \tilde{U}_i \rightarrow U_i$  such that  $\tilde{F}(y_0, t_i) \subseteq \tilde{U}_i$ . After shrinking  $N$ , may assume  $\tilde{F}(N \times \{t_i\}) \subseteq \tilde{U}_i$ . Define  $\tilde{F}$  on  $N \times [t_i, t_{i+1}]$  to be composition of  $F$  with the homeomorphism  $p^{-1}: U_i \rightarrow \tilde{U}_i$ .

After finitely many steps we obtain a lift  $\tilde{F}: N \times I \rightarrow \tilde{X}$ , where  $y_0 \in N \subseteq Y$  is open, so for each  $y \in Y$  there is a neighbourhood  $N_y \subseteq Y$  such that  $F|_{N_y \times I}: N_y \times I \rightarrow X$  lifts. For all  $y \in Y$ ,  $\{y\} \times I$  is connected and can be lifted, so Proposition 1.7 gives that the lift of  $N \times I$  is unique. Thus there is a unique lift  $\tilde{F}: Y \times I \rightarrow \tilde{X}$ .  $\square$

**Example.** Let  $X$  be a topological space and  $A$  be discrete. Then  $p: X \times A \rightarrow X$  is a covering space. This is the **trivial covering**. (Exercise: show the unique lifting property and the homotopy lifting property for the trivial covering)

**Corollary 1.9.** Let  $f: I \rightarrow X$  be a path,  $f(0) = x_0$ , and  $p: \tilde{X} \rightarrow X$  be a covering space. For each  $\tilde{x}_0 \in p^{-1}(x_0)$ , there is a unique lift  $\tilde{f}: I \rightarrow \tilde{X}$  such that  $\tilde{f}(0) = \tilde{x}_0$ .

*Proof.* Proposition 1.8 for  $Y$  a point.  $\square$

**Theorem 1.10.** Let  $x_0 = (1, 0) \in S^1$ .  $\pi_1(S^1, x_0)$  is the infinite cyclic group generated by the homotopy class of the loop

$$\begin{aligned} \omega: I &\rightarrow S^1 \\ s &\mapsto (\cos(2\pi s), \sin(2\pi s)) \end{aligned}$$

*Remark.*

- $[\omega]^n = [\omega_n]$ , where

$$\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns)).$$

- 

$$\begin{aligned} p: \mathbb{R} &\rightarrow S^1 \\ s &\mapsto (\cos(2\pi s), \sin(2\pi s)) \end{aligned}$$

is a covering space.

- $\omega_n$  lifts to

$$\begin{aligned} \tilde{\omega}_n: I &\rightarrow \mathbb{R} \\ s &\mapsto ns \end{aligned}$$

such that  $\tilde{\omega}_n(0) = 0$  and  $\tilde{\omega}_n(1) = n$ .

*Proof of Theorem 1.10.*

- If  $f: I \rightarrow S^1$  be a loop at  $x_0$ , then the homotopy lifting property gives that there exists a lift  $\tilde{f}: I \rightarrow \mathbb{R}$  such that  $\tilde{f}(0) = 0$ . Since  $p(\tilde{f}(1)) = f(1) = x_0$ , then  $\tilde{f}(1) = n$  for some  $n \in \mathbb{Z}$ .  $\tilde{\omega}_n: I \rightarrow \mathbb{R}$  is another path such that  $\tilde{\omega}_n(0) = 0$  and  $\tilde{\omega}_n(1) = n$ , so  $\tilde{f} \cong \tilde{\omega}_n$ . Let  $F: I \times I \rightarrow \mathbb{R}$  be a homotopy equivalence between  $\tilde{f}$  and  $\tilde{\omega}_n$ . Then  $pF: I \times I \rightarrow S^1$  gives a homotopy between  $p\tilde{f} = f$  and  $p\tilde{\omega}_n = \omega_n$ .
- Let  $m, n \in \mathbb{Z}$  and assume  $\omega_m \cong \omega_n$ . Let  $F: I \times I \rightarrow S^1$  be a homotopy.

$$F(0, t) = \omega_m(t), \quad F(1, t) = \omega_n(t), \quad F(s, 0) = F(s, 1) = x_0,$$

for all  $s, t \in I$ . The unique lifting property gives that  $\tilde{\omega}_n, \tilde{\omega}_m: I \rightarrow \mathbb{R}$  are unique lifts such that  $\tilde{\omega}_n(0) = 0 = \tilde{\omega}_m(0)$ . The homotopy lifting property gives that  $F$  lifts uniquely to a homotopy  $\tilde{F}: I \times I \rightarrow \mathbb{R}$  between  $\tilde{\omega}_n$  and  $\tilde{\omega}_m$ , and  $\tilde{F}(s, 1) \in \mathbb{Z}$  for all  $s \in I$ . Thus  $\tilde{F}(s, 1) = n = m$ , so  $\omega_m \cong \omega_n$  if and only if  $n = m$ .  $\square$

Lecture 5 is a problem class.

**Theorem 1.11.** *Every non-constant polynomial  $p \in \mathbb{C}[z]$  has a root in  $\mathbb{C}$ .*

*Proof.* May assume

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n.$$

Assume  $p$  has no roots in  $\mathbb{C}$ . For each  $r \in \mathbb{R}_{\geq 0}$  we obtain a loop

$$\begin{aligned} f_r : I &\rightarrow \mathbb{C} \\ s &\mapsto \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}, \end{aligned}$$

so  $|f_r(s)| = 1$ .  $f_r(0) = 1$  and  $f_r(1) = 1$ , so  $f_r$  is a loop based at 1.  $f_0$  is the constant loop at 1.  $f_r(s)$  depends continuously on  $r$ , so  $f_r \cong f_0$  for all  $r \in \mathbb{R}_{\geq 0}$  and  $[f_r] = [f_0] = 0 \in \pi_1(S^1)$ . Fix  $r \in \mathbb{R}_{\geq 0}$  such that  $r > 1$  and  $r > |a_1| + \cdots + |a_n|$ . For  $|z| = r$  we have

$$|z^n| > (|a_1| + \cdots + |a_n|) |z^{n-1}| \geq |a_1 z^{n-1}| + \cdots + |a_n| \geq |a_1 z^{n-1} + \cdots + a_n|.$$

Hence, for  $0 \leq t \leq 1$  the polynomial  $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$  has no root  $z$  with  $|z| = r$ . Define

$$F_r(t, s) = \frac{p_t(re^{2\pi i s})/p_t(r)}{|p_t(re^{2\pi i s})/p_t(r)|}.$$

$F_r(0, s) = \omega_n(s)$  and  $F_r(1, s) = f_r(s)$ , so  $[\omega_n] = [f_r] = 0 \in \pi_1(S^1)$ . Theorem 1.10 gives that  $n = 0$ , so  $p$  is constant.  $\square$

See Hatcher Theorem 1.9 and Theorem 1.10 for more applications.

**Proposition 1.12.** *Let  $X, Y$  be topological spaces,  $x_0 \in X$ , and  $y_0 \in Y$ . Then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

*Proof.* A map

$$\begin{aligned} f : Z &\rightarrow X \times Y \\ z &\mapsto (g(z), h(z)) \end{aligned}$$

is continuous if and only if  $g : Z \rightarrow X$  and  $h : Z \rightarrow Y$  are continuous. For  $Z = I$ ,

$$\{ \text{loops in } X \times Y \text{ based at } (x_0, y_0) \} \quad \longleftrightarrow \quad \{ \text{loops in } X \text{ based at } x_0 \} \times \{ \text{loops in } Y \text{ based at } y_0 \}.$$

Two loops

$$\begin{aligned} f_1 : I &\rightarrow X \times Y \\ s &\mapsto (g_1(s), h_1(s)) \end{aligned}, \quad \begin{aligned} f_2 : I &\rightarrow X \times Y \\ s &\mapsto (g_2(s), h_2(s)) \end{aligned}$$

are homotopic if and only if  $g_1 \cong g_2$  and  $h_1 \cong h_2$ , so there is a bijection

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

$f_1 \cdot f_2 = (g_1 \cdot g_2, h_1 \cdot h_2)$  and the constant loop is mapped to the constant loop, so this is also a group isomorphism.  $\square$

**Example.** The torus  $S^1 \times S^1$  has

$$\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z}^2.$$

### 1.1.3 Induced homomorphisms

Let  $X, Y$  be topological spaces,  $x_0 \in X$ , and  $\phi : X \rightarrow Y$ . An observation is that  $\phi$  induces a homomorphism

$$\begin{aligned} \phi_* : \pi_1(X, x_0) &\rightarrow \pi_1(Y, \phi(x_0)) \\ [f] &\mapsto [\phi f] \end{aligned} .$$

$\phi_*$  is well-defined, since if  $f_t$  is a homotopy between the loops  $f_0$  and  $f_1$  based at  $x_0$ , then  $\phi f_t$  is a homotopy of loops between  $\phi f_0$  and  $\phi f_1$ . Moreover,

$$\phi(f \cdot g) = (\phi f) \cdot (\phi g),$$

and  $\phi$  maps the constant path at  $x_0$  to the constant path at  $\phi(x_0)$ , so  $\phi$  is a homomorphism.

**Proposition 1.13.**

1. Let  $\psi : X \rightarrow Y$  and  $\phi : Y \rightarrow Z$  be continuous maps between topological spaces,  $x_0 \in X$ , and

$$\begin{aligned} \psi_* : \pi_1(X, x_0) &\rightarrow \pi_1(Y, \psi(x_0)), & \phi_* : \pi_1(Y, \psi(x_0)) &\rightarrow \pi_1(Z, \phi\psi(x_0)), \\ (\phi\psi)_* : \pi_1(X, x_0) &\rightarrow \pi_1(Z, \phi\psi(x_0)). \end{aligned}$$

Then  $(\phi\psi)_* = \phi_*\psi_*$ .

2. Let  $id_X : X \rightarrow X$  be the identity then

$$(id_X)_* : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$$

is the identity.

*Proof.*

1. Let  $f : I \rightarrow X$  be a loop at  $x_0$ , then

$$(\phi\psi)_*([f]) = [(\phi\psi)f] = [\phi(\psi f)] = \phi_*([\psi f]) = \phi_*\psi_*([f]).$$

2.  $(id_X)_*([f]) = [id_X f] = [f]$ .

□

These two observations yield in particular that if  $\phi : X \rightarrow Y$  is a homeomorphism with inverse  $\psi : Y \rightarrow X$ , then

$$\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \phi(x_0))$$

is an isomorphism with inverse  $\psi_*$ .

**Proposition 1.14.** Let  $\phi : X \rightarrow Y$  be a homotopy equivalence. Then

$$\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \phi(x_0))$$

is an isomorphism for all  $x_0 \in X$ .

Recall that if  $x_0, x_1 \in X$  and  $h : I \rightarrow X$  is a path such that  $h(0) = x_0$  and  $h(1) = x_1$ , then we obtain an isomorphism

$$\begin{aligned} \beta_h : \pi_1(X, x_1) &\rightarrow \pi_1(X, x_0) \\ [f] &\mapsto [h \cdot f \cdot h^{-1}] \end{aligned} .$$

Lecture 7  
Friday  
25/01/19

**Lemma 1.15.** Let  $\phi_t : X \rightarrow Y$  be a homotopy and  $x_0 \in X$ . Define the path

$$\begin{aligned} h : I &\rightarrow Y \\ s &\mapsto \phi_s(x_0) \end{aligned} ,$$

where  $h(0) = \phi_0(x_0)$  and  $h(1) = \phi_1(x_0)$ . Then  $(\phi_0)_* = \beta_h(\phi_1)_*$ , that is the following diagram commutes.

$$\begin{array}{ccc} & \pi_1(Y, \phi_1(x_0)) & \\ (\phi_1)_* \nearrow & \downarrow \sim \beta_h & \\ \pi_1(X, x_0) & & \\ (\phi_0)_* \searrow & \downarrow & \\ & \pi_1(Y, \phi_0(x_0)) & \end{array} .$$

*Proof.* For  $t \in I$ , define the path

$$\begin{aligned} h_t : I &\rightarrow X \\ s &\mapsto h(ts) \end{aligned} ,$$

where  $h_t(0) = \phi_0(x_0)$  and  $h_t(1) = h(t) = \phi_t(x_0)$ . Let  $f$  be a loop at  $x_0$ . Define

$$F_t = h_t \cdot (\phi_t f) \cdot h_t^{-1}.$$

Then  $F_t$  is a loop at  $\phi_0(x_0)$ , which is continuous in  $t$ . So  $F_t$  is a homotopy of loops between

$$F_0 = h_0 \cdot (\phi_0 f) \cdot h_0^{-1} \cong \phi_0 f, \quad F_1 = h_1 \cdot (\phi_1 f) \cdot h_1^{-1} = h \cdot (\phi_1 f) \cdot h^{-1}.$$

Hence

$$(\phi_0)_*([f]) = [\phi_0 f] = [h \cdot (\phi_1 f) \cdot h^{-1}] = \beta_h([\phi_1 f]) = \beta_h(\phi_1)_*([f]).$$

□

Lemma 1.15 implies in particular the following.

**Corollary 1.16.** If  $\psi : X \rightarrow X$  is continuous and  $\psi \cong id_X$ , then

$$\psi_* : \pi_1(X, x_0) \rightarrow \pi_1(X, \psi(x_0))$$

is an isomorphism for all  $x_0 \in X$ .

*Proof.* By Lemma 1.15 there is a path  $h$  from  $\psi(x_0)$  to  $x_0$  such that

$$\begin{array}{ccc} & \pi_1(X, x_0) & \\ (id_X)_* \nearrow & \downarrow \sim \beta_h & \\ \pi_1(X, x_0) & & \\ \psi_* \searrow & \downarrow & \\ & \pi_1(X, \psi(x_0)) & \end{array} ,$$

so  $\psi_* = \beta_h$  hence an isomorphism. □

*Proof of Proposition 1.14.* Let  $\phi : X \rightarrow Y$  be a homotopy equivalence. Let  $\psi : Y \rightarrow X$  be a homotopy inverse of  $\phi$ , that is  $\phi\psi \cong id_Y$  and  $\psi\phi \cong id_X$ .

$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\phi(x_0)) \xrightarrow{\phi_*} \pi_1(Y, \psi\phi\psi(x_0)).$$

Have to show that  $\phi_*$  is bijective. The observation above gives that  $(\psi\phi)_* = \psi_*\phi_*$  is an isomorphism, so  $\phi_*$  is injective and  $\psi_*$  is surjective. Similarly  $(\phi\psi)_* = \phi_*\psi_*$  is an isomorphism, so  $\psi_*$  is injective and  $\phi_*$  is surjective. □

**Lemma 1.17.** *Let  $X$  be a topological space and  $x_0 \in X$ . Assume*

$$X = \bigcup_{\alpha \in \Lambda} A_\alpha,$$

*such that*

- *the  $A_\alpha$  are all open and path-connected,*
- *$x_0 \in A_\alpha$  for all  $\alpha \in \Lambda$ , and*
- *all the intersections  $A_\alpha \cap A_\beta$  are path-connected for all  $\alpha, \beta \in \Lambda$ .*

*If  $f$  is a loop in  $X$  at  $x_0$ , then we can write  $[f] = [h_1] \dots [h_m]$ , such that the  $h_i$  are loops at  $x_0$ , and each contained in a single  $A_{\alpha_i}$ .*

*Proof.*  $f$  is continuous, so for all  $s \in I$  there is an open neighbourhood  $V_s$  such that  $f(V_s)$  is contained in some  $A_\alpha$ . We can choose  $V_s$  to be an interval  $(a_s, b_s)$  such that  $f([a_s, b_s]) \subseteq A_\alpha$ .  $I$  is compact gives that a finite number of such intervals cover  $I$ , so there is a partition

$$0 = s_0 < \dots < s_m = 1,$$

such that  $f([s_{i-1}, s_i]) \subseteq A_{\alpha_i}$  for some  $\alpha_i$ . Let  $f_i$  be the path obtained by restricting  $f$  to  $[s_{i-1}, s_i]$ , and rescaling.  $f \cong f_1 \dots f_m$  for  $f_i \subseteq A_{\alpha_i}$  and  $A_{\alpha_i} \cap A_{\alpha_j}$  is path-connected. Let  $g_i$  be a path from  $x_0$  to  $f(s_i)$  in  $A_{\alpha_i} \cap A_{\alpha_{i+1}}$ . Let  $g_0, g_m$  be the constant loops at  $x_0$ .  $h_i = g_{i-1} \cdot f_i \cdot g_i^{-1}$  is a loop based at  $x_0$  and  $h_i \subseteq A_{\alpha_i}$ . Thus

$$f \cong (g_0 \cdot f_1 \cdot g_1^{-1}) \cdot \dots \cdot (g_{m-1} \cdot f_m \cdot g_m^{-1}),$$

so  $[f] = [h_1] \dots [h_m]$ . □

**Example.** Möbius strip  $M$  deformation retracts to  $S^1$ . Thus  $\phi : M \rightarrow S^1$  is a homotopy equivalence, so  $\pi_1(M) \cong \pi_1(S^1) \cong \mathbb{Z}$ .

**Example.** There is no deformation retraction of  $S^1$  to a point  $p \in S^1$  because  $\pi_1(S^1) \not\cong \pi_1(p)$ .

**Example.** There is no retraction of the disc  $D^2$  to its boundary  $S^1 \subseteq D^2$ .

*Proof.* Assume there is a retraction  $r : D^2 \rightarrow S^1$ , consider the embedding  $i : S^1 \hookrightarrow D^2$ . Then  $ri = id_{S^1}$ . Thus

$$\begin{array}{ccccc} \pi_1(S^1) & \xrightarrow{i_*} & \pi_1(D^2) & \xrightarrow{r_*} & \pi_1(S^1) \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array},$$

so  $r_* i_* (\pi_1(S^1)) = 0$  but  $r_* i_* = (ri)_* = id_{\pi_1(S^1)}$ , a contradiction. □

**Theorem 1.18** (Brouwer fixed point theorem). *Let  $h : D^2 \rightarrow D^2$  be a continuous map. Then  $h$  has a fixed point, that is there exists  $x \in D^2$  such that  $h(x) = x$ .*

*Proof.* Assume  $h(x) \neq x$  for all  $x \in D^2$ . Define  $r : D^2 \rightarrow S^1$  by defining  $r(x)$  to be the intersection of the ray starting at  $h(x)$  towards  $x$  with  $S^1$ .  $r$  is continuous, and if  $x \in S^1$ , then  $r(x) = x$ , so  $r$  is a retraction, a contradiction. □

Lemma 1.17 gives that if  $U_1, U_2 \subseteq X$  are open and path-connected such that  $U_1 \cup U_2 = X$  and  $U_1 \cap U_2$  is path-connected and  $x_0 \in U_1 \cap U_2$ , then every  $[f] \in \pi_1(X, x_0)$  can be factorised as  $[f] = [g_1][h_1] \dots [g_n][h_n]$  such that the  $g_i$  are loops at  $x_0$  contained in  $U_1$  and the  $h_i$  are loops at  $x_0$  contained in  $U_2$ . In other words,  $i_1 : U_1 \hookrightarrow X$  and  $i_2 : U_2 \hookrightarrow X$ , so

$$(i_1)_* : \pi_1(U_1, x_0) \rightarrow \pi_1(X, x_0), \quad (i_2)_* : \pi_1(U_2, x_0) \rightarrow \pi_1(X, x_0).$$

Lemma 1.17 gives that  $(i_1)_*(\pi_1(U_1, x_0)) \cup (i_2)_*(\pi_1(U_2, x_0))$  generate  $\pi_1(X, x_0)$ .

**Proposition 1.19.**  $\pi_1(S^n) = 0$  if  $n \geq 2$ .

*Proof.* Let  $U_1 = S^n \setminus \{(1, 0, \dots, 0)\}$  and  $U_2 = S^n \setminus \{(-1, 0, \dots, 0)\}$ . Then  $U_1 \cong \mathbb{R}^n$  and  $U_2 \cong \mathbb{R}^n$ , by stereographic projection.  $U_1 \cup U_2 = S^n$  and  $U_1 \cap U_2$  is path-connected. Let  $x_0 \in U_1 \cap U_2$ .  $\pi_1(U_1, x_0) = 0$  and  $\pi_1(U_2, x_0) = 0$ , so Lemma 1.17 gives that  $\pi_1(S^n, x_0) = 0$ . □

## 1.2 Seifert-van Kampen theorem

### 1.2.1 Free products with amalgamation

**Definition.** If  $S$  is a set, then  $F_S$  is the **free group** on  $S$ . We can write any group  $G$  as a quotient of some free group  $F_S$ ,

$$G = \frac{F}{\langle\langle R \rangle\rangle},$$

where  $\langle\langle R \rangle\rangle$  is the **normal closure** of  $R \subseteq F_S$ , the smallest normal subgroup of  $F_S$  containing  $R$ . We write  $G = \langle S \mid R \rangle$ . This is called a **presentation** of  $G$ .

Let  $G_0, G_1, G_2$  be groups, and  $f_1 : G_0 \rightarrow G_1$  and  $f_2 : G_0 \rightarrow G_2$  be homomorphisms.

**Definition.** A group  $H$  together with homomorphisms  $h_1 : G_1 \rightarrow H$  and  $h_2 : G_2 \rightarrow H$  such that  $h_1 f_1 = h_2 f_2$  is an **amalgamated product** of  $G_1$  and  $G_2$  over  $G_0$  if it satisfies the following universal property. For every group  $G$  and all homomorphisms  $h'_1 : G_1 \rightarrow G$  and  $h'_2 : G_2 \rightarrow G$  such that  $h'_1 f_1 = h'_2 f_2$ , there exists a unique homomorphism  $\alpha : H \rightarrow G$  such that  $h'_1 = \alpha h_1$  and  $h'_2 = \alpha h_2$ .

$$\begin{array}{ccccc} G_0 & \xrightarrow{f_1} & G_1 & & \\ f_2 \downarrow & & \downarrow h_1 & \searrow h'_1 & \\ G_2 & \xrightarrow{h_2} & H & \xrightarrow{\exists! \alpha} & G \\ & \searrow h'_2 & & & \end{array}$$

**Theorem 1.20.** Given  $f_1 : G_0 \rightarrow G_1$  and  $f_2 : G_0 \rightarrow G_2$ . Then there exists an amalgamated product, unique up to isomorphism. We denote it by  $G_1 *_{G_0} G_2$ .

*Proof.* Worksheet 2. □

$G_0 = \{id\}$  is the **free product**. We write  $G_1 * G_2$  instead of  $G_1 *_{\{id\}} G_2$ . Let  $G_1 = \langle S_1 \mid R_1 \rangle$  and  $G_2 = \langle S_2 \mid R_2 \rangle$ . Then  $G_1 * G_2 = \langle S_1 \sqcup S_2 \mid R_1 \cup R_2 \rangle$ , with injections  $G_i \hookrightarrow G_1 * G_2$  for  $i = 1, 2$ . More generally,

$$G_1 * G_2 \cong \frac{G_1 *_{G_0} G_2}{N}.$$

where  $N$  is the normal closure of the set

$$\left\{ f_1(g) f_2(g)^{-1} \mid g \in G_0 \right\} \subseteq G_1 * G_2.$$

### 1.2.2 The Seifert van-Kampen theorem

**Theorem 1.21** (Seifert-van Kampen). Let  $X$  be a topological space and  $U_1, U_2 \subseteq X$  be open and path-connected such that  $X = U_1 \cup U_2$  and  $U_1 \cap U_2$  is path-connected and let  $x_0 \in U_1 \cap U_2$ . Then

$$\pi_1(X, x_0) \cong \pi_1(U_1, x_0) *_{\pi_1(U_1 \cap U_2, x_0)} \pi_2(U_2, x_0) \cong \frac{\pi_1(U_1, x_0) * \pi_1(U_2, x_0)}{N},$$

where  $N$  is the normal closure of the set

$$\left\{ (j_1)_*(\omega) (j_2)_*(\omega)^{-1} \mid \omega \in \pi_1(U_1 \cap U_2, x_0) \right\},$$

and  $j_i : U_i \hookrightarrow X$ .

$$\begin{array}{ccc} U_1 \cap U_2 & \xrightarrow{i_1} & U_1 \\ i_2 \downarrow & & \downarrow j_1 \\ U_2 & \xrightarrow{j_2} & X \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \pi_1(U_1 \cap U_2, x_0) & \xrightarrow{(i_1)_*} & \pi_1(U_1, x_0) \\ (i_2)_* \downarrow & & \downarrow (j_1)_* \\ \pi_1(U_2, x_0) & \xrightarrow{(j_2)_*} & \pi_1(U_1, x_0) *_{\pi_1(U_1 \cap U_2, x_0)} \pi_1(U_2, x_0) \end{array}.$$

*Proof.* Consider the natural homomorphism

$$\Phi : \pi_1(U_1, x_0) * \pi_1(U_2, x_0) \rightarrow \pi_1(X, x_0).$$

$\Phi$  is surjective by Lemma 1.17.  $N \subseteq \text{Ker}(\Phi)$ . Want to show that  $N = \text{Ker}(\Phi)$ . A **factorisation** of an element  $[f] \in \pi_1(X, x_0)$  is a formal product  $[f_1] \dots [f_k]$  such that

- each  $f_i$  is a loop at  $x_0$  in one of the  $U_i$  and  $[f_i] \in \pi_1(U_i, x_0)$  is its homotopy class, and
- the loop  $f_1 \dots f_k$  is homotopic to  $f$  in  $X$ .

A factorisation of  $[f]$  is a word in  $\pi_1(U_1, x_0) * \pi_1(U_2, x_0)$  that is mapped to  $[f]$  by  $\Phi$ . Two factorisations of  $[f]$  are **equivalent** if they are related by finitely many of the following two moves.

- If  $[f_i]$  and  $[f_{i+1}]$  lie in the same group  $\pi_1(U_i, x_0)$ , exchange  $[f_i][f_{i+1}]$  with  $[f_i \cdot f_{i+1}]$ . These are the relations in  $\pi_1(U_i, x_0) * \pi_1(U_i, x_0)$ .
- If  $f_i$  is a loop in  $U_1 \cap U_2$ , consider  $[f_i]$  as an element in  $\pi_1(U_1, x_0)$  instead of  $\pi_1(U_2, x_0)$ , and vice versa. These are the relations in  $\pi_1(U_1, x_0) * \pi_1(U_2, x_0) / N$ .

Given  $[f] \in \pi_1(X, x_0)$ , we want to show that any two factorisations of  $[f]$  are equivalent. Let  $[f_1] \dots [f_k]$  and  $[f'_1] \dots [f'_l]$  be two factorisations of  $[f]$ , so the two loops  $f_1 \dots f_k$  and  $f'_1 \dots f'_l$  are homotopic. Let  $F : I \times I \rightarrow X$  be a homotopy. By compactness, there exist

$$0 = s_0 < \dots < s_m = 1, \quad 0 = t_0 < \dots < t_n = 1,$$

such that  $R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$  and  $F(R_{i,j}) \subseteq U_1$  or  $F(R_{i,j}) \subseteq U_2$ . May assume  $0 = s_0 < \dots < s_m = 1$  subdivides the products  $f_1 \dots f_k$  and  $f'_1 \dots f'_l$ . Relabel the  $R_{i,j}$  to  $R_1, \dots, R_{mn}$ .

$mn - m + 1$	$\dots$	$mn$
$\vdots$	$\ddots$	$\vdots$
1	$\dots$	$m$

A path  $\gamma$  in  $I \times I$  from left to right gives a loop  $F|_\gamma$  in  $X$  at  $x_0$ . Let  $\gamma_r$  be the path separating the first  $r$  rectangles from the others, so

$$F|_{\gamma_0} \cong f_1 \dots f_k, \quad F|_{\gamma_{mn}} = f'_1 \dots f'_l.$$

Let  $v$  be a grid point. Choose a path  $g_v$  in  $X$  from  $x_0$  to  $F(v)$ , such that  $g_v$  is contained in  $U_1 \cap U_2$  if  $F(v) \in U_1 \cap U_2$  and in a single  $U_i$  otherwise. This gives us a factorisation of  $[F|_{\gamma_r}]$  into loops only contained in  $U_1$  or  $U_2$ . The factorisations associated to  $\gamma_r$  and  $\gamma_{r+1}$  are equivalent, because the homotopy between  $F|_{\gamma_r}$  and  $F|_{\gamma_{r+1}}$  by pushing  $\gamma_r$  through  $R_r$  takes place within a single  $U_i$ .  $\square$

**Theorem 1.22** (Seifert-van Kampen, strong version). *Let  $X$  be a path-connected topological space such that*

- $X = \bigcup_\alpha A_\alpha$ ,
- $A_\alpha$ ,  $A_\alpha \cap A_\beta$ , and  $A_\alpha \cap A_\beta \cap A_\gamma$  are open and path-connected for all  $\alpha, \beta, \gamma$ , and
- $x_0 \in \bigcap_\alpha A_\alpha$ .

Then

$$\pi_1(X, x_0) \cong \frac{*_{\alpha} \pi_1(A_\alpha, x_0)}{N},$$

where  $N \subseteq *_{\alpha} \pi_1(A_\alpha, x_0)$  is the normal closure of the set

$$\left\{ (i_{\alpha\beta})_*(\omega) (i_{\beta\alpha})_*(\omega)^{-1} \mid \omega \in \pi_1(A_\alpha \cap A_\beta) \right\},$$

and  $i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$  is the inclusion.

Lecture 10  
Friday  
01/02/19



**Example.** Let  $S^1 \vee S^1$  be the wedge product. Fix  $x \in S^1$  and  $y \in S^1$ . Then

$$S^1 \vee S^1 = \frac{S^1 \sqcup S^1}{x \sim y} = \overset{b}{\underset{a}{\mathbf{O}}} \cdot \overset{a}{\underset{b}{\mathbf{O}}}.$$

Let

$$A = \mathbf{O} \cdot (, \quad B = ) \cdot \mathbf{O}, \quad A \cap B = ) \cdot (.$$

$\pi_1(A) \cong \langle b \rangle \cong \mathbb{Z}$ ,  $\pi_1(B) \cong \langle a \rangle \cong \mathbb{Z}$ , and  $\pi_1(A \cap B) = \{id\}$ .  $A$ ,  $B$ , and  $A \cap B$  are open and path-connected. Van Kampen gives

$$\pi_1(S^1 \vee S^1) \cong \pi_1(A) * \pi_1(B) \cong \mathbb{Z} * \mathbb{Z} \cong F_{\{a,b\}}.$$

More generally, let  $X = S_{a_1}^1 \vee \cdots \vee S_{a_n}^1$ . By induction,

$$\pi_1(X) = \mathbb{Z} * \cdots * \mathbb{Z} \cong F_{\{a_1, \dots, a_n\}}.$$

Similarly, let  $X = \bigvee_{\alpha \in \Lambda} S_{\alpha}^1$ . Strong version of van Kampen gives

$$\pi_1(X) = \bigast_{\alpha \in \Lambda} \mathbb{Z} = F_{\Lambda}.$$

**Example.** Let  $T$  be a torus and  $x_0 \in T$ . Let

$$A = T \setminus \{\text{small closed disc } D\}, \quad B = \{\text{open set that contains } D \text{ and } x_0\}.$$

- $A$  is homotopy equivalent to  $S^1 \vee S^1$ , so  $\pi_1(A) \cong F_{\{a,b\}}$ .
- $B$  is homeomorphic to  $D^2$ , so  $\pi_1(B) = \{id\}$ .
- $A \cap B$  is homotopy equivalent to  $S^1$ , so  $\pi_1(A \cap B) \cong \mathbb{Z}$ .

$A$ ,  $B$ , and  $A \cap B$  are open and path-connected. Van Kampen gives

$$\pi_1(T) \cong \frac{\pi_1(A)}{\langle \langle i_* (\pi_1(A \cap B)) \rangle \rangle},$$

where  $i : A \cap B \hookrightarrow A$ . Then

$$i_* : \begin{array}{ccc} \pi_1(A \cap B) = \langle \omega \rangle & \rightarrow & \pi_1(A) \\ \omega & \mapsto & aba^{-1}b^{-1} \end{array},$$

so

$$\pi_1(T) \cong \frac{F_{\{a,b\}}}{\langle \langle aba^{-1}b^{-1} \rangle \rangle} = \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2.$$

### 1.2.3 Applications to CW-complexes

Let  $X$  be a path-connected topological space. Let  $Y$  be the space obtained by attaching 2-cells  $\{e_{\alpha}^2\}$  to  $X$  along maps  $\phi_{\alpha} : \partial D^2 = S^1 \rightarrow X$ . Consider the loops

$$\phi'_{\alpha} : \begin{array}{ccc} I & \rightarrow & X \\ s & \mapsto & \phi_{\alpha}(\cos(2\pi s), \sin(2\pi s)) \end{array},$$

based at  $\phi'_{\alpha}(0)$ . Let  $\gamma_{\alpha}$  be a path from  $x_0$  to  $\phi'_{\alpha}(0)$  for each  $\alpha$ . Then  $\gamma_{\alpha} \cdot \phi_{\alpha} \cdot \gamma_{\alpha}^{-1}$  is a loop at  $x_0$ . After attaching  $e_{\alpha}^2$ , the loop  $\gamma_{\alpha} \cdot \phi_{\alpha} \cdot \gamma_{\alpha}^{-1}$  is homotopic to the constant loop at  $x_0$ . Let  $N \subseteq \pi_1(X, x_0)$  be the normal closure of all the elements of the form  $[\gamma_{\alpha} \cdot \phi_{\alpha} \cdot \gamma_{\alpha}^{-1}]$ . The inclusion  $i : X \hookrightarrow Y$  yields

$$i_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, x_0),$$

and  $N \subseteq \text{Ker}(i_*)$ .

**Proposition 1.23.** *This inclusion  $i : X \hookrightarrow Y$  induces a surjection*

$$i_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, x_0),$$

and  $\text{Ker}(i_*) = N$ , so

$$\pi_1(Y, x_0) \cong \frac{\pi_1(X, x_0)}{N}.$$

*Proof.* Construct a space  $Z$  from  $Y$  by attaching a strip  $I \times I$  to  $Y$  by identifying the lower edge  $I \times \{0\}$  with the path  $\gamma_\alpha$  and the right edge  $\{1\} \times I$  with an arch on  $e_\alpha^2$ . Attach all the left edges of the strips with each other.  $Z$  deformation retracts to  $Y$ . Choose a point  $y_\alpha \in e_\alpha^2$  for each  $\alpha$ , such that  $y_\alpha$  is not contained in  $X$  or in the attached strip. Let

$$A = Z \setminus \bigcup_{\alpha} \{y_\alpha\}, \quad B = Z \setminus X.$$

- $A$  deformation retracts to  $X$ .
- $B$  is homotopy equivalent to a point.
- $A \cap B$  is homotopy equivalent to

$$\{\text{paths } \gamma_\alpha \text{ from } x_0 \text{ to loops } \phi'_\alpha\} = \overset{\phi'_\alpha}{\underset{\phi'_\alpha}{\text{O}}} \xrightarrow{\gamma_\alpha} x_0 \xleftarrow{\gamma_\alpha} \overset{\phi'_\alpha}{\underset{\phi'_\alpha}{\text{O}}}.$$

$A$ ,  $B$ , and  $A \cap B$  are open and path-connected. Van Kampen gives

$$\pi_1(Y) \cong \pi_1(Z) = \frac{\pi_1(A)}{\langle\langle j_*(\pi_1(A \cap B)) \rangle\rangle},$$

where  $j : A \cap B \hookrightarrow A$  is the inclusion. So  $\langle\langle j_*(\pi_1(A \cap B)) \rangle\rangle$  is exactly  $N$ . Thus  $\pi_1(A) = \pi_1(X)$ .  $\square$

**Corollary 1.24.** *For every group  $G$  there exists a two-dimensional CW-complex  $X_G$  such that  $\pi_1(X_G) = G$ .*

*Proof.* Let  $G = \langle \{g_\alpha\} \mid \{r_\beta\} \rangle$  be a presentation of  $G$ , that is

$$G = \frac{F_{\{g_\alpha\}}}{\langle\langle \{r_\beta\} \rangle\rangle}.$$

Seen last time that  $\pi_1(\bigvee_{g_\alpha} S_{g_\alpha}^1) = F_{\{g_\alpha\}}$ . Each word  $r_\beta$  defines a loop in  $\bigvee_{g_\alpha} S_{g_\alpha}^1$ . Attach 2-cells to  $\bigvee_{g_\alpha} S_{g_\alpha}^1$  along the loops defined by the relations  $\{r_\beta\}$ . Call this new CW-complex  $Y$ . Proposition 1.23 gives that

$$\pi_1(Y, x_0) \cong \frac{\pi_1(X, x_0)}{\langle\langle \{r_\beta\} \rangle\rangle} \cong \frac{F_{\{g_\alpha\}}}{\langle\langle \{r_\beta\} \rangle\rangle} \cong G.$$

$\square$

*Remark.* Let  $X = \bigcup_n X^n$  be a CW-complex, path-connected. Proposition 1.23 can be used to show the following two facts.

- The inclusion  $X^1 \hookrightarrow X$  induces a surjective homomorphism  $\pi_1(X^1) \rightarrow \pi_1(X)$ .
- The inclusion  $X^2 \hookrightarrow X$  induces an isomorphism  $\pi_1(X^2) \rightarrow \pi_1(X)$ .

### 1.3 Covering spaces

#### 1.3.1 Lifting properties

Let  $X$  be a topological space. Recall that a **covering space** is  $p : \tilde{X} \rightarrow X$  such that each  $x \in X$  has an open neighbourhood  $U$  such that

$$p^{-1}(U) = \bigcup_{\alpha} \tilde{U}_{\alpha},$$

where  $U_{\alpha}$  are pairwise disjoint and  $p|_{\tilde{U}_{\alpha}} : \tilde{U}_{\alpha} \rightarrow U$  is a homeomorphism for all  $\alpha$ .

**Example.**

$$\begin{array}{lll} \mathbb{R} & \rightarrow & S^1 \\ s & \mapsto & (\cos(2\pi s), \sin(2\pi s)) \end{array}, \quad \begin{array}{lll} S^1 & \rightarrow & S^1 \\ z & \mapsto & z^n \end{array}, \quad \text{O} \cdot \text{O} \cdot \text{O} \rightarrow S^1 \vee S^1 = \text{O} \cdot \text{O}.$$

Let  $f : Y \rightarrow X$  be a continuous map. A **lift** of  $f$  is a continuous map  $\tilde{f} : Y \rightarrow \tilde{X}$  such that  $p\tilde{f} = f$ , where  $p : \tilde{X} \rightarrow X$  is a covering space. Let  $Y$  be connected.

- **Unique lifting property** states that if two lifts  $\tilde{f}_1$  and  $\tilde{f}_2$  of  $f$  coincide at one point, then they coincide on all of  $Y$ .
- **Homotopy lifting property** states that if  $f_t : Y \rightarrow X$  is a homotopy and  $\tilde{f}_0 : Y \rightarrow \tilde{X}$  is a lift of  $f_0$  then there exists a unique homotopy  $\tilde{f}_t : Y \rightarrow \tilde{X}$  of  $\tilde{f}_0$  that lifts  $f_t$ .

*Remark.*

- If  $Y$  is a point, this is called the **path lifting property**. Let  $f : I \rightarrow X$  be a path with  $f(0) = x_0$ . If  $\tilde{x}_0 \in p^{-1}(x_0)$ , then there is a unique path  $\tilde{f} : I \rightarrow \tilde{X}$  lifting  $f$  and starting at  $\tilde{x}_0$ .
- In particular, the lift of a constant path is constant.
- This implies in particular that the lift of a homotopy of paths is again a homotopy of paths. The endpoints  $f_t(0)$  and  $f_t(1)$  define constant paths as  $t$  varies.

Fix  $x_0 \in X$  and  $\tilde{x}_0 \in \tilde{X}$  such that  $p(\tilde{x}_0) = x_0$ , so

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0).$$

To every element in  $\pi_1(X, x_0)$  we can associate a homotopy class of paths in  $\tilde{X}$  starting at  $\tilde{x}_0$ .

**Proposition 1.25.**

1.  $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$  is injective.
2.  $p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right) \subseteq \pi_1(X, x_0)$  consists of the homotopy classes of loops at  $x_0$  whose lifts to  $\tilde{X}$  starting at  $\tilde{x}_0$  are loops.

*Proof.*

1. Let  $\tilde{f}_0 : I \rightarrow \tilde{X}$  be a loop at  $\tilde{x}_0$  such that  $[\tilde{f}_0] \in \text{Ker}(p_*)$ , so  $p\tilde{f}_0 = f_0$  is homotopic to the constant loop at  $x_0$ . Let  $f_t : I \rightarrow X$  be a homotopy between  $\tilde{f}_0$  and the constant loop. Homotopy lifting property and remark gives that  $f_t$  lifts to a homotopy  $\tilde{f}_t$  of paths between  $\tilde{f}_0$  and the constant loop, so  $[\tilde{f}_0] = \text{id} \in \pi_1(\tilde{X}, \tilde{x}_0)$  and  $p_*$  is injective.
2. Let  $f : I \rightarrow X$  be a loop at  $x_0$  that lifts to a loop  $\tilde{f}$  at  $\tilde{x}_0$ . Then  $p\tilde{f} = f$ , so  $p_*([\tilde{f}]) = [f]$ . On the other hand, if  $f : I \rightarrow X$  is a loop at  $x_0$  such that there exists a loop  $\tilde{f} : I \rightarrow \tilde{X}$  at  $\tilde{x}_0$  with  $p_*([\tilde{f}]) = [f]$ , then  $f$  is homotopic to  $p\tilde{f}$ . Homotopy lifting property gives that there exists a loop  $\tilde{f}' : I \rightarrow \tilde{X}$  at  $\tilde{x}_0$  such that  $p\tilde{f}' = f$ .

□

Lecture 12  
Wednesday  
06/02/19

Let  $p : \tilde{X} \rightarrow X$  be a covering space. Let  $U \subseteq X$  be an evenly covered neighbourhood of  $x \in X$ . Let

$$p^{-1}(U) = \bigsqcup_{\alpha \in \Lambda} \tilde{U}_\alpha.$$

Then the cardinality  $|p^{-1}(x)|$  of  $p^{-1}(x)$  is exactly the cardinality of  $|\Lambda|$ . The set of sheets is in bijection with  $p^{-1}(x)$ . So the cardinality of  $p^{-1}(x)$  is locally constant. If  $X$  is connected, the cardinality of  $p^{-1}(x)$  is constant.

*Notation.* Let  $X, Y$  be topological spaces,  $x \in X$ , and  $y \in Y$ . A continuous map

$$f : (X, x) \rightarrow (Y, y)$$

is a continuous map  $f : X \rightarrow Y$  such that  $f(x) = y$ .

**Proposition 1.26.** *Let  $X, \tilde{X}$  be path-connected and*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

*be a covering space. Then the number of sheets of  $p$  equals the index of  $p_* \left( \pi_1 \left( \tilde{X}, \tilde{x}_0 \right) \right)$  in  $\pi_1(X, x_0)$ .*

*Proof.* Let  $g$  be a loop in  $X$  at  $x_0$  and  $\tilde{g}$  be its lift to  $\tilde{X}$  starting at  $\tilde{x}_0$ . Let  $H = p_* \left( \pi_1 \left( \tilde{X}, \tilde{x}_0 \right) \right)$  and let  $[h] \in H$ . Then  $h \cdot g$  lifts to a path  $\tilde{h} \cdot \tilde{g}$  in  $\tilde{X}$  starting at  $\tilde{x}_0$  with the same endpoint as  $\tilde{g}$ , because  $\tilde{h}$  is a loop, by Proposition 1.25. Define

$$\begin{aligned} \Phi : \{ \text{cosets of } H \text{ in } \pi_1(X, x_0) \} &\rightarrow p^{-1}(x_0) \\ H[g] &\mapsto \tilde{g}(1) \end{aligned}$$

so  $\Phi$  is well-defined. Want to show that  $\Phi$  is bijective.

- $\Phi$  is surjective because  $\tilde{X}$  is path-connected. Let  $\tilde{g}$  be a path in  $\tilde{X}$  from  $\tilde{x}_0$  to any point  $\tilde{x}'_0 \in p^{-1}(x_0)$ , then  $g = p \cdot \tilde{g}$  and  $\Phi(H[g]) = \tilde{x}'_0$ .
- $\Phi$  is injective, since if  $\Phi(H[g_1]) = \Phi(H[g_2])$  then the lift  $\tilde{g}_1 \cdot \tilde{g}_2^{-1}$  of  $g_1 \cdot g_2^{-1}$  defines a loop in  $\tilde{X}$  at  $\tilde{x}_0$ . Proposition 1.25 gives  $[g_1][g_2]^{-1} \in H$ , so  $H[g_1] = H[g_2]$ .

□

We say that a topological space  $X$  has a certain property  $(P)$  **locally** if for each point  $x \in X$  and each neighbourhood  $U$  of  $x$  there is an open neighbourhood  $V \subseteq U$  having this property  $(P)$ .

**Example.**  $X$  is locally path-connected or  $X$  is locally simply-connected.

**Proposition 1.27.** *Let*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

*be a covering space and*

$$f : (Y, y_0) \rightarrow (X, x_0)$$

*a continuous map, where  $Y$  is path-connected and locally path-connected. Then there is a lift*

$$\tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$$

*if and only if  $f_* \left( \pi_1(Y, y_0) \right) \subseteq p_* \left( \pi_1(\tilde{X}, \tilde{x}_0) \right)$ .*

$$\begin{array}{ccc} & & (\tilde{X}, \tilde{x}_0) \\ & \nearrow \tilde{f} & \downarrow p \\ (Y, y_0) & \xrightarrow{f} & (X, x_0) \end{array}.$$

*Proof.*

$\Rightarrow$  Clear, because  $f = p\tilde{f}$  implies  $f_* = p_*\tilde{f}_*$ .

$\Leftarrow$  Assume  $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ . For each  $y \in Y$  choose a path  $\gamma$  from  $y_0$  to  $y$ , so  $f\gamma$  is a path in  $X$  from  $x_0$  to  $f(y)$ . By path lifting, we can lift  $f\gamma$  to a path  $\tilde{f}\gamma$  in  $\tilde{X}$  starting at  $\tilde{x}_0$ . Define the map

$$\begin{array}{ccc} \tilde{f}: (Y, y_0) & \rightarrow & (\tilde{X}, \tilde{x}_0) \\ y & \mapsto & \tilde{f}\gamma(1) \end{array} .$$

$$\begin{array}{ccc} & \tilde{x}_0 & \xrightarrow[\tilde{f}\gamma']{\tilde{f}\gamma} \tilde{f}(y) \\ & \nearrow \tilde{f} & \downarrow p \\ y_0 & \xrightarrow[\gamma']{\gamma} y & \xrightarrow{f} x_0 \xrightarrow[\gamma']{\tilde{f}\gamma} f(y) \end{array} .$$

- This map is well-defined, that is does not depend on the choice of  $\gamma$ . Let  $\gamma'$  be another path from  $y_0$  to  $y$ . Then  $h_0 = (f\gamma') \cdot (f\gamma)^{-1}$  is a loop at  $x_0$  and  $[h_0] \in f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ . Proposition 1.25 gives that can lift  $h_0$  to a loop  $\tilde{h}_0$  at  $\tilde{x}_0$ . The first half of  $\tilde{h}_0$  is  $\tilde{f}\gamma'$  and the second half is  $\tilde{f}\gamma^{-1}$ , so  $\tilde{f}\gamma(1) = \tilde{f}\gamma'(1)$ . Thus  $\tilde{f}$  is well-defined.
- We have  $p\tilde{f} = f$ , so  $\tilde{f}$  lifts  $f$ .
- It remains to show that  $\tilde{f}$  is continuous. Let  $y \in Y$  and let  $U$  be an evenly covered neighbourhood of  $f(y)$ . Let  $\tilde{U}$  be the sheet above  $U$  such that  $\tilde{f}(y) \in \tilde{U}$ , so  $p|_{\tilde{U}}: \tilde{U} \rightarrow U$  is a homeomorphism. Let  $V \subseteq Y$  be a path-connected neighbourhood of  $y$  such that  $f(V) \subseteq U$ . Fix a path  $\gamma$  from  $y_0$  to  $y$ . Let  $y' \in V$  be arbitrary and  $\eta$  be a path from  $y$  to  $y'$ , so  $\gamma \cdot \eta$  is a path from  $y_0$  to  $y'$ . Then  $(f\gamma) \cdot (f\eta)$  is a path in  $U$  from  $x_0$  to  $f(y')$ .  $\tilde{f}\eta = (p|_{\tilde{U}})^{-1}f\eta$ , so  $\tilde{f}|_V = (p|_{\tilde{U}})^{-1}f$ . Thus  $\tilde{f}|_V: V \rightarrow \tilde{U}$  is continuous, so  $\tilde{f}$  is continuous.

□

Lecture 13  
Friday  
08/02/19

### 1.3.2 The classification of covering spaces

**Definition.** A covering space  $p: \tilde{X} \rightarrow X$  is a **universal cover** if  $\tilde{X}$  is simply-connected.

**Definition.** A topological space  $X$  is **semilocally simply-connected** if each  $x \in X$  has a neighbourhood  $U$  such that

$$i_*: \pi_1(U, x) \rightarrow \pi_1(X, x)$$

is trivial, where  $i: U \hookrightarrow X$  is the inclusion.

**Example.** Let  $X = \bigcup_n C_n \subseteq \mathbb{R}^2$  be the Hawaiian earrings, where  $C_n \subseteq \mathbb{R}^2$  is the circle of radius  $1/n$  and centre  $(1/n, 0)$ . Then  $X$  is not semilocally simply-connected.

**Proposition 1.28.** *If  $p: \tilde{X} \rightarrow X$  is a universal cover, then  $X$  is semilocally simply-connected.*

*Proof.* Let  $U \subseteq X$  be an evenly covered neighbourhood of  $x_0 \in X$ ,  $\tilde{U} \subseteq \tilde{X}$  be a sheet over  $U$ , and  $\gamma \subseteq U$  be a loop at  $x_0$ , so  $\gamma$  lifts to a loop  $\tilde{\gamma} \subseteq \tilde{U}$  at  $\tilde{x}_0$ .  $\tilde{\gamma}$  is homotopic to the constant loop at  $\tilde{x}_0$ . Compose this homotopy with  $p$  gives that  $\gamma$  is homotopic to the constant loop at  $x_0$  in  $X$ , so

$$\pi_1(U, x_0) \rightarrow \pi_1(X, x_0)$$

is trivial. □

**Theorem 1.29.** *Let  $X$  be path-connected, locally path-connected, and semilocally simply-connected. Then there exists a universal cover  $p : \tilde{X} \rightarrow X$ .*

*Remark.* If

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

is a universal cover, each point  $\tilde{x} \in \tilde{X}$  can be joined to  $\tilde{x}_0$  by a unique homotopy class of paths, by Proposition 1.6.

$$\{\text{points in } \tilde{X}\} \longleftrightarrow \{[\gamma] \mid \gamma \text{ is a path in } \tilde{X} \text{ starting at } \tilde{x}_0\} \longleftrightarrow \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

by the homotopy lifting property.

*Proof.* Let  $x_0 \in X$ , and

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}, \quad p : \begin{array}{ccc} \tilde{X} & \rightarrow & X \\ [\gamma] & \mapsto & \gamma(1) \end{array}.$$

Have to

1. give  $\tilde{X}$  a topology,
2. show that  $p : \tilde{X} \rightarrow X$  is a covering, and
3. show that  $\tilde{X}$  is simply-connected.

Recall that a **basis** for a topology on a set  $Y$  is a collection  $\mathcal{B}$  of subsets such that

- $Y = \bigcup_{U \in \mathcal{B}} U$ , and
- if  $U_1, U_2 \in \mathcal{B}$  and  $y \in U_1 \cap U_2$  then there exists  $V \in \mathcal{B}$  such that  $y \in V$  and  $V \subseteq U_1 \cap U_2$ .

A basis defines a topology on  $Y$ , by  $A \subseteq Y$  is open if and only if  $A$  is the union of elements of  $\mathcal{B}$ . A map  $f : Z \rightarrow Y$  is continuous if and only if  $f^{-1}(U)$  is open for all  $U \in \mathcal{B}$ .

1. Let  $\mathcal{U}$  be the collection of all path-connected open sets  $U \subseteq X$  such that  $\pi_1(U) \rightarrow \pi_1(X)$  is trivial. Then  $X = \bigcup_{U \in \mathcal{U}} U$  because  $X$  is semilocally simply-connected. Let  $U_1, U_2 \in \mathcal{U}$  and  $y \in U_1 \cap U_2$ , and let  $y \in V \subseteq U_1 \cap U_2$  be path-connected and open.

$$\begin{array}{ccccc} V & \hookrightarrow & U_1 & \hookrightarrow & X \\ & & & & \\ \pi_1(V) & \longrightarrow & \pi_1(U_1) & \xrightarrow{\text{trivial}} & \pi_1(X) \\ & & \searrow & \text{trivial} & \nearrow \end{array},$$

so  $V \in \mathcal{U}$  gives that  $\mathcal{U}$  is a basis for the topology on  $X$ . For  $U \in \mathcal{U}$  and  $\gamma$  a path in  $X$  from  $x_0$  to a point in  $U$ , we define

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ a path in } U \text{ such that } \eta(0) = \gamma(1)\} \subseteq \tilde{X}.$$

$U_{[\gamma]}$  only depends on the class  $[\gamma]$ , so  $p|_{U_{[\gamma]}} : U_{[\gamma]} \rightarrow U$  is bijective. Surjective because  $U$  is path-connected and injective because all paths  $\eta$  in  $U$  with the same endpoint are homotopic. Claim that  $\{U_{[\gamma]}\}$  forms a basis on  $\tilde{X}$ .

- $\bigcup_{U \in \mathcal{U}} U_{[\gamma]} = \tilde{X}$ , because  $\bigcup_{U \in \mathcal{U}} U = X$ .
- Observe that if  $[\gamma'] \in U_{[\gamma]}$  then  $U_{[\gamma]} = U_{[\gamma']}$ . If  $\gamma' = \gamma \cdot \eta$  for  $\eta$  a path in  $U$ , then elements in  $U_{[\gamma']}$  have the form  $[\gamma' \cdot \mu]$ , so  $U_{[\gamma']} \subseteq U_{[\gamma]}$ . Elements in  $U_{[\gamma]}$  have the form  $[\gamma \cdot \mu] = [\gamma \cdot \eta \cdot \eta^{-1} \cdot \mu] = [\gamma' \cdot \eta^{-1} \cdot \mu]$ , so  $U_{[\gamma]} \subseteq U_{[\gamma']}$ . Consider  $U_{[\gamma]}$  and  $U_{[\gamma']}$  and let  $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$ , so  $U_{[\gamma]} = U_{[\gamma']}$  and  $V_{[\gamma']} = V_{[\gamma']}$ . Let  $W \in \mathcal{U}$  such that  $W \subseteq U \cap V$  and such that  $\gamma''(1) \in W$ , so  $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$  and  $[\gamma''] \in W_{[\gamma'']}$ . This proves the claim.

2.  $p|_{U_{[\gamma]}}: U_{[\gamma]} \rightarrow U$  is a homeomorphism. It is bijective, let  $V_{[\gamma']} \subseteq U_{[\gamma]}$  be an element of the basis, so  $p(V_{[\gamma']}) = V \in \mathcal{U}$ .  $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$ . Thus  $p: \tilde{X} \rightarrow X$  is continuous. If  $U \in \mathcal{U}$ , then

$$p^{-1}(U) = \bigsqcup_{[\gamma]} U_{[\gamma]},$$

so  $p: \tilde{X} \rightarrow X$  is a covering space.

3. Let  $\tilde{x}_0 \in \tilde{X}$  be the class of the constant path at  $x_0$ . Let  $[\gamma] \in \tilde{X}$  be arbitrary.  $\gamma: [0, 1] \rightarrow X$  and  $\gamma(0) = x_0$ . Let  $\gamma_t$  be the path in  $X$  defined by

$$\gamma_t(s) = \begin{cases} \gamma(s) & s \in [0, t] \\ \gamma(t) & s \in [t, 1] \end{cases}.$$

Then

$$\begin{array}{ccc} \tilde{\gamma}: & I & \rightarrow \tilde{X} \\ & t & \mapsto [\gamma_t] \end{array}$$

is a path in  $\tilde{X}$  from  $\tilde{x}_0$  to  $[\gamma]$ , so  $\tilde{X}$  is path-connected. Recall that  $p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$  consists of the classes of loops at  $x_0$  in  $X$  that lifts to loops in  $\tilde{X}$  at  $\tilde{x}_0$ . Let  $[\gamma] \in p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$ . Then  $\gamma$  lifts to a loop at  $\tilde{x}_0$  by  $t \mapsto [\gamma_t]$ . Because it is a loop we have  $\tilde{x}_0 = [\gamma_1] = [\gamma]$ , so  $\gamma$  is homotopic to the constant loop. Thus  $p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right) = \{id\}$ , so  $\tilde{X}$  is simply-connected. □

Lecture 14 is a problem class.

Let  $p: \tilde{X} \rightarrow X$  be a covering space, so  $p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right) \subseteq \pi_1(X, x_0)$ .

**Proposition 1.30.** *Let  $X$  be path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup  $H \subseteq \pi_1(X, x_0)$  there is a covering space  $p: X_H \rightarrow X$  such that  $p_*\left(\pi_1(X_H, \tilde{x}_0)\right) = H$  for some basepoint  $x_0$ .*

*Proof.* Let  $\tilde{X}$  be as constructed above. Define  $X_H = \tilde{X}/\sim$ , where  $[\gamma] \sim [\gamma']$  if  $\gamma(1) = \gamma'(1)$  and  $[\gamma \cdot (\gamma')^{-1}] \in H$ . This is an equivalence relation.

- $[\gamma] \sim [\gamma]$  because  $id \in H$ .
- $[\gamma] \sim [\gamma']$  gives  $[\gamma'] \sim [\gamma]$  because  $H$  contains all its inverses.
- $[\gamma] \sim [\gamma']$  and  $[\gamma'] \sim [\gamma'']$  gives  $[\gamma] \sim [\gamma'']$  because  $H$  is closed under product.

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \tilde{X} \\ \downarrow & \swarrow p & \sim \\ X & & X_H \end{array}.$$

Let  $U_{[\gamma]}, U_{[\gamma']}$  be basis neighbourhoods. If  $[\gamma] \sim [\gamma']$  then  $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$ , so  $p$  is a covering space, and  $p^{-1}(U) = \bigcup_{[\gamma]} U_{[\gamma]}$ . Let  $\tilde{x}_0 \in X_H$  be the equivalence class of the constant path  $c_{x_0}$  at  $x_0$ . Let  $\gamma$  be a loop in  $X$  at  $x_0$  such that  $[\gamma] \in p_*\left(\pi_1(X_H, \tilde{x}_0)\right)$ . Again  $t \mapsto [\gamma_t]$  is a lift of  $\gamma$  at  $\tilde{x}_0$ .

$$t \mapsto [\gamma_t] \text{ is a loop in } X_H \iff [\gamma_1] = [\gamma] = [c_{x_0}] \text{ in } X_H \iff [\gamma] \sim [c_{x_0}] \iff \gamma \in H.$$

□

**Definition.** We say that two covering spaces  $p_1 : \widetilde{X}_1 \rightarrow X$  and  $p_2 : \widetilde{X}_2 \rightarrow X$  are **isomorphic** if there exists a homeomorphism  $f : \widetilde{X}_1 \rightarrow \widetilde{X}_2$  such that

$$\begin{array}{ccc} \widetilde{X}_1 & \xrightarrow{f} & \widetilde{X}_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array} .$$

**Proposition 1.31.** Let  $X$  be path-connected and locally path-connected and  $x_0 \in X$ . Two path-connected covering spaces  $p_1 : \widetilde{X}_1 \rightarrow X$  and  $p_2 : \widetilde{X}_2 \rightarrow X$  are isomorphic via an isomorphism  $f : \widetilde{X}_1 \rightarrow \widetilde{X}_2$  mapping a basepoint  $\widetilde{x}_1 \in p_1^{-1}(x_0)$  to a basepoint  $\widetilde{x}_2 \in p_2^{-1}(x_0)$  if and only if

$$(p_1)_* \left( \pi_1 \left( \widetilde{X}_1, \widetilde{x}_1 \right) \right) = (p_2)_* \left( \pi_1 \left( \widetilde{X}_2, \widetilde{x}_2 \right) \right) .$$

*Proof.*

$\Rightarrow$  If

$$f : \left( \widetilde{X}_1, \widetilde{x}_1 \right) \rightarrow \left( \widetilde{X}_2, \widetilde{x}_2 \right)$$

is an isomorphism, then  $p_1 = p_2 f$ , so

$$(p_1)_* \left( \pi_1 \left( \widetilde{X}_1, \widetilde{x}_1 \right) \right) \subseteq (p_2)_* \left( \pi_1 \left( \widetilde{X}_2, \widetilde{x}_2 \right) \right) ,$$

and  $p_2 = p_1 f^{-1}$ , so

$$(p_2)_* \left( \pi_1 \left( \widetilde{X}_2, \widetilde{x}_2 \right) \right) \subseteq (p_1)_* \left( \pi_1 \left( \widetilde{X}_1, \widetilde{x}_1 \right) \right) .$$

$\Leftarrow$  Assume

$$(p_1)_* \left( \pi_1 \left( \widetilde{X}_1, \widetilde{x}_1 \right) \right) = (p_2)_* \left( \pi_1 \left( \widetilde{X}_2, \widetilde{x}_2 \right) \right) .$$

By lifting criterion in Proposition 1.27, we can lift  $p_1$  to a continuous map

$$\widetilde{p}_1 : \left( \widetilde{X}_1, \widetilde{x}_1 \right) \rightarrow \left( \widetilde{X}_2, \widetilde{x}_2 \right) ,$$

and  $p_2$  to a continuous map

$$\widetilde{p}_2 : \left( \widetilde{X}_2, \widetilde{x}_2 \right) \rightarrow \left( \widetilde{X}_1, \widetilde{x}_1 \right) ,$$

so  $p_1 \widetilde{p}_2 = p_2$  and  $p_2 \widetilde{p}_1 = p_1$ .

$$\begin{array}{ccc} \left( \widetilde{X}_1, \widetilde{x}_1 \right) & \xrightarrow{\widetilde{p}_1} & \left( \widetilde{X}_2, \widetilde{x}_2 \right) \\ & \searrow p_1 & \swarrow p_2 \\ & (X, x_0) & \end{array} .$$

$\widetilde{p}_1 \widetilde{p}_2$  fixes the point  $\widetilde{x}_2 \in \widetilde{X}_2$ . By the unique lifting property in Proposition 1.7,  $\widetilde{p}_1 \widetilde{p}_2 = id_{\widetilde{x}_2}$ . Similarly,  $\widetilde{p}_2 \widetilde{p}_1 = id_{\widetilde{x}_1}$ , so  $\widetilde{p}_1$  is an isomorphism.

□

Fix  $x_0 \in X$ ,  $\widetilde{x}_1 \in p_1^{-1}(x_0)$ , and  $\widetilde{x}_2 \in p_2^{-1}(x_0)$ . A **basepoint preserving isomorphism**

$$f : \left( \widetilde{X}_1, \widetilde{x}_1 \right) \rightarrow \left( \widetilde{X}_2, \widetilde{x}_2 \right)$$

is an isomorphism such that  $f(\widetilde{x}_1) = \widetilde{x}_2$ .

Lecture 16  
Friday  
15/02/19



**Theorem 1.32** (Galois correspondence). *Let  $X$  be path-connected, locally path-connected, and semilocally simply-connected, and  $x_0 \in X$ . Then*

1. *there is a bijection*

$$\left\{ \begin{array}{l} \text{path-connected covering spaces } p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0) \\ \text{up to basepoint preserving isomorphisms} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{subgroups} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\},$$

2. *if we ignore the basepoints, this correspondence gives a bijection*

$$\left\{ \begin{array}{l} \text{path-connected covering spaces } p : \tilde{X} \rightarrow X \\ \text{up to isomorphisms} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{l} \text{conjugacy classes of subgroups} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\}.$$

*Proof.*

1. To a covering space  $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  we associate the subgroup  $p_* \left( \pi_1(\tilde{X}, \tilde{x}_0) \right) \subseteq \pi_1(X, x_0)$ . Proposition 1.30 and Proposition 1.31 show that this is well-defined on the isomorphism classes and it is bijective.
2. Let  $p : \tilde{X} \rightarrow X$  be a covering space and  $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ . Let  $H_i = p_* \left( \pi_1(\tilde{X}, \tilde{x}_i) \right) \subseteq \pi_1(X, x_0)$ , for  $i = 1, 2$ . Let  $\tilde{\gamma}$  be a path from  $\tilde{x}_1$  to  $\tilde{x}_2$ . Let  $\gamma = p\tilde{\gamma}$  be a loop at  $x_0$ . Let  $[f] \in \pi_1(X, x_0)$ . Then  $[f] \in H_1$  if and only if the lift  $\tilde{f}$  is a loop at  $\tilde{x}_1$ .  $\tilde{\gamma}^{-1} \cdot \tilde{f} \cdot \tilde{\gamma}$  is a loop at  $\tilde{x}_2$  gives  $p_* \left( \tilde{\gamma}^{-1} \cdot \tilde{f} \cdot \tilde{\gamma} \right) = \gamma^{-1} \cdot f \cdot \gamma$ , so  $[\gamma]^{-1} [f] [\gamma] \in H_2$ . Thus  $[\gamma]^{-1} H_1 [\gamma] \subseteq H_2$ . Similarly,  $[\gamma] H_2 [\gamma]^{-1} \subseteq H_1$ . Conversely, let  $H_1 \subseteq \pi_1(X, x_0)$  as above and  $[\delta] \in \pi_1(X, x_0)$  be an arbitrary element. Let  $\tilde{\delta}$  be a lift of  $\delta$  such that  $\tilde{\delta}(0) = \tilde{x}_0$  and define  $x_3 = \tilde{\delta}(1)$ . Then the same construction yields  $p_* \left( \pi_1(\tilde{X}, \tilde{x}_3) \right) = [\delta]^{-1} H_1 [\delta]$ .

□

### 1.3.3 Deck transformations and group actions

**Definition.** Let  $p : \tilde{X} \rightarrow X$  be a covering space. A **deck-transformation** is an isomorphism from  $\tilde{X}$  to itself.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & \tilde{X} \\ & \searrow p & \swarrow p \\ & X & \end{array}.$$

The group of deck-transformations is denoted by  $G(\tilde{X})$ .

**Example.**

- Let

$$\begin{array}{ccc} p : \mathbb{R} & \rightarrow & S^1 \subset \mathbb{C} \\ t & \mapsto & e^{2\pi i t} \end{array}.$$

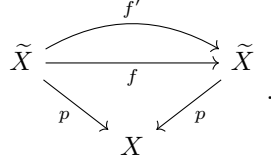
$f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $p(f(t)) = p(t)$  if and only if  $e^{2\pi i f(t)} = e^{2\pi i t}$ , if and only if  $f(t) = t + n$ , so  $G(\mathbb{R}) \cong \mathbb{Z}$ .

- Let

$$\begin{array}{ccc} p : S^1 & \rightarrow & S^1 \\ z & \mapsto & z^n \end{array}.$$

Then  $G(S^1) \cong \mathbb{Z}/n\mathbb{Z}$ .

An observation is that if  $\tilde{X}$  is path-connected then  $f \in G(\tilde{X})$  is uniquely determined by where it sends a single point.



If  $f(x) = f'(x)$  for a single  $x$ , by unique lifting  $f = f'$ . So the identity is the only deck-transformation with a fixed point.

**Definition.** A covering space  $p : \tilde{X} \rightarrow X$  is **normal**, or **regular**, or **Galois**, if for each  $x \in X$  and every pair  $\tilde{x}, \tilde{x}' \in p^{-1}(x)$  there is an  $f \in G(\tilde{X})$  such that  $f(\tilde{x}) = \tilde{x}'$ .

**Example.**

- $p : \mathbb{R} \rightarrow S^1$  is normal.
- $p : S^1 \rightarrow S^1$  is normal.

**Proposition 1.33.** *Let*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

*be a path-connected covering space, and  $X$  be path-connected and locally path-connected. Then  $p : \tilde{X} \rightarrow X$  is normal if and only if  $H = p_* \left( \pi_1(\tilde{X}, \tilde{x}_0) \right) \subseteq \pi_1(X, x_0)$  is a normal subgroup.*

*Proof.* Let  $\tilde{x}_1 \in p^{-1}(x_0)$ , let  $\tilde{\gamma}$  be a path from  $\tilde{x}_0$  to  $\tilde{x}_1$  and  $\gamma = p(\tilde{\gamma})$ . Then  $[\gamma]$  conjugates  $H$  to  $p_* \left( \pi_1(\tilde{X}, \tilde{x}_1) \right)$  so  $[\gamma] H [\gamma]^{-1} = H$ , if and only if  $H = p_* \left( \pi_1(\tilde{X}, \tilde{x}_1) \right)$ , by Proposition 1.31 if and only if  $f(\tilde{x}_0) = \tilde{x}_1$ . So  $G(\tilde{X})$  acts transitively on  $p^{-1}(x_0)$  if and only if  $H \subseteq \pi_1(X, x_0)$  is a normal subgroup. Let  $x'_0 \in X$  be another point and  $h$  a path from  $x_0$  to  $x'_0$ . Let  $\tilde{h}$  be a lift of  $h$  such that  $\tilde{h}(0) = \tilde{x}_0$ . Set  $\tilde{x}'_0 = \tilde{h}(1)$  and  $p(\tilde{x}'_0) = x'_0$ . Then

$$\begin{array}{ccc}
 \pi_1(\tilde{X}, \tilde{x}_0) & \xrightarrow{\beta_{\tilde{h}}} & \pi_1(\tilde{X}, \tilde{x}'_0) \\
 p_* \downarrow & & \downarrow p_* \\
 \pi_1(X, x_0) & \xrightarrow{\beta_h} & \pi_1(X, x'_0)
 \end{array}$$

$H \subseteq \pi_1(X, x_0)$  is normal if and only if  $p_* \left( \pi_1(\tilde{X}, \tilde{x}'_0) \right) \subseteq \pi_1(X, x'_0)$  is normal, as before if and only if  $G(\tilde{X})$  acts transitively on  $p^{-1}(x'_0)$ .  $\square$

**Proposition 1.34.** *Let*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

*be a covering space, and  $X, \tilde{X}$  be path-connected and locally path-connected. Let  $H = p_* \left( \pi_1(\tilde{X}, \tilde{x}_0) \right)$  and  $N(H) \subseteq \pi_1(X, x_0)$  be the normaliser of  $H$ . Then  $G(\tilde{X})$  is isomorphic to  $N(H)/H$ . In particular,*

- if  $\tilde{X}$  is normal, then  $G(\tilde{X}) \cong \pi_1(X, x_0)/H$ , and
- if  $\tilde{X}$  is the universal cover, then  $G(\tilde{X}) \cong \pi_1(X, x_0)$ .

*Proof.* Exercise: read the proof of this in Hatcher.  $\square$

**Example.** Let  $X = S^1 \vee S^1$ , so  $\pi_1(X) = F_{\{a,b\}}$ . Then the following are covering spaces.

- A normal covering space

$$\tilde{X} = \overset{a}{\underset{\cdot}{\circ}} \overset{b}{\underset{\cdot}{\circ}} \overset{a}{\underset{\cdot}{\circ}} \overset{b}{\underset{\cdot}{\circ}} \rightarrow \overset{a}{\underset{\cdot}{\circ}} \overset{b}{\underset{\cdot}{\circ}} = X, \quad p_* \left( \pi_1 \left( \tilde{X}, \widetilde{x_0} \right) \right) = \langle a, b^2, bab^{-1} \rangle \stackrel{2}{\subseteq} F_{\{a,b\}}.$$

In general, a two-oriented graph is a covering space of  $X$ .

- Not a normal covering space

$$\tilde{X} = \overset{a}{\underset{\cdot}{\circ}} \overset{b}{\underset{\cdot}{\circ}} \overset{a}{\underset{\cdot}{\circ}} \overset{b}{\underset{\cdot}{\circ}} \rightarrow \overset{a}{\underset{\cdot}{\circ}} \overset{b}{\underset{\cdot}{\circ}} = X, \quad p_* \left( \pi_1 \left( \tilde{X}, \widetilde{x_0} \right) \right) = \langle b^2, bab^{-1}, a^2, aba^{-1} \rangle.$$

- A normal covering space

$$\tilde{X} = \dots \overset{a}{\underset{b}{\circ}} \overset{a}{\underset{b}{\circ}} \overset{a}{\underset{b}{\circ}} \dots \rightarrow \overset{a}{\underset{\cdot}{\circ}} \overset{b}{\underset{\cdot}{\circ}} = X, \quad p_* \left( \pi_1 \left( \tilde{X}, \widetilde{x_0} \right) \right) = \langle b^n ab^{-n} \mid n \in \mathbb{Z} \rangle.$$

Universal cover is a tree.

**Example.** Let  $T = S^1 \times S^1$ , so  $\pi_1(T) = \mathbb{Z}^2$ . This is abelian, so all covering spaces are normal. Universal cover is

$$\begin{aligned} \mathbb{R}^2 &\rightarrow S^1 \times S^1 \\ (s, t) &\mapsto (e^{2\pi i s}, e^{2\pi i t}) \end{aligned}$$

since  $\mathbb{R}^2$  is simply connected. (Exercise: check that it is a covering space) More generally, if  $p : \tilde{X} \rightarrow X$  and  $q : \tilde{Y} \rightarrow Y$  are covering spaces then

$$\begin{aligned} \tilde{X} \times \tilde{Y} &\rightarrow X \times Y \\ (x, y) &\mapsto (p(x), q(y)) \end{aligned}$$

is again a covering space. For example,

$$\begin{aligned} S^1 \times S^1 &\rightarrow S^1 \times S^1 \\ (z_1, z_2) &\mapsto (z_1^n, z_2^m) \end{aligned}$$

**Example.** Worksheet 3 exercise 7. Let

$$\mathbb{RP}^n = \frac{\mathbb{R}^{n+1} \setminus \{0\}}{\sim} = \frac{S^n}{\sim}$$

be the **projective  $n$ -space**, the space of all lines through the origin in  $\mathbb{R}^{n+1}$ , where  $x \sim -x$ . Let  $p : S^n \rightarrow \mathbb{RP}^n$  be the quotient map. Claim that this is a covering space. Let  $[x] \in \mathbb{RP}^n$ . Then  $p^{-1}([x]) = \{\pm x\}$ . Let  $U$  be an open neighbourhood of  $x$  such that  $U \cap (-U) = \emptyset$ , so  $p(U) = \{[x] \mid x \in U\}$ . Then  $p^{-1}(p(U)) = U \cup (-U)$  is open and disjoint. Thus  $p|_U : U \rightarrow p(U)$  is a homeomorphism, so it is a covering space.

- $n \geq 2$  gives that  $S^n$  is simply-connected, so  $S^n \rightarrow \mathbb{RP}^n$  is a universal cover. Then

$$\{id\} = p_* \left( \pi_1(S^n) \right) \stackrel{2}{\subseteq} \pi_1(\mathbb{RP}^n),$$

so  $|\pi_1(\mathbb{RP}^n)| = 2$ . Thus  $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$ .

- $n = 1$  gives  $\mathbb{RP}^1 = S^1$ , so

$$\begin{aligned} p : S^1 &\rightarrow S^1 \\ z &\mapsto z^2 \end{aligned}$$

is a covering space.

## 2 Homology

Higher homotopy groups  $\pi_n(X, x_0)$  are groups of basepoint preserving homotopies of continuous  $\phi: I^n \rightarrow X$  such that  $\phi(\partial I^n) = x_0$ .

Lecture 18  
Wednesday  
20/02/19

**Example.**

$$\pi_1(S^n) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}, \quad \pi_2(S^n) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases},$$

$$\pi_3(S^n) = \begin{cases} \mathbb{Z} & n = 2, 3 \\ 0 & \text{otherwise} \end{cases}, \quad \pi_i(S^2) = \begin{cases} \mathbb{Z} & i = 4, 5 \\ 2\mathbb{Z} & i = 6 \\ 12\mathbb{Z} & i = 6 \end{cases}.$$

Homology is more suitable. The following is the plan.

- Simplicial homology.
- Singular homology.
- Technical machinery to show that they coincide.
- Applications.

### 2.1 Simplicial and singular homology

#### 2.1.1 $\Delta$ -complexes

**Definition.** Let  $m, n \geq 0$ .

- An  **$n$ -simplex** in  $\mathbb{R}^m$  is the convex hull of a set  $V$  of  $n + 1$  points in  $\mathbb{R}^m$  that are not all contained in an affine  $(n - 1)$ -dimensional subspace of  $\mathbb{R}^m$ .
- The **standard  $n$ -simplex** is the convex hull of the standard basis  $\{e_1, \dots, e_{n+1}\}$  in  $\mathbb{R}^{n+1}$ ,

$$\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, x_0 + \dots + x_n = 1\}.$$

- An **ordered  $n$ -simplex** is an  $n$ -simplex with an ordering on the vertices. We denote it by  $[v_0, \dots, v_n]$ , where  $v_0, \dots, v_n$  are the vertices in ascending order.
- The **standard ordered  $n$ -simplex** is the ordered  $n$ -simplex  $[e_1, \dots, e_{n+1}]$  in  $\mathbb{R}^{n+1}$ . It is denoted by  $\Delta^n$ .
- Let  $[v_0, \dots, v_{n+1}]$  be an  $n$ -simplex in  $\mathbb{R}^m$  and let  $L \subseteq \mathbb{R}^m$  be the affine subspace spanned by  $v_0, \dots, v_n$ . Then there exists a unique affine morphism

$$\begin{aligned} L &\rightarrow \mathbb{R}^{n+1} \\ v_i &\mapsto e_{i+1} \end{aligned},$$

for  $i = 0, \dots, n$ . This gives a homeomorphism from  $[v_0, \dots, v_n]$  to  $\Delta^n$  that preserves this ordering.

- For  $n \geq 1$ , the **faces** of an ordered  $n$ -simplex  $[v_0, \dots, v_n]$  are the ordered  $(n - 1)$ -simplices

$$[v_0, \dots, \widehat{v_i}, \dots, v_n].$$

$\widehat{v_i}$  means we omit the vertex  $v_i$ .

- The union of all the faces of a simplex  $\Delta$  is the **boundary**  $\partial\Delta$ .
- The **interior** of  $\Delta$  is  $\overset{\circ}{\Delta} = \Delta \setminus \partial\Delta$ .

**Example.** Let  $\Delta^2 = [e_1, e_2, e_3]$ . Then  $\partial\Delta^2 = [e_1, e_2] \cup [e_1, e_3] \cup [e_2, e_3]$ .

**Definition.** Let  $X$  be a topological space. A  **$\Delta$ -complex structure** on  $X$  is a collection of continuous maps  $\sigma_\alpha : \Delta^{n(\alpha)} \rightarrow X$  for  $\alpha \in A$  and  $n(\alpha) \in \mathbb{N}$  such that

1. the restriction  $\sigma_\alpha|_{\dot{\Delta}^{n(\alpha)}}$  is injective for all  $\alpha \in A$  and for each  $x \in X$  there is a unique  $\alpha \in A$  such that  $x \in \sigma_\alpha(\dot{\Delta}^{n(\alpha)})$ ,
2. the restriction of  $\sigma_\alpha$  to a face of  $\Delta^{n(\alpha)}$  is equal to  $\sigma_\beta$  for some  $\beta \in A$  and  $n(\beta) = n(\alpha) - 1$ , and
3.  $U \subseteq X$  is open if and only if  $\sigma_\alpha^{-1}(U)$  is open in  $\Delta^{n(\alpha)}$  for all  $\alpha \in A$ .

An observation is that

$$\sigma : \bigsqcup_{\alpha \in A} \Delta^{n(\alpha)} \rightarrow X$$

induced by the  $\sigma_\alpha$  is a quotient map, since it is surjective by 1 and  $U \subseteq X$  is open if and only if  $\sigma^{-1}(U)$  is open by 3.

*Remark.* One can show that an  $X$  with a  $\Delta$ -complex structure is a CW-complex.

**Example.**

- Torus or Klein bottle is two  $\Delta^2$ , three  $\Delta^1$ , and one  $\Delta^0$ .
- $S^2$  is a tetrahedron.

### 2.1.2 Simplicial homology

Let  $X$  be a  $\Delta$ -complex. The group of  **$n$ -chains**  $\Delta_n(X)$  is the free abelian group on the  $n$ -simplices  $\sigma_\alpha : \Delta^{n(\alpha)} \rightarrow X$ , where  $n(\alpha) = n$ . So an element in  $\Delta_n(X)$  is of the form

$$\sum_{\alpha \in A, n(\alpha)=n} c_\alpha \cdot \sigma_\alpha,$$

where  $c_\alpha \in \mathbb{Z}$  and all but finitely many of the  $c_\alpha$  are zero.

**Example.** Let  $K$  be a Klein bottle.

- $\Delta_0(K) = \{n \cdot v \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$ .
- $\Delta_1(K) = \{n_1 \cdot a + n_2 \cdot b + n_3 \cdot c \mid n_1, n_2, n_3 \in \mathbb{Z}\} \cong \mathbb{Z}^3$ .
- $\Delta_2(K) = \{n_1 \cdot U + n_2 \cdot V \mid n_1, n_2 \in \mathbb{Z}\} \cong \mathbb{Z}^2$ .
- $\Delta_n(K) = 0$  for  $n \geq 3$ .