

M3P21 Geometry II: Algebraic Topology

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$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \xrightarrow{\partial} & \begin{array}{c} 0 \\ \downarrow \\ A_{n+1} \end{array} & \xrightarrow{\alpha} & A'_{n+1} & \xrightarrow{\partial} & \dots \\
 & & \downarrow & \nearrow \partial & \downarrow i' & & \\
 \dots & \xrightarrow{\partial} & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} \xrightarrow{\partial} \dots \\
 & & \downarrow i & \nearrow \beta & \downarrow i & \nearrow \beta & \downarrow i \\
 \dots & \xrightarrow{\partial} & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} \xrightarrow{\partial} \dots \\
 & & \downarrow j & \nearrow \gamma & \downarrow j & \nearrow \gamma & \downarrow j \\
 \dots & \xrightarrow{\partial} & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} \xrightarrow{\partial} \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

\Downarrow

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{\partial} & H_{n+1}(A) & \xrightarrow{i_*} & H_{n+1}(B) & \xrightarrow{j_*} & H_{n+1}(C) \\
 & & \downarrow \alpha_* & & \downarrow \beta_* & & \downarrow \gamma_* \\
 \dots & \xrightarrow{\partial} & H_{n+1}(A') & \xrightarrow{i_*} & H_{n+1}(B') & \xrightarrow{j_*} & H_{n+1}(C')
 \end{array}$$

$$\begin{array}{ccccccc}
 & & \swarrow & & \swarrow & & \swarrow \\
 H_n(A) & \xrightarrow{i_*} & H_n(B) & \xrightarrow{j_*} & H_n(C) \\
 \downarrow \alpha_* & \nearrow \partial & \downarrow \beta_* & & \downarrow \gamma_* \\
 H_n(A') & \xrightarrow{i_*} & H_n(B') & \xrightarrow{j_*} & H_n(C')
 \end{array}$$

$$\begin{array}{ccccccc}
 & & \swarrow & & \swarrow & & \swarrow \\
 H_{n-1}(A) & \xrightarrow{i_*} & H_{n-1}(B) & \xrightarrow{j_*} & H_{n-1}(C) & \xrightarrow{\partial} & \dots \\
 \downarrow \alpha_* & \nearrow \partial & \downarrow \beta_* & & \downarrow \gamma_* & & \\
 H_{n-1}(A') & \xrightarrow{i_*} & H_{n-1}(B') & \xrightarrow{j_*} & H_{n-1}(C') & \xrightarrow{\partial} & \dots
 \end{array}$$

Syllabus

Homotopy and homotopy type. Cell complexes. Basic constructions of the fundamental group. Seifert-van Kampen theorem. Covering spaces. Δ -complexes. Simplicial homology. Singular homology. Homotopy invariance. Exact sequences and excision. Mayer-Vietoris sequences. Degree.

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0 Introduction

0.1 Introduction

Lecture 1
Friday
11/01/19

Combines topological spaces with algebraic objects, which are groups.

- How to show that a torus is not homeomorphic to a sphere?
- How to show that $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n \neq m$?

We will follow chapter one and two from

- A Hatcher, Algebraic topology, 2002

The following are prerequisites.

- Point set topology. Topological spaces, continuous maps, product and quotient topologies, Hausdorff spaces, etc.
- Basic group theory. Normal subgroups and quotients, isomorphism theorems, free groups, presentation of groups, etc.

0.2 Some underlying geometric notions

0.2.1 Homotopy and homotopy type

Let X and Y be topological spaces and $I = [0, 1]$.

Definition. A **homotopy** is a continuous map $F : X \times I \rightarrow Y$. For every $t \in I$ we obtain a continuous map

$$\begin{aligned} f_t &: X \longrightarrow Y \\ x &\longmapsto f_t(x) = F(x, t) \end{aligned}$$

Definition. Two continuous maps $f_0, f_1 : X \rightarrow Y$ are **homotopic** if there exists a homotopy $F : X \times I \rightarrow Y$ such that

$$f_0(x) = F(x, 0), \quad f_1(x) = F(x, 1), \quad x \in X.$$

We write $f_0 \cong f_1$. This is an equivalence relation.¹

Definition. Let $A \subseteq X$ be a subspace. A **retraction** of X onto A is a continuous map $r : X \rightarrow A$ such that $r(X) = A$ and $r|_A = \text{id}_A$.

Example. If $X \neq \emptyset$, $p \in X$, then X retracts to p by the constant map $X \rightarrow \{p\}$.

Definition. A **deformation retraction** of X onto $A \subseteq X$ is a retraction that is homotopic to the identity. That is, there is a continuous map

$$\begin{aligned} F &: X \times I \longrightarrow A \\ (x, t) &\longmapsto f_t(x) \end{aligned}$$

such that $f_0 = \text{id}_X$ and $f_1 : X \rightarrow A$ is the deformation retraction.

Example. The closed n -dimensional n -disc

$$D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$

deformation retracts to $(0, \dots, 0) \in \mathbb{R}^n$. Let $f_t(x) = t \cdot x$. Then $t = 1$ implies that $f_1 = \text{id}_{D^n}$ and $t = 0$ implies that $f_0 : D^n \rightarrow (0, \dots, 0)$.

¹Exercise

Example. Let S^n be the n -sphere,

$$\partial D^{n+1} = S^n = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

The cylinder $S^n \times I$ deformation retracts to $S^n \times \{0\}$, by defining $f_t(x, r) = (x, t \cdot r)$.

An observation is that if X is a topological space, and $f : X \rightarrow \{p\}$ for $p \in X$ is a deformation retraction of X to p , then X is path-connected. Indeed, if $F : X \times I \rightarrow X$ is a homotopy from id_X to f and $x \in X$ is a point, then this gives a path

$$\begin{array}{ccc} I & \longrightarrow & X \\ t & \longmapsto & F(x, t) \end{array}$$

that connects x to p . This implies that not all retractions are deformation retractions.

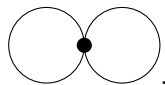
Example. A retraction that is not a deformation retraction. Take a space that is not path-connected and retract it to a point. Let $X = \{0, 1\}$ with discrete topology. Then $x \mapsto 0$ is a retraction, but not a deformation retraction because X is not path-connected.

Definition. A continuous map $f : X \rightarrow Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \rightarrow X$ such that $fg \cong \text{id}_Y$ and $gf \cong \text{id}_X$. If there exists a homotopy equivalence between X and Y , X and Y are **homotopy equivalent** or they have the same **homotopy type**.

Lemma 0.1. A deformation retraction $f : X \rightarrow A$ is a homotopy equivalence.

Proof. Let $i : A \hookrightarrow X$ be the inclusion map. Then $fi = \text{id}_A$ and $if = f \cong \text{id}_X$ by definition. \square

Example. The disc with two holes is equivalent to



Example. \mathbb{R}^n deformation retracts to a point, by $f_t(x) = t \cdot x$.

Definition.

- X is **contractible** if it is homotopy equivalent to a point.
- A continuous map is **nullhomotopic** if it is homotopy equivalent to a constant map.

0.2.2 Cell complexes

Example. The **torus** $S^1 \times S^1$ is the union of a point, two open intervals, and the open disc D^2 .

These are called **cells**. Can think of discs D^n glued together.

Definition. A **CW-complex**, or **cell complex**, is a topological space X such that there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X^n,$$

where the X^n are constructed inductively in the following way.

- X^n is a discrete set.
- For each $n \geq 0$ there is an collection of closed n -discs $\{D_\alpha^n\}$ together with continuous maps $\phi_\alpha : \partial D_\alpha^n \rightarrow X^{n-1}$, such that

$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha} D_\alpha^n / \sim,$$

where $x \sim \phi_\alpha(x)$ for all $x \in \partial D_\alpha^n$ for all α .

- A subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n .

Lecture 2
Tuesday
15/01/19

Remark.

- As a set,

$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha} e_{\alpha}^n,$$

where each e_{α}^n is homeomorphic to an open n -disc. These e_{α}^n are called the n -**cells** of X .

- If $X = X^m$ for some m , then X is called **finite dimensional**. The minimal m such that $X = X^m$ is the **dimension** of X .

Example. The following are CW-complexes.

$$[0, 1], \quad \mathbb{R}, \quad S^1, \quad \text{a graph}, \quad S^n = D^n / \partial D^n.$$

Can also decompose CW-complexes.

- The sphere S^2 is one 0-cell, one 1-cell, and two 2-cells.
- The torus $S^1 \times S^1$ is one 0-cell, two 1-cells, and one 2-cell.
- The Möbius strip is two 0-cells, three 1-cells, and one 2-cell.
- The Klein bottle is one 0-cell, two 1-cells, and one 2-cell.

Definition. If X is a CW-complex with finitely many cells the **Euler characteristic** $\chi(X)$ of X is the number of even cells minus the number of odd cells.

Fact. $\chi(X)$ does not depend of the choice of cells decomposition.

Example.

- $\chi(S^n) = 0$ if n is odd and $\chi(S^n) = 2$ if n is even.
- $\chi(S^1 \times S^1) = 0$.

This is the generalisation of the following observation by Leonhard Euler. Let P be a convex polyhedron, where V is the number of vertices of P , E is the number of edges of P , and F is the number of faces of P . Then $V - E + F = 2$.

Example. A topological space that is not a CW-complex. $X = \{0, 1\}$ with trivial topology does not contain any closed points.

Fact. CW-complexes are always Hausdorff.

1 The fundamental group

1.1 Basic constructions

1.1.1 Paths and homotopy

Let X be a topological space. A **path** is a continuous map $f : I \rightarrow X$, where $I = [0, 1]$.

Definition. Two paths f_0 and f_1 are **homotopic** if there exists a homotopy between f_0 and f_1 preserving the endpoints, that is a continuous map

$$\begin{aligned} F : I \times I &\longrightarrow X \\ (s, t) &\longmapsto f_t(s) \end{aligned}$$

such that

$$\begin{aligned} f_t(0) &= f_0(0), & f_t(1) &= f_0(1), & t &\in I, \\ F(s, 0) &= f_0(s), & F(s, 1) &= f_1(s), & s &\in I. \end{aligned}$$

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set. Then all the paths in X are homotopic if they have the same endpoints. Let $f_0, f_1 : I \rightarrow X$ be two paths such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$. Define

$$f_t(s) = (1 - t)f_0(s) + tf_1(s).$$

Lemma 1.1. *Being homotopic is an equivalence relation on the set of paths with fixed endpoints. We will write $f_0 \cong f_1$ for two homotopic paths f_0 and f_1 .*

Proof.

- f is homotopic to f .
- If f_0 is homotopic to f_1 by a homotopy f_t , then f_1 is homotopic to f_0 by the homotopy f_{1-t} .
- If f_0 is homotopic to f_1 by a homotopy f_t and $f_1 = g_0$ is homotopic to g_1 by a homotopy g_t , then f_0 is homotopic to g_1 by the homotopy

$$h_t = \begin{cases} f_{2t} & 0 \leq t \leq \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Then

$$\begin{aligned} H : I \times I &\longrightarrow X \\ (s, t) &\longmapsto h_t(s) \end{aligned}$$

is continuous because its restriction to the closed subsets $I \times [0, \frac{1}{2}]$ and $I \times [\frac{1}{2}, 1]$ is continuous, since if the restriction to two closed subsets is continuous then the restriction to the union of these subsets is continuous.

□

Let X be a topological space and $I = [0, 1]$. If $f : I \rightarrow X$ is a path, $[f]$ is the class of all paths on X homotopic to f .

Definition. Let $f, g : I \rightarrow X$ be two paths such that $f(1) = g(0)$. The **product path** $f \cdot g$ is the path

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \leq s \leq 1 \end{cases}.$$

A convention is that whenever we write $f \cdot g$ we implicitly assume $f(1) = g(0)$.

Lemma 1.2. *Let f_0, f_1, g_0, g_1 be paths on X such that $f_1 \cong f_0$ and $g_0 \cong g_1$. Then $f_0 \cdot g_0 \cong f_1 \cdot g_1$.*

Proof.

$$\begin{aligned} I \times I &\longrightarrow X \\ (s, t) &\longmapsto (f_t \cdot g_t)(s) \end{aligned}$$

is a homotopy between $f_0 \cdot g_0$ and $f_1 \cdot g_1$.

□

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Wednesday
16/01/19

Remark. Let $\phi : [0, 1] \rightarrow [0, 1]$ be continuous such that $\phi(0) = 0$ and $\phi(1) = 1$. If $f : I \rightarrow X$ is a path, then $f\phi \cong f$. This is a **reparametrisation**. Define

$$\phi_t(s) = (1-t)\phi(s) + ts,$$

then $f\phi_t$ is a homotopy between $f\phi$ and f .

For $x \in X$, let the **constant path** at x be

$$\begin{array}{ccc} c_x & : & I \longrightarrow X \\ & & s \longmapsto x \end{array}.$$

For a path $f : I \rightarrow X$, define

$$\begin{array}{ccc} f^{-1} & : & I \longrightarrow X \\ & & s \longmapsto f(1-s) \end{array}.$$

Lemma 1.3. *Let $f, g, h : I \rightarrow X$ be paths. Then*

1. $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$,
2. $f \cdot c_{f(1)} \cong f$ and $c_{f(0)} \cdot f \cong f$, and
3. $f \cdot f^{-1} \cong c_{f(0)}$ and $f^{-1} \cdot f \cong c_{f(1)}$.

Proof.

1. $((f \cdot g) \cdot h)\phi = f \cdot (g \cdot h)$, where

$$\phi(s) = \begin{cases} \frac{s}{2} & s \in [0, \frac{1}{2}] \\ s - \frac{1}{4} & s \in [\frac{1}{2}, \frac{3}{4}] \\ 2s - 1 & s \in [\frac{3}{4}, 1] \end{cases},$$

so $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$ by reparametrisation.

2. Again reparametrisation, by

$$\psi(s) = \begin{cases} 2s & s \in [0, \frac{1}{2}] \\ 1 & s \in [\frac{1}{2}, 1] \end{cases}, \quad \chi(s) = \begin{cases} 0 & s \in [0, \frac{1}{2}] \\ 2s - 1 & s \in [\frac{1}{2}, 1] \end{cases}.$$

3. Define

$$H(s, t) = \begin{cases} f(\max\{1-2s, t\}) & s \in [0, \frac{1}{2}] \\ f(\max\{2s-1, t\}) & s \in [\frac{1}{2}, 1] \end{cases}.$$

H is continuous, and

$$H(s, 0) = f^{-1} \cdot f, \quad H(s, 1) = c_{f(1)}.$$

The inverse is similar. □

Definition. A **loop** with **basepoint** $x_0 \in X$ is a path $f : I \rightarrow X$ such that $f(0) = f(1) = x_0$.

Definition. Denote by $\pi_1(X, x_0)$ the set of **homotopy classes** $[f]$ of loops $f : I \rightarrow X$ with basepoint x_0 .

Proposition 1.4. $\pi_1(X, x_0)$ is a group with product $[f][g] = [f \cdot g]$ and neutral element $c_{x_0} : I \rightarrow X$, the constant path at x_0 .

Proof. Follows directly from Lemma 1.2 and Lemma 1.3. □

Definition. $\pi_1(X, x_0)$ is the **fundamental group** of X at x_0 .

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set and $x_0 \in X$. Then $\pi_1(X, x_0) = 0$, since X is convex, so all loops are homotopic to each other.

Example.

- The fundamental group of a space X with the trivial topology is trivial, since X is simply-connected, because all maps $f : I \rightarrow X$ are continuous, so path-connected and all paths are homotopic.
- The fundamental group of a space X with the discrete topology is trivial, since $f : I \rightarrow X$ is continuous implies that f is constant.

Assume $x_0, x_1 \in X$ such that x_0 and x_1 are in the same path-component of X . Let $h : I \rightarrow X$ be a path such that $h(0) = x_0$ and $h(1) = x_1$. Define

$$\begin{aligned} \beta_h : \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_0) \\ [f] &\longmapsto [h \cdot f \cdot h^{-1}] \end{aligned} .$$

This is well-defined by Lemma 1.2.

Proposition 1.5. $\beta_h : \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$ is an isomorphism.

Proof. It is a homomorphism, since

$$\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot h^{-1}] = [h \cdot f \cdot h^{-1}] [h \cdot g \cdot h^{-1}] = \beta_h[f] \cdot \beta_h[g],$$

and $\beta_h[c_{x_1}] = [c_{x_1}]$. It is bijective with $(\beta_h)^{-1} = \beta_{h^{-1}}$. □

If X is path-connected, we often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Definition. X is **simply-connected** if it is path-connected and $\pi_1(X) = 0$.

Proposition 1.6. X is simply-connected if and only if there exists a unique homotopy class of paths between any two points of X .

Proof.

\implies There exists a path between any two points. Let f and g be two paths from x_0 to x_1 for $x_0, x_1 \in X$. Then $f \cdot g^{-1} \cong g \cdot g^{-1}$, so

$$f \cong f \cdot g^{-1} \cdot g \cong g \cdot g^{-1} \cdot g \cong g.$$

\impliedby X is path-connected. Then $x_1 = x_0$, so all loops at x_0 are homotopic to each other, so $\pi_1(X) = 0$. □

1.1.2 The fundamental group of the circle

The goal is to show that $\pi_1(S^1) \cong \mathbb{Z}$.

Definition. A **covering space** of a space X is a space \tilde{X} and a continuous map $p : \tilde{X} \rightarrow X$ such that for each $x \in X$ there is an open $U \subseteq X$ such that

- $p^{-1}(U) = \bigcup_{j \in J} \tilde{U}_j$, where $\tilde{U}_j \subseteq \tilde{X}$ is open,
- $\tilde{U}_i \cap \tilde{U}_j = \emptyset$ if $i \neq j$, and
- $p|_{\tilde{U}_j} : \tilde{U}_j \rightarrow U$ is a homeomorphism for all $j \in J$.

Such a U is called **evenly covered**. The \tilde{U}_j are called **sheets**.

Example.

$$\begin{aligned} p : \mathbb{R} &\longrightarrow S^1 \\ s &\longmapsto (\cos 2\pi s, \sin 2\pi s) \end{aligned} .$$

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Friday
18/01/19

Definition. Let $p : \tilde{X} \rightarrow X$ be a covering space. A **lift** of a continuous map $f : Y \rightarrow X$ is a continuous map $\tilde{f} : Y \rightarrow \tilde{X}$ such that $p\tilde{f} = f$, so

$$\begin{array}{ccc} & \tilde{X} & \\ \tilde{f} \nearrow & \downarrow p & \\ Y & \xrightarrow{f} & X \end{array}$$

Proposition 1.7 (Unique lifting property). *Let $p : \tilde{X} \rightarrow X$ be a covering space and $f : Y \rightarrow X$ be a continuous map. If there are two lifts $\tilde{f}_1, \tilde{f}_2 : Y \rightarrow \tilde{X}$ of f such that $\tilde{f}_1(y) = \tilde{f}_2(y)$ for some $y \in Y$ and if Y is connected, then $\tilde{f}_1 = \tilde{f}_2$.*

Proof. Let $y \in Y$ and $U \subseteq X$ be an evenly covered neighbourhood of $f(y)$. Then

$$p^{-1}(U) = \bigcup_j \tilde{U}_j.$$

Let \tilde{U}_1 be the sheet such that $\tilde{f}_1(y) \in \tilde{U}_1$, and let \tilde{U}_2 be the sheet such that $\tilde{f}_2(y) \in \tilde{U}_2$. Let $N \subseteq Y$ be open and $y \in N$ such that $\tilde{f}_1(N) \subseteq \tilde{U}_1$ and $\tilde{f}_2(N) \subseteq \tilde{U}_2$. We have $p\tilde{f}_1 = p\tilde{f}_2$. Then $\tilde{f}_1(y) = \tilde{f}_2(y)$ if and only if $\tilde{U}_1 = \tilde{U}_2$, if and only if $\tilde{f}_1|_N = \tilde{f}_2|_N$. Let

$$A = \left\{ y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y) \right\},$$

so A is open and $Y \setminus A$ is open. Thus $A \neq \emptyset$ implies that $A = Y$. \square

Proposition 1.8 (Homotopy lifting property). *Let $p : \tilde{X} \rightarrow X$ be a covering space and $F : Y \times I \rightarrow X$ be a continuous map such that there exists a lift $\tilde{f}_0 : Y \times \{0\} \rightarrow \tilde{X}$ of $F|_{Y \times \{0\}}$. Then there is a unique lift $\tilde{F} : Y \times I \rightarrow \tilde{X}$ of F such that $\tilde{F}|_{Y \times \{0\}} = \tilde{f}_0$.*

Proof. Let $y_0 \in Y$ and $t \in I$. There are open $y_0 \in N_t \subseteq Y$ and $t \in (a_t, b_t) \subseteq I$ such that $F(N_t \times (a_t, b_t)) \subseteq U \subseteq X$, where $U \subseteq X$ is open and evenly covered. Compactness of I implies that there exist

$$0 = t_0 < \dots < t_m = 1,$$

and there exists $y_0 \in N \subseteq Y$ open such that $F(N \times [t_i, t_{i+1}]) \subseteq U_i \subseteq X$, where $U_i \subseteq X$ is open and evenly covered. We inductively construct a lift $\tilde{F}|_{N \times I}$ of $F|_{N \times I}$.

- $\tilde{F}|_{N \times [0,0]} = \tilde{f}_0|_{N \times [0,0]}$ exists.
- Assume the lift has been constructed on $N \times [0, t_i]$. Let $\tilde{U}_i \subseteq \tilde{X}$ be such that $p|_{\tilde{U}_i} : \tilde{U}_i \rightarrow U_i$ such that $\tilde{F}(y_0, t_i) \in \tilde{U}_i$. After shrinking N , may assume $\tilde{F}(N \times \{t_i\}) \subseteq \tilde{U}_i$. Define \tilde{F} on $N \times [t_i, t_{i+1}]$ to be composition of F with the homeomorphism $p^{-1} : U_i \rightarrow \tilde{U}_i$.

After finitely many steps we obtain a lift $\tilde{F} : N \times I \rightarrow \tilde{X}$, where $y_0 \in N \subseteq Y$ is open, so for each $y \in Y$ there is a neighbourhood $N_y \subseteq Y$ such that $F|_{N_y \times I} : N_y \times I \rightarrow X$ lifts. For all $y \in Y$, $\{y\} \times I$ is connected and can be lifted, so Proposition 1.7 implies that the lift of $N \times I$ is unique. Thus there is a unique lift $\tilde{F} : Y \times I \rightarrow \tilde{X}$. \square

Example. Let X be a topological space and A be discrete. Then $p : X \times A \rightarrow X$ is a covering space. This is the **trivial covering**. Show the unique lifting property and the homotopy lifting property for the trivial covering. ²

Corollary 1.9. *Let $f : I \rightarrow X$ be a path, $f(0) = x_0$, and $p : \tilde{X} \rightarrow X$ be a covering space. For each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \rightarrow \tilde{X}$ such that $\tilde{f}(0) = \tilde{x}_0$.*

Proof. Proposition 1.8 for Y a point. \square

²Exercise

Theorem 1.10. Let $x_0 = (1, 0) \in S^1$. Then $\pi_1(S^1, x_0)$ is the infinite cyclic group generated by the homotopy class of the loop

$$\begin{aligned} \omega &: I \longrightarrow S^1 \\ s &\longmapsto (\cos 2\pi s, \sin 2\pi s) \end{aligned}$$

Remark.

- $[\omega]^n = [\omega_n]$, where

$$\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns).$$

-

$$\begin{aligned} p &: \mathbb{R} \longrightarrow S^1 \\ s &\longmapsto (\cos 2\pi s, \sin 2\pi s) \end{aligned}$$

is a covering space.

- ω_n lifts to

$$\begin{aligned} \widetilde{\omega}_n &: I \longrightarrow \mathbb{R} \\ s &\longmapsto ns \end{aligned}$$

such that $\widetilde{\omega}_n(0) = 0$ and $\widetilde{\omega}_n(1) = n$.

Proof of Theorem 1.10.

- If $f: I \rightarrow S^1$ is a loop at x_0 , then the homotopy lifting property implies that there exists a lift $\tilde{f}: I \rightarrow \mathbb{R}$ such that $\tilde{f}(0) = 0$. Since $p(\tilde{f}(1)) = f(1) = x_0$, then $\tilde{f}(1) = n$ for some $n \in \mathbb{Z}$. Then $\widetilde{\omega}_n: I \rightarrow \mathbb{R}$ is another path such that $\widetilde{\omega}_n(0) = 0$ and $\widetilde{\omega}_n(1) = n$, so $\tilde{f} \cong \widetilde{\omega}_n$. Let $F: I \times I \rightarrow \mathbb{R}$ be a homotopy equivalence between \tilde{f} and $\widetilde{\omega}_n$. Then $pF: I \times I \rightarrow S^1$ gives a homotopy between $p\tilde{f} = f$ and $p\widetilde{\omega}_n = \omega_n$.
- Let $m, n \in \mathbb{Z}$ and assume $\omega_m \cong \omega_n$. Let $F: I \times I \rightarrow S^1$ be a homotopy. Then

$$F(0, t) = \omega_m(t), \quad F(1, t) = \omega_n(t), \quad F(s, 0) = F(s, 1) = x_0, \quad s, t \in I.$$

The unique lifting property implies that $\widetilde{\omega}_n, \widetilde{\omega}_m: I \rightarrow \mathbb{R}$ are unique lifts such that $\widetilde{\omega}_n(0) = 0 = \widetilde{\omega}_m(0)$. The homotopy lifting property implies that F lifts uniquely to a homotopy $\tilde{F}: I \times I \rightarrow \mathbb{R}$ between $\widetilde{\omega}_n$ and $\widetilde{\omega}_m$, and $\tilde{F}(s, 1) \in \mathbb{Z}$ for all $s \in I$. Thus $\tilde{F}(s, 1) = n = m$, so $\omega_m \cong \omega_n$ if and only if $n = m$.

□

Lecture 5 is a problems class.

Theorem 1.11. Every non-constant polynomial $p \in \mathbb{C}[z]$ has a root in \mathbb{C} .

Proof. May assume $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$. Assume p has no roots in \mathbb{C} . For each $r \in \mathbb{R}_{\geq 0}$ we obtain a loop

$$\begin{aligned} f_r &: I \longrightarrow \mathbb{C} \\ s &\longmapsto \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|} \end{aligned}$$

so $|f_r(s)| = 1$. Then $f_r(0) = 1$ and $f_r(1) = 1$, so f_r is a loop based at 1. Then f_0 is the constant loop at 1, and $f_r(s)$ depends continuously on r , so $f_r \cong f_0$ for all $r \in \mathbb{R}_{\geq 0}$ and $[f_r] = [f_0] = 0 \in \pi_1(S^1)$. Fix $r \in \mathbb{R}_{\geq 0}$ such that $r > 1$ and $r > |a_1| + \dots + |a_n|$. For $|z| = r$ we have

$$|z^n| > (|a_1| + \dots + |a_n|)|z^{n-1}| \geq |a_1 z^{n-1}| + \dots + |a_n| \geq |a_1 z^{n-1} + \dots + a_n|.$$

Hence, for $0 \leq t \leq 1$ the polynomial

$$p_t(z) = z^n + t(a_1 z^{n-1} + \dots + a_n)$$

has no root z with $|z| = r$. Define

$$F_r(t, s) = \frac{p_t(re^{2\pi is})/p_t(r)}{|p_t(re^{2\pi is})/p_t(r)|}.$$

Then $F_r(0, s) = \omega_n(s)$ and $F_r(1, s) = f_r(s)$, so $[\omega_n] = [f_r] = 0 \in \pi_1(S^1)$. Theorem 1.10 implies that $n = 0$, so p is constant. □

See Hatcher Theorem 1.9 and Theorem 1.10 for more applications.

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Proposition 1.12. *Let X and Y be path-connected topological spaces, $x_0 \in X$, and $y_0 \in Y$. Then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. A map

$$\begin{aligned} f &: Z \longrightarrow X \times Y \\ z &\longmapsto (g(z), h(z)) \end{aligned}$$

is continuous if and only if $g : Z \rightarrow X$ and $h : Z \rightarrow Y$ are continuous. For $Z = I$,

$$\{ \text{loops in } X \times Y \text{ based } (x_0, y_0) \} \quad \longleftrightarrow \quad \{ \text{loops in } X \text{ based } x_0 \} \times \{ \text{loops in } Y \text{ based } y_0 \}.$$

Two loops

$$\begin{aligned} f_1 &: I \longrightarrow X \times Y & f_2 &: I \longrightarrow X \times Y \\ s &\longmapsto (g_1(s), h_1(s)) & s &\longmapsto (g_2(s), h_2(s)) \end{aligned}$$

are homotopic if and only if $g_1 \cong g_2$ and $h_1 \cong h_2$, so there is a bijection

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Then $f_1 \cdot f_2 = (g_1 \cdot g_2, h_1 \cdot h_2)$ and the constant loop is mapped to the constant loop, so this is also a group isomorphism. \square

Example. The torus $S^1 \times S^1$ has

$$\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z}^2.$$

1.1.3 Induced homomorphisms

Let X and Y be topological spaces, $x_0 \in X$, and $\phi : X \rightarrow Y$. An observation is that ϕ induces a homomorphism

$$\begin{aligned} \phi_* &: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \phi(x_0)) \\ [f] &\longmapsto [\phi f] \end{aligned}$$

ϕ_* is well-defined, since if f_t is a homotopy between the loops f_0 and f_1 based at x_0 , then ϕf_t is a homotopy of loops between ϕf_0 and ϕf_1 . Moreover, $\phi(f \cdot g) = (\phi f) \cdot (\phi g)$ and ϕ maps the constant path at x_0 to the constant path at $\phi(x_0)$, so ϕ is a homomorphism.

Proposition 1.13.

1. Let $\psi : X \rightarrow Y$ and $\phi : Y \rightarrow Z$ be continuous maps between topological spaces, $x_0 \in X$, and

$$\begin{aligned} \psi_* : \pi_1(X, x_0) &\rightarrow \pi_1(Y, \psi(x_0)), & \phi_* : \pi_1(Y, \psi(x_0)) &\rightarrow \pi_1(Z, \phi\psi(x_0)), \\ (\phi\psi)_* : \pi_1(X, x_0) &\rightarrow \pi_1(Z, \phi\psi(x_0)). \end{aligned}$$

Then $(\phi\psi)_* = \phi_* \psi_*$.

2. Let $\text{id}_X : X \rightarrow X$ be the identity then

$$(\text{id}_X)_* : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$$

is the identity.

Proof.

1. Let $f : I \rightarrow X$ be a loop at x_0 , then

$$(\phi\psi)_*([f]) = [(\phi\psi)f] = [\phi(\psi f)] = \phi_*([\psi f]) = \phi_* \psi_*([f]).$$

2. $(\text{id}_X)_*([f]) = [\text{id}_X f] = [f]$.

\square

These two observations yield in particular that if $\phi : X \rightarrow Y$ is a homeomorphism with inverse $\psi : Y \rightarrow X$, then

$$\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \phi(x_0))$$

is an isomorphism with inverse ψ_* .

Proposition 1.14. *Let $\phi : X \rightarrow Y$ be a homotopy equivalence. Then*

$$\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \phi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Recall that if $x_0, x_1 \in X$ and $h : I \rightarrow X$ is a path such that $h(0) = x_0$ and $h(1) = x_1$, then we obtain an isomorphism

$$\begin{aligned} \beta_h : \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_0) \\ [f] &\longmapsto [h \cdot f \cdot h^{-1}] \end{aligned} .$$

Lemma 1.15. *Let $\phi_t : X \rightarrow Y$ be a homotopy and $x_0 \in X$. Define the path*

$$\begin{aligned} h : I &\longrightarrow Y \\ s &\longmapsto \phi_s(x_0) \end{aligned} , \quad h(0) = \phi_0(x_0), \quad h(1) = \phi_1(x_0) .$$

Then $\phi_{0} = \beta_h \phi_{1*}$, that is the following diagram commutes.*

$$\begin{array}{ccc} & \pi_1(Y, \phi_1(x_0)) & \\ \nearrow \phi_{1*} & \downarrow \sim \beta_h & \\ \pi_1(X, x_0) & & \\ \searrow \phi_{0*} & \downarrow & \\ & \pi_1(Y, \phi_0(x_0)) & \end{array} .$$

Proof. For $t \in I$, define the path

$$\begin{aligned} h_t : I &\longrightarrow X \\ s &\longmapsto h(ts) \end{aligned} , \quad h_t(0) = \phi_0(x_0), \quad h_t(1) = h(t) = \phi_t(x_0) .$$

Let f be a loop at x_0 . Define

$$F_t = h_t \cdot (\phi_t f) \cdot h_t^{-1} .$$

Then F_t is a loop at $\phi_0(x_0)$, which is continuous in t . So F_t is a homotopy of loops between

$$F_0 = h_0 \cdot (\phi_0 f) \cdot h_0^{-1} \cong \phi_0 f, \quad F_1 = h_1 \cdot (\phi_1 f) \cdot h_1^{-1} = h \cdot (\phi_1 f) \cdot h^{-1} .$$

Hence

$$\phi_{0*}([f]) = [\phi_0 f] = [h \cdot (\phi_1 f) \cdot h^{-1}] = \beta_h([\phi_1 f]) = \beta_h \phi_{1*}([f]) .$$

□

Lemma 1.15 implies in particular the following.

Corollary 1.16. *If $\psi : X \rightarrow X$ is continuous and $\psi \cong \text{id}_X$, then*

$$\psi_* : \pi_1(X, x_0) \rightarrow \pi_1(X, \psi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Proof. By Lemma 1.15 there is a path h from $\psi(x_0)$ to x_0 such that

$$\begin{array}{ccc} & \pi_1(X, x_0) & \\ \nearrow (\text{id}_X)_* & \downarrow \sim \beta_h & \\ \pi_1(X, x_0) & & \\ \searrow \psi_* & \downarrow & \\ & \pi_1(X, \psi(x_0)) & \end{array} ,$$

so $\psi_* = \beta_h$ hence an isomorphism.

□

Proof of Proposition 1.14. Let $\phi : X \rightarrow Y$ be a homotopy equivalence. Let $\psi : Y \rightarrow X$ be a homotopy inverse of ϕ , that is $\phi\psi \cong \text{id}_Y$ and $\psi\phi \cong \text{id}_X$. Then

$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\phi(x_0)) \xrightarrow{\phi_*} \pi_1(Y, \psi\phi\psi(x_0)).$$

Have to show that ϕ_* is bijective. The observation above implies that $(\psi\phi)_* = \psi_*\phi_*$ is an isomorphism, so ϕ_* is injective and ψ_* is surjective. Similarly $(\phi\psi)_* = \phi_*\psi_*$ is an isomorphism, so ψ_* is injective and ϕ_* is surjective. \square

Lemma 1.17. *Let X be a topological space and $x_0 \in X$. Assume*

$$X = \bigcup_{\alpha \in \Lambda} A_\alpha,$$

such that

- *the A_α are all open and path-connected,*
- *$x_0 \in A_\alpha$ for all $\alpha \in \Lambda$, and*
- *all the intersections $A_\alpha \cap A_\beta$ are path-connected for all $\alpha, \beta \in \Lambda$.*

If f is a loop in X at x_0 , then we can write

$$[f] = [h_1] \dots [h_m],$$

such that the h_i are loops at x_0 , and each contained in a single A_{α_i} .

Proof. f is continuous, so for all $s \in I$ there is an open neighbourhood V_s such that $f(V_s)$ such that $f(V_s) \subseteq A_\alpha$ for some α . We can choose V_s to be an interval (a_s, b_s) such that $f([a_s, b_s]) \subseteq A_\alpha$. Then I is compact, so a finite number of such intervals cover I , so there is a partition

$$0 = s_0 < \dots < s_m = 1,$$

such that $f([s_{i-1}, s_i]) \subseteq A_{\alpha_i}$ for some α_i . Let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$, and rescaling. Then $f \cong f_1 \cdot \dots \cdot f_m$ for $f_i \subseteq A_{\alpha_i}$ and $A_{\alpha_i} \cap A_{\alpha_j}$ is path-connected. Let g_i be a path from x_0 to $f(s_i)$ in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$. Let g_0 and g_m be the constant loops at x_0 . Then $h_i = g_{i-1} \cdot f_i \cdot g_i^{-1}$ is a loop based at x_0 and $h_i \subseteq A_{\alpha_i}$. Thus

$$f \cong (g_0 \cdot f_1 \cdot g_1^{-1}) \cdot \dots \cdot (g_{m-1} \cdot f_m \cdot g_m^{-1}),$$

so $[f] = [h_1] \dots [h_m]$. \square

Example. Möbius strip M deformation retracts to S^1 . Thus $\phi : M \rightarrow S^1$ is a homotopy equivalence, so $\pi_1(M) \cong \pi_1(S^1) \cong \mathbb{Z}$.

Example. There is no deformation retraction of S^1 to a point $p \in S^1$ because $\pi_1(S^1) \not\cong \pi_1(p)$.

Example. There is no retraction of the disc D^2 to its boundary $S^1 \subseteq D^2$. Assume there is a retraction $r : D^2 \rightarrow S^1$, consider the embedding $i : S^1 \hookrightarrow D^2$. Then $ri = \text{id}_{S^1}$. Thus

$$\begin{array}{ccccc} \pi_1(S^1) & \xrightarrow{i_*} & \pi_1(D^2) & \xrightarrow{r_*} & \pi_1(S^1) \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array},$$

so $r_*i_*(\pi_1(S^1)) = 0$ but $r_*i_* = (ri)_* = \text{id}_{\pi_1(S^1)}$, a contradiction.

Theorem 1.18 (Brouwer fixed point theorem). *Let $h : D^2 \rightarrow D^2$ be a continuous map. Then h has a fixed point, that is there exists $x \in D^2$ such that $h(x) = x$.*

Proof. Assume $h(x) \neq x$ for all $x \in D^2$. Define $r : D^2 \rightarrow S^1$ by defining $r(x)$ to be the intersection of the ray starting at $h(x)$ towards x with S^1 . Then r is continuous, and if $x \in S^1$, then $r(x) = x$, so r is a retraction, a contradiction. \square

Lemma 1.17 implies that if $U_1, U_2 \subseteq X$ are open and path-connected such that $U_1 \cup U_2 = X$ and $U_1 \cap U_2$ is path-connected and $x_0 \in U_1 \cap U_2$, then every $[f] \in \pi_1(X, x_0)$ can be factorised as

$$[f] = [g_1][h_1] \dots [g_n][h_n],$$

such that the g_i are loops at x_0 contained in U_1 and the h_i are loops at x_0 contained in U_2 . In other words, $i_1 : U_1 \hookrightarrow X$ and $i_2 : U_2 \hookrightarrow X$, so

$$i_{1*} : \pi_1(U_1, x_0) \rightarrow \pi_1(X, x_0), \quad i_{2*} : \pi_1(U_2, x_0) \rightarrow \pi_1(X, x_0).$$

Lemma 1.17 implies that $i_{1*}(\pi_1(U_1, x_0)) \cup i_{2*}(\pi_1(U_2, x_0))$ generate $\pi_1(X, x_0)$.

Proposition 1.19. $\pi_1(S^n) = 0$ if $n \geq 2$.

Proof. Let

$$U_1 = S^n \setminus \{(1, 0, \dots, 0)\}, \quad U_2 = S^n \setminus \{(-1, 0, \dots, 0)\}.$$

Then $U_1 \cong \mathbb{R}^n$ and $U_2 \cong \mathbb{R}^n$, by stereographic projection. Then $U_1 \cup U_2 = S^n$ and $U_1 \cap U_2$ is path-connected. Let $x_0 \in U_1 \cap U_2$. Then $\pi_1(U_1, x_0) = 0$ and $\pi_1(U_2, x_0) = 0$, so Lemma 1.17 implies that $\pi_1(S^n, x_0) = 0$. \square

1.2 Seifert-van Kampen theorem

1.2.1 Free products with amalgamation

Definition. If S is a set, then F_S is the **free group** on S . We can write any group G as a quotient of some free group F_S , $G = F_S / \langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ is the **normal closure** of $R \subseteq F_S$, the smallest normal subgroup of F_S containing R . We write $G = \langle S \mid R \rangle$. This is called a **presentation** of G .

Let G_0, G_1, G_2 be groups, and $f_1 : G_0 \rightarrow G_1$ and $f_2 : G_0 \rightarrow G_2$ be homomorphisms.

Definition. A group H together with homomorphisms $h_1 : G_1 \rightarrow H$ and $h_2 : G_2 \rightarrow H$ such that $h_1 f_1 = h_2 f_2$ is an **amalgamated product** of G_1 and G_2 over G_0 if it satisfies the following universal property. For every group G and all homomorphisms $h'_1 : G_1 \rightarrow G$ and $h'_2 : G_2 \rightarrow G$ such that $h'_1 f_1 = h'_2 f_2$, there exists a unique homomorphism $\alpha : H \rightarrow G$ such that $h'_1 = \alpha h_1$ and $h'_2 = \alpha h_2$, so

$$\begin{array}{ccccc} G_0 & \xrightarrow{f_1} & G_1 & & \\ f_2 \downarrow & & \downarrow h_1 & \searrow h'_1 & \\ G_2 & \xrightarrow{f_2} & H & \xrightarrow{\exists! \alpha} & G \\ & \searrow h'_2 & & & \end{array}$$

Theorem 1.20. Given $f_1 : G_0 \rightarrow G_1$ and $f_2 : G_0 \rightarrow G_2$. Then there exists an amalgamated product, unique up to isomorphism. We denote it by $G_1 *_{G_0} G_2$.

Proof. Worksheet 2. \square

$G_0 = \{\text{id}\}$ is the **free product**. We write $G_1 * G_2$ instead of $G_1 *_{\{\text{id}\}} G_2$. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Then $G_1 * G_2 = \langle S_1 \sqcup S_2 \mid R_1 \cup R_2 \rangle$, with injections $G_i \hookrightarrow G_1 * G_2$ for $i = 1, 2$. More generally,

$$G_1 *_{G_0} G_2 \cong G_1 * G_2 / N.$$

where N is the normal closure of the set

$$\left\{ f_1(g) f_2(g)^{-1} \mid g \in G_0 \right\} \subseteq G_1 * G_2.$$

1.2.2 The Seifert-van Kampen theorem

Theorem 1.21 (Seifert-van Kampen). *Let X be a topological space and $U_1, U_2 \subseteq X$ be open and path-connected such that $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is path-connected and let $x_0 \in U_1 \cap U_2$. Then*

$$\pi_1(X, x_0) \cong \pi_1(U_1, x_0) *_{\pi_1(U_1 \cap U_2, x_0)} \pi_2(U_2, x_0) \cong \pi_1(U_1, x_0) * \pi_1(U_2, x_0) / N,$$

where N is the normal closure of the set

$$\left\{ j_{1*}(\omega) j_{2*}(\omega)^{-1} \mid \omega \in \pi_1(U_1 \cap U_2, x_0) \right\},$$

and $j_i : U_i \hookrightarrow X$, so

$$\begin{array}{ccc} U_1 \cap U_2 & \xrightarrow{i_1} & U_1 \\ i_2 \downarrow & & \downarrow j_1 \\ U_2 & \xrightarrow{j_2} & X \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \pi_1(U_1 \cap U_2, x_0) & \xrightarrow{i_{1*}} & \pi_1(U_1, x_0) \\ i_{2*} \downarrow & & \downarrow j_{1*} \\ \pi_1(U_2, x_0) & \xrightarrow{j_{2*}} & \pi_1(U_1, x_0) *_{\pi_1(U_1 \cap U_2, x_0)} \pi_1(U_2, x_0) \end{array}.$$

Proof of Theorem 1.21. Appendix A.1. □

Theorem 1.22 (Seifert-van Kampen, strong version). *Let X be a path-connected topological space such that*

- $X = \bigcup_{\alpha} A_{\alpha}$,
- $A_{\alpha}, A_{\alpha} \cap A_{\beta}, A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$ are open and path-connected for all α, β, γ , and
- $x_0 \in \bigcap_{\alpha} A_{\alpha}$.

Then

$$\pi_1(X, x_0) \cong *_{\alpha} \pi_1(A_{\alpha}, x_0) / N,$$

where $N \subseteq *_{\alpha} \pi_1(A_{\alpha}, x_0)$ is the normal closure of the set

$$\left\{ (i_{\alpha\beta})_*(\omega) (i_{\beta\alpha})_*(\omega)^{-1} \mid \omega \in \pi_1(A_{\alpha} \cap A_{\beta}) \right\},$$

and $i_{\alpha\beta} : A_{\alpha} \cap A_{\beta} \hookrightarrow A_{\alpha}$ is the inclusion.

Example. Let $S^1 \vee S^1$ be the wedge product. Fix $x \in S^1$ and $y \in S^1$. Then

$$S^1 \vee S^1 = S^1 \sqcup S^1 / x \sim y = \begin{array}{c} \text{b} \quad \text{a} \\ \bigcirc \quad \bigcirc \end{array}.$$

Let

$$A = \begin{array}{c} \text{ } \\ \bigcirc \end{array}, \quad B = \begin{array}{c} \text{ } \\ \bigcirc \end{array}, \quad A \cap B = \begin{array}{c} \text{ } \\ \times \end{array}.$$

Then $\pi_1(A) \cong \langle b \rangle \cong \mathbb{Z}$, $\pi_1(B) \cong \langle a \rangle \cong \mathbb{Z}$, and $\pi_1(A \cap B) = \{\text{id}\}$, and $A, B, A \cap B$ are open and path-connected. Van Kampen implies that

$$\pi_1(S^1 \vee S^1) \cong \pi_1(A) * \pi_1(B) \cong \mathbb{Z} * \mathbb{Z} \cong F_{\{a, b\}}.$$

More generally, let $X = S_{a_1}^1 \vee \cdots \vee S_{a_n}^1$. Induction implies that

$$\pi_1(X) = \mathbb{Z} * \cdots * \mathbb{Z} \cong F_{\{a_1, \dots, a_n\}}.$$

Similarly, let $X = \bigvee_{\alpha \in \Lambda} S_{\alpha}^1$. Strong version of van Kampen implies that

$$\pi_1(X) = *_{\alpha \in \Lambda} \mathbb{Z} = F_{\Lambda}.$$

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Friday
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Example. Let T be a torus and $x_0 \in T$. Let

$$A = T \setminus \{\text{small closed disc } D\}, \quad B = \{\text{open set that contains } D \text{ and } x_0\}.$$

- A is homotopy equivalent to $S^1 \vee S^1$, so $\pi_1(A) \cong F_{\{a,b\}}$.
- B is homeomorphic to D^2 , so $\pi_1(B) = \{\text{id}\}$.
- $A \cap B$ is homotopy equivalent to S^1 , so $\pi_1(A \cap B) \cong \mathbb{Z}$.

Then $A, B, A \cap B$ are open and path-connected. Van Kampen implies that

$$\pi_1(T) \cong \pi_1(A) / \langle \langle i_* (\pi_1(A \cap B)) \rangle \rangle,$$

where $i : A \cap B \hookrightarrow A$. Then

$$\begin{aligned} i_* : \pi_1(A \cap B) = \langle \omega \rangle &\longrightarrow \pi_1(A) \\ \omega &\longmapsto aba^{-1}b^{-1}, \end{aligned}$$

so

$$\pi_1(T) \cong F_{\{a,b\}} / \langle \langle aba^{-1}b^{-1} \rangle \rangle = \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2.$$

1.2.3 Applications to CW-complexes

Let X be a path-connected topological space. Let Y be the space obtained by attaching 2-cells $\{e_\alpha^2\}$ to X along maps $\phi_\alpha : \partial D^2 = S^1 \rightarrow X$. Consider the loops

$$\begin{aligned} \phi'_\alpha : I &\longrightarrow X \\ s &\longmapsto \phi_\alpha(\cos 2\pi s, \sin 2\pi s), \end{aligned}$$

based at $\phi'_\alpha(0)$. Let γ_α be a path from x_0 to $\phi'_\alpha(0)$ for each α . Then $\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}$ is a loop at x_0 . After attaching e_α^2 , the loop $\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}$ is homotopic to the constant loop at x_0 . Let $N \subseteq \pi_1(X, x_0)$ be the normal closure of all the elements of the form $[\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}]$. The inclusion $i : X \hookrightarrow Y$ yields

$$i_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, x_0),$$

and $N \subseteq \text{Ker } i_*$.

Proposition 1.23. *This inclusion $i : X \hookrightarrow Y$ induces a surjection*

$$i_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, x_0),$$

and $\text{Ker } i_* = N$, so

$$\pi_1(Y, x_0) \cong \pi_1(X, x_0) / N.$$

Proof. Construct a space Z from Y by attaching a strip $I \times I$ to Y by identifying the lower edge $I \times \{0\}$ with the path γ_α and the right edge $\{1\} \times I$ with an arch on e_α^2 . Attach all the left edges of the strips with each other. Then Z deformation retracts to Y . Choose a point $y_\alpha \in e_\alpha^2$ for each α , such that y_α is not contained in X or in the attached strip. Let

$$A = Z \setminus \bigcup_\alpha \{y_\alpha\}, \quad B = Z \setminus X.$$

- A deformation retracts to X .
- B is homotopy equivalent to a point.
- $A \cap B$ is homotopy equivalent to

$$\{\text{paths } \gamma_\alpha \text{ from } x_0 \text{ to loops } \phi'_\alpha\} = \begin{array}{c} \text{---} \bigcirc \phi'_\alpha \text{---} \end{array} \xrightarrow{\gamma_\alpha} x_0 \xrightarrow{\gamma_\alpha} \begin{array}{c} \text{---} \bigcirc \phi'_\alpha \text{---} \end{array}.$$

Then $A, B, A \cap B$ are open and path-connected. Van Kampen implies that

$$\pi_1(Y) \cong \pi_1(Z) = \pi_1(A) / \langle \langle j_* (\pi_1(A \cap B)) \rangle \rangle,$$

where $j : A \cap B \hookrightarrow A$ is the inclusion. So $\langle \langle j_* (\pi_1(A \cap B)) \rangle \rangle$ is exactly N . Thus $\pi_1(A) = \pi_1(X)$. \square

Corollary 1.24. For every group G there exists a two-dimensional CW-complex X_G such that $\pi_1(X_G) = G$.

Proof. Let $G = \langle \{g_\alpha\} \mid \{r_\beta\} \rangle$ be a presentation of G , that is $G = F_{\{g_\alpha\}} / \langle \langle \{r_\beta\} \rangle \rangle$. Seen last time that $\pi_1\left(\bigvee_{g_\alpha} S^1_{g_\alpha}\right) = F_{\{g_\alpha\}}$. Each word r_β defines a loop in $\bigvee_{g_\alpha} S^1_{g_\alpha}$. Attach 2-cells to $\bigvee_{g_\alpha} S^1_{g_\alpha}$ along the loops defined by the relations $\{r_\beta\}$. Call this new CW-complex Y . Proposition 1.23 implies that

$$\pi_1(Y, x_0) \cong \pi_1(X, x_0) / \langle \langle \{r_\beta\} \rangle \rangle \cong F_{\{g_\alpha\}} / \langle \langle \{r_\beta\} \rangle \rangle \cong G.$$

□

Remark. Let $X = \bigcup_n X^n$ be a CW-complex, path-connected. Proposition 1.23 can be used to show the following two facts.

- The inclusion $X^1 \hookrightarrow X$ induces a surjective homomorphism $\pi_1(X^1) \rightarrow \pi_1(X)$.
- The inclusion $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2) \rightarrow \pi_1(X)$.

1.3 Covering spaces

1.3.1 Lifting properties

Let X be a topological space. Recall that a covering space is $p : \tilde{X} \rightarrow X$ such that each $x \in X$ has an open neighbourhood U such that

$$p^{-1}(U) = \bigcup_{\alpha} \tilde{U}_{\alpha},$$

where U_{α} are pairwise disjoint and $p|_{\tilde{U}_{\alpha}} : \tilde{U}_{\alpha} \rightarrow U$ is a homeomorphism for all α .

Example.

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & S^1 \\ s & \longmapsto & (\cos 2\pi s, \sin 2\pi s) \end{array}, \quad \begin{array}{ccc} S^1 & \longrightarrow & S^1 \\ z & \longmapsto & z^n \end{array}, \quad \begin{array}{c} \text{Two circles} \end{array} \rightarrow S^1 \vee S^1 = \begin{array}{c} \text{Two circles} \end{array}.$$

Let $f : Y \rightarrow X$ be a continuous map. A lift of f is a continuous map $\tilde{f} : Y \rightarrow \tilde{X}$ such that $p\tilde{f} = f$, where $p : \tilde{X} \rightarrow X$ is a covering space. Let Y be connected.

- **Unique lifting property** states that if two lifts \tilde{f}_1 and \tilde{f}_2 of f coincide at one point, then they coincide on all of Y .
- **Homotopy lifting property** states that if $f_t : Y \rightarrow X$ is a homotopy and $\tilde{f}_0 : Y \rightarrow \tilde{X}$ is a lift of f_0 then there exists a unique homotopy $\tilde{f}_t : Y \rightarrow \tilde{X}$ of \tilde{f}_0 that lifts f_t .

Remark.

- If Y is a point, this is called the **path lifting property**. Let $f : I \rightarrow X$ be a path with $f(0) = x_0$. If $\tilde{x}_0 \in p^{-1}(x_0)$, then there is a unique path $\tilde{f} : I \rightarrow \tilde{X}$ lifting f and starting at \tilde{x}_0 .
- In particular, the lift of a constant path is constant.
- This implies in particular that the lift of a homotopy of paths is again a homotopy of paths. The endpoints $\tilde{f}_t(0)$ and $\tilde{f}_t(1)$ define constant paths as t varies.

Fix $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$, so

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0).$$

To every element in $\pi_1(X, x_0)$ we can associate a homotopy class of paths in \tilde{X} starting at \tilde{x}_0 .

Proposition 1.25.

1. $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ is injective.
2. $p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right) \subseteq \pi_1(X, x_0)$ consists of the homotopy classes of loops at x_0 whose lifts to \tilde{X} starting at \tilde{x}_0 are loops.

Proof.

1. Let $\tilde{f}_0 : I \rightarrow \tilde{X}$ be a loop at \tilde{x}_0 such that $[\tilde{f}_0] \in \text{Ker } p_*$, so $p\tilde{f}_0 = f_0$ is homotopic to the constant loop at x_0 . Let $f_t : I \rightarrow X$ be a homotopy between f_0 and the constant loop. Homotopy lifting property and remark implies that f_t lifts to a homotopy \tilde{f}_t of paths between \tilde{f}_0 and the constant loop, so $[\tilde{f}_0] = \text{id} \in \pi_1(\tilde{X}, \tilde{x}_0)$ and p_* is injective.
2. Let $f : I \rightarrow X$ be a loop at x_0 that lifts to a loop \tilde{f} at \tilde{x}_0 . Then $p\tilde{f} = f$, so $p_*([\tilde{f}]) = [f]$. On the other hand, if $f : I \rightarrow X$ is a loop at x_0 such that there exists a loop $\tilde{f} : I \rightarrow \tilde{X}$ at \tilde{x}_0 with $p_*([\tilde{f}]) = [f]$, then f is homotopic to $p\tilde{f}$. Homotopy lifting property implies that there exists a loop $\tilde{f}' : I \rightarrow \tilde{X}$ at \tilde{x}_0 such that $p\tilde{f}' = f$.

□

Let $p : \tilde{X} \rightarrow X$ be a covering space. Let $U \subseteq X$ be an evenly covered neighbourhood of $x \in X$. Let

$$p^{-1}(U) = \bigsqcup_{\alpha \in \Lambda} \tilde{U}_\alpha.$$

Then the cardinality $|p^{-1}(x)|$ of $p^{-1}(x)$ is exactly the cardinality of $|\Lambda|$. The set of sheets is in bijection with $p^{-1}(x)$. So the cardinality of $p^{-1}(x)$ is locally constant. If X is connected, the cardinality of $p^{-1}(x)$ is constant.

Notation. Let X and Y be topological spaces, $x \in X$, and $y \in Y$. A continuous map

$$f : (X, x) \rightarrow (Y, y)$$

is a continuous map $f : X \rightarrow Y$ such that $f(x) = y$.

Proposition 1.26. Let X and \tilde{X} be path-connected and

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a covering space. Then the number of sheets of p equals the index of $p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$ in $\pi_1(X, x_0)$.

Proof. Let g be a loop in X at x_0 and \tilde{g} be its lift to \tilde{X} starting at \tilde{x}_0 . Let $H = p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$ and let $[h] \in H$. Then $h \cdot g$ lifts to a path $\tilde{h} \cdot \tilde{g}$ in \tilde{X} starting at \tilde{x}_0 with the same endpoint as \tilde{g} , because \tilde{h} is a loop, by Proposition 1.25. Define

$$\begin{aligned} \Phi : \{\text{cosets of } H \text{ in } \pi_1(X, x_0)\} &\longrightarrow p^{-1}(x_0) \\ H[g] &\longmapsto \tilde{g}(1) \end{aligned},$$

so Φ is well-defined. Want to show that Φ is bijective.

- Φ is surjective because \tilde{X} is path-connected. Let \tilde{g} be a path in \tilde{X} from \tilde{x}_0 to any point $\tilde{x}'_0 \in p^{-1}(x_0)$, then $g = p \cdot \tilde{g}$ and $\Phi(H[g]) = \tilde{x}'_0$.
- Φ is injective, since if $\Phi(H[g_1]) = \Phi(H[g_2])$ then the lift $\tilde{g}_1 \cdot \tilde{g}_2^{-1}$ of $g_1 \cdot g_2^{-1}$ defines a loop in \tilde{X} at \tilde{x}_0 . Proposition 1.25 implies that $[g_1][g_2]^{-1} \in H$, so $H[g_1] = H[g_2]$.

□

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We say that a topological space X has a certain property (P) **locally** if for each point $x \in X$ and each neighbourhood U of x there is an open neighbourhood $V \subseteq U$ having this property (P) .

Example. X is locally path-connected or X is locally simply-connected.

Proposition 1.27. *Let*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a covering space and

$$f : (Y, y_0) \rightarrow (X, x_0)$$

a continuous map, where Y is path-connected and locally path-connected. Then there is a lift

$$\tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$$

if and only if

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0)),$$

so

$$\begin{array}{ccc} & & (\tilde{X}, \tilde{x}_0) \\ & \nearrow \tilde{f} & \downarrow p \\ (Y, y_0) & \xrightarrow{f} & (X, x_0) \end{array} .$$

Proof.

\Rightarrow Clear, because $f = p\tilde{f}$ implies $f_* = p_*\tilde{f}_*$.

\Leftarrow Assume

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

For each $y \in Y$ choose a path γ from y_0 to y , so $f\gamma$ is a path in X from x_0 to $f(y)$. By path lifting, we can lift $f\gamma$ to a path $\tilde{f}\gamma$ in \tilde{X} starting at \tilde{x}_0 . Define the map

$$\begin{array}{ccc} \tilde{f} : (Y, y_0) & \longrightarrow & (\tilde{X}, \tilde{x}_0) \\ y & \longmapsto & \tilde{f}\gamma(1) \end{array} .$$

- This map is well-defined, that is does not depend on the choice of γ . Let γ' be another path from y_0 to y . Then $h_0 = (f\gamma') \cdot (f\gamma)^{-1}$ is a loop at x_0 and

$$[h_0] \in f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Proposition 1.25 implies that can lift h_0 to a loop \tilde{h}_0 at \tilde{x}_0 . The first half of \tilde{h}_0 is $\tilde{f}\gamma'$ and the second half is $\tilde{f}\gamma^{-1}$, so $\tilde{f}\gamma(1) = \tilde{f}\gamma'(1)$. Thus \tilde{f} is well-defined.

- We have $p\tilde{f} = f$, so \tilde{f} lifts f .
- It remains to show that \tilde{f} is continuous. Let $y \in Y$ and let U be an evenly covered neighbourhood of $f(y)$. Let \tilde{U} be the sheet above U such that $\tilde{f}(y) \in \tilde{U}$, so $p|_{\tilde{U}} : \tilde{U} \rightarrow U$ is a homeomorphism. Let $V \subseteq Y$ be a path-connected neighbourhood of y such that $f(V) \subseteq U$. Fix a path γ from y_0 to y . Let $y' \in V$ be arbitrary and η be a path from y to y' , so $\gamma \cdot \eta$ is a path from y_0 to y' . Then $(f\gamma) \cdot (f\eta)$ is a path in U from x_0 to $f(y')$, and $\tilde{f}\eta = (p|_{\tilde{U}})^{-1} f\eta$, so $\tilde{f}|_V = (p|_{\tilde{U}})^{-1} f$. Thus $\tilde{f}|_V : V \rightarrow \tilde{U}$ is continuous, so \tilde{f} is continuous.

□

1.3.2 The classification of covering spaces

Definition. A covering space $p : \tilde{X} \rightarrow X$ is a **universal cover** if \tilde{X} is simply-connected.

Definition. A topological space X is **semilocally simply-connected** if each $x \in X$ has a neighbourhood U such that

$$i_* : \pi_1(U, x) \rightarrow \pi_1(X, x)$$

is trivial, where $i : U \hookrightarrow X$ is the inclusion.

Example. Let $X = \bigcup_n C_n \subseteq \mathbb{R}^2$ be the **Hawaiian earrings**, where $C_n \subseteq \mathbb{R}^2$ is the circle of radius $1/n$ and centre $(1/n, 0)$. Then X is not semilocally simply-connected.

Proposition 1.28. *If $p : \tilde{X} \rightarrow X$ is a universal cover, then X is semilocally simply-connected.*

Proof. Let $U \subseteq X$ be an evenly covered neighbourhood of $x_0 \in X$, $\tilde{U} \subseteq \tilde{X}$ be a sheet over U , and $\gamma \subseteq U$ be a loop at x_0 , so γ lifts to a loop $\tilde{\gamma} \subseteq \tilde{U}$ at \tilde{x}_0 . Then $\tilde{\gamma}$ is homotopic to the constant loop at \tilde{x}_0 . Composing this homotopy with p implies that γ is homotopic to the constant loop at x_0 in X , so

$$\pi_1(U, x_0) \rightarrow \pi_1(X, x_0)$$

is trivial. □

Theorem 1.29. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there exists a universal cover $p : \tilde{X} \rightarrow X$.*

Remark. If

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

is a universal cover, each point $\tilde{x} \in \tilde{X}$ can be joined to \tilde{x}_0 by a unique homotopy class of paths, by Proposition 1.6.

$$\{\text{points in } \tilde{X}\} \rightsquigarrow \{[\gamma] \mid \gamma \text{ is a path in } \tilde{X} \text{ starting at } \tilde{x}_0\} \rightsquigarrow \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

by the homotopy lifting property.

Proof. Let $x_0 \in X$, and

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}, \quad p : \begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ [\gamma] & \longmapsto & \gamma(1) \end{array}.$$

Have to

1. give \tilde{X} a topology,
2. show that $p : \tilde{X} \rightarrow X$ is a covering, and
3. show that \tilde{X} is simply-connected.

Recall that a **basis** for a topology on a set Y is a collection \mathcal{B} of subsets such that

- $Y = \bigcup_{U \in \mathcal{B}} U$, and
- if $U_1, U_2 \in \mathcal{B}$ and $y \in U_1 \cap U_2$ then there exists $V \in \mathcal{B}$ such that $y \in V$ and $V \subseteq U_1 \cap U_2$.

A basis defines a topology on Y , by $A \subseteq Y$ is open if and only if A is the union of elements of \mathcal{B} . A map $f : Z \rightarrow Y$ is continuous if and only if $f^{-1}(U)$ is open for all $U \in \mathcal{B}$.

1. Let \mathcal{U} be the collection of all path-connected open sets $U \subseteq X$ such that $\pi_1(U) \rightarrow \pi_1(X)$ is trivial. Then $X = \bigcup_{U \in \mathcal{U}} U$ because X is semilocally simply-connected. Let $U_1, U_2 \in \mathcal{U}$ and $y \in U_1 \cap U_2$, and let $V \subseteq U_1 \cap U_2$ be path-connected and open. Then

$$\begin{array}{ccccc} V & \hookrightarrow & U_1 & \hookrightarrow & X \\ & & & & \\ \pi_1(V) & \xrightarrow{\quad} & \pi_1(U_1) & \xrightarrow{\text{trivial}} & \pi_1(X) \\ & \searrow & \text{trivial} & \nearrow & \end{array},$$

so $V \in \mathcal{U}$, so \mathcal{U} is a basis for the topology on X . For $U \in \mathcal{U}$ and γ a path in X from x_0 to a point in U , we define

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ a path in } U \text{ such that } \eta(0) = \gamma(1)\} \subseteq \tilde{X}.$$

$U_{[\gamma]}$ only depends on the class $[\gamma]$, so $p|_{U_{[\gamma]}} : U_{[\gamma]} \rightarrow U$ is bijective. Surjective because U is path-connected and injective because all paths η in U with the same endpoint are homotopic. Claim that $\{U_{[\gamma]}\}$ forms a basis on \tilde{X} .

- $\bigcup_{U \in \mathcal{U}} U_{[\gamma]} = \tilde{X}$, because $\bigcup_{U \in \mathcal{U}} U = X$.
- Observe that if $[\gamma'] \in U_{[\gamma]}$ then $U_{[\gamma]} = U_{[\gamma']}$. If $\gamma' = \gamma \cdot \eta$ for η a path in U , then elements in $U_{[\gamma']}$ have the form $[\gamma \cdot \eta \cdot \mu]$, so $U_{[\gamma']} \subseteq U_{[\gamma]}$. The elements in $U_{[\gamma]}$ have the form

$$[\gamma \cdot \mu] = [\gamma \cdot \eta \cdot \eta^{-1} \cdot \mu] = [\gamma' \cdot \eta^{-1} \cdot \mu],$$

so $U_{[\gamma]} \subseteq U_{[\gamma']}$. Consider $U_{[\gamma]}$ and $V_{[\gamma']}$ and let $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, so $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. Let $W \in \mathcal{U}$ such that $W \subseteq U \cap V$ and such that $\gamma''(1) \in W$, so $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$ and $[\gamma''] \in W_{[\gamma'']}$. This proves the claim.

2. $p|_{U_{[\gamma]}} : U_{[\gamma]} \rightarrow U$ is a homeomorphism. It is bijective, let $V_{[\gamma']}$ be an element of the basis, so $p(V_{[\gamma']}) = V \in \mathcal{U}$. Then $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$. Thus $p : \tilde{X} \rightarrow X$ is continuous. If $U \in \mathcal{U}$, then

$$p^{-1}(U) = \bigsqcup_{[\gamma]} U_{[\gamma]},$$

so $p : \tilde{X} \rightarrow X$ is a covering space.

3. Let $\tilde{x}_0 \in \tilde{X}$ be the class of the constant path at x_0 . Let $[\gamma] \in \tilde{X}$ be arbitrary. Then $\gamma : [0, 1] \rightarrow X$ and $\gamma(0) = x_0$. Let γ_t be the path in X defined by

$$\gamma_t(s) = \begin{cases} \gamma(s) & s \in [0, t] \\ \gamma(t) & s \in [t, 1] \end{cases}.$$

Then

$$\begin{array}{ccc} \tilde{\gamma} & : & \mathbf{I} \longrightarrow \tilde{X} \\ & & t \longmapsto [\gamma_t] \end{array}$$

is a path in \tilde{X} from \tilde{x}_0 to $[\gamma]$, so \tilde{X} is path-connected. Recall that $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)$ consists of the classes of loops at x_0 in X that lifts to loops in \tilde{X} at \tilde{x}_0 . Let $[\gamma] \in p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)$. Then γ lifts to a loop at \tilde{x}_0 by $t \mapsto [\gamma_t]$. Because it is a loop we have $\tilde{x}_0 = [\gamma_1] = [\gamma]$, so γ is homotopic to the constant loop. Thus $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right) = \{\text{id}\}$, so \tilde{X} is simply-connected.

□

Let $p : \tilde{X} \rightarrow X$ be a covering space, so $p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0 \right) \right) \subseteq \pi_1 (X, x_0)$.

Proposition 1.30. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \subseteq \pi_1 (X, x_0)$ there is a covering space $p : X_H \rightarrow X$ such that $p_* (\pi_1 (X_H, \tilde{x}_0)) = H$ for some basepoint x_0 .*

Proof. Let \tilde{X} be as constructed above. Define $X_H = \tilde{X} / \sim$, where $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot (\gamma')^{-1}] \in H$. This is an equivalence relation.

- $[\gamma] \sim [\gamma]$ because $\text{id} \in H$.
- $[\gamma] \sim [\gamma']$ implies that $[\gamma'] \sim [\gamma]$ because H contains all its inverses.
- $[\gamma] \sim [\gamma']$ and $[\gamma'] \sim [\gamma'']$ implies that $[\gamma] \sim [\gamma'']$ because H is closed under product.

Then

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \tilde{X} / \sim = X_H \\ \downarrow & \swarrow p & \\ X & & \end{array} .$$

Let $U_{[\gamma]}$ and $U_{[\gamma']}$ be basis neighbourhoods. If $[\gamma] \sim [\gamma']$ then $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$, so p is a covering space, and $p^{-1}(U) = \bigcup_{\gamma} U_{[\gamma]}$. Let $\tilde{x}_0 \in X_H$ be the equivalence class of the constant path c_{x_0} at x_0 . Let γ be a loop in X at x_0 such that $[\gamma] \in p_* (\pi_1 (X_H, \tilde{x}_0))$. Again $t \mapsto [\gamma_t]$ is a lift of γ at \tilde{x}_0 . Then

$$t \mapsto [\gamma_t] \text{ is a loop in } X_H \iff [\gamma_1] = [\gamma] = [c_{x_0}] \text{ in } X_H \iff [\gamma] \sim [c_{x_0}] \iff \gamma \in H.$$

□

Definition. We say that two covering spaces $p_1 : \tilde{X}_1 \rightarrow X$ and $p_2 : \tilde{X}_2 \rightarrow X$ are **isomorphic** if there exists a homeomorphism $f : \tilde{X}_1 \rightarrow \tilde{X}_2$ such that

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{f} & \tilde{X}_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array} .$$

Proposition 1.31. *Let X be path-connected and locally path-connected and $x_0 \in X$. Two path-connected covering spaces $p_1 : \tilde{X}_1 \rightarrow X$ and $p_2 : \tilde{X}_2 \rightarrow X$ are isomorphic via an isomorphism $f : \tilde{X}_1 \rightarrow \tilde{X}_2$ mapping a basepoint $\tilde{x}_1 \in p_1^{-1}(x_0)$ to a basepoint $\tilde{x}_2 \in p_2^{-1}(x_0)$ if and only if*

$$p_{1*} \left(\pi_1 \left(\tilde{X}_1, \tilde{x}_1 \right) \right) = p_{2*} \left(\pi_1 \left(\tilde{X}_2, \tilde{x}_2 \right) \right).$$

Proof.

\implies If

$$f : \left(\tilde{X}_1, \tilde{x}_1 \right) \rightarrow \left(\tilde{X}_2, \tilde{x}_2 \right)$$

is an isomorphism, then $p_1 = p_2 f$, so

$$p_{1*} \left(\pi_1 \left(\tilde{X}_1, \tilde{x}_1 \right) \right) \subseteq p_{2*} \left(\pi_1 \left(\tilde{X}_2, \tilde{x}_2 \right) \right),$$

and $p_2 = p_1 f^{-1}$, so

$$p_{2*} \left(\pi_1 \left(\tilde{X}_2, \tilde{x}_2 \right) \right) \subseteq p_{1*} \left(\pi_1 \left(\tilde{X}_1, \tilde{x}_1 \right) \right).$$

\Leftarrow Assume

$$p_{1*} \left(\pi_1 \left(\widetilde{X}_1, \widetilde{x}_1 \right) \right) = p_{2*} \left(\pi_1 \left(\widetilde{X}_2, \widetilde{x}_2 \right) \right).$$

By lifting criterion in Proposition 1.27, we can lift p_1 to a continuous map

$$\widetilde{p}_1 : \left(\widetilde{X}_1, \widetilde{x}_1 \right) \rightarrow \left(\widetilde{X}_2, \widetilde{x}_2 \right),$$

and p_2 to a continuous map

$$\widetilde{p}_2 : \left(\widetilde{X}_2, \widetilde{x}_2 \right) \rightarrow \left(\widetilde{X}_1, \widetilde{x}_1 \right),$$

so $p_1 \widetilde{p}_2 = p_2$ and $p_2 \widetilde{p}_1 = p_1$.

$$\begin{array}{ccc} \left(\widetilde{X}_1, \widetilde{x}_1 \right) & \xrightleftharpoons{\widetilde{p}_1} & \left(\widetilde{X}_2, \widetilde{x}_2 \right) \\ & \searrow p_1 \quad \swarrow p_2 & \\ & (X, x_0) & \end{array}.$$

Then $\widetilde{p}_1 \widetilde{p}_2$ fixes the point $\widetilde{x}_2 \in \widetilde{X}_2$. By the unique lifting property in Proposition 1.7, $\widetilde{p}_1 \widetilde{p}_2 = \text{id}_{\widetilde{x}_2}$. Similarly, $\widetilde{p}_2 \widetilde{p}_1 = \text{id}_{\widetilde{x}_1}$, so \widetilde{p}_1 is an isomorphism.

□

Fix $x_0 \in X$, $\widetilde{x}_1 \in p_1^{-1}(x_0)$, and $\widetilde{x}_2 \in p_2^{-1}(x_0)$. A **basepoint preserving isomorphism**

$$f : \left(\widetilde{X}_1, \widetilde{x}_1 \right) \rightarrow \left(\widetilde{X}_2, \widetilde{x}_2 \right)$$

is an isomorphism such that $f(\widetilde{x}_1) = \widetilde{x}_2$.

Theorem 1.32 (Galois correspondence). *Let X be path-connected, locally path-connected, and semilocally simply-connected, and $x_0 \in X$. Then*

1. *there is a bijection*

$$\left\{ \begin{array}{l} \text{path-connected covering spaces } p : (\widetilde{X}, \widetilde{x}_0) \rightarrow (X, x_0) \\ \text{up to basepoint preserving isomorphisms} \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \text{subgroups} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\},$$

2. *if we ignore the basepoints, this correspondence gives a bijection*

$$\left\{ \begin{array}{l} \text{path-connected covering spaces } p : \widetilde{X} \rightarrow X \\ \text{up to isomorphisms} \end{array} \right\} \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \text{conjugacy classes of subgroups} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\}.$$

Proof.

1. To a covering space

$$p : (\widetilde{X}, \widetilde{x}_0) \rightarrow (X, x_0),$$

we associate the subgroup

$$p_* \left(\pi_1 \left(\widetilde{X}, \widetilde{x}_0 \right) \right) \subseteq \pi_1(X, x_0).$$

Proposition 1.30 and Proposition 1.31 show that this is well-defined on the isomorphism classes and it is bijective.

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2. Let $p : \tilde{X} \rightarrow X$ be a covering space and $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$. Let

$$H_i = p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_i \right) \right) \subseteq \pi_1 (X, x_0), \quad i = 1, 2.$$

Let $\tilde{\gamma}$ be a path from \tilde{x}_1 to \tilde{x}_2 . Let $\gamma = p\tilde{\gamma}$ be a loop at x_0 . Let $[f] \in \pi_1 (X, x_0)$. Then $[f] \in H_1$ if and only if the lift \tilde{f} is a loop at \tilde{x}_1 . Then $\tilde{\gamma}^{-1} \cdot \tilde{f} \cdot \tilde{\gamma}$ is a loop at \tilde{x}_2 , so

$$p_* \left(\tilde{\gamma}^{-1} \cdot \tilde{f} \cdot \tilde{\gamma} \right) = \gamma^{-1} \cdot f \cdot \gamma,$$

so $[\gamma]^{-1} [f] [\gamma] \in H_2$. Thus $[\gamma]^{-1} H_1 [\gamma] \subseteq H_2$. Similarly, $[\gamma] H_2 [\gamma]^{-1} \subseteq H_1$. Conversely, let $H_1 \subseteq \pi_1 (X, x_0)$ as above and $[\delta] \in \pi_1 (X, x_0)$ be an arbitrary element. Let $\tilde{\delta}$ be a lift of δ such that $\tilde{\delta}(0) = \tilde{x}_0$ and define $\tilde{x}_3 = \tilde{\delta}(1)$. Then the same construction yields

$$p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_3 \right) \right) = [\delta]^{-1} H_1 [\delta].$$

□

1.3.3 Deck transformations and group actions

Definition. Let $p : \tilde{X} \rightarrow X$ be a covering space. A **deck-transformation** is an isomorphism from \tilde{X} to itself.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & \tilde{X} \\ & \searrow p & \swarrow p \\ & X & \end{array}.$$

The group of deck-transformations is denoted by $G(\tilde{X})$.

Example.

- Let

$$\begin{array}{ccc} p & : & \mathbb{R} \longrightarrow \mathbb{S}^1 \subseteq \mathbb{C} \\ & & t \longmapsto e^{2\pi i t} \end{array}.$$

Then $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $p(f(t)) = p(t)$ if and only if $e^{2\pi i f(t)} = e^{2\pi i t}$, if and only if $f(t) = t + n$, so $G(\mathbb{R}) \cong \mathbb{Z}$.

- Let

$$\begin{array}{ccc} p & : & \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \\ & & z \longmapsto z^n \end{array}.$$

Then $G(\mathbb{S}^1) \cong \mathbb{Z}/n\mathbb{Z}$.

An observation is that if \tilde{X} is path-connected then $f \in G(\tilde{X})$ is uniquely determined by where it sends a single point.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f'} & \tilde{X} \\ & \xrightarrow{f} & \\ & \searrow p & \swarrow p \\ & X & \end{array}.$$

If $f(x) = f'(x)$ for a single x , by unique lifting $f = f'$. So the identity is the only deck-transformation with a fixed point.

Definition. A covering space $p : \tilde{X} \rightarrow X$ is **normal**, or **regular**, or **Galois**, if for each $x \in X$ and every pair $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there is an $f \in G(\tilde{X})$ such that $f(\tilde{x}) = \tilde{x}'$.

Example.

- $p : \mathbb{R} \rightarrow S^1$ is normal.
- $p : S^1 \rightarrow S^1$ is normal.

Proposition 1.33. *Let*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a path-connected covering space, and X be path-connected and locally path-connected. Then $p : \tilde{X} \rightarrow X$ is normal if and only if

$$H = p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0 \right) \right) \subseteq \pi_1(X, x_0)$$

is a normal subgroup.

Proof. Let $\tilde{x}_1 \in p^{-1}(x_0)$, let $\tilde{\gamma}$ be a path from \tilde{x}_0 to \tilde{x}_1 and $\gamma = p(\tilde{\gamma})$. Then $[\gamma]$ conjugates H to $p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_1 \right) \right)$ so $[\gamma] H [\gamma]^{-1} = H$, if and only if $H = p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_1 \right) \right)$, by Proposition 1.31 if and only if $f(\tilde{x}_0) = \tilde{x}_1$. So $G(\tilde{X})$ acts transitively on $p^{-1}(x_0)$ if and only if $H \subseteq \pi_1(X, x_0)$ is a normal subgroup. Let $x'_0 \in X$ be another point and h a path from x_0 to x'_0 . Let \tilde{h} be a lift of h such that $\tilde{h}(0) = \tilde{x}_0$. Set $\tilde{x}'_0 = \tilde{h}(1)$ and $p(\tilde{x}'_0) = x'_0$. Then

$$\begin{array}{ccc} \pi_1 \left(\tilde{X}, \tilde{x}_0 \right) & \xrightarrow{\beta_{\tilde{h}}} & \pi_1 \left(\tilde{X}, \tilde{x}'_0 \right) \\ p_* \downarrow & & \downarrow p_* \\ \pi_1(X, x_0) & \xrightarrow{\beta_h} & \pi_1(X, x'_0) \end{array} .$$

Thus $H \subseteq \pi_1(X, x_0)$ is normal if and only if

$$p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}'_0 \right) \right) \subseteq \pi_1(X, x'_0)$$

is normal, as before if and only if $G(\tilde{X})$ acts transitively on $p^{-1}(x'_0)$. □

Proposition 1.34. *Let*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a covering space, X be path-connected and locally path-connected, and \tilde{X} be path-connected. Let $H = p_ \left(\pi_1 \left(\tilde{X}, \tilde{x}_0 \right) \right)$ and $N(H) \subseteq \pi_1(X, x_0)$ be the normaliser of H . Then $G(\tilde{X})$ is isomorphic to $N(H)/H$. In particular,*

- *if \tilde{X} is normal, then*

$$G(\tilde{X}) \cong \pi_1(X, x_0) / H,$$

- *if \tilde{X} is the universal cover, then*

$$G(\tilde{X}) \cong \pi_1(X, x_0) .$$

Proof. Read the proof of this in Hatcher. ³ □

³Exercise

Example. Let $X = S^1 \vee S^1$, so $\pi_1(X) = F_{\{a,b\}}$. Then the following are covering spaces.

- A normal covering space

$$\tilde{X} = \begin{array}{c} \text{Diagram: Three circles arranged horizontally. The first and second circles overlap, and the second and third circles overlap. The first circle is labeled 'a', the second 'b', and the third 'a'. A point \tilde{x}_0 is marked at the intersection of the first and second circles.} \end{array} \quad p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0 \right) \right) = \langle a, b^2, bab^{-1} \rangle \stackrel{2}{\subseteq} F_{\{a,b\}}$$

In general, a two-oriented graph is a covering space of X .

- Not a normal covering space

$$\tilde{X} = \begin{array}{c} \text{Diagram: Four circles arranged horizontally. The first and second circles overlap, the second and third overlap, and the third and fourth overlap. The first circle is labeled 'a', the second 'b', the third 'a', and the fourth 'b'. A point \tilde{x}_0 is marked at the intersection of the second and third circles.} \end{array} \quad p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0 \right) \right) = \langle b^2, bab^{-1}, a^2, aba^{-1} \rangle$$

- A normal covering space

$$\tilde{X} = \begin{array}{c} \text{Diagram: A horizontal dashed line with three circles placed on top of it. The first circle is labeled 'a', the second 'a', and the third 'a'. A point \tilde{x}_0 is marked at the intersection of the first circle and the dashed line. The circles are separated by a distance 'b' between their centers.} \end{array} \quad p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0 \right) \right) = \langle b^n ab^{-n} \mid n \in \mathbb{Z} \rangle$$

The universal cover is a tree.

Example. Let $T = S^1 \times S^1$, so $\pi_1(T) = \mathbb{Z}^2$. This is abelian, so all covering spaces are normal. The universal cover is

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & S^1 \times S^1 \\ (s, t) & \longmapsto & (e^{2\pi i s}, e^{2\pi i t}) \end{array},$$

since \mathbb{R}^2 is simply connected. Check that it is a covering space.⁴ More generally, if $p : \tilde{X} \rightarrow X$ and $q : \tilde{Y} \rightarrow Y$ are covering spaces then

$$\begin{array}{ccc} \tilde{X} \times \tilde{Y} & \longrightarrow & X \times Y \\ (x, y) & \longmapsto & (p(x), q(y)) \end{array}$$

is again a covering space. For example,

$$\begin{array}{ccc} S^1 \times S^1 & \longrightarrow & S^1 \times S^1 \\ (z_1, z_2) & \longmapsto & (z_1^n, z_2^m) \end{array}.$$

Example. Worksheet 3 exercise 7. Let

$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim = S^n / \sim$$

be the **projective n -space**, the space of all lines through the origin in \mathbb{R}^{n+1} , where $x \sim -x$. Let $p : S^n \rightarrow \mathbb{RP}^n$ be the quotient map. Claim that this is a covering space. Let $[x] \in \mathbb{RP}^n$. Then $p^{-1}([x]) = \{\pm x\}$. Let U be an open neighbourhood of x such that $U \cap (-U) = \emptyset$, so $p(U) = \{[x] \mid x \in U\}$. Then $p^{-1}(p(U)) = U \cup (-U)$ is open and disjoint. Thus $p|_U : U \rightarrow p(U)$ is a homeomorphism, so it is a covering space.

- $n \geq 2$ implies that S^n is simply-connected, so $S^n \rightarrow \mathbb{RP}^n$ is a universal cover. Then

$$\{\text{id}\} = p_* \left(\pi_1(S^n) \right) \stackrel{2}{\subseteq} \pi_1(\mathbb{RP}^n),$$

so $|\pi_1(\mathbb{RP}^n)| = 2$. Thus $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$.

- $n = 1$ implies that $\mathbb{RP}^1 = S^1$, so

$$\begin{array}{ccc} p : S^1 & \longrightarrow & S^1 \\ z & \longmapsto & z^2 \end{array}$$

is a covering space.

⁴Exercise

2 Homology

Higher homotopy groups $\pi_n(X, x_0)$ are groups of basepoint preserving homotopies of continuous $\phi : I^n \rightarrow X$ such that $\phi(\partial I^n) = x_0$.

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Example.

$$\pi_1(S^n) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}, \quad \pi_2(S^n) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases},$$

$$\pi_3(S^n) = \begin{cases} \mathbb{Z} & n = 2, 3 \\ 0 & \text{otherwise} \end{cases}, \quad \pi_i(S^2) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 4, 5 \\ \mathbb{Z}/12\mathbb{Z} & i = 6 \end{cases}.$$

Homology is more suitable. The following is the plan.

- Simplicial homology.
- Singular homology.
- Technical machinery to show that they coincide.
- Applications.

2.1 Δ -complexes

Definition. Let $m, n \geq 0$.

- An **n -simplex** in \mathbb{R}^m is the convex hull of a set V of $n + 1$ points in \mathbb{R}^m that are not all contained in an affine $(n - 1)$ -dimensional subspace of \mathbb{R}^m .
- The **standard n -simplex** is the convex hull of the standard basis $\{e_1, \dots, e_{n+1}\}$ in \mathbb{R}^{n+1} ,

$$\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, x_0 + \dots + x_n = 1\}.$$

- An **ordered n -simplex** is an n -simplex with an ordering on the vertices. We denote it by $[v_0, \dots, v_n]$, where v_0, \dots, v_n are the vertices in ascending order.
- The **standard ordered n -simplex** is the ordered n -simplex

$$[e_1, \dots, e_{n+1}]$$

in \mathbb{R}^{n+1} . It is denoted by Δ^n .

- Let $[v_0, \dots, v_{n+1}]$ be an n -simplex in \mathbb{R}^m and let $L \subseteq \mathbb{R}^m$ be the affine subspace spanned by v_0, \dots, v_n . Then there exists a unique affine morphism

$$\begin{array}{ccc} L & \longrightarrow & \mathbb{R}^{n+1} \\ v_i & \longmapsto & e_{i+1} \end{array}, \quad i = 0, \dots, n.$$

This gives a homeomorphism from $[v_0, \dots, v_n]$ to Δ^n that preserves this ordering.

- For $n \geq 1$, the **faces** of an ordered n -simplex $[v_0, \dots, v_n]$ are the ordered $(n - 1)$ -simplices

$$[v_0, \dots, \widehat{v_i}, \dots, v_n].$$

$\widehat{v_i}$ means we omit the vertex v_i .

- The union of all the faces of a simplex Δ is the **boundary** $\partial\Delta$.
- The **interior** of Δ is $\mathring{\Delta} = \Delta \setminus \partial\Delta$.

Example. Let $\Delta^2 = [e_1, e_2, e_3]$. Then $\partial\Delta^2 = [e_1, e_2] \cup [e_1, e_3] \cup [e_2, e_3]$.

Definition. Let X be a topological space. A Δ -complex structure on X is a collection of continuous maps

$$\sigma_\alpha : \Delta^{n(\alpha)} \rightarrow X, \quad \alpha \in A, \quad n(\alpha) \in \mathbb{N},$$

such that

1. the restriction $\sigma_\alpha|_{\dot{\Delta}^{n(\alpha)}}$ is injective for all $\alpha \in A$ and for each $x \in X$ there is a unique $\alpha \in A$ such that $x \in \sigma_\alpha(\dot{\Delta}^{n(\alpha)})$,
2. the restriction of σ_α to a face of $\Delta^{n(\alpha)}$ is equal to σ_β for some $\beta \in A$ and $n(\beta) = n(\alpha) - 1$, and
3. $U \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(U)$ is open in $\Delta^{n(\alpha)}$ for all $\alpha \in A$.

An observation is that $\sigma : \bigsqcup_{\alpha \in A} \Delta^{n(\alpha)} \rightarrow X$ induced by the σ_α is a quotient map, since it is surjective by 1 and $U \subseteq X$ is open if and only if $\sigma^{-1}(U)$ is open by 3.

Remark. One can show that an X with a Δ -complex structure is a CW-complex.

Example.

- Torus or Klein bottle is two Δ^2 , three Δ^1 , and one Δ^0 .
- S^2 is a tetrahedron.
- **Dunce hat**, by identifying all the three faces of the standard 2-simplex with each other, is one Δ^2 , one Δ^1 , and one Δ^0 .

2.2 Simplicial homology

2.2.1 Simplicial homology

Let X be a Δ -complex. The group of n -chains $\Delta_n(X)$ is the free abelian group on the n -simplices $\sigma_\alpha : \Delta^{n(\alpha)} \rightarrow X$, where $n(\alpha) = n$. So an element in $\Delta_n(X)$ is of the form

$$\sum_{\alpha \in A, n(\alpha)=n} c_\alpha \cdot \sigma_\alpha, \quad c_\alpha \in \mathbb{Z},$$

where all but finitely many of the c_α are zero.

Example. Let K be a Klein bottle.

- $\Delta_0(K) = \{n \cdot v \mid n \in \mathbb{Z}\} = \mathbb{Z} \cdot v \cong \mathbb{Z}$.
- $\Delta_1(K) = \{n_1 \cdot a + n_2 \cdot b + n_3 \cdot c \mid n_1, n_2, n_3 \in \mathbb{Z}\} = \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot b \oplus \mathbb{Z} \cdot c \cong \mathbb{Z}^3$.
- $\Delta_2(K) = \{n_1 \cdot U + n_2 \cdot V \mid n_1, n_2 \in \mathbb{Z}\} = \mathbb{Z} \cdot U \oplus \mathbb{Z} \cdot V \cong \mathbb{Z}^2$.
- $\Delta_n(K) = 0$ for $n \geq 3$.

Similarly for a torus T .

Define the **boundary homomorphism** by

$$\begin{aligned} \partial_n : \Delta_n(X) &\longrightarrow \Delta_{n-1}(X) \\ \sigma_\alpha &\longmapsto \sum_{i=0}^n (-1)^i \sigma_\alpha|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]} \end{aligned}$$

Moreover, we define $\partial_0 = 0$.

Example. Let $\sigma : [v_0, v_1, v_2, v_3] \rightarrow X$. Then

$$\partial_3(\sigma) = \sigma|_{[v_1, v_2, v_3]} - \sigma|_{[v_0, v_2, v_3]} + \sigma|_{[v_0, v_1, v_3]} - \sigma|_{[v_0, v_1, v_2]}.$$

Lemma 2.1. *The composition*

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

is the zero map.

Proof. Let $\sigma : [v_0, \dots, v_n] \rightarrow X$ be an n -simplex. Then

$$\partial_n(\sigma) = \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]},$$

so

$$(\partial_{n-1} \circ \partial_n)(\sigma) = \sum_{j < i} (-1)^i (-1)^j \sigma|_{[v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_n]} + \sum_{j > i} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_n]} = 0.$$

If $n = 1$, clear. □

2.2.2 Algebraic situation

A **chain complex** of abelian groups is a diagram (C_\bullet, ∂) of the form

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0,$$

where the C_i are abelian groups and the ∂_n are group homomorphisms such that $\partial_n \circ \partial_{n-1} = 0$ for all n . Then ∂_n are **boundary homomorphisms**. The elements in C_n are **n -chains**. Let

$$Z_n = \text{Ker } \partial_n \subseteq C_n, \quad B_n = \text{Im } \partial_{n+1} \subseteq C_n.$$

The elements in Z_n are **cycles** and the elements in B_n are **boundaries**. Since $\partial_{n+1} \circ \partial_n = 0$, we have that $B_n \subseteq Z_n$. The **n -th homology group** of this chain complex is defined by

$$H_n(C_\bullet, \partial) = Z_n / B_n.$$

So, by Lemma 2.1

$$\dots \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0$$

is a chain complex. The **n -th simplicial homology group** is

$$H_n^\Delta(X) = \text{Ker } \partial_n / \text{Im } \partial_{n+1}.$$

Example. Let $X = S^1$. Then

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_3} & \Delta_2(X) & \xrightarrow{\partial_2} & \Delta_1(X) & \xrightarrow{\partial_1} & \Delta_0(X) \xrightarrow{\partial_0} 0 \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ & & 0 & & \mathbb{Z} & & \mathbb{Z} \end{array}.$$

- $\text{Ker } \partial_0 = \mathbb{Z}$ and $\text{Im } \partial_1 = 0$, so $H_0^\Delta(X) \cong \mathbb{Z}$.
- $\text{Ker } \partial_1 = \Delta_1(X)$ and $\text{Im } \partial_2 = 0$, so $H_1^\Delta(X) \cong \mathbb{Z}$.
- $H_n^\Delta(X) = 0$ if $n \geq 2$.

Example. Let T be a torus. Then

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_4} & \Delta_3(T) & \xrightarrow{\partial_3} & \Delta_2(T) & \xrightarrow{\partial_2} & \Delta_1(T) \xrightarrow{\partial_1} \Delta_0(T) \xrightarrow{\partial_0} 0 \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ & & 0 & & \mathbb{Z} \cdot U \oplus \mathbb{Z} \cdot V & & \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot b \oplus \mathbb{Z} \cdot c \\ & & & & & & \mathbb{Z} \cdot v \end{array}.$$

- $\text{Ker } \partial_0 = \mathbb{Z}$ and $\text{Im } \partial_1 = 0$, so $H_0^\Delta(T) \cong \mathbb{Z}$.
- $\partial_2(U) = a + b - c$ and $\partial_2(V) = a + b - c$, and $\{a, b, a + b - c\}$ is a basis for $\Delta_1(T)$.

$$\text{Ker } \partial_1 = \Delta_1(T), \quad \text{Im } \partial_2 = \mathbb{Z} \cdot (a + b - c),$$

$$\text{so } H_1^\Delta(T) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

- $H_2^\Delta(T) \cong \mathbb{Z}$.⁵

Lecture 20 is a problems class.

⁵Exercise

2.3 Singular homology

2.3.1 Singular homology

A **singular n -simplex** in a topological space X is a continuous map $\sigma : \Delta^n \rightarrow X$. Let $C_n(X)$ be the free abelian group on the set of all singular simplices in X , that is the elements in $C_n(X)$ are finite formal sums

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$$\sum_i n_i \sigma_i, \quad n_i \in \mathbb{Z},$$

where $\sigma_i : \Delta^n \rightarrow X$ are singular n -simplices. The elements in $C_n(X)$ are called **singular n -chains**. Define a **boundary map**

$$\begin{aligned} \partial_n : C_n(X) &\longrightarrow C_{n-1}(X) \\ \sigma &\longmapsto \sum_{i=0}^n (-1)^i \sigma|_{[v_1, \dots, \tilde{v}_i, \dots, v_n]} \end{aligned}$$

for a singular n -simplex σ . Extend it linearly to $C_n(X)$.

Lemma 2.2. $\partial_n \circ \partial_{n+1} = 0$.

Proof. The same proof as for Lemma 2.1. □

We obtain a chain complex

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

Remark. Often we write ∂ instead of ∂_n .

We define the **n -th singular homology group** by

$$H_n(X) = \text{Ker } \partial_n / \text{Im } \partial_{n+1}.$$

An observation is that if X and Y are homeomorphic then $H_n(X) \cong H_n(Y)$.

Proposition 2.3. Let X be a topological space and $X = \bigcup_{\alpha} X_{\alpha}$ be the decomposition into its path-components. Then

$$H_n(X) \cong \bigoplus_{\alpha} H_n(X_{\alpha}).$$

Proof. A singular n -simplex $\sigma : \Delta^n \rightarrow X$ has a path-connected image. So

$$C_n(X) = \bigoplus_{\alpha} C_n(X_{\alpha}).$$

The boundary maps ∂_n preserve this decomposition, so $\partial_n(C_n(X_{\alpha})) \subseteq C_{n-1}(X_{\alpha})$ implies that $\text{Ker } \partial_n$ and $\text{Im } \partial_{n+1}$ split as well as direct sums, so

$$H_n(X) = \text{Ker } \partial_n / \text{Im } \partial_{n+1} \cong \bigoplus_{\alpha} H_n(X_{\alpha}).$$

□

Proposition 2.4. If X is a path-connected, and as always $X \neq \emptyset$, topological space, then

$$H_0(X) \cong \mathbb{Z}.$$

Hence for X arbitrary $H_0(X)$ is a direct sum of \mathbb{Z} 's, one for each path-component.

Proof. $\partial_0 = 0$, so $H_0(X) = C_0(X) / \text{Im } \partial_1$. Define

$$\begin{aligned} \epsilon : C_0(X) &\longrightarrow \mathbb{Z} \\ \sum_i n_i \sigma_i &\longmapsto \sum_i n_i \end{aligned}$$

Then ϵ is surjective. Enough to show that $\text{Ker } \epsilon = \text{Im } \partial_1$. This implies by the isomorphism theorem $H_0(X) \cong \mathbb{Z}$. Let $\sigma : \Delta^1 \rightarrow X$ be a 1-simplex. Then

$$\partial_1(\sigma) = \sigma|_{[v_1]} - \sigma|_{[v_0]},$$

so $\epsilon(\partial_1(\sigma)) = 0$, so $\text{Im } \partial_1 \subseteq \text{Ker } \epsilon$. On the other hand, $\epsilon(\sum_i n_i \sigma_i) = 0$ implies that $\sum_i n_i = 0$. The σ_i correspond to points $\sigma_i([v])$ in X . Choose a basepoint $x_0 \in X$ and let

$$\begin{array}{ccc} \sigma_0 & : & \Delta^0 \longrightarrow X \\ & & \Delta^0 \longmapsto x_0 \end{array}$$

be the singular 0-simplex. Let τ_i be a path from x_0 to $\sigma_i([v])$. Consider τ_i as a singular 1-simplex $\tau_i : [v_0, v_1] \rightarrow X$. We have $\partial_1 \circ \tau_i = \sigma_i - \sigma_0$, so

$$\partial_1 \left(\sum_i n_i \tau_i \right) = \sum_i n_i (\sigma_i - \sigma_0) = \sum_i n_i \sigma_i - \sum_i n_i \sigma_0 = \sum_i n_i \sigma_i.$$

Thus $\text{Ker } \epsilon \subseteq \text{Im } \partial_1$. □

Proposition 2.5. *If X is a point, then*

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}.$$

Proof. For each n there exists a unique singular n -simplex $\partial_n : \Delta^n \rightarrow X$, so $C_n(X) \cong \mathbb{Z}$ for all n . Then

$$\partial_n(\sigma_n) = \sum_{i=0}^n (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases},$$

so $\partial_n = 0$ if n is odd and ∂_n is an isomorphism if n is even, and

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_2} & C_1(X) & \xrightarrow{\partial_1} & C_0(X) & \xrightarrow{\partial_0} & 0 \\ & & \parallel & & \parallel & & \\ \dots & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} & \xrightarrow{0} & 0 \end{array},$$

so $H_n = \text{Ker } \partial_n / \text{Im } \partial_{n+1} = 0$ if $n \geq 1$ and $H_0(X) \cong \mathbb{Z}$. □

2.3.2 Reduced homology groups

The **reduced homology groups** $\widetilde{H}_n(X)$ are the homology groups of the **augmented chain complex**

$$\dots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} \mathbb{Z} \xrightarrow{\epsilon} 0,$$

where ϵ is as in proof of Proposition 2.4. Then

$$H_n(X) \cong \widetilde{H}_n(X), \quad n \geq 1.$$

Seen in the proof of Proposition 2.4 that ϵ is surjective and $\epsilon \circ \partial_1 = 0$, so $\text{Im } \partial_1 \subseteq \text{Ker } \epsilon$, so ϵ induces a surjective homomorphism

$$\phi_\epsilon : H_0(X) = C_0(X) / \text{Im } \partial_1 \rightarrow \mathbb{Z}.$$

Then $\text{Ker } \phi_\epsilon = \text{Ker } \epsilon / \text{Im } \partial_1 = \widetilde{H}_0(X)$, so $H_0(X) / \widetilde{H}_0(X) \cong \mathbb{Z}$, so

$$H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}.$$

2.4 Homotopy invariance

Let (A_\bullet, ∂) and (B_\bullet, ∂) be two chain complexes. A **chain map** $f : (A_\bullet, \partial) \rightarrow (B_\bullet, \partial)$ is a collection of homomorphisms $f_n : A_n \rightarrow B_n$ such that $\partial \circ f_n = f_{n+1} \circ \partial$, that is the following diagram commutes.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial} & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \xrightarrow{\partial} & \dots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ \dots & \xrightarrow{\partial} & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \xrightarrow{\partial} & \dots \end{array}.$$

If X and Y are topological spaces and $f : X \rightarrow Y$ is a continuous map define the homomorphisms

$$f_\# : \begin{array}{ccc} C_n(X) & \longrightarrow & C_n(Y) \\ \sigma : \Delta^n \rightarrow X & \longmapsto & f \circ \sigma : \Delta^n \rightarrow Y \end{array},$$

and extend it linearly to $C_n(X)$. Then

$$(f_\# \circ \partial)(\sigma) = f_\# \left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v}_i, \dots, v_n]} \right) = \sum_{i=0}^n (f \circ \sigma)|_{[v_0, \dots, \widehat{v}_i, \dots, v_n]} = (\partial \circ f_\#)(\sigma),$$

so $f_\# \circ \partial = \partial \circ f_\#$, so $f_\#$ defines a chain map

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial} & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \dots \\ & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & & \\ \dots & \xrightarrow{\partial} & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \xrightarrow{\partial} & \dots \end{array}.$$

$f_\#$ maps cycles to cycles, since $\alpha \in C_n(X)$ such that $\partial \circ \alpha = 0$, so

$$(\partial \circ f_\#)(\alpha) = (f_\# \circ \partial)(\alpha) = 0.$$

$f_\#$ maps boundaries to boundaries, since

$$f_\# \circ (\partial \circ \beta) = \partial \circ (f_\# \circ \beta).$$

$f_\#(\text{Ker } \partial_n) \subseteq \text{Ker } \partial_n$ and $f_\#(\text{Im } \partial_{n+1}) \subseteq \text{Im } \partial_{n+1}$, so $f_\#$ induces a homomorphism

$$f_* : H_n(X) \rightarrow H_n(Y).$$

The following are observations.

- $X \xrightarrow{g} Y \xrightarrow{f} Z$, so $(f \circ g)_\# = f_\# \circ g_\#$, since

$$\Delta^n \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z,$$

so $f \circ (g \circ \sigma) = (f \circ g) \circ \sigma$, so $(f \circ g)_* = f_* \circ g_*$.

- $(\text{id}_X)_* = \text{id}_{H_n(X)}$.

Theorem 2.6. *If two continuous maps $f, g : X \rightarrow Y$ are homotopic, then*

$$f_* = g_* : H_n(X) \rightarrow H_n(Y).$$

Corollary 2.7. *If $f : X \rightarrow Y$ is a homotopy equivalence, then*

$$f_* : H_n(X) \rightarrow H_n(Y)$$

is an isomorphism.

Proof. Let $g : Y \rightarrow X$ be a continuous map such that $f \circ g \cong \text{id}_Y$ and $g \circ f = \text{id}_X$. Then $f_* \circ g_* = (f \circ g)_* = (\text{id}_Y)_* = \text{id}$. Similarly $g_* \circ f_* = \text{id}$, so f_* is an isomorphism. \square

Example.

$$H_n(\mathbb{R}^k) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}, \quad \widetilde{H}_n(\mathbb{R}^k) = 0.$$

Proof of Theorem 2.6. Let $F : X \times I \rightarrow Y$ be a homotopy from f to g and $\sigma : \Delta_n \rightarrow X$ be a singular n -simplex. Consider the map

$$\Delta^n \times I \xrightarrow{\sigma \times I} X \times I \xrightarrow{F} Y.$$

Then $\Delta^n \times I$ is not a simplex. But we can subdivide $\Delta^n \times I$ into $(n+1)$ simplices. In general, we can decompose $\Delta^n \times I$ into $n+1$ $(n+1)$ -simplices

$$[v_0, \dots, v_i, w_i, \dots, w_n], \quad i = 0, \dots, n.$$

Define **prism-operators**

$$\begin{aligned} P : C_n(X) &\longrightarrow C_{n+1}(Y) \\ \sigma &\longmapsto \sum_{i=0}^n (-1)^i F \circ (\sigma \times \text{id})|_{[v_0, \dots, v_i, w_i, \dots, w_n]}, \end{aligned}$$

for $\sigma : \Delta^n \rightarrow X$ a singular n -simplex, so

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial} & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) \xrightarrow{\partial} \dots \\ & & \swarrow P & & \downarrow g_\# & & \swarrow P \\ \dots & \xrightarrow{\partial} & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) \xrightarrow{\partial} \dots \end{array}$$

Claim that

$$\partial \circ P = g_\# - f_\# - P \circ \partial,$$

if and only if $g_\# - f_\# = \partial \circ P + P \circ \partial$. The claim implies the theorem, since if $\alpha \in C_n(X)$ is a cycle, then

$$g_\#(\alpha) - f_\#(\alpha) = (\partial \circ P)(\alpha) + (P \circ \partial)(\alpha) = (\partial \circ P)(\alpha),$$

so $g_\#(\alpha) - f_\#(\alpha)$ is a boundary. Thus $g_\#(\alpha)$ and $f_\#(\alpha)$ are in the same homology class, so $g_*([\alpha]) = f_*([\alpha])$, where $[\alpha]$ is the homology class of α . Let $\sigma : \Delta^n \rightarrow X$ be a singular n -simplex. Then

$$\begin{aligned} (\partial \circ P)(\sigma) &= \partial \left(\sum_{i=0}^n (-1)^i F \circ (\sigma \times \text{id})|_{[v_0, \dots, v_i, w_i, \dots, w_n]} \right) \\ &= \sum_{j \leq i} (-1)^i (-1)^j F \circ (\sigma \times \text{id})|_{[v_0, \dots, \widehat{v_j}, \dots, v_i, w_i, \dots, w_n]} \\ &\quad + \sum_{j \geq i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \text{id})|_{[v_0, \dots, v_i, w_i, \dots, \widehat{w_j}, \dots, w_n]}. \end{aligned}$$

If $i = j$ the two sums cancel except for

$$F \circ (\sigma \times \text{id})|_{[\widehat{v_0}, w_0, \dots, w_n]} = g \circ \sigma = g_\#(\sigma), \quad -F \circ (\sigma \times \text{id})|_{[v_0, \dots, v_n, \widehat{w_n}]} = -f \circ \sigma = -f_\#(\sigma).$$

The terms with $i \neq j$ sum up to $(P \circ \partial)(\sigma)$, since we have

$$\begin{aligned} (P \circ \partial)(\sigma) &= \sum_{j < i} (-1)^i (-1)^j F \circ (\sigma \times \text{id})|_{[v_0, \dots, \widehat{v_j}, \dots, v_i, w_i, \dots, w_n]} \\ &\quad + \sum_{j > i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \text{id})|_{[v_0, \dots, v_i, w_i, \dots, \widehat{w_j}, \dots, w_n]}. \end{aligned}$$

□

Remark. One can show that there are also induced homomorphisms

$$f_* : \widetilde{H}_n(X) \rightarrow \widetilde{H}_n(Y)$$

invariant under homotopy. ⁶

⁶Exercise

2.5 Exact sequences and excision

2.5.1 Exact sequences

Let $A \subseteq X$ be a subspace. What is the relationship between $H_n(A), H_n(X), H_n(X/A)$?

Definition. A sequence of group homomorphisms of abelian groups

$$\dots \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} \dots$$

is **exact** at A_n if $\text{Ker } \alpha_n = \text{Im } \alpha_{n+1}$. The sequence is **exact** if it is exact at A_n for all n .

An observation is if the sequence is exact, then

- $\alpha_n \alpha_{n+1} = 0$, so exact sequences are chain complexes, and
- the homology groups of this chain complex are all trivial.

Example.

- $0 \rightarrow A \xrightarrow{\alpha} B$ is exact if and only if $\text{Ker } \alpha = 0$, if and only if α is injective.
- $A \xrightarrow{\alpha} B \rightarrow 0$ is exact if and only if $\text{Im } \alpha = B$, if and only if α is surjective.
- $0 \rightarrow A \xrightarrow{\alpha} B \rightarrow 0$ is exact if and only if α is an isomorphism.
- $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ is exact if and only if α is injective, β is surjective, and $\text{Ker } \beta = \text{Im } \alpha$, hence β induces an isomorphism

$$C \cong B / \text{Im } \alpha = B/A.$$

This is called a **short exact sequence**.

Definition. Let X be a topological space and $A \subseteq X$. Then A is a **strong deformation retract** of X if there exists a retraction $r : X \rightarrow A$ such that r is homotopic to the identity, and $F : I \times X \rightarrow X$ continuous such that

$$F(0, x) = x, \quad F(1, x) = r(x), \quad F(t, a) = a, \quad x \in X, \quad a \in A, \quad t \in I.$$

Let X be a topological space and $A \subseteq X$ a non-empty closed subspace. Then (X, A) is called a **good pair** if A has a neighbourhood in X that strongly deformation retracts to A .

Example.

- (D^n, S^{n-1}) is a good pair, since S^{n-1} is a deformation retract of $D^n \setminus \{0\}$.
- Let $A = \{1/n \mid n \in \mathbb{N}\} \cup \{0\} \subseteq [0, 1]$ then $([0, 1], A)$ is not a good pair.

Theorem 2.8. Let (X, A) be a good pair, then there is an exact sequence

$$\dots \rightarrow \widetilde{H}_1(A) \xrightarrow{i_*} \widetilde{H}_1(X) \xrightarrow{j_*} \widetilde{H}_1(X/A) \xrightarrow{\partial} \widetilde{H}_0(A) \xrightarrow{i_*} \widetilde{H}_0(X) \xrightarrow{j_*} \widetilde{H}_0(X/A) \rightarrow 0,$$

where $i : A \hookrightarrow X$ is the inclusion and $j : X \rightarrow X/A$ is the quotient.

Corollary 2.9.

$$\widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}.$$

Proof. (D^n, S^{n-1}) is a good pair. Let $n > 0$. Recall that $D^n/S^{n-1} \cong S^n$, so

$$\begin{array}{ccccccccccc} \dots & \rightarrow & \widetilde{H}_i(S^{n-1}) & \xrightarrow{i_*} & \widetilde{H}_i(D^n) & \xrightarrow{j_*} & \widetilde{H}_i(S^n) & \xrightarrow{\partial} & \widetilde{H}_{i-1}(S^{n-1}) & \xrightarrow{i_*} & \widetilde{H}_{i-1}(D^n) & \xrightarrow{j_*} & \widetilde{H}_{i-1}(S^n) & \rightarrow & \dots \\ & & & & \cong & & & & & & \cong & & & & \\ & & & & 0 & & & & & & 0 & & & & \end{array}$$

Then $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1})$ for $i > 0$, so

$$\begin{array}{ccccccccccc} \dots & \rightarrow & \widetilde{H}_1(S^{n-1}) & \xrightarrow{i_*} & \widetilde{H}_1(D^n) & \xrightarrow{j_*} & \widetilde{H}_1(S^n) & \xrightarrow{\partial} & \widetilde{H}_0(S^{n-1}) & \xrightarrow{i_*} & \widetilde{H}_0(D^n) & \xrightarrow{j_*} & \widetilde{H}_0(S^n) & \rightarrow & 0 \\ & & & & \cong & & & & & & \cong & & & & \\ & & & & 0 & & & & & & 0 & & & & \end{array}$$

$n > 0$ and $i > 0$, so $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1})$, and $\widetilde{H}_0(S^n) = 0$. We know that $\widetilde{H}_0(S^0) \cong \mathbb{Z}$ and $\widetilde{H}_n(S^0) = 0$, by Proposition 2.3 and Proposition 2.5. Doing induction on n ,

$$\widetilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}.$$

□

Corollary 2.10. *There exists no retraction $r : D^n \rightarrow \partial D^n$.*

Proof. Assume there exists such an $r : D^n \rightarrow \partial D^n$. Let $i : \partial D^n \rightarrow D^n$. Then $ri = \text{id}_{\partial D^n}$, so $r_*i_* = (ri)_* = \text{id}_*$, so

$$\begin{array}{ccccc} \widetilde{H}_{n-1}(\partial D^n) & \xrightarrow{i_*} & \widetilde{H}_{n-1}(D^n) & \xrightarrow{r_*} & \widetilde{H}_{n-1}(\partial D^n) \\ \cong & & \cong & & \cong \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array}$$

Thus $i_* = 0$ and $r_* = 0$, a contradiction. □

Theorem 2.11 (Brouwer fixed point theorem). *Every continuous map $f : D^n \rightarrow D^n$ has a fixed point.*

Proof. Assume there exists a fixed point then construct as in dimension two a retraction $D^n \rightarrow \partial D^n$, a contradiction to Corollary 2.10. □

2.5.2 Relative homology groups

Let X be a topological space and $A \subseteq X$ be a subspace. Define

$$C_n(X, A) = C_n(X) / C_n(A).$$

Let $\partial : C_n(X) \rightarrow C_{n-1}(X)$ be the boundary map then $\partial(\sigma : \Delta^n \rightarrow A) \in \partial(C_n(A)) \subseteq C_{n-1}(A)$. So ∂ induces a homomorphism

$$\partial : C_n(X, A) \rightarrow C_{n-1}(X, A),$$

such that $\partial \circ \partial = 0$. This gives a chain complex

$$\dots \rightarrow C_{n+1}(X, A) \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \rightarrow \dots$$

- The homology groups $H_n(X, A)$ of this complex are the **relative homology groups**.
- The **relative n -chains** are $C_n(X, A)$.
- The **relative n -cycles** are $\text{Ker } \partial \subseteq C_n(X, A)$, of the form $[\alpha]$ for $\alpha \in C_n(X)$ such that $\partial(\alpha) \in C_{n-1}(A)$.
- The **relative n -boundaries** are $\text{Im } \partial \subseteq C_n(X, A)$, of the form $[\alpha]$ for $\alpha \in C_n(X)$ such that $\alpha = \partial\beta + \gamma$ for $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

A **short exact sequence of chain complexes** is

$$0 \rightarrow (A_\bullet, \partial) \xrightarrow{i} (B_\bullet, \partial) \xrightarrow{j} (C_\bullet, \partial) \rightarrow 0,$$

for i and j chain maps, where

$$0 \rightarrow A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \rightarrow 0$$

is a short exact sequence for all n , so

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{\partial} & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} \xrightarrow{\partial} \dots \\ & & \downarrow i & & \downarrow i & & \downarrow i \\ \dots & \xrightarrow{\partial} & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} \xrightarrow{\partial} \dots \\ & & \downarrow j & & \downarrow j & & \downarrow j \\ \dots & \xrightarrow{\partial} & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} \xrightarrow{\partial} \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

A short exact sequence of chain complexes always yields a **long exact sequence** of homology groups

$$\dots \rightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \rightarrow \dots$$

This is the **zig-zag lemma**. First we construct the **connecting map** $\partial : H_n(C) \rightarrow H_{n-1}(A)$. Let $c \in C_n$ be a cycle.

- j is surjective, so $c = j(b)$ for some $b \in B_n$.
- $j(\partial(b)) = \partial(j(b)) = \partial c = 0$, so $\partial b \in \text{Ker } j \subseteq B_{n-1}$, so $\partial(b) = i(a)$ for some $a \in A_{n-1}$, by exactness.
- $\partial(a) = 0$, since $i(\partial(a)) = \partial(i(a)) = \partial(\partial(b)) = 0$ and i is injective, so $\partial(a) = 0$.

$$\begin{array}{c} a \in A_{n-1} \\ \downarrow i \\ b \in B_n \xrightarrow{\partial} \partial(b) \in B_{n-1} \\ \downarrow j \\ c \in C_n \end{array}$$

Define

$$\partial : H_n(C) \longrightarrow H_{n-1}(A) \\ [c] \longmapsto [a]$$

This is well-defined.

- a is uniquely determined by $\partial(b)$ because i is injective.
- If we choose b' instead of b , then $j(b') = j(b)$, so $j(b' - b) = j(b') - j(b) = 0$, so $b' - b \in \text{Ker } j = \text{Im } i$, hence $b' - b = i(a')$ for some $a' \in A_n$, so $b' = b + i(a')$. If we replace b by $b' = b + i(a')$ this corresponds to replacing a by $a + \partial(a')$, because

$$i(a + \partial(a')) = i(a) + i(\partial(a')) = \partial(b) + \partial(i(a')) = \partial(b + i(a')),$$

$$\text{and } [a] = [a + \partial(a')].$$

- A different choice of c in its homology class has the form $c + \partial(c')$ for some $c' \in C_{n+1}$. Let $b' \in B_{n+1}$ such that $j(b') = c'$. Then

$$c + \partial(c') = c + \partial(j(b')) = j(b) + j(\partial(b')) = j(b + \partial(b')),$$

so b is replaced by $b + \partial(b')$ but $\partial(b) = \partial(b + \partial(b'))$, so $\partial(b)$ is unchanged and hence a is unchanged.

The map $\partial : H_n(C) \rightarrow H_{n-1}(A)$ is a homomorphism, since if $\partial([c_1]) = [a_1]$ and $\partial([c_2]) = [a_2]$ via elements b_1 and b_2 in B_n , then

$$j(b_1 + b_2) = j(b_1) + j(b_2) = c_1 + c_2, \quad i(a_1 + a_2) = i(a_1) + i(a_2) = \partial(b_1) + \partial(b_2) = \partial(b_1 + b_2),$$

so $\partial([c_1] + [c_2]) = [a_1] + [a_2]$.

Theorem 2.12. *The sequence*

$$\cdots \rightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \rightarrow \cdots$$

is exact.

Proof. Diagram chase, see Hatcher. □

Let i be the inclusion and j be the quotient.

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ \cdots & \xrightarrow{\partial} & C_n(A) & \xrightarrow{\partial} & C_{n-1}(A) & \xrightarrow{\partial} & \cdots \\ & & \downarrow i & & \downarrow i & & \\ \cdots & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \cdots \\ & & \downarrow j & & \downarrow j & & \\ \cdots & \xrightarrow{\partial} & C_n(X, A) & \xrightarrow{\partial} & C_{n-1}(X, A) & \xrightarrow{\partial} & \cdots \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

This diagram commutes, so this is a short exact sequence of chain complexes. Zig-zag gives a long exact sequence of homology groups

$$\cdots \rightarrow H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \rightarrow 0.$$

What is $\partial : H_n(X, A) \rightarrow H_{n-1}(A)$? If $[a] \in H_n(X, A)$ is represented by a cycle $\alpha \in C_n(X)$, then $\partial([a])$ is the class of the cycle $\partial(\alpha)$, so $\partial([a]) = [\partial(\alpha)]$. We also obtain a short exact sequence of the augmented chain complex

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & C_1(A) & \longrightarrow & C_0(A) & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & C_1(X) & \longrightarrow & C_0(X) & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & C_1(X, A) & \longrightarrow & C_0(X, A) & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

so if $A \neq \emptyset$, then $\widetilde{H}_n(X, A) = H_n(X, A)$ for all n . We also have a long exact sequence

$$\cdots \rightarrow \widetilde{H}_n(A) \rightarrow \widetilde{H}_n(X) \rightarrow \widetilde{H}_n(X, A) \rightarrow \widetilde{H}_{n-1}(A) \rightarrow \widetilde{H}_{n-1}(X) \rightarrow \widetilde{H}_{n-1}(X, A) \rightarrow \cdots$$

An observation is if $x_0 \in X$ then $H_n(X, x_0) \cong \widetilde{H}_n(X)$ for all n . Another observation is that a continuous map $f : X \rightarrow Y$ such that $f(A) \subseteq B$ induces a chain map

$$f_{\#} : C_n(X, A) \rightarrow C_n(Y, B),$$

since $f_{\#} : C_n(X) \rightarrow C_n(Y)$ maps $C_n(A)$ to $C_n(B)$ so it is well-defined on the quotient, and hence homomorphisms

$$f_* : H_n(X, A) \rightarrow H_n(Y, B).$$

This is functorial, so $(f \circ g)_* = f_* \circ g_*$.

Definition. A **homotopy** between two maps

$$f, g : (X, A) \rightarrow (Y, B)$$

is a continuous map $F : I \times X \rightarrow Y$ such that

$$F(0, x) = f(x), \quad F(1, x) = g(x), \quad F(s, a) \in B, \quad x \in X, \quad s \in I, \quad a \in A.$$

Proposition 2.13. *If*

$$f, g : (X, A) \rightarrow (Y, B)$$

are homotopic, then

$$f_* = g_* : H_n(X, A) \rightarrow H_n(Y, B).$$

Proof. Analogous to proof of Theorem 2.6. Prism operator $P : C_n(X) \rightarrow C_{n+1}(Y)$ maps $C_n(A)$ to $C_n(B)$ so it induces a map

$$P' : C_n(X)/C_n(A) \rightarrow C_{n+1}(Y)/C_{n+1}(B),$$

and $\partial P' + P' \partial = g_{\#} - f_{\#}$, so $f_* = g_*$. □

Let (X, A, B) be a triple for X a topological space and $B \subset A \subset X$, so

$$(A, B) \rightarrow (X, B) \rightarrow (X, A).$$

There is a short exact sequence of chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_n(A, B) & \longrightarrow & C_n(X, B) & \longrightarrow & C_n(X, A) \longrightarrow 0 \\ & & \downarrow \text{IR} & & \downarrow \text{IR} & & \downarrow \text{IR} \\ & & C_n(A)/C_n(B) & & C_n(X)/C_n(B) & & C_n(X)/C_n(A) \end{array},$$

so there is a long exact sequence

$$\cdots \rightarrow H_n(A, B) \rightarrow H_n(X, B) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A, B) \rightarrow H_{n-1}(X, B) \rightarrow H_{n-1}(X, A) \rightarrow \cdots$$

2.5.3 Excision

Theorem 2.14 (Excision). *Let X be a topological space and $Z \subset A \subset X$ be subspaces such that the closure \bar{Z} of Z is contained in the interior \mathring{A} of A . Then the inclusion*

$$(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$$

induces isomorphisms

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_n(X, A),$$

for all n . Equivalently, let $A, B \subseteq X$ such that $\mathring{A} \cup \mathring{B} = X$. Then the inclusion

$$(B, A \cap B) \hookrightarrow (X, A)$$

induces isomorphisms

$$H_n(B, A \cap B) \xrightarrow{\sim} H_n(X, A),$$

for all n .

Why equivalent? Set $B = X \setminus Z$ and $Z = X \setminus B$. Then $A \cap B = A \setminus Z$ and $\bar{Z} = X \setminus \mathring{B}$. Then $\bar{Z} \subseteq \mathring{A}$ if and only if $X = \mathring{A} \cup \mathring{B}$.

Proof. Hatcher page 119 to 124. □

Proposition 2.15. *Let (X, A) be a good pair. Then the quotient map*

$$q : (X, A) \rightarrow (X/A, A/A)$$

induces isomorphisms

$$q_* : H_n(X, A) \xrightarrow{\sim} H_n(X/A, A/A) \cong \widetilde{H}_n(X/A),$$

for all n .

Proof. Let $V \subseteq X$ be a neighbourhood of A that strongly deformation retracts to A . Then (V, A) is homotopy equivalent to (A, A) , so

$$H_n(V, A) \cong H_n(A, A) = 0.$$

The triple (X, V, A) where $A \subset V \subset X$ induces a long exact sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_n(V, A) & \longrightarrow & H_n(X, A) & \longrightarrow & H_n(X, V) \longrightarrow H_{n-1}(V, A) \longrightarrow \dots \\ & & \cong & & & & \cong \\ & & 0 & & & & 0 \end{array},$$

so

$$H_n(X, A) \cong H_n(X, V).$$

The same with the triple $(X/A, V/A, A/A)$, so again

$$H_n(V/A, A/A) \cong H_n(A/A, A/A) = 0.$$

This gives a long exact sequence

$$H_n(X/A, A/A) \cong H_n(X/A, V/A).$$

Consider the diagram

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{\sim} & H_n(X, V) & \xleftarrow{\sim} & H_n(X \setminus A, V \setminus A) \\ \downarrow q_* & & \downarrow q_* & & \sim \downarrow j \\ H_n(X/A, A/A) & \xrightarrow{\sim} & H_n(X/A, V/A) & \xleftarrow{\sim} & H_n(X/A \setminus A/A, V/A \setminus A/A) \end{array}.$$

- This diagram commutes.
- $q : X \rightarrow X/A$ induces a homeomorphism $X \setminus A \rightarrow X/A \setminus A/A$, so j is an isomorphism.
- α and β are isomorphisms by the excision theorem.

Thus

$$q_* : H_n(X, A) \rightarrow H_n(X/A, A/A)$$

is an isomorphism. □

Proof of Theorem 2.8. Long exact sequence of pair (X, A) with reduced homology

$$\dots \rightarrow \widetilde{H}_n(A) \rightarrow \widetilde{H}_n(X) \rightarrow \widetilde{H}_n(X, A) \rightarrow \widetilde{H}_{n-1}(A) \rightarrow \widetilde{H}_{n-1}(X) \rightarrow \widetilde{H}_{n-1}(X, A) \rightarrow \dots,$$

so

$$\widetilde{H}_n(X, A) = H_n(X, A) \cong \widetilde{H}_n(X/A),$$

by last time. □

Corollary 2.16. *Let $\{X_\alpha\}$ for $\alpha \in A$ be a collection of topological spaces and $x_\alpha \in X_\alpha$ such that (X_α, x_α) is a good pair, for all $\alpha \in A$. Let $\bigvee_\alpha X_\alpha$ be the wedge sum with respect to the points x_α . Then there is an isomorphism*

$$\widetilde{H}_n\left(\bigcup_\alpha X_\alpha\right) \cong \bigoplus_\alpha \widetilde{H}_n(X_\alpha) \xrightarrow{\sim} \widetilde{H}_n\left(\bigvee_\alpha X_\alpha\right).$$

Proof. $(X, A) = (\bigsqcup_{\alpha} X_{\alpha}, \bigsqcup_{\alpha} \{x_{\alpha}\})$ is a good pair, so Proposition 2.15 implies that

$$H_n(X, A) \cong H_n\left(\bigvee_{\alpha} X_{\alpha}, \bigsqcup_{\alpha} \{x_{\alpha}\} / \bigsqcup_{\alpha} \{x_{\alpha}\}\right) \cong \widetilde{H}_n\left(\bigvee_{\alpha} X_{\alpha}\right),$$

and

$$H_n(X, A) \cong \bigoplus_{\alpha} H_n(X_{\alpha}, x_{\alpha}) \cong \bigoplus_{\alpha} \widetilde{H}_n(X_{\alpha}).$$

□

Example.

$$\widetilde{H}_n(S^1 \vee S^1) \cong \widetilde{H}_n(S^1) \oplus \widetilde{H}_n(S^1) \cong \begin{cases} 0 & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & n \geq 2 \end{cases}.$$

$$\widetilde{H}_n(S^1 \vee S^1 \vee S^2) \cong \widetilde{H}_n(S^1) \oplus \widetilde{H}_n(S^1) \oplus \widetilde{H}_n(S^2) \cong \begin{cases} 0 & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}.$$

Recall that

$$H_n^{\Delta}(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}.$$

We will see that singular and simplicial homology coincide in Appendix A.2, so $S^1 \vee S^1 \vee S^2$ and $S^1 \times S^1$ have isomorphic homology groups, but they are not homotopy equivalent.

Theorem 2.17 (Invariance of dimension). *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open, non-empty. If U and V are homeomorphic, then $m = n$.*

Proof. For $x \in U$ set $A = \mathbb{R}^m \setminus \{x\}$ and $B = U$. Excision implies that

$$H_k(U, U \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}).$$

Long exact sequence of a pair implies that

$$\begin{array}{ccccccc} \dots & \longrightarrow & \widetilde{H}_k(\mathbb{R}^m) & \longrightarrow & \widetilde{H}_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) & \longrightarrow & \widetilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\}) \longrightarrow \widetilde{H}_{k-1}(\mathbb{R}^m) \longrightarrow \dots \\ & & \downarrow \cong & & & & \downarrow \cong \\ & & 0 & & & & 0 \end{array},$$

so $H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\})$. Then $\mathbb{R}^m \setminus \{x\}$ deformation retracts to S^{m-1} , so

$$H_k(U, U \setminus \{x\}) = \begin{cases} \mathbb{Z} & k = m \\ 0 & \text{otherwise} \end{cases}.$$

Similarly

$$H_k(V, V \setminus \{x\}) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise} \end{cases}.$$

Let $h : U \rightarrow V$ be a homeomorphism then this induces isomorphisms

$$h_* : H_k(U, U \setminus \{x\}) \rightarrow H_k(V, V \setminus \{h(x)\}),$$

for all k , so $m = n$. □

2.5.4 Naturality

Proposition 2.18 (Naturality of connecting homomorphisms). *Let*

$$(A_\bullet, \partial), (B_\bullet, \partial), (C_\bullet, \partial), (A'_\bullet, \partial), (B'_\bullet, \partial), (C'_\bullet, \partial)$$

be chain complexes. Consider a commutative diagram of chain maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_\bullet & \xrightarrow{i} & B_\bullet & \xrightarrow{j} & C_\bullet \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & A'_\bullet & \xrightarrow{i'} & B'_\bullet & \xrightarrow{j'} & C'_\bullet \longrightarrow 0 \end{array},$$

where the rows are short exact sequences. Then the induced diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H_n(A) & \xrightarrow{i_*} & H_n(B) & \xrightarrow{j_*} & H_n(C) & \xrightarrow{\partial} & H_{n-1}(A) & \xrightarrow{i_*} & H_{n-1}(B) & \xrightarrow{j_*} & H_{n-1}(C) \longrightarrow \dots \\ & & \downarrow \alpha_* & & \downarrow \beta_* & & \downarrow \gamma_* & & \downarrow \alpha_* & & \downarrow \beta_* & & \downarrow \gamma_* \\ \dots & \longrightarrow & H_n(A') & \xrightarrow{i'_*} & H_n(B') & \xrightarrow{j'_*} & H_n(C') & \xrightarrow{\partial} & H_{n-1}(A') & \xrightarrow{i'_*} & H_{n-1}(B') & \xrightarrow{j'_*} & H_{n-1}(C') \longrightarrow \dots \end{array}$$

is commutative.

Proof. The first two squares commute by functoriality.

$$\begin{array}{ccc} \partial & : & H_n(C) \longrightarrow H_{n-1}(A) \\ & & [c] \longmapsto [a] \end{array},$$

so

$$\begin{array}{ccc} & & a \in A_{n-1} \\ & & \downarrow i \\ b \in B_n & \xrightarrow{\partial} & \in \partial(b) \in B_{n-1} \\ \downarrow j & & \\ c \in C_n & & \end{array}$$

Then $\gamma(c) = \gamma(j(b)) = j'(\beta(b))$ and $i'(\alpha(a)) = \beta(i(a)) = \beta(\partial(b)) = \partial(\beta(b))$, so

$$\begin{array}{ccc} & & \alpha(a) \in A'_{n-1} \\ & & \downarrow i' \\ \beta(b) \in B'_n & \xrightarrow{\partial} & \in \partial(\beta(b)) \in B'_{n-1} \\ \downarrow j' & & \\ \gamma(c) \in C'_n & & \end{array}$$

so $\partial[\gamma(c)] = [\alpha(a)]$ and hence $\partial(\gamma_*[c]) = \alpha_*[a] = \alpha_*(\partial[c])$. \square

2.6 Mayer-Vietoris sequences

2.6.1 The Mayer-Vietoris sequence

The main ingredient of the proof of the excision theorem is **barycentric subdivision**. Let X be a topological space and $\mathcal{U} = \{U_i\}$ be a collection of subspaces whose interiors form an open cover of X . Define $C_n^{\mathcal{U}} \subseteq C_n(X)$ as the subgroup of all chains of the form $\sum_i n_i \sigma_i$ such that the image of σ_i is contained in some $U_j \in \mathcal{U}$. Then $\partial : C_n(X) \rightarrow C_{n-1}(X)$ satisfies $\partial(C_n^{\mathcal{U}}(X)) \subseteq C_{n-1}^{\mathcal{U}}(X)$ so the $C_n^{\mathcal{U}}(X)$ define a chain complex. Let $H_n^{\mathcal{U}}(X)$ be the homology groups with respect to this chain complex.

Proposition 2.19. *The inclusion $i : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ induces isomorphisms $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all n .*

Proof. Hatcher page 119. \square

Notation. If $\mathcal{U} = \{A, B\}$ we write $C_n(A + B)$ instead of $C_n^{\mathcal{U}}(X)$.

Theorem 2.20 (Mayer-Vietoris sequence). *Let X be a topological space, $A, B \subseteq X$ such that $\mathring{A} \cup \mathring{B} = X$, and*

$$i_1 : A \cap B \hookrightarrow A, \quad i_2 : A \cap B \hookrightarrow B, \quad j_1 : A \hookrightarrow X, \quad j_2 : B \hookrightarrow X$$

be inclusions. Then there is an exact sequence

$$\cdots \rightarrow H_1(A \cap B) \xrightarrow{\Phi} H_1(A) \oplus H_1(B) \xrightarrow{\Psi} H_1(X) \xrightarrow{\partial} H_0(A \cap B) \xrightarrow{\Phi} H_0(A) \oplus H_0(B) \xrightarrow{\Psi} H_0(X) \rightarrow 0,$$

where $\Phi(x) = (i_{1}(x), -i_{2*}(x))$, $\Psi(x, y) = j_{1*}(x) + j_{2*}(y)$, and ∂ is the connecting homomorphism.*

Proof. Let a sequence of chain complexes be

$$0 \rightarrow C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \rightarrow 0,$$

where $\phi(x) = (x, -x)$ and $\psi(x, y) = x + y$.

- ϕ is injective.
- $\text{Im } \phi \subseteq \text{Ker } \psi$.
- If $(x, y) \in \text{Ker } \psi$, then $y = -x$, and $x \in C_n(A)$ and $y \in C_n(B)$, so $x \in C_n(A \cap B)$, so $\text{Ker } \psi \subseteq \text{Im } \phi$.
- ψ is surjective by the definition of $C_n(A + B)$.

So this is a short exact sequence of chain complexes. This induces a long exact sequence of homology groups

$$\cdots \rightarrow H_1(A \cap B) \xrightarrow{\Phi} H_1(A) \oplus H_1(B) \xrightarrow{\Psi} \underset{\cong}{H_1(X)}^{A+B} \xrightarrow{\partial} H_0(A \cap B) \xrightarrow{\Phi} H_0(A) \oplus H_0(B) \xrightarrow{\Psi} \underset{\cong}{H_0(X)}^{A+B} \rightarrow 0,$$

by barycentric division. □

If $A \cap B \neq \emptyset$ we can augment these chain complexes and obtain a short exact sequence between these augmented chain complexes

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & C_0(A \cap B) & \xrightarrow{\phi} & C_0(A) \oplus C_0(B) & \xrightarrow{\psi} & C_0(A + B) \longrightarrow 0 \\ & & \downarrow \epsilon & & \downarrow \epsilon & & \downarrow \epsilon \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \longrightarrow 0 \end{array}$$

This induces a long exact sequence of homology groups

$$\cdots \rightarrow \widetilde{H}_1(A \cap B) \xrightarrow{\Phi} \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \xrightarrow{\Psi} \widetilde{H}_1(X) \xrightarrow{\partial} \widetilde{H}_0(A \cap B) \xrightarrow{\Phi} \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \xrightarrow{\Psi} \widetilde{H}_0(X) \rightarrow 0.$$

This is the Mayer-Vietoris sequence for reduced homology groups.

Note. This is the same as in the non-reduced case, but we need to assume that $A \cap B \neq \emptyset$.

An observation is that if $A \cap B$ is path-connected, then $\widetilde{H}_0(A \cap B) = 0$, so we have an exact sequence

$$\cdots \longrightarrow \widetilde{H}_1(A \cap B) \xrightarrow{\Phi} \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \xrightarrow{\Psi} \widetilde{H}_1(X) \xrightarrow{\partial} \underset{\cong}{\widetilde{H}_0(A \cap B)} \longrightarrow \cdots$$

Thus

$$H_1(X) \cong H_1(A) \oplus H_1(B) / \Phi(H_1(A \cap B)).$$

This is the abelianised version of the theorem of Seifert-van Kampen.

Example. Let $X = S^n \subseteq \mathbb{R}^{n+1}$ and let $x \in S^n$. Define $A = S^n \setminus \{x\}$ and $B = S^n \setminus \{-x\}$. Then A and B are contractible, so $\widetilde{H}_n(A) = \widetilde{H}_n(B) = 0$ for all n , and $A \cap B$ deformation retracts to S^{n-1} . Mayer-Vietoris implies that

$$\begin{array}{ccccccc} \dots & \longrightarrow & \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) & \longrightarrow & \widetilde{H}_i(X) & \longrightarrow & \widetilde{H}_{i-1}(A \cap B) \longrightarrow \widetilde{H}_{i-1}(A) \oplus \widetilde{H}_{i-1}(B) \longrightarrow \dots \\ & & \downarrow \mathbb{R} & & & & \downarrow \mathbb{R} \\ & & 0 & & & & 0 \end{array} \quad ,$$

$$\widetilde{H}_{i-1}(S^{n-1})$$

so $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1})$ for $n \geq 1$. We know $\widetilde{H}_0(S^0) \cong \mathbb{Z}$ and $\widetilde{H}_0(S^n) = 0$ for $n \geq 1$, so induction and knowledge on $H_n(S^0)$ implies that

$$\widetilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise} \end{cases}.$$

Example. Let $U, V \subseteq \mathbb{R}^n$ be two path-connected open subsets such that $\overline{U} \cup \overline{V} = \mathbb{R}^n$. Then $U \cap V$ is path-connected as well. Enough to show that $H_0(U \cap V) \cong \mathbb{Z}$, if and only if $\widetilde{H}_0(U \cap V) = 0$. Then $U \cap V \neq \emptyset$ because \mathbb{R}^n is connected, and U and V are open, so $\overset{\circ}{U} = U$ and $\overset{\circ}{V} = V$, so $\overset{\circ}{U} \cup \overset{\circ}{V} = \mathbb{R}^n$. Mayer-Vietoris long exact sequence for reduced homology groups implies that

$$\begin{array}{ccccccc} \dots & \longrightarrow & \widetilde{H}_1(\mathbb{R}^n) & \longrightarrow & \widetilde{H}_0(U \cap V) & \longrightarrow & \widetilde{H}_0(U) \oplus \widetilde{H}_0(V) \longrightarrow \widetilde{H}_0(\mathbb{R}^n) \longrightarrow 0 \\ & & \downarrow \mathbb{R} & & & & \downarrow \mathbb{R} \\ & & 0 & & & & 0 \end{array} \quad ,$$

since \mathbb{R}^n is contractible, so $\widetilde{H}_k(\mathbb{R}^n) = 0$ for all k , and $\widetilde{H}_0(U) = \widetilde{H}_0(V) = 0$, because U and V are path-connected. Thus $\widetilde{H}_0(U \cap V) = 0$.

2.6.2 Classical applications

Definition. Let X and Y be topological spaces. A continuous map $\phi : X \rightarrow Y$ is an **embedding** if it is a homeomorphism to its image.

Example. If X is compact and Y is Hausdorff, and $\phi : X \rightarrow Y$ is a continuous and injective map, then ϕ is an embedding, since $\phi : X \rightarrow \phi(X)$ is continuous and bijective and $\phi(X)$ is Hausdorff, so worksheet 1 implies that ϕ is a homeomorphism $X \rightarrow \phi(X)$.

Proposition 2.21.

1. Let $h : D^k \rightarrow S^n$ be an embedding, then $\widetilde{H}_i(S^n \setminus h(D^k)) = 0$ for all i .
2. Let $h : S^k \rightarrow S^n$ be an embedding, with $k < n$, then

$$\widetilde{H}_i(S^n \setminus h(S^k)) = \begin{cases} \mathbb{Z} & i = n - k - 1 \\ 0 & \text{otherwise} \end{cases}.$$

Corollary 2.22. Let $h : S^1 \rightarrow S^2$ be an embedding. Then $S^2 \setminus h(S^1)$ consists of exactly two path-components.

Proof. $\widetilde{H}_0(S^2 \setminus h(S^1)) \cong \mathbb{Z}$ by Proposition 2.21. □

Corollary 2.23 (Jordan curve theorem). Let $h : S^1 \rightarrow \mathbb{R}^2$ be an embedding. Then $\mathbb{R}^2 \setminus h(S^1)$ consists of exactly two path-components.

Proof. \mathbb{R}^2 is homeomorphic to $S^2 \setminus \{x\}$, by stereographic projection. □

Similarly, $\mathbb{R}^n \setminus h(S^{n-1})$ consists of exactly two path-components.

Proof of Proposition 2.21.

1. Induction on k .

$k = 0$. $S^n \setminus h(D^0) \cong \mathbb{R}^n$, so $\widetilde{H}_i(S^n \setminus h(D^n)) = 0$ for all n .

$k - 1 \mapsto k$. Let $h : D^k \rightarrow S^n$ be an embedding. Replace D^k by I^k . For a contradiction, assume there is a cycle α in $S^n \setminus h(I^k)$ that is not a boundary in $S^n \setminus h(I^k)$. Claim that there is a nested sequence of intervals

$$[0, 1] = I_0 \supseteq I_1 \supseteq \dots,$$

such that I_i is of length $\frac{1}{2^i}$ and such that α is a cycle in $S^n \setminus h(I^{k-1} \times I_i)$ but not a boundary in $S^n \setminus h(I^{k-1} \times I_i)$. Let $A = S^n \setminus h(I^{k-1} \times [0, \frac{1}{2}])$ and $B = S^n \setminus h(I^{k-1} \times [\frac{1}{2}, 1])$, so $A \cap B = S^n \setminus h(I^k)$ and $A \cup B = S^n \setminus h(I^{k-1} \times \{\frac{1}{2}\})$. Induction hypothesis implies that $\widetilde{H}_j(A \cup B) = 0$ for all j . Mayer-Vietoris implies that

$$\dots \rightarrow \widetilde{H}_{j+1}(A \cup B) \rightarrow \widetilde{H}_j(A \cap B) \xrightarrow{\cong} \widetilde{H}_j(A) \oplus \widetilde{H}_j(B) \rightarrow \widetilde{H}_j(A \cup B) \rightarrow \dots,$$

$\begin{matrix} \mathbb{R} \\ 0 \end{matrix} \qquad \qquad \qquad \begin{matrix} \mathbb{R} \\ 0 \end{matrix}$

so

$$\widetilde{H}_j(S^n \setminus h(I^k)) \cong \widetilde{H}_j(S^n \setminus h(I^{k-1} \times [0, \frac{1}{2}])) \oplus \widetilde{H}_j(S^n \setminus h(I^{k-1} \times [\frac{1}{2}, 1])).$$

Hence α is a cycle but not a boundary in $S^n \setminus h(I^{k-1} \times [0, \frac{1}{2}])$ or $S^n \setminus h(I^{k-1} \times [\frac{1}{2}, 1])$. This gives us I_1 . Iterating, this proves the claim. By induction, α is a boundary of some cycle β in $S^n \setminus h(I^{k-1} \times \{x\})$ for any $x \in I$, so in particular, for $\{x\} = \bigcap_i I_i$. Then $\beta = \sum_i n_i \sigma_i$ is a sum of finitely many singular simplices. The images of the σ_i are compact. But $S^n \setminus h(I^{k-1} \times I_i)$ form an open cover of $S^n \setminus h(I^{k-1} \times \{x\})$. So, by compactness, β is a chain in $S^n \setminus h(I^{k-1} \times I_i)$ for some i . Thus α is a boundary in $S^n \setminus h(I^{k-1} \times I_i)$, a contradiction.

2. Induction on k .

$k = 0$. $S^n \setminus h(S^0) \cong S^{n-1} \times \mathbb{R}$, so

$$\widetilde{H}_i(S^n \setminus h(S^0)) \cong \begin{cases} \mathbb{Z} & i = n - k - 1 \\ 0 & \text{otherwise} \end{cases}.$$

$k - 1 \mapsto k$. Let $h : S^k \rightarrow S^n$ be an embedding and $S^k = D_+^k \cup D_-^k$. Let $A = S^n \setminus h(D_+^k)$ and $B = S^n \setminus h(D_-^k)$, so 1 implies that $\widetilde{H}_i(A) = 0$ and $\widetilde{H}_i(B) = 0$ for all i , and $A \cap B = S^n \setminus h(S^k)$ and $A \cup B = S^n \setminus h(S^{k-1})$. Mayer-Vietoris implies that

$$\dots \rightarrow \widetilde{H}_{i+1}(A) \oplus \widetilde{H}_{i+1}(B) \rightarrow \widetilde{H}_i(A \cup B) \xrightarrow{\cong} \widetilde{H}_i(A \cap B) \rightarrow \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \rightarrow \dots,$$

$\begin{matrix} \mathbb{R} \\ 0 \end{matrix} \qquad \qquad \qquad \begin{matrix} \mathbb{R} \\ 0 \end{matrix}$

by 1, so

$$\widetilde{H}_{i+1}(S^n \setminus h(S^{k-1})) \cong \widetilde{H}_i(S^n \setminus h(S^k)) \cong \begin{cases} \mathbb{Z} & i + 1 = n - (k - 1) - 1 \\ 0 & \text{otherwise} \end{cases},$$

by induction.

□

Lecture 29 is a problems class.

2.7 Degree

Let $n \geq 1$. We have seen that $H_n(S^n) \cong \langle a \rangle \cong \mathbb{Z}$. Let $f : S^n \rightarrow S^n$ be a continuous map, so $f_* : H_n(S^n) \rightarrow H_n(S^n)$ is a homomorphism. Then f_* is given by $f_*(\alpha) = d\alpha$ for some $d \in \mathbb{Z}$ depending only on f . This integer is the **degree** of f .

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Proposition 2.24. *The following are observations.*

1. $\deg \text{id}_{S^n} = 1$.
2. If f is not surjective, then $\deg f = 0$.
3. If $f \cong g$, then $f_* = g_*$, so $\deg f = \deg g$.
4. $\deg fg = \deg f \deg g$. In particular, if f is a homotopy equivalence, then $\deg f = \pm 1$.
5. Let

$$R_i : \begin{array}{ccc} S^n & \longrightarrow & S^n \\ (x_1, \dots, x_i, \dots, x_{n+1}) & \longmapsto & (x_1, \dots, -x_i, \dots, x_{n+1}) \end{array}$$

be the reflection map. Then $\deg R_i = -1$.

6. The antipodal map

$$\begin{array}{ccc} -\text{id}_{S^n} & : & S^n \longrightarrow S^n \\ x & \longmapsto & -x \end{array}$$

has degree $(-1)^{n+1}$.

7. If $f : S^n \rightarrow S^n$ has no fixed points, then $\deg f = (-1)^{n+1}$.

Hopf implies that if $\deg f = \deg g$ then $f \cong g$.

Proof. 1 and 3 are clear.

2. Let $x_0 \in S^n \setminus f(S^n)$. So f factors as $f = i \circ f'$, where

$$S^n \xrightarrow{f'} S^n \setminus \{x_0\} \xrightarrow{i} S^n.$$

$H_n(S^n \setminus \{x_0\}) = 0$ since $S^n \setminus \{x_0\}$ is contractible, so $f_* = i_* \circ f'_* = 0$.

4. $(fg)_* = f_* g_*$, and there exists $g : S^n \rightarrow S^n$ such that $fg \cong \text{id}_{S^n}$, so

$$\deg f \deg g = \deg fg = \deg \text{id}_{S^n} = 1.$$

5. Enough to show it for $i = 1$. Induction on n .

$n = 1$. $R_1(x_1, x_2) = (-x_1, x_2)$. Then $\omega : t \mapsto (\cos 2\pi t, \sin 2\pi t)$ implies that $R_1([\omega]) = -[\omega]$, so $\deg R_1 = -1$.

$n - 1 \mapsto n$. Claim that there is an isomorphism $\phi : H_n(S^n) \xrightarrow{\sim} H_{n-1}(S^{n-1})$ such that

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{\phi} & H_{n-1}(S^{n-1}) \\ \downarrow R_{1*} & & \downarrow R_{1*} \\ H_n(S^n) & \xrightarrow{\phi} & H_{n-1}(S^{n-1}) \end{array}$$

commutes. Let

$$N = (0, \dots, 0, 1), \quad S = (0, \dots, 0, -1), \quad U = S^n \setminus \{N\}, \quad V = S^n \setminus \{S\},$$

so $R_1(U) = U$ and $R_1(V) = V$. There is a commutative diagram of chain maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_\bullet(U \cap V) & \longrightarrow & C_\bullet(U) \oplus C_\bullet(V) & \longrightarrow & C_\bullet(U + V) \longrightarrow 0 \\ & & \downarrow R_{1\#} & & \downarrow R_{1\#} \oplus R_{1\#} & & \downarrow R_{1\#} \\ 0 & \longrightarrow & C_\bullet(U \cap V) & \longrightarrow & C_\bullet(U) \oplus C_\bullet(V) & \longrightarrow & C_\bullet(U + V) \longrightarrow 0 \end{array}.$$

This induces a commutative diagram

$$\begin{array}{ccccc} H_n(S^n) & \xrightarrow{\partial} & H_{n-1}(U \cap V) & \xleftarrow{i_*} & H_{n-1}(S^{n-1}) \\ \downarrow R_{1*} & & \downarrow R_{1*} & & \downarrow R_{1*} \\ H_n(S^n) & \xrightarrow{\partial} & H_{n-1}(U \cap V) & \xleftarrow{i_*} & H_{n-1}(S^{n-1}) \end{array},$$

where

$$\begin{aligned} i : S^{n-1} &\longrightarrow U \cap V \\ (x_1, \dots, x_n) &\longmapsto (x_1, \dots, x_n, 0) \end{aligned}$$

is a homotopy equivalence. Then i_* is an isomorphism because i is a homotopy equivalence and ∂ is an isomorphism as seen last week. The first square commutes by naturality and the second square commutes by functoriality.

6. $-\text{id}_{S^n} = R_1 \dots R_{n+1}$, so

$$\deg -\text{id}_{S^n} = \deg R_1 \dots \deg R_{n+1} = (-1)^{n+1}.$$

7. If $f(x) \neq x$ for all $x \in S^n$, then the line segment from $f(x)$ to $-x$ defined by

$$t \mapsto (1-t)f(x) - tx$$

does not pass through the origin. Define

$$f_t(x) = \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|},$$

so f_t is a homotopy from f to $-\text{id}_{S^n}$. Thus

$$\deg f = \deg -\text{id}_{S^n} = (-1)^{n+1}.$$

□

Proposition 2.25. *If n is even, then $\mathbb{Z}/2\mathbb{Z}$ is the only non-trivial group that can act freely by homeomorphisms on S^n .*

Proof. Let G be a group acting freely by homeomorphisms on S^n , so $G \subseteq \text{Homeo } S^n$. So for $f \in G$, $\deg f = \pm 1$ by 4, and $\deg fg = \deg f \deg g$ for all $f, g \in G$ by 3, so the degree defines a homeomorphism $d : G \rightarrow \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$. The action is free, so if $g \in G \setminus \{\text{id}\}$, then g has no fixed points, so 7 and n even implies that $\deg g = (-1)^{n+1} = -1$. Then $\text{Ker } d = \{\text{id}\}$, so d is injective, so $G = \{\text{id}\}$ or $G \cong \mathbb{Z}/2\mathbb{Z}$. □

Definition. A **vector field** on S^n is a continuous map $v : S^n \rightarrow \mathbb{R}^{n+1}$ such that for each $x \in S^n$, $v(x)$ is **tangent** to S^n at x , that is $v(x)$ and x are orthogonal.

Theorem 2.26 (Hairy ball theorem). *S^n admits a continuous vector field $v : S^n \rightarrow \mathbb{R}^{n+1}$ that is nowhere zero if and only if n is odd.*

Proof. If $v(x) \neq 0$ for all $x \in S^n$, let

$$\begin{aligned} v' : S^n &\longrightarrow \mathbb{R}^{n-1} \\ x &\longmapsto \frac{v(x)}{|v(x)|}. \end{aligned}$$

Define

$$f_t(x) = \cos(t\pi)x + \sin(t\pi)v'(x).$$

Then $f_t(x) \in S^n$ for all $x \in S^n$ and for all $t \in \mathbb{I}$, so f_t is a homotopy from id_{S^n} to $-\text{id}_{S^n}$, so

$$1 = \deg \text{id}_{S^n} = \deg -\text{id}_{S^n} = (-1)^{n+1}.$$

Thus n is odd. Conversely, if $n = 2k - 1$,

$$v(x_1, \dots, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$$

is a vector field on S^n . □

A Proofs

A.1 The Seifert-van Kampen theorem

Proof of Theorem 1.21. Consider the natural homomorphism

$$\Phi : \pi_1(U_1, x_0) * \pi_1(U_2, x_0) \rightarrow \pi_1(X, x_0).$$

Φ is surjective by Lemma 1.17, and $N \subseteq \text{Ker } \Phi$. Want to show that $N = \text{Ker } \Phi$. A **factorisation** of an element $[f] \in \pi_1(X, x_0)$ is a formal product $[f_1] \dots [f_k]$ such that

- each f_i is a loop at x_0 in one of the U_i and $[f_i] \in \pi_1(U_i, x_0)$ is its homotopy class, and
- the loop $f_1 \dots f_k$ is homotopic to f in X .

A factorisation of $[f]$ is a word in $\pi_1(U_1, x_0) * \pi_1(U_2, x_0)$ that is mapped to $[f]$ by Φ . Two factorisations of $[f]$ are **equivalent** if they are related by finitely many of the following two moves.

- If $[f_i]$ and $[f_{i+1}]$ lie in the same group $\pi_1(U_i, x_0)$, exchange $[f_i][f_{i+1}]$ with $[f_i \cdot f_{i+1}]$. These are the relations in $\pi_1(U_i, x_0) * \pi_1(U_i, x_0)$.
- If f_i is a loop in $U_1 \cap U_2$, consider $[f_i]$ as an element in $\pi_1(U_1, x_0)$ instead of $\pi_1(U_2, x_0)$, and vice versa. These are the relations in $\pi_1(U_1, x_0) * \pi_1(U_2, x_0) / N$.

Given $[f] \in \pi_1(X, x_0)$, we want to show that any two factorisations of $[f]$ are equivalent. Let $[f_1] \dots [f_k]$ and $[f'_1] \dots [f'_l]$ be two factorisations of $[f]$, so the two loops $f_1 \dots f_k$ and $f'_1 \dots f'_l$ are homotopic. Let $F : I \times I \rightarrow X$ be a homotopy. By compactness, there exist

$$0 = s_0 < \dots < s_m = 1, \quad 0 = t_0 < \dots < t_n = 1,$$

such that $R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ and $F(R_{i,j}) \subseteq U_1$ or $F(R_{i,j}) \subseteq U_2$. May assume $0 = s_0 < \dots < s_m = 1$ subdivides the products $f_1 \dots f_k$ and $f'_1 \dots f'_l$. Relabel the $R_{i,j}$ to R_1, \dots, R_{mn} .

$mn - m + 1$	\dots	mn
\vdots	\ddots	\vdots
1	\dots	m

A path γ in $I \times I$ from left to right gives a loop $F|_\gamma$ in X at x_0 . Let γ_r be the path separating the first r rectangles from the others, so

$$F|_{\gamma_0} \cong f_1 \dots f_k, \quad F|_{\gamma_{mn}} = f'_1 \dots f'_l.$$

Let v be a grid point. Choose a path g_v in X from x_0 to $F(v)$, such that g_v is contained in $U_1 \cap U_2$ if $F(v) \in U_1 \cap U_2$ and in a single U_i otherwise. This gives us a factorisation of $[F|_{\gamma_r}]$ into loops only contained in U_1 or U_2 . The factorisations associated to γ_r and γ_{r+1} are equivalent, because the homotopy between $F|_{\gamma_r}$ and $F|_{\gamma_{r+1}}$ by pushing γ_r through R_r takes place within a single U_i . \square

A.2 The equivalence of simplicial and singular homology

Lemma A.1 (Five lemma). *Consider the following diagram of abelian groups*

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ A & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}.$$

If the rows are exact and $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is an isomorphism.

Proof. Enough to show

- if β and δ are surjective and ϵ is injective, then γ is surjective, and
- if β and δ are injective and α is surjective, then γ is injective.

□

Let $n \geq 1$. Then

$$H_n(\Delta^n, \partial\Delta^n) \cong \widetilde{H}_n(\Delta^n / \partial\Delta^n) \cong \widetilde{H}_n(S^n) \cong \mathbb{Z},$$

and $H_0(\Delta^0, \partial\Delta^0) \cong \mathbb{Z}$.

Lemma A.2. $H_n(\Delta^n, \partial\Delta^n)$ is generated by the class of the cycle $i_n : \Delta^n \rightarrow \Delta^n$.

Proof. i_n is a cycle. Induction on n .

$n = 0$. $H_0(\Delta^0, \emptyset)$ is generated by $[i_0]$.

$n - 1 \mapsto n$. Let $\Lambda \subseteq \partial\Delta^n$ be the union of all but one of the $(n - 1)$ -dimensional faces of Δ^n . Then Δ^n strongly deformation retracts to Λ , so

$$H_i(\Delta^n, \Lambda) = H_i(\Lambda, \Lambda) = 0.$$

Long exact sequence of the triple $\Lambda \subseteq \partial\Delta^n \subseteq \Delta^n$ implies that

$$\begin{array}{ccccccc} \dots & \rightarrow & H_n(\Delta^n, \Lambda) & \rightarrow & H_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\sim} & H_{n-1}(\partial\Delta^n, \Lambda) \rightarrow H_{n-1}(\Delta^n, \Lambda) \rightarrow \dots \\ & & \parallel & & & & \parallel \\ & & 0 & & & & 0 \end{array}.$$

Note that $\partial\Delta^n / \Lambda$ is homeomorphic to $\Delta^{n-1} / \partial\Delta^{n-1}$, which are good pairs, so

$$\begin{aligned} H_n(\Delta^n, \partial\Delta^n) &\cong H_{n-1}(\partial\Delta^n, \Lambda) \cong \widetilde{H}_{n-1}(\partial\Delta^n / \Lambda) \\ &\cong \widetilde{H}_{n-1}(\Delta^{n-1} / \partial\Delta^{n-1}) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}). \end{aligned}$$

One can check that $[i_n]$ maps to $[\pm i_{n-1}]$ along these isomorphisms, so induction implies that $H_n(\Delta^n, \partial\Delta^n)$ is generated by $[i_n]$.

□

Let X be a topological space with a Δ -complex structure, so there is a simplicial chain complex

$$\cdots \rightarrow \Delta_{n+1}(X) \rightarrow \Delta_n(X) \rightarrow \Delta_{n-1}(X) \rightarrow \cdots$$

Every simplicial chain complex can be viewed as a singular n -chain, so we obtain an inclusion of chain complexes $\Delta_\bullet(X) \rightarrow C_\bullet(X)$.

Theorem A.3. *This inclusion of chain complexes induces an isomorphism $H_n^\Delta(X) \xrightarrow{\sim} H_n(X)$ for all n .*

Proof. We only consider the case, where the Δ -complex structure on X is finite dimensional, that is $\Delta_m(X) = 0$ for all $m > k$, and the maximal such k is $\dim X$. Induction on k , the dimension.

$k = 0$. $H_n^\Delta(X) \cong H_n(X)$ for X points.

$k - 1 \mapsto k$. Let X^l be the l -skeleton of X consisting of all simplices of dimension at most l . Then $H_n^\Delta(X^k, X^{k-1})$ are the homology groups of the chain complex

$$\cdots \rightarrow \Delta_{k+1}(X^k) / \Delta_{k+1}(X^{k-1}) \rightarrow \Delta_k(X^k) / \Delta_k(X^{k-1}) \rightarrow \Delta_{k-1}(X^k) / \Delta_{k-1}(X^{k-1}) \rightarrow \cdots,$$

so

$$H_n^\Delta(X^k, X^{k-1}) = \begin{cases} 0 & n \neq k \\ \text{free abelian group with basis the } k\text{-simplices of } X & n = k \end{cases}.$$

The short exact sequence of chain complexes

$$0 \rightarrow \Delta_n(X^{k-1}) \rightarrow \Delta_n(X^k) \rightarrow \Delta_n(X^k) / \Delta_n(X^{k-1}) \rightarrow 0$$

gives a long exact sequence

$$\begin{array}{ccccccccc} \rightarrow H_{n+1}^\Delta(X^k, X^{k-1}) & \rightarrow & H_n^\Delta(X^{k-1}) & \rightarrow & H_n^\Delta(X^k) & \rightarrow & H_n^\Delta(X^k, X^{k-1}) & \rightarrow & H_{n-1}^\Delta(X^{k-1}) \rightarrow \\ & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\ \rightarrow H_{n+1}(X^k, X^{k-1}) & \rightarrow & H_n(X^{k-1}) & \rightarrow & H_n(X^k) & \rightarrow & H_n(X^k, X^{k-1}) & \rightarrow & H_{n-1}(X^{k-1}) \rightarrow \end{array},$$

which commutes by naturality, where β and ϵ are isomorphisms by induction. Consider the continuous map

$$\Phi : \bigsqcup_\alpha (\Delta_\alpha^k, \partial \Delta_\alpha^k) \rightarrow (X^k, X^{k-1}).$$

This induces an isomorphism

$$H_n(X^k, X^{k-1}) \cong H_n\left(\bigsqcup_\alpha \Delta_\alpha^k, \bigsqcup_\alpha \partial \Delta_\alpha^k\right) = \bigoplus_\alpha H_n(\Delta_\alpha^k, \partial \Delta_\alpha^k),$$

which is the free abelian group on $i_{n\alpha} : \Delta_\alpha^n \rightarrow \Delta_\alpha^n$ by Lemma A.2, so α and δ are isomorphisms. Thus five lemma implies that γ is an isomorphism. □