

M3P21 Geometry II: Algebraic Topology

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0 Introduction

Lecture 1
Friday
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0.1 Introduction

Combines topological spaces with algebraic objects, which are groups.

- How to show that a torus is not homeomorphic to a sphere?
- How to show that $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n \neq m$?

Content is fundamental groups and homology. We will follow chapter one and two from

- A Hatcher, Algebraic topology, 2002

The following are prerequisites.

- Point set topology. Topological spaces, continuous maps, product and quotient topologies, Hausdorff spaces, etc.
- Basic group theory. Normal subgroups and quotients, isomorphism theorems, free groups, presentation of groups, etc.

0.2 Some underlying geometric notions

0.2.1 Homotopy

Let X, Y be topological spaces and $I = [0, 1]$.

Definition. A **homotopy** is a continuous map $F : X \times I \rightarrow Y$. For every $t \in I$ we obtain a continuous map

$$\begin{aligned} f_t : X &\rightarrow Y \\ x &\mapsto f_t(x) = F(x, t) \end{aligned} .$$

Definition. Two continuous maps $f_0, f_1 : X \rightarrow Y$ are **homotopic** if there exists a homotopy $F : X \times I \rightarrow Y$ such that

$$f_0(x) = F(x, 0), \quad f_1(x) = F(x, 1),$$

for all $x \in X$. We write $f_0 \cong f_1$. (Exercise: this is an equivalence relation)

Definition. Let $A \subseteq X$ be a subspace. A **retraction** of X onto A is a continuous map $r : X \rightarrow A$ such that

- $r(X) = A$, and
- $r|_A = id_A$.

Example. If $X \neq \emptyset$, $p \in X$, then X retracts to p by the constant map $X \rightarrow \{p\}$.

Definition. A **deformation retraction** of X onto $A \subseteq X$ is a retraction that is homotopic to the identity. That is, there is a continuous map

$$\begin{aligned} F : X \times I &\rightarrow A \\ (x, t) &\mapsto f_t(x) \end{aligned} ,$$

such that $f_0 = id_X$ and $f_1 : X \rightarrow A$ is the deformation retraction.

Example. The closed n -dimensional n -disc

$$D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$

deformation retracts to $(0, \dots, 0) \in \mathbb{R}^n$. Let $f_t(x) = t \cdot x$. $t = 1$ gives $f_1 = id_{D^n}$ and $t = 0$ gives $f_0 : D^n \rightarrow (0, \dots, 0)$.

Example. Let S^n be the n -sphere,

$$\partial D^{n+1} = S^n = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

The cylinder $S^n \times I$ deformation retracts to $S^n \times \{0\}$, by defining $f_t(x, r) = (x, t \cdot r)$.

An observation is if X is a topological space, and $f : X \rightarrow \{p\}$ for $p \in X$ is a deformation retraction of X to p , then X is path-connected. Indeed, if $F : X \times I \rightarrow X$ is a homotopy from id_X to f and $x \in X$ is a point, then this gives a path

$$\begin{aligned} I &\rightarrow X \\ t &\mapsto F(x, t) \end{aligned}$$

that connects x to p . This implies that not all retractions are deformation retractions.

Example. A retraction that is not a deformation retraction. Take a space that is not path-connected and retract it to a point. Let $X = \{0, 1\}$ with discrete topology. $x \mapsto 0$ is a retraction, but not a deformation retraction because X is not path-connected.

Definition. A continuous map $f : X \rightarrow Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \rightarrow X$ such that $fg \cong id_Y$ and $gf \cong id_X$. If there exists a homotopy equivalence between X and Y , X and Y are **homotopy equivalent** or they have the same **homotopy type**.

Lemma 0.1. A deformation retraction $f : X \rightarrow A$ is a homotopy equivalence.

Proof. Let $i : A \hookrightarrow X$ be the inclusion map. Then $fi = id_A$ and $if = f \cong id_X$ by definition. \square

Example. The disc with two holes is equivalent to $O \cdot O$.

Example. \mathbb{R}^n deformation retracts to a point, by $f_t(x) = t \cdot x$.

Definition.

- X is **contractible** if it is homotopy equivalent to a point.
- A continuous map is **nullhomotopic** if it is homotopy equivalent to a constant map.

0.2.2 Cell complexes

Example. The torus $S^1 \times S^1$ is the union of a point, two open intervals, and the open disc $Int(D^2)$.

These are called **cells**. Can think of discs D^n glued together.

Definition. A **CW-complex**, or **cell complex**, is a topological space X such that there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X^n,$$

where the X^n are constructed inductively in the following way.

- X^n is a discrete set.
- For each $n \geq 0$ there is an collection of closed n -discs $\{D_\alpha^n\}$ together with continuous maps $\phi_\alpha : \partial D_\alpha^n \rightarrow X^{n-1}$, such that

$$X^n = \frac{X^{n-1} \sqcup \bigsqcup_\alpha D_\alpha^n}{\sim},$$

where $x \sim \phi_\alpha(x)$ for all $x \in \partial D_\alpha^n$ for all α .

- A subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n .

Remark.

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- As a set,

$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha} e_{\alpha}^n,$$

where each e_{α}^n is homeomorphic to an open n -disc. These e_{α}^n are called the n -**cells** of X .

- If $X = X^m$ for some m , then X is called **finite dimensional**. The minimal m such that $X = X^m$ is the **dimension** of X .

Example.

- $[0, 1]$ is a CW-complex.
- \mathbb{R} is a CW-complex.
- S^1 is a CW-complex.
- A graph is a CW-complex.
- $S^n = D^n / \partial D^n$ is a CW-complex. See worksheet 1.

Can also decompose CW-complexes.

- The sphere S^2 is one 0-cell, one 1-cell, and two 2-cells.
- The torus $S^1 \times S^1$ is one 0-cell, two 1-cells, and one 2-cell.
- The Möbius strip is two 0-cells, three 1-cells, and one 2-cell.
- The Klein bottle is one 0-cell, two 1-cells, and one 2-cell.

Definition. If X is a CW-complex with finitely many cells the **Euler characteristic** $\chi(X)$ of X is the number of even cells minus the number of odd cells.

Fact. $\chi(X)$ does not depend of the choice of cells decomposition.

Example.

- $\chi(S^n) = 0$ if n is odd and $\chi(S^n) = 2$ if n is even.
- $\chi(S^1 \times S^1) = 0$.

This is the generalisation of the following observation by Leonhard Euler. Let P be a convex polyhedron, where

- V is the number of vertices of P ,
- E is the number of edges of P , and
- F is the number of faces of P .

Then $V - E + F = 2$.

Example. A topological space that is not a CW-complex. $X = \{0, 1\}$ with trivial topology does not contain any closed points.

Fact. CW-complexes are always Hausdorff.

1 The fundamental group

1.1 Basic constructions

1.1.1 Paths and homotopy

Let X be a topological space. A **path** is a continuous map $f : I \rightarrow X$, where $I = [0, 1]$.

Definition. Two paths f_0, f_1 are **homotopic** if there exists a homotopy between f_0 and f_1 preserving the endpoints, that is a continuous map

$$F : I \times I \rightarrow X \\ (s, t) \mapsto f_t(s) ,$$

such that

$$f_t(0) = f_0(0), \quad f_t(1) = f_0(1),$$

for all $t \in I$, and

$$F(s, 0) = f_0(s), \quad F(s, 1) = f_1(s),$$

for all $s \in I$.

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set. Then all the paths in X are homotopic if they have the same endpoints.

Proof. Let $f_0, f_1 : I \rightarrow X$ be two paths such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$. Define

$$f_t(s) = (1 - t)f_0(s) + tf_1(s).$$

□

Lemma 1.1. *Being homotopic is an equivalence relation on the set of paths with fixed endpoints. We will write $f_0 \cong f_1$ for two homotopic paths f_0 and f_1 .*

Proof.

- f is homotopic to f .
- If f_0 is homotopic to f_1 by a homotopy f_t , then f_1 is homotopic to f_0 by the homotopy f_{1-t} .
- If f_0 is homotopic to f_1 by a homotopy f_t and $f_1 = g_0$ is homotopic to g_1 by a homotopy g_t , then f_0 is homotopic to g_1 by the homotopy

$$h_t = \begin{cases} f_{2t} & 0 \leq t \leq \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Then

$$H : I \times I \rightarrow X \\ (s, t) \mapsto h_t(s)$$

is continuous because its restriction to the closed subsets $I \times [0, 1/2]$ and $I \times [1/2, 1]$ is continuous, since if the restriction to two closed subsets is continuous then the restriction to the union of these subsets is continuous.

□

Let X be a topological space and $I = [0, 1]$. If $f : I \rightarrow X$ is a path, $[f]$ is the class of all paths on X homotopic to f .

Definition. Let $f, g : I \rightarrow X$ be two paths such that $f(1) = g(0)$. The **product path** $f \cdot g$ is the path

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \leq s \leq 1 \end{cases}.$$

A convention is that whenever we write $f \cdot g$ we implicitly assume $f(1) = g(0)$.

Lemma 1.2. *Let f_0, f_1, g_0, g_1 be paths on X such that $f_1 \cong f_0$ and $g_0 \cong g_1$. Then $f_0 \cdot g_0 \cong f_1 \cdot g_1$.*

Proof.

$$\begin{aligned} I \times I &\rightarrow X \\ (s, t) &\mapsto (f_t \cdot g_t)(s) \end{aligned}$$

is a homotopy between $f_0 \cdot g_0$ and $f_1 \cdot g_1$. □

Remark. Let $\phi : [0, 1] \rightarrow [0, 1]$ be continuous such that $\phi(0) = 0$ and $\phi(1) = 1$. If $f : I \rightarrow X$ is a path, then $f \circ \phi \cong f$. This is a **reparametrisation**.

Proof. Define

$$\phi_t(s) = (1 - t)\phi(s) + ts,$$

then $f \circ \phi_t$ is a homotopy between $f \circ \phi$ and f . □

For $x \in X$, let the **constant path** at x be

$$\begin{aligned} c_x : I &\rightarrow X \\ s &\mapsto x \end{aligned}.$$

For a path $f : I \rightarrow X$, define

$$\begin{aligned} f^{-1} : I &\rightarrow X \\ s &\mapsto f(1 - s) \end{aligned}.$$

Lemma 1.3. *Let $f, g, h : I \rightarrow X$ be paths. Then*

1. $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$,
2. $f \cdot c_{f(1)} \cong f$ and $c_{f(0)} \cdot f \cong f$, and
3. $f \cdot f^{-1} \cong c_{f(0)}$ and $f^{-1} \cdot f \cong c_{f(1)}$.

Proof.

1. $((f \cdot g) \cdot h) \phi = f \cdot (g \cdot h)$, where

$$\phi(s) = \begin{cases} \frac{s}{2} & s \in [0, \frac{1}{2}] \\ s - \frac{1}{4} & s \in [\frac{1}{2}, \frac{3}{4}] \\ 2s - 1 & s \in [\frac{3}{4}, 1] \end{cases},$$

so $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$ by reparametrisation.

2. Again reparametrisation, by

$$\psi(s) = \begin{cases} 2s & s \in [0, \frac{1}{2}] \\ 1 & s \in [\frac{1}{2}, 1] \end{cases}, \quad \chi(s) = \begin{cases} 0 & s \in [0, \frac{1}{2}] \\ 2s - 1 & s \in [\frac{1}{2}, 1] \end{cases}.$$

3. Define

$$H(s, t) = \begin{cases} f(\max\{1 - 2s, t\}) & s \in [0, \frac{1}{2}] \\ f(\max\{2s - 1, t\}) & s \in [\frac{1}{2}, 1] \end{cases}.$$

H is continuous, and

$$H(s, 0) = f^{-1} \cdot f, \quad H(s, 1) = c_{f(1)}.$$

The inverse is similar. □

Definition. A **loop** with **basepoint** $x_0 \in X$ is a path $f : I \rightarrow X$ such that $f(0) = f(1) = x_0$.

Definition. Denote by $\pi_1(X, x_0)$ the set of homotopy classes $[f]$ of loops $f : I \rightarrow X$ with basepoint x_0 .

Proposition 1.4. $\pi_1(X, x_0)$ is a group with product $[f][g] = [f \cdot g]$ and neutral element $c_{x_0} : I \rightarrow X$, the constant path at x_0 .

Proof. Follows directly from Lemma 1.2 and Lemma 1.3. \square

Definition. $\pi_1(X, x_0)$ is the **fundamental group** of X at x_0 .

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set and $x_0 \in X$. Then $\pi_1(X, x_0) = 0$.

Proof. X is convex gives that all loops are homotopic to each other. \square

Example.

- The fundamental group of a space X with the trivial topology is trivial, since X is simply-connected, because all maps $f : I \rightarrow X$ are continuous, so path-connected and all paths are homotopic.
- The fundamental group of a space X with the discrete topology is trivial, since $f : I \rightarrow X$ continuous gives f constant.

Assume $x_0, x_1 \in X$ such that x_0 and x_1 are in the same path component of X . Let $h : I \rightarrow X$ be a path such that $h(0) = x_0$ and $h(1) = x_1$. Define

$$\begin{aligned} \beta_h : \pi_1(X, x_1) &\rightarrow \pi_1(X, x_0) \\ [f] &\mapsto [h \cdot f \cdot h^{-1}] \end{aligned} .$$

This is well-defined by Lemma 1.2.

Proposition 1.5. $\beta_h : \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$ is an isomorphism.

Proof. It is a homomorphism.

$$\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot h^{-1}] = [h \cdot f \cdot h^{-1}] [h \cdot g \cdot h^{-1}] = \beta_h[f] \cdot \beta_h[g],$$

and $\beta_h[c_{x_1}] = [c_{x_1}]$. It is bijective with $(\beta_h)^{-1} = \beta_{h^{-1}}$. \square

If X is path-connected, we often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Definition. X is **simply-connected** if it is path-connected and $\pi_1(X) = 0$.

Proposition 1.6. X is simply-connected if and only if there exists a unique homotopy class of paths between any two points of X .

Proof.

\implies There exists a path between any two points. Let f, g be two paths from x_0 to x_1 for $x_0, x_1 \in X$. $f \cdot g^{-1} \cong g \cdot g^{-1}$ gives $f \cong f \cdot g^{-1} \cdot g \cong g \cdot g^{-1} \cdot g \cong g$.

\impliedby X is path-connected. $x_1 = x_0$ gives that all loops at x_0 are homotopic to each other, so $\pi_1(X) = 0$. \square

1.1.2 The fundamental group of the circle

Goal is to show that $\pi_1(S^1) \cong \mathbb{Z}$.

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Definition. A **covering space** of a space X is a space \tilde{X} and a continuous map $p : \tilde{X} \rightarrow X$ such that for each $x \in X$ there is an open $U \subseteq X$ such that

- $p^{-1}(U) = \bigcup_{j \in J} \tilde{U}_j$, where $\tilde{U}_j \subseteq \tilde{X}$ is open,
- $\tilde{U}_i \cap \tilde{U}_j = \emptyset$ if $i \neq j$, and
- $p|_{\tilde{U}_j} : \tilde{U}_j \rightarrow U$ is a homeomorphism for all $j \in J$.

Such a U is called **evenly covered**. The \tilde{U}_j are called **sheets**.

Example.

$$\begin{aligned} p : \mathbb{R} &\rightarrow S^1 \\ s &\mapsto (\cos(2\pi s), \sin(2\pi s)) \end{aligned}$$

Definition. Let $p : \tilde{X} \rightarrow X$ be a covering space. A **lift** of a continuous map $f : Y \rightarrow X$ is a continuous map $\tilde{f} : Y \rightarrow \tilde{X}$ such that $p\tilde{f} = f$, so

$$\begin{array}{ccc} & \tilde{X} & \\ & \downarrow p & \\ Y & \xrightarrow{\tilde{f}} & X \\ & \uparrow f & \end{array}$$

Proposition 1.7 (Unique lifting property). *Let $p : \tilde{X} \rightarrow X$ be a covering space and $f : Y \rightarrow X$ be a continuous map. If there are two lifts $\tilde{f}_1, \tilde{f}_2 : Y \rightarrow \tilde{X}$ of f such that $\tilde{f}_1(y) = \tilde{f}_2(y)$ for some $y \in Y$ and if Y is connected, then $\tilde{f}_1 = \tilde{f}_2$.*

Proof. Let $y \in Y$ and $U \subseteq X$ be an evenly covered neighbourhood of $f(y)$. Then

$$p^{-1}(U) = \bigcup_j \tilde{U}_j.$$

Let \tilde{U}_1 be the sheet such that $\tilde{f}_1(y) \in \tilde{U}_1$, and let \tilde{U}_2 be the sheet such that $\tilde{f}_2(y) \in \tilde{U}_2$. Let $N \subseteq Y$ be open and $y \in N$ such that $\tilde{f}_1(N) \subseteq \tilde{U}_1$ and $\tilde{f}_2(N) \subseteq \tilde{U}_2$. We have $p\tilde{f}_1 = p\tilde{f}_2$.

$$\tilde{f}_1(y) = \tilde{f}_2(y) \iff \tilde{U}_1 = \tilde{U}_2 \iff \tilde{f}_1|_N = \tilde{f}_2|_N.$$

Let

$$A = \{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\},$$

so A is open and $Y \setminus A$ is open. Thus $A \neq \emptyset$ gives $A = Y$. \square

Proposition 1.8 (Homotopy lifting property). *Let $p : \tilde{X} \rightarrow X$ be a covering space and $F : Y \times I \rightarrow X$ be a continuous map such that there exists a lift $\tilde{f}_0 : Y \times \{0\} \rightarrow \tilde{X}$ of $F|_{Y \times \{0\}}$. Then there is a unique lift $\tilde{F} : Y \times I \rightarrow \tilde{X}$ of F such that $\tilde{F}|_{Y \times \{0\}} = \tilde{f}_0$.*

Proof. Let $y_0 \in Y$ and $t \in I$. There are open $y_0 \in N_t \subseteq Y$ and $t \in (a_t, b_t) \subseteq I$ such that $F(N_t \times (a_t, b_t)) \subseteq U \subseteq X$, where $U \subseteq X$ is open and evenly covered. Compactness of I gives that there exist

$$0 = t_0 < \dots < t_m = 1,$$

and there exists $y_0 \in N \subseteq Y$ open such that $F(N \times [t_i, t_{i+1}]) \subseteq U_i \subseteq X$, where $U_i \subseteq X$ is open and evenly covered. We inductively construct a lift $\tilde{F}|_{N \times I}$ of $F|_{N \times I}$.

- $\tilde{F}|_{N \times [0,0]} = \tilde{f}_0|_{N \times [0,0]}$ exists.
- Assume the lift has been constructed on $N \times [0, t_i]$. Let $\tilde{U}_i \subseteq \tilde{X}$ be such that $p|_{\tilde{U}_i}: \tilde{U}_i \rightarrow U_i$ such that $\tilde{F}(y_0, t_i) \subseteq \tilde{U}_i$. After shrinking N , may assume $\tilde{F}(N \times \{t_i\}) \subseteq \tilde{U}_i$. Define \tilde{F} on $N \times [t_i, t_{i+1}]$ to be composition of F with the homeomorphism $p^{-1}: U_i \rightarrow \tilde{U}_i$.

After finitely many steps we obtain a lift $\tilde{F}: N \times I \rightarrow \tilde{X}$, where $y_0 \in N \subseteq Y$ is open, so for each $y \in Y$ there is a neighbourhood $N_y \subseteq Y$ such that $F|_{N_y \times I}: N_y \times I \rightarrow X$ lifts. For all $y \in Y$, $\{y\} \times I$ is connected and can be lifted, so Proposition 1.7 gives that the lift of $N \times I$ is unique. Thus there is a unique lift $\tilde{F}: Y \times I \rightarrow \tilde{X}$. \square

Example. Let X be a topological space and A be discrete. Then $p: X \times A \rightarrow X$ is a covering space. This is the **trivial covering**. (Exercise: show the unique lifting property and the homotopy lifting property for the trivial covering)

Corollary 1.9. Let $f: I \rightarrow X$ be a path, $f(0) = x_0$, and $p: \tilde{X} \rightarrow X$ be a covering space. For each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f}: I \rightarrow \tilde{X}$ such that $\tilde{f}(0) = \tilde{x}_0$.

Proof. Proposition 1.8 for Y a point. \square

Theorem 1.10. Let $x_0 = (1, 0) \in S^1$. $\pi_1(S^1, x_0)$ is the infinite cyclic group generated by the homotopy class of the loop

$$\begin{aligned} \omega: I &\rightarrow S^1 \\ s &\mapsto (\cos(2\pi s), \sin(2\pi s)) \end{aligned}$$

Remark.

- $[\omega]^n = [\omega_n]$, where

$$\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns)).$$

-

$$\begin{aligned} p: \mathbb{R} &\rightarrow S^1 \\ s &\mapsto (\cos(2\pi s), \sin(2\pi s)) \end{aligned}$$

is a covering space.

- ω_n lifts to

$$\begin{aligned} \tilde{\omega}_n: I &\rightarrow \mathbb{R} \\ s &\mapsto ns \end{aligned}$$

such that $\tilde{\omega}_n(0) = 0$ and $\tilde{\omega}_n(1) = n$.

Proof of Theorem 1.10.

- If $f: I \rightarrow S^1$ be a loop at x_0 , then the homotopy lifting property gives that there exists a lift $\tilde{f}: I \rightarrow \mathbb{R}$ such that $\tilde{f}(0) = 0$. Since $p(\tilde{f}(1)) = f(1) = x_0$, then $\tilde{f}(1) = n$ for some $n \in \mathbb{Z}$. $\tilde{\omega}_n: I \rightarrow \mathbb{R}$ is another path such that $\tilde{\omega}_n(0) = 0$ and $\tilde{\omega}_n(1) = n$, so $\tilde{f} \cong \tilde{\omega}_n$. Let $F: I \times I \rightarrow \mathbb{R}$ be a homotopy equivalence between \tilde{f} and $\tilde{\omega}_n$. Then $pF: I \times I \rightarrow S^1$ gives a homotopy between $p\tilde{f} = f$ and $p\tilde{\omega}_n = \omega_n$.
- Let $m, n \in \mathbb{Z}$ and assume $\omega_m \cong \omega_n$. Let $F: I \times I \rightarrow S^1$ be a homotopy.

$$F(0, t) = \omega_m(t), \quad F(1, t) = \omega_n(t), \quad F(s, 0) = F(s, 1) = x_0,$$

for all $s, t \in I$. The unique lifting property gives that $\tilde{\omega}_n, \tilde{\omega}_m: I \rightarrow \mathbb{R}$ are unique lifts such that $\tilde{\omega}_n(0) = 0 = \tilde{\omega}_m(0)$. The homotopy lifting property gives that F lifts uniquely to a homotopy $\tilde{F}: I \times I \rightarrow \mathbb{R}$ between $\tilde{\omega}_n$ and $\tilde{\omega}_m$, and $\tilde{F}(s, 1) \in \mathbb{Z}$ for all $s \in I$. Thus $\tilde{F}(s, 1) = n = m$, so $\omega_m \cong \omega_n$ if and only if $n = m$. \square

Lecture 5 is a problem class.

Theorem 1.11. *Every non-constant polynomial $p \in \mathbb{C}[z]$ has a root in \mathbb{C} .*

Proof. May assume

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n.$$

Assume p has no roots in \mathbb{C} . For each $r \in \mathbb{R}_{\geq 0}$ we obtain a loop

$$\begin{aligned} f_r : I &\rightarrow \mathbb{C} \\ s &\mapsto \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}, \end{aligned}$$

so $|f_r(s)| = 1$. $f_r(0) = 1$ and $f_r(1) = 1$, so f_r is a loop based at 1. f_0 is the constant loop at 1. $f_r(s)$ depends continuously on r , so $f_r \cong f_0$ for all $r \in \mathbb{R}_{\geq 0}$ and $[f_r] = [f_0] = 0 \in \pi_1(S^1)$. Fix $r \in \mathbb{R}_{\geq 0}$ such that $r > 1$ and $r > |a_1| + \cdots + |a_n|$. For $|z| = r$ we have

$$|z^n| > (|a_1| + \cdots + |a_n|) |z^{n-1}| \geq |a_1 z^{n-1}| + \cdots + |a_n| \geq |a_1 z^{n-1} + \cdots + a_n|.$$

Hence, for $0 \leq t \leq 1$ the polynomial $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$ has no root z with $|z| = r$. Define

$$F_r(t, s) = \frac{p_t(re^{2\pi i s})/p_t(r)}{|p_t(re^{2\pi i s})/p_t(r)|}.$$

$F_r(0, s) = \omega_n(s)$ and $F_r(1, s) = f_r(s)$, so $[\omega_n] = [f_r] = 0 \in \pi_1(S^1)$. Theorem 1.10 gives that $n = 0$, so p is constant. \square

See Hatcher Theorem 1.9 and Theorem 1.10 for more applications.

Proposition 1.12. *Let X, Y be topological spaces, $x_0 \in X$, and $y_0 \in Y$. Then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. A map

$$\begin{aligned} f : Z &\rightarrow X \times Y \\ z &\mapsto (g(z), h(z)) \end{aligned}$$

is continuous if and only if $g : Z \rightarrow X$ and $h : Z \rightarrow Y$ are continuous. For $Z = I$,

$$\{ \text{loops in } X \times Y \text{ based at } (x_0, y_0) \} \quad \longleftrightarrow \quad \{ \text{loops in } X \text{ based at } x_0 \} \times \{ \text{loops in } Y \text{ based at } y_0 \}.$$

Two loops

$$\begin{aligned} f_1 : I &\rightarrow X \times Y \\ s &\mapsto (g_1(s), h_1(s)) \end{aligned}, \quad \begin{aligned} f_2 : I &\rightarrow X \times Y \\ s &\mapsto (g_2(s), h_2(s)) \end{aligned}$$

are homotopic if and only if $g_1 \cong g_2$ and $h_1 \cong h_2$, so there is a bijection

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

$f_1 \cdot f_2 = (g_1 \cdot g_2, h_1 \cdot h_2)$ and the constant loop is mapped to the constant loop, so this is also a group isomorphism. \square

Example. The torus $S^1 \times S^1$ has

$$\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z}^2.$$

1.1.3 Induced homomorphisms

Let X, Y be topological spaces, $x_0 \in X$, and $\phi : X \rightarrow Y$. An observation is that ϕ induces a homomorphism

$$\begin{aligned} \phi_* : \pi_1(X, x_0) &\rightarrow \pi_1(Y, \phi(x_0)) \\ [f] &\mapsto [\phi f] \end{aligned} .$$

ϕ_* is well-defined, since if f_t is a homotopy between the loops f_0 and f_1 based at x_0 , then ϕf_t is a homotopy of loops between ϕf_0 and ϕf_1 . Moreover,

$$\phi(f \cdot g) = (\phi f) \cdot (\phi g),$$

and ϕ maps the constant path at x_0 to the constant path at $\phi(x_0)$, so ϕ is a homomorphism.

Proposition 1.13.

1. Let $\psi : X \rightarrow Y$ and $\phi : Y \rightarrow Z$ be continuous maps between topological spaces, $x_0 \in X$, and

$$\begin{aligned} \psi_* : \pi_1(X, x_0) &\rightarrow \pi_1(Y, \psi(x_0)), & \phi_* : \pi_1(Y, \psi(x_0)) &\rightarrow \pi_1(Z, \phi\psi(x_0)), \\ (\phi\psi)_* : \pi_1(X, x_0) &\rightarrow \pi_1(Z, \phi\psi(x_0)). \end{aligned}$$

Then $(\phi\psi)_* = \phi_*\psi_*$.

2. Let $id_X : X \rightarrow X$ be the identity then

$$(id_X)_* : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$$

is the identity.

Proof.

1. Let $f : I \rightarrow X$ be a loop at x_0 , then

$$(\phi\psi)_*([f]) = [(\phi\psi)f] = [\phi(\psi f)] = \phi_*([\psi f]) = \phi_*\psi_*([f]).$$

2. $(id_X)_*([f]) = [id_X f] = [f]$.

□

These two observations yield in particular that if $\phi : X \rightarrow Y$ is a homeomorphism with inverse $\psi : Y \rightarrow X$, then

$$\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \phi(x_0))$$

is an isomorphism with inverse ψ_* .

Proposition 1.14. Let $\phi : X \rightarrow Y$ be a homotopy equivalence. Then

$$\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \phi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Recall that if $x_0, x_1 \in X$ and $h : I \rightarrow X$ is a path such that $h(0) = x_0$ and $h(1) = x_1$, then we obtain an isomorphism

$$\begin{aligned} \beta_h : \pi_1(X, x_1) &\rightarrow \pi_1(X, x_0) \\ [f] &\mapsto [h \cdot f \cdot h^{-1}] \end{aligned} .$$

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Lemma 1.15. Let $\phi_t : X \rightarrow Y$ be a homotopy and $x_0 \in X$. Define the path

$$\begin{aligned} h : I &\rightarrow Y \\ s &\mapsto \phi_s(x_0) \end{aligned} ,$$

where $h(0) = \phi_0(x_0)$ and $h(1) = \phi_1(x_0)$. Then $(\phi_0)_* = \beta_h(\phi_1)_*$, that is the following diagram commutes.

$$\begin{array}{ccc} & \pi_1(Y, \phi_1(x_0)) & \\ (\phi_1)_* \nearrow & \downarrow \sim \beta_h & \\ \pi_1(X, x_0) & & \\ (\phi_0)_* \searrow & \downarrow & \\ & \pi_1(Y, \phi_0(x_0)) & \end{array} .$$

Proof. For $t \in I$, define the path

$$\begin{aligned} h_t : I &\rightarrow X \\ s &\mapsto h(ts) \end{aligned} ,$$

where $h_t(0) = \phi_0(x_0)$ and $h_t(1) = h(t) = \phi_t(x_0)$. Let f be a loop at x_0 . Define

$$F_t = h_t \cdot (\phi_t f) \cdot h_t^{-1}.$$

Then F_t is a loop at $\phi_0(x_0)$, which is continuous in t . So F_t is a homotopy of loops between

$$F_0 = h_0 \cdot (\phi_0 f) \cdot h_0^{-1} \cong \phi_0 f, \quad F_1 = h_1 \cdot (\phi_1 f) \cdot h_1^{-1} = h \cdot (\phi_1 f) \cdot h^{-1}.$$

Hence

$$(\phi_0)_*([f]) = [\phi_0 f] = [h \cdot (\phi_1 f) \cdot h^{-1}] = \beta_h([\phi_1 f]) = \beta_h(\phi_1)_*([f]).$$

□

Lemma 1.15 implies in particular the following.

Corollary 1.16. If $\psi : X \rightarrow X$ is continuous and $\psi \cong id_X$, then

$$\psi_* : \pi_1(X, x_0) \rightarrow \pi_1(X, \psi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Proof. By Lemma 1.15 there is a path h from $\psi(x_0)$ to x_0 such that

$$\begin{array}{ccc} & \pi_1(X, x_0) & \\ (id_X)_* \nearrow & \downarrow \sim \beta_h & \\ \pi_1(X, x_0) & & \\ \psi_* \searrow & \downarrow & \\ & \pi_1(X, \psi(x_0)) & \end{array} ,$$

so $\psi_* = \beta_h$ hence an isomorphism. □

Proof of Proposition 1.14. Let $\phi : X \rightarrow Y$ be a homotopy equivalence. Let $\psi : Y \rightarrow X$ be a homotopy inverse of ϕ , that is $\phi\psi \cong id_Y$ and $\psi\phi \cong id_X$.

$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\phi(x_0)) \xrightarrow{\phi_*} \pi_1(Y, \psi\phi\psi(x_0)).$$

Have to show that ϕ_* is bijective. The observation above gives that $(\psi\phi)_* = \psi_*\phi_*$ is an isomorphism, so ϕ_* is injective and ψ_* is surjective. Similarly $(\phi\psi)_* = \phi_*\psi_*$ is an isomorphism, so ψ_* is injective and ϕ_* is surjective. □

Lemma 1.17. *Let X be a topological space and $x_0 \in X$. Assume*

$$X = \bigcup_{\alpha \in \Lambda} A_\alpha,$$

such that

- *the A_α are all open and path-connected,*
- *$x_0 \in A_\alpha$ for all $\alpha \in \Lambda$, and*
- *all the intersections $A_\alpha \cap A_\beta$ are path-connected for all $\alpha, \beta \in \Lambda$.*

If f is a loop in X at x_0 , then we can write $[f] = [h_1] \dots [h_m]$, such that the h_i are loops at x_0 , and each contained in a single A_{α_i} .

Proof. f is continuous, so for all $s \in I$ there is an open neighbourhood V_s such that $f(V_s)$ is contained in some A_α . We can choose V_s to be an interval (a_s, b_s) such that $f([a_s, b_s]) \subseteq A_\alpha$. I is compact gives that a finite number of such intervals cover I , so there is a partition

$$0 = s_0 < \dots < s_m = 1,$$

such that $f([s_{i-1}, s_i]) \subseteq A_{\alpha_i}$ for some α_i . Let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$, and rescaling. $f \cong f_1 \dots f_m$ for $f_i \subseteq A_{\alpha_i}$ and $A_{\alpha_i} \cap A_{\alpha_j}$ is path-connected. Let g_i be a path from x_0 to $f(s_i)$ in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$. Let g_0, g_m be the constant loops at x_0 . $h_i = g_{i-1} \cdot f_i \cdot g_i^{-1}$ is a loop based at x_0 and $h_i \subseteq A_{\alpha_i}$. Thus

$$f \cong (g_0 \cdot f_1 \cdot g_1^{-1}) \cdot \dots \cdot (g_{m-1} \cdot f_m \cdot g_m^{-1}),$$

so $[f] = [h_1] \dots [h_m]$. □

Example. Möbius strip M deformation retracts to S^1 . Thus $\phi : M \rightarrow S^1$ is a homotopy equivalence, so $\pi_1(M) \cong \pi_1(S^1) \cong \mathbb{Z}$.

Example. There is no deformation retraction of S^1 to a point $p \in S^1$ because $\pi_1(S^1) \not\cong \pi_1(p)$.

Example. There is no retraction of the disc D^2 to its boundary $S^1 \subseteq D^2$.

Proof. Assume there is a retraction $r : D^2 \rightarrow S^1$, consider the embedding $i : S^1 \hookrightarrow D^2$. Then $ri = id_{S^1}$. Thus

$$\begin{array}{ccccc} \pi_1(S^1) & \xrightarrow{i_*} & \pi_1(D^2) & \xrightarrow{r_*} & \pi_1(S^1) \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array},$$

so $r_* i_* (\pi_1(S^1)) = 0$ but $r_* i_* = (ri)_* = id_{\pi_1(S^1)}$, a contradiction. □

Theorem 1.18 (Brouwer fixed point theorem). *Let $h : D^2 \rightarrow D^2$ be a continuous map. Then h has a fixed point, that is there exists $x \in D^2$ such that $h(x) = x$.*

Proof. Assume $h(x) \neq x$ for all $x \in D^2$. Define $r : D^2 \rightarrow S^1$ by defining $r(x)$ to be the intersection of the ray starting at $h(x)$ towards x with S^1 . r is continuous, and if $x \in S^1$, then $r(x) = x$, so r is a retraction, a contradiction. □

Lemma 1.17 gives that if $U_1, U_2 \subseteq X$ are open and path-connected such that $U_1 \cup U_2 = X$ and $U_1 \cap U_2$ is path-connected and $x_0 \in U_1 \cap U_2$, then every $[f] \in \pi_1(X, x_0)$ can be factorised as $[f] = [g_1][h_1] \dots [g_n][h_n]$ such that the g_i are loops at x_0 contained in U_1 and the h_i are loops at x_0 contained in U_2 . In other words, $i_1 : U_1 \hookrightarrow X$ and $i_2 : U_2 \hookrightarrow X$, so

$$(i_1)_* : \pi_1(U_1, x_0) \rightarrow \pi_1(X, x_0), \quad (i_2)_* : \pi_1(U_2, x_0) \rightarrow \pi_1(X, x_0).$$

Lemma 1.17 gives that $(i_1)_*(\pi_1(U_1, x_0)) \cup (i_2)_*(\pi_1(U_2, x_0))$ generate $\pi_1(X, x_0)$.

Proposition 1.19. $\pi_1(S^n) = 0$ if $n \geq 2$.

Proof. Let $U_1 = S^n \setminus \{(1, 0, \dots, 0)\}$ and $U_2 = S^n \setminus \{(-1, 0, \dots, 0)\}$. Then $U_1 \cong \mathbb{R}^n$ and $U_2 \cong \mathbb{R}^n$, by stereographic projection. $U_1 \cup U_2 = S^n$ and $U_1 \cap U_2$ is path-connected. Let $x_0 \in U_1 \cap U_2$. $\pi_1(U_1, x_0) = 0$ and $\pi_1(U_2, x_0) = 0$, so Lemma 1.17 gives that $\pi_1(S^n, x_0) = 0$. □

1.2 Seifert-van Kampen theorem

1.2.1 Free products with amalgamation

Definition. If S is a set, then F_S is the **free group** on S . We can write any group G as a quotient of some free group F_S ,

$$G = \frac{F}{\langle\langle R \rangle\rangle},$$

where $\langle\langle R \rangle\rangle$ is the **normal closure** of $R \subseteq F_S$, the smallest normal subgroup of F_S containing R . We write $G = \langle S \mid R \rangle$. This is called a **presentation** of G .

Let G_0, G_1, G_2 be groups, and $f_1 : G_0 \rightarrow G_1$ and $f_2 : G_0 \rightarrow G_2$ be homomorphisms.

Definition. A group H together with homomorphisms $h_1 : G_1 \rightarrow H$ and $h_2 : G_2 \rightarrow H$ such that $h_1 f_1 = h_2 f_2$ is an **amalgamated product** of G_1 and G_2 over G_0 if it satisfies the following universal property. For every group G and all homomorphisms $h'_1 : G_1 \rightarrow G$ and $h'_2 : G_2 \rightarrow G$ such that $h'_1 f_1 = h'_2 f_2$, there exists a unique homomorphism $\alpha : H \rightarrow G$ such that $h'_1 = \alpha h_1$ and $h'_2 = \alpha h_2$.

$$\begin{array}{ccccc} G_0 & \xrightarrow{f_1} & G_1 & & \\ f_2 \downarrow & & \downarrow h_1 & \searrow h'_1 & \\ G_2 & \xrightarrow{h_2} & H & \xrightarrow{\exists! \alpha} & G \\ & \searrow h'_2 & & & \end{array}$$

Theorem 1.20. Given $f_1 : G_0 \rightarrow G_1$ and $f_2 : G_0 \rightarrow G_2$. Then there exists an amalgamated product, unique up to isomorphism. We denote it by $G_1 *_{G_0} G_2$.

Proof. Worksheet 2. □

$G_0 = \{id\}$ is the **free product**. We write $G_1 * G_2$ instead of $G_1 *_{\{id\}} G_2$. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Then $G_1 * G_2 = \langle S_1 \sqcup S_2 \mid R_1 \cup R_2 \rangle$, with injections $G_i \hookrightarrow G_1 * G_2$ for $i = 1, 2$. More generally,

$$G_1 * G_2 \cong \frac{G_1 *_{G_0} G_2}{N}.$$

where N is the normal closure of the set

$$\left\{ f_1(g) f_2(g)^{-1} \mid g \in G_0 \right\} \subseteq G_1 * G_2.$$

1.2.2 The Seifert van-Kampen theorem

Theorem 1.21 (Seifert-van Kampen). Let X be a topological space and $U_1, U_2 \subseteq X$ be open and path-connected such that $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is path-connected and let $x_0 \in U_1 \cap U_2$. Then

$$\pi_1(X, x_0) \cong \pi_1(U_1, x_0) *_{\pi_1(U_1 \cap U_2, x_0)} \pi_2(U_2, x_0) \cong \frac{\pi_1(U_1, x_0) * \pi_1(U_2, x_0)}{N},$$

where N is the normal closure of the set

$$\left\{ (j_1)_*(\omega) (j_2)_*(\omega)^{-1} \mid \omega \in \pi_1(U_1 \cap U_2, x_0) \right\},$$

and $j_i : U_i \hookrightarrow X$.

$$\begin{array}{ccc} U_1 \cap U_2 & \xrightarrow{i_1} & U_1 \\ i_2 \downarrow & & \downarrow j_1 \\ U_2 & \xrightarrow{j_2} & X \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \pi_1(U_1 \cap U_2, x_0) & \xrightarrow{(i_1)_*} & \pi_1(U_1, x_0) \\ (i_2)_* \downarrow & & \downarrow (j_1)_* \\ \pi_1(U_2, x_0) & \xrightarrow{(j_2)_*} & \pi_1(U_1, x_0) *_{\pi_1(U_1 \cap U_2, x_0)} \pi_1(U_2, x_0) \end{array}.$$

Proof. Consider the natural homomorphism

$$\Phi : \pi_1(U_1, x_0) * \pi_1(U_2, x_0) \rightarrow \pi_1(X, x_0).$$

Φ is surjective by Lemma 1.17. $N \subseteq \text{Ker}(\Phi)$. Want to show that $N = \text{Ker}(\Phi)$. A **factorisation** of an element $[f] \in \pi_1(X, x_0)$ is a formal product $[f_1] \dots [f_k]$ such that

- each f_i is a loop at x_0 in one of the U_i and $[f_i] \in \pi_1(U_i, x_0)$ is its homotopy class, and
- the loop $f_1 \dots f_k$ is homotopic to f in X .

A factorisation of $[f]$ is a word in $\pi_1(U_1, x_0) * \pi_1(U_2, x_0)$ that is mapped to $[f]$ by Φ . Two factorisations of $[f]$ are **equivalent** if they are related by finitely many of the following two moves.

- If $[f_i]$ and $[f_{i+1}]$ lie in the same group $\pi_1(U_i, x_0)$, exchange $[f_i][f_{i+1}]$ with $[f_i \cdot f_{i+1}]$. These are the relations in $\pi_1(U_i, x_0) * \pi_1(U_i, x_0)$.
- If f_i is a loop in $U_1 \cap U_2$, consider $[f_i]$ as an element in $\pi_1(U_1, x_0)$ instead of $\pi_1(U_2, x_0)$, and vice versa. These are the relations in $\pi_1(U_1, x_0) * \pi_1(U_2, x_0) / N$.

Given $[f] \in \pi_1(X, x_0)$, we want to show that any two factorisations of $[f]$ are equivalent. Let $[f_1] \dots [f_k]$ and $[f'_1] \dots [f'_l]$ be two factorisations of $[f]$, so the two loops $f_1 \dots f_k$ and $f'_1 \dots f'_l$ are homotopic. Let $F : I \times I \rightarrow X$ be a homotopy. By compactness, there exist

$$0 = s_0 < \dots < s_m = 1, \quad 0 = t_0 < \dots < t_n = 1,$$

such that $R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ and $F(R_{i,j}) \subseteq U_1$ or $F(R_{i,j}) \subseteq U_2$. May assume $0 = s_0 < \dots < s_m = 1$ subdivides the products $f_1 \dots f_k$ and $f'_1 \dots f'_l$. Relabel the $R_{i,j}$ to R_1, \dots, R_{mn} .

$mn - m + 1$	\dots	mn
\vdots	\ddots	\vdots
1	\dots	m

A path γ in $I \times I$ from left to right gives a loop $F|_\gamma$ in X at x_0 . Let γ_r be the path separating the first r rectangles from the others, so

$$F|_{\gamma_0} \cong f_1 \dots f_k, \quad F|_{\gamma_{mn}} = f'_1 \dots f'_l.$$

Let v be a grid point. Choose a path g_v in X from x_0 to $F(v)$, such that g_v is contained in $U_1 \cap U_2$ if $F(v) \in U_1 \cap U_2$ and in a single U_i otherwise. This gives us a factorisation of $[F|_{\gamma_r}]$ into loops only contained in U_1 or U_2 . The factorisations associated to γ_r and γ_{r+1} are equivalent, because the homotopy between $F|_{\gamma_r}$ and $F|_{\gamma_{r+1}}$ by pushing γ_r through R_r takes place within a single U_i . \square

Theorem 1.22 (Seifert-van Kampen, strong version). *Let X be a path-connected topological space such that*

- $X = \bigcup_\alpha A_\alpha$,
- A_α , $A_\alpha \cap A_\beta$, and $A_\alpha \cap A_\beta \cap A_\gamma$ are open and path-connected for all α, β, γ , and
- $x_0 \in \bigcap_\alpha A_\alpha$.

Then

$$\pi_1(X, x_0) \cong \frac{*_{\alpha} \pi_1(A_\alpha, x_0)}{N},$$

where $N \subseteq *_{\alpha} \pi_1(A_\alpha, x_0)$ is the normal closure of the set

$$\left\{ (i_{\alpha\beta})_*(\omega) (i_{\beta\alpha})_*(\omega)^{-1} \mid \omega \in \pi_1(A_\alpha \cap A_\beta) \right\},$$

and $i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ is the inclusion.

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Example. Let $S^1 \vee S^1$ be the wedge product. Fix $x \in S^1$ and $y \in S^1$. Then

$$S^1 \vee S^1 = \frac{S^1 \sqcup S^1}{x \sim y} = \overset{b}{\underset{a}{\mathbf{O}}} \cdot \overset{a}{\underset{b}{\mathbf{O}}}.$$

Let

$$A = \mathbf{O} \cdot (, \quad B =) \cdot \mathbf{O}, \quad A \cap B =) \cdot (.$$

$\pi_1(A) \cong \langle b \rangle \cong \mathbb{Z}$, $\pi_1(B) \cong \langle a \rangle \cong \mathbb{Z}$, and $\pi_1(A \cap B) = \{id\}$. A , B , and $A \cap B$ are open and path-connected. Van Kampen gives

$$\pi_1(S^1 \vee S^1) \cong \pi_1(A) * \pi_1(B) \cong \mathbb{Z} * \mathbb{Z} \cong F_{\{a,b\}}.$$

More generally, let $X = S_{a_1}^1 \vee \cdots \vee S_{a_n}^1$. By induction,

$$\pi_1(X) = \mathbb{Z} * \cdots * \mathbb{Z} \cong F_{\{a_1, \dots, a_n\}}.$$

Similarly, let $X = \bigvee_{\alpha \in \Lambda} S_{\alpha}^1$. Strong version of van Kampen gives

$$\pi_1(X) = \bigast_{\alpha \in \Lambda} \mathbb{Z} = F_{\Lambda}.$$

Example. Let T be a torus and $x_0 \in T$. Let

$$A = T \setminus \{\text{small closed disc } D\}, \quad B = \{\text{open set that contains } D \text{ and } x_0\}.$$

- A is homotopy equivalent to $S^1 \vee S^1$, so $\pi_1(A) \cong F_{\{a,b\}}$.
- B is homeomorphic to D^2 , so $\pi_1(B) = \{id\}$.
- $A \cap B$ is homotopy equivalent to S^1 , so $\pi_1(A \cap B) \cong \mathbb{Z}$.

A , B , and $A \cap B$ are open and path-connected. Van Kampen gives

$$\pi_1(T) \cong \frac{\pi_1(A)}{\langle \langle i_*(\pi_1(A \cap B)) \rangle \rangle},$$

where $i : A \cap B \hookrightarrow A$. Then

$$i_* : \begin{array}{ccc} \pi_1(A \cap B) = \langle \omega \rangle & \rightarrow & \pi_1(A) \\ \omega & \mapsto & aba^{-1}b^{-1} \end{array},$$

so

$$\pi_1(T) \cong \frac{F_{\{a,b\}}}{\langle \langle aba^{-1}b^{-1} \rangle \rangle} = \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2.$$

1.2.3 Applications to CW-complexes

Let X be a path-connected topological space. Let Y be the space obtained by attaching 2-cells $\{e_{\alpha}^2\}$ to X along maps $\phi_{\alpha} : \partial D^2 = S^1 \rightarrow X$. Consider the loops

$$\phi'_{\alpha} : \begin{array}{ccc} I & \rightarrow & X \\ s & \mapsto & \phi_{\alpha}(\cos(2\pi s), \sin(2\pi s)) \end{array},$$

based at $\phi'_{\alpha}(0)$. Let γ_{α} be a path from x_0 to $\phi'_{\alpha}(0)$ for each α . Then $\gamma_{\alpha} \cdot \phi_{\alpha} \cdot \gamma_{\alpha}^{-1}$ is a loop at x_0 . After attaching e_{α}^2 , the loop $\gamma_{\alpha} \cdot \phi_{\alpha} \cdot \gamma_{\alpha}^{-1}$ is homotopic to the constant loop at x_0 . Let $N \subseteq \pi_1(X, x_0)$ be the normal closure of all the elements of the form $[\gamma_{\alpha} \cdot \phi_{\alpha} \cdot \gamma_{\alpha}^{-1}]$. The inclusion $i : X \hookrightarrow Y$ yields

$$i_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, x_0),$$

and $N \subseteq \text{Ker}(i_*)$.

Proposition 1.23. *This inclusion $i : X \hookrightarrow Y$ induces a surjection*

$$i_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, x_0),$$

and $\text{Ker}(i_*) = N$, so

$$\pi_1(Y, x_0) \cong \frac{\pi_1(X, x_0)}{N}.$$

Proof. Construct a space Z from Y by attaching a strip $I \times I$ to Y by identifying the lower edge $I \times \{0\}$ with the path γ_α and the right edge $\{1\} \times I$ with an arch on e_α^2 . Attach all the left edges of the strips with each other. Z deformation retracts to Y . Choose a point $y_\alpha \in e_\alpha^2$ for each α , such that y_α is not contained in X or in the attached strip. Let

$$A = Z \setminus \bigcup_{\alpha} \{y_\alpha\}, \quad B = Z \setminus X.$$

- A deformation retracts to X .
- B is homotopy equivalent to a point.
- $A \cap B$ is homotopy equivalent to

$$\{\text{paths } \gamma_\alpha \text{ from } x_0 \text{ to loops } \phi'_\alpha\} = \overset{\phi'_\alpha}{\underset{\phi'_\alpha}{\text{O}}} \xrightarrow{\gamma_\alpha} x_0 \xleftarrow{\gamma_\alpha} \overset{\phi'_\alpha}{\underset{\phi'_\alpha}{\text{O}}}.$$

A , B , and $A \cap B$ are open and path-connected. Van Kampen gives

$$\pi_1(Y) \cong \pi_1(Z) = \frac{\pi_1(A)}{\langle\langle j_*(\pi_1(A \cap B)) \rangle\rangle},$$

where $j : A \cap B \hookrightarrow A$ is the inclusion. So $\langle\langle j_*(\pi_1(A \cap B)) \rangle\rangle$ is exactly N . Thus $\pi_1(A) = \pi_1(X)$. \square

Corollary 1.24. *For every group G there exists a two-dimensional CW-complex X_G such that $\pi_1(X_G) = G$.*

Proof. Let $G = \langle \{g_\alpha\} \mid \{r_\beta\} \rangle$ be a presentation of G , that is

$$G = \frac{F_{\{g_\alpha\}}}{\langle\langle \{r_\beta\} \rangle\rangle}.$$

Seen last time that $\pi_1(\bigvee_{g_\alpha} S_{g_\alpha}^1) = F_{\{g_\alpha\}}$. Each word r_β defines a loop in $\bigvee_{g_\alpha} S_{g_\alpha}^1$. Attach 2-cells to $\bigvee_{g_\alpha} S_{g_\alpha}^1$ along the loops defined by the relations $\{r_\beta\}$. Call this new CW-complex Y . Proposition 1.23 gives that

$$\pi_1(Y, x_0) \cong \frac{\pi_1(X, x_0)}{\langle\langle \{r_\beta\} \rangle\rangle} \cong \frac{F_{\{g_\alpha\}}}{\langle\langle \{r_\beta\} \rangle\rangle} \cong G.$$

\square

Remark. Let $X = \bigcup_n X^n$ be a CW-complex, path-connected. Proposition 1.23 can be used to show the following two facts.

- The inclusion $X^1 \hookrightarrow X$ induces a surjective homomorphism $\pi_1(X^1) \rightarrow \pi_1(X)$.
- The inclusion $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2) \rightarrow \pi_1(X)$.

1.3 Covering spaces

1.3.1 Lifting properties

Let X be a topological space. Recall that a **covering space** is $p : \tilde{X} \rightarrow X$ such that each $x \in X$ has an open neighbourhood U such that

$$p^{-1}(U) = \bigcup_{\alpha} \tilde{U}_{\alpha},$$

where U_{α} are pairwise disjoint and $p|_{\tilde{U}_{\alpha}} : \tilde{U}_{\alpha} \rightarrow U$ is a homeomorphism for all α .

Example.

$$\begin{array}{lll} \mathbb{R} & \rightarrow & S^1 \\ s & \mapsto & (\cos(2\pi s), \sin(2\pi s)) \end{array}, \quad \begin{array}{lll} S^1 & \rightarrow & S^1 \\ z & \mapsto & z^n \end{array}, \quad \text{O} \cdot \text{O} \cdot \text{O} \rightarrow S^1 \vee S^1 = \text{O} \cdot \text{O}.$$

Let $f : Y \rightarrow X$ be a continuous map. A **lift** of f is a continuous map $\tilde{f} : Y \rightarrow \tilde{X}$ such that $p\tilde{f} = f$, where $p : \tilde{X} \rightarrow X$ is a covering space. Let Y be connected.

- **Unique lifting property** states that if two lifts \tilde{f}_1 and \tilde{f}_2 of f coincide at one point, then they coincide on all of Y .
- **Homotopy lifting property** states that if $f_t : Y \rightarrow X$ is a homotopy and $\tilde{f}_0 : Y \rightarrow \tilde{X}$ is a lift of f_0 then there exists a unique homotopy $\tilde{f}_t : Y \rightarrow \tilde{X}$ of \tilde{f}_0 that lifts f_t .

Remark.

- If Y is a point, this is called the **path lifting property**. Let $f : I \rightarrow X$ be a path with $f(0) = x_0$. If $\tilde{x}_0 \in p^{-1}(x_0)$, then there is a unique path $\tilde{f} : I \rightarrow \tilde{X}$ lifting f and starting at \tilde{x}_0 .
- In particular, the lift of a constant path is constant.
- This implies in particular that the lift of a homotopy of paths is again a homotopy of paths. The endpoints $f_t(0)$ and $f_t(1)$ define constant paths as t varies.

Fix $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$, so

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0).$$

To every element in $\pi_1(X, x_0)$ we can associate a homotopy class of paths in \tilde{X} starting at \tilde{x}_0 .

Proposition 1.25.

1. $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ is injective.
2. $p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right) \subseteq \pi_1(X, x_0)$ consists of the homotopy classes of loops at x_0 whose lifts to \tilde{X} starting at \tilde{x}_0 are loops.

Proof.

1. Let $\tilde{f}_0 : I \rightarrow \tilde{X}$ be a loop at \tilde{x}_0 such that $[\tilde{f}_0] \in \text{Ker}(p_*)$, so $p\tilde{f}_0 = f_0$ is homotopic to the constant loop at x_0 . Let $f_t : I \rightarrow X$ be a homotopy between \tilde{f}_0 and the constant loop. Homotopy lifting property and remark gives that f_t lifts to a homotopy \tilde{f}_t of paths between \tilde{f}_0 and the constant loop, so $[\tilde{f}_0] = \text{id} \in \pi_1(\tilde{X}, \tilde{x}_0)$ and p_* is injective.
2. Let $f : I \rightarrow X$ be a loop at x_0 that lifts to a loop \tilde{f} at \tilde{x}_0 . Then $p\tilde{f} = f$, so $p_*([\tilde{f}]) = [f]$. On the other hand, if $f : I \rightarrow X$ is a loop at x_0 such that there exists a loop $\tilde{f} : I \rightarrow \tilde{X}$ at \tilde{x}_0 with $p_*([\tilde{f}]) = [f]$, then f is homotopic to $p\tilde{f}$. Homotopy lifting property gives that there exists a loop $\tilde{f}' : I \rightarrow \tilde{X}$ at \tilde{x}_0 such that $p\tilde{f}' = f$.

□

Lecture 12
Wednesday
06/02/19

Let $p : \tilde{X} \rightarrow X$ be a covering space. Let $U \subseteq X$ be an evenly covered neighbourhood of $x \in X$. Let

$$p^{-1}(U) = \bigsqcup_{\alpha \in \Lambda} \tilde{U}_\alpha.$$

Then the cardinality $|p^{-1}(x)|$ of $p^{-1}(x)$ is exactly the cardinality of $|\Lambda|$. The set of sheets is in bijection with $p^{-1}(x)$. So the cardinality of $p^{-1}(x)$ is locally constant. If X is connected, the cardinality of $p^{-1}(x)$ is constant.

Notation. Let X, Y be topological spaces, $x \in X$, and $y \in Y$. A continuous map

$$f : (X, x) \rightarrow (Y, y)$$

is a continuous map $f : X \rightarrow Y$ such that $f(x) = y$.

Proposition 1.26. *Let X, \tilde{X} be path-connected and*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a covering space. Then the number of sheets of p equals the index of $p_ \left(\pi_1 \left(\tilde{X}, \tilde{x}_0 \right) \right)$ in $\pi_1(X, x_0)$.*

Proof. Let g be a loop in X at x_0 and \tilde{g} be its lift to \tilde{X} starting at \tilde{x}_0 . Let $H = p_* \left(\pi_1 \left(\tilde{X}, \tilde{x}_0 \right) \right)$ and let $[h] \in H$. Then $h \cdot g$ lifts to a path $\tilde{h} \cdot \tilde{g}$ in \tilde{X} starting at \tilde{x}_0 with the same endpoint as \tilde{g} , because \tilde{h} is a loop, by Proposition 1.25. Define

$$\begin{aligned} \Phi : \{ \text{cosets of } H \text{ in } \pi_1(X, x_0) \} &\rightarrow p^{-1}(x_0) \\ H[g] &\mapsto \tilde{g}(1) \end{aligned},$$

so Φ is well-defined. Want to show that Φ is bijective.

- Φ is surjective because \tilde{X} is path-connected. Let \tilde{g} be a path in \tilde{X} from \tilde{x}_0 to any point $\tilde{x}'_0 \in p^{-1}(x_0)$, then $g = p \cdot \tilde{g}$ and $\Phi(H[g]) = \tilde{x}'_0$.
- Φ is injective, since if $\Phi(H[g_1]) = \Phi(H[g_2])$ then the lift $\tilde{g}_1 \cdot \tilde{g}_2^{-1}$ of $g_1 \cdot g_2^{-1}$ defines a loop in \tilde{X} at \tilde{x}_0 . Proposition 1.25 gives $[g_1][g_2]^{-1} \in H$, so $H[g_1] = H[g_2]$.

□

We say that a topological space X has a certain property (P) **locally** if for each point $x \in X$ and each neighbourhood U of x there is an open neighbourhood $V \subseteq U$ having this property (P) .

Example. X is locally path-connected or X is locally simply-connected.

Proposition 1.27. *Let*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a covering space and

$$f : (Y, y_0) \rightarrow (X, x_0)$$

a continuous map, where Y is path-connected and locally path-connected. Then there is a lift

$$\tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$$

if and only if $f_ (\pi_1(Y, y_0)) \subseteq p_* (\pi_1(\tilde{X}, \tilde{x}_0))$.*

$$\begin{array}{ccc} & & (\tilde{X}, \tilde{x}_0) \\ & \nearrow \tilde{f} & \downarrow p \\ (Y, y_0) & \xrightarrow{f} & (X, x_0) \end{array}.$$

Proof.

\Rightarrow Clear, because $f = p\tilde{f}$ implies $f_* = p_*\tilde{f}_*$.

\Leftarrow Assume $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. For each $y \in Y$ choose a path γ from y_0 to y , so $f\gamma$ is a path in X from x_0 to $f(y)$. By path lifting, we can lift $f\gamma$ to a path $\tilde{f}\gamma$ in \tilde{X} starting at \tilde{x}_0 . Define the map

$$\begin{array}{ccc} \tilde{f}: (Y, y_0) & \rightarrow & (\tilde{X}, \tilde{x}_0) \\ y & \mapsto & \tilde{f}\gamma(1) \end{array} .$$

$$\begin{array}{ccc} & \tilde{x}_0 & \xrightarrow[\tilde{f}\gamma']{\tilde{f}\gamma} \tilde{f}(y) \\ & \nearrow \tilde{f} & \downarrow p \\ y_0 & \xrightarrow[\gamma']{\gamma} y & \xrightarrow{f} x_0 \xrightarrow[\gamma']{\tilde{f}\gamma} f(y) \end{array} .$$

- This map is well-defined, that is does not depend on the choice of γ . Let γ' be another path from y_0 to y . Then $h_0 = (f\gamma') \cdot (f\gamma)^{-1}$ is a loop at x_0 and $[h_0] \in f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Proposition 1.25 gives that can lift h_0 to a loop \tilde{h}_0 at \tilde{x}_0 . The first half of \tilde{h}_0 is $\tilde{f}\gamma'$ and the second half is $\tilde{f}\gamma^{-1}$, so $\tilde{f}\gamma(1) = \tilde{f}\gamma'(1)$. Thus \tilde{f} is well-defined.
- We have $p\tilde{f} = f$, so \tilde{f} lifts f .
- It remains to show that \tilde{f} is continuous. Let $y \in Y$ and let U be an evenly covered neighbourhood of $f(y)$. Let \tilde{U} be the sheet above U such that $\tilde{f}(y) \in \tilde{U}$, so $p|_{\tilde{U}}: \tilde{U} \rightarrow U$ is a homeomorphism. Let $V \subseteq Y$ be a path-connected neighbourhood of y such that $f(V) \subseteq U$. Fix a path γ from y_0 to y . Let $y' \in V$ be arbitrary and η be a path from y to y' , so $\gamma \cdot \eta$ is a path from y_0 to y' . Then $(f\gamma) \cdot (f\eta)$ is a path in U from x_0 to $f(y')$. $\tilde{f}\eta = (p|_{\tilde{U}})^{-1}f\eta$, so $\tilde{f}|_V = (p|_{\tilde{U}})^{-1}f$. Thus $\tilde{f}|_V: V \rightarrow \tilde{U}$ is continuous, so \tilde{f} is continuous.

□

Lecture 13
Friday
08/02/19

1.3.2 The classification of covering spaces

Definition. A covering space $p: \tilde{X} \rightarrow X$ is a **universal cover** if \tilde{X} is simply-connected.

Definition. A topological space X is **semilocally simply-connected** if each $x \in X$ has a neighbourhood U such that

$$i_*: \pi_1(U, x) \rightarrow \pi_1(X, x)$$

is trivial, where $i: U \hookrightarrow X$ is the inclusion.

Example. Let $X = \bigcup_n C_n \subseteq \mathbb{R}^2$ be the Hawaiian earrings, where $C_n \subseteq \mathbb{R}^2$ is the circle of radius $1/n$ and centre $(1/n, 0)$. Then X is not semilocally simply-connected.

Proposition 1.28. *If $p: \tilde{X} \rightarrow X$ is a universal cover, then X is semilocally simply-connected.*

Proof. Let $U \subseteq X$ be an evenly covered neighbourhood of $x_0 \in X$, $\tilde{U} \subseteq \tilde{X}$ be a sheet over U , and $\gamma \subseteq U$ be a loop at x_0 , so γ lifts to a loop $\tilde{\gamma} \subseteq \tilde{U}$ at \tilde{x}_0 . $\tilde{\gamma}$ is homotopic to the constant loop at \tilde{x}_0 . Compose this homotopy with p gives that γ is homotopic to the constant loop at x_0 in X , so

$$\pi_1(U, x_0) \rightarrow \pi_1(X, x_0)$$

is trivial. □

Theorem 1.29. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there exists a universal cover $p : \tilde{X} \rightarrow X$.*

Remark. If

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

is a universal cover, each point $\tilde{x} \in \tilde{X}$ can be joined to \tilde{x}_0 by a unique homotopy class of paths, by Proposition 1.6.

$$\{\text{points in } \tilde{X}\} \longleftrightarrow \{[\gamma] \mid \gamma \text{ is a path in } \tilde{X} \text{ starting at } \tilde{x}_0\} \longleftrightarrow \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

by the homotopy lifting property.

Proof. Let $x_0 \in X$, and

$$\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}, \quad p : \begin{array}{ccc} \tilde{X} & \rightarrow & X \\ [\gamma] & \mapsto & \gamma(1) \end{array}.$$

Have to

1. give \tilde{X} a topology,
2. show that $p : \tilde{X} \rightarrow X$ is a covering, and
3. show that \tilde{X} is simply-connected.

Recall that a **basis** for a topology on a set Y is a collection \mathcal{B} of subsets such that

- $Y = \bigcup_{U \in \mathcal{B}} U$, and
- if $U_1, U_2 \in \mathcal{B}$ and $y \in U_1 \cap U_2$ then there exists $V \in \mathcal{B}$ such that $y \in V$ and $V \subseteq U_1 \cap U_2$.

A basis defines a topology on Y , by $A \subseteq Y$ is open if and only if A is the union of elements of \mathcal{B} . A map $f : Z \rightarrow Y$ is continuous if and only if $f^{-1}(U)$ is open for all $U \in \mathcal{B}$.

1. Let \mathcal{U} be the collection of all path-connected open sets $U \subseteq X$ such that $\pi_1(U) \rightarrow \pi_1(X)$ is trivial. Then $X = \bigcup_{U \in \mathcal{U}} U$ because X is semilocally simply-connected. Let $U_1, U_2 \in \mathcal{U}$ and $y \in U_1 \cap U_2$, and let $y \in V \subseteq U_1 \cap U_2$ be path-connected and open.

$$\begin{array}{ccccc} V & \hookrightarrow & U_1 & \hookrightarrow & X \\ & & & & \\ \pi_1(V) & \longrightarrow & \pi_1(U_1) & \xrightarrow{\text{trivial}} & \pi_1(X) \\ & & \searrow & \text{trivial} & \nearrow \end{array},$$

so $V \in \mathcal{U}$ gives that \mathcal{U} is a basis for the topology on X . For $U \in \mathcal{U}$ and γ a path in X from x_0 to a point in U , we define

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ a path in } U \text{ such that } \eta(0) = \gamma(1)\} \subseteq \tilde{X}.$$

$U_{[\gamma]}$ only depends on the class $[\gamma]$, so $p|_{U_{[\gamma]}} : U_{[\gamma]} \rightarrow U$ is bijective. Surjective because U is path-connected and injective because all paths η in U with the same endpoint are homotopic. Claim that $\{U_{[\gamma]}\}$ forms a basis on \tilde{X} .

- $\bigcup_{U \in \mathcal{U}} U_{[\gamma]} = \tilde{X}$, because $\bigcup_{U \in \mathcal{U}} U = X$.
- Observe that if $[\gamma'] \in U_{[\gamma]}$ then $U_{[\gamma]} = U_{[\gamma']}$. If $\gamma' = \gamma \cdot \eta$ for η a path in U , then elements in $U_{[\gamma']}$ have the form $[\gamma' \cdot \mu]$, so $U_{[\gamma']} \subseteq U_{[\gamma]}$. Elements in $U_{[\gamma]}$ have the form $[\gamma \cdot \mu] = [\gamma \cdot \eta \cdot \eta^{-1} \cdot \mu] = [\gamma' \cdot \eta^{-1} \cdot \mu]$, so $U_{[\gamma]} \subseteq U_{[\gamma']}$. Consider $U_{[\gamma]}$ and $U_{[\gamma']}$ and let $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$, so $U_{[\gamma]} = U_{[\gamma']}$ and $V_{[\gamma']} = V_{[\gamma']}$. Let $W \in \mathcal{U}$ such that $W \subseteq U \cap V$ and such that $\gamma''(1) \in W$, so $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$ and $[\gamma''] \in W_{[\gamma'']}$. This proves the claim.

2. $p|_{U_{[\gamma]}}: U_{[\gamma]} \rightarrow U$ is a homeomorphism. It is bijective, let $V_{[\gamma']} \subseteq U_{[\gamma]}$ be an element of the basis, so $p(V_{[\gamma']}) = V \in \mathcal{U}$. $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$. Thus $p: \tilde{X} \rightarrow X$ is continuous. If $U \in \mathcal{U}$, then

$$p^{-1}(U) = \bigsqcup_{[\gamma]} U_{[\gamma]},$$

so $p: \tilde{X} \rightarrow X$ is a covering space.

3. Let $\tilde{x}_0 \in \tilde{X}$ be the class of the constant path at x_0 . Let $[\gamma] \in \tilde{X}$ be arbitrary. $\gamma: [0, 1] \rightarrow X$ and $\gamma(0) = x_0$. Let γ_t be the path in X defined by

$$\gamma_t(s) = \begin{cases} \gamma(s) & s \in [0, t] \\ \gamma(t) & s \in [t, 1] \end{cases}.$$

Then

$$\begin{array}{ccc} \tilde{\gamma}: & I & \rightarrow \tilde{X} \\ & t & \mapsto [\gamma_t] \end{array}$$

is a path in \tilde{X} from \tilde{x}_0 to $[\gamma]$, so \tilde{X} is path-connected. Recall that $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)$ consists of the classes of loops at x_0 in X that lifts to loops in \tilde{X} at \tilde{x}_0 . Let $[\gamma] \in p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right)$. Then γ lifts to a loop at \tilde{x}_0 by $t \mapsto [\gamma_t]$. Because it is a loop we have $\tilde{x}_0 = [\gamma_1] = [\gamma]$, so γ is homotopic to the constant loop. Thus $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right) = \{id\}$, so \tilde{X} is simply-connected. □

Lecture 14 is a problem class.

Let $p: \tilde{X} \rightarrow X$ be a covering space, so $p_*\left(\pi_1\left(\tilde{X}, \tilde{x}_0\right)\right) \subseteq \pi_1(X, x_0)$.

Proposition 1.30. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \subseteq \pi_1(X, x_0)$ there is a covering space $p: X_H \rightarrow X$ such that $p_*\left(\pi_1(X_H, \tilde{x}_0)\right) = H$ for some basepoint x_0 .*

Proof. Let \tilde{X} be as constructed above. Define $X_H = \tilde{X}/\sim$, where $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot (\gamma')^{-1}] \in H$. This is an equivalence relation.

- $[\gamma] \sim [\gamma]$ because $id \in H$.
- $[\gamma] \sim [\gamma']$ gives $[\gamma'] \sim [\gamma]$ because H contains all its inverses.
- $[\gamma] \sim [\gamma']$ and $[\gamma'] \sim [\gamma'']$ gives $[\gamma] \sim [\gamma'']$ because H is closed under product.

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \tilde{X} \\ \downarrow & \swarrow p & \sim \\ X & & X_H \end{array}.$$

Let $U_{[\gamma]}, U_{[\gamma']}$ be basis neighbourhoods. If $[\gamma] \sim [\gamma']$ then $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$, so p is a covering space, and $p^{-1}(U) = \bigcup_{[\gamma]} U_{[\gamma]}$. Let $\tilde{x}_0 \in X_H$ be the equivalence class of the constant path c_{x_0} at x_0 . Let γ be a loop in X at x_0 such that $[\gamma] \in p_*\left(\pi_1(X_H, \tilde{x}_0)\right)$. Again $t \mapsto [\gamma_t]$ is a lift of γ at \tilde{x}_0 .

$$t \mapsto [\gamma_t] \text{ is a loop in } X_H \iff [\gamma_1] = [\gamma] = [c_{x_0}] \text{ in } X_H \iff [\gamma] \sim [c_{x_0}] \iff \gamma \in H.$$

□

Definition. We say that two covering spaces $p_1 : \widetilde{X}_1 \rightarrow X$ and $p_2 : \widetilde{X}_2 \rightarrow X$ are **isomorphic** if there exists a homeomorphism $f : \widetilde{X}_1 \rightarrow \widetilde{X}_2$ such that

$$\begin{array}{ccc} \widetilde{X}_1 & \xrightarrow{f} & \widetilde{X}_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array} .$$

Proposition 1.31. Let X be path-connected and locally path-connected and $x_0 \in X$. Two path-connected covering spaces $p_1 : \widetilde{X}_1 \rightarrow X$ and $p_2 : \widetilde{X}_2 \rightarrow X$ are isomorphic via an isomorphism $f : \widetilde{X}_1 \rightarrow \widetilde{X}_2$ mapping a basepoint $\widetilde{x}_1 \in p_1^{-1}(x_0)$ to a basepoint $\widetilde{x}_2 \in p_2^{-1}(x_0)$ if and only if

$$(p_1)_* \left(\pi_1 \left(\widetilde{X}_1, \widetilde{x}_1 \right) \right) = (p_2)_* \left(\pi_1 \left(\widetilde{X}_2, \widetilde{x}_2 \right) \right) .$$

Proof.

\Rightarrow If

$$f : \left(\widetilde{X}_1, \widetilde{x}_1 \right) \rightarrow \left(\widetilde{X}_2, \widetilde{x}_2 \right)$$

is an isomorphism, then $p_1 = p_2 f$, so

$$(p_1)_* \left(\pi_1 \left(\widetilde{X}_1, \widetilde{x}_1 \right) \right) \subseteq (p_2)_* \left(\pi_1 \left(\widetilde{X}_2, \widetilde{x}_2 \right) \right) ,$$

and $p_2 = p_1 f^{-1}$, so

$$(p_2)_* \left(\pi_1 \left(\widetilde{X}_2, \widetilde{x}_2 \right) \right) \subseteq (p_1)_* \left(\pi_1 \left(\widetilde{X}_1, \widetilde{x}_1 \right) \right) .$$

\Leftarrow Assume

$$(p_1)_* \left(\pi_1 \left(\widetilde{X}_1, \widetilde{x}_1 \right) \right) = (p_2)_* \left(\pi_1 \left(\widetilde{X}_2, \widetilde{x}_2 \right) \right) .$$

By lifting criterion in Proposition 1.27, we can lift p_1 to a continuous map

$$\widetilde{p}_1 : \left(\widetilde{X}_1, \widetilde{x}_1 \right) \rightarrow \left(\widetilde{X}_2, \widetilde{x}_2 \right) ,$$

and p_2 to a continuous map

$$\widetilde{p}_2 : \left(\widetilde{X}_2, \widetilde{x}_2 \right) \rightarrow \left(\widetilde{X}_1, \widetilde{x}_1 \right) ,$$

so $p_1 \widetilde{p}_2 = p_2$ and $p_2 \widetilde{p}_1 = p_1$.

$$\begin{array}{ccc} \left(\widetilde{X}_1, \widetilde{x}_1 \right) & \xrightarrow{\widetilde{p}_1} & \left(\widetilde{X}_2, \widetilde{x}_2 \right) \\ & \searrow \widetilde{p}_2 & \swarrow \widetilde{p}_1 \\ & (X, x_0) & \end{array} .$$

$\widetilde{p}_1 \widetilde{p}_2$ fixes the point $\widetilde{x}_2 \in \widetilde{X}_2$. By the unique lifting property in Proposition 1.7, $\widetilde{p}_1 \widetilde{p}_2 = id_{\widetilde{x}_2}$. Similarly, $\widetilde{p}_2 \widetilde{p}_1 = id_{\widetilde{x}_1}$, so \widetilde{p}_1 is an isomorphism.

□

Fix $x_0 \in X$, $\widetilde{x}_1 \in p_1^{-1}(x_0)$, and $\widetilde{x}_2 \in p_2^{-1}(x_0)$. A **basepoint preserving isomorphism**

$$f : \left(\widetilde{X}_1, \widetilde{x}_1 \right) \rightarrow \left(\widetilde{X}_2, \widetilde{x}_2 \right)$$

is an isomorphism such that $f(\widetilde{x}_1) = \widetilde{x}_2$.

Theorem 1.32 (Galois correspondence). *Let X be path-connected, locally path-connected, and semilocally simply-connected, and $x_0 \in X$. Then*

1. *there is a bijection*

$$\left\{ \begin{array}{l} \text{path-connected covering spaces } p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0) \\ \text{up to basepoint preserving isomorphisms} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{subgroups} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\},$$

2. *if we ignore the basepoints, this correspondence gives a bijection*

$$\left\{ \begin{array}{l} \text{path-connected covering spaces } p : \tilde{X} \rightarrow X \\ \text{up to isomorphisms} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{l} \text{conjugacy classes of subgroups} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\}.$$

Proof.

1. To a covering space $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ we associate the subgroup $p_* \left(\pi_1(\tilde{X}, \tilde{x}_0) \right) \subseteq \pi_1(X, x_0)$. Proposition 1.30 and Proposition 1.31 show that this is well-defined on the isomorphism classes and it is bijective.
2. Let $p : \tilde{X} \rightarrow X$ be a covering space and $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$. Let $H_i = p_* \left(\pi_1(\tilde{X}, \tilde{x}_i) \right) \subseteq \pi_1(X, x_0)$, for $i = 1, 2$. Let $\tilde{\gamma}$ be a path from \tilde{x}_1 to \tilde{x}_2 . Let $\gamma = p\tilde{\gamma}$ be a loop at x_0 . Let $[f] \in \pi_1(X, x_0)$. Then $[f] \in H_1$ if and only if the lift \tilde{f} is a loop at \tilde{x}_1 . $\tilde{\gamma}^{-1} \cdot \tilde{f} \cdot \tilde{\gamma}$ is a loop at \tilde{x}_2 gives $p_* \left(\tilde{\gamma}^{-1} \cdot \tilde{f} \cdot \tilde{\gamma} \right) = \gamma^{-1} \cdot f \cdot \gamma$, so $[\gamma]^{-1} [f] [\gamma] \in H_2$. Thus $[\gamma]^{-1} H_1 [\gamma] \subseteq H_2$. Similarly, $[\gamma] H_2 [\gamma]^{-1} \subseteq H_1$. Conversely, let $H_1 \subseteq \pi_1(X, x_0)$ as above and $[\delta] \in \pi_1(X, x_0)$ be an arbitrary element. Let $\tilde{\delta}$ be a lift of δ such that $\tilde{\delta}(0) = \tilde{x}_0$ and define $x_3 = \tilde{\delta}(1)$. Then the same construction yields $p_* \left(\pi_1(\tilde{X}, \tilde{x}_3) \right) = [\delta]^{-1} H_1 [\delta]$.

□

1.3.3 Deck transformations and group actions

Definition. Let $p : \tilde{X} \rightarrow X$ be a covering space. A **deck-transformation** is an isomorphism from \tilde{X} to itself.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & \tilde{X} \\ & \searrow p \quad \swarrow p & \\ & X & \end{array}.$$

The group of deck-transformations is denoted by $G(\tilde{X})$.

Example.

- Let

$$p : \begin{array}{ccc} \mathbb{R} & \rightarrow & S^1 \subset \mathbb{C} \\ t & \mapsto & e^{2\pi i t} \end{array}.$$

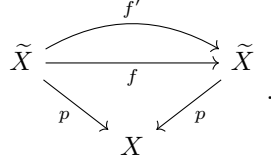
$f : \mathbb{R} \rightarrow \mathbb{R}$ such that $p(f(t)) = p(t)$ if and only if $e^{2\pi i f(t)} = e^{2\pi i t}$, if and only if $f(t) = t + n$, so $G(\mathbb{R}) \cong \mathbb{Z}$.

- Let

$$p : \begin{array}{ccc} S^1 & \rightarrow & S^1 \\ z & \mapsto & z^n \end{array}.$$

Then $G(S^1) \cong \mathbb{Z}/n\mathbb{Z}$.

An observation is that if \tilde{X} is path-connected then $f \in G(\tilde{X})$ is uniquely determined by where it sends a single point.



If $f(x) = f'(x)$ for a single x , by unique lifting $f = f'$. So the identity is the only deck-transformation with a fixed point.

Definition. A covering space $p : \tilde{X} \rightarrow X$ is **normal**, or **regular**, or **Galois**, if for each $x \in X$ and every pair $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there is an $f \in G(\tilde{X})$ such that $f(\tilde{x}) = \tilde{x}'$.

Example.

- $p : \mathbb{R} \rightarrow S^1$ is normal.
- $p : S^1 \rightarrow S^1$ is normal.

Proposition 1.33. *Let*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a path-connected covering space, and X be path-connected and locally path-connected. Then $p : \tilde{X} \rightarrow X$ is normal if and only if $H = p_ \left(\pi_1(\tilde{X}, \tilde{x}_0) \right) \subseteq \pi_1(X, x_0)$ is a normal subgroup.*

Proof. Let $\tilde{x}_1 \in p^{-1}(x_0)$, let $\tilde{\gamma}$ be a path from \tilde{x}_0 to \tilde{x}_1 and $\gamma = p(\tilde{\gamma})$. Then $[\gamma]$ conjugates H to $p_* \left(\pi_1(\tilde{X}, \tilde{x}_1) \right)$ so $[\gamma] H [\gamma]^{-1} = H$, if and only if $H = p_* \left(\pi_1(\tilde{X}, \tilde{x}_1) \right)$, by Proposition 1.31 if and only if $f(\tilde{x}_0) = \tilde{x}_1$. So $G(\tilde{X})$ acts transitively on $p^{-1}(x_0)$ if and only if $H \subseteq \pi_1(X, x_0)$ is a normal subgroup. Let $x'_0 \in X$ be another point and h a path from x_0 to x'_0 . Let \tilde{h} be a lift of h such that $\tilde{h}(0) = \tilde{x}_0$. Set $\tilde{x}'_0 = \tilde{h}(1)$ and $p(\tilde{x}'_0) = x'_0$. Then

$$\begin{array}{ccc}
 \pi_1(\tilde{X}, \tilde{x}_0) & \xrightarrow{\beta_{\tilde{h}}} & \pi_1(\tilde{X}, \tilde{x}'_0) \\
 p_* \downarrow & & \downarrow p_* \\
 \pi_1(X, x_0) & \xrightarrow{\beta_h} & \pi_1(X, x'_0)
 \end{array}$$

$H \subseteq \pi_1(X, x_0)$ is normal if and only if $p_* \left(\pi_1(\tilde{X}, \tilde{x}'_0) \right) \subseteq \pi_1(X, x'_0)$ is normal, as before if and only if $G(\tilde{X})$ acts transitively on $p^{-1}(x'_0)$. \square

Proposition 1.34. *Let*

$$p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

be a covering space, and X, \tilde{X} be path-connected and locally path-connected. Let $H = p_ \left(\pi_1(\tilde{X}, \tilde{x}_0) \right)$ and $N(H) \subseteq \pi_1(X, x_0)$ be the normaliser of H . Then $G(\tilde{X})$ is isomorphic to $N(H)/H$. In particular,*

- if \tilde{X} is normal, then $G(\tilde{X}) \cong \pi_1(X, x_0)/H$, and
- if \tilde{X} is the universal cover, then $G(\tilde{X}) \cong \pi_1(X, x_0)$.

Proof. Exercise: read the proof of this in Hatcher. \square

Example. Let $X = S^1 \vee S^1$, so $\pi_1(X) = F_{\{a,b\}}$. Then the following are covering spaces.

- A normal covering space

$$\tilde{X} = \overset{a}{\underset{\cdot}{\circ}} \cdot \overset{b}{\underset{\cdot}{\circ}} \xrightarrow{\widetilde{x_0}} \overset{a}{\underset{\cdot}{\circ}} \cdot \overset{b}{\underset{\cdot}{\circ}} \rightarrow \overset{a}{\underset{\cdot}{\circ}} \cdot \overset{b}{\underset{\cdot}{\circ}} = X, \quad p_* \left(\pi_1 \left(\tilde{X}, \widetilde{x_0} \right) \right) = \langle a, b^2, bab^{-1} \rangle \stackrel{2}{\subseteq} F_{\{a,b\}}.$$

In general, a two-oriented graph is a covering space of X .

- Not a normal covering space

$$\tilde{X} = \overset{a}{\underset{\cdot}{\circ}} \cdot \overset{b}{\underset{\cdot}{\circ}} \xrightarrow{\widetilde{x_0}} \overset{a}{\underset{\cdot}{\circ}} \cdot \overset{b}{\underset{\cdot}{\circ}} \rightarrow \overset{a}{\underset{\cdot}{\circ}} \cdot \overset{b}{\underset{\cdot}{\circ}} = X, \quad p_* \left(\pi_1 \left(\tilde{X}, \widetilde{x_0} \right) \right) = \langle b^2, bab^{-1}, a^2, aba^{-1} \rangle.$$

- A normal covering space

$$\tilde{X} = \dots \underset{b}{\overset{a}{\underset{\cdot}{\circ}}} \underset{b}{\overset{a}{\underset{\cdot}{\circ}}} \underset{b}{\overset{a}{\underset{\cdot}{\circ}}} \cdots \rightarrow \overset{a}{\underset{\cdot}{\circ}} \cdot \overset{b}{\underset{\cdot}{\circ}} = X, \quad p_* \left(\pi_1 \left(\tilde{X}, \widetilde{x_0} \right) \right) = \langle b^n ab^{-n} \mid n \in \mathbb{Z} \rangle.$$

Universal cover is a tree.

Example. Let $T = S^1 \times S^1$, so $\pi_1(T) = \mathbb{Z}^2$. This is abelian, so all covering spaces are normal. Universal cover is

$$\begin{aligned} \mathbb{R}^2 &\rightarrow S^1 \times S^1 \\ (s, t) &\mapsto (e^{2\pi i s}, e^{2\pi i t}) \end{aligned}$$

since \mathbb{R}^2 is simply connected. (Exercise: check that it is a covering space) More generally, if $p : \tilde{X} \rightarrow X$ and $q : \tilde{Y} \rightarrow Y$ are covering spaces then

$$\begin{aligned} \tilde{X} \times \tilde{Y} &\rightarrow X \times Y \\ (x, y) &\mapsto (p(x), q(y)) \end{aligned}$$

is again a covering space. For example,

$$\begin{aligned} S^1 \times S^1 &\rightarrow S^1 \times S^1 \\ (z_1, z_2) &\mapsto (z_1^n, z_2^m) \end{aligned}$$

Example. Worksheet 3 exercise 7. Let

$$\mathbb{RP}^n = \frac{\mathbb{R}^{n+1} \setminus \{0\}}{\sim} = \frac{S^n}{\sim}$$

be the **projective n -space**, the space of all lines through the origin in \mathbb{R}^{n+1} , where $x \sim -x$. Let $p : S^n \rightarrow \mathbb{RP}^n$ be the quotient map. Claim that this is a covering space. Let $[x] \in \mathbb{RP}^n$. Then $p^{-1}([x]) = \{\pm x\}$. Let U be an open neighbourhood of x such that $U \cap (-U) = \emptyset$, so $p(U) = \{[x] \mid x \in U\}$. Then $p^{-1}(p(U)) = U \cup (-U)$ is open and disjoint. Thus $p|_U : U \rightarrow p(U)$ is a homeomorphism, so it is a covering space.

- $n \geq 2$ gives that S^n is simply-connected, so $S^n \rightarrow \mathbb{RP}^n$ is a universal cover. Then

$$\{id\} = p_* \left(\pi_1(S^n) \right) \stackrel{2}{\subseteq} \pi_1(\mathbb{RP}^n),$$

so $|\pi_1(\mathbb{RP}^n)| = 2$. Thus $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$.

- $n = 1$ gives $\mathbb{RP}^1 = S^1$, so

$$\begin{aligned} p : S^1 &\rightarrow S^1 \\ z &\mapsto z^2 \end{aligned}$$

is a covering space.

2 Homology

Higher homotopy groups $\pi_n(X, x_0)$ are groups of basepoint preserving homotopies of continuous $\phi : I^n \rightarrow X$ such that $\phi(\partial I^n) = x_0$.

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Example.

$$\pi_1(S^n) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}, \quad \pi_2(S^n) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases},$$

$$\pi_3(S^n) = \begin{cases} \mathbb{Z} & n = 2, 3 \\ 0 & \text{otherwise} \end{cases}, \quad \pi_i(S^2) = \begin{cases} \mathbb{Z} & i = 4, 5 \\ 2\mathbb{Z} & i = 6 \\ 12\mathbb{Z} & i = 6 \end{cases}.$$

Homology is more suitable. The following is the plan.

- Simplicial homology.
- Singular homology.
- Technical machinery to show that they coincide.
- Applications.

2.1 Simplicial and singular homology

2.1.1 Δ -complexes

Definition. Let $m, n \geq 0$.

- An **n -simplex** in \mathbb{R}^m is the convex hull of a set V of $n + 1$ points in \mathbb{R}^m that are not all contained in an affine $(n - 1)$ -dimensional subspace of \mathbb{R}^m .
- The **standard n -simplex** is the convex hull of the standard basis $\{e_1, \dots, e_{n+1}\}$ in \mathbb{R}^{n+1} ,

$$\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, x_0 + \dots + x_n = 1\}.$$

- An **ordered n -simplex** is an n -simplex with an ordering on the vertices. We denote it by $[v_0, \dots, v_n]$, where v_0, \dots, v_n are the vertices in ascending order.
- The **standard ordered n -simplex** is the ordered n -simplex $[e_1, \dots, e_{n+1}]$ in \mathbb{R}^{n+1} . It is denoted by Δ^n .
- Let $[v_0, \dots, v_{n+1}]$ be an n -simplex in \mathbb{R}^m and let $L \subseteq \mathbb{R}^m$ be the affine subspace spanned by v_0, \dots, v_n . Then there exists a unique affine morphism

$$\begin{aligned} L &\rightarrow \mathbb{R}^{n+1} \\ v_i &\mapsto e_{i+1} \end{aligned},$$

for $i = 0, \dots, n$. This gives a homeomorphism from $[v_0, \dots, v_n]$ to Δ^n that preserves this ordering.

- For $n \geq 1$, the **faces** of an ordered n -simplex $[v_0, \dots, v_n]$ are the ordered $(n - 1)$ -simplices

$$[v_0, \dots, \widehat{v_i}, \dots, v_n].$$

$\widehat{v_i}$ means we omit the vertex v_i .

- The union of all the faces of a simplex Δ is the **boundary** $\partial\Delta$.
- The **interior** of Δ is $\overset{\circ}{\Delta} = \Delta \setminus \partial\Delta$.

Example. Let $\Delta^2 = [e_1, e_2, e_3]$. Then $\partial\Delta^2 = [e_1, e_2] \cup [e_1, e_3] \cup [e_2, e_3]$.

Definition. Let X be a topological space. A Δ -complex structure on X is a collection of continuous maps $\sigma_\alpha : \Delta^{n(\alpha)} \rightarrow X$ for $\alpha \in A$ and $n(\alpha) \in \mathbb{N}$ such that

1. the restriction $\sigma_\alpha|_{\hat{\Delta}^{n(\alpha)}}$ is injective for all $\alpha \in A$ and for each $x \in X$ there is a unique $\alpha \in A$ such that $x \in \sigma_\alpha(\hat{\Delta}^{n(\alpha)})$,
2. the restriction of σ_α to a face of $\Delta^{n(\alpha)}$ is equal to σ_β for some $\beta \in A$ and $n(\beta) = n(\alpha) - 1$, and
3. $U \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(U)$ is open in $\Delta^{n(\alpha)}$ for all $\alpha \in A$.

An observation is that

$$\sigma : \bigsqcup_{\alpha \in A} \Delta^{n(\alpha)} \rightarrow X$$

induced by the σ_α is a quotient map, since it is surjective by 1 and $U \subseteq X$ is open if and only if $\sigma^{-1}(U)$ is open by 3.

Remark. One can show that an X with a Δ -complex structure is a CW-complex.

Example.

- Torus or Klein bottle is two Δ^2 , three Δ^1 , and one Δ^0 .
- S^2 is a tetrahedron.
- Dunce hat, by identifying all the three faces of the standard 2-simplex with each other, has one Δ^2 , one Δ^1 , and one Δ^0 .

2.1.2 Simplicial homology

Let X be a Δ -complex. The group of n -chains $\Delta_n(X)$ is the free abelian group on the n -simplices $\sigma_\alpha : \Delta^{n(\alpha)} \rightarrow X$, where $n(\alpha) = n$. So an element in $\Delta_n(X)$ is of the form

$$\sum_{\alpha \in A, n(\alpha)=n} c_\alpha \cdot \sigma_\alpha,$$

where $c_\alpha \in \mathbb{Z}$ and all but finitely many of the c_α are zero.

Example. Let K be a Klein bottle.

- $\Delta_0(K) = \{n \cdot v \mid n \in \mathbb{Z}\} = \mathbb{Z} \cdot v \cong \mathbb{Z}$.
- $\Delta_1(K) = \{n_1 \cdot a + n_2 \cdot b + n_3 \cdot c \mid n_1, n_2, n_3 \in \mathbb{Z}\} = \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot b \oplus \mathbb{Z} \cdot c \cong \mathbb{Z}^3$.
- $\Delta_2(K) = \{n_1 \cdot U + n_2 \cdot V \mid n_1, n_2 \in \mathbb{Z}\} = \mathbb{Z} \cdot U \oplus \mathbb{Z} \cdot V \cong \mathbb{Z}^2$.
- $\Delta_n(K) = 0$ for $n \geq 3$.

Similarly for a torus T .

Define the **boundary homomorphism** by

$$\begin{aligned} \partial_n : \Delta_n(X) &\rightarrow \Delta_{n-1}(X) \\ \sigma_\alpha &\mapsto \sum_{i=0}^n (-1)^i \sigma_\alpha|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]} \end{aligned}$$

Moreover, we define $\partial_0 = 0$.

Example. Let $\sigma : [v_0, v_1, v_2, v_3] \rightarrow X$. Then

$$\partial_3(\sigma) = \sigma|_{[v_1, v_2, v_3]} - \sigma|_{[v_0, v_2, v_3]} + \sigma|_{[v_0, v_1, v_3]} - \sigma|_{[v_0, v_1, v_2]}.$$

Lemma 2.1. *The composition*

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

is the zero map.

Proof. Let $\sigma : [v_0, \dots, v_n] \rightarrow X$ be an n -simplex. Then

$$\partial_n(\sigma) = \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]},$$

so

$$(\partial_{n-1} \circ \partial_n)(\sigma) = \sum_{j < i} (-1)^i (-1)^j \sigma|_{[v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_n]} + \sum_{j > i} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_n]} = 0.$$

If $n = 1$, clear. □

The following is the algebraic situation. A **chain complex** of abelian groups is a diagram $(C., \partial)$ of the form

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0,$$

where the C_i are abelian groups and the ∂_n are group homomorphisms such that $\partial_n \circ \partial_{n-1} = 0$ for all n . ∂_n are **boundary homomorphisms**. Elements in C_n are **n -chains**.

$$Z_n = \text{Ker}(\partial_n) \subseteq C_n, \quad B_n = \text{Im}(\partial_{n+1}) \subseteq C_n.$$

Elements in Z_n are **cycles** and elements in B_n are **boundaries**. Since $\partial_{n+1} \circ \partial_n = 0$, we have that $B_n \subseteq Z_n$. The **n -th homology group** of this chain complex is defined by

$$H_n(C., \partial) = \frac{Z_n}{B_n}.$$

So, by Lemma 2.1

$$\dots \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0$$

is a chain complex. The **n -th simplicial homology group** is

$$H_n^\Delta(X) = \frac{\text{Ker}(\partial_n)}{\text{Im}(\partial_{n+1})}.$$

Example. Let $X = S^1$.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_3} & \Delta_2(X) & \xrightarrow{\partial_2} & \Delta_1(X) & \xrightarrow{\partial_1} & \Delta_0(X) \xrightarrow{\partial_0} 0 \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ & & 0 & & \mathbb{Z} & & \mathbb{Z} \end{array}.$$

- $\text{Ker}(\partial_0) = \mathbb{Z}$ and $\text{Im}(\partial_1) = 0$, so $H_0^\Delta(X) \cong \mathbb{Z}$.
- $\text{Ker}(\partial_1) = \Delta_1(X)$ and $\text{Im}(\partial_2) = 0$, so $H_1^\Delta(X) \cong \mathbb{Z}$.
- $H_n^\Delta(X) = 0$ if $n \geq 2$.

Example. Let T be a torus.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_4} & \Delta_3(T) & \xrightarrow{\partial_3} & \Delta_2(T) & \xrightarrow{\partial_2} & \Delta_1(T) \xrightarrow{\partial_1} \Delta_0(T) \xrightarrow{\partial_0} 0 \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ & & 0 & & \mathbb{Z} \cdot U \oplus \mathbb{Z} \cdot V & & \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot b \oplus \mathbb{Z} \cdot c \\ & & & & & & \mathbb{Z} \cdot v \end{array}.$$

- $\text{Ker}(\partial_0) = \mathbb{Z}$ and $\text{Im}(\partial_1) = 0$, so $H_0^\Delta(T) \cong \mathbb{Z}$.
- $\partial_2(U) = a + b - c$ and $\partial_2(V) = a + b - c$, and $\{a, b, a + b - c\}$ is a basis for $\Delta_1(T)$. $\text{Ker}(\partial_1) = \Delta_1(T)$ and $\text{Im}(\partial_2) = \mathbb{Z} \cdot (a + b - c)$, so $H_1^\Delta(T) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- $H_2^\Delta(T) \cong \mathbb{Z}$. (Exercise)

Lecture 20 is a problem class.

2.1.3 Singular homology

A **singular n -simplex** in a topological space X is a continuous map $\sigma : \Delta^n \rightarrow X$. Let $C_n(X)$ be the free abelian group on the set of all singular simplices in X , that is elements in $C_n(X)$ are finite formal sums

$$\sum_i n_i \sigma_i, \quad n_i \in \mathbb{Z},$$

where $\sigma_i : \Delta^n \rightarrow X$ are singular n -simplices. Elements in $C_n(X)$ are called **singular n -chains**. Define a **boundary map**

$$\begin{aligned} \partial_n : C_n(X) &\rightarrow C_{n-1}(X) \\ \sigma &\mapsto \sum_{i=0}^n (-1)^i \sigma|_{[v_1, \dots, \tilde{v}_i, \dots, v_n]}, \end{aligned}$$

for a singular n -simplex σ . Extend it linearly to $C_n(X)$.

Lemma 2.2. $\partial_n \circ \partial_{n+1} = 0$.

Proof. Same proof as for Lemma 2.1. □

We obtain a chain complex

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

Remark. Often we write ∂ instead of ∂_n .

We define the **n -th singular homology group** by

$$H_n(X) = \frac{\text{Ker}(\partial_n)}{\text{Im}(\partial_{n+1})}.$$

An observation is that if X and Y are homeomorphic then $H_n(X) \cong H_n(Y)$.

Proposition 2.3. Let X be a topological space and $X = \bigcup_{\alpha} X_{\alpha}$ be the decomposition into its path-connected components. Then

$$H_n(X) \cong \bigoplus_{\alpha} H_n(X_{\alpha}).$$

Proof. A singular n -simplex $\sigma : \Delta^n \rightarrow X$ has a path-connected image. So

$$C_n(X) = \bigoplus_{\alpha} C_n(X_{\alpha}).$$

The boundary maps ∂_n preserve this decomposition, so $\partial_n(C_n(X_{\alpha})) \subseteq C_{n-1}(X_{\alpha})$ gives that $\text{Ker}(\partial_n)$ and $\text{Im}(\partial_{n+1})$ split as well as direct sums, so

$$H_n(X) = \frac{\text{Ker}(\partial_n)}{\text{Im}(\partial_{n+1})} \cong \bigoplus_{\alpha} H_n(X_{\alpha}).$$

□

Proposition 2.4. If X is a path-connected, and as always $X \neq \emptyset$, topological space, then $H_0(X) \cong \mathbb{Z}$. Hence for X arbitrary $H_0(X)$ is a direct sum of \mathbb{Z} 's, one for each path-connected component.

Proof. $\partial_0 = 0$, so $H_0(X) = C_0(X) / \text{Im}(\partial_1)$. Define

$$\begin{aligned} \epsilon : C_0(X) &\rightarrow \mathbb{Z} \\ \sum_i n_i \sigma_i &\mapsto \sum_i n_i. \end{aligned}$$

ϵ is surjective. Enough to show that $\text{Ker}(\epsilon) = \text{Im}(\partial_1)$. This implies by the isomorphism theorem $H_0(X) \cong \mathbb{Z}$. Let $\sigma : \Delta^1 \rightarrow X$ be a 1-simplex. Then

$$\partial_1(\sigma) = \sigma|_{[v_1]} - \sigma|_{[v_0]},$$

so $\epsilon(\partial_1(\sigma)) = 0$ gives $Im(\partial_1) \subseteq Ker(\epsilon)$. On the other hand, $\epsilon(\sum_i n_i \sigma_i) = 0$ gives $\sum_i n_i = 0$. The σ_i correspond to points $\sigma_i([v])$ in X . Choose a basepoint $x_0 \in X$ and let

$$\begin{array}{ccc} \sigma_0 : \Delta^0 & \rightarrow & X \\ \Delta^0 & \mapsto & x_0 \end{array}$$

be the singular 0-simplex. Let τ_i be a path from x_0 to $\sigma_i([v])$. Consider τ_i as a singular 1-simplex $\tau_i : [v_0, v_1] \rightarrow X$. We have $\partial_1 \circ \tau_i = \sigma_i - \sigma_0$, so

$$\partial_1 \left(\sum_i n_i \tau_i \right) = \sum_i n_i (\sigma_i - \sigma_0) = \sum_i n_i \sigma_i - \sum_i n_i \sigma_0 = \sum_i n_i \sigma_i.$$

Thus $Ker(\epsilon) \subseteq Im(\partial_1)$. □

Proposition 2.5. *If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(\mathbb{Z}) = \mathbb{Z}$.*

Proof. For each n there exists a unique singular n -simplex $\partial_n : \Delta^n \rightarrow X$, so $C_n(X) \cong \mathbb{Z}$ for all n .

$$\partial_n(\sigma_n) = \sum_{i=0}^n (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases},$$

so $\partial_n = 0$ if n is odd and ∂_n is an isomorphism if n is even.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_2} & C_1(X) & \xrightarrow{\partial_1} & C_0(X) & \xrightarrow{\partial_0} & 0 \\ & & \mathbb{Z} & \xrightarrow{\mathbb{Z}} & \mathbb{Z} & \xrightarrow{\mathbb{Z}} & \\ \dots & \xrightarrow{\sim} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\sim} & 0 \end{array},$$

so $H_n = Ker(\partial_n) / Im(\partial_{n+1}) = 0$ if $n \geq 1$ and $H_0(X) \cong \mathbb{Z}$. □

The **reduced homology groups** $\widetilde{H}_n(X)$ are the homology groups of the **augmented chain complex**

$$\dots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} \mathbb{Z} \xrightarrow{\epsilon} 0,$$

where ϵ is as in proof of Proposition 2.4.

$$H_n(X) \cong \widetilde{H}_n(X),$$

for $n \geq 1$. Seen in the proof of Proposition 2.4 that ϵ is surjective and $\epsilon \circ \partial_1 = 0$ gives $Im(\partial_1) \subseteq Ker(\epsilon)$, so ϵ induces a surjective homomorphism

$$\phi_\epsilon : H_0(X) = \frac{C_0(X)}{Im(\partial_1)} \rightarrow \mathbb{Z}.$$

Then $Ker(\phi_\epsilon) = Ker(\epsilon) / Im(\partial_1) = \widetilde{H}_0(X)$ gives $H_0(X) / \widetilde{H}_0(X) \cong \mathbb{Z}$, so

$$H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}.$$