

# M4P54 Differential Topology

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**Syllabus**

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## 0 Introduction

Differential topology is the study of the topology of a manifold using analysis. The topics are

- a review of differential forms,
- de Rham cohomology,
- Morse theory, and
- singular homology.

The following are references.

- J M Lee, Introduction to smooth manifolds, 2000
- L W Tu, Introduction to smooth manifolds, 2008
- J Milnor, Morse theory, 1960
- A Banyaga and D Hurtubise, Lectures on Morse homology, 2004

Lecture 1  
Thursday  
09/01/20

# 1 Differential forms on manifolds

## 1.1 Alternating $p$ -forms on a vector space

Let  $V$  be a vector space over  $\mathbb{R}$ , and let  $p \geq 0$ . Then  $V^p = V \times \cdots \times V$ .

**Definition 1.1.** A multilinear map  $\omega : V^p \rightarrow \mathbb{R}$  is called an **alternating  $p$ -form** if we have

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(p)}) = \epsilon(\sigma) \omega(v_1, \dots, v_p), \quad v_1, \dots, v_p \in V \quad \sigma \in \mathcal{S}_p,$$

where  $\mathcal{S}_p$  is the group of permutations of  $p$  elements and  $\epsilon(\sigma)$  is the signature of  $\sigma$ .

Recall that if  $m$  is the number of transpositions in a decomposition of  $\sigma$ , then  $\epsilon(\sigma) = (-1)^m$ , where a **transposition** is  $(a_i a_j)$  for  $a_i \neq a_j$ .

**Notation 1.2.**  $\Lambda^p V^* = \{\text{alternating } p\text{-forms } \omega \text{ on } V\}$  is called the  **$p$ -th exterior power** of  $V$ .

Check that it is a vector space.<sup>1</sup>

**Example 1.3.**

- $\Lambda^0 V^* = \mathbb{R}$ .
- $\Lambda^1 V^* = V^* = \text{Hom}(V, \mathbb{R})$ , the **dual** of  $V$ .

**Definition 1.4.** Let  $\omega_1 \in \Lambda^p V^*$  and  $\omega_2 \in \Lambda^q V^*$ . We define the **exterior product**  $\omega_1 \wedge \omega_2 \in \Lambda^{p+q} V^*$  of  $\omega_1$  and  $\omega_2$  by

$$\omega_1 \wedge \omega_2(v_1, \dots, v_{p+q}) = \sum_{\sigma \in \mathcal{S}_{p,q}} \epsilon(\sigma) \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \omega_2(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}), \quad v_1, \dots, v_{p+q} \in V,$$

where

$$\mathcal{S}_{p,q} = \{\sigma \in \mathcal{S}_{p+q} \mid \sigma(1) < \cdots < \sigma(p), \sigma(p+1) < \cdots < \sigma(p+q)\}.$$

**Example 1.5.**

- Assume  $\omega_1, \omega_2 \in \Lambda^1 V^*$ . Then

$$\omega_1 \wedge \omega_2(v_1, v_2) = \omega_1(v_1) \omega_2(v_2) - \omega_1(v_2) \omega_2(v_1), \quad v_1, v_2 \in V.$$

- Assume  $\omega_1, \dots, \omega_p \in \Lambda^1 V^*$ . Then

$$\omega_1 \wedge \cdots \wedge \omega_p(v_1, \dots, v_p) = \det(\omega_i(v_j))_{i,j=1,\dots,p}, \quad v_1, \dots, v_p \in V.$$

**Proposition 1.6.** Let  $\omega_i \in \Lambda^{p_i} V^*$  for  $i = 1, 2, 3$ .

- *Associativity*  $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$ .
- *Distributivity*  $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$ , assuming  $p_2 = p_3$ .
- *Supercommutativity*  $\omega_1 \wedge \omega_2 = (-1)^{p_1 p_2} \omega_2 \wedge \omega_1$ .

**Definition 1.7.** Let  $\Phi : V \rightarrow W$  be a linear map between vector spaces over  $\mathbb{R}$ . Let  $\omega \in \Lambda^p W^*$ . Then the **pull-back**  $\Phi^* \omega \in \Lambda^p V^*$  of  $\omega$  is an alternating  $p$ -form on  $V$  defined by

$$\Phi^* \omega(v_1, \dots, v_p) = \omega(\Phi(v_1), \dots, \Phi(v_p)), \quad v_1, \dots, v_p \in V.$$

---

<sup>1</sup>Exercise

**Proposition 1.8.** *Given  $\Phi : V \rightarrow W$  a linear map,*

- *the pull-back*

$$\begin{aligned} \Phi^* &: \Lambda^p W^* \longrightarrow \Lambda^p V^* \\ \omega &\longmapsto \Phi^* \omega \end{aligned}$$

*is a linear map that preserves exterior products, that is*

$$\Phi^* (\omega_1 \wedge \omega_2) = \Phi^* \omega_1 \wedge \Phi^* \omega_2, \quad \omega_1 \in \Lambda^p W^*, \quad \omega_2 \in \Lambda^q W^*,$$

- *if  $\Psi : W \rightarrow Z$  is linear then*

$$(\Psi \circ \Phi)^* \omega = \Phi^* \Psi^* \omega, \quad \omega \in \Lambda^p Z^*,$$

- *assuming  $V = W$  and  $p = \dim V$ , then*

$$\Phi^* \omega = (\det \Phi) \omega, \quad \omega \in \Lambda^p V^*.$$

## 1.2 Differential forms on manifolds

Let  $M$  be a smooth manifold of dimension  $n$ , and let  $x \in M$ . Then the tangent space  $T_x M$  of  $M$  at  $x$  is a vector space of dimension  $n$ .

**Notation 1.9.** Let

$$\Lambda^p T_x^* M = \Lambda^p (T_x M)^*.$$

Consider the set

$$\Lambda^p T^* M = \bigsqcup_{x \in M} \Lambda^p T_x^* M,$$

the  **$p$ -th exterior bundle** on  $M$ . There exists a morphism  $\pi : \Lambda^p T^* M \rightarrow M$  such that for all  $x \in M$ ,  $\pi^{-1}(x) = \Lambda^p T_x^* M$ , so  $\Lambda^p T^* M$  is a vector bundle and it is a smooth manifold, and  $\pi$  is a smooth morphism.

**Example 1.10.**

- $\Lambda^0 T^* M = M \times \mathbb{R}$ .
- $\Lambda^1 T^* M$  is the **cotangent bundle**, the dual of the tangent bundle.

**Definition 1.11.** A **differential  $p$ -form**  $\omega$  on  $M$  is a smooth section of  $\pi$ , that is it is a smooth morphism  $\omega : M \rightarrow \Lambda^p T^* M$  such that  $\pi \circ \omega = \text{id}_M$ .

Thus,  $\omega(x) \in \Lambda^p T_x^* M$ .

**Notation 1.12.**

$$\Omega^p(M) = \{\text{differential } p\text{-forms } \omega \text{ on } M\}, \quad \Omega^\bullet(M) = \bigoplus_p \Omega^p(M).$$

**Example 1.13.**

$$\Omega^0(M) \cong \{f : M \rightarrow \mathbb{R} \text{ } C^\infty\text{-function}\}.$$

**Exercise.** If  $n = \dim M$ , then  $\Omega^{n+1}(M) = 0$ .

The algebra is the same as last week.

**Definition 1.14.** Let  $\omega_1 \in \Omega^p(M)$  and  $\omega_2 \in \Omega^q(M)$ . Then  $\omega_1 \wedge \omega_2 \in \Omega^{p+q}(M)$  is defined by

$$\omega_1 \wedge \omega_2(x) = \omega_1(x) \wedge \omega_2(x) \in \Lambda^{p+q} T_x^* M, \quad x \in M.$$

By Proposition 1.6, associativity, distributivity, and supercommutativity hold for  $\Omega^p(M)$ . Let  $F : M \rightarrow N$  be a smooth morphism between manifolds. Then for all  $x \in M$ , the differential of  $F$  at  $x$  is the linear map

$$DF_x : T_x M \rightarrow T_{F(x)} N.$$

Lecture 2  
Monday  
13/01/20

Thus, for all  $p \geq 0$ , we have a natural map, called the **pull-back**,

$$\begin{aligned} F_x^* : \Lambda^p T_{F(x)}^* N &\longrightarrow \Lambda^p T_x^* M \\ \omega(v_1, \dots, v_p) &\longmapsto \omega(DF_x(v_1), \dots, DF_x(v_p)) \end{aligned}, \quad \omega \in \Lambda^p T_{F(x)}^* N, \quad v_1, \dots, v_p \in T_x^* M.$$

Thus, we can define

$$\begin{aligned} F^* : \Omega^p(N) &\longrightarrow \Omega^p(M) \\ \omega(x) &\longmapsto F^* \omega(F(x)) \end{aligned}, \quad \omega \in \Omega^p(N).$$

By Proposition 1.8, the pull-back preserves the exterior product, so

$$F^*(\omega_1 \wedge \omega_2) = F^* \omega_1 \wedge F^* \omega_2.$$

If  $G : N \rightarrow P$ ,

$$(G \circ F)^* \omega = F^* G^* \omega.$$

### 1.3 Local description of $p$ -forms

Let  $M$  be a manifold of dimension  $n$ , let  $x_0 \in M$ , let  $(U, \phi)$  be a local chart around  $x_0$ , and let  $(x_1, \dots, x_n)$  be local coordinates around  $x_0$ . A basis of  $T_{x_0} M$  is given by

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}.$$

A basis of  $T_{x_0}^* M$  is given by

$$\{dx_1, \dots, dx_n\}, \quad dx_i \left( \frac{\partial}{\partial x_j} \right) = \delta_{ij}.$$

A basis of  $\Lambda^p T_{x_0}^* M$  is

$$dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad i_1 < \dots < i_p.$$

Thus,  $\omega \in \Omega^p(M)$  is locally given by

$$\omega(x) = \sum_{|I|=p} f_I(x) dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad I = (i_1, \dots, i_p), \quad i_1 < \dots < i_p,$$

where  $f_I$  is a  $C^\infty$ -function on  $U$  for all  $I$ .

**Example 1.15.** Let  $F : M \rightarrow N$  be a smooth morphism between manifolds of dimension  $n$ , and let  $\omega \in \Omega^n(N)$ . Locally,

$$\omega(y) = f(y) dy_1 \wedge \dots \wedge dy_n, \quad y \in N,$$

for some  $f \in C^\infty$ . By Proposition 1.8,

$$F^* \omega(x) = (f \circ F)(x) \det DF_x dx_1 \wedge \dots \wedge dx_n, \quad x \in M,$$

where  $y_i = p_i \circ F$  and  $p_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is the  $i$ -th projection.

Let  $f : M \rightarrow \mathbb{R}$  be a smooth function, so  $f \in \Omega^0(M)$ . Locally, the **differential** is

$$\begin{aligned} d : \Omega^0(M) &\longrightarrow \Omega^1(M) \\ f &\longmapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \end{aligned}.$$

Check that  $df \in \Omega^1(M)$ , so  $df$  is a 1-form on  $M$ . Alternatively,  $df = f^* dx$  for  $dx$  a 1-form on  $\mathbb{R}$ , or  $df(X) = X(f)$  for any vector field  $X$  on  $M$ . More in general, let  $\omega \in \Omega^p(M)$ . Locally,

$$\omega = \sum_{|I|=p} f_I dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad f_I \in C^\infty,$$

so  $d\omega \in \Omega^{p+1}(M)$ . Then the **de Rham differential** is

$$\begin{aligned} d : \Omega^p(M) &\longrightarrow \Omega^{p+1}(M) \\ \omega &\longmapsto \sum_{|I|=p} df_I \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p} \end{aligned}.$$

**Proposition 1.16.**

- The Leibnitz rule

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2, \quad \omega_1 \in \Omega^p(M), \quad \omega_2 \in \Omega^q(M).$$

- $d^2 = 0$ , that is

$$d(d\omega) = 0, \quad \omega \in \Omega^p(M).$$

- Let  $F : M \rightarrow N$  be a smooth morphism between manifolds. Then

$$F^*d\omega = d(F^*\omega), \quad \omega \in \Omega^p(M),$$

so

$$\begin{array}{ccc} \Omega^p(M) & \xrightarrow{d} & \Omega^{p+1}(M) \\ F^* \uparrow & & \uparrow F^* \\ \Omega^p(N) & \xrightarrow{d} & \Omega^{p+1}(N) \end{array}.$$

**Definition 1.17.**

- $\omega \in \Omega^p(M)$  is **closed** if  $d\omega = 0$ .
- $\omega \in \Omega^p(M)$  is **exact** if there exists  $\omega' \in \Omega^{p-1}(M)$  such that  $d\omega' = \omega$ .

$\omega$  is exact implies that  $\omega$  is closed, since if  $\omega = d\omega'$  then  $d\omega = d^2\omega' = 0$ .

**1.4 Integrations on manifolds**

Let  $M$  be a manifold of dimension  $n$ , let  $F : M \rightarrow M$  be a smooth morphism, and let  $\omega \in \Omega^n(M)$ . Then

$$F^*\omega(x) = \det DF_x \omega(F(x)).$$

Locally, assume  $\omega = f dy_1 \wedge \cdots \wedge dy_n$  for some coordinates  $(y_1, \dots, y_n)$  and  $f \in C^\infty$ . Let  $\{(U_\alpha, \phi_\alpha)\}$  be an atlas of  $M$ , where  $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n$ . Then

$$h_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \rightarrow \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n,$$

such that

$$h_{\alpha\beta}^* \omega(x) = (f \circ h_{\alpha\beta})(x) \det(Dh_{\alpha\beta})_x dx_1 \wedge \cdots \wedge dx_n.$$

Let  $D \subset \mathbb{R}^n$  be compact such that  $\partial D$  has zero measure, so  $D$  is a domain of integration, let  $f : U \rightarrow \mathbb{R}$  be a  $C^\infty$ -function where  $U \subset \mathbb{R}^n$  is open such that  $D \subset U$ , and let  $h : U \rightarrow h(U)$  be a diffeomorphism. Then

$$\int_{h^{-1}(D)} f(y) dy_1 \cdots dy_n = \int_{h^{-1}(D)} f(y) dy_1 \wedge \cdots \wedge dy_n = \int_D (f \circ h)(x) |\det Dh_x| dx_1 \wedge \cdots \wedge dx_n.$$

Let us assume that  $\omega = f(y) dy_1 \wedge \cdots \wedge dy_n$  on  $U$ . We define

$$\int_D \omega = \int_D f(y) dy_1 \wedge \cdots \wedge dy_n, \quad D \subset U.$$

**Definition 1.18.** Let  $U \subset \mathbb{R}^n$  be an open set. We define the **support** of  $\omega$  as

$$\text{supp } \omega = \overline{\{x \in U \mid \omega(x) \neq 0\}}, \quad \omega(x) \in \Lambda^p T_x^* U.$$

Then  $\omega$  has **compact support**, if  $\text{supp } \omega$  is compact.

**Fact.** Under this assumption, we can define

$$\int_U \omega = \int_D \omega \in \mathbb{R},$$

which is well-defined. Under the same assumption, if  $\phi : V \rightarrow U$  is a diffeomorphism, provided that  $\det D\phi_x > 0$ , since  $\det D\phi_x \neq 0$  for all  $x$ , then

$$\int_U \omega = \int_V \phi^* \omega.$$

Lecture 3  
Tuesday  
14/01/20

## 1.5 Orientation

Let  $V$  be a vector space over  $\mathbb{R}$  of dimension  $n$ , and let  $B = (b_1, \dots, b_n) \subset V$  and  $B' = (b'_1, \dots, b'_n) \subset V$  be ordered bases of  $V$ . Then  $B$  and  $B'$  have the **same orientation** if  $\det T > 0$  where

$$\begin{array}{ccc} T & : & V \longrightarrow V \\ & & b_i \longmapsto b'_i \end{array}$$

is a linear map. Let  $\omega \in \Lambda^n V^*$  for  $\omega \neq 0$ . Then  $B$  and  $B'$  have the same orientation if and only if  $\omega(b_1, \dots, b_n)$  has the same sign as  $\omega(b'_1, \dots, b'_n)$ , by Proposition 1.8. An **orientation**  $\Lambda$  of  $V$  is a set of all the ordered basis of  $V$  with the same orientation. Let  $\phi : V \rightarrow W$  be an isomorphism of vector spaces with fixed orientations  $\Lambda_v$  and  $\Lambda_w$  respectively. We say that  $\phi$  is **orientation preserving** if an ordered basis of  $V$  induces an ordered basis of  $W$ , so  $\Lambda_v$  induces  $\Lambda_w$ . Let  $V = \mathbb{R}^n$ , and let  $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ . Then  $e_1, \dots, e_n$  defines an orientation of  $V$  called **positive**. Let  $M$  be a manifold. The idea is to find an orientation  $\Lambda_x$  of  $T_x M$  for all  $x \in M$ .

Special case. Let  $M = U \subset \mathbb{R}^n$  be open. There exists a natural isomorphism  $\phi_x : T_x U \rightarrow \mathbb{R}^n$ . Let  $\Lambda_x^+$  be an orientation on  $T_x U$  such that  $\phi_x$  is orientation preserving with respect to the positive orientation on  $\mathbb{R}^n$ . Let  $\Lambda^+ = \{\Lambda_x^+\}$ .

General case. Let  $\{(U_\alpha, \phi_\alpha)\}$  be an atlas on  $M$ . On  $U_\alpha$ , we define the orientation so that  $(D\phi_\alpha)_x : T_x U_\alpha \rightarrow T_{\phi_\alpha(x)} \phi_\alpha(U) \subset \mathbb{R}^n$  is orientation preserving. This is called the positive orientation on the chart  $(U_\alpha, \phi_\alpha)$ . We define  $\Lambda$  on  $M$ , which is a collection of  $\Lambda^+$  on  $T_x M$  for all  $x \in M$ . Then  $M$  is **orientable** if there exists an atlas with positive orientation charts. This coincides in assuming that  $\det D(\phi_\beta^{-1} \circ \phi_\alpha) > 0$  for all  $\alpha$  and  $\beta$ .

**Notation 1.19.** For all  $p \geq 0$ ,

$$\Omega_c^p(M) = \{\omega \in \Omega^p(M) \mid \text{supp } \omega \text{ is compact}\}.$$

If  $M$  is compact  $\Omega_c^p(M) = \Omega^p(M)$ . Let  $\omega \in \Omega_c^p(M)$ . Assume  $\text{supp } \omega \subset U$  where  $(U, \phi)$  is a chart of  $M$ , and  $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$ . Assume also that  $(U, \phi)$  is positively oriented. Let  $\phi^{-1} : \phi(U) \rightarrow U$  such that  $(\phi^{-1})^* \omega \in \Omega_c^p(\phi(U))$ , that is  $\text{supp } (\phi^{-1})^* \omega \subset \phi(U)$ . We define

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega. \quad (1)$$

We need to show that, under the assumptions above,  $\int_M \omega$  does not depend on  $(U, \phi)$ . Let  $(\bar{U}, \bar{\phi})$  be also a positively oriented chart such that  $\text{supp } \omega \subset \bar{U}$ . We want to show that

$$\int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\bar{\phi}(\bar{U})} (\bar{\phi}^{-1})^* \omega.$$

Let  $\bar{\phi} \circ \phi^{-1} : \phi(U \cap \bar{U}) \rightarrow \bar{\phi}(U \cap \bar{U})$ , so

$$\begin{array}{ccc} & U \cap \bar{U} & \\ \phi \swarrow & & \searrow \bar{\phi} \\ \mathbb{R}^n \supset \phi(U \cap \bar{U}) & \xrightarrow{\bar{\phi} \circ \phi^{-1}} & \bar{\phi}(U \cap \bar{U}) \subset \mathbb{R}^n \end{array}.$$

Since both charts are positively oriented the determinant of the differential  $D(\bar{\phi} \circ \phi^{-1})$  is positive, so

$$\begin{aligned} \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi}^{-1})^* \omega &= \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi}^{-1})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi} \circ \phi^{-1})^* (\bar{\phi}^{-1})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* \bar{\phi}^* (\bar{\phi}^{-1})^* \omega \\ &= \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* (\bar{\phi}^{-1} \circ \bar{\phi})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* \omega = \int_{\phi(U)} (\phi^{-1})^* \omega, \end{aligned}$$

by a property of the pull-back and since  $(\bar{\phi}^{-1})^* \omega = 0$  outside  $\bar{\phi}(U \cap \bar{U})$ .

Lecture 4  
Thursday  
16/01/20



## 1.6 Partitions of unity

**Definition 1.20.** Let  $M$  be a manifold, and let  $U = \{U_\alpha\}$  be an open covering. A **partition of unity** with respect to  $U$  is a collection of smooth functions  $f_\alpha : M \rightarrow [0, 1]$  such that

1.  $\text{supp } f_\alpha = \overline{\{x \in M \mid f_\alpha(x) > 0\}} \subset U_\alpha$  for all  $\alpha$ ,
2.  $\sum_\alpha f_\alpha(x) = 1$  for all  $x \in M$ , and
3. for all  $x \in M$ , there exists  $U \ni x$  open such that  $\text{supp } f_\alpha \cap U \neq \emptyset$  for only finitely many  $\alpha$ .

**Remark.** 3 implies that 2 is a finite sum.

**Example 1.21.** Let

$$M = S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}, \quad U_1 = S^1 \setminus \{(1, 0)\}, \quad U_2 = S^1 \setminus \{(-1, 0)\},$$

so  $\{U_i\}$  is a cover. Let

$$f_1(\cos \theta, \sin \theta) = \frac{1}{2} - \frac{1}{2} \cos \theta, \quad f_2(\cos \theta, \sin \theta) = \frac{1}{2} + \frac{1}{2} \cos \theta.$$

Then  $f_i$  is a partition of unity.

**Proposition 1.22.** Let  $M$  be a manifold, and let  $U = \{U_\alpha\}$  be an open covering of  $M$ . Then there exists a partition of unity  $f_\alpha$  with respect to  $U$ .

*Proof.* We omit the proof. □

**Proposition 1.23.** Let  $M$  be a manifold, and let  $n = \dim M$ . Then  $M$  is orientable if and only if there exists  $\omega \in \Omega^n(M)$  which is never vanishing on  $M$ , so  $\omega(x) \neq 0$  for all  $x \in M$ .

$\omega$  is called a **volume form** on  $M$ .

*Proof.*

$\Leftarrow$  Assume  $\omega \in \Omega^n(M)$  is a volume form. We want to construct an orientation  $\Lambda$  on  $M$ , that is  $\Lambda_x$  on  $T_x M$  for all  $x \in M$ . Given an oriented basis  $v_1, \dots, v_n$  of  $T_x M$  we say that it is **positively oriented** if  $\omega(x)(v_1, \dots, v_n) > 0$ . For all  $x \in M$ , we define the orientation  $\Lambda_x$  on  $T_x M$  by considering the class of positively oriented ordered basis of  $T_x M$  which is compatible with the choice of an atlas on  $M$ . Take any atlas  $\{(U_\alpha, \phi_\alpha)\}$ , where  $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ . On  $U_\alpha$ ,

$$\omega = g_\alpha \phi_\alpha^* dx_1 \wedge \dots \wedge dx_n.$$

Since  $\omega \neq 0$ ,  $g_\alpha > 0$  or  $g_\alpha < 0$ . If  $g_\alpha < 0$  then switch  $x_1$  with  $x_2$ , so  $g_\alpha > 0$ . After this change of coordinates,  $(U_\alpha, \phi_\alpha)$  is positively oriented, so  $M$  is orientable.

$\Rightarrow$  Assume that  $M$  is orientable, that is there exists an atlas  $\{(U_\alpha, \phi_\alpha)\}$  of positively oriented charts. On  $U_\alpha$ , we consider

$$\omega_\alpha = \phi_\alpha^* dx_1 \wedge \dots \wedge dx_n.$$

Let  $f_\alpha$  be a partition of unity with respect to  $\{U_\alpha\}$ . Let  $\widetilde{\omega}_\alpha = f_\alpha \omega_\alpha \in \Omega^n(U_\alpha)$ . We may assume that  $\widetilde{\omega}_\alpha \in \Omega^n(M)$  by extending equal to zero outside  $U_\alpha$ . We define  $\omega = \sum_\alpha \widetilde{\omega}_\alpha \in \Omega^n(M)$ . For all  $\alpha$ , since  $\sum_\alpha f_\alpha = 1$  there exists  $\alpha$  such that  $\widetilde{\omega}_\alpha \neq 0$ , so  $\omega \neq 0$ . □

Let  $M$  be an orientable manifold of dimension  $n$ , and let  $\omega \in \Omega_c^n(M)$ . We want to define  $\int_M \omega$ . So far we defined for  $\omega$  such that  $\text{supp } \omega \subset U_\alpha$  where  $(U_\alpha, \phi_\alpha)$  is a chart.

**Definition 1.24.** Let  $\{(U_\alpha, \phi_\alpha)\}$  be a positively oriented atlas on  $M$ , and let  $f_\alpha$  be a partition of unity with respect to  $\{U_\alpha\}$ . Then  $\text{supp } f_\alpha \omega \subset U_\alpha$ , so let

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega.$$

**Remark 1.25.** Note that for each  $\alpha$ , we have that the support of  $f_\alpha \omega$  is contained in  $U_\alpha$  and therefore each term of the sum is well-defined as in (1). Indeed, we have

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* f_\alpha.$$

**Lemma 1.26.**  $\int_M \omega$  does not depend on  $\{(U_\alpha, \phi_\alpha)\}$  and  $f_\alpha$ .

*Proof.* Under the assumption that  $\text{supp } \omega \subset U_\alpha$  then we showed  $\int_{U_\alpha} \omega$  does not depend on  $(U_\alpha, \phi_\alpha)$ . Let  $\{(U_\alpha, \phi_\alpha)\}$  and  $\{(\overline{U}_\alpha, \overline{\phi}_\alpha)\}$  be two atlases with positively oriented charts, and let  $f_\alpha$  and  $\overline{f}_\alpha$  be two partitions of unity with respect to  $\{U_\alpha\}$  and  $\{\overline{U}_\alpha\}$  respectively. Then  $\sum_\alpha f_\alpha = \sum_\alpha \overline{f}_\alpha = 1$ , so  $\int_M f_\alpha \omega = \sum_\beta \int_M \overline{f}_\beta f_\alpha \omega$ . Thus

$$\int_M \omega = \sum_\alpha \int_M f_\alpha \omega = \sum_{\alpha, \beta} \int_M \overline{f}_\beta f_\alpha \omega = \sum_\beta \int_M \sum_\alpha f_\alpha \overline{f}_\beta \omega = \sum_\beta \int_M \overline{f}_\beta \omega.$$

□

**Proposition 1.27.** Let  $M$  and  $N$  be orientable manifolds of dimension  $n$ , and let  $\omega, \eta \in \Omega_c^n(M)$ .

1. *Linearity*

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

2. *Orientation reversal.* Let  $\overline{M}$  be the manifold  $M$  with opposite orientation  $\Lambda^- = \{\Lambda_x^- \mid x \in M\}$ , which is the orientation opposite than the one induced by  $M$  with orientation  $\Lambda$ . Then

$$\int_M \omega = - \int_{\overline{M}} \omega.$$

3. *Positivity.* Let  $\omega$  be the volume form on  $M$ . Then

$$\int_M \omega > 0.$$

4. *Diffeomorphism invariance.* Let  $F : N \rightarrow M$  be an orientation preserving diffeomorphism. Then

$$\int_M \omega = \int_N F^* \omega.$$

*Proof.*

1. Exercise. <sup>2</sup>

2. Exercise. <sup>3</sup>

3. Choose a positively oriented chart  $(U_\alpha, \phi_\alpha)$  on  $U_\alpha$ , so

$$\omega = g_\alpha \phi_\alpha^* dx_1 \wedge \cdots \wedge dx_n, \quad g_\alpha > 0.$$

Then  $\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega$  where  $f_\alpha$  is a partition of unity. For all  $x \in M$  there exists  $\alpha$  such that  $x \in U_\alpha$  and  $\int_{U_\alpha} f_\alpha \omega > 0$ , so  $\int_M \omega > 0$ .

4. Let  $(U_\alpha, \phi_\alpha)$  be a positively oriented atlas on  $M$ . Then  $(F^{-1}(U_\alpha), \phi_\alpha \circ F)$  is an atlas on  $N$  which is positively oriented. Let  $f_\alpha$  be a partition of unity with respect to  $\{U_\alpha\}$ . Then  $f_\alpha \circ F$  is a partition of the unity with respect to  $\{F^{-1}(U_\alpha)\}$ , so

$$\int_N F^* \omega = \sum_\alpha \int_N (f_\alpha \circ F) F^* \omega = \sum_\alpha \int_N F^* (f_\alpha \omega) = \sum_\alpha \int_M f_\alpha \omega = \int_M \omega.$$

□

<sup>2</sup>Exercise

<sup>3</sup>Exercise

## 1.7 Manifolds with boundary

Denote

$$\mathbb{R}_{\geq 0}^n = (\mathbb{R}_{\geq 0})^n, \quad \mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Let  $U \subset \mathbb{R}_+^n$  be open, and let  $F : U \rightarrow \mathbb{R}^m$  be a function. Then  $F$  is  $C^\infty$  if it can be extended to a  $C^\infty$ -function  $\tilde{F} : \tilde{U} \rightarrow \mathbb{R}^m$  where  $\tilde{U} \supset U$  and  $\tilde{U}$  is open.

**Definition 1.28.** A **manifold with boundary** of dimension  $n$  is a Hausdorff topological space  $M$  such that there exists an open covering  $\{U_\alpha\}$ , and for all  $\alpha$ , there exists a homeomorphism  $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$  such that for all  $\alpha$  and  $\beta$ ,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n \rightarrow \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n$$

is a diffeomorphism, so

$$\begin{array}{ccc} & U_\alpha \cap U_\beta & \\ \phi_\alpha \swarrow & & \searrow \phi_\beta \\ \mathbb{R}_+^n \supset \phi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow{\phi_\alpha \circ \phi_\beta^{-1}} & \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n \end{array}.$$

The **boundary** of  $M$  is

$$\partial M = \{x \in M \mid \exists \alpha, \phi_\alpha(x) \in \partial \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}\}.$$

Then  $(U_\alpha, \phi_\alpha)$  is called a **chart** and  $\{(U_\alpha, \phi_\alpha)\}$  is called an **atlas**.

**Remark 1.29.**

- $\partial M$  is closed in  $M$ .
- $\mathring{M} = M \setminus \partial M$  is a manifold of dimension  $n$ .

**Example 1.30.**

- $M = [0, 1]$  is a manifold with boundary  $\partial M = \{0, 1\}$ .
- The closed disc  $D = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$  is a manifold with boundary  $\partial D = S^{n-1}$ .
- $M = [0, 1] \times S^1$  is a manifold with boundary  $\partial M = S^1 \sqcup S^1$ .

**Remark 1.31.**

- We can define tangent spaces and differential forms exactly in the same way as usual manifolds.
- The definition of orientability is the same. If  $M$  is orientable, then  $\partial M$  is also orientable. As a convention, the positive orientation on the boundary of  $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}$  is given by  $(-1)^n dx_1 \wedge \dots \wedge dx_{n-1}$ . This induces a positive orientation on  $\partial M$ .
- Also partitions of unity for any open cover  $U_\alpha$  of  $M$  is defined the same way. If  $M$  is orientable, for any manifold with boundary, for all open covering  $U = \{U_\alpha\}$ , there exists a partition of unity  $f_\alpha$ . This implies that if  $\omega \in \Omega_c^n(M)$ , then  $\int_M \omega$  is defined the same way for manifolds.

## 1.8 Stokes' theorem

**Theorem 1.32** (Stokes). *For any manifold with boundary  $M$  of dimension  $n$ , and for any  $\omega \in \Omega_c^{n-1}(M)$  we have*

$$\int_M d\omega = \int_{\partial M} \omega \in \Omega_c^n(M).$$

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*Proof.* Let  $\{(U_\alpha, \phi_\alpha)\}$  be an atlas, and let  $f_\alpha : M \rightarrow \mathbb{R}$  be a partition of unity with respect to this cover. Then  $\sum_\alpha f_\alpha = 1$  on  $M$ , so

$$\int_M d\omega = \int_M d\left(\sum_\alpha f_\alpha \omega\right) = \sum_\alpha \int_M d(f_\alpha \omega) = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* d(f_\alpha \omega).$$

By Proposition 1.16,

$$(\phi_\alpha^{-1})^* d(f_\alpha \omega) = d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right).$$

Then  $(\phi_\alpha^{-1})^* (f_\alpha \omega)$  is an  $(n-1)$ -form on  $\phi_\alpha(U_\alpha)$ . In coordinates,

$$(\phi_\alpha^{-1})^* (f_\alpha \omega) = \sum_{j=1}^n \widetilde{f_\alpha \omega_j} dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

where  $\omega_j$  is a smooth function on  $\phi_\alpha(U_\alpha)$  and

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\widetilde{\phi_\alpha}} & \phi_\alpha(U_\alpha) \\ f_\alpha \downarrow & \swarrow \widetilde{f_\alpha} & \\ [0, 1] & & \end{array}.$$

Then

$$\begin{aligned} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right) &= d\left(\sum_{j=1}^n \widetilde{f_\alpha \omega_j} dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n\right) \\ &= \sum_{j=1}^n \sum_{k=1}^n \frac{\partial}{\partial x_k} \left(\widetilde{f_\alpha \omega_j}\right) dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha \omega_j}\right) dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha \omega_j}\right) dx_1 \wedge \cdots \wedge dx_n, \end{aligned}$$

so

$$\sum_\alpha \int_{\phi_\alpha(U_\alpha)} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right) = \sum_\alpha \int_{\mathbb{R}_+^n} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right),$$

because  $\widetilde{f_\alpha} = 0$  outside  $\phi_\alpha(U_\alpha)$ . Thus

$$\begin{aligned} \int_M d\omega &= \sum_\alpha \int_{\mathbb{R}_+^n} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha \omega_j}\right) dx_1 \wedge \cdots \wedge dx_n \\ &= \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha \omega_j}\right) dx_n dx_{n-1} \cdots dx_1 \\ &= \sum_\alpha \sum_{j=1}^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} (-1)^{j-1} \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha \omega_j}\right) \Big|_{x_n=0} dx_n dx_{n-1} \cdots \widehat{dx_j} \cdots dx_1 \\ &= \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^{n-1} \left(\widetilde{f_\alpha \omega_j}\right) \Big|_{x_n=0} dx_{n-1} \cdots dx_1, \end{aligned}$$

since  $(f_\alpha \omega_j)|_{x_n=0} = 0$  for  $j = 1, \dots, n-1$ , so

$$\int_M d\omega = \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^{n-1} \left(\widetilde{f_\alpha \omega_j}\right) \Big|_{x_n=0} dx_{n-1} \cdots dx_1 = \sum_\alpha \int_{\partial U_\alpha} f_\alpha|_{\partial U_\alpha} \omega = \int_{\partial M} \omega,$$

where  $\partial U_\alpha = U_\alpha \cap \partial M$ . □

## 1.9 Applications of Stokes' theorem

**Theorem 1.33** (Integration by parts). *Let  $M$  be an orientable  $n$ -dimensional manifold with boundary, let  $\omega \in \Omega_c^p(M)$ , let  $\eta \in \Omega_c^{n-p-1}(M)$ , and let  $p \in \{0, \dots, n-1\}$ . Then*

$$\int_{\partial M} \omega \wedge \eta = \int_M d\omega \wedge \eta + (-1)^p \int_M \omega \wedge d\eta.$$

*Proof.*

$$\int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta) = \int_M (d\omega \wedge \eta + (-1)^p \omega \wedge d\eta),$$

by Stokes and the Leibnitz rule.  $\square$

**Theorem 1.34** (Brouwer's fixed point theorem). *Let*

$$D = \{x \in \mathbb{R}^n \mid |x| \leq 1\},$$

*so*

$$\partial D = S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\},$$

*and let  $f : D \rightarrow D$  be a smooth morphism. Then  $f$  admits a fixed point, that is there exists  $x \in D$  such that  $f(x) = x$ .*

*Proof.* Assume that  $f(x) \neq x$  for all  $x \in D$ . For any  $x \in D$ , consider the ray starting from  $f(x)$  and passing through  $x$ . Let  $g(x)$  be the point where this ray intersects  $\partial D$  away from  $f(x)$ . Note that if  $x \in \partial D$  then  $g(x) = x$ . Then  $g : D \rightarrow \partial D$ . It is easy to check that  $g$  is smooth. Since  $\partial D = S^{n-1}$  is orientable by Proposition 1.23 there exists a volume form  $\omega \in \Omega^{n-1}(\partial D)$ , so  $\omega(x) \neq 0$ . Since  $\omega \in \Omega^{n-1}(\partial D)$ ,  $d\omega \in \Omega^n(\partial D)$ , which is an  $n$ -dimensional manifold, so  $d\omega = 0$ . Thus

$$0 < \int_{\partial D} \omega = \int_{\partial D} g^* \omega = \int_D dg^* \omega = \int_D g^* d\omega = 0,$$

by Stokes, a contradiction.  $\square$

**Example 1.35.** Recall any exact form is closed, since  $d^2 = 0$ . But the opposite is not always true. Let  $M = \mathbb{R}^2 \setminus \{0\}$ , and let

$$\omega = \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx \in \Omega^1(M).$$

Then  $\omega$  is closed, since

$$d\omega = \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right) dx \wedge dy - \frac{\partial}{\partial y} \left( \frac{y}{x^2 + y^2} \right) dy \wedge dx = 0.$$

We want to show that  $\omega$  is not exact. Assume that

$$\omega = df, \quad f \in \Omega^0(M) = \{C^\infty\text{-function}\}.$$

In particular  $\omega = df$  on  $S^1 \subset M$ . Let

$$\begin{aligned} \gamma & : [0, 2\pi] \longrightarrow S^1 \\ \theta & \longmapsto (\cos \theta, \sin \theta) \end{aligned}$$

Then

$$\int_{S^1} \omega = \int_0^{2\pi} \gamma^* \omega = \int_0^{2\pi} \left( \left( \frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} \right) \cos \theta d\theta - \left( \frac{\sin \theta}{\cos^2 \theta + \sin^2 \theta} \right) (-\sin \theta) d\theta \right) = \int_0^{2\pi} d\theta = 2\pi,$$

and

$$\int_{S^1} \omega = \int_{S^1} df = \int_{\partial S^1} f = \int_{\emptyset} f = 0,$$

so  $\omega$  is not exact.

**Proposition 1.36.** *Let  $M$  be an orientable manifold of dimension  $n$  without boundary, and let  $\omega \in \Omega_c^n(M)$ . Assume  $\omega$  is exact. Then*

$$\int_M \omega = 0.$$

*Proof.* Easy from Stokes. □

**Proposition 1.37.** *Let  $M$  be an orientable manifold of dimension  $n$  with boundary, and let  $\omega \in \Omega_c^{n-1}(M)$  be a closed form. Then*

$$\int_{\partial M} \omega = 0.$$

*Proof.* Easy from Stokes. □

Let  $M$  be an orientable manifold of dimension  $n$ , let  $\omega \in \Omega_c^k(M)$ , and let  $N \subset M$  be a submanifold of dimension  $k$ . We can define

$$\int_M \omega = \int_N i^* \omega,$$

where  $i : N \hookrightarrow M$  is the inclusion. We will denote

$$\omega|_N = i^* \omega \in \Omega_c^k(N).$$

**Proposition 1.38.** *Let  $M$  be an oriented manifold of dimension  $n$ , let  $\omega \in \Omega_c^k(M)$ , and let  $S \subset M$  be a compact orientable submanifold of dimension  $k$  such that  $\partial S = \emptyset$  and  $\int_S \omega \neq 0$ . Then*

- $\omega$  is not exact,
- $\omega|_S$  is not exact, and
- $S$  is not the boundary of an orientable manifold  $N \subset M$  of dimension  $k + 1$ .

*Proof.* Exercise. <sup>4</sup> □

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<sup>4</sup>Exercise

## 2 De Rham cohomology

### 2.1 De Rham cohomology

**Definition 2.1.** Let  $M$  be a manifold of dimension  $n$ , and let  $p \geq 0$ . Then  $\omega_1, \omega_2 \in \Omega^p(M)$  are said to be **cohomologous** if  $\omega_1 - \omega_2 = d\eta$  where  $\eta \in \Omega^{p-1}(M)$ . In particular  $\omega \in \Omega^p(M)$  is cohomologous to zero if it is exact. Let

$$\mathcal{Z}^p(M) = \text{Ker}(d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)) = \{\omega \in \Omega^p(M) \mid \omega \text{ is closed}\} \subset \Omega^p(M),$$

and let

$$\mathcal{B}^p(M) = \text{Im}(d : \Omega^{p-1}(M) \rightarrow \Omega^p(M)) = \{\omega \in \Omega^p(M) \mid \omega \text{ is exact}\} \subset \Omega^p(M).$$

Then  $\mathcal{B}^p(M) \subset \mathcal{Z}^p(M)$  for all  $p \geq 0$ .

**Notation.** If  $p = 0$ , then  $\mathcal{B}^0(M) = 0$ .

**Note.** If  $\omega_1, \omega_2 \in \mathcal{Z}^p(M)$  then  $\omega_1 - \omega_2 \in \mathcal{B}^p(M)$  if and only if  $\omega_1$  and  $\omega_2$  are cohomologous.

**Definition 2.2.** Denote the  $p$ -th de Rham cohomology group as

$$H^p(M) = \mathcal{Z}^p(M) / \mathcal{B}^p(M) = \{[\omega] \mid \omega \in \mathcal{Z}^p(M)\}, \quad p \geq 0.$$

where

$$[\omega] = \{\omega' \in \Omega^p(M) \text{ cohomologous to } \omega\}$$

is the **de Rham class** of  $\omega$ .

**Remark.**  $H^p(M)$  is a vector space over  $\mathbb{R}$ .

**Definition 2.3.**  $b_p(M) = \dim H^p(M) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$  is the  $p$ -th **Betti number** of  $M$ .

**Proposition 2.4.** If  $M$  is connected then

$$H^0(M) = \mathbb{R},$$

that is  $b_0(M) = 1$ . More in general,  $b_0(M)$  is the number of connected components of  $M$ .

*Proof.* Assume  $M$  is connected. Then  $\mathcal{B}^0(M) = 0$ , so

$$\begin{aligned} H^0(M) &= \mathcal{Z}^0(M) = \{f \in \Omega^0(M) \text{ closed}\} \\ &= \left\{ f \in \Omega^0(M) \mid \text{locally } \forall x \in M, \frac{\partial}{\partial x_i} f(x) = 0 \right\} \\ &= \{f \in \Omega^0(M) \text{ locally constant}\} = \mathbb{R}. \end{aligned}$$

□

**Example.** Let  $M = S^1$ . Then  $H^0(M) = \mathbb{R}$ .

**Proposition 2.5.** Let  $M$  be a manifold of dimension  $n$ . Then

$$H^p(M) = 0, \quad p \geq n+1.$$

*Proof.* Recall  $\Omega^p(M) = 0$  if  $p \geq n+1$  because all alternating  $p$ -forms for  $p \geq n+1$  on an  $n$ -dimensional vector space are zero, so  $\mathcal{Z}^p(M) = 0$ . Thus  $H^p(M) = 0$ . □

**Proposition 2.6.** Let  $M$  be a compact orientable manifold of dimension  $n$  without boundary. Then

$$H^n(M) \neq 0.$$

*Proof.*  $M$  is orientable, so there exists a volume form  $\omega \in \Omega^n(M) = \Omega_c^n(M)$ , by Proposition 1.23. Then  $\omega$  is closed, because  $d\omega$  is an  $(n+1)$ -form on  $M$ , so  $\omega \in \mathcal{Z}^n(M)$ . We want to show that  $[\omega] \neq 0$  in  $H^n(M)$ . Assume  $[\omega] = 0$ , so  $\omega$  is exact. Thus  $\omega = d\eta$  where  $\eta$  is an  $(n-1)$ -form on  $M$ , so

$$0 < \int_M \omega = \int_M d\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0,$$

by Stokes, a contradiction. □

**Proposition 2.7.** *Let  $G : M \rightarrow N$  be a smooth morphism between manifolds. Then*

$$G^* : \Omega^p(N) \rightarrow \Omega^p(M), \quad p \geq 0$$

*takes closed forms of  $N$  to closed forms on  $M$  and exact forms of  $N$  to exact forms on  $M$ .*

*Proof.* By Proposition 1.16,  $G^*d = dG^*$ . If  $\omega$  is closed then  $dG^*\omega = G^*d\omega = G^*0 = 0$ , so  $G^*\omega$  is closed. If  $\omega = d\eta$  is exact then  $G^*\omega = dG^*\eta$  is also exact.  $\square$

Thus  $G^* : \mathcal{Z}^p(N) \rightarrow \mathcal{Z}^p(M)$  and  $G^* : \mathcal{B}^p(N) \rightarrow \mathcal{B}^p(M)$ , so there exists a linear map

$$\begin{array}{ccc} G^* & : & \mathcal{H}^p(N) \longrightarrow \mathcal{H}^p(M) \\ & & [\omega] \longmapsto [G^*\omega] \end{array}.$$

**Corollary 2.8.** *Let  $M$  and  $N$  be diffeomorphic manifolds. Then*

$$\mathcal{H}^p(M) \cong \mathcal{H}^p(N), \quad p \geq 0,$$

*that is  $\mathcal{H}^p(M)$  is a diffeomorphic invariant.*

*Proof.* By Proposition 2.7 there exists  $F^* : \mathcal{H}^p(N) \rightarrow \mathcal{H}^p(M)$  and  $(F^{-1})^* : \mathcal{H}^p(M) \rightarrow \mathcal{H}^p(N)$ . By Proposition 1.8,

$$(F^{-1})^* F^* \omega = (F \circ F^{-1})^* \omega = \text{id}_N^* \omega = \omega, \quad \omega \in \mathcal{H}^p(N)$$

so  $(F^{-1})^* \circ F^* = \text{id}_{\mathcal{H}^p(N)}$ . Similarly  $F^* \circ (F^{-1})^* = \text{id}_{\mathcal{H}^p(M)}$ , so  $F^*$  is an isomorphism.  $\square$

## 2.2 Homotopy invariance

**Definition 2.9.** Let  $M_0$  and  $M_1$  be manifolds, and let  $f_0, f_1 : M_0 \rightarrow M_1$  be smooth morphisms. Then  $f_0$  and  $f_1$  are **smoothly homotopic equivalent** if there exists a smooth morphism  $H : M_0 \times [0, 1] \rightarrow M_1$  such that  $H(x, 0) = f_0(x)$  and  $H(x, 1) = f_1(x)$  for all  $x \in M_0$ . A **homotopy** is a smooth morphism  $H : M_0 \times [0, 1] \rightarrow M_1$  where  $M_0$  and  $M_1$  are smooth manifolds.

**Notation 2.10.** Let  $f_t(x) = H(x, t)$ , so  $f_t : M_0 \rightarrow M_1$  is a smooth morphism. Then  $f_0$  and  $f_1$  are said to be homotopic equivalent, denoted by  $f_0 \sim f_1$ . Then  $\sim$  is an equivalence. <sup>5</sup>

**Definition 2.11.**  $M_0$  and  $M_1$  are **homotopy equivalent** if there exist smooth morphisms  $f : M_0 \rightarrow M_1$  and  $g : M_1 \rightarrow M_0$  such that  $f \circ g \sim \text{id}_{M_1}$  and  $g \circ f \sim \text{id}_{M_0}$ .

**Example 2.12.**

- Let  $M_0 = \mathbb{R}^n$  and  $M_1 = \{0\}$ . Then  $M_0$  and  $M_1$  are homotopy equivalent. Let

$$\begin{array}{ccc} f & : & M_0 \longrightarrow M_1 \\ x & \longmapsto & 0 \end{array}, \quad \begin{array}{ccc} g & : & M_1 \longrightarrow M_0 \\ 0 & \longmapsto & 0 \end{array}.$$

Then

$$\begin{array}{ccc} f \circ g & : & M_1 \longrightarrow M_1 \\ 0 & \longmapsto & 0 \end{array},$$

so  $f \circ g = \text{id}_{M_1}$ , and

$$\begin{array}{ccc} g \circ f & : & M_0 \longrightarrow M_0 \\ x & \longmapsto & 0 \end{array}.$$

We want to show that  $g \circ f \sim \text{id}_{M_0}$ . Define a smooth morphism

$$\begin{array}{ccc} H & : & M_0 \times [0, 1] \longrightarrow M_0 \\ (x, t) & \longmapsto & tx \end{array}.$$

Then  $H(x, 0) = 0 = (g \circ f)(x)$  for all  $x$ , and  $H(x, 1) = x = \text{id}_{M_0}(x)$  for all  $x$ , so  $g \circ f \sim \text{id}_{M_0}$ . More in general  $M \subset \mathbb{R}^n$  is called **convex** if for all  $x, y \in M$  the segment joining  $x$  to  $y$  is contained inside  $M$ . If  $M$  is convex then  $M$  is homotopy equivalent to  $M \times \{0\}$ .

<sup>5</sup>Exercise



- Let  $M_0 = \mathbb{R}^2 \setminus \{0\}$  and  $M_1 = S^1$ . Then  $M_0$  and  $M_1$  are homotopy equivalent. Let

$$\begin{aligned} f &: M_0 \longrightarrow M_1 \\ x &\longmapsto \frac{x}{|x|}, \end{aligned} \quad \begin{aligned} g &: M_1 \longrightarrow M_0 \\ x &\longmapsto x \end{aligned}.$$

Then

$$\begin{aligned} f \circ g &: M_1 \longrightarrow M_1 \\ x &\longmapsto x \end{aligned},$$

so  $f \circ g = \text{id}_{M_1}$ , and

$$\begin{aligned} g \circ f &: M_0 \longrightarrow M_0 \\ x &\longmapsto \frac{x}{|x|}. \end{aligned}$$

Let

$$\begin{aligned} H &: M_0 \times [0, 1] \longrightarrow M_0 \\ (x, t) &\longmapsto tx + (1-t) \frac{x}{|x|} \end{aligned}$$

be smooth. Then  $H(x, 0) = x/|x| = (g \circ f)(x)$  and  $H(x, 1) = x = \text{id}_{M_0}(x)$ , so  $g \circ f \sim \text{id}_{M_0}$ .

**Proposition 2.13.** *Let  $M$  and  $N$  be manifolds, and let  $H : M \times [0, 1] \rightarrow N$  be smooth. Denote*

$$\begin{aligned} f_t &: M \longrightarrow N \\ x &\longmapsto H(x, t), \end{aligned} \quad t \in [0, 1].$$

Then  $f_t^* : H^p(N) \rightarrow H^p(M)$  does not depend on  $t$  for all  $p \geq 0$ .

*Proof.* Let  $t_1, t_2 \in [0, 1]$  such that  $t_1 < t_2$ . The goal is  $f_{t_1}^*[\eta] = f_{t_2}^*[\eta]$  for all  $[\eta] \in H^p(N)$ . Let

$$\begin{aligned} i_k &: M \longrightarrow M \times [0, 1] \\ x &\longmapsto (x, t_k) \end{aligned}, \quad k = 1, 2.$$

Claim that for all  $p$  there exists a linear map  $h : \Omega^p(M \times [t_1, t_2]) \rightarrow \Omega^{p-1}(M)$  such that

$$d(h(\omega)) + h(d\omega) = i_2^*\omega - i_1^*\omega \in \Omega^p(M), \quad \omega \in \Omega^p(M \times [0, 1]). \quad (2)$$

Step 1. The claim implies the proposition. Let  $\eta \in \Omega^p(N)$  be closed, so  $d\eta = 0$ . Then  $H^*\eta$  is also closed, so let  $\omega = H^*\eta \in \Omega^p(M \times [t_1, t_2])$ . Apply  $h$ . Then  $d\omega = 0$ , so  $d(h(\omega)) = i_2^*\omega - i_1^*\omega$  is exact. Thus

$$f_{t_1}^*[\eta] = [f_{t_1}^*\eta] = [i_1^*H^*\eta] = [i_1^*\omega] = [i_2^*\omega] = [i_2^*H^*\eta] = [f_{t_2}^*\eta] = f_{t_2}^*[\eta],$$

so the proposition follows.

Step 2. The proof of the claim. Let  $\omega \in \Omega^p(M \times [t_1, t_2])$ . Then for all  $(x, t) \in M \times [t_1, t_2]$ ,  $\omega(x, t)$  is an alternating  $p$ -form on  $T_{(x,t)}(M \times [t_1, t_2])$ . We want an alternating  $(p-1)$ -form  $h(\omega)(x)$  on  $T_x M$ . Let  $v_1, \dots, v_{p-1} \in T_x M$ . Then

$$h(\omega)(x)(v_1, \dots, v_{p-1}) = \int_{t_1}^{t_2} \omega(x, t) \left( \frac{\partial}{\partial t}, v_1, \dots, v_{p-1} \right) dt$$

is a  $(p-1)$ -form on  $M$ , and  $\frac{\partial}{\partial t}$  is a global vector field. Check  $h$  is linear.<sup>6</sup> It is enough to prove (2) locally. Remark that exactness is not a local property. Fix local coordinates  $(x_1, \dots, x_n, t)$  around a point of  $M \times [0, 1]$ . Then

$$\omega = \sum_{|I|=p} \omega_I + \sum_{|J|=p-1} \omega_J, \quad \omega_I = g_I dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad \omega_J = g_J dx_{j_1} \wedge \dots \wedge dx_{j_{p-1}} \wedge dt,$$

where  $g_I$  and  $g_J$  are smooth functions. Any term of (2) is linear. So we just need to check (2) holds for  $\omega_I$  and  $\omega_J$ .

---

<sup>6</sup>Exercise

$\omega_I$ . Let  $\omega = g(x, t) dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ . Then

$$d \left( h \left( \omega(x, t) \left( \frac{\partial}{\partial t}, v_1, \dots, v_{p-1} \right) \right) \right) = d(h(0)) = 0,$$

and

$$\begin{aligned} h(d\omega) &= h \left( \frac{\partial}{\partial t} g(x, t) dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} + \sum_{j=1}^n \frac{\partial}{\partial x_j} g(x, t) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \right) \\ &= \left( \int_{t_1}^{t_2} \frac{\partial}{\partial t} g(x, t) dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_p} + 0 \\ &= (g(x, t_2) - g(x, t_1)) dx_{i_1} \wedge \cdots \wedge dx_{i_p} = i_2^* \omega - i_1^* \omega, \end{aligned}$$

so (2) holds.

$\omega_J$ . Let  $\omega = g(x, t) dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \wedge dt$ . Then

$$\begin{aligned} d(h(\omega)) &= (-1)^{p-1} d \left( \left( \int_{t_1}^{t_2} g(x, t) dt \right) dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \right) \\ &= (-1)^{p-1} \sum_{j=1}^n \frac{\partial}{\partial x_j} \left( \int_{t_1}^{t_2} g(x, t) dt \right) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \\ &= (-1)^{p-1} \sum_{j=1}^n \left( \int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x, t) dt \right) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}}, \end{aligned}$$

and

$$\begin{aligned} h(d\omega) &= h \left( \sum_{j=1}^n \frac{\partial}{\partial x_j} g(x, t) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \wedge dt + 0 \right) \\ &= (-1)^{p-1} \sum_{j=1}^n \left( \int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x, t) dt \right) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} = -d(h(\omega)), \end{aligned}$$

and  $i_2^* \omega = i_1^* \omega = 0$ , so (2) holds. □

**Corollary 2.14.** *Assume  $M$  and  $N$  are homotopy equivalent. Then there exist isomorphisms*

$$H^p(N) \rightarrow H^p(M), \quad p \geq 0.$$

*Proof.* There exist  $f : M \rightarrow N$  and  $g : N \rightarrow M$  such that  $g \circ f \sim \text{id}_M$  and  $f \circ g \sim \text{id}_N$ . By Proposition 2.13  $(g \circ f)^* : H^p(M) \rightarrow H^p(M)$  coincides with  $\text{id}_M^* = \text{id}_{H^p(M)}$ . Then  $f^* \circ g^* = (g \circ f)^* = \text{id}_{H^p(M)}$ . Similarly  $g^* \circ f^* = \text{id}_{H^p(N)}$ , so  $g^*$  and  $f^*$  are isomorphisms. □

**Definition 2.15.** Let  $M$  be a manifold. Then  $M$  is **smoothly contractible** if  $M$  is homotopy equivalent to a point.

**Example.**  $\mathbb{R}^n$  is contractible, by Example 2.12. If  $M \subset \mathbb{R}^n$  is convex then  $M$  is contractible.

**Theorem 2.16** (Poincaré lemma). *If  $M$  is a contractible manifold then*

$$H^p(M) = 0, \quad p \geq 1.$$

*Proof.* By previous Corollary 2.14, there exists an isomorphism  $H^p(M) \rightarrow H^p(\{\text{point}\})$ . Then  $\{\text{point}\}$  is a zero-dimensional manifold, so by Proposition 2.5,  $H^p(\{\text{point}\}) = 0$  for all  $p > 0$ . □

Thus  $H^p(\mathbb{R}^n) = 0$  for all  $p > 0$ , so  $\mathbb{R}^n$  is not diffeomorphic to any compact orientable manifold.

**Proposition 2.17.** *Let  $M$  be a manifold, and let  $\omega \in \Omega^p(M)$  be a closed  $p$ -form for  $p > 0$ . Then for all  $x \in X$ , there exists a neighbourhood  $U \ni x$  such that  $\omega$  is exact on  $U$ , that is there exists  $\eta \in \Omega^{p-1}(U)$  such that  $\omega = d\eta$  on  $U$ .*

*Proof.* Let  $(U, \phi)$  be a chart around  $x$ . I may assume that  $V = \phi(U)$  is a ball in  $\mathbb{R}^n$ . Then  $U$  is diffeomorphic to  $B = \{z \mid |z - z_0| < r\}$  for some  $z_0 \in \mathbb{R}^n$  and  $r > 0$ , so  $H^p(U) \cong H^p(B)$  for all  $p \geq 0$ . Since  $B$  is contractible,  $H^p(B) = 0$  for all  $p > 0$ . The restriction of  $\omega$  on  $U$  gives a class  $[\omega] \in H^p(U) = 0$ , so  $\omega$  is cohomologous to zero on  $U$ . Thus  $\omega$  is exact on  $U$ .  $\square$

**Definition 2.18.** Let  $M$  be a manifold, let  $\gamma : [0, 1] \rightarrow M$  be a continuous or smooth path, and let  $x = \gamma(0)$  and  $y = \gamma(1)$ . A **homotopy of paths** from  $x$  to  $y$  is a map

$$\begin{aligned} F : [0, 1] \times [0, 1] &\longrightarrow M \\ (0, t) &\longmapsto x \\ (1, t) &\longmapsto y \end{aligned}$$

**Proposition 2.19.** *Let  $\gamma_0$  and  $\gamma_1$  be homotopic paths on a manifold  $M$ , and let  $\omega \in \Omega^1(M)$  be closed. Then*

$$\int_0^1 \gamma_0^* \omega = \int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

*Proof.* Lee's introduction to smooth manifolds. The idea is

$$\int_{\gamma_0} \omega - \int_{\gamma_1} \omega = \int_{\gamma_0 \cup \gamma_1} \omega = \int_{\text{Im } F} \omega = 0,$$

by Stokes' theorem.  $\square$

Recall that  $M$  is **simply connected**, so  $\pi_1(M) = 0$ , if any path  $\gamma$  from  $x$  to  $x$  is homotopic equivalent to a point.

**Proposition 2.20.** *Let  $M$  be a simply connected orientable manifold. Then*

$$H^1(M) = 0.$$

*Proof.* Let  $\omega \in \Omega^1(M)$  be a closed form. Then claim that  $\omega$  is exact if and only if  $\int_\gamma \omega = 0$  for all loops  $\gamma$ , that is paths from  $x$  to  $x$ .

- The proof of the claim. Assume that  $\omega = df$  is exact for  $f \in \Omega^0(M)$ . By Proposition 2.19,

$$\int_\gamma \omega = \int_{\text{trivial loop}} \omega = 0.$$

Assume that  $\int_\gamma \omega = 0$  for all loops  $\gamma$ . Fix  $x$ . Let

$$f(y) = \int_x^y \omega.$$

Since  $\int_{\gamma_1 \cup \gamma_2} \omega = 0$ ,  $f$  is well-defined, that is it does not depend on the choice of the path. Then  $df = \omega$ . This can be checked locally, that is in an open set of  $\mathbb{R}^n$ . Here it follows from the fundamental theorem of calculus.

- The claim implies the proposition. Being simply connected, any loop inside  $M$  is homotopic equivalent to the trivial loop. For all loops  $\gamma$  and for all closed  $\omega$ ,  $\int_\gamma \omega = 0$  by Proposition 2.19, so  $\omega$  is exact. Thus  $[\omega] = 0$  in  $H^1(M)$ .  $\square$

## 2.3 Some homological algebra

Let  $C^\bullet$  be a sequence of vector spaces, that is  $C^k$  is a vector space for  $k \in \mathbb{Z}$ .

**Definition 2.21.**  $(C^\bullet, d^\bullet)$  is a **cochain complex** if  $C^\bullet$  is a sequence of vector spaces and  $d^\bullet$  is a sequence of linear maps  $d^k : C^k \rightarrow C^{k+1}$  such that the composition  $d^{k+1} \circ d^k : C^k \rightarrow C^{k+1} \rightarrow C^{k+2}$  is zero for all  $k$ . Then  $d^\bullet$  is the **differential**.

**Definition 2.22.** The elements of

$$\mathcal{Z}^k(C^\bullet, d^\bullet) = \text{Ker}(d^k : C^k \rightarrow C^{k+1}) \subset C^k$$

are called **cocycles**. The elements of

$$\mathcal{B}^k(C^\bullet, d^\bullet) = \text{Im}(d^k : C^{k-1} \rightarrow C^k) \subset C^k$$

are called **coboundaries**. Then  $d^{k-1} \circ d^k = 0$ , so  $\mathcal{B}^k \subset \mathcal{Z}^k$ . The quotients

$$H^k(C^\bullet, d^\bullet) = \mathcal{Z}^k(C^\bullet, d^\bullet) / \mathcal{B}^k(C^\bullet, d^\bullet)$$

are the  **$k$ -th cohomology groups** of  $(C^\bullet, d^\bullet)$ .

**Definition 2.23.** Let  $(C^\bullet, d^\bullet)$  and  $(D^\bullet, d^\bullet)$  be two cochain complexes. A map  $f : (C^\bullet, d^\bullet) \rightarrow (D^\bullet, d^\bullet)$  is a sequence of linear maps  $f^k : C^k \rightarrow D^k$  such that  $f^{k+1} \circ d^k = d^k \circ f^k$  for all  $k$ , so

$$\begin{array}{ccccccc} \dots & \longrightarrow & C^k & \xrightarrow{d^k} & C^{k+1} & \xrightarrow{d^{k+1}} & C^{k+2} \longrightarrow \dots \\ & & \downarrow f^k & & \downarrow f^{k+1} & & \downarrow f^{k+2} \\ \dots & \longrightarrow & D^k & \xrightarrow{d^k} & D^{k+1} & \xrightarrow{d^{k+1}} & D^{k+2} \longrightarrow \dots \end{array}.$$

**Proposition 2.24.** Let  $f : (C^\bullet, d^\bullet) \rightarrow (D^\bullet, d^\bullet)$  be a map between cochain complexes. Then there exists a natural induced map

$$f^k : H^k(C^\bullet, d^\bullet) \rightarrow H^k(D^\bullet, d^\bullet).$$

*Proof.* Let  $[\omega] \in H^k(C^\bullet, d^\bullet) = \mathcal{Z}^k(C^\bullet, d^\bullet) / \mathcal{B}^k(C^\bullet, d^\bullet)$  for  $\omega \in \mathcal{Z}^k(C^\bullet, d^\bullet)$ , that is  $d^k(\omega) = 0$ . I want to check that  $f^k(\omega) \in \mathcal{Z}^k(D^\bullet, d^\bullet)$ . By definition of maps,  $d^k(f^k(\omega)) = f^{k+1}(d^k(\omega)) = 0$ , so there is a map

$$\mathcal{Z}^k(C^\bullet, d^\bullet) \rightarrow \mathcal{Z}^k(D^\bullet, d^\bullet).$$

Now I need to check that if  $\omega \in \mathcal{B}^k(C^\bullet, d^\bullet)$  then  $f^k(\omega) \in \mathcal{B}^k(D^\bullet, d^\bullet)$ . <sup>7</sup>

□

**Definition 2.25.** A sequence of linear maps

$$C^1 \xrightarrow{f^1} C^2 \rightarrow \dots \rightarrow C^{k-1} \xrightarrow{f^{k-1}} C^k$$

between vector spaces is said to be **exact** if for all  $i$ ,  $\text{Ker } f^i = \text{Im } f^{i-1}$ .

**Example 2.26.**

- A sequence

$$0 \rightarrow C^1 \xrightarrow{f^1} C^2$$

is exact if and only if  $f^1$  is injective.

- A sequence

$$C^1 \xrightarrow{f^1} C^2 \rightarrow 0$$

is exact if and only if  $f^1$  is surjective.

- An exact sequence

$$0 \rightarrow C^1 \xrightarrow{f^1} C^2 \xrightarrow{f^2} C^3 \rightarrow 0$$

is called a **short exact sequence**. In particular  $f^1$  is injective and  $f^2$  is surjective.

<sup>7</sup>Exercise

- Any long exact sequence

$$C^1 \xrightarrow{f^1} C^2 \rightarrow \dots \rightarrow C^{q-1} \xrightarrow{f^{q-1}} C^q$$

can be split into short exact sequences

$$\begin{array}{ccccccc} \dots \rightarrow C^{k-1} & \xrightarrow{f^{k-1}} & C^k & \xrightarrow{f^k} & C^{k+1} & \rightarrow & \dots \\ & \searrow & \nearrow & \searrow & \nearrow & & \\ & \text{Im } f^{k-1} = \text{Ker } f^k & & \text{Im } f^k = \text{Ker } f^{k+1} & & & \\ & \nearrow & \searrow & \nearrow & \searrow & & \\ 0 & & 0 & & 0 & & 0 \end{array}, \quad k = 2, \dots, q-1.$$

**Lemma 2.27** (Snake lemma). *Consider the commutative diagram*

$$\begin{array}{ccccccc} C^1 & \xrightarrow{f^1} & C^2 & \xrightarrow{f^2} & C^3 & \longrightarrow & 0 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \\ 0 & \longrightarrow & D^1 & \xrightarrow{g^1} & D^2 & \xrightarrow{g^2} & D^3 \end{array},$$

such that both the horizontal lines are exact sequences. Then there exists a long exact sequence

$$\text{Ker } \alpha_1 \rightarrow \text{Ker } \alpha_2 \rightarrow \text{Ker } \alpha_3 \xrightarrow{\delta} \text{Coker } \alpha_1 \rightarrow \text{Coker } \alpha_2 \rightarrow \text{Coker } \alpha_3.$$

If

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^1 & \xrightarrow{f^1} & C^2 & \xrightarrow{f^2} & C^3 \longrightarrow 0 \\ & & \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 \\ 0 & \longrightarrow & D^1 & \xrightarrow{g^1} & D^2 & \xrightarrow{g^2} & D^3 \longrightarrow 0 \end{array},$$

then

$$0 \rightarrow \text{Ker } \alpha_1 \rightarrow \text{Ker } \alpha_2 \rightarrow \text{Ker } \alpha_3 \xrightarrow{\delta} \text{Coker } \alpha_1 \rightarrow \text{Coker } \alpha_2 \rightarrow \text{Coker } \alpha_3 \rightarrow 0.$$

*Proof.* We are going to construct  $\delta : \text{Ker } \alpha_3 \rightarrow \text{Coker } \alpha_1$ . Let  $x \in \text{Ker } \alpha_3$ . There exists  $y \in C^2$  such that  $f^2(y) = x$  because  $f^2$  is surjective. Let  $z = \alpha_2(y)$  then

$$g^2(z) = g^2(\alpha_2(y)) = \alpha_3(f^2(y)) = \alpha_3(x) = 0,$$

since  $x \in \text{Ker } \alpha_3$ . Then  $z \in \text{Ker } g^2 = \text{Im } g^1$ , so there exists  $w \in D^1$  such that  $z = g^1(w)$ . The idea is

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{Ker } \alpha_1 & \longrightarrow & \text{Ker } \alpha_2 & \longrightarrow & \text{Ker } \alpha_3 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ C^1 & \xrightarrow{f^1} & y \in C^2 & \xrightarrow{f^2} & x \in C^3 & \longrightarrow & 0 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \\ 0 & \longrightarrow & w \in D^1 & \xrightarrow{g^1} & z \in D^2 & \xrightarrow{g^2} & D^3 \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{Coker } \alpha_1 & \longrightarrow & \text{Coker } \alpha_2 & \longrightarrow & \text{Coker } \alpha_3 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}.$$

Define  $\delta(x) = [w] \in \text{Coker } \alpha_1 = D^1 / \text{Im } \alpha_1$ . Need to check that  $\delta$  is well-defined, so  $[w]$  does not depend on our choice of  $w$  and  $y$ . The rest is an exercise. <sup>8</sup>  $\square$

<sup>8</sup>Exercise



**Example 2.30.** Let  $M = S^1$ , let  $N = (0, 1)$  and  $S = (0, -1)$ , and let  $U = M \setminus \{N\}$  and  $V = M \setminus \{S\}$ , so  $M = U \cup V$  and  $U \cap V = M \setminus \{N, S\}$ . Then

$$H^p(U) \cong H^p(V) \cong H^p((0, 1)) \cong \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \quad (0, 1) \subset \mathbb{R},$$

and

$$H^p(U \cap V) = H^p(U \setminus \{S\}) = H^p\left(\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)\right) = \begin{cases} \mathbb{R}^2 & p = 0 \\ 0 & p > 0 \end{cases}, \quad \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right) \subset \mathbb{R},$$

so

$$\begin{array}{ccccccc} 0 \rightarrow H^0(M) \rightarrow H^0(U) \oplus H^0(V) & \xrightarrow{\phi} & H^0(U \cap V) & \xrightarrow{\delta} & H^1(M) \rightarrow H^1(U) \oplus H^1(V) \rightarrow H^1(U \cap V) \rightarrow \dots \\ \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ \mathbb{R} & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R}^2 & & 0 \oplus 0 & & 0 \end{array}.$$

Then  $\text{Im } \phi = \mathbb{R} \subset H^0(U \cap V) = \mathbb{R}^2$ . Thus

$$H^1(M) = \text{Coker } \phi = \mathbb{R}^2 / \text{Im } \phi \cong \mathbb{R}.$$

**Remark 2.31.** Let

$$0 \rightarrow C^1 \rightarrow \dots \rightarrow C^k \rightarrow 0$$

be an exact sequence. Then

$$\sum_k (-1)^k \dim C^k = 0.^9$$

In our  $M = S^1$  case  $1 - 2 + 2 - \dim H^1(M) = 0$ , so  $\dim H^1(M) = 1$ . Thus  $H^1(M) \cong \mathbb{R}$ .

**Example 2.32.** Let  $M = S^n \subset \mathbb{R}^{n+1}$  be the  $n$ -dimensional sphere. Then

$$H^p(M) = \begin{cases} \mathbb{R} & p = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

By induction on  $n$ .

$n = 1$ . Ok.

$n > 1$ . Let  $U = M \setminus \{N\}$  and  $V = M \setminus \{S\}$ , so  $U \cap V \neq \emptyset$  and  $U \cup V = M$ . Then

$$U \cong V \cong \mathbb{R}^n, \quad U \cap V = V \setminus \{N\} \cong \mathbb{R}^n \setminus \{0\} \cong S^{n-1},$$

so

$$\begin{array}{ccccccc} 0 \rightarrow H^0(M) \rightarrow H^0(U) \oplus H^0(V) \rightarrow H^0(U \cap V) & \xrightarrow{\delta} & H^1(M) \rightarrow H^1(U) \oplus H^1(V) \rightarrow \dots \\ \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ \mathbb{R} & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R} & & 0 \oplus 0 \end{array}.$$

Then  $1 - 2 + 1 - \dim H^1(M) = 0$ , so  $\dim H^1(M) = 0$ . Thus  $H^1(M) = 0$ . Then for  $p > 0$

$$\begin{array}{ccccccc} \dots \rightarrow H^p(U) \oplus H^p(V) \rightarrow H^p(U \cap V) & \xrightarrow{\delta} & H^{p+1}(M) \rightarrow H^{p+1}(U) \oplus H^{p+1}(V) \rightarrow \dots \\ \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ 0 \oplus 0 & & 0 & & 0 \oplus 0 & & 0 \oplus 0 \end{array}$$

is exact, so  $H^p(U \cap V) \cong H^{p+1}(M)$ . By induction

$$H^p(U \cap V) = H^{p+1}(M) = \begin{cases} \mathbb{R} & p = n - 1 \\ 0 & \text{otherwise} \end{cases}.$$

---

<sup>9</sup>Exercise

*Proof of Theorem 2.29.* By Proposition 2.28 for all  $p$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^p(M) & \longrightarrow & \Omega^p(U) \oplus \Omega^p(V) & \longrightarrow & \Omega^p(U \cap V) \longrightarrow 0 \\ & & \downarrow d_M^p & & \downarrow (d_U^p, d_V^p) & & \downarrow d_{U \cap V}^p \\ 0 & \longrightarrow & \Omega^{p+1}(M) & \longrightarrow & \Omega^{p+1}(U) \oplus \Omega^{p+1}(V) & \longrightarrow & \Omega^{p+1}(U \cap V) \longrightarrow 0 \end{array}$$

are exact. Recall  $d$  commutes with the pull-back. By the strong snake lemma,

$$\begin{array}{ccccccc} \text{Coker } d_M^{p-1} & \longrightarrow & \text{Coker } (d_U^{p-1}, d_V^{p-1}) & \longrightarrow & \text{Coker } d_{U \cap V}^{p-1} & \longrightarrow & 0 \\ & & \downarrow \partial_M^p = d_M^p & & \downarrow (\partial_U^p, \partial_V^p) = (d_U^p, d_V^p) & & \downarrow \partial_{U \cap V}^p = d_{U \cap V}^p \\ 0 & \longrightarrow & \text{Ker } d_M^{p+1} & \longrightarrow & \text{Ker } (d_U^{p+1}, d_V^{p+1}) & \longrightarrow & \text{Ker } d_{U \cap V}^{p+1} \end{array},$$

which is well-defined, since  $d^{p+1} \circ d^p = 0$ . By the weak snake lemma again,

$$\text{Ker } \partial_M^p \rightarrow \text{Ker } (\partial_U^p, \partial_V^p) \rightarrow \text{Ker } \partial_{U \cap V}^p \xrightarrow{\delta} \text{Coker } \partial_M^p \rightarrow \text{Coker } (\partial_U^p, \partial_V^p) \rightarrow \text{Coker } \partial_{U \cap V}^p.$$

Then  $\text{Coker } d_M^{p-1} = \Omega^p(M) / \text{Im } d_M^{p-1}$ . There exists

$$H^p(M) = \text{Ker } d_M^p / \text{Im } d_M^{p-1} \xrightarrow{\sim} \text{Ker } (\Omega^p(M) / \text{Im } d_M^{p-1} \rightarrow \text{Ker } d_M^{p+1}) = \text{Ker } \partial_M^p.$$

Similarly,  $\text{Ker } (\partial_U^p, \partial_V^p) \cong H^p(U) \oplus H^p(V)$  and  $\text{Ker } \partial_{U \cap V}^p \cong H^p(U \cap V)$ . There exists

$$H^{p+1}(M) = \text{Ker } d_M^{p+1} / \text{Im } d_M^p \xrightarrow{\sim} \text{Coker } (\Omega^p(M) / \text{Im } d_M^{p-1} \rightarrow \text{Ker } d_M^{p+1}) = \text{Coker } \partial_M^p.$$

Similarly,  $\text{Coker } (\partial_U^p, \partial_V^p) \cong H^{p+1}(U) \oplus H^{p+1}(V)$  and  $\text{Coker } \partial_{U \cap V}^p \cong H^{p+1}(U \cap V)$ . □

**Example 2.33.** Let  $\mathbb{T}^2 = S^1 \times S^1$  be the torus. Then

$$H^p(\mathbb{T}^2) = \begin{cases} \mathbb{R} & p = 0, 2 \\ \mathbb{R} \oplus \mathbb{R} & p = 1 \end{cases}.$$

We leave the proof as an exercise. <sup>10</sup>

**Definition 2.34.** Let  $M$  be a manifold, and let  $U = \{U_i\}$  be an open cover of  $M$ . Then  $U$  is said to be **good** if for all  $I = (i_1, \dots, i_p)$ ,  $U_{i_1} \cap \dots \cap U_{i_p}$  is either  $\emptyset$  or contractible.

**Lemma 2.35.** Let  $M$  be a connected manifold which admits a finite good cover. Then for all  $p \geq 0$ ,  $H^p(M)$  is a finite dimensional vector space.

**Exercise.** Find a counterexample without assuming there exists a finite good cover.

*Proof.* Let  $U$  be a finite good cover. Define  $k = \#U$ . By induction on  $k$ .

$k = 1$ .  $M = U_1$  is contractible, so

$$H^p(M) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & \text{otherwise} \end{cases}.$$

$k > 1$ . Assume ok for covers with at most  $k-1$  elements. Let  $U = \bigcup_{i=1}^{k-1} U_i$  and  $V = U_k$ . Then  $U \cup V = M$  and  $U \cap V \neq \emptyset$ , so Mayer-Vietoris holds. By induction  $H^p(U)$  and  $H^p(V)$  are finite dimensional, since  $H^p(U)$  is covered by  $k-1$  of  $U_i$  and  $H^p(V)$  is contractible. Then  $U \cap V = \bigcup_{i=1}^{k-1} (U_i \cap U_k)$ , and  $\{U_i \cap U_k\}$  is a good cover of  $U \cap V$  with  $k-1$  elements. <sup>11</sup> By induction  $H^p(U \cap V)$  is finite dimensional. Thus  $H^p(M)$  is also finite dimensional. □

<sup>10</sup>Exercise

<sup>11</sup>Exercise



**Fact.** Any manifold admits a good cover.

**Theorem 2.36.** *Let  $M$  be a compact connected manifold. Then  $H^p(M)$  is finite dimensional.*

*Proof.* Follows from the fact and Lemma 2.35. □

## 2.5 Compactly supported de Rham cohomology

Let  $M$  be a manifold, and let  $\omega \in \Omega_c^p(M)$ . Then  $d\omega \in \Omega_c^{p+1}(M)$  and  $d^2 = 0$ , so

$$\Omega_c^p(M) \xrightarrow{d} \Omega_c^{p+1}(M) \xrightarrow{d} \dots$$

**Definition 2.37.** The  $p$ -th compactly supported de Rham cohomology group is

$$H_c^p(M) = \mathcal{Z}_c^p(M) / \mathcal{B}_c^p(M) = \text{Ker}(d : \Omega_c^p(M) \rightarrow \Omega_c^{p+1}(M)) / \text{Im}(d : \Omega_c^{p-1}(M) \rightarrow \Omega_c^p(M)).$$

**Example.** If  $M$  is compact, then

$$H_c^p(M) = H^p(M), \quad p \geq 0.$$

**Proposition 2.38.** *Let  $M$  be a non-compact connected manifold. Then*

$$H_c^0(M) = 0.$$

Recall if  $M$  is connected  $H^0(M) = \mathbb{R}$ , since  $H^0(M) = \{f \text{ constant on } M\}$ .

*Proof.*  $H_c^0(M) = \{f \text{ constant on } M \text{ and with compact support}\}$ . Since  $M$  is non-compact, if  $f \in \Omega_c^0(M)$ , then  $\text{supp } f \subsetneq M$ . Thus there exists  $x \in M$  such that  $f(x) = 0$ , so  $f \equiv 0$ , since  $f$  is constant. □

**Remark 2.39.** Let  $f : M \rightarrow N$  be a smooth morphism between manifolds, and let  $\omega \in \Omega_c^p(N) \subset \Omega^p(N)$ . Then  $f^*\omega \in \Omega^p(M)$ , and  $\text{supp } f^*\omega \subset f^{-1}(\text{supp } \omega)$ , which is not compact in general, so  $f^*\omega \notin \Omega_c^p(M)$  in general. If  $f$  is **proper**, that is  $f^{-1}(K)$  is compact for all compact subsets  $K \subset N$ , then  $f^* : \Omega_c^p(N) \rightarrow \Omega_c^p(M)$  is well-defined.

**Exercise.** If  $f$  is a diffeomorphism then  $f^*$  induces an isomorphism  $H_c^p(N) \rightarrow H_c^p(M)$ .

**Definition 2.40.** Let  $M_0$  and  $M_1$  be manifolds without boundary, and let  $f_i : M_0 \rightarrow M_1$  be smooth morphisms for  $i = 0, 1$ . Then  $f_0$  and  $f_1$  are **properly smoothly homotopic** if there exists a smooth  $H : M_0 \times [0, 1] \rightarrow M_1$  such that  $H(\cdot, i) = f_i(\cdot)$  for  $i = 0, 1$  and  $H$  is proper. Then  $M_0$  and  $M_1$  are **properly smoothly homotopically equivalent** if there exist smooth morphisms  $f : M_0 \rightarrow M_1$  and  $g : M_1 \rightarrow M_0$  such that  $f \circ g \sim \text{id}_{M_1}$  and  $g \circ f \sim \text{id}_{M_0}$ , where the equivalences are properly homotopic.

**Notation.**  $f_t(\cdot) = H(\cdot, t) : M_0 \rightarrow M_1$ .

**Remark 2.41.** To say that  $H$  is proper is not the same as saying  $f_t$  is proper for all  $t$ .

**Exercise.** Find  $H$  such that  $f_t$  is proper but  $H$  is not. A hint is to let  $M_0 = M_1 = \mathbb{R}$  and  $H : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$  such that  $f_t^{-1}(0)$  is bounded for all  $t$  but  $H^{-1}(0)$  is not.

**Proposition 2.42.** *If  $M_0$  and  $M_1$  are properly homotopically equivalent then*

$$H_c^p(M_0) \cong H_c^p(M_1).$$

Let  $M$  be a manifold, and let  $i : U \hookrightarrow M$  be an open set. Then there exist linear **push-forwards**

$$i_* : \Omega_c^p(U) \rightarrow \Omega_c^p(M), \quad p \geq 0.$$

Let  $\omega \in \Omega_c^p(U)$ . Then  $\omega = 0$  outside  $U$ . We can define

$$i_*\omega = \begin{cases} \omega & \text{on } U \\ 0 & \text{on } M \setminus U \end{cases}.$$

If  $j : V \hookrightarrow U$  and  $i : U \hookrightarrow M$ , then  $(i \circ j)_* = i_* \circ j_*$ .

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**Lemma 2.43.** *Let  $M$  be a manifold, and let  $i : U \hookrightarrow M$  be an immersion such that  $U$  is open. Then for all  $p \geq 0$ ,  $i_* : \Omega_c^p(U) \rightarrow \Omega_c^p(M)$  commutes with  $d$ , that is*

$$d(i_*\omega) = i_*d\omega, \quad \omega \in \Omega_c^p(U).$$

*In particular if  $\omega$  is closed then  $i_*\omega$  is closed, and if  $\omega$  is exact then  $i_*\omega$  is exact.*

*Proof.*

$$d(i_*\omega) = \begin{cases} d\omega & \text{on } U \\ 0 & \text{on } M \setminus U \end{cases} = i_*d\omega.$$

Let  $\omega$  be closed, so  $d\omega = 0$ . Then  $d(i_*\omega) = i_*d\omega = 0$ , so  $i_*\omega$  is closed. Similarly for exactness.  $\square$

Let  $U \hookrightarrow M$  be as before. Then there exist

$$i_* : H_c^p(U) \rightarrow H_c^p(M), \quad p \geq 0.$$

**Proposition 2.44** (Punctured manifolds). *Let  $M$  be a manifold of dimension  $n$ , let  $x \in M$ , and let  $i : M \setminus \{x\} \hookrightarrow M$ . Then*

- for all  $p \geq 2$ ,  $i_* : H_c^p(M \setminus \{x\}) \rightarrow H_c^p(M)$  is an isomorphism.
- for all  $p \geq 1$ , if  $M$  is compact  $i_* : H_c^p(M \setminus \{x\}) \rightarrow H_c^p(M) = H^p(M)$  is an isomorphism.

*Proof.*

- Injectivity.

$p \geq 2$ . Let  $\omega \in \Omega_c^p(M \setminus \{x\})$  be closed such that  $i_*[\omega] = 0$ , so  $[i_*\omega] = 0$  in  $H_c^p(M)$ . The goal is  $[\omega] = 0$ . There exists  $\eta \in \Omega_c^{p-1}(M)$  such that  $i_*\omega = d\eta$ . By Poincaré lemma there exists  $U \subset M$  containing  $x$  such that  $H^q(U) = 0$  for all  $q \geq 1$ . Then  $i_*\omega = 0$  in a neighbourhood of  $x$  because  $\text{supp } \omega \subset M \setminus \{x\}$ , so  $d\eta = 0$  in a neighbourhood of  $x$ . By taking  $U$  smaller we can assume  $\eta$  is closed. Since  $p \geq 2$ ,  $[\eta] \in H^{p-1}(U) = 0$ , so  $\eta$  is exact. Then there exists  $\sigma \in \Omega^{p-2}(U)$  such that  $\eta = d\sigma$  on  $U$ . Let  $(U, M \setminus \{x\})$  be an open cover of  $M$ , let  $(f_U, f_{M \setminus \{x\}})$  be a partition of unity, and let  $\eta' = \eta - d(i_*(f_U\sigma))$ . On a neighbourhood of  $x$ ,  $\eta' = 0$  because  $i_*(f_U\sigma) = \sigma$ , so  $\text{supp } \eta' \subset M \setminus \{x\}$ . Thus  $\eta' \in \Omega_c^{p-1}(M \setminus \{x\})$  and  $\omega = d\eta'$ , so  $[\omega] = 0$ .

$p = 1$ . The same proof. Let  $\omega \in \Omega_c^1(M \setminus \{x\})$  be closed such that  $[i_*\omega] = 0$ . There exists  $\eta \in \Omega_c^0(M)$  such that  $i_*\omega = d\eta$ . By taking an open set  $U \subset M$  such that  $x \in U$ , we may assume  $d\eta = 0$ , so  $\eta = c$  is constant on  $U$ . Let  $\eta' = \eta - c$ . Then  $\eta' = 0$  on  $U$ . If  $M$  is compact then  $\eta' \in \Omega_c^0(M \setminus \{x\})$ . Thus  $\omega = d\eta'$ , so  $[\omega] = 0$ .

- Surjectivity.

$p \geq 1$ . Let  $[\omega] \in H_c^p(M)$  such that  $\omega$  is closed. By Poincaré lemma there exists an open  $U \ni x$  such that  $\omega$  is exact, so there exists  $\sigma \in \Omega^{p-1}(U)$  such that  $\omega = d\sigma$ . Let  $(f_U, f_{M \setminus \{x\}})$  be a partition of unity as before, and let  $\omega' = \omega - d(i_*(f_U\sigma))$ . Then  $\omega' = 0$  in a neighbourhood of  $x$  and  $[\omega'] = [\omega]$ , and  $\omega'|_{M \setminus \{x\}} \in \Omega_c^p(M \setminus \{x\})$ . Thus  $[i_*\omega']_{M \setminus \{x\}} = [\omega'] = [\omega]$ .  $\square$

**Exercise.** Compute  $H_c^1(\mathbb{R}^2 \setminus \{0\})$  by hands.

**Example 2.45.**

$$H_c^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}.$$

Recall  $\mathbb{R}^n \cong S^n \setminus \{x\}$  for  $x \in S^n$ . By Proposition 2.44, by  $M = S^n$ ,

$$H_c^p(\mathbb{R}^n) = H_c^p(S^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}, \quad p \geq 1,$$

and  $H_c^0(\mathbb{R}^n) = 0$ .

Let  $M$  be a manifold such that  $M = U \cup V$  for open  $U$  and  $V$  such that  $U \cap V \neq \emptyset$ , and let

$$\begin{array}{ccccc} & U & & \Omega^p(U) & \\ j_U \nearrow & & i_U \searrow & & \nearrow i_{U*} \\ U \cap V & & & \Omega^p(U \cap V) & \longrightarrow \Omega^p(M) \\ j_V \searrow & & i_V \nearrow & \nwarrow j_{V*} & \nearrow i_{V*} \\ & V & & \Omega^p(V) & \end{array}, \quad p \geq 0.$$

**Proposition 2.46.** *We have a short exact sequence*

$$0 \leftarrow \Omega^p(M) \xleftarrow{i} \Omega^p(U) \oplus \Omega^p(V) \xleftarrow{j} \Omega^p(U \cap V) \leftarrow 0,$$

where  $i = i_{U*} + i_{V*}$  and  $j = (-j_{U*}, j_{V*})$ .

*Proof.*

- $j$  is injective. Let  $\omega \in \Omega^p(U \cap V)$  such that  $j(\omega) = 0$ , so  $j_{U*}\omega = j_{V*}\omega = 0$ . Then  $\omega = 0$ , so  $j$  is injective.
- $\text{Ker } i = \text{Im } j$ . Let  $\omega \in \Omega^p(U \cap V)$ . Then  $i(j(\omega)) = i(-j_{U*}\omega, j_{V*}\omega) = -i_{U*}j_{U*}\omega + i_{V*}j_{V*}\omega = 0$ , so  $\text{Ker } i \supset \text{Im } j$ . Let  $(\omega_1, \omega_2) \in \text{Ker } i$ . Then  $i_{U*}\omega_1 + i_{V*}\omega_2 = 0$ , so  $i_{V*}\omega_1 = -i_{V*}\omega_2$ , so  $\text{supp } \omega_1 \subset U \cap V$  and  $\text{supp } \omega_2 \subset U \cap V$ , so there exists  $\eta \in \Omega^p(U \cap V)$  such that  $j_{U*}\eta = -\omega_1$  and  $j_{V*}\eta = \omega_2$ , so  $(\omega_1, \omega_2) = j(\eta)$ , so  $\text{Ker } i \subset \text{Im } j$ .
- $i$  is surjective. Let  $\omega \in \Omega_c^p(M)$ , and let  $\{f_U, f_V\}$  be a partition of unity with respect to  $\{U, V\}$ . Define  $\omega_U = f_U \cdot \omega|_U \in \Omega_c^p(U)$  and  $\omega_V = f_V \cdot \omega|_V \in \Omega_c^p(V)$ . Then  $i(\omega_U, \omega_V) = i_{U*}\omega_U + i_{V*}\omega_V = f_U \cdot \omega + f_V \cdot \omega = (f_U + f_V) \cdot \omega = \omega$ .

□

Thus for all  $p$  we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_c^p(U \cap V) & \longrightarrow & \Omega_c^p(U) \oplus \Omega_c^p(V) & \longrightarrow & \Omega_c^p(M) \longrightarrow 0 \\ & & \downarrow d & & \downarrow (d, d) & & \downarrow d \\ 0 & \longrightarrow & \Omega_c^{p+1}(U \cap V) & \longrightarrow & \Omega_c^{p+1}(U) \oplus \Omega_c^{p+1}(V) & \longrightarrow & \Omega_c^{p+1}(M) \longrightarrow 0 \end{array}.$$

**Theorem 2.47.** *There exists  $\delta : H_c^p(M) \rightarrow H_c^{p+1}(U \cap V)$  such that*

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_c^p(U \cap V) & \longrightarrow & H_c^p(U) \oplus H_c^p(V) & \longrightarrow & H_c^p(M) \longrightarrow \\ & & & & \delta & & \\ & \longleftarrow & H_c^{p+1}(U \cap V) & \longrightarrow & H_c^{p+1}(U) \oplus H_c^{p+1}(V) & \longrightarrow & H_c^{p+1}(M) \longrightarrow \dots \end{array}$$

*Proof.* Same proof as Mayer-Vietoris for  $H^p(M)$ .

□

## 2.6 Poincaré duality

Let  $M$  be an orientable manifold. Then  $H^p(M) \cong H_c^{n-p}(M)^*$ , the dual of  $H_c^{n-p}(M)$ .

**Proposition 2.48.** *Let  $M$  be a manifold. Then the bilinear map*

$$\begin{aligned} \cup : H^p(M) \times H^q(M) &\longrightarrow H^{p+q}(M) \\ ([\omega], [\eta]) &\longmapsto [\omega \wedge \eta] \end{aligned}$$

*is well-defined, and*

$$[\omega] \cup [\eta] = (-1)^{p \cdot q} [\eta] \cup [\omega].$$

*Proof.* Follows from the Leibnitz rule and Proposition 1.6.

□

**Lemma 2.49.** *Let  $M$  be oriented without boundary of dimension  $n$ . Then there exists a linear map*

$$\begin{aligned} I_M : H_c^n(M) &\longrightarrow \mathbb{R} \\ [\omega] &\longmapsto \int_M \omega \end{aligned}$$

and  $I_M$  is surjective.

$I_M$  is called **integration**.

*Proof.* Let  $\omega \in \Omega_c^n(M)$  such that  $[\omega] = 0$ , so  $\omega$  is exact. By Stokes  $\int_M \omega = 0$ , so  $I_M$  is well-defined and it is linear. It is enough to show there exists closed  $\omega \in \Omega_c^n(M)$  such that  $\int_M \omega \neq 0$ . Take a volume form  $\omega_0$ , which exists because  $M$  is oriented. Take  $f \in C^\infty(M)$  for  $f \geq 0$  and with compact support. Let  $\omega = f \cdot \omega_0 \in \Omega_c^n(M)$ . Then  $\omega$  is closed because  $\Omega_c^{n+1}(M) = 0$  and  $\int_M \omega = \int_M (f \cdot \omega_0) > 0$ , by definition of volume forms.  $\square$

**Example 2.50.** Let  $M = S^n$ , and let  $\omega \in \Omega_c^n(M)$  such that  $\int_M \omega = 0$ . We want to show that  $\omega$  is exact. Since  $M$  is compact,  $H_c^n(M) = H^n(M) = \mathbb{R}$ . By Lemma 2.49  $I_M : H_c^n(M) \rightarrow \mathbb{R}$  is surjective, and  $H_c^n(M) = \mathbb{R}$ , so  $I_M$  is injective. Since  $\int_M \omega = 0$ ,  $I_M([\omega]) = 0$ , so  $[\omega] = 0$ . Thus  $\omega$  is exact.

Let  $M$  be a connected manifold of dimension  $n$ . If  $\omega_2 \in H_c^q(M)$  then  $[\omega_1 \wedge \omega_2] \in H_c^{p+q}(M)$ . Then

$$\cup : H^p(M) \times H_c^q(M) \rightarrow H_c^{p+q}(M).$$

Let  $M$  be an oriented manifold without boundary of dimension  $n$ . Then

$$\begin{aligned} I_M : H_c^n(M) &\longrightarrow \mathbb{R} \\ [\omega] &\longmapsto \int_M \omega \end{aligned}$$

Choose  $q = n - p$ . Then

$$I_M \circ \cup : H^p(M) \times H_c^{n-p}(M) \rightarrow H_c^n(M) \rightarrow \mathbb{R}.$$

Recall that if  $\phi : V \times W \rightarrow \mathbb{R}$  is bilinear, then there exists

$$\begin{aligned} V &\longrightarrow W^* = \text{Hom}(W, \mathbb{R}) & \phi_v &: W \longrightarrow \mathbb{R} \\ v &\longmapsto \phi_v & w &\longmapsto \phi(v, w) \end{aligned}$$

Thus, we get

$$H^p(M) \rightarrow H_c^{n-p}(M)^*.$$

Poincaré duality says that this is an isomorphism.

**Example.** Assume  $M$  is compact and oriented. Then  $H^p(M) \xrightarrow{\sim} H^{n-p}(M)$ , so

$$b^p(M) = b^{n-p}(M).$$

**Example 2.51.** Let  $U \subset \mathbb{R}^n$  be an open subset diffeomorphic to  $\mathbb{R}^n$ . Then

$$H^p(U) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \quad H_c^p(U) = \begin{cases} 0 & p < n \\ \mathbb{R} & p = n \end{cases}.$$

We want to show that Poincaré duality holds. We just need to check that Poincaré duality holds for  $p = 0$ . It is enough to show that  $\phi : H^0(U) \rightarrow H_c^n(U)^*$  is injective, that is there exists  $\omega$  such that  $\phi(\omega) \neq 0$ . Given  $\omega \in H^0(U)$ ,

$$\begin{aligned} \phi(\omega) : H_c^n(U) &\longrightarrow \mathbb{R} \\ \eta &\longmapsto \int_U \eta \wedge \omega \end{aligned}$$

Then  $\omega = c$  is a constant function on  $U$ , so

$$\begin{aligned} \phi(\omega) : H_c^n(U) &\longrightarrow \mathbb{R} \\ \eta &\longmapsto \int_U c\omega \end{aligned}$$

If  $c \neq 0$  there exists  $\eta$  such that this map is not zero, so  $\phi(\omega) \neq 0$ . Thus  $\phi$  is an isomorphism.

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We will prove the following.

**Theorem 2.52** (Poincaré duality). *Assume that  $M$  is an oriented manifold, without boundary, such that there exists a finite open cover  $U = \{U_i\}$  such that  $U_{i_1} \cap \cdots \cap U_{i_q}$  is  $\emptyset$  or diffeomorphic to  $\mathbb{R}^n$ . Then*

$$\mu_M : H^p(M) \xrightarrow{\sim} H_c^{n-p}(M)^*, \quad p \geq 0, \quad n = \dim M$$

*is an isomorphism.*

Any compact manifold  $M$  admits such a cover.

**Lemma 2.53.** *Let*

$$C^1 \xrightarrow{f^1} C^2 \xrightarrow{f^2} C^3$$

*be exact, where  $C^i$  are vector spaces of finite dimension. Then there exists*

$$(C^3)^* \xrightarrow{(f^2)^*} (C^2)^* \xrightarrow{(f^1)^*} (C^1)^*,$$

*which is also exact, where  $(f^1)^* \phi = \phi \circ f^1$  and  $(f^2)^* \phi = \phi \circ f^2$ .*

*Proof.* By assumption  $\text{Ker } f^2 = \text{Im } f^1$ . We want to prove  $\text{Ker } (f^1)^* = \text{Im } (f^2)^*$ .

- Let  $\phi \in \text{Im } (f^2)^*$ . Then there exists  $\psi \in (C^3)^*$  such that  $(f^2)^* \psi = \phi$ , so  $\psi \circ f^2 = \phi$ , so  $0 = \psi \circ f^2 \circ f^1 = \phi \circ f^1 = (f^1)^* \phi$ , so  $\phi \in \text{Ker } (f^1)^*$ .
- Let  $\phi \in \text{Ker } (f^1)^*$ . Then  $\phi \circ f^1 = 0$ , so  $\text{Ker } f^2 = \text{Im } f^1 \subset \text{Ker } \phi$ , so there exists  $\bar{\phi} : C^2 / \text{Ker } f^2 \rightarrow \mathbb{R}$ , so there exists  $\psi : C^3 \rightarrow \mathbb{R}$  extending  $\bar{\phi}$  such that  $\psi \circ f^2 = \phi$ , so  $(f^2)^* \psi = \phi$ , so  $\phi \in \text{Im } (f^2)^*$ .

□

**Lemma 2.54** (Five lemma). *Let*

$$\begin{array}{ccccccccc} C^1 & \xrightarrow{f^1} & C^2 & \xrightarrow{f^2} & C^3 & \xrightarrow{f^3} & C^4 & \xrightarrow{f^4} & C^5 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\ D^1 & \xrightarrow{g^1} & D^2 & \xrightarrow{g^2} & D^3 & \xrightarrow{g^3} & D^4 & \xrightarrow{g^4} & D^5 \end{array},$$

*such that the horizontal lines are exact. Suppose*

- $\alpha_1$  is surjective,
- $\alpha_5$  is injective, and
- $\alpha_2$  and  $\alpha_4$  are isomorphisms.

*Then  $\alpha_3$  is an isomorphism.*

*Proof.* Let  $x \in C^3$  such that  $\alpha_3(x) = 0$ , so if  $y = f^3(x)$  then  $\alpha_4(y) = 0$ . Since  $\alpha_4$  is an isomorphism,  $y = 0$ . Then  $x \in \text{Ker } f^3 = \text{Im } f^2$ , so there exists  $z \in C^2$  such that  $f^2(z) = x$ . Let  $w = \alpha_2(z)$  then  $g^2(w) = 0$ , so  $w \in \text{Ker } g^2 = \text{Im } g^1$ . Then there exists  $t \in D^1$  such that  $g^1(t) = w$ . Since  $\alpha_1$  is surjective there exists  $s \in C^1$  such that  $\alpha_1(s) = t$ , so

$$\begin{array}{ccccccccc} s \in C^1 & \xrightarrow{f^1} & z \in C^2 & \xrightarrow{f^2} & x \in C^3 & \xrightarrow{f^3} & y \in C^4 & \xrightarrow{f^4} & C^5 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\ t \in D^1 & \xrightarrow{g^1} & w \in D^2 & \xrightarrow{g^2} & 0 \in D^3 & \xrightarrow{g^3} & 0 \in D^4 & \xrightarrow{g^4} & D^5 \end{array}.$$

We want to show that  $f^1(s) = z$ , and  $\alpha_2(f^1(s)) = g^1(\alpha_1(s)) = g^1(t) = w = \alpha_2(z)$ , so  $f^1(s) = z$ , since  $\alpha_2$  is injective. Thus  $x = f^2(z) = f^2(f^1(s)) = 0$ , so  $\alpha_3$  is injective. Show that  $\alpha_3$  is surjective. <sup>12</sup> □

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<sup>12</sup>Exercise

*Proof of Theorem 2.52.* Let  $N = \#U$ . We proceed by induction on  $N$ . Then  $N = 1$  is ok, so let  $N > 1$ . Let  $U = \bigcup_{i=1}^{N-1} U_i$  and  $V = U_N$ , so  $M = U \cup V$ . Both  $U$  and  $V$ , and  $U \cap V$ , satisfy Poincaré duality by induction. The idea is to use classical Mayer-Vietoris and compact support Mayer-Vietoris, and the five lemma. By Mayer-Vietoris,

$$H^{p-1}(U) \oplus H^{p-1}(V) \xrightarrow{g} H^{p-1}(U \cap V) \xrightarrow{\delta} H^p(M) \xrightarrow{f} H^p(U) \oplus H^p(V) \rightarrow \dots,$$

where  $f = (i_U^*, i_V^*)$  and  $g = j_V^* - j_U^*$ . By compact support Mayer-Vietoris,

$$\dots \rightarrow H_c^{n-p}(U) \oplus H_c^{n-p}(V) \xrightarrow{i} H_c^{n-p}(M) \xrightarrow{\delta_c} H_c^{n-(p-1)}(M) \xrightarrow{j} H_c^{n-(p-1)}(U) \oplus H_c^{n-(p-1)}(V),$$

where  $j = (-j_{U*}, j_{V*})$  and  $i = i_{U*} + i_{V*}$ . Taking the dual, by Lemma 2.53,

$$H_c^{n-(p-1)}(U)^* \oplus H_c^{n-(p-1)}(V)^* \xrightarrow{j^*} H_c^{n-(p-1)}(U \cap V)^* \xrightarrow{\delta_c^*} H_c^{n-p}(M)^* \xrightarrow{i^*} H_c^{n-p}(U)^* \oplus H_c^{n-p}(V)^* \rightarrow \dots$$

We get a diagram

$$\begin{array}{ccccccc} H^{p-1}(U) \oplus H^{p-1}(V) & \xrightarrow{g} & H^{p-1}(U \cap V) & \xrightarrow{\delta} & H^p(M) & \xrightarrow{f} & H^p(U) \oplus H^p(V) \rightarrow \dots \\ \downarrow n_{p-1} \cdot \mu_U \oplus \mu_V & & \downarrow n_{p-1} \cdot \mu_{U \cap V} & & \downarrow n_p \cdot \mu_M & & \downarrow n_p \cdot \mu_U \oplus \mu_V \\ H_c^{n-(p-1)}(U)^* \oplus H_c^{n-(p-1)}(V)^* & \xrightarrow{j^*} & H_c^{n-(p-1)}(U \cap V)^* & \xrightarrow{\delta_c^*} & H_c^{n-p}(M)^* & \xrightarrow{i^*} & H_c^{n-p}(U)^* \oplus H_c^{n-p}(V)^* \rightarrow \dots \end{array},$$

where  $n_0 = 1$  and  $n_p = (-1)^{p-1} n_{p-1}$ . The goal is to show that  $\mu_M$  is an isomorphism. The idea is by the five lemma, it is enough to show that

1. all the other vertical arrows are isomorphisms, and
2. the diagram is commutative.

We know 1 is ok by induction on  $N$ . We need to show 2.

- The first square. We want to show that

$$\mu_{U \cap V} \circ g = j^* \circ (\mu_U \oplus \mu_V).$$

Let  $\omega_U \in \Omega^{p-1}(U)$  and  $\omega_V \in \Omega^{p-1}(V)$  be closed forms. We want to show

$$\mu_{U \cap V}(g([\omega_U], [\omega_V])) = j^*(\mu_U([\omega_U]), \mu_V([\omega_V])),$$

in  $H_c^{n-(p-1)}(U \cap V)^*$ , that is we want to show that on any element of  $H_c^{n-(p-1)}(U \cap V)$  they coincide.

Let  $\eta \in \Omega_c^{n-(p-1)}(U \cap V)$ . Recall  $g = j_V^* - j_U^*$ . Then

$$\int_{U \cap V} g(\omega_U, \omega_V) \wedge \eta = - \int_U \omega_U \wedge j_{U*} \eta + \int_V \omega_V \wedge j_{V*} \eta,$$

since  $g(\omega_U, \omega_V) = j_V^* \omega_V - j_U^* \omega_U$ .

- The second square. We want an explicit construction of  $\delta$  and  $\delta_c$ . Let  $\omega \in \Omega^p(M)$  be a closed form, and let  $\{f_U, f_V\}$  be a partition of the unity with respect to  $\{U, V\}$ . Define

$$\omega_U = f_U \cdot \omega|_U \in \Omega_c^p(U), \quad \omega_V = f_V \cdot \omega|_V \in \Omega_c^p(V),$$

so  $(\omega_U, \omega_V) \in \Omega_c^p(U) \oplus \Omega_c^p(V)$ . Recall  $i = i_{U*} + i_{V*}$ . Then

$$i(\omega_U, \omega_V) = i_{U*} \omega_U + i_{V*} \omega_V = \omega_U + \omega_V = f_U \cdot \omega + f_V \cdot \omega = \omega.$$

If  $\omega$  is closed, then  $i(d\omega_U, d\omega_V) = d(i_{U*}\omega_U) + d(i_{V*}\omega_V) = 0$ , so  $(d\omega_U, d\omega_V) \in \text{Ker } i = \text{Im } j \subset \Omega_c^{p+1}(U) \oplus \Omega_c^{p+1}(V)$ . Since  $j$  is injective there exists a unique  $\delta_c(\omega) \in \Omega_c^{p+1}(U \cap V)$  such that  $j(\delta_c(\omega)) = (d\omega_U, d\omega_V)$ . Since  $f_U + f_V = 1$ ,  $df_U + df_V = 0$ , so  $df_U = -df_V$ . Then

$$j(\delta_c(\omega)) = (d\omega_U, d\omega_V) = (df_U \wedge \omega|_U, df_V \wedge \omega|_V) = (-df_V \wedge \omega|_U, df_V \wedge \omega|_V) = j(df_V \wedge \omega|_{U \cap V}).$$

Since  $j$  is injective,  $\delta_c(\omega) = df_V \wedge \omega|_{U \cap V}$ , so  $\delta_c : \Omega_c^p(M) \rightarrow \Omega_c^{p+1}$ . Let  $\eta$  be a form on  $M$ . Since  $\delta_c(d\eta) = df_V \wedge d\eta|_{U \cap V} = -d\delta_c(\eta)$ ,  $\delta_c$  maps closed forms to closed forms and exact forms to exact forms, so

$$\begin{aligned} \delta_c : \Omega_c^p(M) &\longrightarrow \Omega_c^{p+1}(U \cap V) \\ \omega &\longmapsto df_V \wedge \omega|_{U \cap V}. \end{aligned}$$

By construction, it makes the long exact sequence exact. Similarly

$$\begin{aligned} \delta : \Omega^p(U \cap V) &\longrightarrow \Omega^{p+1}(M) \\ \omega &\longmapsto \begin{cases} df_V \wedge \omega & \text{on } U \cap V \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Now we check that the second square is commutative, that is

$$n_{p-1} \cdot \mu_M(\delta([\omega_1])) = n_p \cdot \delta_c^*(\mu_{U \cap V}([\omega_1])), \quad \omega_1 \in \Omega^{p-1}(U \cap V).$$

That is,

$$n_{p-1} \int_M \delta(\omega_1) \wedge \omega_2 = n_p \int_{U \cap V} \omega_1 \wedge \delta_c(\omega_2), \quad \omega_2 \in \Omega_c^{n-p}(M).$$

For all  $\omega_2 \in \Omega_c^{n-p}(M)$ ,

$$n_{p-1} \int_M \delta(\omega_1) \wedge \omega_2 = n_{p-1} \int_{U \cap V} df_V \wedge \omega_1 \wedge \omega_2 = n_p \int_{U \cap V} \omega_1 \wedge df_V \wedge \omega_2 = n_p \int_{U \cap V} \omega_1 \wedge \delta_c(\omega_2).$$

- The third square. To check

$$(\mu_U \oplus \mu_V) \circ f = i^* \circ \mu_M,$$

so

$$(\mu_U \oplus \mu_V)(f([\omega])) = i^*(\mu_M([\omega])), \quad \omega \in \Omega^p(M),$$

in  $\Omega_c^{n-p}(U)^* \oplus \Omega_c^{n-p}(V)^*$ . Let  $\eta_U \in \Omega_c^{n-p}(U)$  and  $\eta_V \in \Omega_c^{n-p}(V)$ . Then

$$\int_U \omega|_U \wedge \eta_U + \int_V \omega|_V \wedge \eta_V = \int_M \omega \wedge i(\eta_U, \eta_V).$$

□

The following is an easy consequence.

**Corollary 2.55.** *Let  $M$  be an oriented compact connected manifold of dimension  $n$ . Then*

$$H^n(M) = \mathbb{R},$$

and

$$H_c^p(M \setminus \{x\}) = H^p(M), \quad x \in M, \quad 1 \leq p < n.$$

**Definition 2.56.** The **Euler characteristic** of  $M$  is

$$\chi(M) = \sum_{p=0}^n (-1)^p \dim H^p(M).$$

**Corollary 2.57.** *If  $M$  is a compact oriented manifold of odd dimension then  $\chi(M) = 0$ .*

*Proof.* By Poincaré duality,  $\dim H^i(M) = \dim H^{n-i}(M)$ . □

## 2.7 Degree of a morphism

Let  $M$  and  $N$  be connected oriented manifolds of dimension  $n$ , and let  $f : M \rightarrow N$  be a proper smooth morphism. Then  $f^* : H_c^n(N) \cong \mathbb{R} \rightarrow H_c^n(M) \cong \mathbb{R}$  by Poincaré duality and connectedness, so  $f(x) = c \cdot x$  for some  $\deg f = c \in \mathbb{R}$ . For any  $\omega \in \Omega_c^n(M)$ ,

$$\int_M f^* \omega = \deg f \cdot \int_M \omega.$$

**Proposition 2.58.** *Let  $M, N, P$  be connected oriented manifolds of dimension  $n$ .*

- *If  $f : M \rightarrow N$  and  $g : N \rightarrow P$  are smooth and proper morphisms then*

$$\deg(g \circ f) = \deg f \cdot \deg g$$

- *If  $f$  is a diffeomorphism then*

$$\deg f = \begin{cases} 1 & f \text{ is orientation preserving} \\ -1 & \text{otherwise} \end{cases}.$$

- *If  $f, g : M \rightarrow N$  are smooth proper and properly homotopic equivalent then*

$$\deg f = \deg g.$$

**Theorem 2.59** (Mapping degree theorem). *Let  $f : M \rightarrow N$  be a proper smooth morphism between connected oriented manifolds of dimension  $n$ . Then  $\deg f \in \mathbb{Z}$ .*

**Definition 2.60.** Let  $f : M \rightarrow N$  be a smooth morphism. Then  $y \in N$  is **regular** if for all  $x \in f^{-1}(y)$ ,  $Df_x$  has maximal rank.

**Theorem 2.61** (Preimage theorem). *Let  $f : M \rightarrow N$  be a smooth morphism, and let  $y \in N$  be a regular value. Then  $f^{-1}(y)$  is a manifold of dimension  $\dim M - \text{rk } Df_x$  where  $x \in f^{-1}(y)$ .*

**Theorem 2.62** (Implicit function theorem). *Let  $f : M \rightarrow N$  be a smooth morphism, and let  $x \in M$  be such that  $Df_x$  is an isomorphism. Then there exists an open  $x \in U \subset M$  such that  $f|_U : U \rightarrow f(U)$  is an isomorphism.*

**Theorem 2.63** (Sard's theorem). *Let  $f : M \rightarrow N$  be smooth. Then if  $Z \subset N$  is the set of regular values of  $f$  then  $Z \cap f(M)$  is dense in  $f(M)$ .*

*Proof of Theorem 2.59.* Recall  $\dim M = \dim N$ , and if  $\omega \in \Omega_c^n(N)$  and if  $Df_x$  is of rank less than  $n$  for all  $x$ , then  $\deg f = 0$ . We are done. In particular we may assume there exists  $x$  such that  $Df_x$  has rank equal to  $n$ . Let  $y = f(x)$ . By Sard's theorem, we may assume that for all  $x \in f^{-1}(y)$ ,  $Df_x$  has rank  $n$ . By the preimage theorem  $f^{-1}(y)$  is a manifold of dimension zero, such as  $\mathbb{Z} \subset \mathbb{R}$ , so

$$f^{-1}(y) = \{x_1, \dots, x_k\}$$

is a finite set, because  $f$  is proper. By the implicit function theorem, for all  $i$  there exists an open set  $U_i \ni x_i$  such that  $f|_{U_i}$  is a diffeomorphism and  $f(U_i) = U$ . Let  $\omega \in \Omega_c^n(N)$  be such that  $\int_U \omega = 1$  and  $\text{supp } \omega \subset U$ . Since  $f|_{U_i}$  is a diffeomorphism

$$\int_{U_i} f|_{U_i}^* \omega = \text{sgn}(\det Df_{x_i}) \int_U \omega,$$

and  $f|_{U_i}^* \omega$  has support in  $U_i$ . Since  $\text{supp } f^* \omega \subset \bigcup_i U_i$ ,

$$\int_M f^* \omega = \sum_{i=1}^k \int_{U_i} f^* \omega = \sum_{i=1}^k \text{sgn}(\det Df_{x_i}) \int_U \omega = \sum_{i=1}^k \text{sgn}(\det Df_{x_i}) \int_M \omega,$$

so  $\deg f = \sum_{i=1}^k \text{sgn}(\det Df_{x_i}) \in \mathbb{Z}$ , which does not depend on  $y$ , if  $y$  is a regular point.  $\square$

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**Exercise.** Suppose that  $f : M \rightarrow N$  is a proper morphism between oriented connected manifolds. If  $\deg f \neq 0$ , then  $f$  is surjective.

**Example 2.64.** Let  $M = S^n = N$ , and let

$$\begin{array}{ccc} f & : & M \longrightarrow N \\ x & \longmapsto & -x \end{array}$$

be the antipodal map. Claim that  $\deg f = (-1)^{n+1}$ . Let  $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ , let

$$\tilde{\omega} = x_1 dx_2 \wedge \cdots \wedge dx_{n+1} \in \Omega^n(\mathbb{R}^{n+1}),$$

and let  $\omega = i_* \tilde{\omega} \in \Omega^n(S^n)$ . By Stokes and  $S^n = \partial D^{n+1}$ ,

$$\int_{S^n} \omega = \int_{S^n} i^* \tilde{\omega} = \int_{D^{n+1}} d\tilde{\omega} = \int_{D^{n+1}} dx_1 \wedge \cdots \wedge dx_{n+1} \neq 0,$$

so  $f$  can be extended to

$$\begin{array}{ccc} \tilde{f} & : & \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1} \\ x & \longmapsto & -x \end{array}.$$

Then  $\tilde{f} \circ i = i \circ f$  and  $\tilde{f}^* \tilde{\omega} = (-1)^{n+1} \tilde{\omega}$ , so

$$f^* \omega = f^* i^* \tilde{\omega} = (i \circ f)^* \tilde{\omega} = (\tilde{f} \circ i)^* \tilde{\omega} = i^* \tilde{f}^* \tilde{\omega} = (-1)^{n+1} i^* \tilde{\omega} = (-1)^{n+1} \omega.$$

Thus

$$(-1)^{n+1} \int_{S^n} \omega = \int_{S^n} f^* \omega = \deg f \int_{S^n} \omega,$$

so  $\deg f = (-1)^{n+1}$ .

### 3 Morse theory

#### 3.1 Introduction

**Definition 3.1.** Let  $M$  be a manifold of dimension  $n$ , and let  $f : M \rightarrow \mathbb{R}$  be smooth. A **critical point** of  $f$  is a point  $x \in M$  such that  $Df_x = 0$ , that is if  $x_1, \dots, x_n$  are local coordinates at  $x$ , then

$$\frac{\partial}{\partial x_i} f(x) = 0, \quad i = 1, \dots, n.$$

For such  $x$ , we define the Hessian of  $f$  to be

$$H_f(x) = \left( \frac{\partial^2}{\partial x_i \partial x_j} f(x) \right).$$

Then  $x$  is called **non-degenerate** if  $\det H_f(x) \neq 0$ . A function  $f$  such that every critical point of  $f$  is non-degenerate is called a **Morse function**.

**Fact.** By Sard's theorem most of the functions satisfy this property.

#### 3.2 Cell decomposition

**Notation.** Let  $D_n = \{x \mid |x| \leq 1\} \subset \mathbb{R}^n$  be the unit ball, and let  $S^{n-1} = \partial D_n$ .

**Definition 3.2.** An  $n$ -cell is a topological space which is homeomorphic to the open ball  $D_n \setminus \partial D_n$ . A **cell decomposition** of a topological space  $M$  is a family  $F$  of pairwise disjoint subspaces of  $M$  which are  $n$ -cells and such that  $M = \bigsqcup_{e_i \in F} e_i$ . If  $F$  is finite, then this is called a **finite cell decomposition**. For all  $m \geq 0$ ,

$$SK_m M = \bigsqcup_{\dim e_i \leq m} e_i.$$

**Example 3.3.**  $S^1 = (S^1 \setminus \{p\}) \sqcup \{p\}$ , where  $S^1 \setminus \{p\}$  is a 1-cell and  $\{p\}$  is a 0-cell.

**Notation 3.4.** Let  $M$  be a topological space, and let  $f_\partial : S^{n-1} \rightarrow M$  be continuous. We construct a new topological space

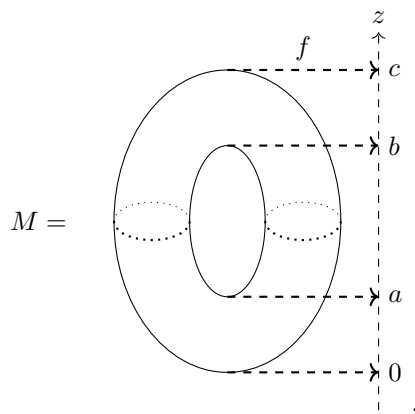
$$M \cup_{f_\partial} D_n = M \sqcup D_n / \sim,$$

where  $M \ni x \sim y \in S^{n-1} \subset D_n$  if  $f_\partial(y) = x$ . Then  $M \cup_{f_\partial} D_n$  is said to be obtained by **attaching** an  $n$ -cell to  $M$  via  $f_\partial$ .

**Example.** Let  $M = \{p\}$ , and let  $f_\partial : S^0 = \partial D_1 = \partial[0, 1] \rightarrow \{p\}$ . Then  $S^1 = M \cup_{f_\partial} D_1$ .

**Exercise.** If  $M$  admits a cell decomposition then also  $M \cup_{f_\partial} D_n$  does.

**Example 3.5.** Let  $M = S^1 \times S^1 \subset \mathbb{R}^3$  be the torus



where

$$f : \begin{array}{ccc} M & \longrightarrow & \mathbb{R} \\ (x, y, z) & \longmapsto & z \end{array}.$$

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For all  $h$  define

$$S_h = \{p \in M \mid f(p) \leq h\} = f^{-1}((-\infty, h]).$$

- If  $h < 0$  then  $S_h = \emptyset$ .
- If  $0 < h < a$  then  $S_h$  is homotopically equivalent to a 0-cell, where

$$S_h = \text{[shaded bowl shape]} \sim \bullet$$

- If  $a < h < b$  then  $S_h$  is homotopically equivalent to a 1-cell attached to the previous  $S_h$ , where

$$S_h = \text{[U-shaped bowl with two shaded top rims]} \sim \text{[shaded bowl with two points connected by a vertical line segment]}$$

- If  $b < h < c$  then  $S_h$  is homotopically equivalent to a 1-cell attached to the previous  $S_h$ , where

$$S_h = \text{[torus with shaded top disk]} \sim \text{[U-shaped bowl with two shaded top rims connected by a horizontal line segment]}$$

- If  $h > c$  then  $S_h$  is homotopically equivalent to a 2-cell attached to the previous  $S_h$ , where

$$S_h = \text{[torus with shaded top disk]} \sim \text{[torus with shaded top disk and a dashed line indicating the boundary of the shaded region]}$$

Thus

$$M = 0\text{-cell} \sqcup \text{two } 1\text{-cells} \sqcup 2\text{-cell}.$$

Given a Morse function  $f : M \rightarrow \mathbb{R}$ , the goal is to study the level sets of  $f$ ,  $S_h = f^{-1}((-\infty, h])$ .

**Definition 3.6.** Let  $M$  be a manifold, let  $f : M \rightarrow \mathbb{R}$  be a Morse function, and let  $x \in M$  be critical. Denote

$$\text{Eig}^- H_f(x) = \{\text{eigenvectors of } H_f \text{ with negative eigenvalues}\}.$$

Recall that  $H_f$  is a symmetric matrix. The **index** of  $f$  at  $x$  is the dimension of  $\text{Eig}^- H_f(x)$ .

**Lemma 3.7** (Morse). *Let  $M$  be a manifold of dimension  $n$ , let  $f : M \rightarrow \mathbb{R}$  be a Morse function, and let  $x_0 \in M$  be a critical point. Then there exist local coordinates  $(x_1, \dots, x_n)$  around  $x_0$  such that  $x_0 = (0, \dots, 0)$  and*

$$f = f(x_0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^n x_i^2,$$

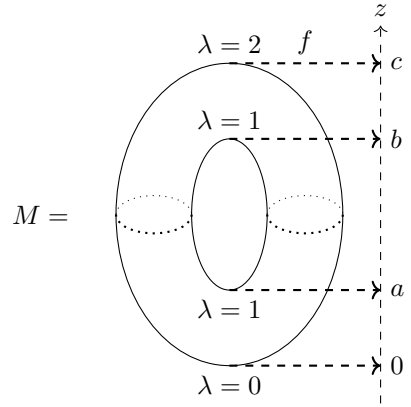
where  $\lambda$  is the index of  $f$  at  $x$ .

Thus the set of critical points of  $f$  is discrete, since locally at critical  $x_0$ ,

$$f = f(x_0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^n x_i^2$$

has no more critical points.

**Example 3.8.** Let  $f : M \rightarrow \mathbb{R}$  be as in the previous example. Then



### 3.3 CW-complexes

**Definition 3.9.** A topological space  $M$  admits a **CW-structure** if there exists a sequence of topological subspaces

$$M^{(0)} \subset \dots \subset M^{(n)},$$

such that

1.  $M^{(0)}$  is a discrete subset of  $M$ ,
2.  $M^{(k)}$  is obtained by attaching  $k$ -cells to  $M^{(k-1)}$ , and
3.  $V \subset M$  is closed if and only if  $V \cap M^{(k)}$  is closed for all  $k$ .

Such  $M$  is called a **CW-complex**. Then  $M$  is a **finite CW-complex** if it is obtained by attaching finitely many cells. In this case 3 is not needed.

**Example 3.10.**

- $S^n = \{p\} \cup D_n = M^{(0)} \cup M^{(1)}$ .
- If  $M = \mathbb{R}^n$ , and  $\Lambda$  is the integral points in  $\mathbb{R}^n$ , then  $\Lambda$  gives a decomposition of  $\mathbb{R}^n$  into  $n$ -cubes, which are  $n$ -cells, where 0-cells are points of  $\Lambda$ , 1-cells are edges of  $\Lambda$ , etc.
- If  $n \neq 4$  and  $M$  is a manifold of dimension  $n$ , then  $M$  is a CW-complex. If  $n = 4$ , then it is open.

A **subcomplex** of a CW-complex  $M$  is a closed subspace of  $M$  which is a union of cells of  $M$ . A **closed cell** is the image of  $D_n$  in a cell. An **open cell** is the image of  $D_n \setminus \partial D_n$  in a cell. Open cells are not open in  $M$  in general.

### 3.4 Flows

**Definition 3.11.** Let  $M$  be a manifold, then a **flow** or a **group of diffeomorphisms** of  $M$  is the collection of diffeomorphisms  $\phi_t : M \rightarrow M$  for  $t \in \mathbb{R}$  such that there exists  $\phi : \mathbb{R} \times M \rightarrow M$  with  $\phi_t = \phi(t, \cdot)$  and

- $\phi_0 = \text{id}_M$ , and
- $\phi_{t+s} = \phi_t \circ \phi_s$ .

For all  $x \in M$ , the **flow line** or the **integral curve** for the flow is

$$\begin{array}{ccc} \gamma_x & : & \mathbb{R} \longrightarrow M \\ & & t \longmapsto \phi(t, x) \end{array},$$

where  $\gamma_x(0) = x$ .

Since  $\frac{d}{dt} \gamma_x|_{t=0} \in T_x M$ , the flow defines a vector field on  $M$ , that is a smooth section of  $TM$ . On the other hand, given a vector field, we can find a flow. Let  $X$  be a section of  $TM$ . There exists  $\phi_t$ , and  $\gamma_x$ , as above such that  $\frac{d}{dt} \gamma_x|_{t=0} \in T_x M$ , locally around  $t = 0$ .

**Example 3.12.** Let  $M = \mathbb{R}$ , and let  $X = x^2 \partial_x$ . Find the flow. <sup>13</sup>

**Definition 3.13.** Let  $M$  be a manifold. A **Riemannian metric**  $g$  on  $M$  is a collection of inner products for  $T_x M$  for  $x \in M$  given by

$$g_x : T_x M \times T_x M \rightarrow \mathbb{R},$$

such that for all vector fields  $X$  and  $Y$  we have that  $g_x(X(x), Y(x)) : M \rightarrow \mathbb{R}$  is smooth. A **Riemannian manifold**  $(M, g)$  is a manifold  $M$  and a Riemannian metric  $g$  on  $M$ .

Then  $g$  gives  $TM \xrightarrow{\sim} \Omega^1(M) = (TM)^*$ .

**Definition 3.14.** Let  $(M, g)$  be a Riemannian manifold, and let  $f : M \rightarrow \mathbb{R}$  be a smooth function. The **gradient vector field**  $\nabla f$  of  $f$  is the unique vector field on  $M$  such that for all vector fields  $X$ ,

$$g(\nabla f, X) = X(f) = Df(X),$$

the derivative of  $f$  with respect to  $X$ .

Thus,

$$\|\nabla f\|^2 = g(\nabla f, \nabla f) = Df(\nabla f).$$

In particular  $\nabla f(x) = 0$  if and only if  $x$  is a critical point of  $f$ , and  $\nabla f$  is orthogonal to any vector tangent to  $f^{-1}(c)$  for all regular values  $c \in \mathbb{R}$ . In particular, for all smooth functions  $f : M \rightarrow \mathbb{R}$ , we can take the flow  $\phi$  associated to  $-\nabla f$  such that if  $\gamma_x(t) = \phi(t, x)$  then  $\frac{d}{dt} \gamma_x(0) = -\nabla f(\gamma_x(0))$  where  $\gamma_x(0) = x$ . This is called the **gradient flow** of  $f$ . The integral curve of this flow are called **gradient flow lines**.

**Lemma 3.15.**  $f$  decreases along the gradient lines, that is  $f(\gamma_x(t))$  is a decreasing function with respect to  $t$ .

*Proof.*

$$0 \leq -\|\nabla f(\gamma_x(t))\|^2 = Df_{\gamma_x(t)}(-\nabla f(\gamma_x(t))) = Df_{\gamma_x(t)}\left(\frac{d}{dt} \gamma_x(t)\right) = \frac{d}{dt} f(\gamma_x(t)).$$

□

**Proposition 3.16.** Let  $M$  be a compact manifold, and let  $f : M \rightarrow \mathbb{R}$  be a Morse function. Then every gradient flow line begins and ends with a critical point of  $f$ , that is  $\lim_{t \rightarrow \pm\infty} \gamma_x(t)$  exist and are critical points of  $f$ .

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<sup>13</sup>Exercise

*Proof.* First we prove that if the limit exists then it is a critical point of  $f$ . For all  $x$ ,  $f(\gamma_x(\cdot)) : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded function. Then

$$0 = \lim_{t \rightarrow \pm\infty} \frac{d}{dt} f(\gamma_x(t)) = \lim_{t \rightarrow \pm\infty} -\|\nabla f(\gamma_x(t))\|^2,$$

so the limit is critical. The goal is to show that limits exist. Since  $M$  is compact and  $f$  is a Morse function, the set of critical points is finite. Fix  $\epsilon > 0$ . Let  $U$  be the union of open balls of radius  $\epsilon > 0$  around each critical point, so  $U$  is open in  $M$ . Then  $M \setminus U$  is compact, so  $\|\nabla f(\cdot)\|^2$  admits a minimum inside  $M \setminus U$ , but it cannot be zero, but  $\lim_{t \rightarrow \pm\infty} \|\nabla f(\gamma_x(t))\|^2 = 0$ . If  $\pm t$  is very large then  $\gamma_x(t) \notin M \setminus U$ , so if  $\epsilon$  is sufficiently small,  $\gamma_x(t)$  is in a ball around a single critical point for  $t \rightarrow \pm\infty$ . This implies that the limit is the critical point.  $\square$

**Definition 3.17.** Let  $X$  be a topological space, and let  $S \subset X$ . Then  $S$  is called a **deformation retract** of  $X$  if there exists  $F : X \times [0, 1] \rightarrow X$  such that  $F(x, 0) = x$  and  $F(x, 1) \in S$  for all  $x \in X$ , and  $F(s, 1) = s$  for all  $s \in S$ .

This implies that  $S$  and  $X$  are homotopy equivalent.

**Theorem 3.18** (First fundamental theorem of Morse theory). *Let  $M$  be a manifold, let  $f : M \rightarrow \mathbb{R}$ , let  $a < b \in \mathbb{R}$  such that  $f^{-1}([a, b])$  does not contain any critical point, and let  $S_t = f^{-1}((-\infty, t])$  for all  $t \in \mathbb{R}$ . Then  $S_a$  is a deformation retract of  $S_b$ .*

The idea is that we will use a perturbation of the gradient flow.

*Proof.* There exists  $\epsilon > 0$  such that  $f^{-1}([a - \epsilon, b + \epsilon])$  does not contain any critical point. Fix a metric  $g$  on  $M$ . Define  $\rho$  smooth on  $M$  such that  $\rho(x) \geq 0$  and

$$\rho(x) = \begin{cases} \frac{1}{\|\nabla f\|^2} & x \in f^{-1}([a, b]) \\ 0 & x \in M \setminus f^{-1}([a - \epsilon, b + \epsilon]) \end{cases}.$$

Define  $X(x) = -\rho(x) \nabla f(x)$ . There exists a flow  $\phi(t, x)$  induced by  $X$ , that is if  $\gamma_x(t) = \phi(t, x)$  then  $\frac{d}{dt} \gamma_x(0) = X(x)$ . By definition of  $\nabla f$ ,

$$\frac{d}{dt} f(\gamma_x(t)) = Df_{\gamma_x(t)} \left( \frac{d}{dt} \gamma_x(t) \right) = g \left( \nabla f, \frac{d}{dt} \gamma_x(t) \right) = g(\nabla f, -\rho(\gamma_x(t)) \nabla f) = -\rho(\gamma_x(t)) \|\nabla f\|^2 \leq 0,$$

so  $f(\gamma_x(t))$  is decreasing. Moreover for all  $t$  such that  $f(\gamma_x(t)) \in [a, b]$  we have that  $\frac{d}{dt} f(\gamma_x(t)) = -1$ , by definition of  $\rho$ . If  $\gamma_x(s) \in f^{-1}([a, b])$  for all  $s \in [0, t]$ , by the fundamental theorem of calculus

$$f(\gamma_x(t)) - f(\gamma_x(0)) = \int_0^t \frac{d}{ds} f(\gamma_x(s)) ds = -t,$$

so  $f(\gamma_x(t)) = f(x) - t$ . Then

1. if  $f(x) \leq b$  then  $f(\phi_{b-a}(x)) = f(\gamma_x(b-a)) \leq a$ , and
2. if  $f(x) > b$  then  $f(\phi_{b-a}(x)) > a$ .

1 implies that  $\phi_{b-a}(S_b) \subset S_a$  and 2 implies that  $\phi_{a-b}(S_a) \subset S_b$ . Recall that  $\phi_{a-b} = \phi_{b-a}^{-1}$ , so  $S_a$  and  $S_b$  are diffeomorphic. Now we define

$$F : S_b \times [0, 1] \longrightarrow S_b \\ (x, t) \longmapsto \begin{cases} x & f(x) \leq a \\ \phi_{t(f(x)-a)}(x) & a \leq f(x) \leq b \end{cases}.$$

Then  $F(x, 0) = x$ , since  $\phi_0(x) = x$ , and

$$F(x, 1) = \begin{cases} x & f(x) \leq a \\ \phi_{f(x)-a}(x) = \gamma_x(f(x)-a) & a \leq f(x) \leq b \end{cases}.$$

In particular if  $x \in S_a$ , then  $F(x, 1) = x$  and for all  $x \in S_b$ ,  $F(x, 1) \in S_a$ .  $\square$

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**Theorem 3.19** (Reeb's theorem). *Let  $M$  be a compact manifold, and let  $f : M \rightarrow \mathbb{R}$  be a Morse function. Assume that  $f$  admits exactly two critical points. Then  $M$  is homeomorphic to a sphere  $S^n$ .*

*Proof.* There exists a unique  $x_{\min}$  such that  $h_{\min} = f(x_{\min})$  is the minimum and there exists a unique  $x_{\max}$  such that  $h_{\max} = f(x_{\max})$  is the maximum. Both  $x_{\min}$  and  $x_{\max}$  are critical points. Then  $\lambda(x_{\min}) = n$  and  $\lambda(x_{\max}) = 0$ . By the Morse lemma, around  $x_{\min}$ , we can write locally

$$f(x) = h_{\min} + \sum_{i=1}^n x_i^2,$$

for some local coordinates  $x_1, \dots, x_n$  such that  $x_{\min} = \underline{0}$ . Let  $a > h_{\min}$  be sufficiently close to  $x_{\min}$ . Then

$$S_a = \left\{ h_{\min} + \sum_{i=1}^n x_i^2 \leq a \right\} = D_n.$$

Similarly there exists  $b < h_{\max}$  sufficiently close such that  $M \setminus S_b \cong D_n$ . By Theorem 3.18, since there does not exist critical points in  $f^{-1}([a, b])$  we know that  $S_b \cong S_a \cong D_n$ . We proved that  $M = H_+ \cup H_- \cong D_n \cup D_n$  where  $H_- = S_b$  and  $H_+ = M \setminus S_b$ . Let  $\phi_+ : D_n \xrightarrow{\sim} H_+$  and  $\phi_- : D_n \xrightarrow{\sim} H_-$ . The problem is that in general  $\phi_+|_{\partial D_n} \neq \phi_-|_{\partial D_n}$ .  $\square$

Lecture 24 is a problems class.

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