M3P21 Geometry II: Algebraic Topology

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Contents

0 Some underlying geometric notions

0.1 Introduction

Lecture 1 Friday 11/01/19

Combines topological spaces with algebraic objects, which are groups.

- How to show that a torus is not homeomorphic to a sphere?
- How to show that $\mathbb{R}^n \ncong \mathbb{R}^m$ if $n \neq m$?

Content is fundamental groups and homology. We will follow chapter one and two from

- A Hatcher, Algebraic topology, 2002

The following are prerequisites.

- Point set topology. Topological spaces, continuous maps, product and quotient topologies, Hausdorff spaces, etc.
- Basic group theory. Normal subgroups and quotients, isomorphism theorems, free groups, presentation of groups, etc.

0.2 Homotopy

Let X, Y be topological spaces and $I = [0, 1]$.

Definition. A **homotopy** is a continuous map $F : X \times I \to Y$. For every $t \in I$ we obtain a continuous map

$$\begin{array}{rccc} f_t : & X & \to & Y \\ & x & \mapsto & f_t(x) = F(x,t) \end{array}.$$

Definition. Two continuous maps $f_0, f_1 : X \to Y$ are **homotopic** if there exists a homotopy $F : X \times I \to Y$ such that

$$f_0(x) = F(x, 0), \qquad f_1(x) = F(x, 1),$$

for all $x \in X$. We write $f_0 \cong f_1$. (Exercise: this is an equivalence relation)

Definition. Let $A \subseteq X$ be a subspace. A **retraction** of X onto A is a continuous map $r : X \to A$ such that

- $r(X) = A$, and
- $r |_A = id_A$.

Example. If $X \neq \emptyset$, $p \in X$, then X retracts to p by the constant map $X \to \{p\}$.

Definition. A **deformation retraction** of X onto $A \subseteq X$ is a retraction that is homotopic to the identity. That is, there is a continuous map

$$\begin{array}{rccc} F : & X \times I & \to & A \\ & (x,t) & \mapsto & f_t(x) \end{array},$$

such that $f_0 = id_X$ and $f_1 : X \to A$ is the deformation retraction.

Example. The closed n-dimensional **n-disc**

$$D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$

deformation retracts to $(0, \ldots, 0) \in \mathbb{R}^n$. Let $f_t(x) = t \cdot x$. $t = 1$ gives $f_1 = id_{D^n}$ and $t = 0$ gives $f_0 : D^n \to (0, \ldots, 0)$.

Example. Let S^n be the **n-sphere**,

$$\partial D^{n+1} = S^n = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

The cylinder $S^n \times I$ deformation retracts to $S^n \times \{0\}$, by defining $f_t(x, r) = (x, t \cdot r)$.

An observation is if X is a topological space, and $f : X \to \{p\}$ for $p \in X$ is a deformation retraction of X to p, then X is path connected. Indeed, if $F : X \times I \to X$ is a homotopy from id_X to f and $x \in X$ is a point, then this gives a path

$$\begin{array}{rcl} I & \to & X \\ t & \mapsto & F(x,t) \end{array}$$

that connects x to p. This implies that not all retractions are deformation retractions.

Example. A retraction that is not a deformation retraction. Take a space that is not path connected and retract it to a point. Let $X = \{0, 1\}$ with discrete topology. $x \mapsto 0$ is a retraction, but not a deformation retraction because X is not path connected.

Definition. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \to X$ such that $fg \cong id_Y$ and $gf \cong id_X$. If there exists a homotopy equivalence between X and Y, X and Y are **homotopy equivalent** or they have the same **homotopy type**.

Lemma 0.1. *A deformation retraction $f : X \to A$ is a homotopy equivalence.*

Proof. Let $i : A \hookrightarrow X$ be the inclusion map. Then $fi = id_A$ and $if = f \cong id_X$ by definition. □

Example. The disc with two holes is equivalent to ∞.

Example. $\mathbb{R}^n$ deformation retracts to a point, by $f_t(x) = t \cdot x$.

Definition.

- X is **contractible** if it is homotopy equivalent to a point.
- A continuous map is **nullhomotopic** if it is homotopy equivalent to a constant map.

0.3 Cell complexes

Example. The torus $S^1 \times S^1$ is the union of a point, two open intervals, and the open disc $Int\left(D^2\right)$.

These are called **cells**. Can think of discs D^n glued together.

Lecture 2
Tuesday
15/01/19

Definition. A **CW-complex**, or **cell complex**, is a topological space X such that there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X^n,$$

where the X^n are constructed inductively in the following way.

- X^n is a discrete set.
- For each $n \geq 0$ there is an collection of closed n-discs $\{D_\alpha^n\}$ together with continuous maps $\phi_\alpha : \partial D_\alpha^n \to X^{n-1}$, such that
$$X^n = \frac{X^{n-1} \sqcup \bigsqcup_\alpha D_\alpha^n}{\sim},$$
where $x \sim \phi_\alpha(x)$ for all $x \in \partial D_\alpha^n$ for all α.
- A subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n.

Remark.

- As a set,
$$X^n = X^{n-1} \sqcup \bigsqcup_\alpha e_\alpha^n,$$
where each e_α^n is homeomorphic to an open n-disc. These e_α^n are called the n-**cells** of X.
- If $X = X^m$ for some m, then X is called **finite dimensional**. The minimal m such that $X = X^m$ is the **dimension** of X.

Example.

- $[0,1]$ is a CW-complex.
- $\mathbb{R}$ is a CW-complex.
- S^1 is a CW-complex.
- A graph is a CW-complex.
- $S^n = D^n / \partial D^n$ is a CW-complex. See worksheet 1.

Can also decompose CW-complexes.

- The sphere S^2 is one 0-cell, one 1-cell, and two 2-cells.
- The torus $S^1 \times S^1$ is one 0-cell, two 1-cells, and one 2-cell.
- The Möbius strip is two 0-cells, three 1-cells, and one 2-cell.
- The Klein bottle is one 0-cell, two 1-cells, and one 2-cell.

Definition. If X is a CW-complex with finitely many cells the **Euler characteristic** $\chi(X)$ of X is the number of even cells minus the number of odd cells.

Fact. $\chi(X)$ does not depend of the choice of cells decomposition.

Example.

- $\chi(S^n) = 0$ if n is odd and $\chi(S^n) = 2$ if n is even.
- $\chi\left(S^1 \times S^1\right) = 0$.

This is the generalisation of the following observation by Leonhard Euler. Let P be a convex polyhedron, where

- V is the number of vertices of P,
- E is the number of edges of P, and
- F is the number of faces of P.

Then $V - E + F = 2$.

Example. A topological space that is not a CW-complex. $X = \{0,1\}$ with trivial topology does not contain any closed points.

Fact. CW-complexes are always Hausdorff.

1 The fundamental group - basic constructions

1.1 Paths and homotopy

Let X be a topological space. A **path** is a continuous map $f : I \to X$, where $I = [0, 1]$.

Definition. Two paths f_0, f_1 are **homotopic** if there exists a homotopy between f_0 and f_1 preserving the endpoints, that is a continuous map

$$\begin{array}{cccc} F : & I \times I & \to & X \\ & (s,t) & \mapsto & f_t(s) \end{array},$$

such that

$$f_t(0) = f_0(0), \qquad f_t(1) = f_0(1),$$

for all $t \in I$, and

$$F(s,0) = f_0(s), \qquad F(s,1) = f_1(s),$$

for all $s \in I$.

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set. Then all the paths in X are homotopic if they have the same endpoints.

Proof. Let $f_0, f_1 : I \to X$ be two paths such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$. Define

$$f_t(s) = (1-t) f_0(s) + t f_1(s).$$

$\square$

Lemma 1.1. *Being homotopic is an equivalence relation on the set of paths with fixed endpoints. We will write $f_0 \cong f_1$ for two homotopic paths f_0 and f_1.*

Proof.

- f is homotopic to f.
- If f_0 is homotopic to f_1 by a homotopy f_t, then f_1 is homotopic to f_0 by the homotopy f_{1-t}.
- If f_0 is homotopic to f_1 by a homotopy f_t and $f_1 = g_0$ is homotopic to g_1 by a homotopy g_t, then f_0 is homotopic to g_1 by the homotopy

  $$h_t = \begin{cases} f_{2t} & 0 \le t \le \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}.$$

  Then

  $$\begin{array}{cccc} H : & I \times I & \to & X \\ & (s,t) & \mapsto & h_t(s) \end{array}$$

  is continuous because its restriction to the closed subsets $I \times [0, 1/2]$ and $I \times [1/2, 1]$ is continuous, since if the restriction to two closed subsets is continuous then the restriction to the union of these subsets is continuous.

$\square$

Lecture 3 Wednesday 16/01/19

Let X be a topological space and $I = [0, 1]$. If $f : I \to X$ is a path, $[f]$ is the class of all paths on X homotopic to f.

Definition. Let $f, g : I \to X$ be two paths such that $f(1) = g(0)$. The **product path** $f \cdot g$ is the path

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}.$$

A convention is that whenever we write $f \cdot g$ we implicitly assume $f(1) = g(0)$.

Lemma 1.2. *Let f_0, f_1, g_0, g_1 be paths on X such that $f_1 \cong f_0$ and $g_0 \cong g_1$. Then $f_0 \cdot g_0 \cong f_1 \cdot g_1$.*

Proof.
$$\begin{array}{ccc} I \times I & \to & X \\ (s,t) & \mapsto & (f_t \cdot g_t)(s) \end{array}$$
is a homotopy between $f_0 \cdot g_0$ and $f_1 \cdot g_1$. ☐

Remark. Let $\phi : [0,1] \to [0,1]$ be continuous such that $\phi(0) = 0$ and $\phi(1) = 1$. If $f : I \to X$ is a path, then $f\phi \cong f$. This is a **reparametrisation**.

Proof. Define
$$\phi_t(s) = (1-t)\phi(s) + ts,$$
then $f\phi_t$ is a homotopy between $f\phi$ and f. ☐

For $x \in X$, let the **constant path** at x be
$$c_x : \begin{array}{ccc} I & \to & X \\ s & \mapsto & x \end{array}.$$

For a path $f : I \to X$, define
$$f^{-1} : \begin{array}{ccc} I & \to & X \\ s & \mapsto & f(1-s) \end{array}.$$

Lemma 1.3. *Let $f, g, h : I \to X$ be paths. Then*

1. *$(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$,*
2. *$f \cdot c_{f(1)} \cong f$ and $c_{f(0)} \cdot f \cong f$, and*
3. *$f \cdot f^{-1} \cong c_{f(0)}$ and $f^{-1} \cdot f \cong c_{f(1)}$.*

Proof.

1. $((f \cdot g) \cdot h)\phi = f \cdot (g \cdot h)$, where
$$\phi(s) = \begin{cases} \frac{s}{2} & s \in \left[0, \frac{1}{2}\right] \\ s - \frac{1}{4} & s \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ 2s - 1 & s \in \left[\frac{3}{4}, 1\right] \end{cases},$$
so $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$ by reparametrisation.
2. Again reparametrisation, by
$$\psi(s) = \begin{cases} 2s & s \in \left[0, \frac{1}{2}\right] \\ 1 & s \in \left[\frac{1}{2}, 1\right] \end{cases}, \qquad \chi(s) = \begin{cases} 0 & s \in \left[0, \frac{1}{2}\right] \\ 2s - 1 & s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$
3. Define
$$H(s,t) = \begin{cases} f(\max\{1 - 2s, t\}) & s \in \left[0, \frac{1}{2}\right] \\ f(\max\{2s - 1, t\}) & s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$
H is continuous, and
$$H(s,0) = f^{-1} \cdot f, \qquad H(s,1) = c_{f(1)}.$$
The inverse is similar.

☐

Definition. A **loop** with **basepoint** $x_0 \in X$ is a path $f : I \to X$ such that $f(0) = f(1) = x_0$.

Definition. Denote by $\pi_1(X, x_0)$ the set of homotopy classes $[f]$ of loops $f : I \to X$ with basepoint x_0.

Proposition 1.4. *$\pi_1(X, x_0)$ is a group with product $[f][g] = [f \cdot g]$ and neutral element $c_{x_0} : I \to X$, the constant path at x_0.*

Proof. Follows directly from Lemma 1.2 and Lemma 1.3. □

Definition. $\pi_1(X, x_0)$ is the **fundamental group** of X at x_0.

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set and $x_0 \in X$. Then $\pi_1(X, x_0) = 0$.

Proof. X is convex gives that all loops are homotopic to each other. □

Example.

- The fundamental group of a space X with the trivial topology is trivial, since X is simply connected, because all maps $f : I \to X$ are continuous, so path connected and all paths are homotopic.
- The fundamental group of a space X with the discrete topology is trivial, since $f : I \to X$ continuous gives f constant.

Assume $x_0, x_1 \in X$ such that x_0 and x_1 are in the same path component of X. Let $h : I \to X$ be a path such that $h(0) = x_0$ and $h(1) = x_1$. Define

$$\begin{array}{cccc} \beta_h : & \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ & [f] & \mapsto & \left[h \cdot f \cdot h^{-1}\right] \end{array}.$$

This is well-defined by Lemma 1.2.

Proposition 1.5. *$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is an isomorphism.*

Proof. It is a homomorphism.

$$\beta_h[f \cdot g] = \left[h \cdot f \cdot g \cdot h^{-1}\right] = \left[h \cdot f \cdot h^{-1}\right]\left[h \cdot g \cdot h^{-1}\right] = \beta_h[f] \cdot \beta_h[g],$$

and $\beta_h[c_{x_1}] = [c_{x_1}]$. It is bijective with $(\beta_h)^{-1} = \beta_{h^{-1}}$. □

If X is path connected, we often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Definition. X is **simply connected** if it is path connected and $\pi_1(X) = 0$.

Proposition 1.6. *X is simply connected if and only if there exists a unique homotopy class of paths between any two points of X.*

Proof.

$\implies$ There exists a path between any two points. Let f, g be two paths from x_0 to x_1 for $x_0, x_1 \in X$. $f \cdot g^{-1} \cong g \cdot g^{-1}$ gives $f \cong f \cdot g^{-1} \cdot g \cong g \cdot g^{-1} \cdot g \cong g$.

$\impliedby$ X is path connected. $x_1 = x_0$ gives that all loops at x_0 are homotopic to each other, so $\pi_1(X) = 0$.

□

1.2 The fundamental group of the circle

Goal is to show that $\pi_1\left(S^1\right) \cong \mathbb{Z}$.

Lecture 4 Friday 18/01/19

Definition. A **covering space** of a space X is a space $\widetilde{X}$ and a continuous map $p : \widetilde{X} \to X$ such that for each $x \in X$ there is an open $x \in U \subseteq X$ such that

- $p^{-1}(U) = \bigcup_{j \in J} \widetilde{U_j}$, where $\widetilde{U_j} \subseteq \widetilde{X}$ is open,
- $\widetilde{U_i} \cap \widetilde{U_j} = \emptyset$ if $i \neq j$, and
- $p|_{\widetilde{U_j}} : \widetilde{U_j} \to U$ is a homeomorphism for all $j \in J$.

Such a U is called **evenly covered**. The $\widetilde{U_j}$ are called **sheets**.

Example.

$$\begin{array}{cccc} p : & \mathbb{R} & \to & S^1 \\ & s & \mapsto & (\cos(2\pi s, \sin(2\pi s))) \end{array}$$

Definition. Let $p : \widetilde{X} \to X$ be a covering space. A **lift** of a continuous map $f : Y \to X$ is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$, so

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd} .$$

Proposition 1.7 (Unique lifting property)**.** *Let $p : \widetilde{X} \to X$ be a covering space and $f : Y \to X$ be a continuous map. If there are two lifts $\widetilde{f_1}, \widetilde{f_2} : Y \to \widetilde{X}$ of f such that $\widetilde{f_1}(y) = \widetilde{f_2}(y)$ for some $y \in Y$ and if Y is connected, then $\widetilde{f_1} = \widetilde{f_2}$.*

Proof. Let $y \in Y$ and $U \subseteq X$ be an evenly covered neighbourhood of $f(y)$. Then

$$p^{-1}(U) = \bigcup_j \widetilde{U_j}.$$

Let $\widetilde{U_1}$ be the sheet such that $\widetilde{f_1}(y) \in \widetilde{U_1}$, and let $\widetilde{U_2}$ be the sheet such that $\widetilde{f_2}(y) \in \widetilde{U_2}$. Let $N \subseteq Y$ be open and $y \in N$ such that $\widetilde{f_1}(N) \subseteq \widetilde{U_1}$ and $\widetilde{f_2}(N) \subseteq \widetilde{U_2}$. We have $p\widetilde{f_1} = p\widetilde{f_2}$.

$$\widetilde{f_1}(y) = \widetilde{f_2}(y) \qquad \iff \qquad \widetilde{U_1} = \widetilde{U_2} \qquad \iff \qquad \widetilde{f_1}|_N = \widetilde{f_2}|_N .$$

Let

$$A = \left\{ y \in Y \mid \widetilde{f_1}(y) = \widetilde{f_2}(y) \right\},$$

so A is open and $Y \setminus A$ is open. Thus $A \neq \emptyset$ gives $A = Y$. ☐

Proposition 1.8 (Homotopy lifting property)**.** *Let $p : \widetilde{X} \to X$ be a covering space and $F : Y \times I \to X$ be a continuous map such that there exists a lift $\widetilde{f_0} : Y \times \{0\} \to \widetilde{X}$ of $F|_{Y \times \{0\}}$. Then there is a unique lift $\widetilde{F} : Y \times I \to \widetilde{X}$ of F such that $\widetilde{F}|_{Y \times \{0\}} = \widetilde{f_0}$.*

Proof. Let $y_0 \in Y$ and $t \in I$. There are open $y_0 \in N_t \subseteq Y$ and $t \in (a_t, b_t) \subseteq I$ such that $F(N_t \times (a_t, b_t)) \subseteq U \subseteq X$, where $U \subseteq X$ is open and evenly covered. Compactness of I gives that there exist

$$0 = t_0 < \cdots < t_m = 1,$$

and there exists $y_0 \in N \subseteq Y$ open such that $F(N \times [t_i, t_{i+1}]) \subseteq U_i \subseteq X$, where $U_i \subseteq X$ is open and evenly covered. We inductively construct a lift $\widetilde{F}|_{N \times I}$ of $F|_{N \times I}$.

- $\widetilde{F} \mid_{N \times [0,0]} = \widetilde{f_0} \mid_{N \times [0,0]}$ exists.
- Assume the lift has been constructed on $N \times [0, t_i]$. Let $\widetilde{U_i} \subseteq \widetilde{X}$ be such that $p \mid_{\widetilde{U_i}} : \widetilde{U_i} \to U_i$ such that $\widetilde{F}(y_0, t_i) \subseteq \widetilde{U_i}$. After shrinking N, may assume $\widetilde{F}(N \times \{t_i\}) \subseteq \widetilde{U_i}$. Define $\widetilde{F}$ on $N \times [t_i, t_{i+1}]$ to be composition of F with the homeomorphism $p^{-1} : U_i \to \widetilde{U_i}$.

After finitely many steps we obtain a lift $\widetilde{F} : N \times I \to \widetilde{X}$, where $y_0 \in N \subseteq Y$ is open, so for each $y \in Y$ there is a neighbourhood $N_y \subseteq Y$ such that $F \mid_{N_y \times I} : N_y \times I \to X$ lifts. For all $y \in Y$, $\{y\} \times I$ is connected and can be lifted, so Proposition 1.7 gives that the lift of $N \times I$ is unique. Thus there is a unique lift $\widetilde{F} : Y \times I \to \widetilde{X}$. □

Example. Let X be a topological space and A be discrete. Then $p : X \times A \to X$ is a covering space. This is the **trivial covering**. (Exercise: show the unique lifting property and the homotopy lifting property for the trivial covering)

Corollary 1.9. *Let $f : I \to X$ be a path, $f(0) = x_0$, and $p : \widetilde{X} \to X$ be a covering space. For each $\widetilde{x_0} \in p^{-1}(x_0)$, there is a unique lift $\widetilde{f} : I \to \widetilde{X}$ such that $\widetilde{f}(0) = \widetilde{x_0}$.*

Proof. Proposition 1.8 for Y a point. □

Theorem 1.10. *Let $x_0 = (1,0) \in S^1$. $\pi_1(S^1, x_0)$ is the infinite cyclic group generated by the homotopy class of the loop*

$$\begin{array}{rccl} \omega : & I & \to & S^1 \\ & s & \mapsto & (\cos(2\pi s), \sin(2\pi s)) \end{array}.$$

Remark.

- $[\omega]^n = [\omega_n]$, where
$$\omega_n(s) = (\cos(2\pi n s), \sin(2\pi n s)).$$
- $$\begin{array}{rccl} p : & \mathbb{R} & \to & S^1 \\ & s & \mapsto & (\cos(2\pi s), \sin(2\pi s)) \end{array}$$
is a covering space.
- ω_n lifts to
$$\begin{array}{rccl} \widetilde{\omega_n} : & I & \to & \mathbb{R} \\ & s & \mapsto & ns \end{array},$$
such that $\widetilde{\omega_n}(0) = 0$ and $\widetilde{\omega_n}(1) = n$.

Proof of Theorem 1.10.

- If $f : I \to S^1$ be a loop at x_0, then the homotopy lifting property gives that there exists a lift $\widetilde{f} : I \to \mathbb{R}$ such that $\widetilde{f}(0) = 0$. Since $p\left(\widetilde{f}(1)\right) = f(1) = x_0$, then $\widetilde{f}(1) = n$ for some $n \in \mathbb{Z}$. $\widetilde{\omega_n} : I \to \mathbb{R}$ is another path such that $\widetilde{\omega_n}(0) = 0$ and $\widetilde{\omega_n}(1) = n$, so $\widetilde{f} \cong \widetilde{\omega_n}$. Let $F : I \times I \to \mathbb{R}$ be a homotopy equivalence between $\widetilde{f}$ and $\widetilde{\omega_n}$. Then $pF : I \times I \to S^1$ gives a homotopy between $p\widetilde{f} = f$ and $p\widetilde{\omega_n} = \omega_n$.
- Let $m, n \in \mathbb{Z}$ and assume $\omega_m \cong \omega_n$. Let $F : I \times I \to S^1$ be a homotopy.
$$F(0,t) = \omega_m(t), \qquad F(1,t) = \omega_n(t), \qquad F(s,0) = F(s,1) = x_0,$$
for all $s, t \in I$. The unique lifting property gives that $\widetilde{\omega_n}, \widetilde{\omega_m} : I \to \mathbb{R}$ are unique lifts such that $\widetilde{\omega_n}(0) = 0 = \widetilde{\omega_m}(0)$. The homotopy lifting property gives that F lifts uniquely to a homotopy $\widetilde{F} : I \times I \to \mathbb{R}$ between $\widetilde{\omega_n}$ and $\widetilde{\omega_m}$, and $\widetilde{F}(s,1) \in \mathbb{Z}$ for all $s \in I$. Thus $\widetilde{F}(s,1) = n = m$, so $\omega_m \cong \omega_n$ if and only if $n = m$.

□

Lecture 5 is a problem class. Lecture 5 Tuesday 22/01/19

Lecture 6 Wednesday 23/01/19

Theorem 1.11. *Every non-constant polynomial $p \in \mathbb{C}[z]$ has a root in $\mathbb{C}$.*

Proof. May assume

$$p(z) = z^n + a_1 z^{n-1} + \dots + a_n.$$

Assume p has no roots in $\mathbb{C}$. For each $r \in \mathbb{R}_{\geq 0}$ we obtain a loop

$$\begin{array}{cccc} f_r : & I & \to & \mathbb{C} \\ & s & \mapsto & \dfrac{p\left(re^{2\pi i s}\right)/p(r)}{\left|p\left(re^{2\pi i s}\right)/p(r)\right|} \end{array},$$

so $|f_r(s)| = 1$. $f_r(0) = 1$ and $f_r(1) = 1$, so f_r is a loop based at 1. f_0 is the constant loop at 1. $f_r(s)$ depends continuously on r, so $f_r \simeq f_0$ for all $r \in \mathbb{R}_{\geq 0}$ and $[f_r] = [f_0] = 0 \in \pi_1\left(S^1\right)$. Fix $r \in \mathbb{R}_{\geq 0}$ such that $r > 1$ and $r > |a_1| + \dots + |a_n|$. For $|z| = r$ we have

$$|z^n| > (|a_1| + \dots + |a_n|)\left|z^{n-1}\right| \geq \left|a_1 z^{n-1}\right| + \dots + |a_n| \geq \left|a_1 z^{n-1} + \dots + a_n\right|.$$

Hence, for $0 \leq t \leq 1$ the polynomial $p_t(z) = z^n + t\left(a_1 z^{n-1} + \dots + a_n\right)$ has no root z with $|z| = r$. Define

$$F_r(t, s) = \frac{p_t\left(re^{2\pi i s}\right)/p_t(r)}{\left|p_t\left(re^{2\pi i s}\right)/p_t(r)\right|}.$$

$F_r(0, s) = \omega_n(s)$ and $F_r(1, s) = f_r(s)$, so $[\omega_n] = [f_r] = 0 \in \pi_1\left(S^1\right)$. Theorem 1.10 gives that $n = 0$, so p is constant. ☐

See Hatcher Theorem 1.9 and Theorem 1.10 for more applications.

Proposition 1.12. *Let X, Y be topological spaces, $x_0 \in X$, and $y_0 \in Y$. Then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. A map

$$\begin{array}{cccc} f : & Z & \to & X \times Y \\ & z & \mapsto & (g(z), h(z)) \end{array}$$

is continuous if and only if $g : Z \to X$ and $h : Z \to Y$ are continuous. For $Z = I$,

$$\{\text{ loops in } X \times Y \text{ based at } (x_0, y_0)\ \} \quad \leftrightsquigarrow \quad \{\text{ loops in } X \text{ based at } x_0\ \} \times \{\text{ loops in } Y \text{ based at } y_0\ \}.$$

Two loops

$$\begin{array}{cccc} f_1 : & I & \to & X \times Y \\ & s & \mapsto & (g_1(s), h_1(s)) \end{array}, \qquad \begin{array}{cccc} f_2 : & I & \to & X \times Y \\ & s & \mapsto & (g_2(s), h_2(s)) \end{array}$$

are homotopic if and only if $g_1 \simeq g_2$ and $h_1 \simeq h_2$, so there is a bijection

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

$f_1 \cdot f_2 = (g_1 \cdot g_2, h_1 \cdot h_2)$ and the constant loop is mapped to the constant loop, so this is also a group isomorphism. ☐

Example. The torus $S^1 \times S^1$ has

$$\pi_1\left(S^1 \times S^1\right) \cong \pi_1\left(S^1\right) \times \pi_1\left(S^1\right) \cong \mathbb{Z}^2.$$

1.3 Induced homomorphisms

Let X, Y be topological spaces, $x_0 \in X$, and $\phi : X \to Y$. An observation is that ϕ induces a homomorphism

$$\begin{array}{cccc} \phi_* : & \pi_1(X, x_0) & \to & \pi_1(Y, \phi(x_0)) \\ & [f] & \mapsto & [\phi f] \end{array} .$$

ϕ_* is well-defined, since if f_t is a homotopy between the loops f_0 and f_1 based at x_0, then ϕf_t is a homotopy of loops between ϕf_0 and ϕf_1. Moreover,

$$\phi(f \cdot g) = (\phi f) \cdot (\phi g),$$

and ϕ maps the constant path at x_0 to the constant path at $\phi(x_0)$, so ϕ is a homomorphism.

Proposition 1.13.

1. *Let $\psi : X \to Y$ and $\phi : Y \to Z$ be continuous maps between topological spaces, $x_0 \in X$, and*

$$\psi_* : \pi_1(X, x_0) \to \pi_1(Y, \psi(x_0)), \qquad \phi_* : \pi_1(Y, \psi(x_0)) \to \pi_1(Z, \phi\psi(x_0)),$$

$$(\phi\psi)_* : \pi_1(X, x_0) \to \pi_1(Z, \phi\psi(x_0)).$$

   Then $(\phi\psi)_* = \phi_*\psi_*$.

2. *Let $id_X : X \to X$ be the identity then*

$$(id_X)_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

   is the identity.

Proof.

1. Let $f : I \to X$ be a loop at x_0, then

$$(\phi\psi)_*([f]) = [(\phi\psi) f] = [\phi(\psi f)] = \phi_*([\psi f]) = \phi_*\psi_*([f]).$$

2. $(id_X)_*([f]) = [id_X f] = [f]$.

$\square$

These two observations yield in particular that if $\phi : X \to Y$ is a homeomorphism with inverse $\psi : Y \to X$, then

$$\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$$

is an isomorphism with inverse ψ_*. Lecture 7 Friday 25/01/19

Proposition 1.14. *Let $\phi : X \to Y$ be a homotopy equivalence. Then*

$$\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Recall that if $x_0, x_1 \in X$ and $h : I \to X$ is a path such that $h(0) = x_0$ and $h(1) = x_1$, then we obtain an isomorphism

$$\begin{array}{cccc} \beta_h : & \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ & [f] & \mapsto & \left[h \cdot f \cdot h^{-1}\right] \end{array} .$$

Lemma 1.15. *Let $\phi_t : X \to Y$ be a homotopy and $x_0 \in X$. Define the path*

$$\begin{array}{cccc} h : & I & \to & Y \\ & s & \mapsto & \phi_s(x_0) \end{array},$$

where $h(0) = \phi_0(x_0)$ and $h(1) = \phi_1(x_0)$. Then $(\phi_0)_* = \beta_h (\phi_1)_*$, that is the following diagram commutes.

$$\begin{tikzcd} & \pi_1(Y, \phi_1(x_0)) \arrow[dd, "\beta_h", "\sim"'] \\ \pi_1(X, x_0) \arrow[ur, "(\phi_1)_*"] \arrow[dr, "(\phi_0)_*"'] & \\ & \pi_1(Y, \phi_0(x_0)) \end{tikzcd}.$$

Proof. For $t \in I$, define the path

$$\begin{array}{cccc} h_t : & I & \to & X \\ & s & \mapsto & h(ts) \end{array},$$

where $h_t(0) = \phi_0(x_0)$ and $h_t(1) = h(t) = \phi_t(x_0)$. Let f be a loop at x_0. Define

$$F_t = h_t \cdot (\phi_t f) \cdot h_t^{-1}.$$

Then F_t is a loop at $\phi_0(x_0)$, which is continuous in t. So F_t is a homotopy of loops between

$$F_0 = h_0 \cdot (\phi_0 f) \cdot h_0^{-1} \cong \phi_0 f, \qquad F_1 = h_1 \cdot (\phi_1 f) \cdot h_1^{-1} = h \cdot (\phi_1 f) \cdot h^{-1}.$$

Hence

$$(\phi_0)_*([f]) = [\phi_0 f] = \left[h \cdot (\phi_1 f) \cdot h^{-1}\right] = \beta_h([\phi_1 f]) = \beta_h (\phi_1)_*([f]).$$

□

Lemma 1.15 implies in particular the following.

Corollary 1.16. *If $\psi : X \to X$ is continuous and $\psi \cong id_X$, then*

$$\psi_* : \pi_1(X, x_0) \to \pi_1(X, \psi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Proof. By Lemma 1.15 there is a path h from $\psi(x_0)$ to x_0 such that

$$\begin{tikzcd} & \pi_1(X, x_0) \arrow[dd, "\beta_h", "\sim"'] \\ \pi_1(X, x_0) \arrow[ur, "(id_X)_*"] \arrow[dr, "\psi_*"'] & \\ & \pi_1(X, \psi(x_0)) \end{tikzcd},$$

so $\psi_* = \beta_h$ hence an isomorphism. □

Proof of Proposition 1.14. Let $\phi : X \to Y$ be a homotopy equivalence. Let $\psi : Y \to X$ be a homotopy inverse of ϕ, that is $\phi\psi \cong id_Y$ and $\psi\phi \cong id_X$.

$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\phi(x_0)) \xrightarrow{\phi_*} \pi_1(Y, \psi\phi\psi(x_0)).$$

Have to show that ϕ_* is bijective. The observation above gives that $(\psi\phi)_* = \psi_*\phi_*$ is an isomorphism, so ϕ_* is injective and ψ_* is surjective. Similarly $(\phi\psi)_* = \phi_*\psi_*$ is an isomorphism, so ψ_* is injective and ϕ_* is surjective. □

Lemma 1.17. *Let X be a topological space and $x_0 \in X$. Assume*

$$X = \bigcup_{\alpha \in \Lambda} A_\alpha,$$

such that

- *the A_α are all open and path connected,*
- *$x_0 \in A_\alpha$ for all $\alpha \in \Lambda$, and*
- *all the intersections $A_\alpha \cap A_\beta$ are path connected for all $\alpha, \beta \in \Lambda$.*

If f is a loop in X at x_0, then we can write $[f] = [h_1] \dots [h_m]$, such that the h_i are loops at x_0, and each contained in a single A_{α_i}.

Proof. f is continuous, so for all $s \in I$ there is an open neighbourhood V_s such that $f(V_s)$ such that $f(V_s) \subseteq A_\alpha$ for some α. We can choose V_s to be an interval (a_s, b_s) such that $f([a_s, b_s]) \subseteq A_\alpha$. I is compact gives that a finite number of such intervals cover I, so there is a partition

$$0 = s_0 < \dots < s_m = 1,$$

such that $f([s_{i-1}, s_i]) \subseteq A_{\alpha_i}$ for some α_i. Let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$, and rescaling. $f \cong f_1 \cdot \dots \cdot f_m$ for $f_i \subseteq A_{\alpha_i}$ and $A_{\alpha_i} \cap A_{\alpha_j}$ is path connected. Let g_i be a path from x_0 to $f(s_i)$ in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$. Let g_0, g_m be the constant loops at x_0. $h_i = g_{i-1} \cdot f_i \cdot g_i^{-1}$ is a loop based at x_0 and $h_i \subseteq A_{\alpha_i}$. Thus

$$f \cong \left(g_0 \cdot f_1 \cdot g_1^{-1}\right) \cdot \dots \cdot \left(g_{m-1} \cdot f_m \cdot g_m^{-1}\right),$$

so $[f] = [h_1] \dots [h_m]$. $\square$

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Example. Möbius strip M deformation retracts to S^1. Thus $\phi : M \to S^1$ is a homotopy equivalence, so $\pi_1(M) \cong \pi_1\left(S^1\right) \cong \mathbb{Z}$.

Example. There is no deformation retraction of S^1 to a point $p \in S^1$ because $\pi_1\left(S^1\right) \ncong \pi_1(p)$.

Example. There is no retraction of the disc D^2 to its boundary $S^1 \subseteq D^2$.

Proof. Assume there is a retraction $r : D^2 \to S^1$, consider the embedding $i : S^1 \hookrightarrow D^2$. Then $ri = id_{S^1}$. Thus

$$\begin{array}{ccccc} \pi_1\left(S^1\right) & \xrightarrow{i_*} & \pi_1\left(D^2\right) & \xrightarrow{r_*} & \pi_1\left(S^1\right) \\ \cong & & \cong & & \cong \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array},$$

so $r_* i_* \left(\pi_1\left(S^1\right)\right) = 0$ but $r_* i_* = (ri)_* = id_{\pi_1(S^1)}$, a contradiction. $\square$

Theorem 1.18 (Brouwer fixed point theorem)**.** *Let $h : D^2 \to D^2$ be a continuous map. Then h has a fixed point, that is there exists $x \in D^2$ such that $h(x) = x$.*

Proof. Assume $h(x) \neq x$ for all $x \in D^2$. Define $r : D^2 \to S^1$ by defining $r(x)$ to be the intersection of the ray starting at $h(x)$ towards x with S^1. r is continuous, and if $x \in S^1$, then $r(x) = x$, so r is a retraction, a contradiction. $\square$

Lemma 1.17 gives that if $U_1, U_2 \subseteq X$ are open and path connected such that $U_1 \cup U_2 = X$ and $U_1 \cap U_2$ is path connected and $x_0 \in U_1 \cap U_2$, then every $[f] \in \pi_1(X, x_0)$ can be factorised as $[f] = [g_1][h_1] \dots [g_n][h_n]$ such that the g_i are loops at x_0 contained in U_1 and the h_i are loops at x_0 contained in U_2. In other words, $i_1 : U_1 \hookrightarrow X$ and $i_2 : U_2 \hookrightarrow X$, so

$$(i_1)_* : \pi_1(U_1, x_0) \to \pi_1(X, x_0), \qquad (i_2)_* : \pi_1(U_2, x_0) \to \pi_1(X, x_0).$$

Lemma 1.17 gives that $(i_1)_*(\pi_1(U_1, x_0)) \cup (i_2)_*(\pi_1(U_2, x_0))$ generate $\pi_1(X, x_0)$.

Proposition 1.19. $\pi_1(S^n) = 0$ *if* $n \geq 2$.

Proof. Let $U_1 = S^n \setminus \{(1, 0, \dots, 0)\}$ and $U_2 = S^n \setminus \{(-1, 0, \dots, 0)\}$. Then $U_1 \cong \mathbb{R}^n$ and $U_2 \cong \mathbb{R}^n$, by stereographic projection. $U_1 \cup U_2 = S^n$ and $U_1 \cap U_2$ is path connected. Let $x_0 \in U_1 \cap U_2$. $\pi_1(U_1, x_0) = 0$ and $\pi_1(U_2, x_0) = 0$, so Lemma 1.17 gives that $\pi_1(S^n, x_0)$. $\square$

2 The fundamental group - Seifert-van Kampen theorem

2.1 Free products with amalgamation

Definition. If S is a set, then F_S is the **free group** on S. We can write any group G as a quotient of some free group F_S,

$$G = \frac{F}{\langle\langle R \rangle\rangle},$$

where $\langle\langle R \rangle\rangle$ is the **normal closure** of $R \subseteq F_S$, the smallest normal subgroup of F_S containing R. We write $G = \langle S \mid R \rangle$. This is called a **presentation** of G.

Let G_0, G_1, G_2 be groups, and $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$ be homomorphisms.

Definition. A group H together with homomorphisms $h_1 : G_1 \to H$ and $h_2 : G_2 \to H$ such that $h_1 f_1 = h_2 f_2$ is an **amalgamated product** of G_1 and G_2 over G_0 if it satisfies the following universal property. For every group G and all homomorphisms $h_1' : G_1 \to G$ and $h_2' : G_2 \to G$ such that $h_1' f_1 = h_2' f_2$, there exists a unique homomorphism $\alpha : H \to G$ such that $h_1' = \alpha h_1$ and $h_2' = \alpha h_2$.

$$\begin{tikzcd}
G_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & G_1 \arrow[d, "h_1"] \arrow[ddr, bend left, "h_1'"] & \\
G_2 \arrow[r, "h_2"'] \arrow[drr, bend right, "h_2'"'] & H \arrow[dr, "\exists ! \alpha"] & \\
& & G.
\end{tikzcd}$$

Theorem 2.1. *Given $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$. Then there exists an amalgamated product, unique up to isomorphism. We denote it by $G_1 \underset{G_0}{*} G_2$.*

Proof. Worksheet 2. $\square$

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$G_0 = \{id\}$ is the **free product**. We write $G_1 * G_2$ instead of $G_1 \underset{\{id\}}{*} G_2$. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Then $G_1 * G_2 = \langle S_1 \sqcup S_2 \mid R_1 \cup R_2 \rangle$, with injections $G_i \hookrightarrow G_1 * G_2$ for $i = 1, 2$. More generally,

$$G_1 * G_2 \cong \frac{G_1 \underset{G_0}{*} G_2}{N}.$$

where N is the normal closure of the set

$$\left\{ f_1(g) f_2(g)^{-1} \mid g \in G_0 \right\} \subseteq G_1 * G_2.$$

2.2 The Seifert van-Kampen theorem

Theorem 2.2 (Seifert-van Kampen)**.** *Let X be a topological space and $U_1, U_2 \subseteq X$ be open and path connected such that $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is path connected and let $x_0 \in U_1 \cap U_2$. Then*

$$\pi_1(X, x_0) \cong \pi_1(U_1, x_0) \underset{\pi_1(U_1 \cap U_2, x_0)}{*} \pi_2(U_2, x_0) \cong \frac{\pi_1(U_1, x_0) * \pi_1(U_2, x_0)}{N},$$

where N is the normal closure of the set

$$\left\{ (j_1)_*(\omega) (j_2)_*(\omega)^{-1} \mid \omega \in \pi_1(U_1 \cap U_2, x_0) \right\},$$

and $j_i : U_1 \cap U_2 \hookrightarrow U_i$.

$$\begin{tikzcd}
U_1 \cap U_2 \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"'] & U_1 \arrow[d, hook, "j_1"] \\
U_2 \arrow[r, hook, "j_2"'] & X
\end{tikzcd}
\quad \implies \quad
\begin{tikzcd}
\pi_1(U_1 \cap U_2, x_0) \arrow[r, "(i_1)_*"] \arrow[d, "(i_2)_*"'] & \pi_1(U_1, x_0) \arrow[d, "(j_1)_*"] \\
\pi_1(U_2, x_0) \arrow[r, "(j_2)_*"'] & \pi_1(U_1, x_0) \underset{\pi_1(U_1 \cap U_2, x_0)}{*} \pi_2(U_2, x_0)
\end{tikzcd}.$$

Proof. Consider the natural homomorphism

$$\Phi : \pi_1(U_1, x_0) * \pi_1(U_2, x_0) \to \pi_1(X, x_0).$$

Φ is surjective by Lemma 1.17. $N \subseteq Ker(\Phi)$. Want to show that $N = Ker(\Phi)$. A **factorisation** of an element $[f] \in \pi_1(X, x_0)$ is a formal product $[f_1] \dots [f_k]$ such that

- each f_i is a loop at x_0 in one of the U_i and $[f_i] \in \pi_1(U_i, x_0)$ is its homotopy class, and
- the loop $f_1 \cdot \dots \cdot f_k$ is homotopic to f in X.

A factorisation of $[f]$ is a word in $\pi_1(U_1, x_0) * \pi_1(U_2, x_0)$ that is mapped to $[f]$ by Φ. Two factorisations of $[f]$ are **equivalent** if they are related by finitely many of the following two moves.

- If $[f_i]$ and $[f_{i+1}]$ lie in the same group $\pi_1(U_i, x_0)$, exchange $[f_i][f_{i+1}]$ with $[f_i \cdot f_{i+1}]$. These are the relations in $\pi_1(U_i, x_0) * \pi_1(U_i, x_0)$.
- If f_i is a loop in $U_1 \cap U_2$, consider $[f_i]$ as an element in $\pi_1(U_1, x_0)$ instead of $\pi_1(U_2, x_0)$, and vice versa. These are the relations in $\pi_1(U_1, x_0) * \pi(U_2, x_0)/N$.

Given $[f] \in \pi_1(X, x_0)$, we want to show that any two factorisations of $[f]$ are equivalent. Let $[f_1] \dots [f_k]$ and $[f_1'] \dots [f_l']$ be two factorisations of $[f]$, so the two loops $f_1 \cdot \dots \cdot f_k$ and $f_1' \cdot \dots \cdot f_k'$ are homotopic. Let $F : I \times I \to X$ be a homotopy. By compactness, there exist

$$0 = s_0 < \dots < s_m = 1, \qquad 0 = t_0 < \dots < t_n = 1,$$

such that $R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ and $F(R_{ij}) \subseteq U_1$ or $F(R_{ij}) \subseteq U_2$. May assume $0 = s_0 < \dots < s_m = 1$ subdivides the products $f_1 \cdot \dots \cdot f_k$ and $f_1' \cdot \dots \cdot f_l'$. Relabel the R_{ij} to $R_1, \dots, R_{mn}$.

$mn - m + 1$	$\dots$	mn
$\vdots$	$\ddots$	$\vdots$
1	$\dots$	m

A path γ in $I \times I$ from left to right gives a loop $F\mid_\gamma$ in X at x_0. Let γ_r be the path separating the first r rectangles from the others, so

$$F\mid_{\gamma_0} \cong f_1 \cdot \dots \cdot f_k, \qquad F\mid_{\gamma_{mn}} = f_1' \cdot \dots \cdot f_l'.$$

Let v be a grid point. Choose a path g_v in X from x_0 to $F(v)$, such that g_v is contained in $U_1 \cap U_2$ if $F(v) \in U_1 \cap U_2$ and in a single U_i otherwise. This gives us a factorisation of $[F\mid_{\gamma_r}]$ into loops only contained in U_1 or U_2. The factorisations associated to γ_r and γ_{r+1} are equivalent, because the homotopy between $F\mid_{\gamma_r}$ and $F\mid_{\gamma_{r+1}}$ by pushing γ_r through R_r takes place within a single U_i. $\square$

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Theorem 2.3 (Seifert-van Kampen, strong version)**.** *Let X be a path connected topological space such that*

- $X = \bigcup_\alpha A_\alpha$,
- A_α, $A_\alpha \cap A_\beta$, *and* $A_\alpha \cap A_\beta \cap A_\gamma$ *are open and path connected for all* α, β, γ, *and*
- $x_0 \in \cap_\alpha A_\alpha$.

Then

$$\pi_1(X, x_0) \cong \frac{\underset{\alpha}{*} \pi_1(A_\alpha, x_0)}{N},$$

where $N \subseteq \underset{\alpha}{*} \pi_1(A_\alpha, x_0)$ *is the normal closure of the set*

$$\left\{ (i_{\alpha\beta})_* (\omega) (i_{\beta\alpha})_* (\omega)^{-1} \mid \omega \in \pi_1(A_\alpha \cap A_\beta) \right\},$$

and $i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ *is the inclusion.*

Example. Let $S^1 \vee S^1$ be the wedge product. Fix $x \in S^1$ and $y \in S^1$. Then

$$S^1 \vee S^1 = \frac{S^1 \sqcup S^1}{x \sim y} = \overset{b}{\mathrm{O}} \cdot \overset{a}{\mathrm{O}}.$$

Let

$$A = \mathrm{O} \cdot (, \qquad B =) \cdot \mathrm{O}, \qquad A \cap B =) \cdot (.$$

$\pi_1(A) \cong \langle b \rangle \cong \mathbb{Z}$, $\pi_1(B) \cong \langle a \rangle \cong \mathbb{Z}$, and $\pi(A \cap B) = \{id\}$. A, B, and $A \cap B$ are open and path connected. Van Kampen gives

$$\pi_1\left(S^1 \vee S^1\right) \cong \pi_1(A) * \pi_1(B) \cong \mathbb{Z} * \mathbb{Z} \cong F_{\{a,b\}}.$$

More generally, let $X = S^1_{a_1} \vee \cdots \vee S^1_{a_n}$. By induction,

$$\pi_1(X) = \mathbb{Z} * \cdots * \mathbb{Z} \cong F_{\{a_1,\dots,a_n\}}.$$

Similarly, let $X = \bigvee_{\alpha \in \Lambda} S^1_\alpha$. Strong version of van Kampen gives

$$\pi_1(X) = \mathop{*}_{\alpha \in \Lambda} \mathbb{Z} = F_\Lambda.$$

Example. Let T be a torus and $x_0 \in T$. Let

$$A = T \setminus \{\text{small closed disc } D\}, \qquad B = \{\text{open set that contains } D \text{ and } x_0\}.$$

- A is homotopy equivalent to $S^1 \vee S^1$, so $\pi_1(A) \cong F_{\{a,b\}}$.
- B is homeomorphic to D^2, so $\pi_1(B) = \{id\}$.
- $A \cap B$ is homotopy equivalent to S^1, so $\pi_1(A \cap B) \cong \mathbb{Z}$.

A, B, and $A \cap B$ are open and path connected. Van Kampen gives

$$\pi_1(T) \cong \frac{\pi_1(A)}{\langle\langle i_*(\pi_1(A \cap B)) \rangle\rangle},$$

where $i : A \cap B \hookrightarrow A$. Then

$$\begin{array}{cccc} i_* : & \pi_1(A \cap B) = \langle \omega \rangle & \to & \pi_1(A) \\ & \omega & \mapsto & aba^{-1}b^{-1} \end{array},$$

so

$$\pi_1(T) \cong \frac{F_{\{a,b\}}}{\langle\langle aba^{-1}b^{-1} \rangle\rangle} = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle \cong \mathbb{Z}^2.$$

2.3 Applications to CW-complexes

Let X be a path connected topological space. Let Y be the space obtained by attaching 2-cells $\left\{e^2_\alpha\right\}$ to X along maps $\phi_\alpha : \partial D^2 = S^1 \to X$. Consider the loops

$$\begin{array}{cccc} \phi'_\alpha : & I & \to & X \\ & s & \mapsto & \phi_\alpha(\cos(2\pi s), \sin(2\pi s)) \end{array},$$

based at $\phi'_\alpha(0)$. Let γ_α be a path from x_0 to $\phi'_\alpha(0)$ for each α. Then $\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}$ is a loop at x_0. After attaching e^2_α, the loop $\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}$ is homotopic to the constant loop at x_0. Let $N \subseteq \pi_1(X, x_0)$ be the normal closure of all the elements of the form $\left[\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}\right]$. The inclusion $i : X \hookrightarrow Y$ yields

$$i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0),$$

and $N \subseteq Ker(i_*)$.

Proposition 2.4. *This inclusion $i : X \hookrightarrow Y$ induces a surjection $i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0)$ and $Ker(i_*) = N$, so*

$$\pi_1(Y, x_0) \cong \frac{\pi_1(X, x_0)}{N}.$$

Proof. Construct a space Z from Y by attaching a strip $I \times I$ to Y by identifying the lower edge $I \times \{0\}$ with the path γ_α and the right edge $\{1\} \times I$ with an arch on e^2_α. Attach all the left edges of the strips with each other. Z deformation retracts to Y. Choose a point $y_\alpha \in e^2_\alpha$ for each α, such that y_α is not contained in X or in the attached strip. Let

$$A = Z \setminus \bigcup_\alpha \{y_\alpha\}, \qquad B = Z \setminus X.$$

- A deformation retracts to X.
- B is homotopy equivalent to a point.
- $A \cap B$ is homotopy equivalent to

$$\{\text{paths } \gamma_\alpha \text{ from } x_0 \text{ to loops } \phi'_\alpha\} = \overset{\phi'_\alpha}{\mathrm{O}} \overset{\gamma_\alpha}{-} \overset{x_0}{\cdot} \overset{\gamma_\alpha}{-} \overset{\phi'_\alpha}{\mathrm{O}}.$$

A, B, and $A \cap B$ are open and path connected. Van Kampen gives

$$\pi_1(Y) \cong \pi_1(Z) = \frac{\pi_1(A)}{\langle\langle j_*(\pi_1(A \cap B))\rangle\rangle},$$

where $j : A \cap B \hookrightarrow A$ is the inclusion. So $\langle\langle j_*(\pi(A \cap B))\rangle\rangle$ is exactly N. Thus $\pi_1(A) = \pi_1(X)$. $\square$

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Corollary 2.5. *For every group G there exists a two-dimensional CW-complex X_G such that $\pi_1(X_G) = G$.*

Proof. Let $G = \langle \{g_\alpha\} \mid \{r_\beta\}\rangle$ be a presentation of G, that is

$$G = \frac{F_{\{g_\alpha\}}}{\langle\langle\{r_\beta\}\rangle\rangle}.$$

Seen last time that $\pi_1\left(\bigvee_{g_\alpha} S^1_{g_\alpha}\right) = F_{\{g_\alpha\}}$. Each word r_β defines a loop in $\bigvee_{g_\alpha} S^1_{g_\alpha}$. Attach 2-cells to $\bigvee_{g_\alpha} S^1_{g_\alpha}$ along the loops defined by the relations $\{r_\beta\}$. Call this new CW-complex Y. Proposition 2.4 gives that

$$\pi_1(Y, x_0) \cong \frac{\pi_1(X, x_0)}{\langle\langle\{r_\beta\}\rangle\rangle} \cong \frac{F_{\{g_\alpha\}}}{\langle\langle\{r_\beta\}\rangle\rangle} \cong G.$$

$\square$

Remark. Let $X = \bigcup_n X^n$ be a CW-complex, path connected. Proposition 2.4 can be used to show the following two facts.

- The inclusion $X^1 \hookrightarrow X$ induces a surjective homomorphism $\pi_1(X^1) \to \pi_1(X)$.
- The inclusion $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2) \to \pi_1(X)$.

3 The fundamental group - covering spaces

3.1 Lifting properties

Let X be a topological space. Recall that a **covering space** is $p : \widetilde{X} \to X$ such that each $x \in X$ has an open neighbourhood U such that $p^{-1}(U) = \bigcup_\alpha \widetilde{U_\alpha}$, where U_α are pairwise disjoint and $p \mid_{\widetilde{U_\alpha}} : \widetilde{U_\alpha} \to U$ is a homeomorphism for all α.

Example.

$$\begin{array}{cccc} p : & \mathbb{R} & \to & S^1 \\ & s & \mapsto & (\cos(2\pi s), \sin(2\pi s)) \end{array}, \qquad \begin{array}{cccc} p : & S^1 & \to & S^1 \\ & z & \mapsto & z^n \end{array}, \qquad \mathrm{O}\cdot\mathrm{O}\cdot\mathrm{O} \to S^1 \vee S^1.$$

Let $f : Y \to X$ be a continuous map. A **lift** of f is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$, where $p : \widetilde{X} \to X$ is a covering space. Let Y be connected.

- **Unique lifting property** states that if two lifts $\widetilde{f_1}$ and $\widetilde{f_2}$ of f coincide at one point, then they coincide on all of Y.
- **Homotopy lifting property** states that if $f_t : Y \to X$ is a homotopy and $\widetilde{f_0} : Y \to \widetilde{X}$ is a lift of f_0 then there exists a unique homotopy $\widetilde{f_t} : Y \to \widetilde{X}$ of $\widetilde{f_0}$ that lifts f_t.

Remark.

- If Y is a point, this is called the **path lifting property**. Let $f : I \to X$ be a path with $f(0) = x_0$. If $\widetilde{x_0} \in p^{-1}(x_0)$, then there is a unique path $\widetilde{f} : I \to \widetilde{X}$ lifting f and starting at $\widetilde{x_0}$.
- In particular, the lift of a constant path is constant.
- This implies in particular that the lift of a homotopy of paths is again a homotopy of paths. The endpoints $f_t(0)$ and $f_t(1)$ define constant paths as t varies.

Fix $x_0 \in X$ and $\widetilde{x_0} \in \widetilde{X}$ such that $p(\widetilde{x_0}) = x_0$, so $p_* : \pi_1\left(\widetilde{X}, \widetilde{x_0}\right) \to \pi_1(X, x_0)$. To every element in $\pi_1(X, x_0)$ we can associate a homotopy class of paths in $\widetilde{X}$ starting at $\widetilde{x_0}$.

Proposition 3.1.

1. *$p_* : \pi_1\left(\widetilde{X}, \widetilde{x_0}\right) \to \pi_1(X, x_0)$ is injective.*
2. *$p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) \subseteq \pi_1(X, x_0)$ consists of the homotopy classes of loops at x_0 whose lifts to $\widetilde{X}$ starting at $\widetilde{x_0}$ are loops.*

Proof.

1. Let $\widetilde{f_0} : I \to \widetilde{X}$ be a loop at $\widetilde{x_0}$ such that $\left[\widetilde{f_0}\right] \in Ker(p_*)$, so $p\widetilde{f_0} = f_0$ is homotopic to the constant loop at x_0. Let $f_t : I \to X$ be a homotopy between f_0 and the constant loop. Homotopy lifting property and remark gives that f_t lifts to a homotopy $\widetilde{f_t}$ of paths between $\widetilde{f_0}$ and the constant loop, so $\left[\widetilde{f_0}\right] = id \in \pi_1\left(\widetilde{X}, \widetilde{x_0}\right)$ and p_* is injective.
2. Let $f : I \to X$ be a loop at x_0 that lifts to a loop $\widetilde{f}$ at $\widetilde{x_0}$. Then $p\widetilde{f} = f$, so $p_*\left(\left[\widetilde{f}\right]\right) = [f]$. On the other hand, if $f : I \to X$ is a loop at x_0 such that there exists a loop $\widetilde{f} : I \to \widetilde{X}$ at $\widetilde{x_0}$ with $p_*\left(\left[\widetilde{f}\right]\right) = [f]$, then f is homotopic to $p\widetilde{f}$. Homotopy lifting property gives that there exists a loop $\widetilde{f}' : I \to \widetilde{X}$ at x_0 such that $p\widetilde{f}' = f$.

∎