

# M4P58 Modular Forms

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Autumn 2019

**Syllabus**

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## 0 Introduction

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The following are textbooks.

- Serre, A course in arithmetic, 1973
- J Shurman and F Diamond, A first course in modular forms, 2005

Let

$$f = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 = \sum_{n=1} b_n q^n = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \dots,$$

and let  $a_n$  be the number of solutions modulo  $n$  to the elliptic curve

$$E = \{(x, y) \in \mathbb{Z} \mid y^2 + y = x^3 - x^2 - 10x - 20\}.$$

- Modulo 2, there are  $a_2 = 4$  solutions  $(0, 0), (0, 1), (1, 0), (1, 1)$ .
- Modulo 3, there are  $a_3 = 4$  solutions  $(1, 0), (1, -1), (-1, 0), (-1, -1)$ .
- Modulo 5, there are  $a_5 = 4$  solutions  $(0, 0), (0, -1), (1, 0), (-1, -1)$ .
- Modulo 7, there are  $a_7 = 9$  solutions  $(1, 3), (2, 2), (2, -3), (-1, 1), (-1, -2), (-2, 1), (-2, -2), (-3, 1), (-3, -2)$ .

If  $p \neq 11$ , then

$$a_p - p = -b_p.$$

The following are some questions.

- What is the relationship between  $E$  and  $f$ ?
- Can we find similar relationships for other  $E$ ?
- How does one prove something like this?

Let

$$\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\} \subseteq \mathbb{C}.$$

Then  $\mathbb{H}$  has an action of

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

Modular forms are complex functions on  $\mathbb{H}$  with a high degree of symmetry. These functions are symmetric under the action of large discrete subgroups of  $\mathrm{SL}_2(\mathbb{R})$ , in particular

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} \subseteq \mathrm{SL}_2(\mathbb{R}).$$

Why are these interesting to number theorists? Power series expansions often involve expressions of interest to number theorists. For example,

- Bernoulli numbers,
- divisor functions  $\sigma_k(n) = \sum_{d|n} d^k$ ,
- number of points on elliptic curves, and
- traces of Galois representations.

# 1 Modular forms of level one

## 1.1 Modular forms

### 1.1.1 Modular actions

Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

Then  $\mathrm{SL}_2(\mathbb{R})$  acts on  $\mathbb{C} \cup \{\infty\}$  by

$$\gamma \cdot z = \begin{cases} \frac{az+b}{cz+d} & z \neq -\frac{d}{c} \\ \infty & z = -\frac{d}{c} \end{cases} \quad \gamma \cdot \infty = \frac{a}{c}.$$

One checks that this gives a bijection from  $\mathbb{C} \cup \{\infty\}$  to  $\mathbb{C} \cup \{\infty\}$ , where inverse is given by the inverse matrix

$$\gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and  $\gamma \cdot (\gamma' \cdot z) = \gamma\gamma' \cdot z$ . One obtains a left action of  $\mathrm{SL}_2(\mathbb{R})$  on  $\mathbb{C} \cup \{\infty\}$ . An observation is

$$\mathrm{Im} \gamma z = \mathrm{Im} \frac{az+b}{cz+d} = \mathrm{Im} \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{\mathrm{Im}(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{(ad-bc)\mathrm{Im} z}{|cz+d|^2}.$$

In particular, if  $\gamma \in \mathrm{SL}_2(\mathbb{R})$ , then

$$\mathrm{Im} \gamma z = \frac{\mathrm{Im} z}{|cz+d|^2}.$$

So  $\mathrm{SL}_2(\mathbb{R})$  preserves  $\mathbb{H} \cup \{\infty\}$ . More generally, if  $\gamma \in \mathrm{GL}_2(\mathbb{R})$ , then

$$\mathrm{Im} \gamma z = \frac{\det \gamma \mathrm{Im} z}{|cz+d|^2}.$$

So

$$\mathrm{GL}_2(\mathbb{R})_+ = \{\gamma \in \mathrm{GL}_2(\mathbb{R}) \mid \det \gamma > 0\}$$

preserves  $\mathbb{H} \cup \{\infty\}$ . Define

$$\begin{aligned} f|_{k,\gamma} &: \mathbb{H} \longrightarrow \mathbb{C} \\ z &\longmapsto \det \gamma^{k-1} f(\gamma z) (cz+d)^{-k}, \quad f: \mathbb{H} \rightarrow \mathbb{C}, \quad \gamma \in \mathrm{GL}_2(\mathbb{R})_+, \quad k \in \mathbb{Z}, \end{aligned}$$

where  $\det \gamma^{k-1}$  is the fudge factor, which is one for  $\gamma \in \mathrm{SL}_2(\mathbb{R})$ , and  $(cz+d)^{-k}$  is the twisted action on functions. Check that

$$f|_{k,\mathrm{id}} = f, \quad \left(f|_{k,\gamma}\right)|_{k,\gamma'} = f|_{k,\gamma'\gamma}.$$

This gives, for each  $k$ , a left action of  $\mathrm{GL}_2(\mathbb{R})_+$  on functions  $\mathbb{H} \rightarrow \mathbb{C}$ , a **modular action of weight  $k$** . A modular form of weight  $k$  will be a sufficiently nice function  $f: \mathbb{H} \rightarrow \mathbb{C}$  such that  $f|_{k,\gamma} = f$  for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ . That is, for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  and  $z \in \mathbb{H}$ ,

$$f(\gamma z) (cz+d)^{-k} = f(z), \quad \implies \quad f(\gamma z) = f(z) (cz+d)^k,$$

the **modular transformation law of weight  $k$** . The following are some observations.

- Let  $k = 0$ . Then constant functions satisfy  $f(\gamma z) = f(z)$ . It will turn out that all functions of weight zero are constant.
- Let  $k$  be odd, and  $\gamma = -\mathrm{id}$ . Then  $\gamma z = z$  for all  $z$  and  $cz+d = -1$ , so  $f(\gamma z) = f(z) (cz+d)^k$  gives  $f(z) = f(z) (-1)^k$ , so  $f(z) = -f(z)$ , so  $f(z) = 0$  for all  $z$ . So no non-zero functions  $f: \mathbb{H} \rightarrow \mathbb{C}$  satisfy the modular transformation law of weight  $k$ , for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , when  $k$  is odd.

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### 1.1.2 Review of complex analysis

Let  $f : U \rightarrow \mathbb{C}$ , for  $U \subseteq \mathbb{C}$  open, and let  $p \in U$ .

**Definition 1.1.1.**  $f$  is **holomorphic** at  $p$  if

$$f'(p) = \lim_{\epsilon \rightarrow 0, \epsilon \in \mathbb{C}} \frac{f(p' + \epsilon) - f(p')}{\epsilon}$$

exists for all  $p'$  in a neighbourhood of  $p$ .

**Proposition 1.1.2.**  $f$  is holomorphic at  $p$  implies that  $f$  is continuous.

**Proposition 1.1.3.**  $f$  is holomorphic at  $p$  implies that  $f$  is infinitely differentiable at  $p$ , that is  $f^{(n)}(p)$  exists for all  $n \geq 0$ . Moreover, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (z-p)^n = f(p) + f'(p)(z-p) + \frac{f''(p)}{2} (z-p)^2 + \dots,$$

for all  $z$  in a neighbourhood of  $p$ .

**Corollary 1.1.4.** If  $f$  is holomorphic and not identically zero on an open set  $U$ , then the zeroes of  $f$  are isolated on  $U$ .

More generally is the following.

**Definition 1.1.5.**  $f$  is **meromorphic** at  $p$  if there exists a neighbourhood  $U$  of  $p$  and  $g, h : U \rightarrow \mathbb{C}$  holomorphic on  $U$  such that  $f = g/h$  on  $U \setminus \{p\}$ . Such an  $f$  has a **Laurent series expansion** at  $p$ ,

$$f(z) = \sum_{i=-N}^{\infty} c_i (z-p)^i.$$

The smallest  $i$  such that  $c_i \neq 0$  is denoted by  $\text{ord}_p f$ , the **order of vanishing** of  $f$  at  $p$ .

- If  $\text{ord}_p f = -n$  for  $n > 0$ , we say  $f$  has a **pole of order  $n$** .
- If  $\text{ord}_p f = n$  for  $n > 0$ , we say  $f$  has a **zero of order  $n$** .

**Proposition 1.1.6.**

- $\text{ord}_p fg = \text{ord}_p f + \text{ord}_p g$ .
- $\text{ord}_p (f + g) \geq \min \{\text{ord}_p f, \text{ord}_p g\}$ , with equality if  $\text{ord}_p f \neq \text{ord}_p g$ .

If  $f$  is holomorphic on  $U \setminus \{p\}$  for  $U$  a neighbourhood of  $p$ , then  $f$  may or may not be meromorphic at  $p$ .

**Example.**  $f(z) = e^{-1/z^2}$  is holomorphic on  $\mathbb{C} \setminus \{0\}$ , but not meromorphic at zero.

**Theorem 1.1.7.** Let  $f$  be holomorphic on  $U \setminus \{p\}$ , and there exists  $n > 0$  such that

$$\lim_{x \rightarrow p} (x-p)^n f(x)$$

exists. Then  $f$  is meromorphic on  $U$ , and  $\text{ord}_p f \geq -n$ .

### 1.1.3 Modular forms

**Definition 1.1.8.**  $f : \mathbb{H} \rightarrow \mathbb{C}$  is a **weakly modular function of weight  $k$**  if

- $f$  is meromorphic on  $\mathbb{H}$ , and
- $f$  satisfies the modular transformation law of weight  $k$ .

Consider

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so  $\gamma z = z + 1$  and  $cz + d = 1$ . The modular transformation law gives  $f(z + 1) = f(z)$ . Let

$$D = \{q \mid |q| < 1\}.$$

Can define a function

$$\begin{aligned} g : D \setminus \{0\} &\longrightarrow \mathbb{C} \\ q &\longmapsto f\left(\frac{\log q}{2\pi i}\right), \end{aligned}$$

that is  $f(z) = g(e^{2\pi iz})$  for  $z \in \mathbb{H}$ , where  $g$  is holomorphic or meromorphic on  $\{z \mid 0 < |z| < 1\}$  if and only if  $f$  is holomorphic or meromorphic on  $\mathbb{H}$ .

**Definition 1.1.9.**  $f : \mathbb{H} \rightarrow \mathbb{C}$  is a **modular form of weight  $k$**  if

1.  $f$  satisfies the modular transformation law of weight  $k$ ,
2.  $f$  is holomorphic on  $\mathbb{H}$ , and
3.  $f$  is holomorphic at  $\infty$ , so the function  $g : D \setminus \{0\} \rightarrow \mathbb{C}$ , which is holomorphic on  $D \setminus \{0\}$  by 2, extends to a holomorphic function on  $D$ .

Then  $q \rightarrow 0$  in  $D$  if and only if  $\text{Im } z \rightarrow +\infty$ . Then 3 means  $g(q)$  is bounded as  $q \rightarrow 0$  so  $f(z)$  is bounded as  $\text{Im } z \rightarrow +\infty$ . For  $f$  satisfying 3,  $g : D \setminus \{0\} \rightarrow \mathbb{C}$  has a series expansion

$$g(q) = \sum_n a_n q^n = a_0 + a_1 q + \dots$$

in  $q = e^{2\pi iz}$ . We call this the  **$q$ -expansion** for  $f$ .

**Definition 1.1.10.**  $f : \mathbb{H} \rightarrow \mathbb{C}$  is a **meromorphic modular form of weight  $k$**  if the same conditions 1 to 3 hold, but with holomorphic weakened to meromorphic.

**Note.** If  $f$  is only meromorphic at  $\infty$  then a finite number of negative powers of  $q$  can appear.

**Example.**

- The **modular discriminant**

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

is a modular form of weight 12.

- The  **$j$ -invariant**

$$j(z) = \frac{1}{q} + 744 + 196844q + 21493760q^2 + \dots$$

is a meromorphic modular form of weight 0.

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Monday  
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### 1.1.4 Lattice functions

How can we construct modular forms?

**Definition 1.1.11.** A **lattice** in  $\mathbb{C}$  is an abelian subgroup of  $\mathbb{C}$  of the form  $\mathbb{Z}w_1 + \mathbb{Z}w_2$ , where  $w_1, w_2 \in \mathbb{C}$  are  $\mathbb{R}$ -linearly independent. More generally if  $V$  is an  $\mathbb{R}$ -vector space, a **lattice**  $L$  in  $V$  is a discrete abelian subgroup of  $V$  that spans  $V$  over  $\mathbb{R}$ . For  $L \subseteq \mathbb{C}$  a lattice and  $\lambda \in \mathbb{C}^\times$ , let

$$\lambda L = \{\lambda x \mid x \in L\} \subseteq \mathbb{C}.$$

We say that  $L$  and  $\lambda L$  are **homothetic**. For  $z \in \mathbb{H}$ , let

$$L_{z,1} = \mathbb{Z} + \mathbb{Z}z = \{az + b \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}.$$

A question is when is  $L_{z,1}$  homothetic to  $L_{z',1}$ , and what is a homothety factor?

- Suppose  $L_{z,1} = \lambda L_{z',1}$ . Then there exist  $a, b, c, d$  such that  $\lambda z' = az + b$  and  $\lambda = cz + d$ , so

$$\begin{pmatrix} \lambda z' \\ \lambda \end{pmatrix} = \gamma \begin{pmatrix} z \\ 1 \end{pmatrix}. \quad (1)$$

On the other hand there exist  $a', b', c', d'$  such that  $z = a'\lambda z' + b'\lambda$  and  $1 = c'\lambda z' + d'\lambda$ , so

$$\gamma' \begin{pmatrix} \lambda z' \\ \lambda \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix}. \quad (2)$$

Then (1) and (2) imply that

$$\gamma' \gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix},$$

so  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ . Moreover (1) implies that  $z' = (az + b) / (cz + d)$ .

- Conversely, if  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , then  $\gamma z = (az + b) / (cz + d)$ , so

$$L_{\gamma z,1} = (cz + d)^{-1} L_{az+b, cz+d}.$$

But certainly  $L_{az+b, cz+d} \subseteq L_{z,1}$ . On the other hand if  $\gamma'$  is inverse to  $\gamma$ ,

$$\begin{pmatrix} z \\ 1 \end{pmatrix} = \gamma' \gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \gamma \begin{pmatrix} az + b \\ cz + d \end{pmatrix} = \begin{pmatrix} a'(az + b) + b'(cz + d) \\ c'(az + b) + d'(cz + d) \end{pmatrix},$$

so  $z \in L_{az+b, cz+d}$  and  $1 \in L_{az+b, cz+d}$ . So  $L_{az+b, cz+d} = L_{z,1}$ , so  $L_{\gamma z,1} = (cz + d)^{-1} L_{z,1}$ .

**Definition 1.1.12.** A **lattice function of weight  $k$**  is a function  $F : \{\text{lattices in } \mathbb{C}\} \rightarrow \mathbb{C}$  such that

$$F(\lambda L) = \lambda^{-k} F(L),$$

for all lattices  $L$ . Given such an  $F$ , can define

$$\begin{array}{ccc} f & : & \mathbb{H} \longrightarrow \mathbb{C} \\ z & \longmapsto & F(L_{z,1}) \end{array}.$$

If  $F$  has weight  $k$ , then

$$f(\gamma z) = F(L_{\gamma z,1}) = F((cz + d)^{-1} L_{z,1}) = (cz + d)^k F(L_{z,1}) = (cz + d)^k f(z).$$

## 1.2 Eisenstein series

### 1.2.1 Eisenstein series

**Definition 1.2.1.** For  $L \in \mathbb{C}$ , define the **Eisenstein series**

$$G_k(L) = \sum_{w \in L, w \neq 0} \frac{1}{w^k}, \quad g_k(z) = G_k(L_{z,1}) = \sum_{\substack{m=-\infty \\ (m,n) \neq (0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k}.$$

Then

$$G_k(\lambda L) = \sum_{w' \in \lambda L, w' \neq 0} \frac{1}{w'^k} = \sum_{w \in L, w \neq 0} \frac{1}{(\lambda w)^k} = \lambda^{-k} G_k(L).$$

**Corollary 1.2.2.**  $g_k$  satisfies the modular transformation law of weight  $k$ .

The following are some questions.

- Does  $G_k$ , or  $g_k$ , converge?
- Is  $g_k$  holomorphic or meromorphic on  $\mathbb{H}$ ?
- Is  $g_k$  holomorphic at  $\infty$ ?
- What is the  $q$ -expansion of  $g_k$ ?

### 1.2.2 Convergence and holomorphy on $\mathbb{H}$

**Definition 1.2.3.** Let  $U \subseteq \mathbb{C}$  be open. A sequence of functions  $f_n : U \rightarrow \mathbb{C}$  **converges uniformly on compact sets** to  $f$  if for all  $C \subseteq U$  compact and  $\epsilon > 0$ , there exists  $N \in \mathbb{Z}$  such that for all  $n > N$ ,

$$|f(z) - f_n(z)| < \epsilon, \quad z \in C.$$

**Theorem 1.2.4.** A uniform limit of holomorphic functions is holomorphic. If  $f_n$  converges to  $f$  uniformly on compact sets and  $f_n$  is holomorphic on  $U$ , then  $f$  is holomorphic on  $U$ .

**Theorem 1.2.5.** Let  $k \geq 4$ . The series  $g_k(z)$  converges absolutely and uniformly on compact subsets of  $\mathbb{H}$ .

*Proof.* Let

$$P_{z,r} = \{az + b \mid a, b \in \mathbb{R}, \max(|a|, |b|) = r\} \subseteq \mathbb{C},$$

so  $P_{z,r} = rP_{z,1}$ , and there are  $8r$  points on  $P_{z,r} \cap L_{z,1}$ . Then

$$g_k(z) = \sum_{r=1}^{\infty} \sum_{w \in L_{z,1} \cap P_{z,r}} \frac{1}{w^k}.$$

The function  $z \mapsto |z|$  attains a non-zero minimum  $\delta(z)$  on  $P_{z,1}$ , so on  $P_{z,1}$ , have  $|z| > \delta(z)$ , so  $1/|z|^k < 1/\delta(z)^k$ . On  $P_{z,r}$ , have  $|z| > r\delta(z)$ , so  $1/|z|^k < 1/r^k \delta(z)^k$ . Let  $C \subseteq \mathbb{H}$  be compact. Then  $z \mapsto \delta(z)$  is a continuous function on  $C$  and attains a minimum  $\delta_C$ . For all  $z \in C$  and all  $w \in P_{z,r}$ , get  $|w| > r\delta_C$ , so

$$\frac{1}{|w|^k} < \frac{1}{r^k \delta_C^k}.$$

Thus for  $z \in C$ ,  $g_k(z)$  is dominated by

$$\sum_{r=1}^{\infty} \frac{8r}{r^k \delta_C^k} = \frac{8}{\delta_C^k} \sum_{r=1}^{\infty} \frac{1}{r^{k-1}},$$

which converges absolutely for  $k \geq 4$ . □

**Corollary 1.2.6.**  $g_k(z)$  is holomorphic on  $\mathbb{H}$ .

Lecture 4  
Friday  
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### 1.2.3 $q$ -expansion and holomorphy at $\infty$

The idea is to understand series of the form

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k}.$$

**Theorem 1.2.7.** *A bounded holomorphic function on all of  $\mathbb{C}$  is constant.*

**Lemma 1.2.8.**

1.

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

2.

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

*Proof.*

1. The right hand side converges absolutely and uniformly on compact subsets of  $\mathbb{C} \setminus \mathbb{Z}$ , so the right hand side is holomorphic on  $\mathbb{C} \setminus \mathbb{Z}$ . Locally around  $z = n$ , the series looks like

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \cdots + \frac{1}{(z-n+1)^2} + \frac{1}{(z-n)^2} + \frac{1}{(z-n-1)^2} + \cdots = \frac{1}{(z-n)^2} + h_1(z),$$

where  $h_1(z)$  is holomorphic in a neighbourhood of  $z = n$ . Similarly, the left hand side is meromorphic on  $\mathbb{C}$ , and the Laurent series near  $z = n$  is

$$\frac{\pi^2}{\sin^2 \pi z} = \pi \left( \frac{1}{\pi^2 (z-n)^2} + \frac{1}{3} + \frac{1}{15} \pi^2 (z-n)^2 + \cdots \right) = \frac{1}{(z-n)^2} + h_2(z),$$

where  $h_2(z)$  is a holomorphic function. So the difference

$$g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} - \frac{\pi^2}{\sin^2 \pi z}$$

is meromorphic on  $\mathbb{C}$  and holomorphic on  $\mathbb{C} \setminus \mathbb{Z}$ , and the Laurent expression around  $z = n$  is

$$g(z) = \frac{1}{(z-n)^2} + h_1(z) - \left( \frac{1}{(z-n)^2} + h_2(z) \right) = h_1(z) - h_2(z),$$

so  $g(z)$  is holomorphic at  $z = n$  for all  $n$ . Consider  $t \rightarrow \pm\infty$  for  $z = a + it$ . The right hand side is

$$R = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=a-N}^{a+N} \frac{1}{(z-n)^2} + \sum_{n=-\infty}^{a-N-1} \frac{1}{(z-n)^2} + \sum_{n=a+N+1}^{\infty} \frac{1}{(z-n)^2} = R_0 + R_- + R_+,$$

where  $R_0$  has finitely many terms that converge to less than  $\epsilon/2$  as  $t \rightarrow \pm\infty$  and  $R_- + R_+ < \epsilon/2$  for  $N \gg 0$  independent of  $t$ , so  $R < \epsilon$  converges to zero. Similarly, the left hand side is

$$\left| \frac{\pi^2}{\sin^2 \pi z} \right| = \left| \frac{2\pi^2}{e^{\pi i z} - e^{-\pi i z}} \right| \rightarrow 0,$$

so  $\lim_{t \rightarrow \infty} g(a + it) = 0$ . Moreover,  $g(z+1) = g(z)$  for all  $z$ . Then

$$S = \{z \in \mathbb{C} \mid n-1 \leq \operatorname{Re} z \leq n, -N \leq \operatorname{Im} z \leq N\}, \quad n \in \mathbb{Z}$$

is compact, so  $|g(z)|$  attains a maximum in  $S$ , so  $g(z)$  is bounded in  $S$ . Since  $g(z)$  is also bounded in  $R_- + R_+$ ,  $g(z)$  is bounded in  $\mathbb{C}$ , so  $g$  is constant. Since  $\lim_{t \rightarrow \infty} g(a + it) = 0$ ,  $g = 0$ .

2. Check that the right hand side converges absolutely and uniformly on compact subsets of  $\mathbb{C} \setminus \mathbb{Z}$ , so the right hand side is meromorphic on  $\mathbb{C} \setminus \mathbb{Z}$ . Similarly, the left hand side is also meromorphic on  $\mathbb{C} \setminus \mathbb{Z}$ . Comparing derivatives,

$$-\frac{\pi^2}{\sin^2 \pi z} = -\frac{1}{z^2} - \sum_{n=1}^{\infty} \left( \frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} \right),$$

so the difference is constant. Let  $z = \frac{1}{2}$ . The left hand side is  $\pi \cot \pi/2 = 0$  and the right hand side is

$$\frac{2}{1} + \left( -\frac{2}{1} + \frac{2}{3} \right) + \left( -\frac{2}{3} + \frac{2}{5} \right) + \cdots \rightarrow 0, \quad n \rightarrow \infty,$$

so the difference is zero. □

Thus

$$\frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z-n} + \frac{1}{z+n} \right) = \pi \cot \pi z = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = \pi i \frac{q+1}{q-1} = \pi i - \frac{2\pi i}{1-q} = \pi i - 2\pi i \sum_{n=0}^{\infty} q^n.$$

Take  $\frac{d^{k-1}}{dz^{k-1}}$ . For  $k \geq 2$  even, get

$$-(k-1)! \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = -(2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} q^n,$$

so

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n.$$

Collecting powers of  $q$ ,

$$\begin{aligned} g_k(z) &= \sum_{\substack{m=-\infty \\ (m,n) \neq (0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{n^k} + 2 \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{k-1} q^{nm} \\ &= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \end{aligned} \quad \begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} n^{-s} \\ \sigma_{k-1}(n) &= \sum_{d|n, d>0} d^{k-1}. \end{aligned}$$

**Corollary 1.2.9.**  $g_k(z)$  is holomorphic at  $\infty$ . In particular,  $g_k$  is a modular form of weight  $k$ .

### 1.2.4 Bernoulli numbers

**Definition 1.2.10.** The **Bernoulli numbers**  $b_k$  are defined by

$$\sum_{k=0}^{\infty} b_k \frac{x^k}{k!} = \frac{x}{e^x - 1},$$

a formal power series with rational coefficients.

Then

$$b_0 = 1, \quad b_1 = -\frac{1}{2}, \quad b_2 = \frac{1}{6}, \quad b_3 = 0, \quad b_4 = -\frac{1}{20}, \quad \dots, \quad b_{2k} \in \mathbb{Q}, \quad b_{2k+1} = 0, \quad \dots$$

**Proposition 1.2.11.** *For all even  $k$ ,*

$$\zeta(k) = -b_k \frac{(2\pi i)^k}{2k!}.$$

*Proof.* On one hand,

$$\pi z \cot \pi z = \pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1} = \pi i z + \sum_{k=0}^{\infty} b_k \frac{(2\pi i z)^k}{k!}.$$

On the other hand,

$$\begin{aligned} \pi \cot \pi z &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \frac{1}{z} - \frac{2z}{n^2} \sum_{n=1}^{\infty} \frac{1}{1 - z^2/n^2} \\ &= \frac{1}{z} - \sum_{n=1}^{\infty} \frac{2}{z} \sum_{k=1}^{\infty} \left(\frac{z^2}{n^2}\right)^k = \frac{1}{z} - \frac{2}{z} \sum_{k=1}^{\infty} z^{2k} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{1}{z} - \frac{2}{z} \sum_{k=1}^{\infty} \zeta(2k) z^{2k}, \end{aligned}$$

so

$$\pi i z + \sum_{k=0}^{\infty} b_k \frac{(2\pi i z)^k}{k!} = \pi z \cot \pi z = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k}.$$

Comparing,

$$b_{2k} \frac{(2\pi i)^{2k}}{(2k)!} = -2\zeta(2k),$$

get the desired formula. □

So

$$g_k(z) = \frac{-b_k (2\pi i)^k}{k!} + \frac{2 (2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Set the **normalised Eisenstein series**

$$E_k = \frac{g_k}{2\zeta(k)} = 1 - \frac{2k}{b_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

**Example.**

$$\begin{aligned} E_4 &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, & E_6 &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \\ E_8 &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, & E_{12} &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n. \end{aligned}$$

An observation is if  $f$  is modular of weight  $k$  and  $g$  is modular of weight  $k'$ , then  $fg$  is modular of weight  $k + k'$ , and if  $k = k'$ , then  $f + g$  is modular of weight  $k$ .

**Example.** Important examples.

- The **modular discriminant**

$$\Delta(z) = \frac{E_4 - E_6^2}{1728} = q - 24q^2 + 252q^3 + \dots$$

is a modular form of weight 12.

- The **j-invariant**

$$j(z) = \frac{E_4^3}{\Delta} = \frac{1}{q} + 744 + 196844q + \dots$$

is a meromorphic modular form of weight 0.

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### 1.3 Controlling modular forms

#### 1.3.1 The fundamental domain

The idea is to control the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathbb{H}$ . If  $f : \mathbb{H} \rightarrow \mathbb{C}$  satisfies  $f(\gamma z) = (cz + d)^k f(z)$  for all  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , and if  $D \subseteq \mathbb{H}$  such that  $D$  meets every  $\mathrm{SL}_2(\mathbb{Z})$ -orbit in  $\mathbb{H}$ , then  $f$  is determined by its values on  $D$ .

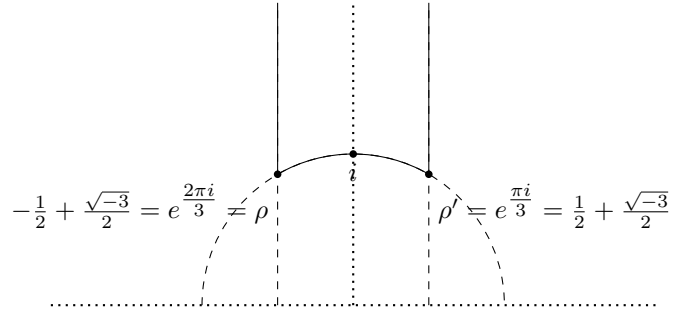
**Definition 1.3.1.** Let  $G$  be a group acting continuously on a complex analytic space  $X$ , such as  $X = \mathbb{H}$ . A subset  $D \subseteq X$  is a **fundamental domain** for the action of  $G$  if

- $D$  meets every  $G$ -orbit in  $X$ ,
- the subset  $\{x \in D \mid \exists g \in G, gx \in D, gx \neq x\}$  has measure zero, and
- $D$  is closed in  $X$ .

Define

$$\mathcal{D} = \{z \in \mathbb{H} \mid \tfrac{1}{2} \leq \operatorname{Re} z \leq \tfrac{1}{2}, |z| \geq 1\} \subseteq \mathbb{H},$$

so



Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : z \mapsto -\frac{1}{z}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : z \mapsto z + 1,$$

and let  $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$  be the subgroup generated by  $S$  and  $T$ . We will see later that  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ .

**Theorem 1.3.2.**

1. For all  $z \in \mathbb{H}$ , there exists  $\gamma \in \Gamma$  such that  $\gamma z \in \mathcal{D}$ .
2. Suppose  $z, z' \in \mathcal{D}$  and  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  with  $\gamma z = z'$ . Then either
  - $z = z'$ ,
  - $\operatorname{Re} z = \pm \frac{1}{2}$  and  $z' = z \mp 1$ , or
  - $|z| = 1$  and  $z' = -1/z$ .

In particular, if  $z \neq z'$ , then  $z$  and  $z'$  are on the boundary of  $\mathcal{D}$ .

3. For  $z \in \mathcal{D}$ , let  $I_z$  be the stabiliser of  $z$  in  $\mathrm{SL}_2(\mathbb{Z})$ , that is

$$I_z = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma z = z\}.$$

Then  $I_z = \{\pm I\}$  unless

- $z = i$ , where  $I_z = \{\pm I, \pm S\}$ ,
- $z = \rho$ , where  $I_z = \{\pm I, \pm (ST), \pm (T^{-1}S)\}$ , or
- $z = \rho'$ , where  $I_z = \{\pm I, \pm (TS), \pm (ST^{-1})\}$ .

**Corollary 1.3.3.**  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ .

*Proof.* Fix  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  and  $z \in \mathring{\mathcal{D}}$  so  $\mathrm{SL}_2(\mathbb{Z})z \cap \mathcal{D} = \{z\}$  and  $I_z = \{\pm I\}$ . Consider  $\gamma z$ . There exists  $\gamma' \in \Gamma$  such that  $\gamma'\gamma z \in \mathcal{D}$ , so  $\gamma'\gamma z = z$ . So  $\gamma'\gamma = \pm I$ , so  $\gamma = \pm \gamma'^{-1}$ . But  $\gamma'^{-1} \in \Gamma$  and  $-I = S^2 \in \Gamma$ , so  $\gamma \in \Gamma$ .  $\square$

*Proof of Theorem 1.3.2.* Recall  $\operatorname{Im} \gamma z = \operatorname{Im} z / |cz + d|^2$  for  $\gamma \in \operatorname{SL}_2(\mathbb{Z})$ .

1. As  $c$  and  $d$  vary,  $\{cz + d\}$  forms a lattice in  $\mathbb{C}$ , so there exist only finitely many  $c$  and  $d$  such that  $|cz + d| < 1$ . So  $\operatorname{Im} \gamma z$  attains a maximum as  $\gamma$  varies over  $\Gamma$ , so there exists  $\gamma \in \Gamma$  such that  $\operatorname{Im} \gamma z$  is maximal. There exists  $n \in \mathbb{Z}$  such that  $T^n \gamma z$  has real part between  $-\frac{1}{2}$  and  $\frac{1}{2}$ . Consider  $|T^n \gamma z|$ . If this is less than one, then

$$\operatorname{Im} ST^n \gamma z = \operatorname{Im} \frac{-1}{T^n \gamma z} > \operatorname{Im} T^n \gamma z = \operatorname{Im} \gamma z.$$

Since  $ST^n \gamma \in \Gamma$ , this contradicts maximality so  $|T^n \gamma z| \geq 1$ , so  $T^n \gamma z \in \mathcal{D}$ .

- 2, 3. Let  $z, z' \in \mathcal{D}$  such that  $\gamma z = z'$ . Without loss of generality  $\operatorname{Im} z' \geq \operatorname{Im} z$ , so  $|cz + d| \leq 1$ . Note that  $|cz + d| \geq \operatorname{Im}(cz + d) \geq \frac{\sqrt{3}}{2}c$ , so  $c = -1, 0, 1$ . Note that can replace  $\gamma$  with  $-\gamma$  if convenient.

$c = 0$ . Then  $ad = 1$ , so can assume  $a = d = 1$ , so  $\gamma z = z + b$ . Since  $z, z + b \in \mathcal{D}$ ,  $b = \pm 1$  and  $\operatorname{Re} z = \mp \frac{1}{2}$ .

$c = 1$ . Have  $|z + d| \leq 1$  and  $|z| \geq 1$ , so  $d = -1, 0, 1$ .

$d = 0$ . Then  $|z| = 1$ , and  $\gamma z = (az - 1)/z = a - 1/z$ . The only possibilities are

- \*  $a = 0$  and  $\gamma = S$ ,
- \*  $a = 1$  and  $\gamma = TS$ , so  $z = \rho'$ , or
- \*  $a = -1$  and  $\gamma = T^{-1}S$ , so  $z = \rho$ .

$d = 1$ . Then  $z = \rho$ , and  $\gamma z = ((b + 1)z + b)/(z + 1) = b + 1 - 1/(z + 1)$ , so  $b = 0$  or  $b = -1$ .

$d = -1$ . Then  $z = \rho'$  is similar.

$c = -1$ . Similar.

□

### 1.3.2 Further review of complex analysis

Recall that on any compact set, a meromorphic function has only finitely many zeroes and poles. If  $f(z) = g(e^{2\pi iz})$  is meromorphic at  $\infty$  and  $g$  is meromorphic on  $D = \{|q| < 1\}$ , zeroes and poles of  $g$  are discrete with respect to  $q$ , and  $\operatorname{Im} z \gg 0$  if and only if  $|q| < \epsilon$ .

**Definition 1.3.4.** Let  $U \subseteq \mathbb{C}$  be open, and let  $f : U \rightarrow \mathbb{C}$  be meromorphic on  $U$ . If  $f$  has a pole at  $p$ , can write

$$f(z) = \sum_{n=\operatorname{ord}_p f < 0}^{\infty} a_n (z - p)^n.$$

The coefficient  $a_{-1}$  is called the **residue**  $\operatorname{Res}_p f$  of  $f$  at  $p$ .

**Theorem 1.3.5** (Residue theorem). *Let  $V$  be a region in  $\mathbb{C}$  whose boundary  $\partial V$  is a simple closed curve. Then*

$$\frac{1}{2\pi} \int_{\partial V} f(z) dz = \sum_{p \in V \text{ pole of } f} \operatorname{Res}_p f.$$

**Definition 1.3.6.** Let  $f$  be meromorphic on  $U \subseteq \mathbb{C}$  open. Then the **logarithmic derivative**  $d \log f$  is the function  $f'/f$ .

If  $f(z) = c_n (z - p)^n + c_{n+1} (z - p)^{n+1} + \dots$ , then if  $n \neq 0$ , then the leading term of  $f'$  is  $nc_n (z - p)^{n-1}$  and the leading term of  $f$  is  $c_n (z - p)^n$ , so the leading term of  $f'/f$  is  $n(z - p)^{-1}$ . If  $n = 0$ , then  $f'/f$  is holomorphic. So  $f'/f$  is meromorphic with simple poles precisely at the points where  $\operatorname{ord}_p f \neq 0$ , and  $\operatorname{Res}_p f'/f$  at such  $p$  is  $\operatorname{ord}_p f$ .

**Theorem 1.3.7** (Argument principle).

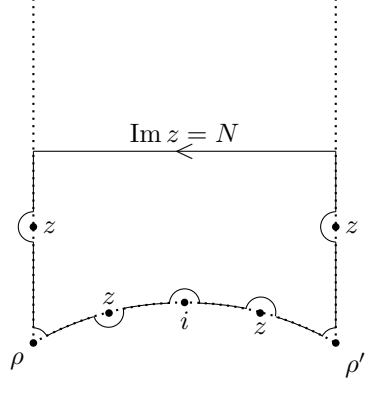
$$\frac{1}{2\pi i} \int_{\partial V} \frac{f'(z)}{f(z)} dz = \sum_{p \in V} \operatorname{ord}_p f.$$

### 1.3.3 Controlling modular forms

**Theorem 1.3.8** ( $k/12$ -formula). *Let  $f$  be a non-zero meromorphic modular form of weight  $k$ . Then*

$$\text{ord}_\infty f + \frac{\text{ord}_\rho f}{3} + \frac{\text{ord}_i f}{2} + \sum_{p \in \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}, p \sim \{i, \rho\}} \text{ord}_p f = \frac{k}{12}.$$

*Proof.* Consider the closed curve  $C_{N, \epsilon}$ ,



where the  $z$ 's are zeroes or poles of  $f$ , and the circles are of radius  $\epsilon$ . Consider

$$\frac{1}{2\pi i} \int_{C_{N, \epsilon}} \frac{f'(z)}{f(z)} dz = \sum_{p \in \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}, p \sim \{i, \rho\}} \text{ord}_p f, \quad \epsilon \rightarrow 0.$$

So it suffices to show

$$\lim_{\epsilon \rightarrow 0, N \rightarrow \infty} \frac{1}{2\pi i} \int_{C_{N, \epsilon}} \frac{f'(z)}{f(z)} dz = -\text{ord}_\infty f - \frac{\text{ord}_\rho f}{3} - \frac{\text{ord}_i f}{2} + \frac{k}{12}.$$

The vertical parts of the boundary cancel. The integral over the circular part of  $\partial \mathcal{D}$  approaches

$$\frac{1}{2\pi i} \int_\rho^i \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i} \int_i^{\rho'} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \left( \int_\rho^i \frac{f'(z)}{f(z)} dz - \int_\rho^i \frac{f'(-1/z)}{f(-1/z)} dz \right)$$

Since  $f(-1/z) = z^k f(z)$ ,

$$d(z^k f(z)) = (kz^{k-1} f(z) + z^k f'(z)) dz,$$

so

$$\frac{1}{2\pi i} \int_\rho^i \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i} \int_i^{\rho'} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_\rho^i \frac{f'(z)}{f(z)} dz - \frac{kz^{k-1} f(z) + z^k f'(z)}{z^k f(z)} dz = -\frac{1}{2\pi i} \int_\rho^i \frac{k}{z} dz = \frac{k}{12}.$$

Since  $dq = 2\pi i q dz$ , the top part is

$$\frac{1}{2\pi i} \int_{\frac{1}{2} - iN}^{\frac{1}{2} - iN} \frac{f'(z)}{f(z)} dz = -\frac{1}{2\pi i} \int_{\text{circle of radius } \epsilon} \frac{g'(q)}{g(q)} dq = -\text{ord}_\infty f.$$

Near  $i$ ,  $f'/f = \text{ord}_i f (z - i)^{-1} + h(z)$ , where  $h(z)$  is holomorphic and  $h(z) \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Then the circle  $C_{\epsilon, i}$  of radius  $\epsilon$  centered at  $i$  is

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{C_{\epsilon, i}} \frac{f'(z)}{f(z)} dz = \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\text{arc of half circle centered at } i} \frac{\text{ord}_i f}{z - i} dz = -\frac{\text{ord}_i f}{2}.$$

Similarly, at  $\rho$  and  $\rho'$ , get that the circles  $C_{\epsilon, \rho}$  and  $C_{\epsilon, \rho'}$  of radius  $\epsilon$  centered at  $\rho$  and  $\rho'$  are

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{C_{\epsilon, \rho}} \frac{f'(z)}{f(z)} dz = \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{C_{\epsilon, \rho'}} \frac{f'(z)}{f(z)} dz = -\frac{\text{ord}_\rho f}{6},$$

which gives  $-\text{ord}_\rho f/3$ . □

### 1.3.4 The space of holomorphic modular forms

Let

$$M_k = \{\text{holomorphic modular forms of weight } k\},$$

and let

$$S_k = \{\text{cusp forms of weight } k\} = \{f \in M_k \mid \text{ord}_\infty f > 0\} \subseteq M_k.$$

**Corollary 1.3.9.**

- $M_k = 0$  if  $k < 0$ ,  $k = 2$ , or  $k$  odd.
- $M_0$  are constants.
- $M_4 = \mathbb{C}E_4$ , where  $\text{ord}_p E_4 = 1$  and no other zeroes.
- $M_6 = \mathbb{C}E_6$ , where  $\text{ord}_i E_6 = 1$  and no other zeroes.
- $M_8 = \mathbb{C}E_8$ , where  $\text{ord}_p E_8 = 2$  and no other zeroes.
- $M_{10} = \mathbb{C}E_{10}$ , where  $\text{ord}_p E_{10} = \text{ord}_i E_{10} = 1$  and no other zeroes.
- $M_{12} = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta$ , where  $\text{ord}_\infty \Delta = 1$  and no other zeroes.

**Corollary 1.3.10.**  $\Delta : M_k \rightarrow S_{k+12}$  is an isomorphism. On the other hand,

$$M_k \cong \mathbb{C}E_k \oplus S_k, \quad k \geq 4 \text{ even},$$

so

$$M_k \cong \mathbb{C}E_k \oplus \cdots \oplus \mathbb{C}E_{k-12r}\Delta^r, \quad k - 12r \in \{0, 4, 6, 8, 10, 14\}.$$

So for  $k \geq 4$ , the set

$$\begin{cases} E_k, \dots, E_{k-12\lfloor k/12 \rfloor} \Delta^{\lfloor k/12 \rfloor} & k \not\equiv 2 \pmod{12} \\ E_k, \dots, E_{14} \Delta^{\lfloor k/12 \rfloor - 1} & k \equiv 2 \pmod{12} \end{cases}$$

is a basis for  $M_k$ .

**Corollary 1.3.11.**  $E_4^2 = E_8$  and  $E_4 E_6 = E_{10}$ .

A variant is to write  $k = 4n + 6m$  with  $m = 0, 1$  and  $n \geq 0$ , for  $k \geq 4$ . Then  $M_k = \mathbb{C}E_4^n E_6^m \oplus S_k$  gives a basis

$$E_4^n E_6^m, \dots, E_4^{n-3\lfloor n/3 \rfloor} E_6^m \Delta^{\lfloor n/3 \rfloor}$$

for  $M_k$ . Since  $\Delta = (E_4^3 - E_6^2)/1728$ , we see every modular form of weight  $k$  is a polynomial in  $E_4$  and  $E_6$ , and

$$\Delta \in q + q^2 \mathbb{Z}[[q]], \quad E_4^n E_6^m \in 1 + q\mathbb{Z}[[q]], \quad E_4^{n-3\lfloor n/3 \rfloor} E_6^m \Delta \in q + q^2 \mathbb{Z}[[q]], \quad \dots$$

have integer coefficients. The upshot is if the  $q$ -expansion of  $f$  has integer coefficients, then  $f$  is an integer combination of

$$E_4^n E_6^m, \dots, E_4^{n-3\lfloor n/3 \rfloor} E_6^m \Delta^{\lfloor n/3 \rfloor}.$$

**Notation.**  $M_k(\mathbb{Z}) \subseteq M_k$  consists of modular forms with integer  $q$ -expansions.

**Theorem 1.3.12.**  $M_k(\mathbb{Z})$  spans  $M_k$ , and  $f \in M_k$  lies in  $M_k(\mathbb{Z})$  if and only if  $f$  is an integral polynomial in  $E_4, E_6, \Delta$ .

**Definition 1.3.13.** A **graded ring** is a ring  $R$ , together with a direct sum decomposition, as abelian groups,

$$R = \bigoplus_{i \in \mathbb{Z}} R_i,$$

such that  $R_i \cdot R_j \subseteq R_{i+j}$  for all  $i, j \in \mathbb{Z}$ .

**Example.**

- $R = \mathbb{C}[X, Y]$ , where  $R_i$  are polynomials homogeneous of degree  $i$ .
- $R = \bigoplus_{k \in \mathbb{Z}} M_k$ .

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Let  $\mathbb{C}[X, Y]$  be graded with  $\deg X = 4$  and  $\deg Y = 6$ . Have a homomorphism of graded rings

$$\begin{aligned} \mathbb{C}[X, Y] &\longrightarrow \bigoplus_{k \in \mathbb{Z}} M_k \\ (X, Y) &\longmapsto (E_4, E_6) \end{aligned}.$$

**Theorem 1.3.14.** *This is an isomorphism of graded rings.*

*Proof.* This map is surjective, since every  $f \in M_k$  is a polynomial in  $E_4$  and  $E_6$ . It remains to show this map is injective. Suppose not. There exists  $P(X, Y)$ , homogeneous of degree  $k$ , such that  $P(E_4, E_6) = 0$ . Write  $k = 4n + 6m$  with  $m = 0, 1$ . If  $P = c_0 X^n Y^m + \cdots + c_r X^{n-3r} Y^{m+2r}$  where  $r = \lfloor n/3 \rfloor$ , then

$$c_0 E_4^n E_6^m + \cdots + c_r E_4^{n-3r} E_6^{m+2r} = 0.$$

Dividing by  $E_4^{n-3r} E_6^{m+2r}$ , get  $Q(E_4^3/E_6^2) = 0$  where  $Q(X) = c_0 X^r + \cdots + c_r$ . Since the roots of  $Q$  are discrete, and  $E_4^3/E_6^2$  is non-constant, this is impossible.  $\square$

### 1.3.5 The space of meromorphic modular forms

**Note.** The meromorphic modular forms of weight zero form a field. For example,  $j(z) = E_4^3/\Delta = 1728E_4^3/(E_4^3 - E_6^2)$  is a non-constant meromorphic modular form, with a pole of order one at  $\infty$ , a zero of order three at  $\rho$ , and no other zeroes or poles.

**Theorem 1.3.15.**  *$j$  gives a bijection between  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$  and  $\mathbb{C}$ .*

*Proof.* Given  $\lambda \in \mathbb{C}$ , want  $z \in \mathbb{H}$  such that  $j(z) = \lambda$ . Consider  $g = j - \lambda$ . This is meromorphic of weight zero. There is a pole at  $\infty$ , and no other poles, and

$$\mathrm{ord}_\infty g + \frac{\mathrm{ord}_\rho g}{3} + \frac{\mathrm{ord}_i g}{2} + \sum_{p \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}, p \neq \{i, \rho\}} \mathrm{ord}_p g = 0.$$

The only possibilities are

- $g$  has a zero at  $\rho$  of order three, and no other zeroes,
- $g$  has a zero at  $i$  of order two, and no other zeroes, or
- $g$  has a simple zero somewhere else, and no others.

In each case, the zero of  $g$  is a unique  $\mathrm{SL}_2(\mathbb{Z})$ -orbit on which  $j(z) = \lambda$ . So  $j$  is bijective.  $\square$

**Theorem 1.3.16.** *Every meromorphic modular form of weight zero is a rational function in  $j$ . That is, the field of meromorphic modular forms is  $\mathbb{C}(j)$ .*

*Proof.* Let  $g$  be meromorphic of weight zero. Then  $g$  has finitely many  $\mathrm{SL}_2(\mathbb{Z})$ -orbits worth of poles in  $\mathbb{H}$ . Saw last time that  $j$  is holomorphic in  $\mathbb{H}$ . If  $p$  is a pole of  $g$ , then  $(j(z) - j(p))^{n_p}$  is holomorphic on  $\mathbb{H}$  and zero at  $z = p$ . Doing this for all poles, there exists  $P \in \mathbb{C}[X]$  such that  $P(j)g(z)$  is holomorphic on  $\mathbb{H}$ . Then for some  $m$ ,  $P(j)g(z)\Delta^m$  is holomorphic of weight  $12m$ . So it suffices to show if  $h$  is holomorphic of weight  $12m$ , then  $h/\Delta^m$  is a rational function in  $j$ , since if  $P(j)g(z)\Delta^m = h$  then  $P(j)g(z) \in \mathbb{C}(j)$ , so  $g(z) \in \mathbb{C}(j)$ . Then  $h$  is a sum of terms

$$h = \sum_{a,b} c_{a,b} E_4^a E_6^b, \quad c_{a,b} \in \mathbb{C}, \quad 4a + 6b = 12m.$$

Considering this equation modulo four and modulo three, find  $3 \mid a$  and  $2 \mid b$ , so

$$\frac{h}{\Delta^m} = \sum_{a,b} c_{a,b} \left( \frac{E_4^3}{\Delta} \right)^{\frac{a}{3}} \left( \frac{E_6^2}{\Delta} \right)^{\frac{b}{2}}.$$

So it suffices to show  $E_4^3/\Delta$  and  $E_6^2/\Delta$  are rational functions in  $j$ . Then  $j = E_4^3/\Delta$ , and

$$\frac{E_6^2}{\Delta} = \frac{1728E_6^2}{E_4^3 - E_6^2} = \frac{1728(E_6^2 - E_4^3) + 1728E_4^3}{E_4^3 - E_6^2} = -1728 + \frac{1728E_4^3}{E_4^3 - E_6^2} = j - 1728.$$

$\square$

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## 1.4 Theta series

Let  $L \subseteq \mathbb{R}^n$  be a lattice. For  $x, y \in L$ ,  $x \cdot y \in \mathbb{R}$ . Suppose  $x \cdot y \in \mathbb{Z}$  for all  $x, y \in L$ . A question is for  $n \in \mathbb{Z}$ , how many  $x \in L$  have  $x \cdot x = n$ ? The rough idea is to form the series

$$\sum_{x \in L} q^{x \cdot x} = \sum_{n=0}^{\infty} a_n q^n, \quad a_n = \# \{x \in L \mid x \cdot x = n\}.$$

We will show, with some slight modifications, and extra hypotheses on  $L$ , this generating function turns out to be a modular form.

### 1.4.1 Quadratic forms

Fix a lattice  $L \subseteq \mathbb{R}^n$ , so

$$L = \mathbb{Z} \cdot e_1 \oplus \cdots \oplus \mathbb{Z} \cdot e_n.$$

Given these  $e_i$ , form a matrix  $A$  such that  $A_{ij} = e_i \cdot e_j$ .

**Note.**  $A = B^T B$ , where  $B$  is the matrix whose columns are the  $e_i$ , and  $|\det B|$  is the volume of the parallelogram spanned by  $e_i$ , so  $\det A = (\det B)^2 > 0$ .

**Definition 1.4.1.** The **dual lattice**  $L^\vee$  is the set of  $y \in \mathbb{R}^n$  such that  $y \cdot x \in \mathbb{Z}$  for all  $x \in L$ .

Let  $f_1, \dots, f_n$  be the dual basis to  $e_1, \dots, e_n$ , that is the unique set of solutions  $f_1, \dots, f_n$  such that

$$f_i \cdot e_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then  $L^\vee$  is spanned by the  $f_i$ . Clearly  $f_i \in L^\vee$  for all  $i$ . Conversely, if  $y \in L^\vee$ , then  $y \cdot e_i = a_i \in \mathbb{Z}$ , then  $y = \sum_{i=1}^n a_i f_i$ .

**Proposition 1.4.2.** Let  $C = A^{-1}$ . Then

$$f_i = \sum_{j=1}^n C_{ij} e_j.$$

*Proof.*

$$f_i \cdot e_k = \sum_{j=1}^n C_{ij} e_j \cdot e_k = \sum_{j=1}^n C_{ij} A_{jk} = (CA)_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}.$$

□

**Definition 1.4.3.** A lattice  $L$  is **self-dual** if  $L^\vee = L$  as subsets of  $\mathbb{R}^n$ .

**Proposition 1.4.4.**  $L$  is self-dual if and only if the associated matrix  $A$  has integer entries and determinant 1.

*Proof.* Clearly if  $L = L^\vee$ , then  $e_i \cdot e_j \in \mathbb{Z}$ , so  $A$  has integer entries. Since  $L^\vee \subseteq L$ ,  $f_i$  is an integer combination of the  $e_j$ , so  $C = A^{-1}$  has integer entries. So  $\det A = \pm 1$ , but already saw  $\det A > 0$ . Conversely if  $A$  has integer entries and determinant one,  $C = A^{-1}$  has integer entries. Then  $A$  has integer entries implies that  $e_i \cdot e_j \in \mathbb{Z}$  for all  $i$  and  $j$ , so  $e_i \in L^\vee$  for all  $i$ , so  $L \subseteq L^\vee$ . Similarly,  $C$  has integer entries implies that  $L^\vee \subseteq L$ . □

If  $L$  is self-dual, get an integer-valued **quadratic form**

$$\begin{aligned} Q_L : \quad \mathbb{Z}^n &\longrightarrow \mathbb{Z} \\ (a_1, \dots, a_n) &\longmapsto (a_1 e_1 + \cdots + a_n e_n) \cdot (a_1 e_1 + \cdots + a_n e_n) = (a_1 \ \dots \ a_n) A \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}. \end{aligned}$$

A question is given  $m$ , how often does  $Q_L$  represent  $m$ ?

### 1.4.2 Fourier analysis

Let  $f$  be a  $C^\infty$  function on  $\mathbb{R}^n \rightarrow \mathbb{C}$ .

**Definition 1.4.5.** We will say  $f$  is **rapidly decreasing** if for all  $m$ ,

$$\|x\|^m \cdot |f(x)| \rightarrow 0, \quad |x| \rightarrow \infty,$$

where  $|x| = (x \cdot x)^{1/2}$ . For  $f \in C^\infty$ , rapidly decreasing, define

$$\widehat{f}(y) = \int_{\mathbb{R}^n} e^{-2\pi i(x \cdot y)} dx : \mathbb{R}^n \rightarrow \mathbb{C}.$$

**Fact.** If  $f$  is smooth and rapidly decreasing, so is  $\widehat{f}$ .

**Fact.** If  $f(x) = e^{-\pi(x \cdot x)}$ , then  $\widehat{f}(x) = f(x)$ .

**Fact.** If  $f$  is smooth and rapidly decreasing, and  $\mathbb{R}^n$  is a lattice with volume  $V$ , then

$$\sum_{x \in L} f(x) = \frac{1}{V} \sum_{x \in L^\vee} \widehat{f}(x).$$

### 1.4.3 Theta series

A crucial assumption is that  $L$  is self-dual. An assumption that can be removed is that  $L$  is even, so for all  $x \in L$ ,  $Q_L(x) \in 2\mathbb{Z}$ .

**Definition 1.4.6.** The **theta series**  $\Theta_L$  is defined by

$$\Theta_L(z) = \sum_{x \in L} q^{\frac{1}{2}x \cdot x} = \sum_{m=0}^{\infty} a_m q^m, \quad a_m = \# \{x \in \mathbb{Z}^n \mid Q_L(x) = 2m\}.$$

**Theorem 1.4.7.**  $\Theta_L$  is modular of weight  $n/2$ .

**Example.** Let  $\Gamma_8 \subseteq \mathbb{R}^8$  be spanned by

$$\begin{aligned} e_1 &= \left( \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right), & e_2 &= (1, 1, 0, 0, 0, 0, 0, 0), \\ e_3 &= (1, -1, 0, 0, 0, 0, 0, 0), & e_4 &= (0, 1, -1, 0, 0, 0, 0, 0), & e_5 &= (0, 0, 1, -1, 0, 0, 0, 0), \\ e_6 &= (0, 0, 0, 1, -1, 0, 0, 0), & e_7 &= (0, 0, 0, 0, 1, -1, 0, 0), & e_8 &= (0, 0, 0, 0, 0, 1, -1, 0). \end{aligned}$$

Then

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

and

$$Q_L(z_1, \dots, z_8) = 2(z_1^2 + \dots + z_8^2 - z_1 z_3 - z_2 z_4 - z_3 z_4 - z_4 z_5 - z_6 z_7 - z_7 z_8).$$

If  $L \subseteq \mathbb{R}^n$  is even and self-dual, and  $\Theta_L$  is modular of weight  $n/2$ , then dimension is  $\sim 24$ .

**Fact.**  $L \subseteq \mathbb{R}^n$  even and self-dual implies that  $8 \mid n$ .

*Proof.* Serre V.2.1 Corollary 2. □

*Proof of Theorem 1.4.7.* Know, since  $L$  is even, that  $\Theta_L(z+1) = \Theta_L(z)$ . It suffices to show  $\Theta_L(-1/z) = z^{n/2}\Theta_L(z)$ . Both sides are holomorphic on  $\mathbb{H}$ , so it suffices to show

$$\Theta_L\left(-\frac{1}{it}\right) = (it)^{\frac{n}{2}} \Theta_L(it).$$

For  $t \in \mathbb{R}^\times$ , let  $L_t = t^{1/2} \cdot L$  and  $L_t^\vee = t^{-1/2} \cdot L = L_{t^{-1}}$ , so  $\text{vol } L_t = t^{n/2}$ . By the facts,

$$\sum_{x \in L_t} e^{-\pi(x \cdot x)} = t^{-\frac{n}{2}} \sum_{x \in L_{t^{-1}}} e^{-\pi(x \cdot x)},$$

so

$$\sum_{x \in L} e^{-\pi(x \cdot x)t} = t^{-\frac{n}{2}} \sum_{x \in L} e^{-\frac{\pi(x \cdot x)}{t}}.$$

Now return to  $\Theta_L$ . The left hand side is

$$\Theta_L\left(-\frac{1}{it}\right) = \sum_{x \in L} e^{\frac{1}{2} \cdot 2\pi i \cdot \left(-\frac{1}{it}\right) \cdot (x \cdot x)} = \sum_{x \in L} e^{-\frac{\pi(x \cdot x)}{t}},$$

and the right hand side is

$$\Theta_L(it) = \sum_{x \in L} e^{\frac{1}{2} \cdot 2\pi i \cdot (it) \cdot (x \cdot x)} = \sum_{x \in L} e^{\pi(x \cdot x)t},$$

so the result follows.  $\square$

#### 1.4.4 Asymptotic analysis

Let  $\Theta_L = \sum_{m=1}^{\infty} a_m q^m$ , where  $a_m$  is the number of ways  $Q_L$  represents  $2m$ , so  $a_0 = 1$ . Then

$$\Theta_L = E_{\frac{n}{2}} + g, \quad E_{\frac{n}{2}} \sim \sigma_{\frac{n}{2}-1}(m) \sim m^{\frac{n}{2}-1},$$

where  $g$  is a cusp form.

Lecture 12 is a problem class.

**Proposition 1.4.8.** *Let*

$$E_k = \sum_{n=0}^{\infty} a_n q^n = 1 + C \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

*Then there exist  $A, B \in \mathbb{R}_{>0}$  such that*

$$A n^{k-1} \leq a_n \leq B n^{k-1}.$$

*Proof.* Set  $A = C$ . Then

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1} \geq n^{k-1},$$

so  $a_n = C \sigma_{k-1}(n) \geq C n^{k-1}$ . Consider

$$\frac{\sigma_{k-1}(n)}{n^{k-1}} = \sum_{d|n} \frac{d^{k-1}}{n^{k-1}} = \sum_{d'|n} \frac{1}{d'^{k-1}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{k-1}} = \zeta(k-1),$$

so  $\sigma_{k-1}(n) \leq \zeta(k-1) n^{k-1}$ . So set  $B = C \cdot \zeta(k-1)$ , so  $a_n \leq B n^{k-1}$ .  $\square$

**Theorem 1.4.9** (Hasse). *Let  $f = \sum_{n=1}^{\infty} a_n q^n$  be a cusp form of weight  $k$ . Then*

$$|a_n| = O\left(n^{\frac{k}{2}}\right),$$

*that is  $|a_n| n^{-k/2}$  is bounded as  $n \rightarrow \infty$ .*

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*Proof.*  $f/q$  is holomorphic on  $\mathbb{H}$ , so  $|f/q|$  is bounded as  $q \rightarrow 0$ , so  $|f(z)|/e^{-2\pi \operatorname{Im} z}$  is bounded as  $\operatorname{Im} z \rightarrow \infty$ . That is, there exist  $M \in \mathbb{R}$  such that  $|f(z)| \leq M e^{-2\pi \operatorname{Im} z}$ . Consider

$$\phi(z) = |f(z)| \operatorname{Im} z^{\frac{k}{2}},$$

so  $\lim_{\operatorname{Im} z \rightarrow \infty} \phi(z) = 0$ . Note that

$$\phi(\gamma z) = |f(\gamma z)| \operatorname{Im} \gamma z^{\frac{k}{2}} = |f(z)| |cz + d|^k \frac{\operatorname{Im} z^{\frac{k}{2}}}{|cz + d|^{2\frac{k}{2}}} = |f(z)| \operatorname{Im} z^{\frac{k}{2}} = \phi(z), \quad \gamma \in \operatorname{SL}_2(\mathbb{Z}).$$

Then  $\phi(z)$  is determined by its values on the standard fundamental domain, so  $\phi(z)$  is bounded on  $\mathbb{H}$ , so  $|f(z)| < M' \operatorname{Im} z^{-k/2}$  for some  $M' \in \mathbb{R}$ . If  $z = x + iy$  for  $y$  fixed, then the residue theorem implies that

$$a_m = \frac{1}{2\pi i} \int_C \frac{f(q)}{q^{m+1}} dq = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f(x + iy)}{e^{2\pi i(x+iy)m}} dx,$$

so

$$|a_m| \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|f(x + iy)|}{e^{-2\pi ym}} dx \leq \frac{|f(x + iy)|}{e^{-2\pi ym}} \leq e^{2\pi ym} M' y^{-\frac{k}{2}}.$$

Set  $y = 1/m$ . Get  $|a_n| \leq e^{2\pi} M' m^{k/2}$ , so  $|a_m|/m^{k/2}$  is bounded.  $\square$

Had

$$\Theta_L = E_{\frac{n}{2}} + g, \quad E_{\frac{n}{2}} \sim m^{\frac{n}{2}-1}, \quad g = O\left(m^{\frac{n}{4}}\right).$$

**Theorem 1.4.10** (Deligne). *Let  $f = \sum_{n=1}^{\infty} a_n q^n$  be a cusp form of weight  $k$ . Then*

$$|a_n| = O\left(n^{\frac{k-1}{2}} \sigma_0(n)\right).$$

*Proof.* Very rough sketch of argument.

Ramanujan 1910s. Conjectured by Ramanujan for  $f = \Delta$ .

Weil 1940s. For an algebraic variety  $V$  over  $\mathbb{F}_q$ , what can we say about  $\#V(\mathbb{F}_{q^n})$  for various  $n$ ? Weil associated to  $V$  and  $\mathbb{F}_q$  a generating function called the **zeta function**  $\zeta_{V,q}(t)$  of  $V$  over  $\mathbb{F}_q$ , conjectured several things about  $\zeta_{V,q}$ , and proved in the case of curves.

- $\zeta_{V,q}$  is a rational function in  $t$ .
- $\zeta_{V,q}$  satisfies a certain symmetry under  $t \mapsto 1/t$ .
- The Riemann hypothesis

$$\zeta_{V,q}(t) = \frac{P_1(t) \dots P_{2d-1}(t)}{P_0(t) \dots P_{2d}(t)}, \quad \dim V = d,$$

where the roots of  $P_i(t)$  have absolute value  $q^{i/2}$ .

Eichler-Shimura 1950s. Let  $\Gamma \subseteq \operatorname{SL}_2(\mathbb{Z})$  be a nice **congruence subgroup**. Then  $X_\Gamma = \Gamma \backslash \mathbb{H}$  has the structure of an algebraic curve over  $\mathbb{Q}$ , with **good reduction** at primes  $p$  not dividing  $[\operatorname{SL}_2(\mathbb{Z}) : \Gamma]$ . Eichler, Shimura, and others studied  $\zeta_{V,p}$  for  $V = X_\Gamma$ , and related  $\zeta_{V,p}$  to the  $p$ -th Fourier coefficients of a basis for forms of weight two and **level**  $\Gamma$ . The Weil conjectures bound  $a_p$  in terms of  $q^{1/2}$ .

Deligne 1960s. Deligne showed that in weight  $k$ , there exists a **Kuga-Sato variety**, of dimension  $k - 1$ , whose zeta function has a factor coming from modular forms of weight  $k$  and level  $\Gamma$ , and showed that if the Weil conjectures, particularly the Riemann hypothesis, holds, then get the coefficient bound.

Deligne 1970s. Riemann hypothesis in higher dimensions.  $\square$

## 1.5 Hecke operators

Let  $\Delta = (E_4^3 - E_6^2) / 1728 = \sum_{n=1}^{\infty} \tau(n) q^n$ . Then  $\tau(n)$  grows roughly like  $n^6$  or  $n^{11/2+\epsilon}$ . Mordell proved

- $\tau(mn) = \tau(n) \tau(m)$  if  $(m, n) = 1$ , and
- $\tau(p^{n+1}) = \tau(p) \tau(p^n) - p^{11} \tau(p^{n-1})$ .

If  $E_k = 1 + C \sum_n \sigma_{k-1}(n) q^n$ , set

$$E'_k = \frac{1}{C} + \sum_n \sigma_{k-1}(n) q^n.$$

**Note.**

- If  $(m, n) = 1$ , then

$$\sigma_{k-1}(nm) = \sum_{d|n} \sum_{d'|m} (dd')^{k-1} = \left( \sum_{d|n} d^{k-1} \right) \left( \sum_{d'|m} d'^{k-1} \right) = \sigma_{k-1}(n) \sigma_{k-1}(m).$$

- Since  $\sigma_{k-1}(p^n) = 1 + \dots + p^{n(k-1)}$ ,

$$\begin{aligned} \sigma_{k-1}(p) \sigma_{k-1}(p^n) &= (1 + p^{k-1}) (1 + \dots + p^{n(k-1)}) \\ &= 1 + 2p^{k-1} + \dots + 2p^{n(k-1)} + p^{(n+1)(k-1)} \\ &= \sigma_{k-1}(p^{n+1}) + p^{k-1} \sigma_{k-1}(p^{n-1}), \end{aligned}$$

so

$$\sigma_{k-1}(p^{n+1}) = \sigma_{k-1}(p) \sigma_{k-1}(p^n) - p^{k-1} \sigma_{k-1}(p^{n-1}).$$

### 1.5.1 Correspondences

**Definition 1.5.1.** Let  $X$  be a set. The **free abelian group on  $X$** , denoted  $\mathbb{Z}X$ , is the set of finite formal sums

$$\sum_{i=1}^r a_i x_i, \quad a_i \in \mathbb{Z}, \quad x_i \in X,$$

where  $x_i$  are distinct. Add by combining like terms.

**Definition 1.5.2.** A **correspondence** on  $X$  is a homomorphism  $\mathbb{Z}X \rightarrow \mathbb{Z}X$ . Let

$$\text{Corr } X = \{\text{correspondences on } X\}.$$

Equivalently, a correspondence associates to each  $x \in X$ , a finite formal sum

$$\sum_{i=1}^r a_i y_i, \quad a_i \in \mathbb{Z}, \quad y_i \in X.$$

If  $X$  is a finite set  $X = \{x_1, \dots, x_r\}$ , any correspondence  $T$  can be represented, in a unique way, by the matrix  $M_T$  such that

$$Tx_i = \sum_{j=1}^r (M_T)_{ij} x_j,$$

and composition of correspondences is matrix multiplication. Let  $X$  be a set, and let

$$\text{Fun}_{\mathbb{C}} X = \{\text{functions } X \rightarrow \mathbb{C}\}.$$

Then  $T \in \text{Corr } X$  acts on  $\text{Fun}_{\mathbb{C}} X$  as follows. If  $Tx = \sum_i a_i x_i$  then  $(Tf)x = \sum_i a_i f(x_i)$ . Check  $(T \circ T')f = T(T'f)$ , etc. Let

$$\mathcal{L} = \{\text{lattices in } \mathbb{C}\}.$$

**Example.** The following are correspondences in  $\mathcal{L}$ .

- For  $\lambda \in \mathbb{C}^\times$ , have

$$\begin{array}{ccc} R_\lambda & : & \mathbb{Z}\mathcal{L} \longrightarrow \mathbb{Z}\mathcal{L} \\ & & L \longmapsto \lambda L \end{array}.$$

- For  $n \in \mathbb{Z}_{>0}$ , have

$$\begin{array}{ccc} T_n & : & \mathbb{Z}\mathcal{L} \longrightarrow \mathbb{Z}\mathcal{L} \\ & & L \longmapsto \sum_{L' \subseteq_n L} L' \end{array},$$

the  $n$  **Hecke operators**. Note that there are only finitely many  $L' \subseteq L$  of index  $n$ , since if  $L'$  has index  $n$  in  $L$ , then  $L'$  contains  $R_n L$ . Then  $L/R_n L \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ . The image of  $L'$  in  $L/R_n L$  is a subgroup  $H$  of  $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$  of order  $n$ . The preimage of  $H$  in  $L$  is  $L'$ . Thus there is a bijection

$$\{ \text{subgroups of } L/R_n L \text{ of order } n \} \longleftrightarrow \{ \text{sublattices of index } n \}.$$

**Proposition 1.5.3.**

1.  $R_\lambda R_\mu = R_{\lambda\mu}$ .
2.  $R_\lambda T_n = T_n R_\lambda$ .
3.  $T_n T_m = T_{nm}$  if  $(m, n) = 1$ .
4.  $T_p T_{p^n} = T_{p^{n+1}} + p T_{p^{n+1}} R_p$ .

**Corollary 1.5.4.**  $T_p$  commute with each other for  $p$  prime, also with  $R_\lambda$ , and every  $T_n$  is a polynomial in  $T_p$  and  $R_p$  for  $p \mid n$ , so all  $T_n$  and  $R_\lambda$  commute.

**Proposition 1.5.5.** If  $A$  is an abelian group of order  $nm$ , with  $(n, m) = 1$ , then  $A$  factors uniquely as  $B \times C$ , where  $B$  has order  $n$  and  $C$  has order  $m$ . In particular  $B$  is the unique subgroup of  $A$  of order  $n$ .

*Proof.* Write  $1 = an + bm$  for  $a, b \in \mathbb{Z}$ . Have a map

$$\begin{array}{ccc} A & \longleftrightarrow & mA \times nA \\ x & \longmapsto & (mbx, nax) \\ x + y & \longmapsto & (x, y) \end{array}$$

Then  $mA$  has order  $n$  and  $nA$  has order  $m$ . Clearly inverses on one side, so counting implies isomorphism.  $\square$

*Proof of Proposition 1.5.3.*

1. Easy.
2. If  $L \in \mathcal{L}$ , then

$$R_\lambda T_n L = R_\lambda \sum_{L' \subseteq_n L} L' = \sum_{L' \subseteq_n L} R_\lambda L' = \sum_{L' \subseteq_n R_\lambda L} L' = T_n R_\lambda L.$$

3. If  $L \in \mathcal{L}$ , then

$$T_n T_m L = T_n \sum_{L' \subseteq_m L} L' = \sum_{L' \subseteq_m L} T_n L' = \sum_{L' \subseteq_m L} \sum_{L'' \subseteq_n L'} L''.$$

An observation is  $L'' \subseteq_n L' \subseteq_m L$ , so  $L''$  has index  $nm$  in  $L$ . Let

$$T_n T_m L = \sum_{L'' \subseteq_{nm} L} c_{n,m}(L'', L) L'', \quad c_{n,m}(L'', L) = \# \{ L' \in \mathcal{L} \mid L'' \subseteq_n L' \subseteq_m L \}.$$

An observation is that there is a bijection

$$\begin{array}{ccc} \{ \text{lattices } L' \mid L'' \subseteq_n L' \subseteq_m L \} & \longleftrightarrow & \{ \text{subgroups } H \text{ of } L/L'' \text{ of order } n \} \\ L' & \longmapsto & L'/L'' \subseteq L/L'' \\ \text{preimage of } H \text{ under } L \rightarrow L/L'' & \longleftarrow & H \end{array}.$$

Have  $(n, m) = 1$ , so  $c_{n,m}(L'', L) = 1$  so

$$T_n T_m L = \sum_{L'' \subseteq_{nm} L} c_{n,m}(L'', L) L'' = \sum_{L'' \subseteq_{nm} L} L'' = T_{nm} L.$$

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4. If  $L \in \mathcal{L}$ , then

$$T_p T_{p^r} L = \sum_{L'' \subseteq_{p^{r+1}} L} c_{p,p^r}(L'', L) L'', \quad c_{p,p^r}(L'', L) = \#\{L' \in \mathcal{L} \mid L'' \subseteq_p L' \subseteq_{p^r} L\}.$$

What is

$$c_{p,p^r}(L'', L) = \#\{\text{subgroups of order } p \text{ in } L/L''\}?$$

$L/L''$  is abelian of order  $p^{r+1}$  and generated by two elements. The classification of finite abelian groups implies that every finite abelian group can be written uniquely as  $\mathbb{Z}/a_1\mathbb{Z} \times \cdots \times \mathbb{Z}/a_r\mathbb{Z}$  where  $a_1 \mid \cdots \mid a_r$ , up to isomorphism, and  $r$  is the minimal number of generators for such a group. So

$$L/L'' \cong \mathbb{Z}/p^a\mathbb{Z} \times \mathbb{Z}/p^b\mathbb{Z}, \quad a, b \geq 0, \quad a + b = r + 1.$$

Case 1.  $L/L'' \cong \mathbb{Z}/p^{r+1}\mathbb{Z}$  is cyclic. In this case  $c_{p,p^r}(L'', L) = 1$ .

Case 2.  $L/L'' \cong \mathbb{Z}/p^a\mathbb{Z} \times \mathbb{Z}/p^b\mathbb{Z}$  with  $a, b > 0$ . Any subgroup of order  $p$  is contained in the subgroup killed by  $p$ ,

$$p^{a-1}\mathbb{Z}/p^a\mathbb{Z} \times p^{b-1}\mathbb{Z}/p^b\mathbb{Z} \cong (\mathbb{Z}/p\mathbb{Z})^2.$$

The  $p^2 - 1$  elements of  $(\mathbb{Z}/p\mathbb{Z})^2 \setminus \{0\}$  each spans a subgroup of order  $p$ , and two elements span the same group if and only if they differ by a scalar in  $(\mathbb{Z}/p\mathbb{Z})^\times$ , so there are  $(p^2 - 1) / (p - 1) = p + 1$  subgroups of order  $p$  in  $(\mathbb{Z}/p\mathbb{Z})^2$ . In this case  $c_{p,p^r}(L'', L) = p + 1$ .

The latter case occurs if and only if  $L/L''$  maps surjectively to  $(\mathbb{Z}/p\mathbb{Z})^2 \cong L/R_p L$ , if and only if  $R_p L \supseteq L''$ . Thus

$$\begin{aligned} T_p T_{p^r} L &= \sum_{L'' \subseteq_{p^{r+1}} L} c_{p,p^r}(L'', L) L'' = \sum_{L'' \subseteq_{p^{r+1}} L \text{ cyclic}} L'' + \sum_{L'' \subseteq_{p^{r+1}} L \text{ not cyclic}} (p + 1) L'' \\ &= T_{p^{r+1}} L + p \sum_{L'' \subseteq_{p^{r+1}} L \text{ not cyclic}} L'' = T_{p^{r+1}} L + p \sum_{L'' \subseteq_{p^{r-1}} R_p L} L'' = T_{p^{r+1}} L + p T_{p^{r-1}} R_p L. \end{aligned}$$

□

### 1.5.2 Hecke operators

If  $F : \mathcal{L} \rightarrow \mathbb{C}$ , then

$$T_n F(L) = \sum_{L' \subseteq_n L} F(L'), \quad R_\lambda F(L) = F(R_\lambda L).$$

Recall that  $F$  has weight  $k$  if  $F(R_\lambda L) = \lambda^{-k} F(L)$  for all  $\lambda \in \mathbb{C}^\times$ , if and only if  $R_\lambda F = \lambda^{-k} F$  for all  $\lambda \in \mathbb{C}^\times$ , so

$$R_\lambda T_n F = T_n R_\lambda F = T_n \lambda^{-k} F = \lambda^{-k} T_n F.$$

So the  $T_n$  and  $R_\lambda$  preserve lattice functions of weight  $k$ . Have a bijection

$$\begin{aligned} \left\{ f : \mathbb{H} \rightarrow \mathbb{C} \mid f(\gamma z) = (cz + d)^k f(z) \right\} &\longrightarrow \{\text{lattice functions } F \text{ of weight } k\} \\ f(z) &\longmapsto F(L_{z,1}) \end{aligned}$$

On lattice functions of weight  $k$ , have

$$T_p T_{p^r} = T_{p^{r+1}} + p^{1-k} T_{p^{r-1}}.$$

**Definition 1.5.6.** For  $f : \mathbb{H} \rightarrow \mathbb{C}$  corresponding to  $F : \mathcal{L} \rightarrow \mathbb{C}$  of weight  $k$ , define  $T_n f$  by

$$(T_n f)(z) = n^{k-1} (T_n F)(L_{z,1}) = n^{k-1} \sum_{L' \subseteq_n L_{z,1}} F(L').$$

On  $f : \mathbb{H} \rightarrow \mathbb{C}$ ,  $T_n$  satisfy

$$T_p T_{p^r} = T_{p^{r+1}} + p^{k-1} T_{p^{r-1}}.$$

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Need to rewrite  $\sum_{L' \subseteq_n L_{z,1}} F(L')$  in terms of  $f$ . Let

$$S_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \text{Mat}_{2 \times 2} \mathbb{Z} \mid ad = n, a, d > 0, 0 \leq b < d \right\}, \quad s_n = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in S_n.$$

**Lemma 1.5.7.** *The map*

$$\begin{aligned} S_n &\longrightarrow \{ \text{sublattices of } L_{z,1} \text{ of index } n \} \\ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} &\longmapsto L_{az+b,d} \end{aligned}$$

*is a bijection.*

*Proof.* For surjectivity, let  $L \subseteq_n L_{z,1}$ . Then  $L_{z,1}/L$  is a group of order  $n$ . Can consider  $1 + L \in L_{z,1}/L$ . Let  $d$  be the order of  $1 + L$ , that is  $d$  is the smallest positive integer such that  $d \in L$ . Then  $d \mid n$ , so set  $a = n/d$ . Let  $L' = \mathbb{Z} + L$  be the lattice generated by 1 and  $L$ . Then  $L \subseteq_d L'$  and  $L \subseteq_n L_{z,1}$ , so  $L' \subseteq_a L_{z,1}$ , so  $az \in L'$ , so there exists  $b \in \mathbb{Z}$  such that  $az + b \in L$ . Since  $d \in L$ , without loss of generality can arrange  $0 \leq b < d$ . Now  $d \in L$  and  $az + b \in L$ , so  $L \subseteq_n L_{z,1}$  and  $L_{az+b,d} \subseteq_n L_{z,1}$ , so  $L = L_{az+b,d}$ . Thus surjective, and for injectivity, can recover  $a, b, d$  from  $L_{az+b,d} \subseteq L_{z,1}$ .  $\square$

Thus

$$\begin{aligned} T_n f &= n^{k-1} \sum_{L' \subseteq_n L_{z,1}} F(L') = n^{k-1} \sum_{s_n \in S_n} F(L_{az+b,d}) \\ &= n^{k-1} \sum_{s_n \in S_n} d^{-k} F\left(L_{\frac{az+b}{d},1}\right) = n^{k-1} \sum_{s_n \in S_n} d^{-k} f\left(\frac{az+b}{d}\right). \end{aligned}$$

**Theorem 1.5.8.** *If  $f = \sum_{m=0}^{\infty} c(m) q^m$  is modular of weight  $k$ , then*

$$T_n f = \sum_{m=0}^{\infty} \gamma(m) q^m, \quad \gamma(m) = \sum_{a \mid (m,n), a \geq 1} d^{k-1} c\left(\frac{mn}{a^2}\right).$$

*Proof.*

$$\begin{aligned} T_n f &= n^{k-1} \sum_{s_n \in S_n} d^{-k} f\left(\frac{az+b}{d}\right) = n^{k-1} \sum_{s_n \in S_n} \sum_{m=0}^{\infty} d^{-k} c(m) e^{2\pi i m \left(\frac{az+b}{d}\right)} \\ &= n^{k-1} \sum_{ad=n, a>0} \sum_{b=0}^{d-1} \sum_{m=0}^{\infty} d^{-k} c(m) q^{\frac{ma}{d}} e^{\frac{2\pi i mb}{d}} = n^{k-1} \sum_{m=0}^{\infty} \sum_{ad=n, a>0} d^{-k} c(m) q^{\frac{ma}{d}} \sum_{b=0}^{d-1} e^{\frac{2\pi i mb}{d}}. \end{aligned}$$

Then

$$\sum_{b=0}^{d-1} e^{\frac{2\pi i mb}{d}} = \begin{cases} d & d \mid m \\ 0 & d \nmid m \end{cases},$$

so

$$T_n f = n^{k-1} \sum_{m=0}^{\infty} \sum_{d \mid m, ad=n, a>0} d^{1-k} c(m) q^{\frac{ma}{d}} = \sum_{a \mid n, a>0} \sum_{m'=0}^{\infty} a^{k-1} c\left(\frac{m'n}{a}\right) q^{m'a}.$$

Which  $m'$  and  $a$  give  $q^m$ ? Need  $a \mid (m,n)$  for  $a > 0$  and  $m'a = m$ , so the coefficient is  $a^{k-1} c(mn/a^2)$ . The sum of these is  $\gamma(m)$ .  $\square$

**Corollary 1.5.9.**  $T_n$  preserves  $M_k$  and  $S_k$ .

In the case  $n = p$ ,

$$T_p f = \sum_{m=0}^{\infty} \gamma(m) q^m, \quad \gamma(m) = \begin{cases} c(mp) + p^{k-1} c\left(\frac{m}{p}\right) & p \mid m \\ c(mp) & p \nmid m \end{cases}.$$



### 1.5.3 Eigenforms

An observation is that the dimensions of  $M_4, M_6, M_8, M_{10}, S_{12}$  are one, so  $E_4, E_6, E_8, E_{10}, \Delta$  are eigenvectors for  $T_n$  for all  $n$ .

**Definition 1.5.10.** A function  $f \in M_k$  is an **eigenform** if there exists  $\lambda_n \in \mathbb{C}^\times$  such that  $T_n f = \lambda_n f$  for all  $n \in \mathbb{Z}_{>0}$ .

**Proposition 1.5.11.** Let  $f \in M_k$  be an eigenform, with  $k > 0$ , so  $T_n f = \lambda_n f$  for all  $n$ . Then if  $f = \sum_m c_m q^m$ , we have  $c_1 \neq 0$  and  $\lambda_n c_1 = c_n$  for all  $n \geq 1$ . In particular, if  $c_1 = 1$ , then  $c_n = \lambda_n$  for all  $n$ .

*Proof.*

$$\sum_{m=0}^{\infty} \lambda_n c_m q^m = \lambda_n f = T_n f = \sum_{m=0}^{\infty} \gamma(m) q^m, \quad \gamma(1) = \sum_{a|(1,n)} a^{k-1} c(n) = c(n),$$

so  $\lambda_n c_1 = c_n$ . Suppose  $c_1 = 0$ . Then  $c_n = 0$  for all  $n \geq 1$ , so  $f$  is constant. Since  $k \neq 0$ , this does not happen.  $\square$

**Corollary 1.5.12.** Recall  $\Delta(z) = \sum_n \tau(n) q^n$ . Then

- $\tau(mn) = \tau(n) \tau(m)$  if  $(m, n) = 1$ , and
- $\tau(p^{r+1}) = \tau(p) \tau(p^r) - p^{11} \tau(p^{r-1})$ .

*Proof.*  $\Delta \in S_{12}$  is one-dimensional, so there exists  $\lambda_n$  such that  $T_n \Delta = \lambda_n \Delta$ . Proposition 1.5.11 implies that  $\lambda_n = \tau(n)$  for all  $n$ . Thus

- $\tau(mn) \Delta = \lambda_{mn} \Delta = T_{mn} \Delta = T_m T_n \Delta = \lambda_m \lambda_n \Delta = \tau(m) \tau(n) \Delta$ , and
- $\tau(p^{r+1}) \Delta = T_{p^{r+1}} \Delta = T_p T_{p^r} \Delta - p^{11} T_{p^{r-1}} \Delta = (\tau(p) \tau(p^r) - p^{11} \tau(p^{r-1})) \Delta$ .

$\square$

In fact, the same argument shows if  $f \in M_k$  for  $k > 0$  is an eigenform, with  $q$ -coefficient one, a **normalised eigenform**, and  $f = \sum_{n=0}^{\infty} c_n q^n$ , then

- $c_{nm} = c_n c_m$  if  $(n, m) = 1$ , and
- $c_{p^{r+1}} = c_p c_{p^r} - p^{k-1} c_{p^{r-1}}$ .

**Proposition 1.5.13.**  $E_k$  is an eigenform for all  $k$ .

*Proof.* It suffices to show  $T_p E_k = \lambda_p E_k$  for all primes  $p$ . Recall  $E_k$  is a constant multiple of  $G_k$ , where  $G_k(L) = \sum_{w \in L, w \neq 0} 1/w^k$ . Now

$$(T_p f)(L) = \sum_{L' \subseteq_p L} \sum_{w \in L', w \neq 0} \frac{1}{w^k} = \sum_{w \in L, w \neq 0} c_w \frac{1}{w^k}, \quad c_w = \# \{L' \subseteq_p L \mid w \in L'\}.$$

Note that  $pL \subseteq L' \subseteq L$ . If  $w \in pL$ , then  $w \in L'$  for all  $L' \subseteq_p L$ , and there are  $p+1$  of these. If  $w \notin pL$ , then  $pL \subseteq_{p^2} L$  and  $pL \subsetneq pL + \mathbb{Z}w \subsetneq L$ , so  $pL \subsetneq_p pL + \mathbb{Z}w$  and  $pL + \mathbb{Z}w \subsetneq_p L$ . In this case there exists a unique lattice of index  $p$  containing  $w$ . Thus

$$\begin{aligned} T_p G_k(L) &= \sum_{w \in L \setminus pL} \frac{1}{w^k} + \sum_{w \in pL, w \neq 0} (p+1) \frac{1}{w^k} = \sum_{w \in L, w \neq 0} \frac{1}{w^k} + p \sum_{w \in pL, w \neq 0} \frac{1}{w^k} \\ &= G_k(L) + p \sum_{w \in L, w \neq 0} \frac{1}{(pw)^k} = G_k(L) + p^{1-k} \sum_{w \in L} \frac{1}{w^k} = (1 + p^{1-k}) G_k(L), \end{aligned}$$

so  $T_p E_k = (1 + p^{k-1}) E_k$ .  $\square$

A question is does  $M_k$  have a basis of eigenforms for all  $k$ ? By linear algebra, there exist nice classes of operators that are guaranteed to admit bases of eigenvectors, such as self-adjoint, or more generally, normal operators.

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### 1.5.4 Hermitian pairings

Let  $V$  be a  $\mathbb{C}$ -vector space and  $\langle -, - \rangle : V \times V \rightarrow \mathbb{C}$  a **Hermitian pairing**. That is,

- $\langle \lambda v + w, x \rangle = \lambda \langle v, x \rangle + \langle w, x \rangle$ ,
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$ , and
- $\langle x, x \rangle > 0$  for all  $x \neq 0$ .

**Example.** The standard pairing

$$\begin{aligned} \mathbb{C}^n \times \mathbb{C}^n &\longrightarrow \mathbb{C} \\ \langle z, w \rangle &\longmapsto \sum_{i=1}^n z_i \overline{w_i} . \end{aligned}$$

**Definition 1.5.14.** Let  $A : V \rightarrow V$  be  $\mathbb{C}$ -linear, and  $\langle -, - \rangle : V \times V \rightarrow \mathbb{C}$  Hermitian. Then the **adjoint**  $A^* : V \rightarrow V$  is the unique linear map  $V \rightarrow V$  such that

$$\langle Av, w \rangle = \langle v, A^*w \rangle .$$

We say  $A$  is **self-adjoint** if  $A^* = A$ , and **normal** if  $A^*$  commutes with  $A$ .

**Theorem 1.5.15.** *If  $A$  is normal, then  $A$  has a basis of eigenvectors.*

**Lemma 1.5.16.**  $A^{**} = A$ .

*Proof.* For all  $v, w \in V$ ,

$$\langle v, A^{**}w \rangle = \langle A^*v, w \rangle = \overline{\langle w, A^*v \rangle} = \overline{\langle Aw, v \rangle} = \langle v, Aw \rangle ,$$

so  $A^{**}w = Aw$  for all  $w \in V$ . □

**Definition 1.5.17.** If  $W \subseteq V$ , let

$$W^\perp = \{v \in V \mid \forall w \in W, \langle v, w \rangle = 0\} .$$

**Proposition 1.5.18.**  $\text{Im } A^* = (\text{Ker } A)^\perp$ .

*Proof.*  $\langle v, A^*w \rangle = \langle Av, w \rangle = 0$  if  $v \in \text{Ker } A$ . So  $\text{Im } A^* \subseteq (\text{Ker } A)^\perp$ , so  $\text{rk } A^* \leq \text{rk } A$ . The same argument with  $A^*$  in place of  $A$  implies that  $\text{rk } A = \text{rk } A^{**} \leq \text{rk } A^*$ . So  $\text{rk } A^* = \text{rk } A$ , so  $\text{Im } A^* = (\text{Ker } A)^\perp$ . □

In particular,  $\text{Im } A^* \cap \text{Ker } A = \{0\}$  and  $\dim \text{Im } A^* + \dim \text{Ker } A = \text{rk } A^* + n - \text{rk } A = n$ . So  $V = \text{Im } A^* \oplus \text{Ker } A$ .

**Theorem 1.5.19** (Spectral theorem for normal operators). *If  $A$  and  $A^*$  commute, then  $A^*$  is diagonalisable.*

*Proof.* Induction on  $\dim V$ . Then  $\dim V = 1$  is clear. Let  $\lambda$  be an eigenvalue of  $A$ , and let  $A' = A - \lambda I_V$ , so  $V = \text{Ker } A' \oplus \text{Im } A'^*$ , where  $\dim \text{Ker } A' > 0$ . Then  $A$  commutes with  $A'$ , and  $A'^* = A^* - \overline{\lambda} I_V$ , so  $A$  commutes with  $A'^*$ . So  $AA'^*v = A'^*Av$ , so  $A$  preserves the image of  $A'^*$ . The restriction of  $\langle -, - \rangle$  to  $\text{Im } A'^*$  is still Hermitian on  $\text{Im } A'^*$  and the restriction of  $A$  to  $\text{Im } A'^*$  is still normal, since its adjoint is the restriction of  $A^*$  to  $\text{Im } A'^*$ . By induction  $A$  is diagonalisable on  $\text{Im } A'^*$  and scalar on  $\text{Ker } A'$ , so diagonalisable. □

Also the need the following observation.

**Proposition 1.5.20.** *If  $A : V \rightarrow V$  and  $B : V \rightarrow V$  commute, and  $V_\lambda = \text{Ker } (A - \lambda I_V)$ , then  $BV_\lambda = V_\lambda$ .*

*Proof.* If  $v \in V_\lambda$ , then  $ABv = BAv = B\lambda v = \lambda Bv$ , so  $Bv \in V_\lambda$ . □

### 1.5.5 The Petersson inner product

To apply this to modular forms, we need a bilinear pairing on  $M_k$  or  $S_k$ . The idea is to show that there exists a pairing  $\langle -, - \rangle_k : S_k \times S_k \rightarrow \mathbb{C}$  such that  $\langle T_n f, g \rangle = \langle f, T_n g \rangle$  for all  $n$ , so  $T_n$  are self-adjoint, hence diagonalisable.

**Definition 1.5.21.** Let  $f, g \in S_k$ . The **Petersson inner product**  $\langle f, g \rangle_k$  is

$$\langle f, g \rangle_k = \iint_{\mathcal{D}} f(z) \overline{g(z)} \frac{(\text{Im } z)^k}{(\text{Im } z)^2} dx dy = \frac{i}{2} \iint_{\mathcal{D}} f(z) \overline{g(z)} \frac{(\text{Im } z)^k}{(\text{Im } z)^2} dz d\bar{z} .$$

Here  $z = x + iy$  and  $\bar{z} = x - iy$ , so  $dz d\bar{z} = (dx + idy) \wedge (dx - idy) = -2i(dx \wedge dy)$ .

Then

$$f(\gamma z) \overline{g(\gamma z)} (\operatorname{Im} \gamma z)^k = f(z) (cz + d)^k \overline{g(z) (cz + d)^k} \frac{\operatorname{Im} z}{|cz + d|^{2k}} = f(z) \overline{g(z)} (\operatorname{Im} z)^k,$$

and

$$\frac{1}{(\operatorname{Im} \gamma z)^2} d(\gamma z) (\gamma \bar{z}) = \frac{1}{(\operatorname{Im} \gamma z)^2 |cz + d|^4} dz d\bar{z} = \frac{1}{(\operatorname{Im} z)^2} dz d\bar{z},$$

so for all  $U \subseteq \mathbb{H}$ ,

$$\iint_{\gamma(U)} f(z) \overline{g(z)} \frac{(\operatorname{Im} z)^k}{(\operatorname{Im} z)^2} dz d\bar{z} = \iint_U f(z) \overline{g(z)} \frac{(\operatorname{Im} z)^k}{(\operatorname{Im} z)^2} dz d\bar{z}.$$

**Note.** This converges for  $f, g \in S_k$ , since  $f(a + it)$  goes like  $e^{-t}$  as  $t \rightarrow \pm\infty$ , and the same for  $g$ . If  $\langle f, f \rangle = 0$ , the integrand vanishes identically, since it lives in  $\mathbb{R}_{\geq 0}$ . So  $f = 0$  on  $\mathcal{D}$ , hence everywhere. Then

$$\langle \lambda f, g \rangle_k = \lambda \langle f, g \rangle_k, \quad \langle f, \lambda g \rangle_k = \bar{\lambda} \langle f, g \rangle_k, \quad \langle f, g \rangle_k = \overline{\langle g, f \rangle_k}.$$

So  $\langle -, - \rangle_k$  is Hermitian.

**Theorem 1.5.22.**  $\langle T_n f, g \rangle_k = \langle f, T_n g \rangle_k$  for all  $f, g \in S_k$  and  $n \in \mathbb{Z}_{\geq 1}$ .

**Corollary 1.5.23.** Each  $T_n$  is diagonalisable on  $S_k$ . Since  $T_n$  and  $T_m$  commute for all  $n$  and  $m$ ,  $T_m$  preserves eigenspaces of  $T_n$  for all  $m$ . By induction,  $T_m$  preserves the simultaneous eigenspaces of  $T_n$  for all  $n < m$ .

**Proposition 1.5.24.** Let  $n > \lfloor k/12 \rfloor + 1$ . Fix  $\lambda_2, \dots, \lambda_n \in \mathbb{C}$ . The subspace  $V$  of  $S_k$  on which  $T_i = \lambda_i$  for  $i = 2, \dots, n$  is zero or one-dimensional.

*Proof.* Let  $f \in V$ , so  $f = c_1 q + c_2 q^2 + \dots$ . Seen if  $T_i f = \lambda_i f$ , then  $c_i = \lambda_i c_1$ . Also seen that if the first  $n$  Fourier coefficients of  $f$  vanishes, then  $f = 0$ , by the  $k/12$ -formula. So  $c_1 \neq 0$  unless  $f = 0$ . Now if  $f, g \in V \setminus \{0\}$ , there exists  $\lambda \in \mathbb{C}$  such that  $f$  and  $\lambda g$  have the same  $q$ -coefficient, and thus the same first  $n$  Fourier coefficients. But then  $f - \lambda g = 0$ .  $\square$

**Corollary 1.5.25.**  $S_k$  admits a basis of eigenforms for all  $k$ .

*Proof.* Let  $n \geq \lfloor k/12 \rfloor + 1$ . Can diagonalise  $S_k$  with respect to the first  $n$  Hecke operators. Any simultaneous eigenspace for these is at most one-dimensional, and preserved by all  $T_n$ . So each of these is actually an eigenspace for all  $T_n$ .  $\square$

**Note.** If  $f$  and  $g$  are eigenforms, and  $f$  is not a scalar multiple of  $g$ , there exists  $T_n$  such that  $T_n f = \lambda_n f$  and  $T_n g = \mu_n g$  with  $\lambda_n \neq \mu_n$ . Then

$$\langle T_n f, g \rangle_k = \langle \lambda_n f, g \rangle_k = \lambda_n \langle f, g \rangle_k, \quad \langle f, T_n g \rangle_k = \langle f, \mu_n g \rangle_k = \mu_n \langle f, g \rangle_k,$$

$$\lambda_n \langle f, f \rangle_k = \langle T_n f, f \rangle_k = \langle f, T_n f \rangle_k = \overline{\langle T_n f, f \rangle_k} = \bar{\lambda}_n \langle f, f \rangle_k.$$

So  $\lambda_n = \bar{\lambda}_n$  and  $\mu_n = \bar{\mu}_n$ . Then  $(\lambda_n - \mu_n) \langle f, g \rangle_k = 0$ , so  $\langle f, g \rangle_k = 0$ .

The formula for  $T_n$  on  $q$ -expansions implies that  $T_n$  takes a  $q$ -expansion with  $\mathbb{Z}$  coefficients to another such. Saw that the space of modular forms with integral  $q$ -expansions is spanned by

$$E_4^n E_6^m, \dots, E_4^{n-3\lfloor n/3 \rfloor} E_6^m \Delta^{\lfloor n/3 \rfloor}, \quad k = 4n + 6m, \quad n, m > 0,$$

where  $m \in \{0, 1\}$  is minimal, so the matrix of  $T_n$  with respect to this basis has integer entries. Thus the characteristic polynomial of  $T_n$  on  $S_k$  has integer coefficients, so the eigenvalues of  $T_n$  are algebraic integers.

**Example.** Can ask when modular forms are congruent modulo  $p$ . In fact  $E_{12} \equiv \Delta \pmod{691}$ .

Ribet 1970s proved that when an Eisenstein series of suitable weight is congruent modulo  $p$  to a cusp form, can use the Galois representation attached to that cusp form to construct elements of ideal class groups of cyclotomic fields.

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## 1.6 L-functions

### 1.6.1 Dirichlet L-functions

**Definition 1.6.1.** Let  $\{a_n\}_{n \geq 1}$  be a sequence of complex numbers, usually algebraic integers. The **Dirichlet series** attached to  $a_n$  is the formal series  $\sum_{n=1}^{\infty} a_n n^{-s}$ , thought of as a function of  $s \in \mathbb{C}$ .

**Example.**  $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ .

In general, if  $|a_n| \leq Cn^k$ , then the corresponding series converges absolutely for  $\operatorname{Re} s > k + 1$ .

**Example.** Let  $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  be a **primitive character**, that is does not factor through  $(\mathbb{Z}/N\mathbb{Z})^\times \rightarrow (\mathbb{Z}/m\mathbb{Z})^\times$  for  $m \mid N$  such that  $m \neq N$ . Set

$$a_n = \begin{cases} \chi(n) & (n, N) = 1 \\ 0 & (n, N) \neq 1 \end{cases}.$$

Then

$$L(s, \chi) = \sum_n a_n n^{-s}$$

is the **Dirichlet L-function** attached to  $\chi$ .

In both these examples, and many others,

- these series have meromorphic, and often analytic, continuations to all of  $\mathbb{C}$ ,
- there is a **functional equation** relating values at  $s$  and  $k - s$  for some  $k$ , and
- there is an **Euler product**.

**Example.**

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s), \quad \zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \quad L(s, \chi) = \prod_{p \nmid N} \frac{1}{1-\chi(p)p^{-s}}.$$

### 1.6.2 Hecke L-functions

Let  $f = \sum_{n=0}^{\infty} a_n q^n \in M_k$ . Define

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

**Example.** Let  $f = E'_k = (-1)^{k/2} b_k/2k + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$ . Then

$$L(s, f) = \sum_{n=1}^{\infty} \sigma_{k-1}(n) n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - \sigma_{k-1}(p) p^{-s}} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \cdot \frac{1}{1 - p^{k-1} p^{-s}} = \zeta(s) \zeta(s - k + 1),$$

since  $\sigma_{k-1}(mn) = \sigma_{k-1}(m) \sigma_{k-1}(n)$  for  $(m, n) = 1$  and  $\sigma_{k-1}(p^r) = 1 + \dots + p^{r(k-1)}$ .

Let  $f = \sum_{n=1}^{\infty} a_n q^n$  be a cusp form. Recall that Hasse implies that  $|a_n| \leq Cn^{k/2}$ , so gives absolute convergence of  $L(s, f)$  for  $\operatorname{Re} s > k/2 + 1$ .

**Theorem 1.6.2.**

1.  $L(s, f)$  extends to a holomorphic function on all of  $\mathbb{C}$ .
2. Set  $R(s, f) = (2\pi)^{-s} \Gamma(s) L(s, f)$ . Then

$$R(s, f) = (-1)^{\frac{k}{2}} R(k - s, f).$$

3. If  $f$  is a normalised eigenform, then

$$L(s, f) = \prod_{p \text{ prime}} \frac{1}{1 - a_p p^{-s} + p^{k-1} p^{-2s}}.$$

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**Definition 1.6.3.** The infinite product  $\prod_{n=1}^{\infty} (1 + c_n)$  **converges** if  $\lim_{N \rightarrow \infty} \prod_{n=1}^N (1 + c_n)$  converges to a non-zero number, if and only if  $\sum_{n=1}^{\infty} \log(1 + c_n)$  converges. Then  $\prod_{n=1}^{\infty} (1 + c_n)$  **converges absolutely** if  $\prod_{n=1}^{\infty} (1 + |c_n|)$  converges.

**Lemma 1.6.4.**  $\prod_{n=1}^{\infty} (1 + c_n)$  converges absolutely if and only if  $\sum_{n=1}^{\infty} |c_n|$  converges.

*Proof.*

$$\sum_{n=1}^N |c_n| \leq \prod_{n=1}^N (1 + |c_n|) \leq \prod_{n=1}^N e^{|c_n|} \leq e^{\sum_{n=1}^N |c_n|}.$$

□

*Proof of Theorem 1.6.2.* Recall that  $\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$  is meromorphic on  $\mathbb{H}$ , with poles at  $\mathbb{Z}_{\leq 0}$  and never zero, and satisfies  $\Gamma(s+1) = s\Gamma(s)$  so  $\Gamma(n) = (n-1)!$ . Substituting  $t \mapsto 2\pi nt$  in  $\Gamma(s)$ ,

$$\Gamma(s) = \int_0^{\infty} (2\pi nt)^{s-1} e^{-2\pi nt} (2\pi n) dt = (2\pi n)^s \int_0^{\infty} t^{s-1} e^{-2\pi nt} dt,$$

so

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} a_n \frac{(2\pi)^s}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-2\pi nt} dt.$$

Then

$$\begin{aligned} R(s, f) &= \frac{\Gamma(s)}{(2\pi)^s} L(s, f) = \sum_{n=1}^{\infty} a_n \int_0^{\infty} t^{s-1} e^{-2\pi nt} dt = \int_0^{\infty} t^{s-1} \sum_{n=1}^{\infty} a_n e^{-2\pi nt} dt = \int_0^{\infty} t^{s-1} f(it) dt \\ &= \int_0^1 t^{s-1} f(it) dt + \int_1^{\infty} t^{s-1} f(it) dt = \int_1^{\infty} \left(\frac{1}{t}\right)^{s-1} f\left(\frac{i}{t}\right) d\left(\frac{1}{t}\right) + \int_1^{\infty} t^{s-1} f(it) dt \\ &= \int_1^{\infty} \left(t^{-s-1} (it)^k f(it) + t^{s-1} f(it)\right) dt = \int_1^{\infty} f(it) \left((-1)^{\frac{k}{2}} t^{k-s-1} + t^{s-1}\right) dt, \end{aligned}$$

1.  $R(s, f)$  converges independently of  $s$  uniformly for  $s$  in a compact subset of  $\mathbb{C}$ , so it is holomorphic in  $s$ , and extends to a holomorphic function on  $\mathbb{C}$ . Then  $L(s, f) = (2\pi)^s \Gamma(s)^{-1} R(s, f)$ , so  $L(s, f)$  is holomorphic since  $\Gamma(s)$  is non-vanishing.
2.  $R(s, f)$  is symmetric up to a sign under  $s \mapsto k - s$ , so  $R(s, f) = (-1)^{k/2} R(k - s, f)$ .
3. Now assume  $f$  is a normalised eigenform, so  $f = \sum_{n=1}^{\infty} a_n q^n$  with  $a_1 = 1$  and  $T_n f = a_n f$ . Then  $a_{nm} = a_n a_m$  if  $(n, m) = 1$ , so

$$L(s, f) = \sum_n a_n n^{-s} = \prod_p \sum_{k=0}^{\infty} a_{p^k} p^{-ks},$$

a power series in  $p^{-s}$ . Fix  $p$ , and consider

$$(1 - a_p p^{-s} + p^{k-1} p^{-2s}) \sum_{k=0}^{\infty} a_{p^k} p^{-ks}.$$

The  $p^0$  coefficient is  $a_1 = 1$ , the  $p^1$  coefficient is  $a_p p^{-s} - a_p p^{-s} = 0$ , and the  $p^{r+1}$  coefficient is

$$a_{p^{r+1}} p^{-(r+1)s} - a_p a_{p^r} p^{-(r+1)s} + p^{k-1} a_{p^{r-1}} p^{-(r+1)s} = (a_{p^{r+1}} - a_p a_{p^r} + p^{k-1} a_{p^{r-1}}) p^{-(r+1)s} = 0,$$

since  $a_{p^{r+1}} = a_p a_{p^r} - p^{k-1} a_{p^{r-1}}$ . So

$$L(s, f) = \prod_p \sum_{k=0}^{\infty} a_{p^k} p^{-ks} = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1} p^{-2s}}.$$

□

Lecture 21 is a problem class.

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## 2 Modular forms of higher level

### 2.1 Modular forms

#### 2.1.1 Congruence subgroups

$\mathrm{GL}_2(\mathbb{Q})_+$  acts on  $\mathbb{H}$  by fractional linear transformations.

**Definition 2.1.1.**  $\Gamma(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$  is the kernel of  $\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$  for  $N \in \mathbb{Z}_{>0}$ . Alternatively,

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{N}, b \equiv c \equiv 0 \pmod{N} \right\}.$$

**Note.**  $\Gamma(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$  has finite index.

**Definition 2.1.2.**  $\Gamma \subseteq \mathrm{GL}_2(\mathbb{Q})_+$  is a **congruence subgroup** if  $\Gamma$  contains  $\Gamma(N)$  with finite index for some  $N \in \mathbb{Z}_{>0}$ .

**Example.**  $\mathrm{SL}_2(\mathbb{Z})$  and  $\Gamma(N)$  are congruence subgroups. Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\},$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{N}, c \equiv 0 \pmod{N} \right\},$$

so  $\Gamma_1(N)$  is the preimage of

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \subseteq \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$$

in  $\mathrm{SL}_2(\mathbb{Z})$ . Then  $\Gamma_0(N)$  and  $\Gamma_1(N)$  are congruence subgroups such that

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

**Proposition 2.1.3.** Let  $\alpha \in \mathrm{GL}_2(\mathbb{Q})_+$ , and let  $\Gamma$  be a congruence subgroup. Then  $\alpha\Gamma\alpha^{-1}$  is also a congruence subgroup.

*Proof.* Need that there exists  $M$  with  $\Gamma(M) \subseteq \alpha\Gamma\alpha^{-1}$  with finite index. There exists  $N$  such that  $\Gamma(N) \subseteq \Gamma$ . Note that  $\Gamma(N) = \mathrm{SL}_2(\mathbb{Q}) \cap (\mathrm{I}_2 + N \mathrm{Mat}_2 \mathbb{Z})$ . Consider

$$\alpha\Gamma(N)\alpha^{-1} = \mathrm{SL}_2(\mathbb{Q}) \cap (\mathrm{I}_2 + N\alpha \mathrm{Mat}_2 \mathbb{Z}\alpha^{-1}).$$

Choose  $n \in \mathbb{Z}$  such that  $n\alpha$  and  $n\alpha^{-1}$  have entries in  $\mathbb{Z}$ . Then  $n^2\alpha^{-1} \mathrm{Mat}_2 \mathbb{Z}\alpha \subseteq \mathrm{Mat}_2 \mathbb{Z}$ , so  $n^2 \mathrm{Mat}_2 \mathbb{Z} \subseteq \alpha \mathrm{Mat}_2 \mathbb{Z}\alpha^{-1}$ , so  $Nn^2 \mathrm{Mat}_2 \mathbb{Z} \subseteq N\alpha \mathrm{Mat}_2 \mathbb{Z}\alpha^{-1}$ , so

$$\Gamma(n^2N) = \mathrm{SL}_2(\mathbb{Q}) \cap (\mathrm{I}_2 + Nn^2 \mathrm{Mat}_2 \mathbb{Z}) \subseteq \mathrm{SL}_2(\mathbb{Q}) \cap (\mathrm{I}_2 + N\alpha \mathrm{Mat}_2 \mathbb{Z}\alpha^{-1}) = \alpha\Gamma(N)\alpha^{-1}.$$

Similarly, show

$$\alpha\Gamma(n^4N)\alpha^{-1} \subseteq \Gamma(n^2N) \subseteq \alpha\Gamma(N)\alpha^{-1}.$$

Since  $\Gamma(n^4N)$  has finite index in  $\Gamma(N)$ ,  $\Gamma(n^2N)$  has finite index in  $\alpha\Gamma(N)\alpha^{-1}$ . □

**Note.** If  $T = \mathrm{lcm}(M, N)$  then  $\Gamma(T) \subseteq \Gamma(M) \cap \Gamma(N)$ , so the intersection of two congruence subgroups is a congruence subgroup.

**Example.** Let

$$\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\alpha^{-1} \mathrm{SL}_2(\mathbb{Z}) \alpha = \left\{ \begin{pmatrix} a & p^{-1}b \\ pc & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\},$$

and

$$\alpha^{-1} \mathrm{SL}_2(\mathbb{Z}) \alpha \cap \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \mid ad - pbc = 1 \right\} = \Gamma_0(p).$$

### 2.1.2 Modular forms

Recall that for  $f : \mathbb{H} \rightarrow \mathbb{C}$  and  $\alpha \in \mathrm{GL}_2(\mathbb{Q})_+$ , we defined  $f|_{k,\alpha}$  by

$$f|_{k,\alpha}(z) = \det \alpha^{k-1} f(\alpha z) (cz + d)^{-k}.$$

Suppose we have a  $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Q})$  and  $f : \mathbb{H} \rightarrow \mathbb{C}$  such that  $f|_{k,\gamma} = f$  for all  $\gamma \in \Gamma$ . Then if  $g = f|_{k,\alpha}$ , then  $g|_{k,\gamma} = g$  for all  $\gamma \in \alpha^{-1}\Gamma\alpha$ , since

$$(f|_{k,\alpha})|_{k,\gamma} = f|_{k,\gamma\alpha} = (f|_{k,\gamma})|_{k,\alpha} = f|_{k,\alpha}.$$

**Definition 2.1.4.** Fix  $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Q})$  a congruence subgroup. A function  $f : \mathbb{H} \rightarrow \mathbb{C}$  is a **weakly holomorphic or meromorphic modular form of weight  $k$  and level  $\Gamma$**  if

- $f|_{k,\gamma} = f$  for all  $\gamma \in \Gamma$ , and
- $f$  is holomorphic or meromorphic on  $\mathbb{H}$ .

A question is what condition should we impose at  $\infty$  to get a good theory?

**Example.** Let  $k \geq 4$  and  $N \in \mathbb{Z}$ , and let

$$E_k^{0,1}(z) = \sum_{(m,n) \in S^{0,1}} \frac{1}{(mz + n)^k}, \quad S^{0,1} = \{(m,n) \in \mathbb{Z}^2 \setminus \{0\} \mid m \equiv 1 \pmod{N}, n \equiv 0 \pmod{N}\}.$$

Claim that  $E_k(\gamma z) = E_k(z)$  for  $\gamma \in \Gamma(N)$ . Let  $\gamma \in \Gamma(N)$ . Then

$$\begin{aligned} E_k^{0,1}(\gamma z) &= \sum_{(m,n) \in S^{0,1}} \frac{1}{\left(m \left(\frac{az+b}{cz+d}\right) + n\right)^k} \\ &= (cz + d)^k \sum_{(m,n) \in S^{0,1}} \frac{1}{(m(az + b) + n(cz + d))^k} \\ &= (cz + d)^k \sum_{(m,n) \in S^{0,1}} \frac{1}{((ma + nc)z + (mb + nd))^k}, \end{aligned}$$

so  $m \equiv a \equiv d \equiv 1 \pmod{N}$  and  $n \equiv b \equiv c \equiv 0 \pmod{N}$ , so  $ma + nc \equiv 1 \pmod{N}$  and  $mb + nd \equiv 0 \pmod{N}$ . So  $(ma + nc, mb + nd) \in S^{0,1}$ . Moreover, the map

$$\begin{array}{ccc} S^{0,1} & \longleftrightarrow & S^{0,1} \\ (m,n) & \longmapsto & (ma + nc, mb + nd) \\ (m'a' + n'c', m'b' + n'd') & \longleftarrow & (m', n') \end{array}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \gamma^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

is a bijection. So

$$E_k^{0,1}(\gamma z) = E_k^{0,1}(z) (cz + d)^k.$$

Every congruence subgroup is conjugate to a subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ , but

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

need not be in  $\Gamma$ . On the other hand, if  $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ , then  $\Gamma$  has finite index in  $\mathrm{SL}_2(\mathbb{Z})$ , so there exists a minimal  $n_\Gamma > 0$  such that

$$\begin{pmatrix} 1 & n_\Gamma \\ 0 & 1 \end{pmatrix} \in \Gamma.$$

Then if  $f$  is weakly modular of weight  $k$  and level  $\Gamma$ , know  $f(z + n_\Gamma) = f(z)$  for all  $z$ , so  $f$  is a function of  $q^{1/n_\Gamma}$ . Let  $g(q^{1/n_\Gamma})$  be a function on  $\mathbb{D} \setminus \{0\}$  such that  $f(z) = g(e^{2\pi iz/n_\Gamma})$ . Then if  $g$  is meromorphic on  $\mathbb{D}$ , can express  $g$  as a Laurent series in  $q^{1/n_\Gamma}$ . We say  $f$  is **meromorphic at  $\infty$** , and the series for  $q$  is its  **$q$ -expansion**.

**Example.** For  $\Gamma = \Gamma_0(N)$  or  $\Gamma = \Gamma_1(N)$ ,  $n_\Gamma = 1$ . For  $\Gamma = \Gamma(N)$ ,  $n_\Gamma = N$ .





**2.1.4 The space of holomorphic modular forms**

Let

$$M_k(\Gamma) = \{\text{holomorphic modular forms of weight } k \text{ and level } \Gamma\},$$

and let

$$S_k(\Gamma) = \{f \in M_k(\Gamma) \mid f \text{ vanishes at all cusps}\}.$$

**Note.** For any  $\gamma \in \text{GL}_2(\mathbb{Q})_+$ , if  $f \in M_k(\Gamma)$ , then  $f|_{k,\gamma} \in M_k(\gamma^{-1}\Gamma\gamma)$ . If we consider the  $\mathbb{C}$ -vector space  $\widetilde{M}_k = \bigcup_{\Gamma} M_k(\Gamma)$ , then  $\gamma$  acts on  $\widetilde{M}_k$  by  $\gamma \cdot f = f|_{k,\gamma}$ . In fact,  $\text{GL}_2(\mathbb{Q})_+ \subseteq \text{GL}_2(\mathbb{A}_{\mathbb{Q}}^f)$  and the action extends to this larger group. If we enlarge  $\widetilde{M}_k$  in a suitable way, the correct group that acts is  $\text{GL}_2(\mathbb{A}_{\mathbb{Q}})$ .

A question is what can we say about  $\dim_{\mathbb{C}} M_k(\Gamma)$ ? Assume  $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ , and fix  $f \in M_k(\Gamma)$ . Write  $d = [\text{SL}_2(\mathbb{Z}) : \Gamma]$ , and write  $\text{SL}_2(\mathbb{Z}) = \bigsqcup_{j=1}^d \Gamma \cdot \alpha_j$ . Let

$$g = \prod_{j=1}^d f|_{k,\alpha_j}.$$

**Proposition 2.1.9.**  *$g$  is independent of the choice of  $\alpha_i$ .*

*Proof.* Suppose I replace  $\alpha'_j$  such that  $\Gamma \cdot \alpha_j = \Gamma \cdot \alpha'_j$ . Then there exists  $\gamma \in \Gamma$  such that  $\gamma\alpha_j = \alpha'_j$ , so

$$f|_{k,\alpha'_j} = \left(f|_{k,\alpha_j}\right)\Big|_{k,\alpha_j} = f|_{k,\alpha_j}.$$

So the product defining  $g$  does not change.  $\square$

**Proposition 2.1.10.**  *$g \in M_{kd}$ .*

*Proof.* For  $\alpha \in \text{SL}_2(\mathbb{Z})$ ,

$$g|_{kd,\alpha} = \prod_{j=1}^d \left(f|_{k,\alpha_j}\right)\Big|_{k,\alpha} = \prod_{j=1}^d f|_{k,\alpha_j\alpha}.$$

Since  $\text{SL}_2(\mathbb{Z}) = \bigsqcup_{j=1}^d \Gamma \cdot \alpha_j$ ,  $\text{SL}_2(\mathbb{Z}) \cdot \alpha = \bigsqcup_{j=1}^d \Gamma \cdot \alpha_j\alpha$ . So the elements  $\alpha_i\alpha$  are another set of coset representatives for  $\Gamma$  in  $\text{SL}_2(\mathbb{Z})$ . Since  $g$  was independent of the choice of representatives,  $g|_{kd,\alpha} = g$ .  $\square$

Have

$$\sum_{p \in \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}} \frac{1}{e_p} \text{ord}_p g = \frac{kd}{12}, \quad e_p = \begin{cases} \frac{1}{2} (\# \text{Stab}_{\text{SL}_2(\mathbb{Z})} p) & p \in \mathbb{H} \\ 1 & p \in \mathbb{Q} \cup \{\infty\} \end{cases},$$

so

$$\frac{kd}{12} = \sum_{p \in \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}} \frac{1}{e_p} \sum_{j=1}^d \text{ord}_p f|_{k,\alpha_j} = \sum_{p \in \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}} \frac{1}{e_p} \sum_{j=1}^d \text{ord}_{\alpha_j^{-1}p} f.$$

As  $p$  runs over a set of representatives for  $\text{SL}_2(\mathbb{Z})$ -orbits, and  $\alpha_j$  runs over the coset representatives for  $\Gamma$  in  $\text{SL}_2(\mathbb{Z})$ ,  $\alpha_j^{-1}p$  runs over the representatives for  $\Gamma$ -orbits, so

$$\sum_{q \in \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}} \frac{n_q}{e_q} \text{ord}_q g = \frac{kd}{12}, \quad n_q = \#\{j \mid \alpha_j^{-1}q \in \Gamma \cdot q\} \geq 1.$$

**Corollary 2.1.11.** *If  $\text{ord}_{\infty} f \geq kd/12n_{\infty} + 1$  for  $f \in M_k(\Gamma)$ , then  $f = 0$ .*

Then

$$\begin{aligned} n_{\infty} &= \#\{j \mid \alpha_j^{-1}\infty \in \Gamma \cdot \infty\} = \#\{j \mid \exists \gamma \in \Gamma, \alpha_j^{-1}\infty = \gamma\infty\} = \#\{j \mid \exists \gamma \in \Gamma, \alpha_j\gamma \in \text{Stab}_{\text{SL}_2(\mathbb{Z})}\infty\} \\ &= \#\{j \mid \alpha_j \in \text{Stab}_{\text{SL}_2(\mathbb{Z})}\infty\Gamma\} = \#\text{Stab}_{\text{SL}_2(\mathbb{Z})}\infty/\Gamma = \#\text{Stab}_{\text{SL}_2(\mathbb{Z})}\infty/\text{Stab}_{\Gamma}\infty, \end{aligned}$$

so  $f$  is a power series in  $q^{1/n_{\infty}}$ , and  $f$  is determined by its terms of order at most  $kd/12n_{\infty}$ . So  $f$  is determined by the first  $1 + kd/12$  terms of its  $q$ -expansion. Thus

$$\dim_{\mathbb{C}} M_k(\Gamma) \leq 1 + \frac{kd}{12}.$$

### 2.1.5 The space of meromorphic modular forms

Let  $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$  be a congruence subgroup. Let  $F_\Gamma$  be the field of meromorphic modular forms of weight zero and level  $\Gamma$ , and let  $F_N = F_{\Gamma(N)}$ , so  $F_1 = F_{\mathrm{SL}_2(\mathbb{Z})} = \mathbb{C}(j)$ . If  $M \mid N$ , then  $\Gamma(N) \subseteq \Gamma(M)$ , so  $F_M \subseteq F_N$ . Then  $\mathrm{SL}_2(\mathbb{Z})$  normalises  $\Gamma(N)$  so if  $f \in F_N$ , then  $f|_{0,\alpha}$  is modular for  $\alpha^{-1}\Gamma(N)\alpha = \Gamma(N)$  if  $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ .

**Note.**  $(fg)|_{0,\alpha} = f|_{0,\alpha} \cdot g|_{0,\alpha}$  and  $(f+g)|_{0,\alpha} = f|_{0,\alpha} + g|_{0,\alpha}$ .

Then  $\alpha \in \mathrm{SL}_2(\mathbb{Z})$  gives an automorphism of  $F_N$  fixing  $F_1$ . Get an action of  $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N)$  on  $F_N$  by field automorphisms and  $F_1$  is the fixed field.

**Theorem 2.1.12** (Galois theory). *Let  $F$  be a field and  $G$  a finite group acting faithfully on  $F$  by automorphisms, that is no  $g \in G$  acts on  $F$  as the identity except  $g = \mathrm{id}_G$ . Then  $F$  is a Galois extension of  $F^G = \{x \in F \mid \forall g \in G, gx = x\}$  with Galois group  $G$ . In particular  $[F : F^G] = \#G$ .*

**Proposition 2.1.13.**  $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$  acts faithfully on  $F_N$ .

*Proof.* Use the dimension formulae for  $M_k(\Gamma)$  to show that for  $k \gg 0$  even,  $\dim M_k(\Gamma(N)) > \dim M_k(\Gamma)$  for  $\Gamma \supsetneq \Gamma(N)$ , so there exists  $f \in M_k(\Gamma(N))$  such that the only elements of  $\mathrm{SL}_2(\mathbb{Z})$  fixing  $f$  lie in  $\Gamma(N)$ . Then  $f/E_k$  lies in  $F_N$  but not in  $F_\Gamma$  for  $\Gamma \supsetneq \Gamma(N)$ . So  $f/E_k$  is not fixed by non-trivial elements of  $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N)$ .  $\square$

The upshot is that  $F_N/F_1$  is Galois with Galois group  $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ . Then  $F_N$  is a finite and algebraic extension of  $\mathbb{C}(j)$ , of transcendence degree one over  $\mathbb{C}$ . For  $\Gamma$  arbitrary in  $\mathrm{SL}_2(\mathbb{Z})$ ,  $\Gamma \supseteq \Gamma(N)$  for some  $N$ , so  $F_\Gamma$  is the fixed field of  $\Gamma/\Gamma(N)$  in  $F_N$ , and  $F_\Gamma/F_1$  is not Galois in general, but is algebraic of degree  $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ .

**Proposition 2.1.14.** *There exists a unique smooth and projective algebraic curve  $X(\Gamma)$  over  $\mathbb{C}$ , whose field of rational functions is  $F_\Gamma$ .*

*Proof.* Fix  $\Gamma$ , and let  $f$  be a primitive element of  $F_\Gamma$ , that is  $f$  generates  $F_\Gamma$  over  $F_1$ . Consider the polynomial

$$\begin{aligned} P(X) &= \prod_{\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_j \Gamma \cdot \alpha_j} (X - f|_{0,\alpha_j}) \in F_1[X] \\ &= X^d + \frac{G_1(j)}{H_1(j)} X^{d-1} + \cdots + \frac{G_d(j)}{H_d(j)}, \quad G_i, H_i \in \mathbb{C}[Y]. \end{aligned}$$

Let

$$Q(X, Y) = H_1(Y) \cdots H_d(Y) \left( X^d + \frac{G_1(Y)}{H_1(Y)} X^{d-1} + \cdots + \frac{G_d(Y)}{H_d(Y)} \right) \in \mathbb{C}[X, Y].$$

Then  $Q(X, j) = H_1(j) \cdots H_d(j) \cdot P(X)$ . Since  $P(f) = 0$ ,  $Q(f, j) = 0$ . Consider the map

$$\begin{aligned} \phi : \mathbb{H} &\longrightarrow \mathbb{C}^2 \\ z &\longmapsto (f(z), j(z)) \end{aligned}$$

The image is contained in the zero locus of  $Q(X, Y)$ , and factors through  $\Gamma \backslash \mathbb{H}$ . The following are some issues.

- This map is not necessarily defined everywhere. To fix, replace  $\mathbb{C}^2$  with  $\mathbb{CP}^2$ . Then  $\phi$  extends to  $\Gamma \backslash \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \rightarrow \mathbb{CP}^2$ .
- This map is not necessarily injective on  $\Gamma \backslash \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ , but will be generically injective since  $f$  is primitive.
- This image might be singular. There are standard ways to fix, such as normalisation. When these are fixed, the map becomes injective.

The upshot is to get a complex algebraic curve  $X(\Gamma)$  whose function field is  $F_\Gamma$ , whose complex points are in bijection with  $\Gamma \backslash \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ .  $\square$

$M_k(\Gamma)$  is the space of sections of certain line bundles on  $X(\Gamma)$ .