M4P54 Differential Topology

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Syllabus

Contents

0   Introduction

Differential topology is the study of the topology of a manifold using analysis. The topics are

Lecture 1
Thursday
09/01/20

- a review of differential forms,

- de Rham cohomology,

- Morse theory, and

- singular homology.

The following are references.

- J M Lee, Introduction to smooth manifolds, 2000

- L W Tu, Introduction to smooth manifolds, 2008

- J Milnor, Morse theory, 1960

- A Banyaga and D Hurtubise, Lectures on Morse homology, 2004

1   Differential forms on manifolds

1.1   Alternating p-forms on a vector space

Let V be a vector space over $\mathbb{R}$, and let $p \geq 0$. Then $V^p = V \times \cdots \times V$.

Definition 1.1. A multilinear map $\omega : V^p \to \mathbb{R}$ is called an **alternating p-form** if we have

$$\omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(p)}\right) = \epsilon\left(\sigma\right) \omega\left(v_1, \ldots, v_p\right), \qquad v_1, \ldots, v_p \in V \qquad \sigma \in \mathcal{S}_p,$$

where $\mathcal{S}_p$ is the group of permutations of p elements and $\epsilon\left(\sigma\right)$ is the signature of σ.

Recall that if m is the number of transpositions in a decomposition of σ, then $\epsilon\left(\sigma\right) = \left(-1\right)^m$, where a **transposition** is $(a_i a_j)$ for $a_i \neq a_j$.

Notation 1.2. $\Lambda^p V^* = \{\text{alternating } p\text{-forms } \omega \text{ on } V\}$ is called the **p-th exterior power** of V.

Check that it is a vector space. [1]

Example 1.3.

- $\Lambda^0 V^* = \mathbb{R}$.

- $\Lambda^1 V^* = V^* = \mathrm{Hom}\left(V, \mathbb{R}\right)$, the **dual** of V.

Definition 1.4. Let $\omega_1 \in \Lambda^p V^*$ and $\omega_2 \in \Lambda^q V^*$. We define the **exterior product** $\omega_1 \wedge \omega_2 \in \Lambda^{p+q} V^*$ of ω_1 and ω_2 by

$$\omega_1 \wedge \omega_2\left(v_1, \ldots, v_{p+q}\right) = \sum_{\sigma \in \mathcal{S}_{p,q}} \epsilon\left(\sigma\right) \omega_1\left(v_{\sigma(1)}, \ldots, v_{\sigma(p)}\right) \omega_2\left(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}\right), \qquad v_1, \ldots, v_{p+q} \in V,$$

where

$$\mathcal{S}_{p,q} = \left\{\sigma \in \mathcal{S}_{p+q} \mid \sigma\left(1\right) < \cdots < \sigma\left(p\right), \ \sigma\left(p+1\right) < \cdots < \sigma\left(p+q\right)\right\}.$$

Example 1.5.

- Assume $\omega_1, \omega_2 \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \omega_2\left(v_1, v_2\right) = \omega_1\left(v_1\right) \omega_2\left(v_2\right) - \omega_1\left(v_2\right) \omega_2\left(v_1\right), \qquad v_1, v_2 \in V.$$

- Assume $\omega_1, \ldots, \omega_p \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \cdots \wedge \omega_p\left(v_1, \ldots, v_p\right) = \det\left(\omega_i\left(v_j\right)\right)_{i,j=1,\ldots,p}, \qquad v_1, \ldots, v_p \in V.$$

Proposition 1.6. *Let $\omega_i \in \Lambda^{p_i} V^*$ for $i = 1, 2, 3$.*

- *Associativity $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$.*

- *Distributivity $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$, assuming $p_2 = p_3$.*

- *Supercommutativity $\omega_1 \wedge \omega_2 = (-1)^{p_1 \cdot p_2} \omega_2 \wedge \omega_1$.*

Definition 1.7. Let $\Phi : V \to W$ be a linear map between vector spaces over $\mathbb{R}$. Let $\omega \in \Lambda^p W^*$. Then the **pull-back** $\Phi^* \omega \in \Lambda^p V^*$ of ω is an alternating p-form on V defined by

$$\Phi^* \omega\left(v_1, \ldots, v_p\right) = \omega\left(\Phi\left(v_1\right), \ldots, \Phi\left(v_p\right)\right), \qquad v_1, \ldots, v_p \in V.$$

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[1]Exercise

Proposition 1.8. *Given $\Phi : V \to W$ a linear map,*

- *the pull-back*

$$\Phi^* \quad : \quad \Lambda^p W^* \quad \longrightarrow \quad \Lambda^p V^*$$
$$\omega \quad \longmapsto \quad \Phi^* \omega$$

is a linear map that preserves exterior products, that is

$$\Phi^* (\omega_1 \wedge \omega_2) = \Phi^* \omega_1 \wedge \Phi^* \omega_2, \qquad \omega_1 \in \Lambda^p W^*, \qquad \omega_2 \in \Lambda^q W^*,$$

- *if $\Psi : W \to Z$ is linear then*

$$(\Psi \circ \Phi)^* \omega = \Phi^* \Psi^* \omega, \qquad \omega \in \Lambda^p Z^*,$$

- *assuming $V = W$ and $p = \dim V$, then*

$$\Phi^* \omega = (\det \Phi) \, \omega, \qquad \omega \in \Lambda^p V^*.$$

1.2 Differential forms on manifolds

Let M be a smooth manifold of dimension n, and let $x \in M$. Then the tangent space $\mathrm{T}_x M$ of M at x is a vector space of dimension n.

Notation 1.9. Let

$$\Lambda^p \mathrm{T}_x^* M = \Lambda^p (\mathrm{T}_x M)^* .$$

Consider the set

$$\Lambda^p \mathrm{T}^* M = \bigsqcup_{x \in M} \Lambda^p \mathrm{T}_x^* M,$$

the *p*-**th exterior bundle** on M. There exists a morphism $\pi : \Lambda^p \mathrm{T}^* M \to M$ such that for all $x \in M$, $\pi^{-1}(x) = \Lambda^p \mathrm{T}_x^* M$, so $\Lambda^p \mathrm{T}^* M$ is a vector bundle and it is a smooth manifold, and π is a smooth morphism.

Example 1.10.

- $\Lambda^0 \mathrm{T}^* M = M \times \mathbb{R}$.

- $\Lambda^1 \mathrm{T}^* M$ is the **cotangent bundle**, the dual of the tangent bundle.

Definition 1.11. A **differential p-form** ω on M is a smooth section of π. That is, it is a smooth morphism $\omega : M \to \Lambda^p \mathrm{T}^* M$ such that $\pi \circ \omega = \mathrm{id}_M$.

Thus, $\omega(x) \in \Lambda^p \mathrm{T}_x^* M$.

Lecture 2
Monday
13/01/20

Notation 1.12.

$$\Omega^p (M) = \{\text{differential } p\text{-forms } \omega \text{ on } M\}, \qquad \Omega^\bullet (M) = \bigoplus_p \Omega^p (M).$$

Example 1.13.

$$\Omega^0 (M) \cong \{f : M \to \mathbb{R} \; C^\infty\text{-function}\}.$$

Exercise. If $n = \dim M$, then $\Omega^{n+1} (M) = 0$.

The algebra is the same as last week.

Definition 1.14. Let $\omega_1 \in \Omega^p (M)$ and $\omega_2 \in \Omega^q (M)$. Then $\omega_1 \wedge \omega_2 \in \Omega^{p+q} (M)$ is defined by

$$\omega_1 \wedge \omega_2 (x) = \omega_1 (x) \wedge \omega_2 (x) \in \Lambda^{p+q} \mathrm{T}_x^* M, \qquad x \in M.$$

By Proposition 1.6, associativity, distributivity, and supercommutativity hold for $\Omega^p (M)$. Let $F : M \to N$ be a smooth morphism between manifolds. Then for all $x \in M$, the differential of F at x is the linear map

$$\mathrm{D}F_x : \mathrm{T}_x M \to \mathrm{T}_{F(x)} N.$$

Thus, for all $p \geq 0$, we have a natural map, called the **pull-back**,

$$F_x^* : \begin{array}{ccc} \Lambda^p T^*_{F(x)} N & \longrightarrow & \Lambda^p T^*_x M \\ \omega(v_1, \ldots, v_p) & \longmapsto & \omega(DF_x(v_1), \ldots, DF_x(v_p)) \end{array}, \qquad \omega \in \Lambda^p T^*_{F(x)} N, \qquad v_1, \ldots, v_p \in T^*_x M.$$

Thus, we can define

$$F^* : \begin{array}{ccc} \Omega^p(N) & \longrightarrow & \Omega^p(M) \\ \omega(x) & \longmapsto & F^*\omega(F(x)) \end{array}, \qquad \omega \in \Omega^p(N).$$

By Proposition 1.8, the pull-back preserves the exterior product, so

$$F^*(\omega_1 \wedge \omega_2) = F^*\omega_1 \wedge F^*\omega_2.$$

If $G : N \to P$,

$$(G \circ F)^* \omega = F^* G^* \omega.$$

1.3 Local description of p-forms

Let M be a manifold of dimension n, let $x_0 \in M$, let (U, ϕ) be a local chart around x_0, and let $(x_1, \ldots, x_n)$ be local coordinates around x_0. A basis of $T_{x_0} M$ is given by

$$\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}.$$

A basis of $T^*_{x_0} M$ is given by

$$\{dx_1, \ldots, dx_n\}, \qquad dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}.$$

A basis of $\Lambda^p T^*_{x_0} M$ is

$$dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \qquad i_1 < \cdots < i_p.$$

Thus, $\omega \in \Omega^p(M)$ is locally given by

$$\omega(x) = \sum_{|I|=p} f_I(x) \, dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \qquad I = (i_1, \ldots, i_p), \qquad i_1 < \cdots < i_p,$$

where f_I is a C^∞-function on U for all I.

Example 1.15. Let $F : M \to N$ be a smooth morphism between manifolds of dimension n, and let $\omega \in \Omega^n(N)$. Locally,

$$\omega(y) = f(y) \, dy_1 \wedge \cdots \wedge dy_n, \qquad y \in N,$$

for some $f \in C^\infty$. Proposition 1.8 implies that

$$F^*\omega(x) = (f \circ F)(x) \det DF_x \, dx_1 \wedge \cdots \wedge dx_n, \qquad x \in M,$$

where $y_i = p_i \circ F$ and $p_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th projection.

Let $f : M \to \mathbb{R}$ be a smooth function, so $f \in \Omega^0(M)$. Locally, the **differential** is

$$d : \begin{array}{ccc} \Omega^0(M) & \longrightarrow & \Omega^1(M) \\ f & \longmapsto & \sum_{i=1}^n \frac{\partial f}{\partial x_i} \, dx_i \end{array}.$$

Check that $df \in \Omega^1(M)$, so df is a 1-form on M. Alternatively, $df = f^* dx$ for dx a 1-form on $\mathbb{R}$, or $df(X) = X(f)$ for any vector field X on M. More in general, let $\omega \in \Omega^p(M)$. Locally,

$$\omega = \sum_{|I|=p} f_I dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \qquad f_I \in C^\infty,$$

so $d\omega \in \Omega^{p+1}(M)$. Then the **de Rham differential** is

$$d : \begin{array}{ccc} \Omega^p(M) & \longrightarrow & \Omega^{p+1}(M) \\ \omega & \longmapsto & \sum_{|I|=p} df_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \end{array}.$$

Proposition 1.16.

- *The Leibnitz rule*

$$\mathrm{d}\left(\omega_1 \wedge \omega_2\right) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \mathrm{d}\omega_2, \qquad w_1 \in \Omega^p\left(M\right), \qquad \omega_2 \in \Omega^q\left(M\right).$$

- $\mathrm{d}^2 = 0$, *that is*

$$\mathrm{d}\left(\mathrm{d}\omega\right) = 0, \qquad \omega \in \Omega^p\left(M\right).$$

- *Let $F : M \to N$ be a smooth morphism between manifolds. Then*

$$F^* \mathrm{d}\omega = \mathrm{d}\left(F^* \omega\right), \qquad \omega \in \Omega^p\left(M\right)$$

  so

$$
\begin{array}{ccc}
\Omega^p\left(M\right) & \xrightarrow{\ \mathrm{d}\ } & \Omega^{p+1}\left(M\right) \\
{\scriptstyle F^*}\big\uparrow & & \big\uparrow{\scriptstyle F^*} \\
\Omega^p\left(N\right) & \xrightarrow[\ \mathrm{d}\ ]{} & \Omega^{p+1}\left(N\right)
\end{array}
\quad .
$$

Definition 1.17.

- $\omega \in \Omega^p\left(M\right)$ is **closed** if $\mathrm{d}\omega = 0$.

- $\omega \in \Omega^p\left(M\right)$ is **exact** if there exists $\omega' \in \Omega^{p-1}\left(M\right)$ such that $\mathrm{d}\omega' = \omega$.

ω is exact implies that ω is closed, since if $\omega = \mathrm{d}\omega'$ then $\mathrm{d}\omega = \mathrm{d}^2\omega' = 0$.

1.4 Integrations on manifolds

Let M be a manifold of dimension n, let $F : M \to M$ be a smooth morphism, and let $\omega \in \Omega^n\left(M\right)$. Then

$$F^*\omega\left(x\right) = \det \mathrm{D}F_x \omega\left(F\left(x\right)\right).$$

Locally, assume $\omega = f\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$ for some coordinates $y_1, \ldots, y_n$ and $f \in \mathrm{C}^\infty$. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of M, where $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$. Then

$$h_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha\left(U_\alpha \cap U_\beta\right) \subset \mathbb{R}^n \to \phi_\beta\left(U_\alpha \cap U_\beta\right) \subset \mathbb{R}^n,$$

such that

$$h_{\alpha\beta}^*\omega\left(x\right) = \left(f \circ h_{\alpha\beta}\right)\left(x\right)\det\left(\mathrm{D}h_{\alpha\beta}\right)_x \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

Let $D \subset \mathbb{R}^n$ be compact such that ∂D has zero measure, so D is a domain of integration, let $f : U \to \mathbb{R}$ be a C^∞-function where $U \subset \mathbb{R}^n$ is open such that $D \subset U$, and let $h : U \to h\left(U\right)$ be a diffeomorphism. Then

$$\int_{h^{-1}(D)} f\left(y\right) \mathrm{d}y_1 \ldots \mathrm{d}y_n = \int_{h^{-1}(D)} f\left(y\right) \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n = \int_D \left(f \circ h\right)\left(x\right)|\det \mathrm{D}h_x| \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

Let us assume that $\omega = f\left(y\right)\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$ on U. We define

$$\int_D \omega = \int_D f\left(y\right) \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n, \qquad D \subset U.$$

Definition 1.18. Let $U \subset \mathbb{R}^n$ be an open set. We define the **support** of ω as

$$\mathrm{supp}\,\omega = \overline{\{x \in U \mid \omega\left(x\right) \neq 0\}}, \qquad \omega\left(x\right) \in \Lambda^p \mathrm{T}_x^* U.$$

Then ω has **compact support**, if $\mathrm{supp}\,\omega$ is compact.

Fact. Under this assumption, we can define

$$\int_U \omega = \int_D \omega \in \mathbb{R},$$

which is well-defined. Under the same assumption, if $\phi : V \to U$ is a diffeomorphism, provided that $\det \mathrm{D}\phi_x > 0$, since $\det \mathrm{D}\phi_x \neq 0$ for all x, then

$$\int_U \omega = \int_V \phi^*\omega.$$

1.5 Orientation

Let V be a vector space over $\mathbb{R}$ of dimension n, and let $B = (b_1, \ldots, b_n) \subset V$ and $B' = (b'_1, \ldots, b'_n) \subset V$ be ordered bases of V. Then B and B' have the **same orientation** if $\det T > 0$ where

$$
\begin{array}{rrcl}
T & : & V & \longrightarrow & V \\
& & b_i & \longmapsto & b'_i
\end{array}
$$

is a linear map. Let $\omega \in \Lambda^n V^*$ for $\omega \neq 0$. Then B and B' have the same orientation if and only if $\omega(b_1, \ldots, b_n)$ has the same sign as $\omega(b'_1, \ldots, b'_n)$, by Proposition 1.8. An **orientation** Λ of V is a set of all the ordered basis of V with the same orientation. Let $\phi : V \to W$ be an isomorphism of vector spaces with fixed orientations Λ_v and Λ_w respectively. We say that ϕ is **orientation preserving** if an ordered basis of V induces an ordered basis of W, so Λ_v induces Λ_w. Let $V = \mathbb{R}^n$, and let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. Then $e_1, \ldots, e_n$ defines an orientation of V called **positive**. Let M be a manifold. The idea is to find an orientation Λ_x of $\mathrm{T}_x M$ for all $x \in M$.

Special case. Let $M = U \subset \mathbb{R}^n$ be open. There exists a natural isomorphism $\phi_x : \mathrm{T}_x U \to \mathbb{R}^n$. Let Λ_x^+ be an orientation on $\mathrm{T}_x U$ such that ϕ_x is orientation preserving with respect to the positive orientation on $\mathbb{R}^n$. Let $\Lambda^+ = \{\Lambda_x^+\}$.

General case. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M. On U_α, we define the orientation so that $(\mathrm{D}\phi_\alpha)_x : \mathrm{T}_x U_\alpha \to \mathrm{T}_{\phi_\alpha(x)} \phi_\alpha(U) \subset \mathbb{R}^n$ is orientation preserving. This is called the positive orientation on the chart (U_α, ϕ_α). We define Λ on M, which is a collection of Λ^+ on $\mathrm{T}_x M$ for all $x \in M$. Then M is **orientable** if there exists an atlas with positive orientation charts. This coincides in assuming that $\det \mathrm{D}\left(\phi_\beta^{-1} \circ \phi_\alpha\right) > 0$ for all α and β.

Notation 1.19. For all $p \geq 0$,

$$
\Omega_c^p(M) = \{\omega \in \Omega^p(M) \mid \operatorname{supp} M \text{ is compact}\}.
$$

If M is compact $\Omega_c^p(M) = \Omega^p(M)$. Let $\omega \in \Omega_c^r(M)$. Assume $\operatorname{supp} \omega \subset U$ where (U, ϕ) is a chart of M, and $\phi : U \to \phi(U) \subset \mathbb{R}^n$. Assume also that (U, ϕ) is positively oriented. Let $\phi^{-1} : \phi(U) \to U$ such that $\left(\phi^{-1}\right)^* \omega \in \Omega_c^n(\phi(U))$, that is $\operatorname{supp}\left(\phi^{-1}\right)^* \omega \subset \phi(U)$. We define

$$
\int_M \omega = \int_{\phi(U)} \left(\phi^{-1}\right)^* \omega. \tag{1}
$$

We need to show that, under the assumptions above, $\int_M \omega$ does not depend on (U, ϕ). Let $\left(\overline{U}, \overline{\phi}\right)$ be also a positively oriented chart such that $\operatorname{supp} \omega \subset \overline{U}$. We want to show that

$$
\int_{\phi(U)} \left(\phi^{-1}\right)^* \omega = \int_{\overline{\phi}(\overline{U})} \left(\overline{\phi}^{-1}\right)^* \omega.
$$

Let $\overline{\phi} \circ \phi^{-1} : \phi\left(U \cap \overline{U}\right) \to \overline{\phi}\left(U \cap \overline{U}\right)$, so

$$
\begin{array}{ccc}
& U \cap \overline{U} & \\
{}^{\phi}\swarrow & & \searrow^{\overline{\phi}} \\
\mathbb{R}^n \supset \phi\left(U \cap \overline{U}\right) \xrightarrow{\ \overline{\phi} \circ \phi^{-1}\ } & & \overline{\phi}\left(U \cap \overline{U}\right) \subset \mathbb{R}^n
\end{array} \quad .
$$

Since both charts are positively oriented the determinant of the differential $\mathrm{D}\left(\overline{\phi} \circ \phi^{-1}\right)$ is positive, so

$$
\int_{\overline{\phi}(U)} \left(\overline{\phi}^{-1}\right)^* \omega = \int_{\overline{\phi}(U \cap \overline{U})} \left(\overline{\phi}^{-1}\right)^* \omega = \int_{\overline{\phi}(U \cap \overline{U})} \left(\overline{\phi} \circ \phi^{-1}\right)^* \left(\overline{\phi}^{-1}\right)^* \omega = \int_{\overline{\phi}(U \cap \overline{U})} \left(\phi^{-1}\right)^* \overline{\phi}^* \left(\overline{\phi}^{-1}\right)^* \omega
$$

$$
= \int_{\overline{\phi}(U \cap \overline{U})} \left(\phi^{-1}\right)^* \left(\overline{\phi}^{-1} \circ \overline{\phi}\right)^* \omega = \int_{\overline{\phi}(U \cap \overline{U})} \left(\phi^{-1}\right)^* \omega = \int_{\overline{\phi}(U)} \left(\phi^{-1}\right)^* \omega,
$$

by a property of the pull-back and since $\left(\overline{\phi}^{-1}\right)^* \omega = 0$ outside $\overline{\phi}\left(U \cap \overline{U}\right)$.

1.6   Partitions of unity

Definition 1.20. Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering. A **partition of unity** with respect to U is a collection of smooth functions $f_\alpha : M \to [0,1]$ such that

1. $\operatorname{supp} f_\alpha = \overline{\{x \in M \mid f_\alpha(x) = 0\}} \subset U_\alpha$ for all α,

2. $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$, and

3. for all $x \in M$, there exists $U \ni x$ open such that $\operatorname{supp} f_\alpha \cap U \neq \emptyset$ for only finitely many α.

Remark. 3 implies that 2 is a finite sum.

Example 1.21. Let

$$M = \mathrm{S}^1 = \left\{ x \in \mathbb{R}^2 \mid |x| = 1 \right\}, \qquad U_1 = \mathrm{S}^1 \setminus \{(1,0)\}, \qquad U_2 = \mathrm{S}^1 \setminus \{(-1,0)\},$$

so $\{U_i\}$ is a cover. Let

$$f_1(\cos\theta, \sin\theta) = \frac{1}{2} - \frac{1}{2}\cos\theta, \qquad f_2(\cos\theta, \sin\theta) = \frac{1}{2} + \frac{1}{2}\cos\theta.$$

Then f_i is a partition of unity.

Theorem 1.22. *Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering of M. Then there exists a partition of unity f_α with respect to U.*

Proof. We omit the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 1.23. *Let M be a manifold, and let $n = \dim M$. Then M is orientable if and only if there exists $\omega \in \Omega^n(M)$ which is never vanishing on M, so $\omega(x) \neq 0$ for all $x \in M$.*

ω is called a **volume form** on M.

Proof.

$\Longleftarrow$ Assume $\omega \in \Omega^n(M)$ is a volume form. We want to construct an orientation Λ on M, that is Λ_x on $\mathrm{T}_x M$ for all $x \in M$. Given an oriented basis $v_1, \dots, v_n$ of $\mathrm{T}_x M$ we say that it is **positively oriented** if $\omega(x)(v_1, \dots, v_n) > 0$. For all $x \in M$, we define the orientation Λ_x on $\mathrm{T}_x M$ by considering the class of positively oriented ordered basis of $\mathrm{T}_x M$ which is compatible with the choice of an atlas on M. Take any atlas $\{(U_\alpha, \phi_\alpha)\}$, where $\phi_\alpha : U_\alpha \to \mathbb{R}^n$. On U_α,

$$\omega = g_\alpha \phi_\alpha^* \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

Since $\omega \neq 0$, $g_\alpha > 0$ or $g_\alpha < 0$. If $g_\alpha < 0$ then switch x_1 with x_2, so $g_\alpha > 0$. After this change of coordinates, (U_α, ϕ_α) is positively oriented, so M is orientable.

$\Longrightarrow$ Assume that M is orientable, that is there exists an atlas $\{(U_\alpha, \phi_\alpha)\}$ of positively oriented charts. On U_α, we consider

$$\omega_\alpha = \phi_\alpha^* \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Let $\widetilde{\omega_\alpha} = f_\alpha \omega_\alpha \in \Omega^n(U_\alpha)$. We may assume that $\widetilde{\omega_\alpha} \in \Omega^n(M)$ by extending equal to zero outside U_α. We define $\omega = \sum_\alpha \widetilde{\omega_\alpha} \in \Omega^n(M)$. For all α, since $\sum_\alpha f_\alpha = 1$ there exists α such that $\widetilde{\omega_\alpha} \neq 0$, so $\omega \neq 0$.

$\qquad\square$

Let M be an orientable manifold of dimension n, and let $\omega \in \Omega_{\mathrm{c}}^n(M)$. We want to define $\int_M \omega$. So far we defined for ω such that $\operatorname{supp} \omega \subset U_\alpha$ where (U_α, ϕ_α) is a chart.

Definition 1.24. Let $\{(U_\alpha, \phi_\alpha)\}$ be a positively oriented atlas on M, and let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $\operatorname{supp} f_\alpha \omega \subset U_\alpha$, so let

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega.$$

Remark 1.25. Note that for each α, we have that the support of $f_\alpha \omega$ is contained in U_α and therefore each term of the sum is well-defined as in (1). Indeed, we have

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} \left(\phi^{-1}\right)^* f_\alpha.$$

Lemma 1.26. $\int_M \omega$ *does not depend on* $\{(U_\alpha, \phi_\alpha)\}$ *and* f_α.

Proof. Under the assumption that $\operatorname{supp} \omega \subset U_\alpha$ then we showed $\int_{U_\alpha} \omega$ does not depend on (U_α, ϕ_α). Let $\{(U_\alpha, \phi_\alpha)\}$ and $\{(\overline{U_\alpha}, \overline{\phi_\alpha})\}$ be two atlases with positively oriented charts, and let f_α and $\overline{f_\alpha}$ be two partitions of unity with respect to $\{U_\alpha\}$ and $\{\overline{U_\alpha}\}$ respectively. Then $\sum_\alpha f_\alpha = \sum_\alpha \overline{f_\alpha} = 1$, so $\int_M f_\alpha \omega = \sum_\beta \int_M \overline{f_\beta} f_\alpha \omega$. Thus

$$\int_M \omega = \sum_\alpha \int_M f_\alpha \omega = \sum_{\alpha,\beta} \int_M \overline{f_\beta} f_\alpha \omega = \sum_\beta \int_M \sum_\alpha f_\alpha \overline{f_\beta} \omega = \sum_\beta \int_M \overline{f_\beta} \omega.$$

$\square$

Proposition 1.27. *Let M and N be orientable manifolds of dimension n, and let $\omega, \eta \in \Omega^n_c(M)$.*

1. *Linearity*

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

2. *Orientation reversal. Let $\overline{M}$ be the manifold M with opposite orientation $\Lambda^- = \{\Lambda_x^- \mid x \in M\}$, which is the orientation opposite than the one induced by M with orientation Λ. Then*

$$\int_M \omega = -\int_{\overline{M}} \omega.$$

3. *Positivity. Let ω be the volume form on M. Then*

$$\int_M \omega > 0.$$

4. *Diffeomorphism invariance. Let $F : N \to M$ be an orientation preserving diffeomorphism. Then*

$$\int_M \omega = \int_N F^* \omega.$$

Proof.

1. Exercise. [2]

2. Exercise. [3]

3. Choose a positively oriented chart (U_α, ϕ_α) on U_α, so

$$\omega = g_\alpha \phi_\alpha^* \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n, \qquad g_\alpha > 0.$$

   Then $\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega$ where f_α is a partition of unity. For all $x \in M$ there exists α such that $x \in U_\alpha$ and $\int_{U_\alpha} f_\alpha \omega > 0$, so $\int_M \omega > 0$.

4. Let (U_α, ϕ_α) be a positively oriented atlas on M. Then $\left(F^{-1}(U_\alpha), \phi_\alpha \circ F\right)$ is an atlas on N which is positively oriented. Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $f_\alpha \circ F$ is a partition of the unity with respect to $\{F^{-1}(U_\alpha)\}$, so

$$\int_N F^* \omega = \sum_\alpha \int_N (f_\alpha \circ F) F^* \omega = \sum_\alpha \int_N F^* (f_\alpha \omega) = \sum_\alpha \int_M f_\alpha \omega = \int_M \omega.$$

$\square$

---

[2]Exercise
[3]Exercise

1.7   Manifolds with boundary

Denote
$$\mathbb{R}^n_{\geq 0} = (\mathbb{R}_{\geq 0})^n, \qquad \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Let $U \subset \mathbb{R}^n_+$ be open, and let $F : U \to \mathbb{R}^m$ be a function. Then F is C^∞ if it can be extended to a C^∞-function $\widetilde{F} : \widetilde{U} \to \mathbb{R}^m$ where $\widetilde{U} \supset U$ and $\widetilde{U}$ is open.

Definition 1.28. A **manifold with boundary** of dimension n is a Hausdorff topological space M such that there exists an open covering $\{U_\alpha\}$, and for all α, there exists a homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ such that for all α and β,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta (U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+ \to \phi_\alpha (U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+$$

is a diffeomorphism, so

The **boundary** of M is

$$\partial M = \left\{ x \in M \mid \exists \alpha, \ \phi_\alpha (x) \in \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\} \right\}.$$

Then (U_α, ϕ_α) is called a **chart** and $\{(U_\alpha, \phi_\alpha)\}$ is called an **atlas**.

Remark 1.29.

- ∂M is closed in M.

- $\mathring{M} = M \setminus \partial M$ is a manifold of dimension n.

Example 1.30.

- $M = [0, 1]$ is a manifold with boundary $\partial M = \{0, 1\}$.

- The closed disc $D = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is a manifold with boundary $\partial D = S^{n-1}$.

- $M = [0, 1] \times S^1$ is a manifold with boundary $\partial M = S^1 \sqcup S^1$.

Remark 1.31.

- We can define tangent spaces and differential forms exactly in the same way as usual manifolds.

- The definition of orientability is the same. If M is orientable, then ∂M is also orientable. As a convention, the positive orientation on the boundary of $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$ is given by $(-1)^n \, dx_1 \wedge \cdots \wedge dx_{n-1}$. This induces a positive orientation on ∂M.

- Also partitions of unity for any open cover U_α of M is defined the same way. If M is orientable, for any manifold with boundary, for all open covering $U = \{U_\alpha\}$, there exists a partition of unity f_α. This implies that if $\omega \in \Omega^n_c (M)$, then $\int_M \omega$ is defined the same way for manifolds.

1.8   Stokes' theorem

Theorem 1.32 (Stokes)**.** *For any manifold with boundary M of dimension n, and for any $\omega \in \Omega^{n-1}_c (M)$ we have*

$$\int_M d\omega = \int_{\partial M} \omega \in \Omega^n_c (M).$$

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas, and let $f_\alpha : M \to \mathbb{R}$ be a partition of unity with respect to this cover. Then $\sum_\alpha f_\alpha = 1$ on M, so

$$\int_M \mathrm{d}\omega = \int_M \mathrm{d}\left(\sum_\alpha f_\alpha \omega\right) = \sum_\alpha \int_M \mathrm{d}(f_\alpha \omega) = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} \left(\phi_\alpha^{-1}\right)^* \mathrm{d}\left(f_\alpha \omega\right).$$

Proposition 1.16 implies that

$$\left(\phi_\alpha^{-1}\right)^* \mathrm{d}\left(f_\alpha \omega\right) = \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right).$$

Then $\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)$ is an $(n-1)$-form on $\phi_\alpha(U_\alpha)$. In coordinates,

$$\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right) = \sum_{j=1}^{n} \widetilde{f_\alpha} \omega_j \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_n,$$

where ω_j is a smooth function on $\phi_\alpha(U_\alpha)$ and

$$
\begin{array}{ccc}
U_\alpha & \xrightarrow{\widetilde{\phi_\alpha}} & \phi_\alpha(U_\alpha) \\
{\scriptstyle f_\alpha}\downarrow & \swarrow {\scriptstyle \widetilde{f_\alpha}} & \\
[0,1] & &
\end{array}
\quad .
$$

Then

$$\mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right) = \mathrm{d}\left(\sum_{j=1}^{n} \widetilde{f_\alpha} \omega_j \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_n\right)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_k \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_n$$

$$= \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_j \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_n$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n,$$

so

$$\sum_\alpha \int_{\phi_\alpha(U_\alpha)} \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right) = \sum_\alpha \int_{\mathbb{R}^n_+} \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right),$$

because $\widetilde{f_\alpha} = 0$ outside $\phi_\alpha(U_\alpha)$. Thus

$$\int_M \mathrm{d}\omega = \sum_\alpha \int_{\mathbb{R}^n_+} \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

$$= \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_n \, \mathrm{d}x_{n-1} \ldots \mathrm{d}x_1$$

$$= \sum_\alpha \sum_{j=1}^{n} \int_{-\infty}^{\infty} \cdots \widehat{\int_{-\infty}^{\infty}} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right)\Big|_{x_n=0} \mathrm{d}x_n \, \mathrm{d}x_{n-1} \ldots \widehat{\mathrm{d}x_j} \ldots \mathrm{d}x_1$$

$$= \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^{n-1} \left(\widetilde{f_\alpha} \omega_j\right)\Big|_{x_n=0} \mathrm{d}x_{n-1} \ldots \mathrm{d}x_1,$$

since $(f_\alpha \omega_j)|_{x_n=0} = 0$ for $j = 1, \ldots, n-1$, so

$$\int_M \mathrm{d}\omega = \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^{n-1} \left(\widetilde{f_\alpha} \omega_j\right)\Big|_{x_n=0} \mathrm{d}x_{n-1} \ldots \mathrm{d}x_1 = \sum_\alpha \int_{\partial U_\alpha} f_\alpha|_{\partial U_\alpha} \omega = \int_{\partial M} \omega,$$

where $\partial U_\alpha = U_\alpha \cap \partial M$. $\square$

1.9 Applications of Stokes' theorem

Theorem 1.33 (Integration by parts)**.** *Let M be an orientable n-dimensional manifold with boundary, let $\omega \in \Omega_c^p(M)$, let $\eta \in \Omega_c^{n-p-1}(M)$, and let $p \in \{0, \ldots, n-1\}$. Then*

$$\int_{\partial M} \omega \wedge \eta = \int_M d\omega \wedge \eta + (-1)^p \int_M \omega \wedge d\eta.$$

Proof.

$$\int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta) = \int_M \left(d\omega \wedge \eta + (-1)^p \omega \wedge d\eta \right),$$

by Stokes and the Leibnitz rule. $\qquad\square$

Theorem 1.34 (Brouwer's fixed point theorem)**.** *Let*

$$D = \{ x \in \mathbb{R}^n \mid |x| \le 1 \},$$

so

$$\partial D = S^{n-1} = \{ x \in \mathbb{R}^n \mid |x| = 1 \},$$

and let $f : D \to D$ be a smooth morphism. Then f admits a fixed point, that is there exists $x \in D$ such that $f(x) = x$.

Proof. Assume that $f(x) \ne x$ for all $x \in D$. For any $x \in D$, consider the ray starting from $f(x)$ and passing through x. Let $g(x)$ be the point where this ray intersects ∂D away from $f(x)$. Note that if $x \in \partial D$ then $g(x) = x$. Then $g : D \to \partial D$. It is easy to check that g is smooth. Since $\partial D = S^{n-1}$ is orientable by Proposition 1.23 there exists a volume form $\omega \in \Omega^{n-1}(\partial D)$, so $\omega(x) \ne 0$. Since $\omega \in \Omega^{n-1}(\partial D)$, $d\omega \in \Omega^n(\partial D)$, which is an n-dimensional manifold, so $d\omega = 0$. Thus

$$0 < \int_{\partial D} \omega = \int_{\partial D} g^* \omega = \int_D dg^* \omega = \int_D g^* d\omega = 0,$$

by Stokes, a contradiction. $\qquad\square$

Example 1.35. Recall any exact form is closed, since $d^2 = 0$. But the opposite is not always true. Let $M = \mathbb{R}^2 \setminus \{0\}$, and let

$$\omega = \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx \in \Omega^1(M).$$

Then ω is closed, since

$$d\omega = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2} \right) dx \wedge dy - \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2} \right) dy \wedge dx = 0.$$

We want to show that ω is not exact. Assume that

$$\omega = df, \qquad f \in \Omega^0(M) = \{ C^\infty\text{-function} \}.$$

In particular $\omega = df$ on $S^1 \subset M$. Let

$$\gamma \;:\; \begin{array}{ccc} [0, 2\pi] & \longrightarrow & S^1 \\ \theta & \longmapsto & (\cos\theta, \sin\theta) \end{array}.$$

Then

$$\int_{S^1} \omega = \int_0^{2\pi} \gamma^* \omega = \int_0^{2\pi} \left(\left(\frac{\cos\theta}{\cos^2\theta + \sin^2\theta} \right) \cos\theta \, d\theta - \left(\frac{\sin\theta}{\cos^2\theta + \sin^2\theta} \right) (-\sin\theta) \, d\theta \right) = \int_0^{2\pi} d\theta = 2\pi,$$

and

$$\int_{S^1} \omega = \int_{S^1} df = \int_{\partial S^1} f = \int_\emptyset f = 0,$$

so ω is not exact.

Theorem 1.36. *Let M be an orientable manifold of dimension n without boundary, and let $\omega \in \Omega_{\mathrm{c}}^n(M)$. Assume ω is exact. Then*

$$\int_M \omega = 0.$$

Proof. Easy from Stokes.   □

Theorem 1.37. *Let M be an orientable manifold of dimension n with boundary, and let $\omega \in \Omega_{\mathrm{c}}^{n-1}(M)$ be a closed form. Then*

$$\int_{\partial M} \omega = 0.$$

Proof. Easy from Stokes.   □

Definition 1.38. Let M be an orientable manifold of dimension n, let $\omega \in \Omega_{\mathrm{c}}^k(M)$, and let $N \subset M$ be a submanifold of dimension k. We can define

$$\int_M \omega = \int_N i^*\omega,$$

where $i : N \hookrightarrow M$ is the inclusion. We will denote

$$\omega|_N = i^*\omega \in \Omega_{\mathrm{c}}^k(N).$$

Theorem 1.39. *Let M be an oriented manifold of dimension n, let $\omega \in \Omega_{\mathrm{c}}^k(M)$, and let $S \subset M$ be a compact orientable submanifold of dimension k such that $\partial S = \emptyset$ and $\int_S \omega \neq 0$. Then*

- *ω is not exact,*

- *$\omega|_S$ is not exact, and*

- *S is not the boundary of an orientable manifold $N \subset M$ of dimension $k+1$.*

Proof. Exercise. [4]   □

---

[4]Exercise

2 De Rham cohomology

2.1 De Rham cohomology

Let M be a manifold of dimension n, and let $p \geq 0$. Then $\omega_1, \omega_2 \in \Omega^p(M)$ are said to be **cohomologous** if $\omega_1 - \omega_2 = \mathrm{d}\eta$ where $\eta \in \Omega^{p-1}(M)$. In particular $\omega \in \Omega^p(M)$ is cohomologous to zero if it is exact. Let

$$\mathrm{Z}^p(M) = \mathrm{Ker}\left(\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)\right) = \{\omega \in \Omega^p(M) \mid \omega \text{ is closed}\} \subset \Omega^p(M),$$

and let

$$\mathrm{B}^p(M) = \mathrm{Im}\left(\mathrm{d} : \Omega^{p-1}(M) \to \Omega^p(M)\right) = \{\omega \in \Omega^p(M) \mid \omega \text{ is exact}\} \subset \Omega^p(M).$$

Then $\mathrm{B}^p(M) \subset \mathrm{Z}^p(M)$ for all $p \geq 0$.

Notation 2.1. If $p = 0$, then $\mathrm{B}^0(M) = 0$.

Note. If $\omega_1, \omega_2 \in \mathrm{Z}^p(M)$ then $\omega_1 - \omega_2 \in \mathrm{B}^p(M)$ if and only if ω_1 and ω_2 are cohomologous.

Denote the p-**th de Rham cohomology group** as

$$\mathrm{H}^p(M) = \mathrm{Z}^p(M)/\mathrm{B}^p(M) = \{[\omega] \mid \omega \in \mathrm{Z}^p(M)\}, \qquad p \geq 0.$$

where

$$[\omega] = \{\omega' \in \Omega^p(M) \text{ cohomologous to } \omega\}$$

is the **de Rham class** of ω.

Remark 2.2. $\mathrm{H}^p(M)$ is a vector space over $\mathbb{R}$.

Definition 2.3. $\mathrm{b}_p(M) = \dim \mathrm{H}^p(M) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is the p-**th Betti number** of M.

Proposition 2.4. *If M is connected then*

$$\mathrm{H}^0(M) = \mathbb{R},$$

that is $\mathrm{b}_0(M) = 1$. *More in general,* $\mathrm{b}_0(M)$ *is the number of connected components of M.*

Proof. Assume M is connected. Then $\mathrm{B}^0(M) = 0$, so

$$\mathrm{H}^0(M) = \mathrm{Z}^0(M) = \left\{f \in \Omega^0(M) \text{ closed}\right\}$$
$$= \left\{f \in \Omega^0(M) \,\middle|\, \text{locally } \forall x \in M, \ \frac{\partial}{\partial x_i} f(x) = 0\right\}$$
$$= \left\{f \in \Omega^0(M) \text{ is locally constant}\right\} = \mathbb{R}.$$

$\square$

Example. Let $M = \mathrm{S}^1$. Then $\mathrm{H}^0(M) = \mathbb{R}$.

Proposition 2.5. *Let M be a manifold of dimension n. Then*

$$\mathrm{H}^p(M) = 0, \qquad p \geq n + 1.$$

Proof. Recall $\Omega^p(M) = 0$ if $p \geq n + 1$ because all alternating p-forms for $p \geq n + 1$ on an n-dimensional vector space are zero, so $\mathrm{Z}^p(M) = 0$. Thus $\mathrm{H}^p(M) = 0$. $\square$

Proposition 2.6. *Let M be a compact orientable manifold of dimension n without boundary. Then*

$$\mathrm{H}^n(M) \neq 0.$$

Proof. M is orientable, so there exists a volume form $\omega \in \Omega^n(M) = \Omega_{\mathrm{c}}^n(M)$, by Proposition 1.23. Then ω is closed, because $\mathrm{d}\omega$ is an $(n+1)$-form on M, so $\omega \in \mathrm{Z}^n(M)$. We want to show that $[\omega] \neq 0$ in $\mathrm{H}^n(M)$. Assume $[\omega] = 0$, so ω is exact. Thus $\omega = \mathrm{d}\eta$ where η is an $(n-1)$-form on M, so

$$0 < \int_M \omega = \int_M \mathrm{d}\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0,$$

by Stokes, a contradiction. $\square$

Proposition 2.7. *Let $G : M \to N$ be a smooth morphism between manifolds. Then for all $p \geq 0$, G^* : $\Omega^p(N) \to \Omega^p(M)$ takes closed forms of N to closed forms on M and exact forms of N to exact forms on M.*

Proof. Proposition 1.16 implies that $G^* \mathrm{d} = \mathrm{d} G^*$. If ω is closed then $\mathrm{d} G^* \omega = G^* \mathrm{d} \omega = G^* 0 = 0$, so $G^* \omega$ is closed. If $\omega = \mathrm{d} \eta$ is exact then $G^* \omega = \mathrm{d} G^* \eta$ is also exact. □

Thus $G^* : \mathrm{Z}^p(N) \to \mathrm{Z}^p(M)$ and $G^* : \mathrm{B}^p(N) \to \mathrm{B}^p(M)$, so there exists a linear map

$$G^* \quad : \quad \begin{array}{ccc} \mathrm{H}^p(N) & \longrightarrow & \mathrm{H}^p(M) \\ [\omega] & \longmapsto & [G^* \omega] \end{array} \quad .$$

Proposition 2.8. *Let M and N be diffeomorphic manifolds. Then for all $p \geq 0$, $\mathrm{H}^p(M) \cong \mathrm{H}^p(N)$, that is $\mathrm{H}^p(M)$ is a diffeomorphic invariant.*

Proof. By Proposition 2.7 there exists $F^* : \mathrm{H}^p(N) \to \mathrm{H}^p(M)$ and $(F^{-1})^* : \mathrm{H}^p(M) \to \mathrm{H}^p(N)$. By Proposition 1.8,
$$(F^{-1})^* F^* \omega = (F \circ F^{-1})^* \omega = \mathrm{id}_N^* \omega = \omega, \qquad \omega \in \mathrm{H}^p(N)$$
so $(F^{-1})^* \circ F^* = \mathrm{id}_{\mathrm{H}^p(N)}$. Similarly $F^* \circ (F^{-1})^* = \mathrm{id}_{\mathrm{H}^p(M)}$, so F^* is an isomorphism. □

Definition 2.9. Let M_0 and M_1 be manifolds, and let $f_0, f_1 : M_0 \to M_1$ be smooth morphisms. Then f_0 and f_1 are smoothly **homotopic equivalent** if there exists a smooth morphism $H : M_0 \times [0,1] \to M_1$ such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$ for all $x \in M_0$.

A **homotopy** is a smooth morphism $H : M_0 \times [0,1] \to M_1$ where M_0 and M_1 are smooth manifolds.

Notation 2.10. Let $f_t(x) = H(x,t)$, so $f_t : M_0 \to M_1$ is a smooth morphism. Then f_0 and f_1 are said to be homotopic equivalent, denoted by $f_0 \sim f_1$, and $\sim$ is an equivalence. [5]

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Definition 2.11. M_0 and M_1 are **homotopy equivalent** if there exist smooth morphisms $f : M_0 \to M_1$ and $g : M_1 \to M_0$ such that $f \circ g \sim \mathrm{id}_{M_1}$ and $g \circ f \sim \mathrm{id}_{M_0}$.

Example 2.12.

- Let $M_0 = \mathbb{R}^n$ and $M_1 = \{0\}$. Then M_0 and M_1 are homotopy equivalent. Let

$$f \quad : \quad \begin{array}{ccc} M_0 & \longrightarrow & M_1 \\ x & \longmapsto & 0 \end{array} \quad , \qquad g \quad : \quad \begin{array}{ccc} M_1 & \longrightarrow & M_0 \\ 0 & \longmapsto & 0 \end{array} \quad .$$

  Then

$$f \circ g \quad : \quad \begin{array}{ccc} M_1 & \longrightarrow & M_1 \\ 0 & \longmapsto & 0 \end{array} \quad ,$$

  so $f \circ g = \mathrm{id}_{M_1}$, and

$$g \circ f \quad : \quad \begin{array}{ccc} M_0 & \longrightarrow & M_0 \\ x & \longmapsto & 0 \end{array} \quad .$$

  We want to show that $g \circ f \sim \mathrm{id}_{M_0}$. Define a smooth morphism

$$H \quad : \quad \begin{array}{ccc} M_0 \times [0,1] & \longrightarrow & M_0 \\ (x,t) & \longmapsto & tx \end{array} \quad .$$

  Then $H(x,0) = 0 = (g \circ f)(x)$ for all x, and $H(x,1) = x = \mathrm{id}_{M_0}(x)$ for all x, so $g \circ f \sim \mathrm{id}_{M_0}$. More in general $M \subset \mathbb{R}^n$ is called **convex** if for all $x, y \in M$ the segment joining x to y is contained inside M. If M is convex then M is homotopy equivalent to $M \times \{0\}$.

---

[5]Exercise

- Let $M_0 = \mathbb{R}^2 \setminus \{0\}$ and $M_1 = S^1$. Then M_0 and M_1 are homotopy equivalent. Let

$$
f \; : \; \begin{array}{ccc} M_0 & \longrightarrow & M_1 \\ x & \longmapsto & \dfrac{x}{|x|} \end{array} \; , \qquad g \; : \; \begin{array}{ccc} M_1 & \longrightarrow & M_0 \\ x & \longmapsto & x \end{array} \; .
$$

Then

$$
f \circ g \; : \; \begin{array}{ccc} M_1 & \longrightarrow & M_1 \\ x & \longmapsto & x \end{array} \; ,
$$

so $f \circ g = \mathrm{id}_{M_1}$, and

$$
g \circ f \; : \; \begin{array}{ccc} M_0 & \longrightarrow & M_0 \\ x & \longmapsto & \dfrac{x}{|x|} \end{array} \; .
$$

Let

$$
H \; : \; \begin{array}{ccc} M_0 \times [0,1] & \longrightarrow & M_0 \\ (x,t) & \longmapsto & tx + (1-t)\dfrac{x}{|x|} \end{array}
$$

be smooth. Then $H(x,0) = x/|x| = (g \circ f)(x)$ and $H(x,1) = x = \mathrm{id}_{M_0}(x)$, so $g \circ f \sim \mathrm{id}_{M_0}$.

Proposition 2.13. *Let M and N be manifolds, and let $H : M \times [0,1] \to N$. Denote*

$$
f_t \; : \; \begin{array}{ccc} M & \longrightarrow & N \\ x & \longmapsto & H(x,t) \end{array} \; , \qquad t \in [0,1] .
$$

Then for all $p \geq 0$, $f_t^* : \mathrm{H}^p(N) \to \mathrm{H}^p(M)$ does not depend on t.

Proof. Let $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$. The goal is $f_{t_1}^*[\eta] = f_{t_2}^*[\eta]$ for all $[\eta] \in \mathrm{H}^p(N)$. Let

$$
i_k \; : \; \begin{array}{ccc} M & \longrightarrow & M \times [0,1] \\ x & \longmapsto & (x, t_k) \end{array} \; , \qquad k = 1,2.
$$

Claim that for all p there exists a linear map $h : \Omega^p(M \times [t_1, t_2]) \to \Omega^{p-1}(M)$ such that

$$
\mathrm{d}(h(\omega)) + h(\mathrm{d}\omega) = i_2^* \omega - i_1^* \omega \in \Omega^p(M), \qquad \omega \in \Omega^p(M \times [0,1]) .
$$

Step 1. The claim implies the proposition. Let $\eta \in \Omega^p(N)$ be closed, so $\mathrm{d}\eta = 0$. Then $H^*\eta$ is also closed, so let $\omega = H^*\eta \in \Omega^p(M \times [t_1, t_2])$. Apply h. Then $\mathrm{d}\omega = 0$, so $\mathrm{d}(h(\omega)) = i_2^*\omega - i_1^*\omega$ is exact. Thus

$$
f_{t_1}^*[\eta] = [f_{t_1}^*\eta] = [i_1^* H^*\eta] = [i_1^*\omega] = [i_2^*\omega] = [i_2^* H^*\eta] = [f_{t_2}^*\eta] = f_{t_2}^*[\eta] ,
$$

so the proposition follows.

Step 2. The proof of the claim. Let $\omega \in \Omega^p(M \times [t_1, t_2])$. Then for all $(x,t) \in M \times [t_1, t_2]$, $\omega(x,t)$ is an alternating p-form on $\mathrm{T}_{(x,t)}(M \times [t_1, t_2])$. We want an alternating $(p-1)$-form $h\omega(x)$ on $\mathrm{T}_x M$. Let $v_1, \ldots, v_{p-1} \in \mathrm{T}_x M$. Then

$$
h\omega(x)(v_1, \ldots, v_{p-1}) = \int_{t_1}^{t_2} \omega(x,t)\left(v_1, \ldots, v_{p-1}, \frac{\partial}{\partial t}\right) \mathrm{d}t
$$

is a $(p-1)$-form on M. Check h is linear. [6]

$\square$

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[6] Exercise