M3P20 Geometry I: Algebraic Curves

Lectured by Dr Mattia Talpo
Typeset by David Kurniadi Angdinata

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Contents

Lecture 1
Monday
08/10/18

1 Introduction

This course is intended as a first course in algebraic geometry, the area of mathematics that studies spaces defined by polynomial equations using algebra. It will focus on one dimensional algebraic varieties. The reference books for the course are:

1. F Kirwan, Complex algebraic curves, 1992
2. W Fulton, Algebraic curves: an introduction to algebraic geometry, 1969

Geometry is the study of shapes in suitable spaces, such as sets of points on the real line $\mathbb{R}$, lines and circles in $\mathbb{R}^2$, spheres in higher dimensional Euclidean spaces $\mathbb{R}^n$, etc. One way to think about shapes is to see them as the locus of zeroes defined by

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid f(x_1, \dots, x_n) = 0\} \qquad \Longleftrightarrow \qquad \{f(x_1, \dots, x_n) = 0\} \subset \mathbb{R}^n$$

for some suitable function f.

Example 1.1.

1. Circles $\left\{f_1(x, y) = x^2 + y^2 - R^2 = 0\right\}$ in $\mathbb{R}^2$ for some $R \in \mathbb{R}$.
2. The unit square with vertices at $\{(\pm 1, 0), (0, \pm 1)\}$ in $\mathbb{R}^2$ defined by $\{f_2(x, y) = |x| + |y| - 1 = 0\}$.
3. Spheres $\left\{f_3(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2 - R^2\right\}$ in $\mathbb{R}^n$ for some $R \in \mathbb{R}$.

Remark 1.2. Every subset $S \subseteq \mathbb{R}^n$ is the zero-set of some function. We can define $\chi_S(x) = 1$ if $x \notin S$, $\chi_S(x) = 0$ if $x \in S$.

The class of functions used to defined our shapes has great consequences on their geometry. For the circle, f_1 is a polynomial so that it is differentiable and also C^∞. For the square, f is continuous but not differentiable at $\{(0, \pm 1), (\pm 1, 0)\}$, the vertices of the square. The function χ_S is not even continuous, unless S is empty, or the whole $\mathbb{R}^n$. As these examples illustrate, an underlying principle is the following equivalence.

Fact 1. Regularity properties of f are regularity properties of $\{f = 0\}$.

Such shapes are called **algebraic varieties**. Their geometric properties are intimately related to the algebraic properties of the defining polynomial equations.

Example 1.3.

1. Let $f(x)$ be a polynomial. Then the zero set of $f(x)$, $\{f(x) = 0\} \subseteq \mathbb{R}$ is a finite set of points in $\mathbb{R}$, and every finite set of points arises in this manner.
2. The circle is an algebraic variety.
3. Spheres in higher dimensions are algebraic varieties.

Exercise 1.

1. Is $\mathbb{Z} \subseteq \mathbb{R}$ an algebraic variety?
2. Is the unit square an algebraic variety?

Definition 1.4. Let K be a field, such as $K = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ a multi-index, denote by $|\alpha| = \sum_{i=1}^n \alpha_i$ and by $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, a **monomial**. A **polynomial of degree** d in n variables with coefficients in K is a finite sum.

$$P(x_1, \dots, x_n) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha,$$

where $a_\alpha \in K$, $a_\alpha = 0$ for all $|\alpha| > d$ and $a_\alpha \neq 0$ for some α with $|\alpha| = d$. The set of polynomials of arbitrary degree in n variables with coefficients in K is denoted $K[x_1, \dots, x_n]$.

Example 1.5. Let $n=3$. $P(x_1,x_2,x_3)=3+x_1^2x_2+x_3^{10}$ for $\alpha=(2,1,0)$ and $\alpha=(0,0,10)$ has degree ten.

Exercise 2.

1. Show that $K[x_1,\dots,x_n]$ is a ring, and that if P,Q are polynomials of degrees p,q respectively, then the degree of $\lambda P+\mu Q$ for $\lambda,\mu\in K$ is at most $\max\{p,q\}$. Give an example of polynomials $P,Q\in K[x]$ such that $\deg(P+Q)<\max\{\deg(P),\deg(Q)\}$.

2. Show that $(P\cdot Q)(x_1,\dots,x_n)=P(x_1,\dots,x_n)Q(x_1,\dots,x_n)$ is a polynomial $P\cdot Q\in K[x_1,\dots,x_n]$ with $\deg(P\cdot Q)=\deg(P)+\deg(Q)$. What if $P=0$? What is $\deg(0)$?

Definition 1.6. An **affine plane curve** defined over K is

$$C=\left\{(x,y)\in K^2\mid P(x,y)=0\right\}\subset K^2,$$

where $P=K[x,y]$. More generally, an **algebraic variety** $V\subset K^n$ is a subset of K^n defined as the locus

$$\{f_1=\dots=f_k=0\}\subset K^n,$$

where $f_1,\dots,f_k\in K[x_1,\dots,x_n]$ are polynomials in n variables with coefficients in K.

Example 1.7.

1. Let $a,b,c\in\mathbb{R}$ with $(a,b)\neq(0,0)$, and let

   $$f(x,y)=ax+by+c.$$

   Then $\left\{(x,y)\in\mathbb{R}^2\mid f(x,y)=0\right\}$ is a line.

2. Let $a,b\in\mathbb{R}^*=\mathbb{R}\setminus\{0\}$ and

   $$f(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1.$$

   The curve $\left\{(x,y)\in\mathbb{R}^2\mid f(x,y)=0\right\}$ is an ellipse.

3. Let $a,b\in\mathbb{R}^*$ and

   $$g(x,y)=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1.$$

   The curve $\left\{(x,y)\in\mathbb{R}^2\mid g(x,y)=0\right\}$ is a hyperbola.

4. Spheres, quadrics such as ellipsoids, paraboloids, and hyperboloids in $\mathbb{R}^3$ are all defined via a single polynomial equation of degree two. A line in $\mathbb{R}^3$ can be defined by two equations in degree one.

The first property of algebraic curves is the following.

Lemma 1.8. The union of two affine plane curves is again an affine plane curve.

Proof. Let $f_1,f_2\in K[x,y]$ and let $C_1=\{f_1=0\}$ and $C_2=\{f_2=0\}$. Then $f_1\cdot f_2\in K[x,y]$ is a polynomial and

$$C_1\cup C_2=\{f_1\cdot f_2=0\},$$

so that $C_1\cup C_2$ is an affine plane curve. □

Exercise 3. Write down an equation for the plane curve that is the union of the lines through any two vertices of the unit square.

Recall the following.

Definition 1.9. A polynomial $P\in K[x_1,\dots,x_n]$ is **reducible** over K if there are non-constant polynomials $Q,R\in K[x_1,\dots,x_n]$, so $\deg(Q),\deg(R)>0$, such that $P=Q\cdot R$. A polynomial P is **irreducible** if it is not reducible.

Example 1.10. x_1x_2 is reducible, x_1+x_2 is irreducible.

Fact 2. Recall also that every polynomial $P \in K[x_1,\dots,x_n]$ can be written as a product of irreducible factors $P = f_1 \dots f_k$ in an essentially unique way up to multiplication by constants. We have

$$\{P=0\} = \{f_1=0\} \cup \dots \cup \{f_k=0\} \subseteq K^n,$$

so in particular every algebraic curve is a union of algebraic curves defined by irreducible polynomials.

In the course, we will consider questions such as:

1. When do polynomials $f,g \in K[x,y]$ define the same affine plane curve?
2. What can be said about the intersection $\{f=0\} \cap \{g=0\} \subset K^2$?

Very different questions can be approached through algebraic curves. For example, we can study integer solutions to some Diophantine equations.

Example 1.11. The unit circle is the curve

$$C = \left\{x^2+y^2=1\right\} \subset \mathbb{R}^2.$$

Several parametrisations are known, such as

$$t \in [0,2\pi) \mapsto (\cos t, \sin t) \in \mathbb{R}^2.$$

We can write down another parametrisation of C by considering lines through the point $P=(-1,0)$ using a stereographic projection. A line through P with slope $t \in \mathbb{R}$ has equation

$$L_t = \{y = t(x+1)\} \subset \mathbb{R}^2$$

and meets C in two points, P and $P_t = (x(t), y(t))$. We can determine the coordinate of P_t by solving the system

$$L_t \cap C = \begin{cases} y = t(x+1) \\ x^2+y^2=1 \end{cases}.$$

Replacing the value of y given by the first equation into the second yields two solutions for $x(t)$. The first one is $x=-1$ and corresponds to the point $P=(-1,0)$. The second is $(x(t),y(t))$, where

$$x(t) = \frac{1-t^2}{1+t^2}, \qquad y(t) = \frac{2t}{1+t^2}.$$

Note that when $t \to \infty$, $(x(t),y(t)) \to (-1,0)$, so that $t \mapsto (x(t),y(t))$ is a parametrisation of C that identifies it with $\mathbb{R} \cup \{\infty\}$. The advantage of this parametrisation is that it is given by rational functions, that is $x(t)$ and $y(t)$ are of the form

$$t \mapsto \frac{p(t)}{q(t)},$$

where p,q are polynomials. One can use this parametrisation to get the general solution of the equation

$$x^2+y^2=z^2 \tag{1}$$

for $x,y,z \in \mathbb{Z}$ coprime. If $t=p/q \in \mathbb{Q}$, where $p,q \in \mathbb{Z}$ are coprime, then $x(t), y(t) \in \mathbb{Q}$ becomes

$$x(t) = \frac{p^2-q^2}{p^2+q^2}, \qquad y(t) = \frac{2pq}{p+q^2}.$$

If $x=p^2-q^2$, $y=2pq$, and $z=p^2+q^2$, $x,y,z \in \mathbb{Z}$ satisfy (1). They are coprime precisely when p,q are coprime and not both odd. When p,q are coprime and both odd, then

$$x = \frac{p^2-q^2}{2}, \qquad y = pq, \qquad \frac{p^2+q^2}{2}$$

satisfy (1). Conversely, this is the general form of solutions in (1). Indeed, given $x, y, z \in \mathbb{Z}$ coprime that satisfy (1), $z \neq 0$ and

$$\frac{x^2}{z^2} + \frac{y^2}{z^2} = 1,$$

so that $(x/z, y/z) \in \mathbb{C}$ and if $(x, y, z) \neq (-1, 0, 1)$, there is $t \in \mathbb{R}$ such that $(x/z, y/z) = (x(t), y(t))$. But then since $x/z, y/z \in \mathbb{Q}$, we can take $t \in \mathbb{Q}$ and x, y, z have the form above.

Lecture 2 Thursday 11/10/18

Definition 1.12. Let $f \in \mathbb{R}[x, y]$ and let $C = \{f = 0\}$. A **rational point** of C is a point $(x, y) \in C$, that is $f(x, y) = 0$, such that $x, y \in \mathbb{Q}$.

Example 1.13. There are infinitely many rational points on the circle $\left\{x^2 + y^2 = 1\right\} \subseteq \mathbb{R}^2$, which can be described explicitly, and can be used to solve $a^2 + b^2 = c^2$ for $a, b, c \in \mathbb{Z}$, a problem in number theory. Now take $n \geq 3$ and consider

$$C = \{x^n + y^n - 1 = 0\}.$$

What are the rational points of C? Write

$$x = \frac{a}{c}, \qquad y = \frac{b}{c}, \qquad a, b, c \in \mathbb{Z}, \qquad c \neq 0.$$

Then

$$(x, y) \in C \qquad \Longleftrightarrow \qquad a^n + b^n = c^n.$$

Fermat's Last Theorem by Wiles then states that there exists no solution with $a, b \neq 0$.

2 Complex plane curves

Let $P \in \mathbb{R}[x, y]$ be a polynomial with coefficients in $\mathbb{R}$. A priori, it is natural to study the real plane curve $C_{\mathbb{R}} = \left\{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\right\}$. However, P can also been seen as a polynomial with coefficients in $\mathbb{C}$, and it will often be simpler to study the complex plane curve $C_{\mathbb{C}} = \left\{(x, y) \in \mathbb{C}^2 \mid P(x, y) = 0\right\}$. We first explain some of the properties of algebraic curves that we would like to hold and explain why these properties do not necessarily hold for real plane curves and some unpleasant things happen.

Fact 3. Many real curves are so degenerate that they do not even have points, that is $C_{\mathbb{R}} = \emptyset$. If $C_{\mathbb{R}} \neq \emptyset$, the dimension of $C_{\mathbb{R}}$ is difficult to determine.

Example 2.1. Let $t \in \mathbb{R}$ and consider $f_t(x, y) = x^2 + y^2 - t$ and the real plane curve $C_t = \{f_t(x, y) = 0\} \subseteq \mathbb{R}^2$. If $t > 0$, C_t is a circle with radius $\sqrt{t}$, if $t = 0$, has $C_0 = \{(0, 0)\}$, and if $t < 0$, $C_t = \emptyset$.

Fact 4. In general, it is not clear when two polynomials $f, g \in \mathbb{R}[x, y]$ define the same real plane curve, that is when

$$\left\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\right\} = \left\{(x, y) \in \mathbb{R}^2 \mid g(x, y) = 0\right\}.$$

Example 2.2. Let f, g denote the polynomials

$$f(x, y) = x^2 y + y^2 + x^3 + x, \qquad g(x, y) = x^2 + 2xy + y^2.$$

Then, since $f(x, y) = (x + 1) \cdot \left(x^2 + 1\right)$ and $g(x, y) = (x + y)^2$,

$$\left\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\right\} = \left\{(x, y) \in \mathbb{R}^2 \mid g(x, y) = 0\right\}.$$

Fact 5. In general, it is hard to predict when a curve intersects a fixed line, or more generally when two real curves intersect.

Example 2.3. In the notation of Example 2.1, let $C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 1\right\} \subset R^2$ be the unit circle. Consider the line $\{ax + by + c = 0\}$ for $(a, b) \neq (0, 0)$. Then, depending on $(a, b, c) \in \mathbb{R}^3$, $L \cap C$ consists of two points, one point, or is empty.

Most of these difficulties disappear when working with curves $C_{\mathbb{C}} \subset \mathbb{C}^2$, essentially because $\mathbb{C}$ is algebraically closed, in other words the following theorem holds.

Theorem 2.4 (Fundamental theorem of algebra)**.** Let $P \in \mathbb{C}[x]$ be a non-constant polynomial. Then P has at least one complex root, that is there exists $\alpha \in \mathbb{C}$ such that $P(\alpha) = 0$.

A consequence of the fundamental theorem of algebra is that if $P \in \mathbb{C}[x, y]$ is non-constant, then $C = \{P = 0\}$ has infinitely many points. Assume without loss of generality that the polynomials in one variable $P(\cdot, y)$ and $P(x, \cdot)$ are not constant. This means that if

$$P(x, y) = \sum_{(r,s) \in \mathbb{N}^2} c_{r,s} x^r y^s,$$

there exist (r, s) and (r', s') in $\mathbb{N}^2$, which may be equal, such that $r \neq 0$ and $s' \neq 0$, and $c_{r,s} \neq 0$ and $c_{r',s'} \neq 0$. When $y_0 \neq 0$ the polynomial $P(x, y_0) \in \mathbb{C}[x]$ is non-constant and by Theorem 2.4, there exists $x_0 \in \mathbb{C}$ such that $P(x_0, y_0) = 0$. In fact, for most choices of $y_0 \neq y_0'$, the polynomials $P(\cdot, y_0)$ and $P(\cdot, y_0')$ are different, and have some distinct roots. It follows that $\{P = 0\}$ contains infinitely many points.

Example 2.5. Let $a, b, c \in \mathbb{C}$ with $(a, b) \neq (0, 0)$, and let $f(x, y) = ax + by + c$. If $a \neq 0$, for each $y \in \mathbb{C}$, there is precisely one solution of $f(x, y) = 0$, namely

$$x = -\frac{b}{a} y - \frac{c}{a}.$$

Thus $\mathbb{C}^2 \supset \{f = 0\} = C \to \mathbb{C} \cong \mathbb{R}^2$ is an isomorphism. We will call C a **complex line**.

Remark 2.6. It is difficult to draw complex curves. Our intuition is for real vector spaces, and this makes complex curves hard to visualise. They are objects of real dimension two in $\mathbb{C}^2 \cong \mathbb{R}^4$, a four-dimensional real vector space.

Example 2.7. Let $f(x, y) = x^2 + y^2$. Then $f(x, y) = (x + iy) \cdot (x - iy)$ and, as in Lemma 1.8, $C = \{f = 0\} \subset \mathbb{C}^2$ is the union of the two complex lines $\{x + iy = 0\}$ and $\{x - iy = 0\}$. When seen as $\mathbb{R}$-vector spaces, these two planes meet at exactly one point corresponding to $(0, 0) \in \mathbb{R}^2 \subseteq \mathbb{C}^2$, the only real point of C. It is difficult to imagine two planes meeting in one point, because our intuition relies on three-dimensional space $\mathbb{R}^3$, while $\mathbb{C}^2 = \mathbb{R}^4$.

Describing intersections is also easier.

Example 2.8. Consider $C = \left\{x^2 + y^2 - 1 = 0\right\} \subseteq \mathbb{C}^2$ and $L = \{ax + by + c = 0\} \subseteq \mathbb{C}^2$. If $b \neq 0$, we determine the intersection $C \cap L$ by solving the equation

$$x^2 + y^2 = 1, \qquad y = -\frac{a}{b} x - \frac{c}{b}.$$

If $a^2 = -b^2$ or $c = 0$, there are one or two solutions. Again, it is hard to imagine a two-dimensional real surface which meets a real plane in two points.

We now turn to the question of recognising when two polynomials define the same plane curve. Here again, working in $\mathbb{C}$ is a simplification.

Theorem 2.9 (Consequence of Hilbert's Nullstellensatz)**.** Let $f, g \in \mathbb{C}[x, y]$ be two polynomials. Then

$$\{f = 0\} = \{g = 0\}$$

if and only if there exist $P_1, \dots, P_k \in \mathbb{C}[x, y]$, $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{Z}_{>0}$ and $\lambda_1, \lambda_2 \in \mathbb{C}^*$ such that

$$f(x, y) = \lambda_1 P_1^{a_1} \dots P_k^{a_k}, \qquad g(x, y) = \lambda_2 P_1^{b_1} \dots P_k^{b_k}. \tag{2}$$

Lecture 3
Friday
12/10/18

Proof. Assume that (2) holds. Then by the proof of Lemma 1.8,

$$\{f=0\} = \{P_1^{a_1}=0\} \cup \cdots \cup \{P_k^{a_k}=0\} = \{P_1=0\} \cup \cdots \cup \{P_k=0\},$$

because if $\alpha \in \mathbb{C}$ is such that $\alpha^n = 0$, then $\alpha = 0$. The same holds for $\{g=0\}$. Therefore $\{f=0\} = \{g=0\}$. The second half of the proof needs tools of commutative algebra and is omitted. □

Remark 2.10. The theorem fails over $\mathbb{R}$. Let $f(x,y) = x^2+1$ and $g(x,y)=1$. Then $\{f=0\} = \{g=0\} = \emptyset$ but the conclusion in (2) is not true.

Thus, the relation between the geometric shape $C = \{f=0\}$ in $\mathbb{C}^2$ and the polynomial $f \in \mathbb{C}[x,y]$ is more transparent than in $\mathbb{R}$. We will always work in $\mathbb{C}$. Let us introduce some important notions for the study of polynomials.

Definition 2.11. A polynomial $f \in K[x,y]$ has **no repeated factors** over K if it cannot be written as a product of the form

$$f(x,y) = g(x,y)^2 \cdot h(x,y),$$

where $g,h \in K[x,y]$ and g is non-constant.

Exercise 4. Show that this is equivalent to

$$f = P_1 \cdots P_k,$$

where $P_1, \cdot, P_k$ are distinct irreducible polynomials.

Corollary 2.12. Let $f,g \in \mathbb{C}[x,y]$ be polynomials with no repeated factors. Then f,g define the same complex plane curve

$$\{f=0\} = \{g=0\}$$

if and only if there is a non-zero constant $\lambda \in \mathbb{C}^*$ such that $f = \lambda g$.

Proof. Follows immediately from Theorem 2.9. □

Remark 2.13. If $f = P_1^{a_1} \cdot P_k^{a_k}$ with P_i irreducible for all i and $a_i \in \mathbb{N}$, then $\{f=0\} = \{g=0\}$ where $g = P_1 \cdot P_k$. We do not lose anything by only looking at f with no repeated factors.

Let $C \subseteq \mathbb{C}^2$ be a complex plane curve. We have proved that, up to multiplication by $\lambda \in \mathbb{C}^*$, there is a unique non-constant polynomial $f \in \mathbb{C}$ with no repeated factors such that

$$C = \{f=0\}.$$

It makes sense to define the following.

Definition 2.14. The **degree** of an affine curve $C \subseteq \mathbb{C}^2$ is the degree of any polynomial with no repeated factors f such that $C = \{f=0\}$, that is

$$\deg(C) = \deg(f).$$

Example 2.15. Lines have degree one, since they are defined by a linear polynomial. Conics have degree two. $\left\{x^2y + y^2 + x + 1 = 0\right\}$ has degree 3, assuming it has no repeated factors.

Unless mentioned otherwise, in the first few weeks, we will assume that polynomials have no repeated factors.

Definition 2.16. Let $f_1, f_2 \in \mathbb{C}[x,y]$ be polynomials with no repeated factors and let $C_1 = \{f=0\}$ and $C_2 = \{g=0\}$ be the associated complex curves. The curves C_1 and C_2 have **no common component** if there is no non-constant polynomial P that divides both f and g.

This is equivalent to saying that if $f = P_1^{a_1} \dots P_k^{a_k}$ and $g = Q_1^{b_1} \dots Q_k^{b_k}$ with P_i, Q_i irreducibles, P_i distinct distinct, and Q_i distinct, then $\lambda P_i \neq Q_j$ for all $i, j, \lambda \in \mathbb{C}^*$.

Exercise 5. Show that if C_1 and C_2 have no common component, then $\deg(C_1 \cup C_2) = \deg(C_1) + \deg(C_2)$.

Exercise 6. Let L, L' be the lines

$$L = \{ax + by + c = 0\} \subset \mathbb{C}^2, \qquad L' = \{a'x + b'y + c' = 0\} \subset \mathbb{C}^2.$$

1. Show that L and L' meet at exactly one point if and only if $ab' - a'b \neq 0$.
2. Show that $L = L'$ if and only if there exists $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$ and

$$a' = \lambda a, \qquad b' = \lambda b, \qquad c' = \lambda c.$$

Remark 2.17 (First aid topology)**.**

1. A **topological space** is a set X with a collection of open subsets $\{U_i \subset X\}$ such that
   (a) $\emptyset$ and X are open,
   (b) any union $\cup_{i \in I} U_i$ of open sets U_i is open, and
   (c) any finite intersection $\cap_{i=1}^{k} U_i$ of open sets U_i is open.
2. A **metric space** X, such as $(\mathbb{C}^n, ||.||)$, is a topological space. The open sets are given by arbitrary unions and finite intersections of the familiar open balls $B(x, \epsilon) = \{z \in X \mid ||z - x||\} < \epsilon$.
3. A subset $X \subset Y$ of a topological space Y inherits a topology from Y. The open sets of X are the sets $X \cap U$, where $U \subset Y$ is an open set of Y.
4. X is **compact** if for all open covering $X = \sum_{i \in I}$ where U_i are open, there exists a finite subcovering $\Cup_{i_1, \dots, i_k} U_{i_j}$ for $\{i_1, \dots, i_k\} \subseteq I$.
5. The **Heine-Borel theorem** states that a subset X of $\mathbb{R}^n$ or of $\mathbb{C}^m$ is compact if and only if X is closed, that is its complement is open, and bounded for the usual norm.
6. A closed subset of a compact space is compact.
7. A map $f : X \to Y$ between topological spaces is **continuous** if and only if $f^{-1}(U)$ is open in X whenever $U \subset Y$ is open. It follows that $f^{-1}(F)$ is closed whenever $F \subset Y$ is closed. In particular, if $f \in \mathbb{C}[x_1, \dots, x_n]$ is a polynomial, f defines a map $f : \mathbb{C}^n \to \mathbb{C}$ that is continuous, and

$$f^{-1}(\{0\}) = \{f = 0\} \subset \mathbb{C}^n$$

is closed because $\{0\}$ is a closed subset of $\mathbb{C}$.

In particular, $\mathbb{C}^2$ is a topological space with the Euclidean distance in $\mathbb{R}^4$ and the affine plane curve $C = \{f = 0\} \subseteq \mathbb{C}^2 \cong \mathbb{R}^4$ inherits a topology. Open sets of C are $U \cup C$ where $U \subseteq \mathbb{C}^2$ is open. So algebraic curves have a natural topology.

Lemma 2.18. Let $C \subset \mathbb{C}^2$ be an affine plane curve, then C is not compact.

Proof. Since f is a continuous function $\mathbb{C}^2 \to \mathbb{C}$, $C = \{f = 0\} = f^{-1}(\{0\})$, and $\{0\}$ is closed in $\mathbb{C}$, C is closed in $\mathbb{C}^2$. We check that $C \subset \mathbb{C}^2$ is not bounded. Assume that it is, then there is a constant $M > 0$ such that $C \subset B(0, M)$, where the open ball is

$$B(0, M) = \left\{ |x|^2 + |y|^2 < M \right\}.$$

Want to show that some points in $\mathbb{C}$ are outside this open ball. Let $y_0 \in \mathbb{C}$ be such that $|y_0| > M$ and assume we can arrange for $g = f(\cdot, y_0)$ to be a non-constant polynomial of x. By the fundamental theorem of algebra, g has a root $x_0 \in \mathbb{C}$ and the point $(x_0, y_0) \in C$. This is a contradiction, as $(x_0, y_0) \notin B(0, M)$. $\square$

TODO Exercise: what if $f(x, y)$ happens to be a polynomial of y alone, so that this cannot be arranged? Lecture 4 Monday 15/10/18

3 Projective space

Recall that it is difficult to determine when two affine plane curves $C, C' \in \mathbb{C}^2$ intersect, and some curves do not in fact intersect even over $\mathbb{C}$. We want to fix that, and the key is adding points at infinity.

Example 3.1.

1. Consider two distinct lines

$$L_1 = \{ax + by + c = 0\}, \qquad L_2 = \{a'x + b'y + c' = 0\}.$$

Then L_1 and L_2 meet at exactly one point if and only if

$$\det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \neq 0.$$

But we can pretend that parallel lines meet at a point at infinity corresponding to the direction vector.

2. Consider the asymptotic curve and line

$$C = \{xy - 1 = 0\}, \qquad L = \{x = 0\}.$$

Then C and L do not meet, but again we can pretend that they meet at a point at infinity.

A heuristic trick is to introduce a variable z.

1. Replace x by x/z and y by y/z.
2. Solve for $z = 0$.

Example 3.2.

1. Consider the lines

$$L_1 = \{x + y + 1 = 0\}, \qquad L_2 = \{x + y - 1 = 0\}.$$

Clearly L_1 and L_2 do not meet. Let us apply the trick. By 1 and 2 we get

$$\begin{cases} x/z + y/z + 1 = 0 \\ x/z + y/z - 1 = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x + y + z = 0 \\ x + y - z = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x + y = 0 \\ x + y = 0 \end{cases}.$$

We get that the point $(1, -1, 0)$ is a common solution. This will be called the point at infinity.

2. Consider the asymptotic curve and line

$$C = \{xy - 1 = 0\}, \qquad L = \{x = 0\}.$$

Apply 1 and 2 to get

$$\begin{cases} xy - z^2 = 0 \\ x/z = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} xy = 0 \\ x = 0 \end{cases}.$$

We get that $(0, 1, 0)$ is a common solution. Again, this will be called the point at infinity.

To make this formal, we introduce the projective plane $\mathbb{P}^2$. We will add points at infinity to $\mathbb{C}^2$, in such a way that asymptotic curves meet at infinity. We will then compactify an affine plane curve C so that the two compactifications are compatible, that is

$$\left(C \subseteq \mathbb{C}^2\right) \hookrightarrow \left(\overline{C} \subseteq \mathbb{P}^2\right).$$

Notation 3.3. Fix $n \geq 0$ and $\mathbb{C}^{n+1}$. Let $\underline{0} = (0, \ldots, 0) \in \mathbb{C}^{n+1}$ be the origin of the $(n+1)$-dimensional complex Euclidean space. We will denote

$$W = \mathbb{C}^{n+1} \setminus \{0\},$$

that is a point $x \in W$ is given by $x = (x_0, \ldots, x_n)$ where $x_0, \ldots, x_n \in \mathbb{C}$ are not all zero. We define the equivalence relation on W, for any $x, y \in W$ by

$$x \sim y \qquad \iff \qquad \exists \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \ x = \lambda y.$$

Exercise 7. Show that $\sim$ is an equivalence relation on W.

Notation 3.4. Given $x \in W$, we denote

$$[x] = \{y \in W \mid x \sim y\}.$$

For simplicity, if $x = (x_0, \ldots, x_n)$ we will denote $[x] = [x_0, \ldots, x_n]$ instead of $x = [(x_0, \ldots, x_n)]$.

Definition 3.5. The n**-dimensional projective space** $\mathbb{P}^n_{\mathbb{C}}$ or $\mathbb{P}^n(\mathbb{C})$ or simply $\mathbb{P}^n$ is defined as the quotient of W by $\sim$, that is

$$\mathbb{P}^n = \frac{W}{\sim} = \left\{[x] \mid x \in W = \mathbb{C}^{n+1} \setminus \{0\}\right\}.$$

The coordinates of $\mathbb{P}^n$ are $[x] \in \mathbb{P}^n$ except $[0, \ldots, 0]$ and $[\lambda x_0, \ldots, \lambda x_n] = [x_0, \ldots, x_n]$ for $\lambda \in \mathbb{C}^*$. In other words, in $\mathbb{P}^n$, two points $[x_0, \ldots, x_n]$ and $[y_0, \ldots, y_n]$ are the same point if and only if there exists a non-zero constant λ such that

$$x_0 = \lambda y_0, \qquad \ldots, \qquad x_n = \lambda y_n.$$

Exercise 8. Show that $[x] = [y]$ if and only if $x \sim y$. Show that if $y \notin [x]$ then $[x] \cap [y] = \emptyset$.

Example 3.6. The point $[1, 2, i]$ is the same as the point $[i, 2i, -1]$.

Exercise 9. Show that there exists a bijection between $\mathbb{P}^n$ and the set of all the one-dimensional subspaces of $\mathbb{C}^{n+1}$. In fact, if V is a finite-dimensional vector space over $\mathbb{C}$ without the choice of a basis, we can defined the associated projective space $\mathbb{P}(V)$ as the set of one-dimensional linear subspaces of V.

Example 3.7. For any non-zero $x \in \mathbb{C}$ we have $[x] = [1]$. So $\mathbb{P}^0 = \mathbb{C}^1 \setminus \{0\} / \sim$ is a point $\mathbb{C}^0 = \{[1]\}$.

Notation 3.8. For any $i = 0, \ldots, n$, denote

$$U_i = \{[x] = [x_1, \ldots, x_n] \in \mathbb{P}^n \mid x_i \neq 0\} \subseteq \mathbb{P}^n.$$

Lemma 3.9. $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$.

Proof. Take $[x] = [x_0, \ldots, x_n] \in \mathbb{P}^n$ in $\mathbb{P}^n$ then $x \in W$ and in particular $x = (x_0, \ldots, x_n)$ where at least one of the coefficients is non-zero, say $x_i \neq 0$. Then $[x] \in U_i$. Thus any $[x] \in \mathbb{P}^n$ is contained in the union of $U_0, \ldots, U_n$. $\square$

Lemma 3.10. Pick $i = 0, \ldots, n$. Define $\phi_i : \mathbb{C}^n \to U_i$ by

$$\phi_i(y_1, \ldots, y_n) = [y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n].$$

Then ϕ_i is a bijection and its inverse $\rho_i : U_i \to \mathbb{C}^n$ is given by

$$\rho_i[x_0, \ldots, x_n] = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right).$$

Proof. First note that both ϕ_i and ρ_i is well-defined, indeed, if

$$(y_1, \ldots, y_n) \in \mathbb{C}^n$$

then

$$(y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n) \in W$$

and therefore $[y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n] \in \mathbb{P}^n$. Similarly, if $[x_0, \ldots, x_n] = [x'_0, \ldots, x'_n]$ then it follows that $\rho_i [x_0, \ldots, x_n] = \rho_i [x'_0, \ldots, x'_n]$. Thus, it is enough to show that both $\phi_i \circ \rho_i$ and $\rho_i \circ \phi_i$ coincide with the identity. We have

$$\rho_i (\phi_i (y_1, \ldots, y_n)) = \rho_i [y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n] = \left(\frac{y_1}{1}, \ldots, \frac{y_{i-1}}{1}, \frac{y_i}{1}, \ldots, \frac{y_n}{1} \right) = (y_1, \ldots, y_n).$$

Similarly,

$$\phi_i (\rho_i [x_0, \ldots, x_n]) = \phi_i \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right) = \left[\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right] = [x_0, \ldots, x_n].$$

Thus they are inverses. ∎

Example 3.11. Let $n = 2$ and $[x_0, x_1, x_2] \in \mathbb{P}^2$. Let $\phi_1 : \mathbb{C}^2 \to U_1$ be defined by $(0, 0) \mapsto [0, 1, 0]$ and $\phi_2 : \mathbb{C}^2 \to U_2$ be defined by $(0, 0) \mapsto [0, 0, 1]$. Check that

$$\rho_i (\phi_i (y_1, \ldots, y_n)) = \rho_i ([y_1, \ldots, 1, \ldots, y_n]) = (y_1, \ldots, y_n).$$

The previous two lemmas can be used to define a topology on $\mathbb{P}^n$. Let $U \subseteq \mathbb{P}^n$, then U is open if and only if $\phi_i^{-1} (U \cap U_i) \subseteq \mathbb{C}^n$ is open in $\mathbb{C}^n$ for any $i = 0, \ldots, n$.

Exercise 10. Show that $U_i \subseteq \mathbb{P}^n$ is open in $\mathbb{P}^n$ for all $i = 0, \ldots, n$.

Exercise 11. We can define another topology on $\mathbb{P}^n$, as follows. A subset $U \subseteq \mathbb{P}^n$ is open if and only if its preimage $\pi^{-1} (U)$ is open in W, where $\pi : W \to \mathbb{P}^n$ is the map defined by $(x_0, \ldots, x_n) \mapsto [x_0, \ldots, x_n]$. Show that this indeed defines a topology, and that this topology coincides with the one defined above using the maps ϕ_i.

Exercise 12. Prove that $\mathbb{P}^n$ is compact. Restrict the projection π of the previous exercise to the $(n + 1)$-dimensional sphere $S^{n+1} \subseteq W$, and check that this restriction is surjective and continuous. Since S^{n+1} is compact, it follows that $\mathbb{P}^n$ is compact as well.

Example 3.12.

1. $\mathbb{P}^1 = \left\{ [x_0, x_1] \mid (x_0, x_1) \in \mathbb{C}^2 \setminus \left\{ \overline{0} \right\} \right\}$ is the union of two copies U_0 and U_1 of $\mathbb{C}^1$. The intersection of U_0 and U_1 is $U_0 \cap U_1 = \{[x_0, x_1] \mid x_0 \neq 0, x_1 \neq 0\}$ can be identified with $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ via the map $[x_0, x_1] \mapsto x_1/x_0$. Using this identification and the maps ρ_i, the inclusion $U_0 \cap U_1 \subseteq U_0$ is the map $\mathbb{C}^* \to \mathbb{C}^1$ sending z to itself, and the inclusion $U_0 \cap U_1 \subseteq U_1$ is the map $\mathbb{C}^* \to \mathbb{C}^1$ given by $z \mapsto 1/z$. $\mathbb{P}^1$ is glued together from two copies of $\mathbb{C}$ along $U_0 \cap U_1$ by these inclusions, so $\mathbb{P}^1 = \mathbb{C}^1 \cup \{\infty\}$. Over the real numbers, $\mathbb{P}^1 (\mathbb{R})$ is built up in the same way from two copies of $\mathbb{R}^1$, and can be identified with the circle S^1.

2. P^2 is the union of three copies of $U_0 \cong U_1 \cong U_2 \cong \mathbb{C}^2$, and the intersection can be described similarly. At infinity we will have a line. More generally, $\mathbb{P}^n$ can be described similarly.

In practice, we will view the affine complex plane $\mathbb{C}^2$ as being embedded in $\mathbb{P}^2$ as one of the open sets U_i. For example, we identify

$$(x, y) \in \mathbb{C}^2 \qquad \Longleftrightarrow \qquad [x, y, 1] \in \mathbb{P}^2.$$

What is the complement of this embedding, that is the points at infinity?

Notation 3.13. For any $i = 0, \ldots, n$, denote

$$P_i = \{[x_0, \ldots, x_n] \in \mathbb{P}^n \mid x_i = 0\} \subseteq \mathbb{P}^n.$$

Lemma 3.14. For any $i = 0, \dots, n$, we have

$$\mathbb{P}^n = U_i \sqcup P_i.$$

Moreover if we define $f_i : \mathbb{P}^{n-1} \to P_i$ by

$$f_i \left[z_0, \dots, z_{n-1}\right] \mapsto \left[z_0, \dots, z_{i-1}, 0, z_i, \dots, z_{n-1}\right]$$

then f_i is a bijection.

Proof. TODO Exercise: both statements are easy to check. □

In conclusion, we have that

$$\mathbb{P}^n = U_i \sqcup P_i \cong \mathbb{C}^n \cup \mathbb{P}^{n-1} = \dots = \mathbb{C}^n \cup \dots \cup \mathbb{C}^0.$$

P_i is called the hyperplane at infinity.

Example 3.15. In $\mathbb{R}$, $\mathbb{P}^1 (\mathbb{R}) = \mathbb{R} \cup \mathbb{R}$ by the identification $r \mapsto 1/r$, so $\mathbb{P}^1 (\mathbb{R}) \cong S^1$, while $\mathbb{P}^2 (\mathbb{R}) = \mathbb{R}^2 \cup \mathbb{P}^1 (\mathbb{R})$ has a circle at infinity identified by directions.