M3P21 Geometry II: Algebraic Topology

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Spring 2019

Contents

0 Introduction

Lecture 1 Friday 11/01/19

0.1 Introduction

Combines topological spaces with algebraic objects, which are groups.

- How to show that a torus is not homeomorphic to a sphere?
- How to show that $\mathbb{R}^n \ncong \mathbb{R}^m$ if $n \neq m$?

Content is fundamental groups and homology. We will follow chapter one and two from

- A Hatcher, Algebraic topology, 2002

The following are prerequisites.

- Point set topology. Topological spaces, continuous maps, product and quotient topologies, Hausdorff spaces, etc.
- Basic group theory. Normal subgroups and quotients, isomorphism theorems, free groups, presentation of groups, etc.

0.2 Some underlying geometric notions

0.2.1 Homotopy and homotopy type

Let X, Y be topological spaces and $I = [0, 1]$.

Definition. A **homotopy** is a continuous map $F : X \times I \to Y$. For every $t \in I$ we obtain a continuous map

$$\begin{array}{cccc} f_t : & X & \to & Y \\ & x & \mapsto & f_t(x) = F(x,t) \end{array}.$$

Definition. Two continuous maps $f_0, f_1 : X \to Y$ are **homotopic** if there exists a homotopy $F : X \times I \to Y$ such that

$$f_0(x) = F(x, 0), \qquad f_1(x) = F(x, 1),$$

for all $x \in X$. We write $f_0 \cong f_1$. (Exercise: this is an equivalence relation)

Definition. Let $A \subseteq X$ be a subspace. A **retraction** of X onto A is a continuous map $r : X \to A$ such that

- $r(X) = A$, and
- $r|_A = id_A$.

Example. If $X \neq \emptyset$, $p \in X$, then X retracts to p by the constant map $X \to \{p\}$.

Definition. A **deformation retraction** of X onto $A \subseteq X$ is a retraction that is homotopic to the identity. That is, there is a continuous map

$$\begin{array}{cccc} F : & X \times I & \to & A \\ & (x,t) & \mapsto & f_t(x) \end{array},$$

such that $f_0 = id_X$ and $f_1 : X \to A$ is the deformation retraction.

Example. The closed n-dimensional n**-disc**

$$D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$

deformation retracts to $(0, \ldots, 0) \in \mathbb{R}^n$. Let $f_t(x) = t \cdot x$. $t = 1$ gives $f_1 = id_{D^n}$ and $t = 0$ gives $f_0 : D^n \to (0, \ldots, 0)$.

Example. Let S^n be the n-**sphere**,

$$\partial D^{n+1} = S^n = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

The cylinder $S^n \times I$ deformation retracts to $S^n \times \{0\}$, by defining $f_t(x, r) = (x, t \cdot r)$.

An observation is if X is a topological space, and $f : X \to \{p\}$ for $p \in X$ is a deformation retraction of X to p, then X is path-connected. Indeed, if $F : X \times I \to X$ is a homotopy from id_X to f and $x \in X$ is a point, then this gives a path

$$\begin{array}{ccc} I & \to & X \\ t & \mapsto & F(x, t) \end{array}$$

that connects x to p. This implies that not all retractions are deformation retractions.

Example. A retraction that is not a deformation retraction. Take a space that is not path-connected and retract it to a point. Let $X = \{0, 1\}$ with discrete topology. $x \mapsto 0$ is a retraction, but not a deformation retraction because X is not path-connected.

Definition. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \to X$ such that $fg \cong id_Y$ and $gf \cong id_X$. If there exists a homotopy equivalence between X and Y, X and Y are **homotopy equivalent** or they have the same **homotopy type**.

Lemma 0.1. *A deformation retraction $f : X \to A$ is a homotopy equivalence.*

Proof. Let $i : A \hookrightarrow X$ be the inclusion map. Then $fi = id_A$ and $if = f \cong id_X$ by definition. □

Example. The disc with two holes is equivalent to O · O.

Example. $\mathbb{R}^n$ deformation retracts to a point, by $f_t(x) = t \cdot x$.

Definition.

- X is **contractible** if it is homotopy equivalent to a point.
- A continuous map is **nullhomotopic** if it is homotopy equivalent to a constant map.

0.2.2 Cell complexes

Example. The torus $S^1 \times S^1$ is the union of a point, two open intervals, and the open disc $Int\left(D^2\right)$.

These are called **cells**. Can think of discs D^n glued together.

Lecture 2 Tuesday 15/01/19

Definition. A **CW-complex**, or **cell complex**, is a topological space X such that there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X^n,$$

where the X^n are constructed inductively in the following way.

- X^n is a discrete set.
- For each $n \geq 0$ there is an collection of closed n-discs $\{D_\alpha^n\}$ together with continuous maps $\phi_\alpha : \partial D_\alpha^n \to X^{n-1}$, such that
$$X^n = \frac{X^{n-1} \sqcup \bigsqcup_\alpha D_\alpha^n}{\sim},$$
where $x \sim \phi_\alpha(x)$ for all $x \in \partial D_\alpha^n$ for all α.
- A subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n.

Remark.

- As a set,
$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha} e_{\alpha}^n,$$
where each e_{α}^n is homeomorphic to an open n-disc. These e_{α}^n are called the **n-cells** of X.

- If $X = X^m$ for some m, then X is called **finite dimensional**. The minimal m such that $X = X^m$ is the **dimension** of X.

Example.

- $[0, 1]$ is a CW-complex.
- $\mathbb{R}$ is a CW-complex.
- S^1 is a CW-complex.
- A graph is a CW-complex.
- $S^n = D^n/\partial D^n$ is a CW-complex. See worksheet 1.

Can also decompose CW-complexes.

- The sphere S^2 is one 0-cell, one 1-cell, and two 2-cells.
- The torus $S^1 \times S^1$ is one 0-cell, two 1-cells, and one 2-cell.
- The Möbius strip is two 0-cells, three 1-cells, and one 2-cell.
- The Klein bottle is one 0-cell, two 1-cells, and one 2-cell.

Definition. If X is a CW-complex with finitely many cells the **Euler characteristic** $\chi(X)$ of X is the number of even cells minus the number of odd cells.

Fact. $\chi(X)$ does not depend of the choice of cells decomposition.

Example.

- $\chi(S^n) = 0$ if n is odd and $\chi(S^n) = 2$ if n is even.
- $\chi\left(S^1 \times S^1\right) = 0$.

This is the generalisation of the following observation by Leonhard Euler. Let P be a convex polyhedron, where

- V is the number of vertices of P,
- E is the number of edges of P, and
- F is the number of faces of P.

Then $V - E + F = 2$.

Example. A topological space that is not a CW-complex. $X = \{0, 1\}$ with trivial topology does not contain any closed points.

Fact. CW-complexes are always Hausdorff.

1 The fundamental group

1.1 Basic constructions

1.1.1 Paths and homotopy

Let X be a topological space. A **path** is a continuous map $f : I \to X$, where $I = [0,1]$.

Definition. Two paths f_0, f_1 are **homotopic** if there exists a homotopy between f_0 and f_1 preserving the endpoints, that is a continuous map

$$\begin{array}{cccc} F : & I \times I & \to & X \\ & (s,t) & \mapsto & f_t(s) \end{array},$$

such that

$$f_t(0) = f_0(0), \qquad f_t(1) = f_0(1),$$

for all $t \in I$, and

$$F(s,0) = f_0(s), \qquad F(s,1) = f_1(s),$$

for all $s \in I$.

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set. Then all the paths in X are homotopic if they have the same endpoints.

Proof. Let $f_0, f_1 : I \to X$ be two paths such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$. Define

$$f_t(s) = (1-t) f_0(s) + t f_1(s).$$

□

Lemma 1.1. *Being homotopic is an equivalence relation on the set of paths with fixed endpoints. We will write $f_0 \cong f_1$ for two homotopic paths f_0 and f_1.*

Proof.

- f is homotopic to f.
- If f_0 is homotopic to f_1 by a homotopy f_t, then f_1 is homotopic to f_0 by the homotopy f_{1-t}.
- If f_0 is homotopic to f_1 by a homotopy f_t and $f_1 = g_0$ is homotopic to g_1 by a homotopy g_t, then f_0 is homotopic to g_1 by the homotopy

$$h_t = \begin{cases} f_{2t} & 0 \le t \le \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}.$$

  Then

$$\begin{array}{cccc} H : & I \times I & \to & X \\ & (s,t) & \mapsto & h_t(s) \end{array}$$

  is continuous because its restriction to the closed subsets $I \times [0, 1/2]$ and $I \times [1/2, 1]$ is continuous, since if the restriction to two closed subsets is continuous then the restriction to the union of these subsets is continuous.

□

Lecture 3 Wednesday 16/01/19

Let X be a topological space and $I = [0,1]$. If $f : I \to X$ is a path, $[f]$ is the class of all paths on X homotopic to f.

Definition. Let $f, g : I \to X$ be two paths such that $f(1) = g(0)$. The **product path** $f \cdot g$ is the path

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}.$$

A convention is that whenever we write $f \cdot g$ we implicitly assume $f(1) = g(0)$.

Lemma 1.2. *Let f_0, f_1, g_0, g_1 be paths on X such that $f_1 \cong f_0$ and $g_0 \cong g_1$. Then $f_0 \cdot g_0 \cong f_1 \cdot g_1$.*

Proof.

$$\begin{array}{ccc} I \times I & \to & X \\ (s,t) & \mapsto & (f_t \cdot g_t)(s) \end{array}$$

is a homotopy between $f_0 \cdot g_0$ and $f_1 \cdot g_1$. □

Remark. Let $\phi : [0,1] \to [0,1]$ be continuous such that $\phi(0) = 0$ and $\phi(1) = 1$. If $f : I \to X$ is a path, then $f\phi \cong f$. This is a **reparametrisation**.

Proof. Define

$$\phi_t(s) = (1-t)\phi(s) + ts,$$

then $f\phi_t$ is a homotopy between $f\phi$ and f. □

For $x \in X$, let the **constant path** at x be

$$c_x : \begin{array}{ccc} I & \to & X \\ s & \mapsto & x \end{array}.$$

For a path $f : I \to X$, define

$$f^{-1} : \begin{array}{ccc} I & \to & X \\ s & \mapsto & f(1-s) \end{array}.$$

Lemma 1.3. *Let $f, g, h : I \to X$ be paths. Then*

1. *$(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$,*
2. *$f \cdot c_{f(1)} \cong f$ and $c_{f(0)} \cdot f \cong f$, and*
3. *$f \cdot f^{-1} \cong c_{f(0)}$ and $f^{-1} \cdot f \cong c_{f(1)}$.*

Proof.

1. $((f \cdot g) \cdot h)\phi = f \cdot (g \cdot h)$, where
$$\phi(s) = \begin{cases} \frac{s}{2} & s \in \left[0, \frac{1}{2}\right] \\ s - \frac{1}{4} & s \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ 2s - 1 & s \in \left[\frac{3}{4}, 1\right] \end{cases},$$
so $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$ by reparametrisation.
2. Again reparametrisation, by
$$\psi(s) = \begin{cases} 2s & s \in \left[0, \frac{1}{2}\right] \\ 1 & s \in \left[\frac{1}{2}, 1\right] \end{cases}, \qquad \chi(s) = \begin{cases} 0 & s \in \left[0, \frac{1}{2}\right] \\ 2s - 1 & s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$
3. Define
$$H(s,t) = \begin{cases} f(\max\{1-2s, t\}) & s \in \left[0, \frac{1}{2}\right] \\ f(\max\{2s-1, t\}) & s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$
H is continuous, and
$$H(s,0) = f^{-1} \cdot f, \qquad H(s,1) = c_{f(1)}.$$
The inverse is similar.

□

Definition. A **loop** with **basepoint** $x_0 \in X$ is a path $f : I \to X$ such that $f(0) = f(1) = x_0$.

Definition. Denote by $\pi_1(X, x_0)$ the set of homotopy classes $[f]$ of loops $f : I \to X$ with basepoint x_0.

Proposition 1.4. *$\pi_1(X, x_0)$ is a group with product $[f][g] = [f \cdot g]$ and neutral element $c_{x_0} : I \to X$, the constant path at x_0.*

Proof. Follows directly from Lemma 1.2 and Lemma 1.3. □

Definition. $\pi_1(X, x_0)$ is the **fundamental group** of X at x_0.

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set and $x_0 \in X$. Then $\pi_1(X, x_0) = 0$.

Proof. X is convex gives that all loops are homotopic to each other. □

Example.

- The fundamental group of a space X with the trivial topology is trivial, since X is simply-connected, because all maps $f : I \to X$ are continuous, so path-connected and all paths are homotopic.
- The fundamental group of a space X with the discrete topology is trivial, since $f : I \to X$ continuous gives f constant.

Assume $x_0, x_1 \in X$ such that x_0 and x_1 are in the same path component of X. Let $h : I \to X$ be a path such that $h(0) = x_0$ and $h(1) = x_1$. Define

$$\begin{array}{rrcl} \beta_h : & \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ & [f] & \mapsto & \left[h \cdot f \cdot h^{-1}\right] \end{array}.$$

This is well-defined by Lemma 1.2.

Proposition 1.5. *$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is an isomorphism.*

Proof. It is a homomorphism.

$$\beta_h[f \cdot g] = \left[h \cdot f \cdot g \cdot h^{-1}\right] = \left[h \cdot f \cdot h^{-1}\right]\left[h \cdot g \cdot h^{-1}\right] = \beta_h[f] \cdot \beta_h[g],$$

and $\beta_h[c_{x_1}] = [c_{x_1}]$. It is bijective with $(\beta_h)^{-1} = \beta_{h^{-1}}$. □

If X is path-connected, we often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Definition. X is **simply-connected** if it is path-connected and $\pi_1(X) = 0$.

Proposition 1.6. *X is simply-connected if and only if there exists a unique homotopy class of paths between any two points of X.*

Proof.

$\implies$ There exists a path between any two points. Let f, g be two paths from x_0 to x_1 for $x_0, x_1 \in X$. $f \cdot g^{-1} \cong g \cdot g^{-1}$ gives $f \cong f \cdot g^{-1} \cdot g \cong g \cdot g^{-1} \cdot g \cong g$.

$\impliedby$ X is path-connected. $x_1 = x_0$ gives that all loops at x_0 are homotopic to each other, so $\pi_1(X) = 0$.

□

1.1.2 The fundamental group of the circle

Goal is to show that $\pi_1\left(S^1\right) \cong \mathbb{Z}$. Lecture 4 Friday 18/01/19

Definition. A **covering space** of a space X is a space $\widetilde{X}$ and a continuous map $p : \widetilde{X} \to X$ such that for each $x \in X$ there is an open $x \in U \subseteq X$ such that

- $p^{-1}(U) = \bigcup_{j \in J} \widetilde{U_j}$, where $\widetilde{U_j} \subseteq \widetilde{X}$ is open,
- $\widetilde{U_i} \cap \widetilde{U_j} = \emptyset$ if $i \neq j$, and
- $p \mid_{\widetilde{U_j}} : \widetilde{U_j} \to U$ is a homeomorphism for all $j \in J$.

Such a U is called **evenly covered**. The $\widetilde{U_j}$ are called **sheets**.

Example.

$$\begin{array}{rccc} p : & \mathbb{R} & \to & S^1 \\ & s & \mapsto & \left(\cos(2\pi s), \sin(2\pi s)\right) \end{array}.$$

Definition. Let $p : \widetilde{X} \to X$ be a covering space. A **lift** of a continuous map $f : Y \to X$ is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$, so

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}.$$

Proposition 1.7 (Unique lifting property)**.** *Let $p : \widetilde{X} \to X$ be a covering space and $f : Y \to X$ be a continuous map. If there are two lifts $\widetilde{f_1}, \widetilde{f_2} : Y \to \widetilde{X}$ of f such that $\widetilde{f_1}(y) = \widetilde{f_2}(y)$ for some $y \in Y$ and if Y is connected, then $\widetilde{f_1} = \widetilde{f_2}$.*

Proof. Let $y \in Y$ and $U \subseteq X$ be an evenly covered neighbourhood of $f(y)$. Then

$$p^{-1}(U) = \bigcup_j \widetilde{U_j}.$$

Let $\widetilde{U_1}$ be the sheet such that $\widetilde{f_1}(y) \in \widetilde{U_1}$, and let $\widetilde{U_2}$ be the sheet such that $\widetilde{f_2}(y) \in \widetilde{U_2}$. Let $N \subseteq Y$ be open and $y \in N$ such that $\widetilde{f_1}(N) \subseteq \widetilde{U_1}$ and $\widetilde{f_2}(N) \subseteq \widetilde{U_2}$. We have $p\widetilde{f_1} = p\widetilde{f_2}$.

$$\widetilde{f_1}(y) = \widetilde{f_2}(y) \qquad \iff \qquad \widetilde{U_1} = \widetilde{U_2} \qquad \iff \qquad \widetilde{f_1} \mid_N = \widetilde{f_2} \mid_N .$$

Let

$$A = \left\{ y \in Y \mid \widetilde{f_1}(y) = \widetilde{f_2}(y) \right\},$$

so A is open and $Y \setminus A$ is open. Thus $A \neq \emptyset$ gives $A = Y$. □

Proposition 1.8 (Homotopy lifting property)**.** *Let $p : \widetilde{X} \to X$ be a covering space and $F : Y \times I \to X$ be a continuous map such that there exists a lift $\widetilde{f_0} : Y \times \{0\} \to \widetilde{X}$ of $F \mid_{Y \times \{0\}}$. Then there is a unique lift $\widetilde{F} : Y \times I \to \widetilde{X}$ of F such that $\widetilde{F} \mid_{Y \times \{0\}} = \widetilde{f_0}$.*

Proof. Let $y_0 \in Y$ and $t \in I$. There are open $y_0 \in N_t \subseteq Y$ and $t \in (a_t, b_t) \subseteq I$ such that $F(N_t \times (a_t, b_t)) \subseteq U \subseteq X$, where $U \subseteq X$ is open and evenly covered. Compactness of I gives that there exist

$$0 = t_0 < \cdots < t_m = 1,$$

and there exists $y_0 \in N \subseteq Y$ open such that $F(N \times [t_i, t_{i+1}]) \subseteq U_i \subseteq X$, where $U_i \subseteq X$ is open and evenly covered. We inductively construct a lift $\widetilde{F} \mid_{N \times I}$ of $F \mid_{N \times I}$.

- $\widetilde{F}\mid_{N\times[0,0]}=\widetilde{f_0}\mid_{N\times[0,0]}$ exists.
- Assume the lift has been constructed on $N\times[0,t_i]$. Let $\widetilde{U_i}\subseteq\widetilde{X}$ be such that $p\mid_{\widetilde{U_i}}:\widetilde{U_i}\to U_i$ such that $\widetilde{F}(y_0,t_i)\subseteq\widetilde{U_i}$. After shrinking N, may assume $\widetilde{F}(N\times\{t_i\})\subseteq\widetilde{U_i}$. Define $\widetilde{F}$ on $N\times[t_i,t_{i+1}]$ to be composition of F with the homeomorphism $p^{-1}:U_i\to\widetilde{U_i}$.

After finitely many steps we obtain a lift $\widetilde{F}:N\times I\to\widetilde{X}$, where $y_0\in N\subseteq Y$ is open, so for each $y\in Y$ there is a neighbourhood $N_y\subseteq Y$ such that $F\mid_{N_y\times I}:N_y\times I\to X$ lifts. For all $y\in Y$, $\{y\}\times I$ is connected and can be lifted, so Proposition 1.7 gives that the lift of $N\times I$ is unique. Thus there is a unique lift $\widetilde{F}:Y\times I\to\widetilde{X}$. □

Example. Let X be a topological space and A be discrete. Then $p:X\times A\to X$ is a covering space. This is the **trivial covering**. (Exercise: show the unique lifting property and the homotopy lifting property for the trivial covering)

Corollary 1.9. *Let $f:I\to X$ be a path, $f(0)=x_0$, and $p:\widetilde{X}\to X$ be a covering space. For each $\widetilde{x_0}\in p^{-1}(x_0)$, there is a unique lift $\widetilde{f}:I\to\widetilde{X}$ such that $\widetilde{f}(0)=\widetilde{x_0}$.*

Proof. Proposition 1.8 for Y a point. □

Theorem 1.10. *Let $x_0=(1,0)\in S^1$. $\pi_1\left(S^1,x_0\right)$ is the infinite cyclic group generated by the homotopy class of the loop*

$$\begin{array}{rccl}\omega: & I & \to & S^1 \\ & s & \mapsto & (\cos(2\pi s),\sin(2\pi s))\end{array}.$$

Remark.

- $[\omega]^n=[\omega_n]$, where
$$\omega_n(s)=(\cos(2\pi ns),\sin(2\pi ns)).$$
- $$\begin{array}{rccl}p: & \mathbb{R} & \to & S^1 \\ & s & \mapsto & (\cos(2\pi s),\sin(2\pi s))\end{array}$$
is a covering space.
- ω_n lifts to
$$\begin{array}{rccl}\widetilde{\omega_n}: & I & \to & \mathbb{R} \\ & s & \mapsto & ns\end{array},$$
such that $\widetilde{\omega_n}(0)=0$ and $\widetilde{\omega_n}(1)=n$.

Proof of Theorem 1.10.

- If $f:I\to S^1$ be a loop at x_0, then the homotopy lifting property gives that there exists a lift $\widetilde{f}:I\to\mathbb{R}$ such that $\widetilde{f}(0)=0$. Since $p\left(\widetilde{f}(1)\right)=f(1)=x_0$, then $\widetilde{f}(1)=n$ for some $n\in\mathbb{Z}$. $\widetilde{\omega_n}:I\to\mathbb{R}$ is another path such that $\widetilde{\omega_n}(0)=0$ and $\widetilde{\omega_n}(1)=n$, so $\widetilde{f}\simeq\widetilde{\omega_n}$. Let $F:I\times I\to\mathbb{R}$ be a homotopy equivalence between $\widetilde{f}$ and $\widetilde{\omega_n}$. Then $pF:I\times I\to S^1$ gives a homotopy between $p\widetilde{f}=f$ and $p\widetilde{\omega_n}=\omega_n$.
- Let $m,n\in\mathbb{Z}$ and assume $\omega_m\simeq\omega_n$. Let $F:I\times I\to S^1$ be a homotopy.
$$F(0,t)=\omega_m(t),\qquad F(1,t)=\omega_n(t),\qquad F(s,0)=F(s,1)=x_0,$$
for all $s,t\in I$. The unique lifting property gives that $\widetilde{\omega_n},\widetilde{\omega_m}:I\to\mathbb{R}$ are unique lifts such that $\widetilde{\omega_n}(0)=0=\widetilde{\omega_m}(0)$. The homotopy lifting property gives that F lifts uniquely to a homotopy $\widetilde{F}:I\times I\to\mathbb{R}$ between $\widetilde{\omega_n}$ and $\widetilde{\omega_m}$, and $\widetilde{F}(s,1)\in\mathbb{Z}$ for all $s\in I$. Thus $\widetilde{F}(s,1)=n=m$, so $\omega_m\simeq\omega_n$ if and only if $n=m$.

□

Lecture 5 Tuesday 22/01/19

Lecture 6 Wednesday 23/01/19

Lecture 5 is a problem class.

Theorem 1.11. *Every non-constant polynomial $p \in \mathbb{C}[z]$ has a root in $\mathbb{C}$.*

Proof. May assume
$$p(z) = z^n + a_1 z^{n-1} + \dots + a_n.$$
Assume p has no roots in $\mathbb{C}$. For each $r \in \mathbb{R}_{\geq 0}$ we obtain a loop
$$\begin{array}{rccc} f_r : & I & \to & \mathbb{C} \\ & s & \mapsto & \dfrac{p\left(re^{2\pi i s}\right)/p(r)}{\left|p\left(re^{2\pi i s}\right)/p(r)\right|} \end{array},$$
so $|f_r(s)| = 1$. $f_r(0) = 1$ and $f_r(1) = 1$, so f_r is a loop based at 1. f_0 is the constant loop at 1. $f_r(s)$ depends continuously on r, so $f_r \simeq f_0$ for all $r \in \mathbb{R}_{\geq 0}$ and $[f_r] = [f_0] = 0 \in \pi_1\left(S^1\right)$. Fix $r \in \mathbb{R}_{\geq 0}$ such that $r > 1$ and $r > |a_1| + \dots + |a_n|$. For $|z| = r$ we have
$$|z^n| > (|a_1| + \dots + |a_n|)\left|z^{n-1}\right| \geq \left|a_1 z^{n-1}\right| + \dots + |a_n| \geq \left|a_1 z^{n-1} + \dots + a_n\right|.$$
Hence, for $0 \leq t \leq 1$ the polynomial $p_t(z) = z^n + t\left(a_1 z^{n-1} + \dots + a_n\right)$ has no root z with $|z| = r$. Define
$$F_r(t,s) = \frac{p_t\left(re^{2\pi i s}\right)/p_t(r)}{\left|p_t\left(re^{2\pi i s}\right)/p_t(r)\right|}.$$
$F_r(0,s) = \omega_n(s)$ and $F_r(1,s) = f_r(s)$, so $[\omega_n] = [f_r] = 0 \in \pi_1\left(S^1\right)$. Theorem 1.10 gives that $n = 0$, so p is constant. ☐

See Hatcher Theorem 1.9 and Theorem 1.10 for more applications.

Proposition 1.12. *Let X, Y be topological spaces, $x_0 \in X$, and $y_0 \in Y$. Then*
$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. A map
$$\begin{array}{rccc} f : & Z & \to & X \times Y \\ & z & \mapsto & (g(z), h(z)) \end{array}$$
is continuous if and only if $g : Z \to X$ and $h : Z \to Y$ are continuous. For $Z = I$,
$$\{\text{ loops in } X \times Y \text{ based at } (x_0, y_0) \} \quad \leftrightsquigarrow \quad \{\text{ loops in } X \text{ based at } x_0 \} \times \{\text{ loops in } Y \text{ based at } y_0 \}.$$
Two loops
$$\begin{array}{rccc} f_1 : & I & \to & X \times Y \\ & s & \mapsto & (g_1(s), h_1(s)) \end{array}, \qquad \begin{array}{rccc} f_2 : & I & \to & X \times Y \\ & s & \mapsto & (g_2(s), h_2(s)) \end{array}$$
are homotopic if and only if $g_1 \simeq g_2$ and $h_1 \simeq h_2$, so there is a bijection
$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$
$f_1 \cdot f_2 = (g_1 \cdot g_2, h_1 \cdot h_2)$ and the constant loop is mapped to the constant loop, so this is also a group isomorphism. ☐

Example. The torus $S^1 \times S^1$ has
$$\pi_1\left(S^1 \times S^1\right) \cong \pi_1\left(S^1\right) \times \pi_1\left(S^1\right) \cong \mathbb{Z}^2.$$

1.1.3 Induced homomorphisms

Let X, Y be topological spaces, $x_0 \in X$, and $\phi : X \to Y$. An observation is that ϕ induces a homomorphism

$$\begin{array}{rccc} \phi_* : & \pi_1(X, x_0) & \to & \pi_1(Y, \phi(x_0)) \\ & [f] & \mapsto & [\phi f] \end{array}.$$

ϕ_* is well-defined, since if f_t is a homotopy between the loops f_0 and f_1 based at x_0, then ϕf_t is a homotopy of loops between ϕf_0 and ϕf_1. Moreover,

$$\phi(f \cdot g) = (\phi f) \cdot (\phi g),$$

and ϕ maps the constant path at x_0 to the constant path at $\phi(x_0)$, so ϕ is a homomorphism.

Proposition 1.13.

1. *Let $\psi : X \to Y$ and $\phi : Y \to Z$ be continuous maps between topological spaces, $x_0 \in X$, and*

$$\psi_* : \pi_1(X, x_0) \to \pi_1(Y, \psi(x_0)), \qquad \phi_* : \pi_1(Y, \psi(x_0)) \to \pi_1(Z, \phi\psi(x_0)),$$

$$(\phi\psi)_* : \pi_1(X, x_0) \to \pi_1(Z, \phi\psi(x_0)).$$

   Then $(\phi\psi)_* = \phi_*\psi_*$.

2. *Let $id_X : X \to X$ be the identity then*

$$(id_X)_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

   is the identity.

Proof.

1. Let $f : I \to X$ be a loop at x_0, then

$$(\phi\psi)_*([f]) = [(\phi\psi) f] = [\phi(\psi f)] = \phi_*([\psi f]) = \phi_*\psi_*([f]).$$

2. $(id_X)_*([f]) = [id_X f] = [f]$.

□

These two observations yield in particular that if $\phi : X \to Y$ is a homeomorphism with inverse $\psi : Y \to X$, then

$$\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$$

is an isomorphism with inverse ψ_*. Lecture 7 Friday 25/01/19

Proposition 1.14. *Let $\phi : X \to Y$ be a homotopy equivalence. Then*

$$\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Recall that if $x_0, x_1 \in X$ and $h : I \to X$ is a path such that $h(0) = x_0$ and $h(1) = x_1$, then we obtain an isomorphism

$$\begin{array}{rccc} \beta_h : & \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ & [f] & \mapsto & \left[h \cdot f \cdot h^{-1}\right] \end{array}.$$

Lemma 1.15. *Let $\phi_t : X \to Y$ be a homotopy and $x_0 \in X$. Define the path*

$$\begin{array}{cccc} h: & I & \to & Y \\ & s & \mapsto & \phi_s(x_0) \end{array},$$

where $h(0) = \phi_0(x_0)$ and $h(1) = \phi_1(x_0)$. Then $(\phi_0)_* = \beta_h (\phi_1)_*$, that is the following diagram commutes.

$$\begin{tikzcd} & \pi_1(Y, \phi_1(x_0)) \arrow[dd, "\beta_h", "\sim"'] \\ \pi_1(X, x_0) \arrow[ur, "(\phi_1)_*"] \arrow[dr, "(\phi_0)_*"'] & \\ & \pi_1(Y, \phi_0(x_0)) \end{tikzcd}.$$

Proof. For $t \in I$, define the path

$$\begin{array}{cccc} h_t: & I & \to & X \\ & s & \mapsto & h(ts) \end{array},$$

where $h_t(0) = \phi_0(x_0)$ and $h_t(1) = h(t) = \phi_t(x_0)$. Let f be a loop at x_0. Define

$$F_t = h_t \cdot (\phi_t f) \cdot h_t^{-1}.$$

Then F_t is a loop at $\phi_0(x_0)$, which is continuous in t. So F_t is a homotopy of loops between

$$F_0 = h_0 \cdot (\phi_0 f) \cdot h_0^{-1} \cong \phi_0 f, \qquad F_1 = h_1 \cdot (\phi_1 f) \cdot h_1^{-1} = h \cdot (\phi_1 f) \cdot h^{-1}.$$

Hence

$$(\phi_0)_*([f]) = [\phi_0 f] = \left[h \cdot (\phi_1 f) \cdot h^{-1}\right] = \beta_h([\phi_1 f]) = \beta_h (\phi_1)_*([f]).$$

□

Lemma 1.15 implies in particular the following.

Corollary 1.16. *If $\psi : X \to X$ is continuous and $\psi \cong id_X$, then*

$$\psi_* : \pi_1(X, x_0) \to \pi_1(X, \psi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Proof. By Lemma 1.15 there is a path h from $\psi(x_0)$ to x_0 such that

$$\begin{tikzcd} & \pi_1(X, x_0) \arrow[dd, "\beta_h", "\sim"'] \\ \pi_1(X, x_0) \arrow[ur, "(id_X)_*"] \arrow[dr, "\psi_*"'] & \\ & \pi_1(X, \psi(x_0)) \end{tikzcd},$$

so $\psi_* = \beta_h$ hence an isomorphism. □

Proof of Proposition 1.14. Let $\phi : X \to Y$ be a homotopy equivalence. Let $\psi : Y \to X$ be a homotopy inverse of ϕ, that is $\phi\psi \cong id_Y$ and $\psi\phi \cong id_X$.

$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\phi(x_0)) \xrightarrow{\phi_*} \pi_1(Y, \psi\phi\psi(x_0)).$$

Have to show that ϕ_* is bijective. The observation above gives that $(\psi\phi)_* = \psi_*\phi_*$ is an isomorphism, so ϕ_* is injective and ψ_* is surjective. Similarly $(\phi\psi)_* = \phi_*\psi_*$ is an isomorphism, so ψ_* is injective and ϕ_* is surjective. □

Lemma 1.17. *Let X be a topological space and $x_0 \in X$. Assume*

$$X = \bigcup_{\alpha \in \Lambda} A_\alpha,$$

such that

- *the A_α are all open and path-connected,*
- *$x_0 \in A_\alpha$ for all $\alpha \in \Lambda$, and*
- *all the intersections $A_\alpha \cap A_\beta$ are path-connected for all $\alpha, \beta \in \Lambda$.*

If f is a loop in X at x_0, then we can write $[f] = [h_1] \dots [h_m]$, such that the h_i are loops at x_0, and each contained in a single A_{α_i}.

Proof. f is continuous, so for all $s \in I$ there is an open neighbourhood V_s such that $f(V_s)$ such that $f(V_s) \subseteq A_\alpha$ for some α. We can choose V_s to be an interval (a_s, b_s) such that $f([a_s, b_s]) \subseteq A_\alpha$. I is compact gives that a finite number of such intervals cover I, so there is a partition

$$0 = s_0 < \dots < s_m = 1,$$

such that $f([s_{i-1}, s_i]) \subseteq A_{\alpha_i}$ for some α_i. Let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$, and rescaling. $f \cong f_1 \cdot \dots \cdot f_m$ for $f_i \subseteq A_{\alpha_i}$ and $A_{\alpha_i} \cap A_{\alpha_j}$ is path-connected. Let g_i be a path from x_0 to $f(s_i)$ in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$. Let g_0, g_m be the constant loops at x_0. $h_i = g_{i-1} \cdot f_i \cdot g_i^{-1}$ is a loop based at x_0 and $h_i \subseteq A_{\alpha_i}$. Thus

$$f \cong \left(g_0 \cdot f_1 \cdot g_1^{-1}\right) \cdot \dots \cdot \left(g_{m-1} \cdot f_m \cdot g_m^{-1}\right),$$

so $[f] = [h_1] \dots [h_m]$. □

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Example. Möbius strip M deformation retracts to S^1. Thus $\phi : M \to S^1$ is a homotopy equivalence, so $\pi_1(M) \cong \pi_1(S^1) \cong \mathbb{Z}$.

Example. There is no deformation retraction of S^1 to a point $p \in S^1$ because $\pi_1(S^1) \ncong \pi_1(p)$.

Example. There is no retraction of the disc D^2 to its boundary $S^1 \subseteq D^2$.

Proof. Assume there is a retraction $r : D^2 \to S^1$, consider the embedding $i : S^1 \hookrightarrow D^2$. Then $ri = id_{S^1}$. Thus

$$\begin{array}{ccccc} \pi_1\left(S^1\right) & \xrightarrow{i_*} & \pi_1\left(D^2\right) & \xrightarrow{r_*} & \pi_1\left(S^1\right) \\ \cong & & \cong & & \cong \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array},$$

so $r_* i_* \left(\pi_1\left(S^1\right)\right) = 0$ but $r_* i_* = (ri)_* = id_{\pi_1(S^1)}$, a contradiction. □

Theorem 1.18 (Brouwer fixed point theorem)**.** *Let $h : D^2 \to D^2$ be a continuous map. Then h has a fixed point, that is there exists $x \in D^2$ such that $h(x) = x$.*

Proof. Assume $h(x) \neq x$ for all $x \in D^2$. Define $r : D^2 \to S^1$ by defining $r(x)$ to be the intersection of the ray starting at $h(x)$ towards x with S^1. r is continuous, and if $x \in S^1$, then $r(x) = x$, so r is a retraction, a contradiction. □

Lemma 1.17 gives that if $U_1, U_2 \subseteq X$ are open and path-connected such that $U_1 \cup U_2 = X$ and $U_1 \cap U_2$ is path-connected and $x_0 \in U_1 \cap U_2$, then every $[f] \in \pi_1(X, x_0)$ can be factorised as $[f] = [g_1][h_1] \dots [g_n][h_n]$ such that the g_i are loops at x_0 contained in U_1 and the h_i are loops at x_0 contained in U_2. In other words, $i_1 : U_1 \hookrightarrow X$ and $i_2 : U_2 \hookrightarrow X$, so

$$(i_1)_* : \pi_1(U_1, x_0) \to \pi_1(X, x_0), \qquad (i_2)_* : \pi_1(U_2, x_0) \to \pi_1(X, x_0).$$

Lemma 1.17 gives that $(i_1)_*(\pi_1(U_1, x_0)) \cup (i_2)_*(\pi_1(U_2, x_0))$ generate $\pi_1(X, x_0)$.

Proposition 1.19. *$\pi_1(S^n) = 0$ if $n \geq 2$.*

Proof. Let $U_1 = S^n \setminus \{(1, 0, \dots, 0)\}$ and $U_2 = S^n \setminus \{(-1, 0, \dots, 0)\}$. Then $U_1 \cong \mathbb{R}^n$ and $U_2 \cong \mathbb{R}^n$, by stereographic projection. $U_1 \cup U_2 = S^n$ and $U_1 \cap U_2$ is path-connected. Let $x_0 \in U_1 \cap U_2$. $\pi_1(U_1, x_0) = 0$ and $\pi_1(U_2, x_0) = 0$, so Lemma 1.17 gives that $\pi_1(S^n, x_0)$. □

1.2 Seifert-van Kampen theorem

1.2.1 Free products with amalgamation

Definition. If S is a set, then F_S is the **free group** on S. We can write any group G as a quotient of some free group F_S,

$$G = \frac{F}{\langle\langle R \rangle\rangle},$$

where $\langle\langle R \rangle\rangle$ is the **normal closure** of $R \subseteq F_S$, the smallest normal subgroup of F_S containing R. We write $G = \langle S \mid R \rangle$. This is called a **presentation** of G.

Let G_0, G_1, G_2 be groups, and $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$ be homomorphisms.

Definition. A group H together with homomorphisms $h_1 : G_1 \to H$ and $h_2 : G_2 \to H$ such that $h_1 f_1 = h_2 f_2$ is an **amalgamated product** of G_1 and G_2 over G_0 if it satisfies the following universal property. For every group G and all homomorphisms $h_1' : G_1 \to G$ and $h_2' : G_2 \to G$ such that $h_1' f_1 = h_2' f_2$, there exists a unique homomorphism $\alpha : H \to G$ such that $h_1' = \alpha h_1$ and $h_2' = \alpha h_2$.

$$\begin{tikzcd}
G_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & G_1 \arrow[d, "h_1"] \arrow[ddr, bend left, "h_1'"] & \\
G_2 \arrow[r, "h_2"'] \arrow[drr, bend right, "h_2'"'] & H \arrow[dr, "\exists!\alpha"] & \\
& & G.
\end{tikzcd}$$

Theorem 1.20. *Given $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$. Then there exists an amalgamated product, unique up to isomorphism. We denote it by $G_1 \underset{G_0}{*} G_2$.*

Proof. Worksheet 2. $\square$

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$G_0 = \{id\}$ is the **free product**. We write $G_1 * G_2$ instead of $G_1 \underset{\{id\}}{*} G_2$. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Then $G_1 * G_2 = \langle S_1 \sqcup S_2 \mid R_1 \cup R_2 \rangle$, with injections $G_i \hookrightarrow G_1 * G_2$ for $i = 1, 2$. More generally,

$$G_1 * G_2 \cong \frac{G_1 \underset{G_0}{*} G_2}{N}.$$

where N is the normal closure of the set

$$\left\{ f_1(g) f_2(g)^{-1} \mid g \in G_0 \right\} \subseteq G_1 * G_2.$$

1.2.2 The Seifert-vanKampen theorem

Theorem 1.21 (Seifert-van Kampen)**.** *Let X be a topological space and $U_1, U_2 \subseteq X$ be open and path-connected such that $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is path-connected and let $x_0 \in U_1 \cap U_2$. Then*

$$\pi_1(X, x_0) \cong \pi_1(U_1, x_0) \underset{\pi_1(U_1 \cap U_2, x_0)}{*} \pi_2(U_2, x_0) \cong \frac{\pi_1(U_1, x_0) * \pi_1(U_2, x_0)}{N},$$

where N is the normal closure of the set

$$\left\{ (j_1)_*(\omega) (j_2)_*(\omega)^{-1} \mid \omega \in \pi_1(U_1 \cap U_2, x_0) \right\},$$

and $j_i : U_1 \cap U_2 \hookrightarrow U_i$.

$$\begin{tikzcd}
U_1 \cap U_2 \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"'] & U_1 \arrow[d, hook, "j_1"] \\
U_2 \arrow[r, hook, "j_2"'] & X
\end{tikzcd}
\implies
\begin{tikzcd}
\pi_1(U_1 \cap U_2, x_0) \arrow[r, "(i_1)_*"] \arrow[d, "(i_2)_*"'] & \pi_1(U_1, x_0) \arrow[d, "(j_1)_*"] \\
\pi_1(U_2, x_0) \arrow[r, "(j_2)_*"'] & \pi_1(U_1, x_0) \underset{\pi_1(U_1 \cap U_2, x_0)}{*} \pi_2(U_2, x_0)
\end{tikzcd}.$$

Proof. Consider the natural homomorphism

$$\Phi : \pi_1(U_1, x_0) * \pi_1(U_2, x_0) \to \pi_1(X, x_0).$$

Φ is surjective by Lemma 1.17. $N \subseteq Ker(\Phi)$. Want to show that $N = Ker(\Phi)$. A **factorisation** of an element $[f] \in \pi_1(X, x_0)$ is a formal product $[f_1] \dots [f_k]$ such that

- each f_i is a loop at x_0 in one of the U_i and $[f_i] \in \pi_1(U_i, x_0)$ is its homotopy class, and
- the loop $f_1 \cdot \dots \cdot f_k$ is homotopic to f in X.

A factorisation of $[f]$ is a word in $\pi_1(U_1, x_0) * \pi_1(U_2, x_0)$ that is mapped to $[f]$ by Φ. Two factorisations of $[f]$ are **equivalent** if they are related by finitely many of the following two moves.

- If $[f_i]$ and $[f_{i+1}]$ lie in the same group $\pi_1(U_i, x_0)$, exchange $[f_i][f_{i+1}]$ with $[f_i \cdot f_{i+1}]$. These are the relations in $\pi_1(U_i, x_0) * \pi_1(U_i, x_0)$.
- If f_i is a loop in $U_1 \cap U_2$, consider $[f_i]$ as an element in $\pi_1(U_1, x_0)$ instead of $\pi_1(U_2, x_0)$, and vice versa. These are the relations in $\pi_1(U_1, x_0) * \pi_1(U_2, x_0) / N$.

Given $[f] \in \pi_1(X, x_0)$, we want to show that any two factorisations of $[f]$ are equivalent. Let $[f_1] \dots [f_k]$ and $[f_1'] \dots [f_l']$ be two factorisations of $[f]$, so the two loops $f_1 \cdot \dots \cdot f_k$ and $f_1' \cdot \dots \cdot f_k'$ are homotopic. Let $F : I \times I \to X$ be a homotopy. By compactness, there exist

$$0 = s_0 < \dots < s_m = 1, \qquad 0 = t_0 < \dots < t_n = 1,$$

such that $R_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ and $F(R_{ij}) \subseteq U_1$ or $F(R_{ij}) \subseteq U_2$. May assume $0 = s_0 < \dots < s_m = 1$ subdivides the products $f_1 \cdot \dots \cdot f_k$ and $f_1' \cdot \dots \cdot f_l'$. Relabel the R_{ij} to $R_1, \dots, R_{mn}$.

$mn - m + 1$	$\dots$	mn
$\vdots$	$\ddots$	$\vdots$
1	$\dots$	m

A path γ in $I \times I$ from left to right gives a loop $F|_\gamma$ in X at x_0. Let γ_r be the path separating the first r rectangles from the others, so

$$F|_{\gamma_0} \cong f_1 \cdot \dots \cdot f_k, \qquad F|_{\gamma_{mn}} = f_1' \cdot \dots \cdot f_l'.$$

Let v be a grid point. Choose a path g_v in X from x_0 to $F(v)$, such that g_v is contained in $U_1 \cap U_2$ if $F(v) \in U_1 \cap U_2$ and in a single U_i otherwise. This gives us a factorisation of $[F|_{\gamma_r}]$ into loops only contained in U_1 or U_2. The factorisations associated to γ_r and γ_{r+1} are equivalent, because the homotopy between $F|_{\gamma_r}$ and $F|_{\gamma_{r+1}}$ by pushing γ_r through R_r takes place within a single U_i. □

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Theorem 1.22 (Seifert-van Kampen, strong version)**.** *Let X be a path-connected topological space such that*

- $X = \bigcup_\alpha A_\alpha$,
- A_α, $A_\alpha \cap A_\beta$, *and* $A_\alpha \cap A_\beta \cap A_\gamma$ *are open and path-connected for all* α, β, γ, *and*
- $x_0 \in \cap_\alpha A_\alpha$.

Then

$$\pi_1(X, x_0) \cong \frac{\underset{\alpha}{*}\pi_1(A_\alpha, x_0)}{N},$$

where $N \subseteq \underset{\alpha}{*}\pi_1(A_\alpha, x_0)$ *is the normal closure of the set*

$$\left\{ (i_{\alpha\beta})_* (\omega) (i_{\beta\alpha})_* (\omega)^{-1} \mid \omega \in \pi_1(A_\alpha \cap A_\beta) \right\},$$

and $i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ *is the inclusion.*

Example. Let $S^1 \vee S^1$ be the wedge product. Fix $x \in S^1$ and $y \in S^1$. Then

$$S^1 \vee S^1 = \frac{S^1 \sqcup S^1}{x \sim y} = \overset{b}{\mathrm{O}} \cdot \overset{a}{\mathrm{O}}.$$

Let

$$A = \mathrm{O} \cdot (, \qquad B =) \cdot \mathrm{O}, \qquad A \cap B =) \cdot (.$$

$\pi_1(A) \cong \langle b \rangle \cong \mathbb{Z}$, $\pi_1(B) \cong \langle a \rangle \cong \mathbb{Z}$, and $\pi_1(A \cap B) = \{id\}$. A, B, and $A \cap B$ are open and path-connected. Van Kampen gives

$$\pi_1\left(S^1 \vee S^1\right) \cong \pi_1(A) * \pi_1(B) \cong \mathbb{Z} * \mathbb{Z} \cong F_{\{a,b\}}.$$

More generally, let $X = S^1_{a_1} \vee \dots \vee S^1_{a_n}$. By induction,

$$\pi_1(X) = \mathbb{Z} * \dots * \mathbb{Z} \cong F_{\{a_1,\dots,a_n\}}.$$

Similarly, let $X = \bigvee_{\alpha \in \Lambda} S^1_\alpha$. Strong version of van Kampen gives

$$\pi_1(X) = \mathop{*}_{\alpha \in \Lambda} \mathbb{Z} = F_\Lambda.$$

Example. Let T be a torus and $x_0 \in T$. Let

$$A = T \setminus \{\text{small closed disc } D\}, \qquad B = \{\text{open set that contains } D \text{ and } x_0\}.$$

- A is homotopy equivalent to $S^1 \vee S^1$, so $\pi_1(A) \cong F_{\{a,b\}}$.
- B is homeomorphic to D^2, so $\pi_1(B) = \{id\}$.
- $A \cap B$ is homotopy equivalent to S^1, so $\pi_1(A \cap B) \cong \mathbb{Z}$.

A, B, and $A \cap B$ are open and path-connected. Van Kampen gives

$$\pi_1(T) \cong \frac{\pi_1(A)}{\langle\langle i_*(\pi_1(A \cap B))\rangle\rangle},$$

where $i : A \cap B \hookrightarrow A$. Then

$$\begin{array}{rrcl} i_* : & \pi_1(A \cap B) = \langle \omega \rangle & \to & \pi_1(A) \\ & \omega & \mapsto & aba^{-1}b^{-1} \end{array},$$

so

$$\pi_1(T) \cong \frac{F_{\{a,b\}}}{\langle\langle aba^{-1}b^{-1}\rangle\rangle} = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle \cong \mathbb{Z}^2.$$

1.2.3 Applications to CW-complexes

Let X be a path-connected topological space. Let Y be the space obtained by attaching 2-cells $\left\{e^2_\alpha\right\}$ to X along maps $\phi_\alpha : \partial D^2 = S^1 \to X$. Consider the loops

$$\begin{array}{rrcl} \phi'_\alpha : & I & \to & X \\ & s & \mapsto & \phi_\alpha(\cos(2\pi s), \sin(2\pi s)) \end{array},$$

based at $\phi'_\alpha(0)$. Let γ_α be a path from x_0 to $\phi'_\alpha(0)$ for each α. Then $\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}$ is a loop at x_0. After attaching e^2_α, the loop $\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}$ is homotopic to the constant loop at x_0. Let $N \subseteq \pi_1(X, x_0)$ be the normal closure of all the elements of the form $\left[\gamma_\alpha \cdot \phi_\alpha \cdot \gamma_\alpha^{-1}\right]$. The inclusion $i : X \hookrightarrow Y$ yields

$$i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0),$$

and $N \subseteq Ker(i_*)$.

Proposition 1.23. *This inclusion $i : X \hookrightarrow Y$ induces a surjection*

$$i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0),$$

and $Ker(i_*) = N$, so

$$\pi_1(Y, x_0) \cong \frac{\pi_1(X, x_0)}{N}.$$

Proof. Construct a space Z from Y by attaching a strip $I \times I$ to Y by identifying the lower edge $I \times \{0\}$ with the path γ_α and the right edge $\{1\} \times I$ with an arch on e_α^2. Attach all the left edges of the strips with each other. Z deformation retracts to Y. Choose a point $y_\alpha \in e_\alpha^2$ for each α, such that y_α is not contained in X or in the attached strip. Let

$$A = Z \setminus \bigcup_\alpha \{y_\alpha\}, \qquad B = Z \setminus X.$$

- A deformation retracts to X.
- B is homotopy equivalent to a point.
- $A \cap B$ is homotopy equivalent to

$$\{\text{paths } \gamma_\alpha \text{ from } x_0 \text{ to loops } \phi'_\alpha\} = \overset{\phi'_\alpha}{\mathrm{O}} \overset{\gamma_\alpha}{-} \overset{x_0}{\cdot} \overset{\gamma_\alpha}{-} \overset{\phi'_\alpha}{\mathrm{O}}.$$

A, B, and $A \cap B$ are open and path-connected. Van Kampen gives

$$\pi_1(Y) \cong \pi_1(Z) = \frac{\pi_1(A)}{\langle\langle j_*(\pi_1(A \cap B))\rangle\rangle},$$

where $j : A \cap B \hookrightarrow A$ is the inclusion. So $\langle\langle j_*(\pi_1(A \cap B))\rangle\rangle$ is exactly N. Thus $\pi_1(A) = \pi_1(X)$. $\square$

Lecture 11 Tuesday 05/02/19

Corollary 1.24. *For every group G there exists a two-dimensional CW-complex X_G such that $\pi_1(X_G) = G$.*

Proof. Let $G = \langle \{g_\alpha\} \mid \{r_\beta\} \rangle$ be a presentation of G, that is

$$G = \frac{F_{\{g_\alpha\}}}{\langle\langle \{r_\beta\} \rangle\rangle}.$$

Seen last time that $\pi_1\left(\bigvee_{g_\alpha} S_{g_\alpha}^1\right) = F_{\{g_\alpha\}}$. Each word r_β defines a loop in $\bigvee_{g_\alpha} S_{g_\alpha}^1$. Attach 2-cells to $\bigvee_{g_\alpha} S_{g_\alpha}^1$ along the loops defined by the relations $\{r_\beta\}$. Call this new CW-complex Y. Proposition 1.23 gives that

$$\pi_1(Y, x_0) \cong \frac{\pi_1(X, x_0)}{\langle\langle \{r_\beta\} \rangle\rangle} \cong \frac{F_{\{g_\alpha\}}}{\langle\langle \{r_\beta\} \rangle\rangle} \cong G.$$

$\square$

Remark. Let $X = \bigcup_n X^n$ be a CW-complex, path-connected. Proposition 1.23 can be used to show the following two facts.

- The inclusion $X^1 \hookrightarrow X$ induces a surjective homomorphism $\pi_1(X^1) \to \pi_1(X)$.
- The inclusion $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2) \to \pi_1(X)$.

1.3 Covering spaces

1.3.1 Lifting properties

Let X be a topological space. Recall that a **covering space** is $p : \widetilde{X} \to X$ such that each $x \in X$ has an open neighbourhood U such that

$$p^{-1}(U) = \bigcup_{\alpha} \widetilde{U_\alpha},$$

where U_α are pairwise disjoint and $p\mid_{\widetilde{U_\alpha}} : \widetilde{U_\alpha} \to U$ is a homeomorphism for all α.

Example.

$$\begin{array}{ccc} \mathbb{R} & \to & S^1 \\ s & \mapsto & (\cos(2\pi s), \sin(2\pi s)) \end{array}, \qquad \begin{array}{ccc} S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}, \qquad \mathrm{O} \cdot \mathrm{O} \cdot \mathrm{O} \to S^1 \vee S^1 = \mathrm{O} \cdot \mathrm{O}.$$

Let $f : Y \to X$ be a continuous map. A **lift** of f is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$, where $p : \widetilde{X} \to X$ is a covering space. Let Y be connected.

- **Unique lifting property** states that if two lifts $\widetilde{f_1}$ and $\widetilde{f_2}$ of f coincide at one point, then they coincide on all of Y.
- **Homotopy lifting property** states that if $f_t : Y \to X$ is a homotopy and $\widetilde{f_0} : Y \to \widetilde{X}$ is a lift of f_0 then there exists a unique homotopy $\widetilde{f_t} : Y \to \widetilde{X}$ of $\widetilde{f_0}$ that lifts f_t.

Remark.

- If Y is a point, this is called the **path lifting property**. Let $f : I \to X$ be a path with $f(0) = x_0$. If $\widetilde{x_0} \in p^{-1}(x_0)$, then there is a unique path $\widetilde{f} : I \to \widetilde{X}$ lifting f and starting at $\widetilde{x_0}$.
- In particular, the lift of a constant path is constant.
- This implies in particular that the lift of a homotopy of paths is again a homotopy of paths. The endpoints $f_t(0)$ and $f_t(1)$ define constant paths as t varies.

Fix $x_0 \in X$ and $\widetilde{x_0} \in \widetilde{X}$ such that $p(\widetilde{x_0}) = x_0$, so

$$p_* : \pi_1\left(\widetilde{X}, \widetilde{x_0}\right) \to \pi_1(X, x_0).$$

To every element in $\pi_1(X, x_0)$ we can associate a homotopy class of paths in $\widetilde{X}$ starting at $\widetilde{x_0}$.

Proposition 1.25.

1. *$p_* : \pi_1\left(\widetilde{X}, \widetilde{x_0}\right) \to \pi_1(X, x_0)$ is injective.*
2. *$p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) \subseteq \pi_1(X, x_0)$ consists of the homotopy classes of loops at x_0 whose lifts to $\widetilde{X}$ starting at $\widetilde{x_0}$ are loops.*

Proof.

1. Let $\widetilde{f_0} : I \to \widetilde{X}$ be a loop at $\widetilde{x_0}$ such that $\left[\widetilde{f_0}\right] \in Ker(p_*)$, so $p\widetilde{f_0} = f_0$ is homotopic to the constant loop at x_0. Let $f_t : I \to X$ be a homotopy between f_0 and the constant loop. Homotopy lifting property and remark gives that f_t lifts to a homotopy $\widetilde{f_t}$ of paths between $\widetilde{f_0}$ and the constant loop, so $\left[\widetilde{f_0}\right] = id \in \pi_1\left(\widetilde{X}, \widetilde{x_0}\right)$ and p_* is injective.
2. Let $f : I \to X$ be a loop at x_0 that lifts to a loop $\widetilde{f}$ at $\widetilde{x_0}$. Then $p\widetilde{f} = f$, so $p_*\left(\left[\widetilde{f}\right]\right) = [f]$. On the other hand, if $f : I \to X$ is a loop at x_0 such that there exists a loop $\widetilde{f} : I \to \widetilde{X}$ at $\widetilde{x_0}$ with $p_*\left(\left[\widetilde{f}\right]\right) = [f]$, then f is homotopic to $p\widetilde{f}$. Homotopy lifting property gives that there exists a loop $\widetilde{f}' : I \to \widetilde{X}$ at x_0 such that $p\widetilde{f}' = f$.

$\square$

Lecture 12 Wednesday 06/02/19

Let $p : \widetilde{X} \to X$ be a covering space. Let $U \subseteq X$ be an evenly covered neighbourhood of $x \in X$. Let

$$p^{-1}(U) = \bigsqcup_{\alpha \in \Lambda} \widetilde{U_\alpha}.$$

Then the cardinality $\left|p^{-1}(x)\right|$ of $p^{-1}(x)$ is exactly the cardinality of $|\Lambda|$. The set of sheets is in bijection with $p^{-1}(x)$. So the cardinality of $p^{-1}(x)$ is locally constant. If X is connected, the cardinality of $p^{-1}(x)$ is constant.

Notation. Let X, Y be topological spaces, $x \in X$, and $y \in Y$. A continuous map

$$f : (X, x) \to (Y, y)$$

is a continuous map $f : X \to Y$ such that $f(x) = y$.

Proposition 1.26. *Let $X, \widetilde{X}$ be path-connected and*

$$p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$$

be a covering space. Then the number of sheets of p equals the index of $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ in $\pi_1(X, x_0)$.

Proof. Let g be a loop in X at x_0 and $\widetilde{g}$ be its lift to $\widetilde{X}$ starting at $\widetilde{x_0}$. Let $H = p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ and let $[h] \in H$. Then $h \cdot g$ lifts to a path $\widetilde{h} \cdot \widetilde{g}$ in $\widetilde{X}$ starting at $\widetilde{x_0}$ with the same endpoint as $\widetilde{g}$, because $\widetilde{h}$ is a loop, by Proposition 1.25. Define

$$\begin{array}{cccc} \Phi : & \{\text{cosets of } H \text{ in } \pi_1(X, x_0)\} & \to & p^{-1}(x_0) \\ & H[g] & \mapsto & \widetilde{g}(1) \end{array},$$

so Φ is well-defined. Want to show that Φ is bijective.

- Φ is surjective because $\widetilde{X}$ is path-connected. Let $\widetilde{g}$ be a path in $\widetilde{X}$ from $\widetilde{x_0}$ to any point $\widetilde{x_0'} \in p^{-1}(x_0)$, then $g = p \cdot \widetilde{g}$ and $\Phi(H[g]) = \widetilde{x_0'}$.
- Φ is injective, since if $\Phi(H[g_1]) = \Phi(H[g_2])$ then the lift $\widetilde{g_1} \cdot \widetilde{g_2}^{-1}$ of $g_1 \cdot g_2^{-1}$ defines a loop in $\widetilde{X}$ at $\widetilde{x_0}$. Proposition 1.25 gives $[g_1][g_2]^{-1} \in H$, so $H[g_1] = H[g_2]$.

$\square$

We say that a topological space X has a certain property (P) **locally** if for each point $x \in X$ and each neighbourhood U of x there is an open neighbourhood $V \subseteq U$ having this property (P).

Example. X is locally path-connected or X is locally simply-connected.

Proposition 1.27. *Let*

$$p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$$

be a covering space and

$$f : (Y, y_0) \to (X, x_0)$$

a continuous map, where Y is path-connected and locally path-connected. Then there is a lift

$$\widetilde{f} : (Y, y_0) \to \left(\widetilde{X}, \widetilde{x_0}\right)$$

if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$.

$$\begin{tikzcd} & \left(\widetilde{X}, \widetilde{x_0}\right) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd}.$$

Proof.

$\implies$ Clear, because $f = p\widetilde{f}$ implies $f_* = p_* \widetilde{f}_*$.

$\impliedby$ Assume $f_* \left(\pi_1 \left(Y, y_0\right)\right) \subseteq p_* \left(\pi_1 \left(\widetilde{X}, \widetilde{x_0}\right)\right)$. For each $y \in Y$ choose a path γ from y_0 to y, so $f\gamma$ is a path in X from x_0 to $f(y)$. By path lifting, we can lift $f\gamma$ to a path $\widetilde{f\gamma}$ in $\widetilde{X}$ starting at $\widetilde{x_0}$. Define the map

$$\begin{array}{cccc} \widetilde{f} : & (Y, y_0) & \to & \left(\widetilde{X}, \widetilde{x_0}\right) \\ & y & \mapsto & \widetilde{f\gamma}(1) \end{array}.$$

$$\begin{tikzcd} & \widetilde{x_0} \overset{\widetilde{f\gamma}}{\underset{\widetilde{f\gamma'}}{=}} \widetilde{f}(y) \arrow[d, "p"] \\ y_0 \overset{\gamma}{\underset{\gamma'}{=}} y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & x_0 \overset{f\gamma}{\underset{f\gamma'}{=}} f(y) \end{tikzcd}.$$

- This map is well-defined, that is does not depend on the choice of γ. Let γ' be another path from y_0 to y. Then $h_0 = (f\gamma') \cdot (f\gamma)^{-1}$ is a loop at x_0 and $[h_0] \in f_* \left(\pi_1 \left(Y, y_0\right)\right) \subseteq p_* \left(\pi_1 \left(\widetilde{X}, \widetilde{x_0}\right)\right)$. Proposition 1.25 gives that can lift h_0 to a loop $\widetilde{h_0}$ at $\widetilde{x_0}$. The first half of $\widetilde{h_0}$ is $\widetilde{f\gamma'}$ and the second half is $\widetilde{f\gamma}^{-1}$, so $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$. Thus $\widetilde{f}$ is well-defined.
- We have $p\widetilde{f} = f$, so $\widetilde{f}$ lifts f.
- It remains to show that $\widetilde{f}$ is continuous. Let $y \in Y$ and let U be an evenly covered neighbourhood of $f(y)$. Let $\widetilde{U}$ be the sheet above U such that $\widetilde{f}(y) \in \widetilde{U}$, so $p \mid_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism. Let $V \subseteq Y$ be a path-connected neighbourhood of y such that $f(V) \subseteq U$. Fix a path γ from y_0 to y. Let $y' \in V$ be arbitrary and η be a path from y to y', so $\gamma \cdot \eta$ is a path from y_0 to y'. Then $(f\gamma) \cdot (f\eta)$ is a path in U from x_0 to $f(y')$. $\widetilde{f\eta} = \left(p \mid_{\widetilde{U}}\right)^{-1} f\eta$, so $\widetilde{f} \mid_V = \left(p \mid_{\widetilde{U}}\right)^{-1} f$. Thus $\widetilde{f} \mid_V : V \to \widetilde{U}$ is continuous, so $\widetilde{f}$ is continuous.

$\square$

Lecture 13
Friday
08/02/19

1.3.2 The classification of covering spaces

Definition. A covering space $p : \widetilde{X} \to X$ is a **universal cover** if $\widetilde{X}$ is simply-connected.

Definition. A topological space X is **semilocally simply-connected** if each $x \in X$ has a neighbourhood U such that

$$i_* : \pi_1 (U, x) \to \pi_1 (X, x)$$

is trivial, where $i : U \hookrightarrow X$ is the inclusion.

Example. Let $X = \bigcup_n C_n \subseteq \mathbb{R}^2$ be the Hawaiian earrings, where $C_n \subseteq \mathbb{R}^2$ is the circle of radius $1/n$ and centre $(1/n, 0)$. Then X is not semilocally simply-connected.

Proposition 1.28. *If $p : \widetilde{X} \to X$ is a universal cover, then X is semilocally simply-connected.*

Proof. Let $U \subseteq X$ be an evenly covered neighbourhood of $x_0 \in X$, $\widetilde{U} \subseteq \widetilde{X}$ be a sheet over U, and $\gamma \subseteq U$ be a loop at x_0, so γ lifts to a loop $\widetilde{\gamma} \subseteq \widetilde{U}$ at $\widetilde{x_0}$. $\widetilde{\gamma}$ is homotopic to the constant loop at $\widetilde{x_0}$. Compose this homotopy with p gives that γ is homotopic to the constant loop at x_0 in X, so

$$\pi_1 (U, x_0) \to \pi_1 (X, x_0)$$

is trivial. $\square$

Theorem 1.29. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there exists a universal cover $p : \widetilde{X} \to X$.*

Remark. If

$$p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$$

is a universal cover, each point $\widetilde{x} \in \widetilde{X}$ can be joined to $\widetilde{x_0}$ by a unique homotopy class of paths, by Proposition 1.6.

$$\left\{\text{points in } \widetilde{X}\right\} \leftrightsquigarrow \left\{[\gamma] \mid \gamma \text{ is a path in } \widetilde{X} \text{ starting at } \widetilde{x_0}\right\} \leftrightsquigarrow \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\},$$

by the homotopy lifting property.

Proof. Let $x_0 \in X$, and

$$\widetilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}, \qquad p : \begin{array}{ccc} \widetilde{X} & \to & X \\ [\gamma] & \mapsto & \gamma(1) \end{array}.$$

Have to

1. give $\widetilde{X}$ a topology,
2. show that $p : \widetilde{X} \to X$ is a covering, and
3. show that $\widetilde{X}$ is simply-connected.

Recall that a **basis** for a topology on a set Y is a collection $\mathcal{B}$ of subsets such that

- $Y = \bigcup_{U \in \mathcal{B}} U$, and
- if $U_1, U_2 \in \mathcal{B}$ and $y \in U_1 \cap U_2$ then there exists $V \in \mathcal{B}$ such that $y \in V$ and $V \subseteq U_1 \cap U_2$.

A basis defines a topology on Y, by $A \subseteq Y$ is open if and only if A is the union of elements of $\mathcal{B}$. A map $f : Z \to Y$ is continuous if and only if $f^{-1}(U)$ is open for all $U \in \mathcal{B}$.

1. Let $\mathcal{U}$ be the collection of all path-connected open sets $U \subseteq X$ such that $\pi_1(U) \to \pi_1(X)$ is trivial. Then $X = \bigcup_{U \in \mathcal{U}} U$ because X is semilocally simply-connected. Let $U_1, U_2 \in \mathcal{U}$ and $y \in U_1 \cap U_2$, and let $y \in V \subseteq U_1 \cap U_2$ be path-connected and open.

$$\begin{tikzcd} V \arrow[r, hook] & U_1 \arrow[r, hook] & X \end{tikzcd}, \qquad \begin{tikzcd} \pi_1(V) \arrow[r] \arrow[rr, bend right, "\text{trivial}"'] & \pi_1(U_1) \arrow[r, "\text{trivial}"] & \pi_1(X) \end{tikzcd}$$

so $V \in \mathcal{U}$ gives that $\mathcal{U}$ is a basis for the topology on X. For $U \in \mathcal{U}$ and γ a path in X from x_0 to a point in U, we define

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ a path in } U \text{ such that } \eta(0) = \gamma(1)\} \subseteq \widetilde{X}.$$

$U_{[\gamma]}$ only depends on the class $[\gamma]$, so $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is bijective. Surjective because U is path-connected and injective because all paths η in U with the same endpoint are homotopic. Claim that $\left\{U_{[\gamma]}\right\}$ forms a basis on $\widetilde{X}$.

- $\bigcup_{U \in \mathcal{U},\ \gamma} U_{[\gamma]} = \widetilde{X}$, because $\bigcup_{U \in \mathcal{U}} U = X$.
- Observe that if $[\gamma'] \in U_{[\gamma]}$ then $U_{[\gamma]} = U_{[\gamma']}$. If $\gamma' = \gamma \cdot \eta$ for η a path in U, then elements in $U_{[\gamma']}$ have the form $[\gamma \cdot \eta \cdot \mu]$, so $U_{[\gamma']} \subseteq U_{[\gamma]}$. Elements in $U_{[\gamma]}$ have the form $[\gamma \cdot \mu] = \left[\gamma \cdot \eta \cdot \eta^{-1} \cdot \mu\right] = \left[\gamma' \cdot \eta^{-1} \cdot \mu\right]$, so $U_{[\gamma]} \subseteq U_{[\gamma']}$. Consider $U_{[\gamma]}$ and $V_{[\gamma']}$ and let $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, so $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. Let $W \in \mathcal{U}$ such that $W \subseteq U \cap V$ and such that $\gamma''(1) \in W$, so $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$ and $[\gamma''] \in W_{[\gamma'']}$. This proves the claim.

2. $p\mid_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is a homeomorphism. It is bijective, let $V_{[\gamma']} \subseteq U_{[\gamma]}$ be an element of the basis, so $p\left(V_{[\gamma']}\right) = V \in \mathcal{U}$. $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$. Thus $p : \widetilde{X} \to X$ is continuous. If $U \in \mathcal{U}$, then
$$p^{-1}(U) = \bigsqcup_{[\gamma]} U_{[\gamma]},$$
so $p : \widetilde{X} \to X$ is a covering space.

3. Let $\widetilde{x_0} \in \widetilde{X}$ be the class of the constant path at x_0. Let $[\gamma] \in \widetilde{X}$ be arbitrary. $\gamma : [0,1] \to X$ and $\gamma(0) = x_0$. Let γ_t be the path in X defined by
$$\gamma_t(s) = \begin{cases} \gamma(s) & s \in [0,t] \\ \gamma(t) & s \in [t,1] \end{cases}.$$
Then
$$\begin{array}{rccc} \widetilde{\gamma} : & I & \to & \widetilde{X} \\ & t & \mapsto & [\gamma_t] \end{array}$$
is a path in $\widetilde{X}$ from $\widetilde{x_0}$ to $[\gamma]$, so $\widetilde{X}$ is path-connected. Recall that $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ consists of the classes of loops at x_0 in X that lifts to loops in $\widetilde{X}$ at $\widetilde{x_0}$. Let $[\gamma] \in p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$. Then γ lifts to a loop at $\widetilde{x_0}$ by $t \mapsto [\gamma_t]$. Because it is a loop we have $\widetilde{x_0} = [\gamma_1] = [\gamma]$, so γ is homotopic to the constant loop. Thus $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) = \{id\}$, so $\widetilde{X}$ is simply-connected.

$\square$

Lecture 14 Tuesday 12/02/19

Lecture 14 is a problem class.

Let $p : \widetilde{X} \to X$ be a covering space, so $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) \subseteq \pi_1(X, x_0)$.

Lecture 15 Wednesday 13/02/19

Proposition 1.30. *Let X be path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \subseteq \pi_1(X, x_0)$ there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \widetilde{x_0})) = H$ for some basepoint x_0.*

Proof. Let $\widetilde{X}$ be as constructed above. Define $X_H = \widetilde{X}/\sim$, where $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $\left[\gamma \cdot (\gamma')^{-1}\right] \in H$. This is an equivalence relation.

- $[\gamma] \sim [\gamma]$ because $id \in H$.
- $[\gamma] \sim [\gamma']$ gives $[\gamma'] \sim [\gamma]$ because H contains all its inverses.
- $[\gamma] \sim [\gamma']$ and $[\gamma'] \sim [\gamma'']$ gives $[\gamma] \sim [\gamma'']$ because H is closed under product.

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \frac{\widetilde{X}}{\sim} = X_H \arrow[dl, "p"] \\ X & \end{tikzcd}.$$

Let $U_{[\gamma]}, U_{[\gamma']}$ be basis neighbourhoods. If $[\gamma] \sim [\gamma']$ then $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$, so p is a covering space, and $p^{-1}(U) = \bigcup_\gamma U_{[\gamma]}$. Let $\widetilde{x_0} \in X_H$ be the equivalence class of the constant path c_{x_0} at x_0. Let γ be a loop in X at x_0 such that $[\gamma] \in p_*(\pi_1(X_H, \widetilde{x_0}))$. Again $t \mapsto [\gamma_t]$ is a lift of γ at $\widetilde{x_0}$.

$$t \mapsto [\gamma_t] \text{ is a loop in } X_H \iff [\gamma_1] = [\gamma] = [c_{x_0}] \text{ in } X_H \iff [\gamma] \sim [c_{x_0}] \iff \gamma \in H.$$

$\square$

Definition. We say that two covering spaces $p_1 : \widetilde{X_1} \to X$ and $p_2 : \widetilde{X_2} \to X$ are **isomorphic** if there exists a homeomorphism $f : \widetilde{X_1} \to \widetilde{X_2}$ such that

$$\begin{tikzcd} \widetilde{X_1} \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \widetilde{X_2} \arrow[dl, "p_2"] \\ & X & \end{tikzcd}.$$

Proposition 1.31. *Let X be path-connected and locally path-connected and $x_0 \in X$. Two path-connected covering spaces $p_1 : \widetilde{X_1} \to X$ and $p_2 : \widetilde{X_2} \to X$ are isomorphic via an isomorphism $f : \widetilde{X_1} \to \widetilde{X_2}$ mapping a basepoint $\widetilde{x_1} \in p_1^{-1}(x_0)$ to a basepoint $\widetilde{x_2} \in p_2^{-1}(x_0)$ if and only if*

$$(p_1)_* \left(\pi_1 \left(\widetilde{X_1}, \widetilde{x_1}\right)\right) = (p_2)_* \left(\pi_1 \left(\widetilde{X_2}, \widetilde{x_2}\right)\right).$$

Proof.

$\implies$ If

$$f : \left(\widetilde{X_1}, \widetilde{x_1}\right) \to \left(\widetilde{X_2}, \widetilde{x_2}\right)$$

is an isomorphism, then $p_1 = p_2 f$, so

$$(p_1)_* \left(\pi_1 \left(\widetilde{X_1}, \widetilde{x_1}\right)\right) \subseteq (p_2)_* \left(\pi_1 \left(\widetilde{X_2}, \widetilde{x_2}\right)\right),$$

and $p_2 = p_1 f^{-1}$, so

$$(p_2)_* \left(\pi_1 \left(\widetilde{X_2}, \widetilde{x_2}\right)\right) \subseteq (p_1)_* \left(\pi_1 \left(\widetilde{X_1}, \widetilde{x_1}\right)\right).$$

$\impliedby$ Assume

$$(p_1)_* \left(\pi_1 \left(\widetilde{X_1}, \widetilde{x_1}\right)\right) = (p_2)_* \left(\pi_1 \left(\widetilde{X_2}, \widetilde{x_2}\right)\right).$$

By lifting criterion in Proposition 1.27, we can lift p_1 to a continuous map

$$\widetilde{p_1} : \left(\widetilde{X_1}, \widetilde{x_1}\right) \to \left(\widetilde{X_2}, \widetilde{x_2}\right),$$

and p_2 to a continuous map

$$\widetilde{p_2} : \left(\widetilde{X_2}, \widetilde{x_2}\right) \to \left(\widetilde{X_1}, \widetilde{x_1}\right),$$

so $p_1 \widetilde{p_2} = p_2$ and $p_2 \widetilde{p_1} = p_1$.

$$\begin{tikzcd} \left(\widetilde{X_1}, \widetilde{x_1}\right) \arrow[rr, bend left, "\widetilde{p_1}"] \arrow[dr, "p_1"'] & & \left(\widetilde{X_2}, \widetilde{x_2}\right) \arrow[ll, bend left, "\widetilde{p_2}"] \arrow[dl, "p_2"] \\ & (X, x_0) & \end{tikzcd}.$$

$\widetilde{p_1} \widetilde{p_2}$ fixes the point $\widetilde{x_2} \in \widetilde{X_2}$. By the unique lifting property in Proposition 1.7, $\widetilde{p_1} \widetilde{p_2} = id_{\widetilde{x_2}}$. Similarly, $\widetilde{p_2} \widetilde{p_1} = id_{\widetilde{x_1}}$, so $\widetilde{p_1}$ is an isomorphism.

$\square$

Lecture 16 Friday 15/02/19

Fix $x_0 \in X$, $\widetilde{x_1} \in p_1^{-1}(x_0)$, and $\widetilde{x_2} \in p_2^{-1}(x_0)$. A **basepoint preserving isomorphism**

$$f : \left(\widetilde{X_1}, \widetilde{x_1}\right) \to \left(\widetilde{X_2}, \widetilde{x_2}\right)$$

is an isomorphism such that $f(\widetilde{x_1}) = \widetilde{x_2}$.

Theorem 1.32 (Galois correspondence)**.** *Let X be path-connected, locally path-connected, and semilocally simply-connected, and $x_0 \in X$. Then*

1. *there is a bijection*

$$\left\{ \begin{array}{c} \text{path-connected covering spaces } p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0) \\ \text{up to basepoint preserving isomorphisms} \end{array} \right\} \qquad \leftrightsquigarrow \qquad \left\{ \begin{array}{c} \text{subgroups} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\},$$

2. *if we ignore the basepoints, this correspondence gives a bijection*

$$\left\{ \begin{array}{c} \text{path-connected covering spaces } p : \widetilde{X} \to X \\ \text{up to isomorphisms} \end{array} \right\} \qquad \leftrightsquigarrow \qquad \left\{ \begin{array}{c} \text{conjugacy classes of subgroups} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\}.$$

Proof.

1. To a covering space $p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$ we associate the subgroup $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) \subseteq \pi_1(X, x_0)$. Proposition 1.30 and Proposition 1.31 show that this is well-defined on the isomorphism classes and it is bijective.

2. Let $p : \widetilde{X} \to X$ be a covering space and $\widetilde{x_1}, \widetilde{x_2} \in p^{-1}(x_0)$. Let $H_i = p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_i}\right)\right) \subseteq \pi_1(X, x_0)$, for $i = 1, 2$. Let $\widetilde{\gamma}$ be a path from $\widetilde{x_1}$ to $\widetilde{x_2}$. Let $\gamma = p\widetilde{\gamma}$ be a loop at x_0. Let $[f] \in \pi_1(X, x_0)$. Then $[f] \in H_1$ if and only if the lift $\widetilde{f}$ is a loop at $\widetilde{x_1}$. $\widetilde{\gamma}^{-1} \cdot \widetilde{f} \cdot \widetilde{\gamma}$ is a loop at $\widetilde{x_2}$ gives $p_*\left(\widetilde{\gamma}^{-1} \cdot \widetilde{f} \cdot \widetilde{\gamma}\right) = \gamma^{-1} \cdot f \cdot \gamma$, so $[\gamma]^{-1}[f][\gamma] \in H_2$. Thus $[\gamma]^{-1} H_1 [\gamma] \subseteq H_2$. Similarly, $[\gamma] H_2 [\gamma]^{-1} \subseteq H_1$. Conversely, let $H_1 \subseteq \pi_1(X, x_0)$ as above and $[\delta] \in \pi_1(X, x_0)$ be an arbitrary element. Let $\widetilde{\delta}$ be a lift of δ such that $\widetilde{\delta}(0) = \widetilde{x_0}$ and define $x_3 = \widetilde{\delta}(1)$. Then the same construction yields $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_3}\right)\right) = [\delta]^{-1} H_1 [\delta]$.

□

1.3.3 Deck transformations and group actions

Definition. Let $p : \widetilde{X} \to X$ be a covering space. A **deck-transformation** is an isomorphism from $\widetilde{X}$ to itself.

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}.$$

The group of deck-transformations is denoted by $G\left(\widetilde{X}\right)$.

Example.

- Let

$$\begin{array}{rccc} p : & \mathbb{R} & \to & S^1 \subset \mathbb{C} \\ & t & \mapsto & e^{2\pi i t} \end{array}.$$

  $f : \mathbb{R} \to \mathbb{R}$ such that $p(f(t)) = p(t)$ if and only if $e^{2\pi i f(t)} = e^{2\pi i t}$, if and only if $f(t) = t + n$, so $G(\mathbb{R}) \cong \mathbb{Z}$.

- Let

$$\begin{array}{rccc} p : & S^1 & \to & S^1 \\ & z & \mapsto & z^n \end{array}.$$

  Then $G\left(S^1\right) \cong \mathbb{Z}/n\mathbb{Z}$.

An observation is that if $\widetilde{X}$ is path-connected then $f \in G\left(\widetilde{X}\right)$ is uniquely determined by where it sends a single point.

$$
\begin{tikzcd}
\widetilde{X} \arrow[rr, bend left, "f'"] \arrow[rr, "f"] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\
& X &
\end{tikzcd}.
$$

If $f(x) = f'(x)$ for a single x, by unique lifting $f = f'$. So the identity is the only deck-transformation with a fixed point.

Definition. A covering space $p : \widetilde{X} \to X$ is **normal**, or **regular**, or **Galois**, if for each $x \in X$ and every pair $\widetilde{x}, \widetilde{x}' \in p^{-1}(x)$ there is an $f \in G\left(\widetilde{X}\right)$ such that $f(\widetilde{x}) = \widetilde{x}'$.

Example.

- $p : \mathbb{R} \to S^1$ is normal.
- $p : S^1 \to S^1$ is normal.

Proposition 1.33. *Let*
$$p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$$
be a path-connected covering space, and X be path-connected and locally path-connected. Then $p : \widetilde{X} \to X$ is normal if and only if $H = p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) \subseteq \pi_1(X, x_0)$ is a normal subgroup.

Proof. Let $\widetilde{x_1} \in p^{-1}(x_0)$, let $\widetilde{\gamma}$ be a path from $\widetilde{x_0}$ to $\widetilde{x_1}$ and $\gamma = p(\widetilde{\gamma})$. Then $[\gamma]$ conjugates H to $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_1}\right)\right)$ so $[\gamma] H [\gamma]^{-1} = H$, if and only if $H = p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_1}\right)\right)$, by Proposition 1.31 if and only if $f(\widetilde{x_0}) = \widetilde{x_1}$. So $G\left(\widetilde{X}\right)$ acts transitively on $p^{-1}(x_0)$ if and only if $H \subseteq \pi_1(X, x_0)$ is a normal subgroup. Let $x_0' \in X$ be another point and h a path from x_0 to $\widetilde{x_0}$. Let $\widetilde{h}$ be a lift of h such that $\widetilde{h}(0) = \widetilde{x_0}$. Set $\widetilde{x_0} = \widetilde{h}(1)$ and $p\left(\widetilde{x_0'}\right) = x_0'$. Then

$$
\begin{tikzcd}
\pi_1\left(\widetilde{X}, \widetilde{x_0}\right) \arrow[r, "\beta_{\widetilde{h}}"] \arrow[d, "p_*"'] & \pi_1\left(\widetilde{X}, \widetilde{x_0'}\right) \arrow[d, "p_*"] \\
\pi_1(X, x_0) \arrow[r, "\beta_h"'] & \pi_1(X, x_0')
\end{tikzcd}.
$$

$H \subseteq \pi_1(X, x_0)$ is normal if and only if $p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0'}\right)\right) \subseteq \pi_1(X, x_0')$ is normal, as before if and only if $G\left(\widetilde{X}\right)$ acts transitively on $p^{-1}(x_0')$. $\square$

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Proposition 1.34. *Let*
$$p : \left(\widetilde{X}, \widetilde{x_0}\right) \to (X, x_0)$$
be a covering space, and $X, \widetilde{X}$ be path-connected and locally path-connected. Let $H = p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right)$ and $N(H) \subseteq \pi_1(X, x_0)$ be the normaliser of H. Then $G\left(\widetilde{X}\right)$ is isomorphic to $N(H)/H$. In particular,

- *if $\widetilde{X}$ is normal, then $G\left(\widetilde{X}\right) \cong \pi_1(X, x_0)/H$, and*
- *if $\widetilde{X}$ is the universal cover, then $G\left(\widetilde{X}\right) \cong \pi_1(X, x_0)$.*

Proof. Exercise: read the proof of this in Hatcher. $\square$

Example. Let $X = S^1 \vee S^1$, so $\pi_1(X) = F_{\{a,b\}}$. Then the following are covering spaces.

- A normal covering space
$$\widetilde{X} = \overset{a}{\mathrm{O}} \overset{\widetilde{x_0}}{\cdot} \overset{b}{\mathrm{O}} \cdot \overset{a}{\mathrm{O}} \to \overset{a}{\mathrm{O}} \cdot \overset{b}{\mathrm{O}} = X, \qquad p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) = \left\langle a, b^2, bab^{-1}\right\rangle \overset{2}{\subseteq} F_{\{a,b\}}.$$
In general, a two-oriented graph is a covering space of X.

- Not a normal covering space
$$\widetilde{X} = \overset{a}{\mathrm{O}} \cdot \overset{b}{\mathrm{O}} \overset{\widetilde{x_0}}{\cdot} \overset{a}{\mathrm{O}} \cdot \overset{b}{\mathrm{O}} \to \overset{a}{\mathrm{O}} \cdot \overset{b}{\mathrm{O}} = X, \qquad p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) = \left\langle b^2, bab^{-1}, a^2, aba^{-1}\right\rangle.$$

- A normal covering space
$$\widetilde{X} = \dots \underset{b}{-}\overset{a}{\mathrm{O}}\underset{b}{-}\underset{\widetilde{x_0}}{\overset{a}{\mathrm{O}}}\underset{b}{-}\overset{a}{\mathrm{O}}\underset{b}{-} \dots \to \overset{a}{\mathrm{O}} \cdot \overset{b}{\mathrm{O}} = X, \qquad p_*\left(\pi_1\left(\widetilde{X}, \widetilde{x_0}\right)\right) = \left\langle b^n a b^{-n} \mid n \in \mathbb{Z}\right\rangle.$$
Universal cover is a tree.

Example. Let $T = S^1 \times S^1$, so $\pi_1(T) = \mathbb{Z}^2$. This is abelian, so all covering spaces are normal. Universal cover is
$$\begin{array}{ccc} \mathbb{R}^2 & \to & S^1 \times S^1 \\ (s,t) & \mapsto & \left(e^{2\pi i s}, e^{2\pi i t}\right) \end{array},$$
since $\mathbb{R}^2$ is simply connected. (Exercise: check that it is a covering space) More generally, if $p : \widetilde{X} \to X$ and $q : \widetilde{X} \to X$ are covering spaces then
$$\begin{array}{ccc} \widetilde{X} \times \widetilde{Y} & \to & X \times Y \\ (x,y) & \mapsto & (p(x), q(y)) \end{array}$$
is again a covering space. For example,
$$\begin{array}{ccc} S^1 \times S^1 & \to & S^1 \times S^1 \\ (z_1, z_2) & \mapsto & (z_1^n, z_2^m) \end{array}.$$

Example. Worksheet 3 exercise 7. Let
$$\mathbb{RP}^n = \frac{\mathbb{R}^{n+1} \setminus \{0\}}{\sim} = \frac{S^n}{\sim}$$
be the **projective n-space**, the space of all lines through the origin in $\mathbb{R}^{n+1}$, where $x \sim -x$. Let $p : S^n \to \mathbb{RP}^n$ be the quotient map. Claim that this is a covering space. Let $[x] \in \mathbb{RP}^n$. Then $p^{-1}([x]) = \{\pm x\}$. Let U be an open neighbourhood of x such that $U \cap (-U) = \emptyset$, so $p(U) = \{[x] \mid x \in U\}$. Then $p^{-1}(p(U)) = U \cup (-U)$ is open and disjoint. Thus $p|_U : U \to p(U)$ is a homeomorphism, so it is a covering space.

- $n \geq 2$ gives that S^n is simply-connected, so $S^n \to \mathbb{RP}^n$ is a universal cover. Then
$$\{id\} = p_*(\pi_1(S^n)) \overset{2}{\subseteq} \pi_1(\mathbb{RP}^n),$$
so $|\pi_1(\mathbb{RP}^n)| = 2$. Thus $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$.

- $n = 1$ gives $\mathbb{RP}^1 = S^1$, so
$$\begin{array}{cccc} p : & S^1 & \to & S^1 \\ & z & \mapsto & z^2 \end{array}$$
is a covering space.

2 Homology

Lecture 18 Wednesday 20/02/19

Higher homotopy groups $\pi_n(X, x_0)$ are groups of basepoint preserving homotopies of continuous $\phi : I^n \to X$ such that $\phi(\partial I^n) = x_0$.

Example.

$$\pi_1(S^n) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \pi_2(S^n) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \text{otherwise} \end{cases},$$

$$\pi_3(S^n) = \begin{cases} \mathbb{Z} & n = 2, 3 \\ 0 & \text{otherwise} \end{cases}, \qquad \pi_i(S^2) = \begin{cases} \frac{\mathbb{Z}}{2\mathbb{Z}} & i = 4, 5 \\ \frac{\mathbb{Z}}{12\mathbb{Z}} & i = 6 \end{cases}.$$

Homology is more suitable. The following is the plan.

- Simplicial homology.
- Singular homology.
- Technical machinery to show that they coincide.
- Applications.

2.1 Simplicial and singular homology

2.1.1 Δ-complexes

Definition. Let $m, n \geq 0$.

- An n**-simplex** in $\mathbb{R}^m$ is the convex hull of a set V of $n+1$ points in $\mathbb{R}^m$ that are not all contained in an affine $(n-1)$-dimensional subspace of $\mathbb{R}^m$.
- The **standard** n**-simplex** is the convex hull of the standard basis $\{e_1, \dots, e_{n+1}\}$ in $\mathbb{R}^{n+1}$,

$$\left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0,\ x_0 + \dots + x_n = 1\right\}.$$

- An **ordered** n**-simplex** is an n-simplex with an ordering on the vertices. We denote it by $[v_0, \dots, v_n]$, where $v_0, \dots, v_n$ are the vertices in ascending order.
- The **standard ordered** n**-simplex** is the ordered n-simplex $[e_1, \dots, e_{n+1}]$ in $\mathbb{R}^{n+1}$. It is denoted by Δ^n.
- Let $[v_0, \dots, v_{n+1}]$ be an n-simplex in $\mathbb{R}^m$ and let $L \subseteq \mathbb{R}^m$ be the affine subspace spanned by $v_0, \dots, v_n$. Then there exists a unique affine morphism

$$\begin{array}{ccc} L & \to & \mathbb{R}^{n+1} \\ v_i & \mapsto & e_{i+1} \end{array},$$

  for $i = 0, \dots, n$. This gives a homeomorphism from $[v_0, \dots, v_n]$ to Δ^n that preserves this ordering.
- For $n \geq 1$, the **faces** of an ordered n-simplex $[v_0, \dots, v_n]$ are the ordered $(n-1)$-simplices

$$[v_0, \dots, \widehat{v_i}, \dots, v_n].$$

  $\widehat{v_i}$ means we omit the vertex v_i.
- The union of all the faces of a simplex Δ is the **boundary** $\partial\Delta$.
- The **interior** of Δ is $\mathring{\Delta} = \Delta \setminus \partial\Delta$.

Example. Let $\Delta^2 = [e_1, e_2, e_3]$. Then $\partial\Delta^2 = [e_1, e_2] \cup [e_1, e_3] \cup [e_2, e_3]$.

Definition. Let X be a topological space. A **Δ-complex structure** on X is a collection of continuous maps $\sigma_\alpha : \Delta^{n(\alpha)} \to X$ for $\alpha \in A$ and $n(\alpha) \in \mathbb{N}$ such that

1. the restriction $\sigma_\alpha |_{\mathring{\Delta}^{n(\alpha)}}$ is injective for all $\alpha \in A$ and for each $x \in X$ there is a unique $\alpha \in A$ such that $x \in \sigma_\alpha\left(\mathring{\Delta}^{n(\alpha)}\right)$,
2. the restriction of σ_α to a face of $\Delta^{n(\alpha)}$ is equal to σ_β for some $\beta \in A$ and $n(\beta) = n(\alpha) - 1$, and
3. $U \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(U)$ is open in $\Delta^{n(\alpha)}$ for all $\alpha \in A$.

An observation is that

$$\sigma : \bigsqcup_{\alpha \in A} \Delta^{n(\alpha)} \to X$$

induced by the σ_α is a quotient map, since it is surjective by 1 and $U \subseteq X$ is open if and only if $\sigma^{-1}(U)$ is open by 3.

Remark. One can show that an X with a Δ-complex structure is a CW-complex.

Example.

- Torus or Klein bottle is two Δ^2, three Δ^1, and one Δ^0.
- S^2 is a tetrahedron.
- Dunce hat, by identifying all the three faces of the standard 2-simplex with each other, has one Δ^2, one Δ^1, and one Δ^0.

2.1.2 Simplicial homology

Let X be a Δ-complex. The group of **n-chains** $\Delta_n(X)$ is the free abelian group on the n-simplices $\sigma_\alpha : \Delta^{n(\alpha)} \to X$, where $n(\alpha) = n$. So an element in $\Delta_n(X)$ is of the form

$$\sum_{\alpha \in A,\ n(\alpha)=n} c_\alpha \cdot \sigma_\alpha,$$

where $c_\alpha \in \mathbb{Z}$ and all but finitely many of the c_α are zero.

Example. Let K be a Klein bottle.

- $\Delta_0(K) = \{n \cdot v \mid n \in \mathbb{Z}\} = \mathbb{Z} \cdot v \cong \mathbb{Z}$.
- $\Delta_1(K) = \{n_1 \cdot a + n_2 \cdot b + n_3 \cdot c \mid n_1, n_2, n_3 \in \mathbb{Z}\} = \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot b \oplus \mathbb{Z} \cdot c \cong \mathbb{Z}^3$.
- $\Delta_2(K) = \{n_1 \cdot U + n_2 \cdot V \mid n_1, n_2 \in \mathbb{Z}\} = \mathbb{Z} \cdot U \oplus \mathbb{Z} \cdot V \cong \mathbb{Z}^2$.
- $\Delta_n(K) = 0$ for $n \geq 3$.

Similarly for a torus T.

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Define the **boundary homomorphism** by

$$\begin{array}{rccc} \partial_n : & \Delta_n(X) & \to & \Delta_{n-1}(X) \\ & \sigma_\alpha & \mapsto & \sum_{i=0}^{n} (-1)^i \sigma_\alpha |_{[v_0, \dots, \widehat{v_i}, \dots, v_n]} \end{array}.$$

Moreover, we define $\partial_0 = 0$.

Example. Let $\sigma : [v_0, v_1, v_2, v_3] \to X$. Then

$$\partial_3(\sigma) = \sigma |_{[v_1, v_2, v_3]} - \sigma |_{[v_0, v_2, v_3]} + \sigma |_{[v_0, v_1, v_3]} - \sigma |_{[v_0, v_1, v_2]}.$$

Lemma 2.1. *The composition*

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

is the zero map.

Proof. Let $\sigma : [v_0, \dots, v_n] \to X$ be an n-simplex. Then

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma |_{[v_0,\dots,\widehat{v_i},\dots,v_n]},$$

so

$$(\partial_{n-1} \circ \partial_n)(\sigma) = \sum_{j<i} (-1)^i (-1)^j \sigma |_{[v_0,\dots,\widehat{v_j},\dots,\widehat{v_i},\dots,v_n]} + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma |_{[v_0,\dots,\widehat{v_i},\dots,\widehat{v_j},\dots,v_n]} = 0.$$

If $n = 1$, clear. ∎

The following is the algebraic situation. A **chain complex** of abelian groups is a diagram $(C., \partial)$ of the form

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0,$$

where the C_i are abelian groups and the ∂_n are group homomorphisms such that $\partial_n \circ \partial_{n-1} = 0$ for all n. ∂_n are **boundary homomorphisms**. Elements in C_n are **n-chains**.

$$Z_n = Ker(\partial_n) \subseteq C_n, \qquad B_n = Im(\partial_{n+1}) \subseteq C_n.$$

Elements in Z_n are **cycles** and elements in B_n are **boundaries**. Since $\partial_{n+1} \circ \partial_n = 0$, we have that $B_n \subset Z_n$. The **n-th homology group** of this chain complex is defined by

$$H_n(C., \partial) = \frac{Z_n}{B_n}.$$

So, by Lemma 2.1

$$\dots \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0$$

is a chain complex. The **n-th simplicial homology group** is

$$H_n^{\Delta}(X) = \frac{Ker(\partial_n)}{Im(\partial_{n+1})}.$$

Example. Let $X = S^1$.

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\partial_3} & \Delta_2(X) & \xrightarrow{\partial_2} & \Delta_1(X) & \xrightarrow{\partial_1} & \Delta_0(X) & \xrightarrow{\partial_0} & 0 \\ & & \cong & & \cong & & \cong & & \\ & & 0 & & \mathbb{Z} & & \mathbb{Z} & & \end{array}.$$

- $Ker(\partial_0) = \mathbb{Z}$ and $Im(\partial_1) = 0$, so $H_0^{\Delta}(X) \cong \mathbb{Z}$.
- $Ker(\partial_1) = \Delta_1(X)$ and $Im(\partial_2) = 0$, so $H_1^{\Delta}(X) \cong \mathbb{Z}$.
- $H_n^{\Delta}(X) = 0$ if $n \geq 2$.

Example. Let T be a torus.

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{\partial_4} & \Delta_3(T) & \xrightarrow{\partial_3} & \Delta_2(T) & \xrightarrow{\partial_2} & \Delta_1(T) & \xrightarrow{\partial_1} & \Delta_0(T) & \xrightarrow{\partial_0} & 0 \\ & & \cong & & \cong & & \cong & & \cong & & \\ & & 0 & & \mathbb{Z}\cdot U \oplus \mathbb{Z}\cdot V & & \mathbb{Z}\cdot a \oplus \mathbb{Z}\cdot b \oplus \mathbb{Z}\cdot c & & \mathbb{Z}\cdot v & & \end{array}.$$

- $Ker(\partial_0) = \mathbb{Z}$ and $Im(\partial_1) = 0$, so $H_0^{\Delta}(T) \cong \mathbb{Z}$.
- $\partial_2(U) = a + b - c$ and $\partial_2(V) = a + b - c$, and $\{a, b, a + b - c\}$ is a basis for $\Delta_1(T)$. $Ker(\partial_1) = \Delta_1(T)$ and $Im(\partial_2) = \mathbb{Z} \cdot (a + b - c)$, so $H_1^{\Delta}(T) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- $H_2^{\Delta}(T) \cong \mathbb{Z}$. (Exercise)

Lecture 20 is a problem class.

Lecture 20
Tuesday
26/02/19

Lecture 21
Wednesday
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2.1.3 Singular homology

A **singular n-simplex** in a topological space X is a continuous map $\sigma : \Delta^n \to X$. Let $C_n(X)$ be the free abelian group on the set of all singular simplices in X, that is elements in $C_n(X)$ are finite formal sums

$$\sum_i n_i \sigma_i, \qquad n_i \in \mathbb{Z},$$

where $\sigma_i : \Delta^n \to X$ are singular n-simplices. Elements in $C_n(X)$ are called **singular n-chains**. Define a **boundary map**

$$\begin{array}{cccc} \partial_n : & C_n(X) & \to & C_{n-1}(X) \\ & \sigma & \mapsto & \sum_{i=0}^n (-1)^i \sigma |_{[v_1,\dots,\widetilde{v_i},\dots,v_n]} \end{array},$$

for a singular n-simplex σ. Extend it linearly to $C_n(X)$.

Lemma 2.2. $\partial_n \circ \partial_{n+1} = 0$.

Proof. Same proof as for Lemma 2.1. □

We obtain a chain complex

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

Remark. Often we write ∂ instead of ∂_n.

We define the **n-th singular homology group** by

$$H_n(X) = \frac{Ker(\partial_n)}{Im(\partial_{n+1})}.$$

An observation is that if X and Y are homeomorphic then $H_n(X) \cong H_n(Y)$.

Proposition 2.3. *Let X be a topological space and $X = \bigcup_\alpha X_\alpha$ be the decomposition into its path-connected components. Then*

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha).$$

Proof. A singular n-simplex $\sigma : \Delta^n \to X$ has a path-connected image. So

$$C_n(X) = \bigoplus_\alpha C_n(X_\alpha).$$

The boundary maps ∂_n preserve this decomposition, so $\partial_n(C_n(X_\alpha)) \subseteq C_{n-1}(X_\alpha)$ gives that $Ker(\partial_n)$ and $Im(\partial_{n+1})$ split as well as direct sums, so

$$H_n(X) = \frac{Ker(\partial_n)}{Im(\partial_{n+1})} \cong \bigoplus_\alpha H_n(X_\alpha).$$

□

Proposition 2.4. *If X is a path-connected, and as always $X \neq \emptyset$, topological space, then $H_0(X) \cong \mathbb{Z}$. Hence for X arbitrary $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path-connected component.*

Proof. $\partial_0 = 0$, so $H_0(X) = C_0(X)/Im(\partial_1)$. Define

$$\begin{array}{cccc} \epsilon : & C_0(X) & \to & \mathbb{Z} \\ & \sum_i n_i \sigma_i & \mapsto & \sum_i n_i \end{array}.$$

ϵ is surjective. Enough to show that $Ker(\epsilon) = Im(\partial_1)$. This implies by the isomorphism theorem $H_0(X) \cong \mathbb{Z}$. Let $\sigma : \Delta^1 \to X$ be a 1-simplex. Then

$$\partial_1(\sigma) = \sigma |_{[v_1]} - \sigma |_{[v_0]},$$

so $\epsilon(\partial_1(\sigma)) = 0$ gives $Im(\partial_1) \subseteq Ker(\epsilon)$. On the other hand, $\epsilon\left(\sum_i n_i \sigma_i\right) = 0$ gives $\sum_i n_i = 0$. The σ_i correspond to points $\sigma_i([v])$ in X. Choose a basepoint $x_0 \in X$ and let

$$\begin{array}{rccc} \sigma_0 : & \Delta^0 & \to & X \\ & \Delta^0 & \mapsto & x_0 \end{array}$$

be the singular 0-simplex. Let τ_i be a path from x_0 to $\sigma_i([v])$. Consider τ_i as a singular 1-simplex $\tau_i : [v_0, v_1] \to X$. We have $\partial_1 \circ \tau_i = \sigma_i - \sigma_0$, so

$$\partial_1\left(\sum_i n_i \tau_i\right) = \sum_i n_i(\sigma_i - \sigma_0) = \sum_i n_i \sigma_i - \sum_i n_i \sigma_0 = \sum_i n_i \sigma_i.$$

Thus $Ker(\epsilon) \subseteq Im(\partial_1)$. $\square$

Proposition 2.5. *If X is a point, then*

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}.$$

Proof. For each n there exists a unique singular n-simplex $\partial_n : \Delta^n \to X$, so $C_n(X) \cong \mathbb{Z}$ for all n.

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases},$$

so $\partial_n = 0$ if n is odd and ∂_n is an isomorphism if n is even.

$$\begin{array}{ccccccc} \dots & \xrightarrow{\partial_2} & C_1(X) & \xrightarrow{\partial_1} & C_0(X) & \xrightarrow{\partial_0} & 0 \\ & & \wr\Vert & & \wr\Vert & & \\ \dots & \xrightarrow{\sim} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\sim} & 0 \end{array},$$

so $H_n = Ker(\partial_n)/Im(\partial_{n+1}) = 0$ if $n \geq 1$ and $H_0(X) \cong \mathbb{Z}$. $\square$

The **reduced homology groups** $\widetilde{H_n}(X)$ are the homology groups of the **augmented chain complex**

$$\dots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} \mathbb{Z} \xrightarrow{\epsilon} 0,$$

where ϵ is as in proof of Proposition 2.4.

$$H_n(X) \cong \widetilde{H_n}(X), \qquad n \geq 1.$$

Seen in the proof of Proposition 2.4 that ϵ is surjective and $\epsilon \circ \partial_1 = 0$ gives $Im(\partial_1) \subseteq Ker(\epsilon)$, so ϵ induces a surjective homomorphism

$$\phi_\epsilon : H_0(X) = \frac{C_0(X)}{Im(\partial_1)} \to \mathbb{Z}.$$

Then $Ker(\phi_\epsilon) = Ker(\epsilon)/Im(\partial_1) = \widetilde{H_0}(X)$ gives $H_0(X)/\widetilde{H_0}(X) \cong \mathbb{Z}$, so

$$H_0(X) \cong \widetilde{H_0}(X) \oplus \mathbb{Z}.$$

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2.1.4 Homotopy invariance

Let $(A_., \partial)$ and $(B_., \partial)$ be two chain complexes. A **chain map** $f : (A_., \partial) \to (B_., \partial)$ is a collection of homomorphisms $f_n : A_n \to B_n$ such that $\partial \circ f_n = f_{n+1} \circ \partial$, that is the following diagram commutes.

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\partial} & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \xrightarrow{\partial} & \dots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \dots & \xrightarrow{\partial} & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \xrightarrow{\partial} & \dots \end{array} .$$

If X and Y are topological spaces and $f : X \to Y$ is a continuous map define the homomorphisms

$$f_\# : \begin{array}{ccc} C_n(X) & \to & C_n(Y) \\ \sigma : \Delta^n \to X & \mapsto & f \circ \sigma : \Delta^n \to Y \end{array},$$

and extend it linearly to $C_n(X)$.

$$(f_\# \circ \partial)(\sigma) = f_\# \left(\sum_{i=0}^n (-1)^i \sigma \mid_{[v_0,\dots,\widehat{v_i},\dots,v_n]} \right) = \sum_{i=0}^n (f \circ \sigma) \mid_{[v_0,\dots,\widehat{v_i},\dots,v_n]} = (\partial \circ f_\#)(\sigma)$$

gives $f_\# \circ \partial = \partial \circ f_\#$, so $f_\#$ defines a chain map

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\partial} & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \dots \\ & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \\ \dots & \xrightarrow{\partial} & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \xrightarrow{\partial} & \dots \end{array} .$$

$f_\#$ maps cycles to cycles, since $\alpha \in C_n(X)$ such that $\partial \circ \alpha = 0$ gives

$$(\partial \circ f_\#)(\alpha) = (f_\# \circ \partial)(\alpha) = 0.$$

$f_\#$ maps boundaries to boundaries, since

$$f_\# \circ (\partial \circ \beta) = \partial \circ (f_\# \circ \beta).$$

$f_\#(Ker(\partial_n)) \subseteq Ker(\partial_n)$ and $f_\#(Im(\partial_{n+1})) \subseteq Im(\partial_{n+1})$ gives that $f_\#$ induces a homomorphism

$$f_* : H_n(X) \to H_n(Y).$$

The following are observations.

- $X \xrightarrow{g} Y \xrightarrow{f} Z$ gives $(f \circ g)_\# = f_\# \circ g_\#$, since

$$\Delta^n \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z$$

  gives $f \circ (g \circ \sigma) = (f \circ g) \circ \sigma$, so $(f \circ g)_* = f_* \circ g_*$.

- $(id_X)_* = id_{H_n(X)}$.

Theorem 2.6. *If two continuous maps $f, g : X \to Y$ are homotopic, then $f_* = g_* : H_n(X) \to H_n(Y)$.*

Corollary 2.7. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism.*

Proof. Let $g : Y \to X$ be a continuous map such that $f \circ g \cong id_Y$ and $g \circ f = id_X$. Then $f_* \circ g_* = (f \circ g)_* = (id_Y)_* = id$. Similarly $g_* \circ f_* = id$, so f_* is an isomorphism. ☐

Example.

$$H_n\left(\mathbb{R}^k\right) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}, \qquad \widetilde{H}_n\left(\mathbb{R}^k\right) = 0.$$

Proof of Theorem 2.6. Let $F : X \times I \to Y$ be a homotopy from f to g and $\sigma : \Delta_n \to X$ be a singular n-simplex. Consider the map

$$\Delta^n \times I \xrightarrow{\sigma \times I} X \times I \xrightarrow{F} Y.$$

$\Delta^n \times I$ is not a simplex. But we can subdivide $\Delta^n \times I$ into $(n+1)$ simplices. In general, we can decompose $\Delta^n \times I$ into $n+1$ $(n+1)$-simplices

$$[v_0, \dots, v_i, w_i, \dots, w_n], \qquad i = 0, \dots, n.$$

Define **prism-operators**

$$\begin{array}{rccc} P: & C_n(X) & \to & C_{n+1}(Y) \\ & \sigma & \mapsto & \sum_{i=0}^n (-1)^i F \circ (\sigma \times id)\,|_{[v_0,\dots,v_i,w_i,\dots,w_n]} \end{array},$$

for $\sigma : \Delta^n \to X$ a singular n-simplex.

$$\begin{tikzcd} \dots \arrow[r, "\partial"] & C_{n+1}(X) \arrow[r, "\partial"] & C_n(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, "g_\# \ f_\#"] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[dl, "P"] & \dots \\ \dots \arrow[r, "\partial"] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r, "\partial"] & \dots \end{tikzcd}.$$

Claim that

$$\partial \circ P = g_\# - f_\# - P \circ \partial,$$

if and only if $g_\# - f_\# = \partial \circ P + P \circ \partial$. The claim implies the theorem, since if $\alpha \in C_n(X)$ is a cycle, then

$$g_\#(\alpha) - f_\#(\alpha) = (\partial \circ P)(\alpha) + (P \circ \partial)(\alpha) = (\partial \circ P)(\alpha),$$

so $g_\#(\alpha) - f_\#(\alpha)$ is a boundary. Thus $g_\#(\alpha)$ and $f_\#(\alpha)$ are in the same homology class, so $g_*([\alpha]) = f_*([\alpha])$, where $[\alpha]$ is the homology class of α. Let $\sigma : \Delta^n \to X$ be a singular n-simplex.

$$\begin{aligned} (\partial \circ P)(\sigma) &= \partial\left(\sum_{i=0}^n (-1)^i F \circ (\sigma \times id)\,|_{[v_0,\dots,v_i,w_i,\dots,w_n]}\right) \\ &= \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times id)\,|_{[v_0,\dots,\widehat{v_j},\dots,v_i,w_i,\dots,w_n]} \\ &\quad + \sum_{j \ge i} (-1)^i (-1)^{j+1} F \circ (\sigma \times id)\,|_{[v_0,\dots,v_i,w_i,\dots,\widehat{w_j},\dots,w_n]}. \end{aligned}$$

If $i = j$ the two sums cancel except for

$$F \circ (\sigma \times id)\,|_{[\widehat{v_0},w_0,\dots,w_n]} = g \circ \sigma = g_\#(\sigma),$$

and

$$-F \circ (\sigma \times id)\,|_{[v_0,\dots,v_n,\widehat{w_n}]} = -f \circ \sigma = -f_\#(\sigma).$$

The terms with $i \ne j$ sum up to $(P \circ \partial)(\sigma)$, since we have

$$\begin{aligned} (P \circ \partial)(\sigma) = &\sum_{j < i} (-1)^i (-1)^j F \circ (\sigma \times id)\,|_{[v_0,\dots,\widehat{v_j},\dots,v_i,w_i,\dots,w_n]} \\ &+ \sum_{j > i} (-1)^i (-1)^{j+1} F \circ (\sigma \times id)\,|_{[v_0,\dots,v_i,w_i,\dots,\widehat{w_j},\dots,w_n]}. \end{aligned}$$

∎

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Remark. One can show that there are also induced homomorphisms $f_* : \widetilde{H_n}(X) \to \widetilde{H_n}(Y)$ invariant under homotopy. (Exercise)

2.1.5 Exact sequences and excision

Let $A \subseteq X$ be a subspace. What is the relationship between $H_n(A), H_n(X), H_n(X/A)$?

Definition. A sequence of group homomorphisms of abelian groups

$$\dots \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} \dots$$

is **exact at** A_n if $Ker(\alpha_n) = Im(\alpha_{n+1})$. The sequence is **exact** if it is exact at A_n for all n.

An observation is if the sequence is exact, then

- $\alpha_n \alpha_{n+1} = 0$, so exact sequences are chain complexes, and
- the homology groups of this chain complex are all trivial.

Example.

1. $0 \to A \xrightarrow{\alpha} B$ is exact if and only if $Ker(\alpha) = 0$, if and only if α is injective.
2. $A \xrightarrow{\alpha} B \to 0$ is exact if and only if $Im(\alpha) = B$, if and only if α is surjective.
3. $0 \to A \xrightarrow{\alpha} B \to 0$ is exact if and only if α is an isomorphism.
4. $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact if and only if α is injective, β is surjective, and $Ker(\beta) = Im(\alpha)$, hence β induces an isomorphism $C \cong B/Im(\alpha) = B/A$.

An exact sequence as in 4 is called a **short exact sequence**.

Definition. Let X be a topological space, $A \subseteq X$ a non-empty closed subspace. Then (X, A) is called a **good pair** if A has a neighbourhood in X that deformation retracts to A.

Example.

- $\left(D^n, S^{n-1}\right)$ is a good pair, since S^{n-1} is a deformation retract of $D^n \setminus \{0\}$.
- Let $A = \{1/n \mid n \in \mathbb{N}\} \cup \{0\} \subseteq [0,1]$ then $([0,1], A)$ is not a good pair.

Theorem 2.8. *Let (X, A) be a good pair, then there is an exact sequence*

$$\dots \xrightarrow{\partial} \widetilde{H_1}(A) \xrightarrow{i_*} \widetilde{H_1}(X) \xrightarrow{j_*} \widetilde{H_1}(X/A) \xrightarrow{\partial} \widetilde{H_0}(A) \xrightarrow{i_*} \widetilde{H_0}(X) \xrightarrow{j_*} \widetilde{H_0}(X/A) \to 0,$$

where $i : A \hookrightarrow X$ is the inclusion and $j : X \to X/A$ is the quotient.

Corollary 2.9. $\widetilde{H_n}(S^n) \cong \mathbb{Z}$ *and* $\widetilde{H_i}(S^n) = 0$ *if* $i \neq 0$.

Proof. $\left(D^n, S^{n-1}\right)$ is a good pair. Let $n > 0$. Recall that $D^n/S^{n-1} \cong S^n$, so

$$\dots \xrightarrow{\partial} \widetilde{H_i}\left(S^{n-1}\right) \xrightarrow{i_*} \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{H_i}(D^n)} \xrightarrow{j_*} \widetilde{H_i}(S^n) \xrightarrow{\partial} \widetilde{H_{i-1}}\left(S^{n-1}\right) \xrightarrow{i_*} \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{H_{i-1}}(D^n)} \xrightarrow{j_*} \widetilde{H_{i-1}}(S^n) \xrightarrow{\partial} \dots .$$

Then $\widetilde{H_i}(S^n) \cong \widetilde{H_{i-1}}\left(S^{n-1}\right)$ for $i > 0$, so

$$\dots \xrightarrow{\partial} \widetilde{H_1}\left(S^{n-1}\right) \xrightarrow{i_*} \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{H_1}(D^n)} \xrightarrow{j_*} \widetilde{H_1}(S^n) \xrightarrow{\partial} \widetilde{H_0}\left(S^{n-1}\right) \xrightarrow{i_*} \underset{\substack{\rotatebox{90}{$\cong$} \\ 0}}{\widetilde{H_0}(D^n)} \xrightarrow{j_*} \widetilde{H_0}(S^n) \longrightarrow 0 .$$

$n > 0$ and $i > 0$, so $\widetilde{H_i}(S^n) \cong \widetilde{H_{i-1}}\left(S^{n-1}\right)$, and $\widetilde{H_0}(S^n) = 0$. We know that $\widetilde{H_0}\left(S^0\right) \cong \mathbb{Z}$ and $\widetilde{H_n}\left(S^0\right) = 0$, by Proposition 2.3 and Proposition 2.5. Doing induction on n, $\widetilde{H_n}(S^n) \cong \mathbb{Z}$ and $\widetilde{H_i}(S^n) = 0$ if $i \neq n$. $\square$

Corollary 2.10. *There exists no retraction $r : D^n \to \partial D^n$.*

Proof. Assume there exists such an $r : D^n \to \partial D^n$. Let $i : \partial D^n \to D^n$. Then $ri = id_{\partial D^n}$ gives $r_* i_* = (ri)_* = id$, so

$$\begin{array}{ccccc} \widetilde{H_{n-1}}(\partial D^n) & \xrightarrow{i_*} & \widetilde{H_{n-1}}(D^n) & \xrightarrow{r_*} & \widetilde{H_{n-1}}(\partial D^n) \\ \cong & & \cong & & \cong \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array}.$$

Thus $i_* = 0$ and $r_* = 0$, a contradiction. ☐

Theorem 2.11 (Brouwer fixed point theorem)**.** *Every continuous map $f : D^n \to D^n$ has a fixed point.*

Proof. Assume there exists a fixed point then construct as in dimension two a retraction $D^n \to \partial D^n$, a contradiction to Corollary 2.10. ☐

Let X be a topological space and $A \subseteq X$ be a subspace. Define $C_n(X, A) = C_n(X) / C_n(A)$. Let $\partial : C_n(X) \to C_{n-1}(X)$ be the boundary map then $\partial(\sigma : \Delta^n \to A) \in \partial(C_n(A)) \subseteq C_{n-1}(A)$. So ∂ induces a homomorphism

$$\partial : C_n(X, A) \to C_{n-1}(X, A),$$

such that $\partial \circ \partial = 0$. This gives a chain complex

$$\dots \to C_{n+1}(X, A) \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \to \dots.$$

- The homology groups $H_n(X, A)$ of this complex are the **relative homology groups**.
- The **relative n-chains** are $C_n(X, A)$.
- The **relative n-cycles** are $Ker(\partial) \subseteq C_n(X, A)$, of the form $[\alpha]$, for $\alpha \in C_n(X)$ such that $\partial(\alpha) \in C_{n-1}(A)$.
- The **relative n-boundaries** are $Im(\partial) \subseteq C_n(X, A)$, of the form $[\alpha]$, for $\alpha \in C_n(X)$ such that $\alpha = \partial\beta + \gamma$ for $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.