

M4P54 Differential Topology

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Syllabus

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0 Introduction

Differential topology is the study of the topology of a manifold using analysis. The topics are

- a review of differential forms,
- de Rham cohomology,
- Morse theory, and
- singular homology.

The following are references.

- J M Lee, Introduction to smooth manifolds, 2000
- L W Tu, Introduction to smooth manifolds, 2008
- J Milnor, Morse theory, 1960
- A Banyaga and D Hurtubise, Lectures on Morse homology, 2004

Lecture 1
Thursday
09/01/20

1 Differential forms on manifolds

1.1 Alternating p -forms on a vector space

Let V be a vector space over \mathbb{R} , and let $p \geq 0$. Then $V^p = V \times \cdots \times V$.

Definition 1.1. A multilinear map $\omega : V^p \rightarrow \mathbb{R}$ is called an **alternating p -form** if we have

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(p)}) = \epsilon(\sigma) \omega(v_1, \dots, v_p), \quad v_1, \dots, v_p \in V \quad \sigma \in \mathcal{S}_p,$$

where \mathcal{S}_p is the group of permutations of p elements and $\epsilon(\sigma)$ is the signature of σ .

Recall that if m is the number of transpositions in a decomposition of σ , then $\epsilon(\sigma) = (-1)^m$, where a **transposition** is $(a_i a_j)$ for $a_i \neq a_j$.

Notation 1.2. $\Lambda^p V^* = \{\text{alternating } p\text{-forms } \omega \text{ on } V\}$ is called the **p -th exterior power** of V .

Check that it is a vector space.¹

Example 1.3.

- $\Lambda^0 V^* = \mathbb{R}$.
- $\Lambda^1 V^* = V^* = \text{Hom}(V, \mathbb{R})$, the **dual** of V .

Definition 1.4. Let $\omega_1 \in \Lambda^p V^*$ and $\omega_2 \in \Lambda^q V^*$. We define the **exterior product** $\omega_1 \wedge \omega_2 \in \Lambda^{p+q} V^*$ of ω_1 and ω_2 by

$$\omega_1 \wedge \omega_2(v_1, \dots, v_{p+q}) = \sum_{\sigma \in \mathcal{S}_{p,q}} \epsilon(\sigma) \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \omega_2(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}), \quad v_1, \dots, v_{p+q} \in V,$$

where

$$\mathcal{S}_{p,q} = \{\sigma \in \mathcal{S}_{p+q} \mid \sigma(1) < \cdots < \sigma(p), \sigma(p+1) < \cdots < \sigma(p+q)\}.$$

Example 1.5.

- Assume $\omega_1, \omega_2 \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \omega_2(v_1, v_2) = \omega_1(v_1) \omega_2(v_2) - \omega_1(v_2) \omega_2(v_1), \quad v_1, v_2 \in V.$$

- Assume $\omega_1, \dots, \omega_p \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \cdots \wedge \omega_p(v_1, \dots, v_p) = \det(\omega_i(v_j))_{i,j=1,\dots,p}, \quad v_1, \dots, v_p \in V.$$

Proposition 1.6. Let $\omega_i \in \Lambda^{p_i} V^*$ for $i = 1, 2, 3$.

- **Associativity** $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$.
- **Distributivity** $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$, assuming $p_2 = p_3$.
- **Supercommutativity** $\omega_1 \wedge \omega_2 = (-1)^{p_1 p_2} \omega_2 \wedge \omega_1$.

Definition 1.7. Let $\Phi : V \rightarrow W$ be a linear map between vector spaces over \mathbb{R} . Let $\omega \in \Lambda^p W^*$. Then the **pull-back** $\Phi^* \omega \in \Lambda^p V^*$ of ω is an alternating p -form on V defined by

$$\Phi^* \omega(v_1, \dots, v_p) = \omega(\Phi(v_1), \dots, \Phi(v_p)), \quad v_1, \dots, v_p \in V.$$

¹Exercise

Proposition 1.8. Given $\Phi : V \rightarrow W$ a linear map,

- the pull-back

$$\begin{aligned} \Phi^* &: \Lambda^p W^* \longrightarrow \Lambda^p V^* \\ \omega &\longmapsto \Phi^* \omega \end{aligned}$$

is a linear map that preserves exterior products, that is

$$\Phi^* (\omega_1 \wedge \omega_2) = \Phi^* \omega_1 \wedge \Phi^* \omega_2, \quad \omega_1 \in \Lambda^p W^*, \quad \omega_2 \in \Lambda^q W^*,$$

- if $\Psi : W \rightarrow Z$ is linear then

$$(\Psi \circ \Phi)^* \omega = \Phi^* \Psi^* \omega, \quad \omega \in \Lambda^p Z^*,$$

- assuming $V = W$ and $p = \dim V$, then

$$\Phi^* \omega = (\det \Phi) \omega, \quad \omega \in \Lambda^p V^*.$$

1.2 Differential forms on manifolds

Let M be a smooth manifold of dimension n , and let $x \in M$. Then the tangent space $T_x M$ of M at x is a vector space of dimension n .

Notation 1.9. Let

$$\Lambda^p T_x^* M = \Lambda^p (T_x M)^*.$$

Consider the set

$$\Lambda^p T^* M = \bigsqcup_{x \in M} \Lambda^p T_x^* M,$$

the **p -th exterior bundle** on M . There exists a morphism $\pi : \Lambda^p T^* M \rightarrow M$ such that for all $x \in M$, $\pi^{-1}(x) = \Lambda^p T_x^* M$, so $\Lambda^p T^* M$ is a vector bundle and it is a smooth manifold, and π is a smooth morphism.

Example 1.10.

- $\Lambda^0 T^* M = M \times \mathbb{R}$.
- $\Lambda^1 T^* M$ is the **cotangent bundle**, the dual of the tangent bundle.

Definition 1.11. A **differential p -form** ω on M is a smooth section of π . That is, it is a smooth morphism $\omega : M \rightarrow \Lambda^p T^* M$ such that $\pi \circ \omega = \text{id}_M$.

Thus, $\omega(x) \in \Lambda^p T_x^* M$.

Notation 1.12.

$$\Omega^p(M) = \{\text{differential } p\text{-forms } \omega \text{ on } M\}, \quad \Omega^\bullet(M) = \bigoplus_p \Omega^p(M).$$

Example 1.13.

$$\Omega^0(M) \cong \{f : M \rightarrow \mathbb{R} \text{ } C^\infty\text{-function}\}.$$

Exercise. If $n = \dim M$, then $\Omega^{n+1}(M) = 0$.

The algebra is the same as last week.

Definition 1.14. Let $\omega_1 \in \Omega^p(M)$ and $\omega_2 \in \Omega^q(M)$. Then $\omega_1 \wedge \omega_2 \in \Omega^{p+q}(M)$ is defined by

$$\omega_1 \wedge \omega_2(x) = \omega_1(x) \wedge \omega_2(x) \in \Lambda^{p+q} T_x^* M, \quad x \in M.$$

By Proposition 1.6, associativity, distributivity, and supercommutativity hold for $\Omega^p(M)$. Let $F : M \rightarrow N$ be a smooth morphism between manifolds. Then for all $x \in M$, the differential of F at x is the linear map

$$DF_x : T_x M \rightarrow T_{F(x)} N.$$

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Monday
13/01/20

Thus, for all $p \geq 0$, we have a natural map, called the **pull-back**,

$$\begin{aligned} F_x^* : \Lambda^p T_{F(x)}^* N &\longrightarrow \Lambda^p T_x^* M \\ \omega(v_1, \dots, v_p) &\longmapsto \omega(DF_x(v_1), \dots, DF_x(v_p)) \end{aligned} \quad , \quad \omega \in \Lambda^p T_{F(x)}^* N, \quad v_1, \dots, v_p \in T_x^* M.$$

Thus, we can define

$$\begin{aligned} F^* : \Omega^p(N) &\longrightarrow \Omega^p(M) \\ \omega(x) &\longmapsto F^* \omega(F(x)) \end{aligned} \quad , \quad \omega \in \Omega^p(N).$$

By Proposition 1.8, the pull-back preserves the exterior product, so

$$F^*(\omega_1 \wedge \omega_2) = F^* \omega_1 \wedge F^* \omega_2.$$

If $G : N \rightarrow P$,

$$(G \circ F)^* \omega = F^* G^* \omega.$$

1.3 Local description of p -forms

Let M be a manifold of dimension n , let $x_0 \in M$, let (U, ϕ) be a local chart around x_0 , and let (x_1, \dots, x_n) be local coordinates around x_0 . A basis of $T_{x_0} M$ is given by

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}.$$

A basis of $T_{x_0}^* M$ is given by

$$\{dx_1, \dots, dx_n\}, \quad dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij}.$$

A basis of $\Lambda^p T_{x_0}^* M$ is

$$dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad i_1 < \dots < i_p.$$

Thus, $\omega \in \Omega^p(M)$ is locally given by

$$\omega(x) = \sum_{|I|=p} f_I(x) dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad I = (i_1, \dots, i_p), \quad i_1 < \dots < i_p,$$

where f_I is a C^∞ -function on U for all I .

Example 1.15. Let $F : M \rightarrow N$ be a smooth morphism between manifolds of dimension n , and let $\omega \in \Omega^n(N)$. Locally,

$$\omega(y) = f(y) dy_1 \wedge \dots \wedge dy_n, \quad y \in N,$$

for some $f \in C^\infty$. Proposition 1.8 implies that

$$F^* \omega(x) = (f \circ F)(x) \det DF_x dx_1 \wedge \dots \wedge dx_n, \quad x \in M,$$

where $y_i = p_i \circ F$ and $p_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is the i -th projection.

Let $f : M \rightarrow \mathbb{R}$ be a smooth function, so $f \in \Omega^0(M)$. Locally, the **differential** is

$$\begin{aligned} d : \Omega^0(M) &\longrightarrow \Omega^1(M) \\ f &\longmapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \end{aligned}.$$

Check that $df \in \Omega^1(M)$, so df is a 1-form on M . Alternatively, $df = f^* dx$ for dx a 1-form on \mathbb{R} , or $df(X) = X(f)$ for any vector field X on M . More in general, let $\omega \in \Omega^p(M)$. Locally,

$$\omega = \sum_{|I|=p} f_I dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad f_I \in C^\infty,$$

so $d\omega \in \Omega^{p+1}(M)$. Then the **de Rham differential** is

$$\begin{aligned} d : \Omega^p(M) &\longrightarrow \Omega^{p+1}(M) \\ \omega &\longmapsto \sum_{|I|=p} df_I \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p} \end{aligned}.$$

Proposition 1.16.

- The Leibnitz rule

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2, \quad \omega_1 \in \Omega^p(M), \quad \omega_2 \in \Omega^q(M).$$

- $d^2 = 0$, that is

$$d(d\omega) = 0, \quad \omega \in \Omega^p(M).$$

- Let $F : M \rightarrow N$ be a smooth morphism between manifolds. Then

$$F^*d\omega = d(F^*\omega), \quad \omega \in \Omega^p(M),$$

so

$$\begin{array}{ccc} \Omega^p(M) & \xrightarrow{d} & \Omega^{p+1}(M) \\ F^* \uparrow & & \uparrow F^* \\ \Omega^p(N) & \xrightarrow{d} & \Omega^{p+1}(N) \end{array}.$$

Definition 1.17.

- $\omega \in \Omega^p(M)$ is **closed** if $d\omega = 0$.
- $\omega \in \Omega^p(M)$ is **exact** if there exists $\omega' \in \Omega^{p-1}(M)$ such that $d\omega' = \omega$.

ω is exact implies that ω is closed, since if $\omega = d\omega'$ then $d\omega = d^2\omega' = 0$.

1.4 Integrations on manifolds

Let M be a manifold of dimension n , let $F : M \rightarrow M$ be a smooth morphism, and let $\omega \in \Omega^n(M)$. Then

$$F^*\omega(x) = \det DF_x \omega(F(x)).$$

Locally, assume $\omega = f dy_1 \wedge \cdots \wedge dy_n$ for some coordinates (y_1, \dots, y_n) and $f \in C^\infty$. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of M , where $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n$. Then

$$h_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \rightarrow \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n,$$

such that

$$h_{\alpha\beta}^*\omega(x) = (f \circ h_{\alpha\beta})(x) \det(Dh_{\alpha\beta})_x dx_1 \wedge \cdots \wedge dx_n.$$

Let $D \subset \mathbb{R}^n$ be compact such that ∂D has zero measure, so D is a domain of integration, let $f : U \rightarrow \mathbb{R}$ be a C^∞ -function where $U \subset \mathbb{R}^n$ is open such that $D \subset U$, and let $h : U \rightarrow h(U)$ be a diffeomorphism. Then

$$\int_{h^{-1}(D)} f(y) dy_1 \cdots dy_n = \int_{h^{-1}(D)} f(y) dy_1 \wedge \cdots \wedge dy_n = \int_D (f \circ h)(x) |\det Dh_x| dx_1 \wedge \cdots \wedge dx_n.$$

Let us assume that $\omega = f(y) dy_1 \wedge \cdots \wedge dy_n$ on U . We define

$$\int_D \omega = \int_D f(y) dy_1 \wedge \cdots \wedge dy_n, \quad D \subset U.$$

Definition 1.18. Let $U \subset \mathbb{R}^n$ be an open set. We define the **support** of ω as

$$\text{supp } \omega = \overline{\{x \in U \mid \omega(x) \neq 0\}}, \quad \omega(x) \in \Lambda^p T_x^*U.$$

Then ω has **compact support**, if $\text{supp } \omega$ is compact.

Fact. Under this assumption, we can define

$$\int_U \omega = \int_D \omega \in \mathbb{R},$$

which is well-defined. Under the same assumption, if $\phi : V \rightarrow U$ is a diffeomorphism, provided that $\det D\phi_x > 0$, since $\det D\phi_x \neq 0$ for all x , then

$$\int_U \omega = \int_V \phi^*\omega.$$

Lecture 3
Tuesday
14/01/20

1.5 Orientation

Let V be a vector space over \mathbb{R} of dimension n , and let $B = (b_1, \dots, b_n) \subset V$ and $B' = (b'_1, \dots, b'_n) \subset V$ be ordered bases of V . Then B and B' have the **same orientation** if $\det T > 0$ where

$$\begin{array}{ccc} T & : & V \longrightarrow V \\ & & b_i \longmapsto b'_i \end{array}$$

is a linear map. Let $\omega \in \Lambda^n V^*$ for $\omega \neq 0$. Then B and B' have the same orientation if and only if $\omega(b_1, \dots, b_n)$ has the same sign as $\omega(b'_1, \dots, b'_n)$, by Proposition 1.8. An **orientation** Λ of V is a set of all the ordered basis of V with the same orientation. Let $\phi : V \rightarrow W$ be an isomorphism of vector spaces with fixed orientations Λ_v and Λ_w respectively. We say that ϕ is **orientation preserving** if an ordered basis of V induces an ordered basis of W , so Λ_v induces Λ_w . Let $V = \mathbb{R}^n$, and let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. Then e_1, \dots, e_n defines an orientation of V called **positive**. Let M be a manifold. The idea is to find an orientation Λ_x of $T_x M$ for all $x \in M$.

Special case. Let $M = U \subset \mathbb{R}^n$ be open. There exists a natural isomorphism $\phi_x : T_x U \rightarrow \mathbb{R}^n$. Let Λ_x^+ be an orientation on $T_x U$ such that ϕ_x is orientation preserving with respect to the positive orientation on \mathbb{R}^n . Let $\Lambda^+ = \{\Lambda_x^+\}$.

General case. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M . On U_α , we define the orientation so that $(D\phi_\alpha)_x : T_x U_\alpha \rightarrow T_{\phi_\alpha(x)} \phi_\alpha(U) \subset \mathbb{R}^n$ is orientation preserving. This is called the positive orientation on the chart (U_α, ϕ_α) . We define Λ on M , which is a collection of Λ^+ on $T_x M$ for all $x \in M$. Then M is **orientable** if there exists an atlas with positive orientation charts. This coincides in assuming that $\det D(\phi_\beta^{-1} \circ \phi_\alpha) > 0$ for all α and β .

Notation 1.19. For all $p \geq 0$,

$$\Omega_c^p(M) = \{\omega \in \Omega^p(M) \mid \text{supp } \omega \text{ is compact}\}.$$

If M is compact $\Omega_c^p(M) = \Omega^p(M)$. Let $\omega \in \Omega_c^p(M)$. Assume $\text{supp } \omega \subset U$ where (U, ϕ) is a chart of M , and $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$. Assume also that (U, ϕ) is positively oriented. Let $\phi^{-1} : \phi(U) \rightarrow U$ such that $(\phi^{-1})^* \omega \in \Omega_c^p(\phi(U))$, that is $\text{supp } (\phi^{-1})^* \omega \subset \phi(U)$. We define

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega. \quad (1)$$

We need to show that, under the assumptions above, $\int_M \omega$ does not depend on (U, ϕ) . Let $(\bar{U}, \bar{\phi})$ be also a positively oriented chart such that $\text{supp } \omega \subset \bar{U}$. We want to show that

$$\int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\bar{\phi}(\bar{U})} (\bar{\phi}^{-1})^* \omega.$$

Let $\bar{\phi} \circ \phi^{-1} : \phi(U \cap \bar{U}) \rightarrow \bar{\phi}(U \cap \bar{U})$, so

$$\begin{array}{ccc} & U \cap \bar{U} & \\ \phi \swarrow & & \searrow \bar{\phi} \\ \mathbb{R}^n \supset \phi(U \cap \bar{U}) & \xrightarrow{\bar{\phi} \circ \phi^{-1}} & \bar{\phi}(U \cap \bar{U}) \subset \mathbb{R}^n \end{array}.$$

Since both charts are positively oriented the determinant of the differential $D(\bar{\phi} \circ \phi^{-1})$ is positive, so

$$\begin{aligned} \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi}^{-1})^* \omega &= \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi}^{-1})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi} \circ \phi^{-1})^* (\bar{\phi}^{-1})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* \bar{\phi}^* (\bar{\phi}^{-1})^* \omega \\ &= \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* (\bar{\phi}^{-1} \circ \bar{\phi})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* \omega = \int_{\phi(U)} (\phi^{-1})^* \omega, \end{aligned}$$

by a property of the pull-back and since $(\bar{\phi}^{-1})^* \omega = 0$ outside $\bar{\phi}(U \cap \bar{U})$.

Lecture 4
Thursday
16/01/20

1.6 Partitions of unity

Definition 1.20. Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering. A **partition of unity** with respect to U is a collection of smooth functions $f_\alpha : M \rightarrow [0, 1]$ such that

1. $\text{supp } f_\alpha = \overline{\{x \in M \mid f_\alpha(x) > 0\}} \subset U_\alpha$ for all α ,
2. $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$, and
3. for all $x \in M$, there exists $U \ni x$ open such that $\text{supp } f_\alpha \cap U \neq \emptyset$ for only finitely many α .

Remark. 3 implies that 2 is a finite sum.

Example 1.21. Let

$$M = S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}, \quad U_1 = S^1 \setminus \{(1, 0)\}, \quad U_2 = S^1 \setminus \{(-1, 0)\},$$

so $\{U_i\}$ is a cover. Let

$$f_1(\cos \theta, \sin \theta) = \frac{1}{2} - \frac{1}{2} \cos \theta, \quad f_2(\cos \theta, \sin \theta) = \frac{1}{2} + \frac{1}{2} \cos \theta.$$

Then f_i is a partition of unity.

Proposition 1.22. Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering of M . Then there exists a partition of unity f_α with respect to U .

Proof. We omit the proof. □

Proposition 1.23. Let M be a manifold, and let $n = \dim M$. Then M is orientable if and only if there exists $\omega \in \Omega^n(M)$ which is never vanishing on M , so $\omega(x) \neq 0$ for all $x \in M$.

ω is called a **volume form** on M .

Proof.

\Leftarrow Assume $\omega \in \Omega^n(M)$ is a volume form. We want to construct an orientation Λ on M , that is Λ_x on $T_x M$ for all $x \in M$. Given an oriented basis v_1, \dots, v_n of $T_x M$ we say that it is **positively oriented** if $\omega(x)(v_1, \dots, v_n) > 0$. For all $x \in M$, we define the orientation Λ_x on $T_x M$ by considering the class of positively oriented ordered basis of $T_x M$ which is compatible with the choice of an atlas on M . Take any atlas $\{(U_\alpha, \phi_\alpha)\}$, where $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$. On U_α ,

$$\omega = g_\alpha \phi_\alpha^* dx_1 \wedge \dots \wedge dx_n.$$

Since $\omega \neq 0$, $g_\alpha > 0$ or $g_\alpha < 0$. If $g_\alpha < 0$ then switch x_1 with x_2 , so $g_\alpha > 0$. After this change of coordinates, (U_α, ϕ_α) is positively oriented, so M is orientable.

\Rightarrow Assume that M is orientable, that is there exists an atlas $\{(U_\alpha, \phi_\alpha)\}$ of positively oriented charts. On U_α , we consider

$$\omega_\alpha = \phi_\alpha^* dx_1 \wedge \dots \wedge dx_n.$$

Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Let $\widetilde{\omega}_\alpha = f_\alpha \omega_\alpha \in \Omega^n(U_\alpha)$. We may assume that $\widetilde{\omega}_\alpha \in \Omega^n(M)$ by extending equal to zero outside U_α . We define $\omega = \sum_\alpha \widetilde{\omega}_\alpha \in \Omega^n(M)$. For all α , since $\sum_\alpha f_\alpha = 1$ there exists α such that $\widetilde{\omega}_\alpha \neq 0$, so $\omega \neq 0$. □

Let M be an orientable manifold of dimension n , and let $\omega \in \Omega_c^n(M)$. We want to define $\int_M \omega$. So far we defined for ω such that $\text{supp } \omega \subset U_\alpha$ where (U_α, ϕ_α) is a chart.

Definition 1.24. Let $\{(U_\alpha, \phi_\alpha)\}$ be a positively oriented atlas on M , and let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $\text{supp } f_\alpha \omega \subset U_\alpha$, so let

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega.$$

Remark 1.25. Note that for each α , we have that the support of $f_\alpha \omega$ is contained in U_α and therefore each term of the sum is well-defined as in (1). Indeed, we have

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* f_\alpha.$$

Lemma 1.26. $\int_M \omega$ does not depend on $\{(U_\alpha, \phi_\alpha)\}$ and f_α .

Proof. Under the assumption that $\text{supp } \omega \subset U_\alpha$ then we showed $\int_{U_\alpha} \omega$ does not depend on (U_α, ϕ_α) . Let $\{(U_\alpha, \phi_\alpha)\}$ and $\{(\overline{U}_\alpha, \overline{\phi}_\alpha)\}$ be two atlases with positively oriented charts, and let f_α and \overline{f}_α be two partitions of unity with respect to $\{U_\alpha\}$ and $\{\overline{U}_\alpha\}$ respectively. Then $\sum_\alpha f_\alpha = \sum_\alpha \overline{f}_\alpha = 1$, so $\int_M f_\alpha \omega = \sum_\beta \int_M \overline{f}_\beta f_\alpha \omega$. Thus

$$\int_M \omega = \sum_\alpha \int_M f_\alpha \omega = \sum_{\alpha, \beta} \int_M \overline{f}_\beta f_\alpha \omega = \sum_\beta \int_M \sum_\alpha f_\alpha \overline{f}_\beta \omega = \sum_\beta \int_M \overline{f}_\beta \omega.$$

□

Proposition 1.27. Let M and N be orientable manifolds of dimension n , and let $\omega, \eta \in \Omega_c^n(M)$.

1. *Linearity*

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

2. *Orientation reversal.* Let \overline{M} be the manifold M with opposite orientation $\Lambda^- = \{\Lambda_x^- \mid x \in M\}$, which is the orientation opposite than the one induced by M with orientation Λ . Then

$$\int_M \omega = - \int_{\overline{M}} \omega.$$

3. *Positivity.* Let ω be the volume form on M . Then

$$\int_M \omega > 0.$$

4. *Diffeomorphism invariance.* Let $F : N \rightarrow M$ be an orientation preserving diffeomorphism. Then

$$\int_M \omega = \int_N F^* \omega.$$

Proof.

1. Exercise. ²

2. Exercise. ³

3. Choose a positively oriented chart (U_α, ϕ_α) on U_α , so

$$\omega = g_\alpha \phi_\alpha^* dx_1 \wedge \cdots \wedge dx_n, \quad g_\alpha > 0.$$

Then $\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega$ where f_α is a partition of unity. For all $x \in M$ there exists α such that $x \in U_\alpha$ and $\int_{U_\alpha} f_\alpha \omega > 0$, so $\int_M \omega > 0$.

4. Let (U_α, ϕ_α) be a positively oriented atlas on M . Then $(F^{-1}(U_\alpha), \phi_\alpha \circ F)$ is an atlas on N which is positively oriented. Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $f_\alpha \circ F$ is a partition of the unity with respect to $\{F^{-1}(U_\alpha)\}$, so

$$\int_N F^* \omega = \sum_\alpha \int_N (f_\alpha \circ F) F^* \omega = \sum_\alpha \int_N F^* (f_\alpha \omega) = \sum_\alpha \int_M f_\alpha \omega = \int_M \omega.$$

□

²Exercise

³Exercise

1.7 Manifolds with boundary

Denote

$$\mathbb{R}_{\geq 0}^n = (\mathbb{R}_{\geq 0})^n, \quad \mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Let $U \subset \mathbb{R}_+^n$ be open, and let $F : U \rightarrow \mathbb{R}^m$ be a function. Then F is C^∞ if it can be extended to a C^∞ -function $\tilde{F} : \tilde{U} \rightarrow \mathbb{R}^m$ where $\tilde{U} \supset U$ and \tilde{U} is open.

Definition 1.28. A **manifold with boundary** of dimension n is a Hausdorff topological space M such that there exists an open covering $\{U_\alpha\}$, and for all α , there exists a homeomorphism $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ such that for all α and β ,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n \rightarrow \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n$$

is a diffeomorphism, so

$$\begin{array}{ccc} & U_\alpha \cap U_\beta & \\ \phi_\alpha \swarrow & & \searrow \phi_\beta \\ \mathbb{R}_+^n \supset \phi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow{\phi_\alpha \circ \phi_\beta^{-1}} & \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n \end{array}.$$

The **boundary** of M is

$$\partial M = \{x \in M \mid \exists \alpha, \phi_\alpha(x) \in \partial \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}\}.$$

Then (U_α, ϕ_α) is called a **chart** and $\{(U_\alpha, \phi_\alpha)\}$ is called an **atlas**.

Remark 1.29.

- ∂M is closed in M .
- $\mathring{M} = M \setminus \partial M$ is a manifold of dimension n .

Example 1.30.

- $M = [0, 1]$ is a manifold with boundary $\partial M = \{0, 1\}$.
- The closed disc $D = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is a manifold with boundary $\partial D = S^{n-1}$.
- $M = [0, 1] \times S^1$ is a manifold with boundary $\partial M = S^1 \sqcup S^1$.

Remark 1.31.

- We can define tangent spaces and differential forms exactly in the same way as usual manifolds.
- The definition of orientability is the same. If M is orientable, then ∂M is also orientable. As a convention, the positive orientation on the boundary of $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}$ is given by $(-1)^n dx_1 \wedge \dots \wedge dx_{n-1}$. This induces a positive orientation on ∂M .
- Also partitions of unity for any open cover U_α of M is defined the same way. If M is orientable, for any manifold with boundary, for all open covering $U = \{U_\alpha\}$, there exists a partition of unity f_α . This implies that if $\omega \in \Omega_c^n(M)$, then $\int_M \omega$ is defined the same way for manifolds.

1.8 Stokes' theorem

Theorem 1.32 (Stokes). *For any manifold with boundary M of dimension n , and for any $\omega \in \Omega_c^{n-1}(M)$ we have*

$$\int_M d\omega = \int_{\partial M} \omega \in \Omega_c^n(M).$$

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Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas, and let $f_\alpha : M \rightarrow \mathbb{R}$ be a partition of unity with respect to this cover. Then $\sum_\alpha f_\alpha = 1$ on M , so

$$\int_M d\omega = \int_M d\left(\sum_\alpha f_\alpha \omega\right) = \sum_\alpha \int_M d(f_\alpha \omega) = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* d(f_\alpha \omega).$$

Proposition 1.16 implies that

$$(\phi_\alpha^{-1})^* d(f_\alpha \omega) = d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right).$$

Then $(\phi_\alpha^{-1})^* (f_\alpha \omega)$ is an $(n-1)$ -form on $\phi_\alpha(U_\alpha)$. In coordinates,

$$(\phi_\alpha^{-1})^* (f_\alpha \omega) = \sum_{j=1}^n \widetilde{f}_\alpha \omega_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

where ω_j is a smooth function on $\phi_\alpha(U_\alpha)$ and

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\phi_\alpha} & \phi_\alpha(U_\alpha) \\ f_\alpha \downarrow & \swarrow \widetilde{f}_\alpha & \\ [0, 1] & & \end{array}.$$

Then

$$\begin{aligned} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right) &= d\left(\sum_{j=1}^n \widetilde{f}_\alpha \omega_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n\right) \\ &= \sum_{j=1}^n \sum_{k=1}^n \frac{\partial}{\partial x_k} (\widetilde{f}_\alpha \omega_j) dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \sum_{j=1}^n \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) dx_1 \wedge \cdots \wedge dx_n, \end{aligned}$$

so

$$\sum_\alpha \int_{\phi_\alpha(U_\alpha)} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right) = \sum_\alpha \int_{\mathbb{R}_+^n} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right),$$

because $\widetilde{f}_\alpha = 0$ outside $\phi_\alpha(U_\alpha)$. Thus

$$\begin{aligned} \int_M d\omega &= \sum_\alpha \int_{\mathbb{R}_+^n} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) dx_1 \wedge \cdots \wedge dx_n \\ &= \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) dx_n dx_{n-1} \cdots dx_1 \\ &= \sum_\alpha \sum_{j=1}^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} (-1)^{j-1} \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) \Big|_{x_n=0} dx_n dx_{n-1} \cdots \widehat{dx_j} \cdots dx_1 \\ &= \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^{n-1} (\widetilde{f}_\alpha \omega_j) \Big|_{x_n=0} dx_{n-1} \cdots dx_1, \end{aligned}$$

since $(f_\alpha \omega_j)|_{x_n=0} = 0$ for $j = 1, \dots, n-1$, so

$$\int_M d\omega = \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^{n-1} (\widetilde{f}_\alpha \omega_j) \Big|_{x_n=0} dx_{n-1} \cdots dx_1 = \sum_\alpha \int_{\partial U_\alpha} f_\alpha|_{\partial U_\alpha} \omega = \int_{\partial M} \omega,$$

where $\partial U_\alpha = U_\alpha \cap \partial M$. □

1.9 Applications of Stokes' theorem

Theorem 1.33 (Integration by parts). *Let M be an orientable n -dimensional manifold with boundary, let $\omega \in \Omega_c^p(M)$, let $\eta \in \Omega_c^{n-p-1}(M)$, and let $p \in \{0, \dots, n-1\}$. Then*

$$\int_{\partial M} \omega \wedge \eta = \int_M d\omega \wedge \eta + (-1)^p \int_M \omega \wedge d\eta.$$

Proof.

$$\int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta) = \int_M (d\omega \wedge \eta + (-1)^p \omega \wedge d\eta),$$

by Stokes and the Leibnitz rule. \square

Theorem 1.34 (Brouwer's fixed point theorem). *Let*

$$D = \{x \in \mathbb{R}^n \mid |x| \leq 1\},$$

so

$$\partial D = S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\},$$

and let $f : D \rightarrow D$ be a smooth morphism. Then f admits a fixed point, that is there exists $x \in D$ such that $f(x) = x$.

Proof. Assume that $f(x) \neq x$ for all $x \in D$. For any $x \in D$, consider the ray starting from $f(x)$ and passing through x . Let $g(x)$ be the point where this ray intersects ∂D away from $f(x)$. Note that if $x \in \partial D$ then $g(x) = x$. Then $g : D \rightarrow \partial D$. It is easy to check that g is smooth. Since $\partial D = S^{n-1}$ is orientable by Proposition 1.23 there exists a volume form $\omega \in \Omega^{n-1}(\partial D)$, so $\omega(x) \neq 0$. Since $\omega \in \Omega^{n-1}(\partial D)$, $d\omega \in \Omega^n(\partial D)$, which is an n -dimensional manifold, so $d\omega = 0$. Thus

$$0 < \int_{\partial D} \omega = \int_{\partial D} g^* \omega = \int_D dg^* \omega = \int_D g^* d\omega = 0,$$

by Stokes, a contradiction. \square

Example 1.35. Recall any exact form is closed, since $d^2 = 0$. But the opposite is not always true. Let $M = \mathbb{R}^2 \setminus \{0\}$, and let

$$\omega = \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx \in \Omega^1(M).$$

Then ω is closed, since

$$d\omega = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) dx \wedge dy - \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) dy \wedge dx = 0.$$

We want to show that ω is not exact. Assume that

$$\omega = df, \quad f \in \Omega^0(M) = \{C^\infty\text{-function}\}.$$

In particular $\omega = df$ on $S^1 \subset M$. Let

$$\begin{aligned} \gamma & : [0, 2\pi] \longrightarrow S^1 \\ \theta & \longmapsto (\cos \theta, \sin \theta) \end{aligned}$$

Then

$$\int_{S^1} \omega = \int_0^{2\pi} \gamma^* \omega = \int_0^{2\pi} \left(\left(\frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} \right) \cos \theta d\theta - \left(\frac{\sin \theta}{\cos^2 \theta + \sin^2 \theta} \right) (-\sin \theta) d\theta \right) = \int_0^{2\pi} d\theta = 2\pi,$$

and

$$\int_{S^1} \omega = \int_{S^1} df = \int_{\partial S^1} f = \int_{\emptyset} f = 0,$$

so ω is not exact.

Proposition 1.36. *Let M be an orientable manifold of dimension n without boundary, and let $\omega \in \Omega_c^n(M)$. Assume ω is exact. Then*

$$\int_M \omega = 0.$$

Proof. Easy from Stokes. □

Proposition 1.37. *Let M be an orientable manifold of dimension n with boundary, and let $\omega \in \Omega_c^{n-1}(M)$ be a closed form. Then*

$$\int_{\partial M} \omega = 0.$$

Proof. Easy from Stokes. □

Let M be an orientable manifold of dimension n , let $\omega \in \Omega_c^k(M)$, and let $N \subset M$ be a submanifold of dimension k . We can define

$$\int_M \omega = \int_N i^* \omega,$$

where $i : N \hookrightarrow M$ is the inclusion. We will denote

$$\omega|_N = i^* \omega \in \Omega_c^k(N).$$

Proposition 1.38. *Let M be an oriented manifold of dimension n , let $\omega \in \Omega_c^k(M)$, and let $S \subset M$ be a compact orientable submanifold of dimension k such that $\partial S = \emptyset$ and $\int_S \omega \neq 0$. Then*

- ω is not exact,
- $\omega|_S$ is not exact, and
- S is not the boundary of an orientable manifold $N \subset M$ of dimension $k + 1$.

Proof. Exercise. ⁴ □

⁴Exercise

2 De Rham cohomology

2.1 De Rham cohomology

Definition 2.1. Let M be a manifold of dimension n , and let $p \geq 0$. Then $\omega_1, \omega_2 \in \Omega^p(M)$ are said to be **cohomologous** if $\omega_1 - \omega_2 = d\eta$ where $\eta \in \Omega^{p-1}(M)$. In particular $\omega \in \Omega^p(M)$ is cohomologous to zero if it is exact. Let

$$\mathcal{Z}^p(M) = \text{Ker}(d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)) = \{\omega \in \Omega^p(M) \mid \omega \text{ is closed}\} \subset \Omega^p(M),$$

and let

$$\mathcal{B}^p(M) = \text{Im}(d : \Omega^{p-1}(M) \rightarrow \Omega^p(M)) = \{\omega \in \Omega^p(M) \mid \omega \text{ is exact}\} \subset \Omega^p(M).$$

Then $\mathcal{B}^p(M) \subset \mathcal{Z}^p(M)$ for all $p \geq 0$.

Notation. If $p = 0$, then $\mathcal{B}^0(M) = 0$.

Note. If $\omega_1, \omega_2 \in \mathcal{Z}^p(M)$ then $\omega_1 - \omega_2 \in \mathcal{B}^p(M)$ if and only if ω_1 and ω_2 are cohomologous.

Definition 2.2. Denote the p -th de Rham cohomology group as

$$H^p(M) = \mathcal{Z}^p(M) / \mathcal{B}^p(M) = \{[\omega] \mid \omega \in \mathcal{Z}^p(M)\}, \quad p \geq 0.$$

where

$$[\omega] = \{\omega' \in \Omega^p(M) \text{ cohomologous to } \omega\}$$

is the **de Rham class** of ω .

Remark. $H^p(M)$ is a vector space over \mathbb{R} .

Definition 2.3. $b_p(M) = \dim H^p(M) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is the p -th **Betti number** of M .

Proposition 2.4. If M is connected then

$$H^0(M) = \mathbb{R},$$

that is $b_0(M) = 1$. More in general, $b_0(M)$ is the number of connected components of M .

Proof. Assume M is connected. Then $\mathcal{B}^0(M) = 0$, so

$$\begin{aligned} H^0(M) &= \mathcal{Z}^0(M) = \{f \in \Omega^0(M) \text{ closed}\} \\ &= \left\{ f \in \Omega^0(M) \mid \text{locally } \forall x \in M, \frac{\partial}{\partial x_i} f(x) = 0 \right\} \\ &= \{f \in \Omega^0(M) \text{ locally constant}\} = \mathbb{R}. \end{aligned}$$

□

Example. Let $M = S^1$. Then $H^0(M) = \mathbb{R}$.

Proposition 2.5. Let M be a manifold of dimension n . Then

$$H^p(M) = 0, \quad p \geq n+1.$$

Proof. Recall $\Omega^p(M) = 0$ if $p \geq n+1$ because all alternating p -forms for $p \geq n+1$ on an n -dimensional vector space are zero, so $\mathcal{Z}^p(M) = 0$. Thus $H^p(M) = 0$. □

Proposition 2.6. Let M be a compact orientable manifold of dimension n without boundary. Then

$$H^n(M) \neq 0.$$

Proof. M is orientable, so there exists a volume form $\omega \in \Omega^n(M) = \Omega_c^n(M)$, by Proposition 1.23. Then ω is closed, because $d\omega$ is an $(n+1)$ -form on M , so $\omega \in \mathcal{Z}^n(M)$. We want to show that $[\omega] \neq 0$ in $H^n(M)$. Assume $[\omega] = 0$, so ω is exact. Thus $\omega = d\eta$ where η is an $(n-1)$ -form on M , so

$$0 < \int_M \omega = \int_M d\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0,$$

by Stokes, a contradiction. □

Proposition 2.7. *Let $G : M \rightarrow N$ be a smooth morphism between manifolds. Then*

$$G^* : \Omega^p(N) \rightarrow \Omega^p(M), \quad p \geq 0$$

takes closed forms of N to closed forms on M and exact forms of N to exact forms on M .

Proof. Proposition 1.16 implies that $G^*d = dG^*$. If ω is closed then $dG^*\omega = G^*d\omega = G^*0 = 0$, so $G^*\omega$ is closed. If $\omega = d\eta$ is exact then $G^*\omega = dG^*\eta$ is also exact. \square

Thus $G^* : \mathcal{Z}^p(N) \rightarrow \mathcal{Z}^p(M)$ and $G^* : \mathcal{B}^p(N) \rightarrow \mathcal{B}^p(M)$, so there exists a linear map

$$\begin{array}{ccc} G^* & : & \mathcal{H}^p(N) \longrightarrow \mathcal{H}^p(M) \\ & & [\omega] \longmapsto [G^*\omega] \end{array}.$$

Corollary 2.8. *Let M and N be diffeomorphic manifolds. Then*

$$\mathcal{H}^p(M) \cong \mathcal{H}^p(N), \quad p \geq 0,$$

that is $\mathcal{H}^p(M)$ is a diffeomorphic invariant.

Proof. By Proposition 2.7 there exists $F^* : \mathcal{H}^p(N) \rightarrow \mathcal{H}^p(M)$ and $(F^{-1})^* : \mathcal{H}^p(M) \rightarrow \mathcal{H}^p(N)$. By Proposition 1.8,

$$(F^{-1})^* F^* \omega = (F \circ F^{-1})^* \omega = \text{id}_N^* \omega = \omega, \quad \omega \in \mathcal{H}^p(N)$$

so $(F^{-1})^* \circ F^* = \text{id}_{\mathcal{H}^p(N)}$. Similarly $F^* \circ (F^{-1})^* = \text{id}_{\mathcal{H}^p(M)}$, so F^* is an isomorphism. \square

2.2 Homotopy invariance

Definition 2.9. Let M_0 and M_1 be manifolds, and let $f_0, f_1 : M_0 \rightarrow M_1$ be smooth morphisms. Then f_0 and f_1 are **smoothly homotopic equivalent** if there exists a smooth morphism $H : M_0 \times [0, 1] \rightarrow M_1$ such that $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$ for all $x \in M_0$. A **homotopy** is a smooth morphism $H : M_0 \times [0, 1] \rightarrow M_1$ where M_0 and M_1 are smooth manifolds.

Notation 2.10. Let $f_t(x) = H(x, t)$, so $f_t : M_0 \rightarrow M_1$ is a smooth morphism. Then f_0 and f_1 are said to be homotopic equivalent, denoted by $f_0 \sim f_1$. Then \sim is an equivalence. ⁵

Definition 2.11. M_0 and M_1 are **homotopy equivalent** if there exist smooth morphisms $f : M_0 \rightarrow M_1$ and $g : M_1 \rightarrow M_0$ such that $f \circ g \sim \text{id}_{M_1}$ and $g \circ f \sim \text{id}_{M_0}$.

Example 2.12.

- Let $M_0 = \mathbb{R}^n$ and $M_1 = \{0\}$. Then M_0 and M_1 are homotopy equivalent. Let

$$\begin{array}{ccc} f & : & M_0 \longrightarrow M_1 \\ x & \longmapsto & 0 \end{array}, \quad \begin{array}{ccc} g & : & M_1 \longrightarrow M_0 \\ 0 & \longmapsto & 0 \end{array}.$$

Then

$$\begin{array}{ccc} f \circ g & : & M_1 \longrightarrow M_1 \\ 0 & \longmapsto & 0 \end{array},$$

so $f \circ g = \text{id}_{M_1}$, and

$$\begin{array}{ccc} g \circ f & : & M_0 \longrightarrow M_0 \\ x & \longmapsto & 0 \end{array}.$$

We want to show that $g \circ f \sim \text{id}_{M_0}$. Define a smooth morphism

$$\begin{array}{ccc} H & : & M_0 \times [0, 1] \longrightarrow M_0 \\ (x, t) & \longmapsto & tx \end{array}.$$

Then $H(x, 0) = 0 = (g \circ f)(x)$ for all x , and $H(x, 1) = x = \text{id}_{M_0}(x)$ for all x , so $g \circ f \sim \text{id}_{M_0}$. More in general $M \subset \mathbb{R}^n$ is called **convex** if for all $x, y \in M$ the segment joining x to y is contained inside M . If M is convex then M is homotopy equivalent to $M \times \{0\}$.

⁵Exercise

- Let $M_0 = \mathbb{R}^2 \setminus \{0\}$ and $M_1 = S^1$. Then M_0 and M_1 are homotopy equivalent. Let

$$\begin{aligned} f &: M_0 \longrightarrow M_1 \\ x &\longmapsto \frac{x}{|x|}, \end{aligned} \quad \begin{aligned} g &: M_1 \longrightarrow M_0 \\ x &\longmapsto x \end{aligned}.$$

Then

$$\begin{aligned} f \circ g &: M_1 \longrightarrow M_1 \\ x &\longmapsto x \end{aligned},$$

so $f \circ g = \text{id}_{M_1}$, and

$$\begin{aligned} g \circ f &: M_0 \longrightarrow M_0 \\ x &\longmapsto \frac{x}{|x|}. \end{aligned}$$

Let

$$\begin{aligned} H &: M_0 \times [0, 1] \longrightarrow M_0 \\ (x, t) &\longmapsto tx + (1-t) \frac{x}{|x|} \end{aligned}$$

be smooth. Then $H(x, 0) = x/|x| = (g \circ f)(x)$ and $H(x, 1) = x = \text{id}_{M_0}(x)$, so $g \circ f \sim \text{id}_{M_0}$.

Proposition 2.13. *Let M and N be manifolds, and let $H : M \times [0, 1] \rightarrow N$ be smooth. Denote*

$$\begin{aligned} f_t &: M \longrightarrow N \\ x &\longmapsto H(x, t), \end{aligned} \quad t \in [0, 1].$$

Then $f_t^* : H^p(N) \rightarrow H^p(M)$ does not depend on t for all $p \geq 0$.

Proof. Let $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$. The goal is $f_{t_1}^*[\eta] = f_{t_2}^*[\eta]$ for all $[\eta] \in H^p(N)$. Let

$$\begin{aligned} i_k &: M \longrightarrow M \times [0, 1] \\ x &\longmapsto (x, t_k) \end{aligned}, \quad k = 1, 2.$$

Claim that for all p there exists a linear map $h : \Omega^p(M \times [t_1, t_2]) \rightarrow \Omega^{p-1}(M)$ such that

$$d(h(\omega)) + h(d\omega) = i_2^*\omega - i_1^*\omega \in \Omega^p(M), \quad \omega \in \Omega^p(M \times [0, 1]). \quad (2)$$

Step 1. The claim implies the proposition. Let $\eta \in \Omega^p(N)$ be closed, so $d\eta = 0$. Then $H^*\eta$ is also closed, so let $\omega = H^*\eta \in \Omega^p(M \times [t_1, t_2])$. Apply h . Then $d\omega = 0$, so $d(h(\omega)) = i_2^*\omega - i_1^*\omega$ is exact. Thus

$$f_{t_1}^*[\eta] = [f_{t_1}^*\eta] = [i_1^*H^*\eta] = [i_1^*\omega] = [i_2^*\omega] = [i_2^*H^*\eta] = [f_{t_2}^*\eta] = f_{t_2}^*[\eta],$$

so the proposition follows.

Step 2. The proof of the claim. Let $\omega \in \Omega^p(M \times [t_1, t_2])$. Then for all $(x, t) \in M \times [t_1, t_2]$, $\omega(x, t)$ is an alternating p -form on $T_{(x,t)}(M \times [t_1, t_2])$. We want an alternating $(p-1)$ -form $h(\omega)(x)$ on $T_x M$. Let $v_1, \dots, v_{p-1} \in T_x M$. Then

$$h(\omega)(x)(v_1, \dots, v_{p-1}) = \int_{t_1}^{t_2} \omega(x, t) \left(\frac{\partial}{\partial t}, v_1, \dots, v_{p-1} \right) dt$$

is a $(p-1)$ -form on M , and $\frac{\partial}{\partial t}$ is a global vector field. Check h is linear.⁶ It is enough to prove (2) locally. Remark that exactness is not a local property. Fix local coordinates (x_1, \dots, x_n, t) around a point of $M \times [0, 1]$. Then

$$\omega = \sum_{|I|=p} \omega_I + \sum_{|J|=p-1} \omega_J, \quad \omega_I = g_I dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad \omega_J = g_J dx_{j_1} \wedge \dots \wedge dx_{j_{p-1}} \wedge dt,$$

where g_I and g_J are smooth functions. Any term of (2) is linear. So we just need to check (2) holds for ω_I and ω_J .

⁶Exercise

ω_I . Let $\omega = g(x, t) dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Then

$$d \left(h \left(\omega(x, t) \left(\frac{\partial}{\partial t}, v_1, \dots, v_{p-1} \right) \right) \right) = d(h(0)) = 0,$$

and

$$\begin{aligned} h(d\omega) &= h \left(\frac{\partial}{\partial t} g(x, t) dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} + \sum_{j=1}^n \frac{\partial}{\partial x_j} g(x, t) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \right) \\ &= \left(\int_{t_1}^{t_2} \frac{\partial}{\partial t} g(x, t) dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_p} + 0 \\ &= (g(x, t_2) - g(x, t_1)) dx_{i_1} \wedge \cdots \wedge dx_{i_p} = i_2^* \omega - i_1^* \omega, \end{aligned}$$

so (2) holds.

ω_J . Let $\omega = g(x, t) dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \wedge dt$. Then

$$\begin{aligned} d(h(\omega)) &= (-1)^{p-1} d \left(\left(\int_{t_1}^{t_2} g(x, t) dt \right) dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \right) \\ &= (-1)^{p-1} \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\int_{t_1}^{t_2} g(x, t) dt \right) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \\ &= (-1)^{p-1} \sum_{j=1}^n \left(\int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x, t) dt \right) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}}, \end{aligned}$$

and

$$\begin{aligned} h(d\omega) &= h \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} g(x, t) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \wedge dt + 0 \right) \\ &= (-1)^{p-1} \sum_{j=1}^n \left(\int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x, t) dt \right) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} = -d(h(\omega)), \end{aligned}$$

and $i_2^* \omega = i_1^* \omega = 0$, so (2) holds. □

Corollary 2.14. *Assume M and N are homotopy equivalent. Then there exist isomorphisms*

$$H^p(N) \rightarrow H^p(M), \quad p \geq 0.$$

Proof. There exist $f : M \rightarrow N$ and $g : N \rightarrow M$ such that $g \circ f \sim \text{id}_M$ and $f \circ g \sim \text{id}_N$. By Proposition 2.13 $(g \circ f)^* : H^p(M) \rightarrow H^p(M)$ coincides with $\text{id}_M^* = \text{id}_{H^p(M)}$. Then $f^* \circ g^* = (g \circ f)^* = \text{id}_{H^p(M)}$. Similarly $g^* \circ f^* = \text{id}_{H^p(N)}$, so g^* and f^* are isomorphisms. □

Definition 2.15. Let M be a manifold. Then M is **smoothly contractible** if M is homotopy equivalent to a point.

Example. \mathbb{R}^n is contractible, by Example 2.12. If $M \subset \mathbb{R}^n$ is convex then M is contractible.

Theorem 2.16 (Poincaré lemma). *If M is a contractible manifold then*

$$H^p(M) = 0, \quad p \geq 1.$$

Proof. By previous Corollary 2.14, there exists an isomorphism $H^p(M) \rightarrow H^p(\{\text{point}\})$. Then $\{\text{point}\}$ is a zero-dimensional manifold, so by Proposition 2.5, $H^p(\{\text{point}\}) = 0$ for all $p > 0$. □

Thus $H^p(\mathbb{R}^n) = 0$ for all $p > 0$, so \mathbb{R}^n is not diffeomorphic to any compact orientable manifold.

Proposition 2.17. *Let M be a manifold, and let $\omega \in \Omega^p(M)$ be a closed p -form for $p > 0$. Then for all $x \in X$, there exists a neighbourhood $U \ni x$ such that ω is exact on U , that is there exists $\eta \in \Omega^{p-1}(U)$ such that $\omega = d\eta$ on U .*

Proof. Let (U, ϕ) be a chart around x . I may assume that $V = \phi(U)$ is a ball in \mathbb{R}^n . Then U is diffeomorphic to $B = \{z \mid |z - z_0| < r\}$ for some $z_0 \in \mathbb{R}^n$ and $r > 0$, so $H^p(U) \cong H^p(B)$ for all $p \geq 0$. Since B is contractible, $H^p(B) = 0$ for all $p > 0$. The restriction of ω on U gives a class $[\omega] \in H^p(U) = 0$, so ω is cohomologous to zero on U . Thus ω is exact on U . \square

Definition 2.18. Let M be a manifold, let $\gamma : [0, 1] \rightarrow M$ be a continuous or smooth path, and let $x = \gamma(0)$ and $y = \gamma(1)$. A **homotopy of paths** from x to y is a map

$$\begin{aligned} F : [0, 1] \times [0, 1] &\longrightarrow M \\ (0, t) &\longmapsto x \\ (1, t) &\longmapsto y \end{aligned}$$

Proposition 2.19. *Let γ_0 and γ_1 be homotopic paths on a manifold M , and let $\omega \in \Omega^1(M)$ be closed. Then*

$$\int_0^1 \gamma_0^* \omega = \int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

Proof. Lee's introduction to smooth manifolds. The idea is

$$\int_{\gamma_0} \omega - \int_{\gamma_1} \omega = \int_{\gamma_0 \cup \gamma_1} \omega = \int_{\text{Im } F} \omega = 0,$$

by Stokes' theorem. \square

Recall that M is **simply connected**, so $\pi_1(M) = 0$, if any path γ from x to x is homotopic equivalent to a point.

Proposition 2.20. *Let M be a simply connected orientable manifold. Then*

$$H^1(M) = 0.$$

Proof. Let $\omega \in \Omega^1(M)$ be a closed form. Then claim that ω is exact if and only if $\int_\gamma \omega = 0$ for all loops γ , that is paths from x to x .

- The proof of the claim. Assume that $\omega = df$ is exact for $f \in \Omega^0(M)$. By Proposition 2.19,

$$\int_\gamma \omega = \int_{\text{trivial loop}} \omega = 0.$$

Assume that $\int_\gamma \omega = 0$ for all loops γ . Fix x . Let

$$f(y) = \int_x^y \omega.$$

Since $\int_{\gamma_1 \cup \gamma_2} \omega = 0$, f is well-defined, that is it does not depend on the choice of the path. Then $df = \omega$. This can be checked locally, that is in an open set of \mathbb{R}^n . Here it follows from the fundamental theorem of calculus.

- The claim implies the proposition. Being simply connected, any loop inside M is homotopic equivalent to the trivial loop. For all loops γ and for all closed ω , $\int_\gamma \omega = 0$ by Proposition 2.19, so ω is exact. Thus $[\omega] = 0$ in $H^1(M)$. \square

2.3 Some homological algebra

Let C^\bullet be a sequence of vector spaces, that is C^k is a vector space for $k \in \mathbb{Z}$.

Definition 2.21. (C^\bullet, d^\bullet) is a **cochain complex** if C^\bullet is a sequence of vector spaces and d^\bullet is a sequence of linear maps $d^k : C^k \rightarrow C^{k+1}$ such that the composition $d^{k+1} \circ d^k : C^k \rightarrow C^{k+1} \rightarrow C^{k+2}$ is zero for all k . Then d^\bullet is the **differential**.

Definition 2.22. The elements of

$$\mathcal{Z}^k(C^\bullet, d^\bullet) = \text{Ker}(d^k : C^k \rightarrow C^{k+1}) \subset C^k$$

are called **cocycles**. The elements of

$$\mathcal{B}^k(C^\bullet, d^\bullet) = \text{Im}(d^k : C^{k-1} \rightarrow C^k) \subset C^k$$

are called **coboundaries**. Then $d^{k-1} \circ d^k = 0$, so $\mathcal{B}^k \subset \mathcal{Z}^k$. The quotients

$$H^k(C^\bullet, d^\bullet) = \mathcal{Z}^k(C^\bullet, d^\bullet) / \mathcal{B}^k(C^\bullet, d^\bullet)$$

are the **k -th cohomology groups** of (C^\bullet, d^\bullet) .

Definition 2.23. Let (C^\bullet, d^\bullet) and (D^\bullet, d^\bullet) be two cochain complexes. A map $f : (C^\bullet, d^\bullet) \rightarrow (D^\bullet, d^\bullet)$ is a sequence of linear maps $f^k : C^k \rightarrow D^k$ such that $f^{k+1} \circ d^k = d^k \circ f^k$ for all k , so

$$\begin{array}{ccccccc} \dots & \longrightarrow & C^k & \xrightarrow{d^k} & C^{k+1} & \xrightarrow{d^{k+1}} & C^{k+2} \longrightarrow \dots \\ & & \downarrow f^k & & \downarrow f^{k+1} & & \downarrow f^{k+2} \\ \dots & \longrightarrow & D^k & \xrightarrow{d^k} & D^{k+1} & \xrightarrow{d^{k+1}} & D^{k+2} \longrightarrow \dots \end{array}.$$

Proposition 2.24. Let $f : (C^\bullet, d^\bullet) \rightarrow (D^\bullet, d^\bullet)$ be a map between cochain complexes. Then there exists a natural induced map

$$f^k : H^k(C^\bullet, d^\bullet) \rightarrow H^k(D^\bullet, d^\bullet).$$

Proof. Let $[\omega] \in H^k(C^\bullet, d^\bullet) = \mathcal{Z}^k(C^\bullet, d^\bullet) / \mathcal{B}^k(C^\bullet, d^\bullet)$ for $\omega \in \mathcal{Z}^k(C^\bullet, d^\bullet)$, that is $d^k(\omega) = 0$. I want to check that $f^k(\omega) \in \mathcal{Z}^k(D^\bullet, d^\bullet)$. By definition of maps, $d^k(f^k(\omega)) = f^{k+1}(d^k(\omega)) = 0$, so there is a map

$$\mathcal{Z}^k(C^\bullet, d^\bullet) \rightarrow \mathcal{Z}^k(D^\bullet, d^\bullet).$$

Now I need to check that if $\omega \in \mathcal{B}^k(C^\bullet, d^\bullet)$ then $f^k(\omega) \in \mathcal{B}^k(D^\bullet, d^\bullet)$. ⁷

□

Definition 2.25. A sequence of linear maps

$$C^1 \xrightarrow{f^1} C^2 \rightarrow \dots \rightarrow C^{k-1} \xrightarrow{f^{k-1}} C^k$$

between vector spaces is said to be **exact** if for all i , $\text{Ker } f^i = \text{Im } f^{i-1}$.

Example 2.26.

- A sequence

$$0 \rightarrow C^1 \xrightarrow{f^1} C^2$$

is exact if and only if f^1 is injective.

- A sequence

$$C^1 \xrightarrow{f^1} C^2 \rightarrow 0$$

is exact if and only if f^1 is surjective.

- An exact sequence

$$0 \rightarrow C^1 \xrightarrow{f^1} C^2 \xrightarrow{f^2} C^3 \rightarrow 0$$

is called a **short exact sequence**. In particular f^1 is injective and f^2 is surjective.

⁷Exercise

- Any long exact sequence

$$C^1 \xrightarrow{f^1} C^2 \rightarrow \dots \rightarrow C^{q-1} \xrightarrow{f^{q-1}} C^q$$

can be split into short exact sequences

$$\begin{array}{ccccccc} \dots \rightarrow C^{k-1} & \xrightarrow{f^{k-1}} & C^k & \xrightarrow{f^k} & C^{k+1} & \rightarrow & \dots \\ & \searrow & \nearrow & \searrow & \nearrow & & \\ & \text{Im } f^{k-1} = \text{Ker } f^k & & \text{Im } f^k = \text{Ker } f^{k+1} & & & \\ & \nearrow & \searrow & \nearrow & \searrow & & \\ 0 & & 0 & & 0 & & 0 \end{array}, \quad k = 2, \dots, q-1.$$

Lemma 2.27 (Snake lemma). *Consider the commutative diagram*

$$\begin{array}{ccccccc} C^1 & \xrightarrow{f^1} & C^2 & \xrightarrow{f^2} & C^3 & \longrightarrow & 0 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \\ 0 & \longrightarrow & D^1 & \xrightarrow{g^1} & D^2 & \xrightarrow{g^2} & D^3 \end{array},$$

such that both the horizontal lines are exact sequences. Then there exists a long exact sequence

$$\text{Ker } \alpha_1 \rightarrow \text{Ker } \alpha_2 \rightarrow \text{Ker } \alpha_3 \xrightarrow{\delta} \text{Coker } \alpha_1 \rightarrow \text{Coker } \alpha_2 \rightarrow \text{Coker } \alpha_3.$$

If

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^1 & \xrightarrow{f^1} & C^2 & \xrightarrow{f^2} & C^3 \longrightarrow 0 \\ & & \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 \\ 0 & \longrightarrow & D^1 & \xrightarrow{g^1} & D^2 & \xrightarrow{g^2} & D^3 \longrightarrow 0 \end{array},$$

then

$$0 \rightarrow \text{Ker } \alpha_1 \rightarrow \text{Ker } \alpha_2 \rightarrow \text{Ker } \alpha_3 \xrightarrow{\delta} \text{Coker } \alpha_1 \rightarrow \text{Coker } \alpha_2 \rightarrow \text{Coker } \alpha_3 \rightarrow 0.$$

Proof. We are going to construct $\delta : \text{Ker } \alpha_3 \rightarrow \text{Coker } \alpha_1$. Let $x \in \text{Ker } \alpha_3$. There exists $y \in C^2$ such that $f^2(y) = x$ because f^2 is surjective. Let $z = \alpha_2(y)$ then

$$g^2(z) = g^2(\alpha_2(y)) = \alpha_3(f^2(y)) = \alpha_3(x) = 0,$$

since $x \in \text{Ker } \alpha_3$. Then $z \in \text{Ker } g^2 = \text{Im } g^1$, so there exists $w \in D^1$ such that $z = g^1(w)$. The idea is

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{Ker } \alpha_1 & \longrightarrow & \text{Ker } \alpha_2 & \longrightarrow & \text{Ker } \alpha_3 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ C^1 & \xrightarrow{f^1} & y \in C^2 & \xrightarrow{f^2} & x \in C^3 & \longrightarrow & 0 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \\ 0 & \longrightarrow & w \in D^1 & \xrightarrow{g^1} & z \in D^2 & \xrightarrow{g^2} & D^3 \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{Coker } \alpha_1 & \longrightarrow & \text{Coker } \alpha_2 & \longrightarrow & \text{Coker } \alpha_3 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}.$$

Define $\delta(x) = [w] \in \text{Coker } \alpha_1 = D^1 / \text{Im } \alpha_1$. Need to check that δ is well-defined, so $[w]$ does not depend on our choice of w and y . The rest is an exercise. ⁸ \square

⁸Exercise

2.4 The Mayer-Vietoris sequence

The idea is given a manifold M , we may write $M = U \cup V$ with open U and V so that $H^i(U)$, $H^i(V)$, and $H^i(U \cap V)$ are easy to compute, so this will give us $H^i(M)$. Let M be a manifold, and let U and V be open such that $M = U \cup V$. Assume $U \cap V \neq \emptyset$. Let

$$i_U : U \rightarrow M, \quad i_V : V \rightarrow M, \quad j_U : U \cap V \rightarrow U, \quad j_V : U \cap V \rightarrow V$$

be inclusions, and let $i_U^*, i_V^*, j_U^*, j_V^*$ be pull-backs.

Proposition 2.28. *For all p there exist short exact sequences*

$$0 \rightarrow \Omega^p(M) \xrightarrow{f} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{g} \Omega^p(U \cap V) \rightarrow 0,$$

where $f = (i_U^*, i_V^*)$ and $g = j_V^* - j_U^*$. More precisely, if $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^p(V)$ then $g(\omega_1, \omega_2) = j_V^* \omega_2 - j_U^* \omega_1$.

Proof.

- f is injective. Assume $\omega \in \Omega^p(M)$ such that $f(\omega) = 0$, so $i_U^* \omega = i_V^* \omega = 0$. Since $M = U \cup V$ then $\omega = 0$ on M , so f is injective.
- $\text{Im } f = \text{Ker } g$. Let $f(\omega) \in \text{Im } f$, so $f(\omega) = (i_U^* \omega, i_V^* \omega)$. Then $g(f(\omega)) = j_V^* i_V^* \omega - j_U^* i_U^* \omega = l^* \omega - l^* \omega = 0$, where

$$\begin{array}{ccccc} & & U & & \\ & j_U \nearrow & & \searrow i_U & \\ U \cap V & \xrightarrow{l} & M & & \\ & j_V \searrow & & \nearrow i_V & \\ & & V & & \end{array}$$

so $\text{Im } f \subset \text{Ker } g$. Now let $(\omega_1, \omega_2) \in \text{Ker } g$, so $j_V^* \omega_2 = j_U^* \omega_1$ for $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^p(V)$. The restriction of ω_2 on $U \cap V$ coincides with the restriction of ω_1 on $U \cap V$. Then define

$$\omega = \begin{cases} \omega_1 & \text{on } U \\ \omega_2 & \text{on } V \end{cases}.$$

Then $f(\omega) = (\omega_1, \omega_2)$, so $\text{Ker } g \subset \text{Im } f$.

- g is surjective. Let $\eta \in \Omega^p(U \cap V)$, and let $\{f_U, f_V\}$ be a partition of unity with respect to $\{U, V\}$. Then $\text{supp } f_U \subset U$ and $f_U + f_V = 1$. Let $\eta_1 \in \Omega^p(U)$ be defined by

$$\eta_1 = \begin{cases} f_V \cdot \eta & \text{on } U \cap V \\ 0 & \text{outside } \text{supp } f_V \end{cases},$$

and let $\eta_2 \in \Omega^p(V)$ be defined by

$$\eta_2 = \begin{cases} f_U \cdot \eta & \text{on } U \cap V \\ 0 & \text{outside } \text{supp } f_U \end{cases}.$$

Then $g(-\eta_2, \eta_1) = \eta_1|_{U \cap V} + \eta_2|_{U \cap V} = (f_U + f_V) \cdot \eta = \eta$, so $\eta \in \text{Im } g$.

□

Theorem 2.29 (Mayer-Vietoris). *Let M be a manifold, and let U and V be open in M such that $M = U \cup V$ and $U \cap V \neq \emptyset$. Then for all $p \geq 0$ there exists a linear $\delta : H^p(U \cap V) \rightarrow H^{p+1}(M)$ such that*

$$\begin{array}{c} \dots \longrightarrow H^p(M) \xrightarrow{(i_U^*, i_V^*)} H^p(U) \oplus H^p(V) \xrightarrow{j_V^* - j_U^*} H^p(U \cap V) \longrightarrow \\ \xrightarrow{\delta} H^{p+1}(M) \xrightarrow{(i_U^*, i_V^*)} H^{p+1}(U) \oplus H^{p+1}(V) \xrightarrow{j_V^* - j_U^*} H^{p+1}(U \cap V) \longrightarrow \dots \end{array}$$

is exact.

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Example 2.30. Let $M = S^1$, let $N = (0, 1)$ and $S = (0, -1)$, and let $U = M \setminus \{N\}$ and $V = M \setminus \{S\}$, so $M = U \cup V$ and $U \cap V = M \setminus \{N, S\}$. Then

$$H^p(U) \cong H^p(V) \cong H^p((0, 1)) \cong \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \quad (0, 1) \subset \mathbb{R},$$

and

$$H^p(U \cap V) = H^p(U \setminus \{S\}) = H^p\left(\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)\right) = \begin{cases} \mathbb{R}^2 & p = 0 \\ 0 & p > 0 \end{cases}, \quad \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right) \subset \mathbb{R},$$

so

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(M) & \rightarrow & H^0(U) \oplus H^0(V) & \xrightarrow{\phi} & H^0(U \cap V) \xrightarrow{\delta} H^1(M) \rightarrow H^1(U) \oplus H^1(V) \rightarrow H^1(U \cap V) \rightarrow \dots \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ & & \mathbb{R} & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R}^2 \\ & & & & & & \mathbb{R} \\ & & & & & & 0 \oplus 0 \\ & & & & & & 0 \end{array}.$$

Then $\text{Im } \phi = \mathbb{R} \subset H^0(U \cap V) = \mathbb{R}^2$. Thus

$$H^1(M) = \text{Coker } \phi = \mathbb{R}^2 / \text{Im } \phi \cong \mathbb{R}.$$

Remark 2.31. Let

$$0 \rightarrow C^1 \rightarrow \dots \rightarrow C^k \rightarrow 0$$

be an exact sequence. Then

$$\sum_k (-1)^k \dim C^k = 0.^9$$

In our $M = S^1$ case $1 - 2 + 2 - \dim H^1(M) = 0$, so $\dim H^1(M) = 1$. Thus $H^1(M) \cong \mathbb{R}$.

Example 2.32. Let $M = S^n \subset \mathbb{R}^{n+1}$ be the n -dimensional sphere. Then

$$H^p(M) = \begin{cases} \mathbb{R} & p = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

By induction on n .

$n = 1$. Ok.

$n > 1$. Let $U = M \setminus \{N\}$ and $V = M \setminus \{S\}$, so $U \cap V \neq \emptyset$ and $U \cup V = M$. Then

$$U \cong V \cong \mathbb{R}^n, \quad U \cap V = V \setminus \{N\} \cong \mathbb{R}^n \setminus \{0\} \cong S^{n-1},$$

so

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(M) & \rightarrow & H^0(U) \oplus H^0(V) & \rightarrow & H^0(U \cap V) \xrightarrow{\delta} H^1(M) \rightarrow H^1(U) \oplus H^1(V) \rightarrow \dots \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ & & \mathbb{R} & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R} \\ & & & & & & 0 \oplus 0 \end{array}.$$

Then $1 - 2 + 1 - \dim H^1(M) = 0$, so $\dim H^1(M) = 0$. Thus $H^1(M) = 0$. Then for $p > 0$

$$\begin{array}{ccccccc} \dots & \rightarrow & H^p(U) \oplus H^p(V) & \rightarrow & H^p(U \cap V) & \xrightarrow{\delta} & H^{p+1}(M) \rightarrow H^{p+1}(U) \oplus H^{p+1}(V) \rightarrow \dots \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} \\ & & 0 \oplus 0 & & & & 0 \oplus 0 \end{array}$$

is exact, so $H^p(U \cap V) \cong H^{p+1}(M)$. By induction

$$H^p(U \cap V) = H^{p+1}(M) = \begin{cases} \mathbb{R} & p = n - 1 \\ 0 & \text{otherwise} \end{cases}.$$

⁹Exercise

Proof of Theorem 2.29. By Proposition 2.28 for all p

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^p(M) & \longrightarrow & \Omega^p(U) \oplus \Omega^p(V) & \longrightarrow & \Omega^p(U \cap V) \longrightarrow 0 \\ & & \downarrow d_M^p & & \downarrow d_U^p \oplus d_V^p & & \downarrow d_{U \cap V}^p \\ 0 & \longrightarrow & \Omega^{p+1}(M) & \longrightarrow & \Omega^{p+1}(U) \oplus \Omega^{p+1}(V) & \longrightarrow & \Omega^{p+1}(U \cap V) \longrightarrow 0 \end{array}$$

are exact. Recall d commutes with the pull-back. By the strong snake lemma,

$$\begin{array}{ccccccc} \text{Coker } d_M^{p-1} & \longrightarrow & \text{Coker } (d_U^{p-1}, d_V^{p-1}) & \longrightarrow & \text{Coker } d_{U \cap V}^{p-1} & \longrightarrow & 0 \\ & & \downarrow \partial_M^p = d_M^p & & \downarrow (\partial_U^p, \partial_V^p) = (d_U^p, d_V^p) & & \downarrow \partial_{U \cap V}^p = d_{U \cap V}^p \\ 0 & \longrightarrow & \text{Ker } d_M^{p+1} & \longrightarrow & \text{Ker } (d_U^{p+1}, d_V^{p+1}) & \longrightarrow & \text{Ker } d_{U \cap V}^{p+1} \end{array},$$

which is well-defined, since $d^{p+1} \circ d^p = 0$. By the weak snake lemma again,

$$\text{Ker } \partial_M^p \rightarrow \text{Ker } (\partial_U^p, \partial_V^p) \rightarrow \text{Ker } \partial_{U \cap V}^p \xrightarrow{\delta} \text{Coker } \partial_M^p \rightarrow \text{Coker } (\partial_U^p, \partial_V^p) \rightarrow \text{Coker } \partial_{U \cap V}^p.$$

Then $\text{Coker } d_M^{p-1} = \Omega^p(M) / \text{Im } d_M^{p-1}$. There exists

$$H^p(M) = \text{Ker } d_M^p / \text{Im } d_M^{p-1} \xrightarrow{\sim} \text{Ker } (\Omega^p(M) / \text{Im } d_M^{p-1} \rightarrow \text{Ker } d_M^{p+1}) = \text{Ker } \partial_M^p.$$

Similarly, $\text{Ker } (\partial_U^p, \partial_V^p) \cong H^p(U) \oplus H^p(V)$ and $\text{Ker } \partial_{U \cap V}^p \cong H^p(U \cap V)$. There exists

$$H^{p+1}(M) = \text{Ker } d_M^{p+1} / \text{Im } d_M^p \xrightarrow{\sim} \text{Coker } (\Omega^p(M) / \text{Im } d_M^{p-1} \rightarrow \text{Ker } d_M^{p+1}) = \text{Coker } \partial_M^p.$$

Similarly, $\text{Coker } (\partial_U^p, \partial_V^p) \cong H^{p+1}(U) \oplus H^{p+1}(V)$ and $\text{Coker } \partial_{U \cap V}^p \cong H^{p+1}(U \cap V)$. □

Example 2.33. Let $\mathbb{T}^2 = S^1 \times S^1$ be the torus. Then

$$H^p(\mathbb{T}^2) = \begin{cases} \mathbb{R} & p = 0, 2 \\ \mathbb{R} \oplus \mathbb{R} & p = 1 \end{cases}.$$

We leave the proof as an exercise. ¹⁰

Definition 2.34. Let M be a manifold, and let $U = \{U_i\}$ be an open cover of M . Then U is said to be **good** if for all $I = (i_1, \dots, i_p)$, $U_{i_1} \cap \dots \cap U_{i_p}$ is either \emptyset or contractible.

Lemma 2.35. Let M be a connected manifold which admits a finite good cover. Then for all $p \geq 0$, $H^p(M)$ is a finite dimensional vector space.

Exercise. Find a counterexample without assuming there exists a finite good cover.

Proof. Let U be a finite good cover. Define $k = \#U$. By induction on k .

$k = 1$. $M = U_1$ is contractible, so

$$H^p(M) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & \text{otherwise} \end{cases}.$$

$k > 1$. Assume ok for covers with at most $k-1$ elements. Let $U = \bigcup_{i=1}^{k-1} U_i$ and $V = U_k$. Then $U \cup V = M$ and $U \cap V \neq \emptyset$, so Mayer-Vietoris holds. By induction $H^p(U)$ and $H^p(V)$ are finite dimensional, since $H^p(U)$ is covered by $k-1$ of U_i and $H^p(V)$ is contractible. Then $U \cap V = \bigcup_{i=1}^{k-1} (U_i \cap U_k)$, and $\{U_i \cap U_k\}$ is a good cover of $U \cap V$ with $k-1$ elements. ¹¹ By induction $H^p(U \cap V)$ is finite dimensional. Thus $H^p(M)$ is also finite dimensional. □

¹⁰Exercise

¹¹Exercise

Fact. Any manifold admits a good cover.

Theorem 2.36. *Let M be a compact connected manifold. Then $H^p(M)$ is finite dimensional.*

Proof. Follows from the fact and Lemma 2.35. □

2.5 Compactly supported de Rham cohomology

Let M be a manifold, and let $\omega \in \Omega_c^p(M)$. Then $d\omega \in \Omega_c^{p+1}(M)$ and $d^2 = 0$, so

$$\Omega_c^p(M) \xrightarrow{d} \Omega_c^{p+1}(M) \xrightarrow{d} \dots$$

Let

$$H_c^p(M) = \mathcal{Z}_c^p(M) / \mathcal{B}_c^p(M) = \text{Ker}(d : \Omega_c^p(M) \rightarrow \Omega_c^{p+1}(M)) / \text{Im}(d : \Omega_c^{p-1}(M) \rightarrow \Omega_c^p(M)).$$

Example. If M is compact, then

$$H_c^p(M) = H^p(M), \quad p \geq 0.$$

Lemma 2.37. *Let M be a non-compact connected manifold. Then*

$$H_c^0(M) = 0.$$

Recall if M is connected $H^0(M) = \mathbb{R}$.

Proof. $H^0(M) = \{f \text{ constant on } M\}$ and $H_c^0(M) = \{f \text{ constant on } M \text{ and with compact support}\}$. Since M is non-compact, if $f \in \Omega_c^0(M)$, then $\text{supp } f \subsetneq M$. Thus there exists $x \in M$ such that $f(x) = 0$, so $f \equiv 0$, since f is constant. □

Let $f : M \rightarrow N$ be a smooth morphism between manifolds, and let $\omega \in \Omega_c^p(N) \subset \Omega^p(N)$. Then $f^*\omega \in \Omega^p(M)$, and $\text{supp } f^*\omega \subset f^{-1}(\text{supp } \omega)$, which is not compact in general, so $f^*\omega \notin \Omega_c^p(M)$ in general.

Definition 2.38. f is called **proper** if $f^{-1}(K)$ is compact for all compact subsets $K \subset N$.

If f is proper, then $f^* : \Omega_c^p(N) \rightarrow \Omega_c^p(M)$ is well-defined.

Exercise. If f is a diffeomorphism then f^* induces an isomorphism $H_c^p(N) \rightarrow H_c^p(M)$.

Definition 2.39. Let M_0 and M_1 be manifolds without boundary, and let $f_i : M_0 \rightarrow M_1$ be smooth morphisms for $i = 0, 1$. Then f_0 and f_1 are **smoothly properly homotopic** if there exists a smooth $H : M_0 \times [0, 1] \rightarrow M_1$ such that $H(\cdot, i) = f_i(\cdot)$ for $i = 0, 1$ and H is proper.

Notation. $f_t(\cdot) = H(\cdot, t) : M_0 \rightarrow M_1$.

Remark. To say that H is proper is not the same as saying f_t is proper for all t .

Exercise. Find H such that f_t is proper but H is not. A hint is to let $M_0 = M_1 = \mathbb{R}$ and $H : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ such that $f_t^{-1}(0)$ is bounded for all t but $H^{-1}(0)$ is not.

Definition 2.40. M_0 and M_1 are **properly smoothly homotopically equivalent** if there exist smooth morphisms $f : M_0 \rightarrow M_1$ and $g : M_1 \rightarrow M_0$ such that $f \circ g \sim \text{id}_{M_1}$ and $g \circ f \sim \text{id}_{M_0}$, where the equivalences are properly homotopic.

Proposition 2.41. *If M_0 and M_1 are properly homotopically equivalent then*

$$H_c^p(M_0) \cong H_c^p(M_1).$$

Let M be a manifold, and let $i : U \hookrightarrow M$ be an open set. Then there exist linear **push-forwards**

$$i_* : \Omega_c^p(U) \rightarrow \Omega_c^p(M), \quad p \geq 0.$$

Let $\omega \in \Omega_c^p(U)$. Then $\omega = 0$ outside $K \subset U$. We can define

$$i_*\omega = \begin{cases} \omega & \text{on } U \\ 0 & \text{on } M \setminus U \end{cases}.$$

If $j : V \hookrightarrow U$ and $i : U \hookrightarrow M$, then $(i \circ j)_* = i_* \circ j_*$.

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Lemma 2.42. *Let M be a manifold, and let $i : U \hookrightarrow M$ be an immersion such that U is open. Then for all $p \geq 0$, $i_* : \Omega_c^p(U) \rightarrow \Omega_c^p(M)$ commutes with d , that is*

$$d(i_*\omega) = i_*d\omega, \quad \omega \in \Omega_c^p(U).$$

In particular if ω is closed then $i_*\omega$ is closed, and if ω is exact then $i_*\omega$ is exact.

Proof.

$$d(i_*\omega) = \begin{cases} d\omega & \text{on } U \\ 0 & \text{otherwise} \end{cases} = i_*d\omega.$$

Let ω be closed, so $d\omega = 0$. Then $d(i_*\omega) = i_*d\omega = 0$, so $i_*\omega$ is closed. Similarly for exactness. \square

Let $U \hookrightarrow M$ be as before. Then there exist

$$i_* : H_c^p(U) \rightarrow H_c^p(M), \quad p \geq 0.$$

Theorem 2.43 (Punctured manifolds). *Let M be a manifold of dimension n , let $x \in M$, and let $i : M \setminus \{x\} \hookrightarrow M$. Then*

- for all $p \geq 2$, $i_* : H_c^p(M \setminus \{x\}) \rightarrow H_c^p(M)$ is an isomorphism.
- for all $p \geq 1$, if M is compact $i_* : H_c^p(M \setminus \{x\}) \rightarrow H_c^p(M) = H^p(M)$ is an isomorphism.

Proof. Injectivity.

$p \geq 2$. Let $\omega \in \Omega_c^p(M \setminus \{x\})$ be closed such that $i_*[\omega] = 0$, so $[i_*\omega] = 0$ in $H_c^p(M)$. The goal is $[\omega] = 0$. There exists $\eta \in \Omega_c^{p-1}(M)$ such that $i_*\omega = d\eta$. By Poincaré lemma there exists $U \subset M$ containing x such that $H^q(U) = 0$ for all $q \geq 1$. Then $i_*\omega = 0$ in a neighbourhood of x because $\text{supp } \omega \subset M \setminus \{x\}$, so $d\eta = 0$ in a neighbourhood of x . By taking U smaller we can assume η is closed. Since $p \geq 2$, $[\eta] \in H^{p-1}(U) = 0$, so η is exact. Then there exists $\sigma \in \Omega^{p-2}(U)$ such that $\eta = d\sigma$ on U . Let $(U, M \setminus \{x\})$ be an open cover of M , let $(f_U, f_{M \setminus \{x\}})$ be a partition of unity, and let $\eta' = \eta - d(i_*(f_U\sigma))$. On a neighbourhood of x , $\eta' = 0$ because $i_*(f_U\sigma) = \sigma$, so $\text{supp } \eta' \subset M \setminus \{x\}$. Thus $\eta' \in \Omega_c^{p-1}(M \setminus \{x\})$ and $\omega = d\eta'$, so $[\omega] = 0$.

$p = 1$. The same proof. Let $\omega \in \Omega_c^1(M \setminus \{x\})$ be closed such that $[i_*\omega] = 0$. There exists $\eta \in \Omega_c^0(M)$ such that $i_*\omega = d\eta$. By taking an open set $U \subset M$ such that $x \in U$, we may assume $d\eta = 0$. Then η is constant on U , so $\eta = c$. Let $\eta' = \eta - c$. Then $\eta' = 0$ on U . If M is compact then $\eta' \in \Omega_c^0(M \setminus \{x\})$. Thus $\omega = d\eta'$, so $[\omega] = 0$.

Surjectivity.

$p \geq 1$. Let $[\omega] \in \Omega_c^p(M)$ such that ω is closed. By Poincaré lemma there exists open $U \ni x$ such that ω is exact, so there exists $\sigma \in \Omega^{p-1}(U)$ such that $\omega = d\sigma$. Let $(f_U, f_{M \setminus \{x\}})$ be a partition of unity as before, and let $\omega' = \omega - d(i_*(f_U\sigma))$. Then $\omega' = 0$ in a neighbourhood of x and $[\omega'] = [\omega]$, and $\omega'|_{M \setminus \{x\}} \in \Omega_c^p(M \setminus \{x\})$. Thus $[i_*\omega']_{M \setminus \{x\}} = [\omega'] = [\omega]$. \square

Exercise. Compute $H_c^1(\mathbb{R}^2 \setminus \{0\})$ by hands.

Example 2.44.

$$H_c^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}.$$

Recall $\mathbb{R}^n \cong S^n \setminus \{x\}$ for $x \in S^n$. By Theorem 2.43, by $M = S^n$,

$$H_c^p(\mathbb{R}^n) = H_c^p(S^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}, \quad p \geq 1,$$

and $H_c^0(\mathbb{R}^n) = 0$.