

# M4P54 Differential Topology

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**Syllabus**

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## 0 Introduction

Differential topology is the study of the topology of a manifold using analysis. The topics are

- a review of differential forms,
- de Rham cohomology,
- Morse theory, and
- singular homology.

The following are references.

- J M Lee, Introduction to smooth manifolds, 2000
- L W Tu, Introduction to smooth manifolds, 2008
- J Milnor, Morse theory, 1960
- A Banyaga and D Hurtubise, Lectures on Morse homology, 2004

Lecture 1  
Thursday  
09/01/20

# 1 Differential forms on manifolds

## 1.1 Alternating $p$ -forms on a vector space

Let  $V$  be a vector space over  $\mathbb{R}$ , and let  $p \geq 0$ . Then  $V^p = V \times \cdots \times V$ .

**Definition 1.1.** A multilinear map  $\omega : V^p \rightarrow \mathbb{R}$  is called an **alternating  $p$ -form** if we have

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(p)}) = \epsilon(\sigma) \omega(v_1, \dots, v_p), \quad v_1, \dots, v_p \in V \quad \sigma \in \mathcal{S}_p,$$

where  $\mathcal{S}_p$  is the group of permutations of  $p$  elements and  $\epsilon(\sigma)$  is the signature of  $\sigma$ .

Recall that if  $m$  is the number of transpositions in a decomposition of  $\sigma$ , then  $\epsilon(\sigma) = (-1)^m$ , where a **transposition** is  $(a_i a_j)$  for  $a_i \neq a_j$ .

**Notation 1.2.**  $\Lambda^p V^* = \{\text{alternating } p\text{-forms } \omega \text{ on } V\}$  is called the  **$p$ -th exterior power** of  $V$ .

Check that it is a vector space.<sup>1</sup>

**Example 1.3.**

- $\Lambda^0 V^* = \mathbb{R}$ .
- $\Lambda^1 V^* = V^* = \text{Hom}(V, \mathbb{R})$ , the **dual** of  $V$ .

**Definition 1.4.** Let  $\omega_1 \in \Lambda^p V^*$  and  $\omega_2 \in \Lambda^q V^*$ . We define the **exterior product**  $\omega_1 \wedge \omega_2 \in \Lambda^{p+q} V^*$  of  $\omega_1$  and  $\omega_2$  by

$$\omega_1 \wedge \omega_2(v_1, \dots, v_{p+q}) = \sum_{\sigma \in \mathcal{S}_{p,q}} \epsilon(\sigma) \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \omega_2(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}), \quad v_1, \dots, v_{p+q} \in V,$$

where

$$\mathcal{S}_{p,q} = \{\sigma \in \mathcal{S}_{p+q} \mid \sigma(1) < \cdots < \sigma(p), \sigma(p+1) < \cdots < \sigma(p+q)\}.$$

**Example 1.5.**

- Assume  $\omega_1, \omega_2 \in \Lambda^1 V^*$ . Then

$$\omega_1 \wedge \omega_2(v_1, v_2) = \omega_1(v_1) \omega_2(v_2) - \omega_1(v_2) \omega_2(v_1), \quad v_1, v_2 \in V.$$

- Assume  $\omega_1, \dots, \omega_p \in \Lambda^1 V^*$ . Then

$$\omega_1 \wedge \cdots \wedge \omega_p(v_1, \dots, v_p) = \det(\omega_i(v_j))_{i,j=1,\dots,p}, \quad v_1, \dots, v_p \in V.$$

**Proposition 1.6.** Let  $\omega_i \in \Lambda^{p_i} V^*$  for  $i = 1, 2, 3$ .

- *Associativity*  $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$ .
- *Distributivity*  $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$ , assuming  $p_2 = p_3$ .
- *Supercommutativity*  $\omega_1 \wedge \omega_2 = (-1)^{p_1 p_2} \omega_2 \wedge \omega_1$ .

**Definition 1.7.** Let  $\Phi : V \rightarrow W$  be a linear map between vector spaces over  $\mathbb{R}$ . Let  $\omega \in \Lambda^p W^*$ . Then the **pull-back**  $\Phi^* \omega \in \Lambda^p V^*$  of  $\omega$  is an alternating  $p$ -form on  $V$  defined by

$$\Phi^* \omega(v_1, \dots, v_p) = \omega(\Phi(v_1), \dots, \Phi(v_p)), \quad v_1, \dots, v_p \in V.$$

---

<sup>1</sup>Exercise

**Proposition 1.8.** *Given  $\Phi : V \rightarrow W$  a linear map,*

- *the pull-back*

$$\begin{aligned} \Phi^* &: \Lambda^p W^* \longrightarrow \Lambda^p V^* \\ \omega &\longmapsto \Phi^* \omega \end{aligned}$$

*is a linear map that preserves exterior products, that is*

$$\Phi^* (\omega_1 \wedge \omega_2) = \Phi^* \omega_1 \wedge \Phi^* \omega_2, \quad \omega_1 \in \Lambda^p W^*, \quad \omega_2 \in \Lambda^q W^*,$$

- *if  $\Psi : W \rightarrow Z$  is linear then*

$$(\Psi \circ \Phi)^* \omega = \Phi^* \Psi^* \omega, \quad \omega \in \Lambda^p Z^*,$$

- *assuming  $V = W$  and  $p = \dim V$ , then*

$$\Phi^* \omega = (\det \Phi) \omega, \quad \omega \in \Lambda^p V^*.$$

## 1.2 Differential forms on manifolds

Let  $M$  be a smooth manifold of dimension  $n$ , and let  $x \in M$ . Then the tangent space  $T_x M$  of  $M$  at  $x$  is a vector space of dimension  $n$ .

**Notation 1.9.** Let

$$\Lambda^p T_x^* M = \Lambda^p (T_x M)^*.$$

Consider the set

$$\Lambda^p T^* M = \bigsqcup_{x \in M} \Lambda^p T_x^* M,$$

the  **$p$ -th exterior bundle** on  $M$ . There exists a morphism  $\pi : \Lambda^p T^* M \rightarrow M$  such that for all  $x \in M$ ,  $\pi^{-1}(x) = \Lambda^p T_x^* M$ , so  $\Lambda^p T^* M$  is a vector bundle and it is a smooth manifold, and  $\pi$  is a smooth morphism.

**Example 1.10.**

- $\Lambda^0 T^* M = M \times \mathbb{R}$ .
- $\Lambda^1 T^* M$  is the **cotangent bundle**, the dual of the tangent bundle.

**Definition 1.11.** A **differential  $p$ -form**  $\omega$  on  $M$  is a smooth section of  $\pi$ . That is, it is a smooth morphism  $\omega : M \rightarrow \Lambda^p T^* M$  such that  $\pi \circ \omega = \text{id}_M$ .

Thus,  $\omega(x) \in \Lambda^p T_x^* M$ .

**Notation 1.12.**

$$\Omega^p(M) = \{\text{differential } p\text{-forms } \omega \text{ on } M\}, \quad \Omega^\bullet(M) = \bigoplus_p \Omega^p(M).$$

**Example 1.13.**

$$\Omega^0(M) \cong \{f : M \rightarrow \mathbb{R} \text{ } C^\infty\text{-function}\}.$$

**Exercise.** If  $n = \dim M$ , then  $\Omega^{n+1}(M) = 0$ .

The algebra is the same as last week.

**Definition 1.14.** Let  $\omega_1 \in \Omega^p(M)$  and  $\omega_2 \in \Omega^q(M)$ . Then  $\omega_1 \wedge \omega_2 \in \Omega^{p+q}(M)$  is defined by

$$\omega_1 \wedge \omega_2(x) = \omega_1(x) \wedge \omega_2(x) \in \Lambda^{p+q} T_x^* M, \quad x \in M.$$

By Proposition 1.6, associativity, distributivity, and supercommutativity hold for  $\Omega^p(M)$ . Let  $F : M \rightarrow N$  be a smooth morphism between manifolds. Then for all  $x \in M$ , the differential of  $F$  at  $x$  is the linear map

$$DF_x : T_x M \rightarrow T_{F(x)} N.$$

Lecture 2  
Monday  
13/01/20

Thus, for all  $p \geq 0$ , we have a natural map, called the **pull-back**,

$$\begin{aligned} F_x^* : \Lambda^p T_{F(x)}^* N &\longrightarrow \Lambda^p T_x^* M \\ \omega(v_1, \dots, v_p) &\longmapsto \omega(DF_x(v_1), \dots, DF_x(v_p)) \end{aligned}, \quad \omega \in \Lambda^p T_{F(x)}^* N, \quad v_1, \dots, v_p \in T_x^* M.$$

Thus, we can define

$$\begin{aligned} F^* : \Omega^p(N) &\longrightarrow \Omega^p(M) \\ \omega(x) &\longmapsto F^* \omega(F(x)) \end{aligned}, \quad \omega \in \Omega^p(N).$$

By Proposition 1.8, the pull-back preserves the exterior product, so

$$F^*(\omega_1 \wedge \omega_2) = F^* \omega_1 \wedge F^* \omega_2.$$

If  $G : N \rightarrow P$ ,

$$(G \circ F)^* \omega = F^* G^* \omega.$$

### 1.3 Local description of $p$ -forms

Let  $M$  be a manifold of dimension  $n$ , let  $x_0 \in M$ , let  $(U, \phi)$  be a local chart around  $x_0$ , and let  $(x_1, \dots, x_n)$  be local coordinates around  $x_0$ . A basis of  $T_{x_0} M$  is given by

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}.$$

A basis of  $T_{x_0}^* M$  is given by

$$\{dx_1, \dots, dx_n\}, \quad dx_i \left( \frac{\partial}{\partial x_j} \right) = \delta_{ij}.$$

A basis of  $\Lambda^p T_{x_0}^* M$  is

$$dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad i_1 < \dots < i_p.$$

Thus,  $\omega \in \Omega^p(M)$  is locally given by

$$\omega(x) = \sum_{|I|=p} f_I(x) dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad I = (i_1, \dots, i_p), \quad i_1 < \dots < i_p,$$

where  $f_I$  is a  $C^\infty$ -function on  $U$  for all  $I$ .

**Example 1.15.** Let  $F : M \rightarrow N$  be a smooth morphism between manifolds of dimension  $n$ , and let  $\omega \in \Omega^n(N)$ . Locally,

$$\omega(y) = f(y) dy_1 \wedge \dots \wedge dy_n, \quad y \in N,$$

for some  $f \in C^\infty$ . Proposition 1.8 implies that

$$F^* \omega(x) = (f \circ F)(x) \det DF_x dx_1 \wedge \dots \wedge dx_n, \quad x \in M,$$

where  $y_i = p_i \circ F$  and  $p_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is the  $i$ -th projection.

Let  $f : M \rightarrow \mathbb{R}$  be a smooth function, so  $f \in \Omega^0(M)$ . Locally, the **differential** is

$$\begin{aligned} d : \Omega^0(M) &\longrightarrow \Omega^1(M) \\ f &\longmapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i. \end{aligned}$$

Check that  $df \in \Omega^1(M)$ , so  $df$  is a 1-form on  $M$ . Alternatively,  $df = f^* dx$  for  $dx$  a 1-form on  $\mathbb{R}$ , or  $df(X) = X(f)$  for any vector field  $X$  on  $M$ . More in general, let  $\omega \in \Omega^p(M)$ . Locally,

$$\omega = \sum_{|I|=p} f_I dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad f_I \in C^\infty,$$

so  $d\omega \in \Omega^{p+1}(M)$ . Then the **de Rham differential** is

$$\begin{aligned} d : \Omega^p(M) &\longrightarrow \Omega^{p+1}(M) \\ \omega &\longmapsto \sum_{|I|=p} df_I \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}. \end{aligned}$$

**Proposition 1.16.**

- The Leibnitz rule

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2, \quad \omega_1 \in \Omega^p(M), \quad \omega_2 \in \Omega^q(M).$$

- $d^2 = 0$ , that is

$$d(d\omega) = 0, \quad \omega \in \Omega^p(M).$$

- Let  $F : M \rightarrow N$  be a smooth morphism between manifolds. Then

$$F^*d\omega = d(F^*\omega), \quad \omega \in \Omega^p(M)$$

so

$$\begin{array}{ccc} \Omega^p(M) & \xrightarrow{d} & \Omega^{p+1}(M) \\ F^* \uparrow & & \uparrow F^* \\ \Omega^p(N) & \xrightarrow{d} & \Omega^{p+1}(N) \end{array}.$$

**Definition 1.17.**

- $\omega \in \Omega^p(M)$  is **closed** if  $d\omega = 0$ .
- $\omega \in \Omega^p(M)$  is **exact** if there exists  $\omega' \in \Omega^{p-1}(M)$  such that  $d\omega' = \omega$ .

$\omega$  is exact implies that  $\omega$  is closed, since if  $\omega = d\omega'$  then  $d\omega = d^2\omega' = 0$ .

**1.4 Integrations on manifolds**

Let  $M$  be a manifold of dimension  $n$ , let  $F : M \rightarrow M$  be a smooth morphism, and let  $\omega \in \Omega^n(M)$ . Then

$$F^*\omega(x) = \det DF_x \omega(F(x)).$$

Locally, assume  $\omega = f dy_1 \wedge \cdots \wedge dy_n$  for some coordinates  $y_1, \dots, y_n$  and  $f \in C^\infty$ . Let  $\{(U_\alpha, \phi_\alpha)\}$  be an atlas of  $M$ , where  $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n$ . Then

$$h_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \rightarrow \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n,$$

such that

$$h_{\alpha\beta}^* \omega(x) = (f \circ h_{\alpha\beta})(x) \det(Dh_{\alpha\beta})_x dx_1 \wedge \cdots \wedge dx_n.$$

Let  $D \subset \mathbb{R}^n$  be compact such that  $\partial D$  has zero measure, so  $D$  is a domain of integration, let  $f : U \rightarrow \mathbb{R}$  be a  $C^\infty$ -function where  $U \subset \mathbb{R}^n$  is open such that  $D \subset U$ , and let  $h : U \rightarrow h(U)$  be a diffeomorphism. Then

$$\int_{h^{-1}(D)} f(y) dy_1 \wedge \cdots \wedge dy_n = \int_{h^{-1}(D)} f(y) dy_1 \wedge \cdots \wedge dy_n = \int_D (f \circ h)(x) |\det Dh_x| dx_1 \wedge \cdots \wedge dx_n.$$

Let us assume that  $\omega = f(y) dy_1 \wedge \cdots \wedge dy_n$  on  $U$ . We define

$$\int_D \omega = \int_D f(y) dy_1 \wedge \cdots \wedge dy_n, \quad D \subset U.$$

**Definition 1.18.** Let  $U \subset \mathbb{R}^n$  be an open set. We define the **support** of  $\omega$  as

$$\text{supp } \omega = \overline{\{x \in U \mid \omega(x) \neq 0\}}, \quad \omega(x) \in \Lambda^p T_x^* U.$$

Then  $\omega$  has **compact support**, if  $\text{supp } \omega$  is compact.

**Fact.** Under this assumption, we can define

$$\int_U \omega = \int_D \omega \in \mathbb{R},$$

which is well-defined. Under the same assumption, if  $\phi : V \rightarrow U$  is a diffeomorphism, provided that  $\det D\phi_x > 0$ , since  $\det D\phi_x \neq 0$  for all  $x$ , then

$$\int_U \omega = \int_V \phi^* \omega.$$

Lecture 3  
Tuesday  
14/01/20

## 1.5 Orientation

Let  $V$  be a vector space over  $\mathbb{R}$  of dimension  $n$ , and let  $B = (b_1, \dots, b_n) \subset V$  and  $B' = (b'_1, \dots, b'_n) \subset V$  be ordered bases of  $V$ . Then  $B$  and  $B'$  have the **same orientation** if  $\det T > 0$  where

$$\begin{array}{ccc} T & : & V \longrightarrow V \\ & & b_i \longmapsto b'_i \end{array}$$

is a linear map. Let  $\omega \in \Lambda^n V^*$  for  $\omega \neq 0$ . Then  $B$  and  $B'$  have the same orientation if and only if  $\omega(b_1, \dots, b_n)$  has the same sign as  $\omega(b'_1, \dots, b'_n)$ , by Proposition 1.8. An **orientation**  $\Lambda$  of  $V$  is a set of all the ordered basis of  $V$  with the same orientation. Let  $\phi : V \rightarrow W$  be an isomorphism of vector spaces with fixed orientations  $\Lambda_v$  and  $\Lambda_w$  respectively. We say that  $\phi$  is **orientation preserving** if an ordered basis of  $V$  induces an ordered basis of  $W$ , so  $\Lambda_v$  induces  $\Lambda_w$ . Let  $V = \mathbb{R}^n$ , and let  $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ . Then  $e_1, \dots, e_n$  defines an orientation of  $V$  called **positive**. Let  $M$  be a manifold. The idea is to find an orientation  $\Lambda_x$  of  $T_x M$  for all  $x \in M$ .

Special case. Let  $M = U \subset \mathbb{R}^n$  be open. There exists a natural isomorphism  $\phi_x : T_x U \rightarrow \mathbb{R}^n$ . Let  $\Lambda_x^+$  be an orientation on  $T_x U$  such that  $\phi_x$  is orientation preserving with respect to the positive orientation on  $\mathbb{R}^n$ . Let  $\Lambda^+ = \{\Lambda_x^+\}$ .

General case. Let  $\{(U_\alpha, \phi_\alpha)\}$  be an atlas on  $M$ . On  $U_\alpha$ , we define the orientation so that  $(D\phi_\alpha)_x : T_x U_\alpha \rightarrow T_{\phi_\alpha(x)} \phi_\alpha(U) \subset \mathbb{R}^n$  is orientation preserving. This is called the positive orientation on the chart  $(U_\alpha, \phi_\alpha)$ . We define  $\Lambda$  on  $M$ , which is a collection of  $\Lambda^+$  on  $T_x M$  for all  $x \in M$ . Then  $M$  is **orientable** if there exists an atlas with positive orientation charts. This coincides in assuming that  $\det D(\phi_\beta^{-1} \circ \phi_\alpha) > 0$  for all  $\alpha$  and  $\beta$ .

**Notation 1.19.** For all  $p \geq 0$ ,

$$\Omega_c^p(M) = \{\omega \in \Omega^p(M) \mid \text{supp } \omega \text{ is compact}\}.$$

If  $M$  is compact  $\Omega_c^p(M) = \Omega^p(M)$ . Let  $\omega \in \Omega_c^p(M)$ . Assume  $\text{supp } \omega \subset U$  where  $(U, \phi)$  is a chart of  $M$ , and  $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$ . Assume also that  $(U, \phi)$  is positively oriented. Let  $\phi^{-1} : \phi(U) \rightarrow U$  such that  $(\phi^{-1})^* \omega \in \Omega_c^p(\phi(U))$ , that is  $\text{supp } (\phi^{-1})^* \omega \subset \phi(U)$ . We define

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega. \quad (1)$$

We need to show that, under the assumptions above,  $\int_M \omega$  does not depend on  $(U, \phi)$ . Let  $(\bar{U}, \bar{\phi})$  be also a positively oriented chart such that  $\text{supp } \omega \subset \bar{U}$ . We want to show that

$$\int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\bar{\phi}(\bar{U})} (\bar{\phi}^{-1})^* \omega.$$

Let  $\bar{\phi} \circ \phi^{-1} : \phi(U \cap \bar{U}) \rightarrow \bar{\phi}(U \cap \bar{U})$ , so

$$\begin{array}{ccc} & U \cap \bar{U} & \\ \phi \swarrow & & \searrow \bar{\phi} \\ \mathbb{R}^n \supset \phi(U \cap \bar{U}) & \xrightarrow{\bar{\phi} \circ \phi^{-1}} & \bar{\phi}(U \cap \bar{U}) \subset \mathbb{R}^n \end{array}.$$

Since both charts are positively oriented the determinant of the differential  $D(\bar{\phi} \circ \phi^{-1})$  is positive, so

$$\begin{aligned} \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi}^{-1})^* \omega &= \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi}^{-1})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi} \circ \phi^{-1})^* (\bar{\phi}^{-1})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* \bar{\phi}^* (\bar{\phi}^{-1})^* \omega \\ &= \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* (\bar{\phi}^{-1} \circ \bar{\phi})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* \omega = \int_{\phi(U)} (\phi^{-1})^* \omega, \end{aligned}$$

by a property of the pull-back and since  $(\bar{\phi}^{-1})^* \omega = 0$  outside  $\bar{\phi}(U \cap \bar{U})$ .

Lecture 4  
Thursday  
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## 1.6 Partitions of unity

**Definition 1.20.** Let  $M$  be a manifold, and let  $U = \{U_\alpha\}$  be an open covering. A **partition of unity** with respect to  $U$  is a collection of smooth functions  $f_\alpha : M \rightarrow [0, 1]$  such that

1.  $\text{supp } f_\alpha = \overline{\{x \in M \mid f_\alpha(x) > 0\}} \subset U_\alpha$  for all  $\alpha$ ,
2.  $\sum_\alpha f_\alpha(x) = 1$  for all  $x \in M$ , and
3. for all  $x \in M$ , there exists  $U \ni x$  open such that  $\text{supp } f_\alpha \cap U \neq \emptyset$  for only finitely many  $\alpha$ .

**Remark.** 3 implies that 2 is a finite sum.

**Example 1.21.** Let

$$M = S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}, \quad U_1 = S^1 \setminus \{(1, 0)\}, \quad U_2 = S^1 \setminus \{(-1, 0)\},$$

so  $\{U_i\}$  is a cover. Let

$$f_1(\cos \theta, \sin \theta) = \frac{1}{2} - \frac{1}{2} \cos \theta, \quad f_2(\cos \theta, \sin \theta) = \frac{1}{2} + \frac{1}{2} \cos \theta.$$

Then  $f_i$  is a partition of unity.

**Theorem 1.22.** Let  $M$  be a manifold, and let  $U = \{U_\alpha\}$  be an open covering of  $M$ . Then there exists a partition of unity  $f_\alpha$  with respect to  $U$ .

*Proof.* We omit the proof. □

**Proposition 1.23.** Let  $M$  be a manifold, and let  $n = \dim M$ . Then  $M$  is orientable if and only if there exists  $\omega \in \Omega^n(M)$  which is never vanishing on  $M$ , so  $\omega(x) \neq 0$  for all  $x \in M$ .

$\omega$  is called a **volume form** on  $M$ .

*Proof.*

$\Leftarrow$  Assume  $\omega \in \Omega^n(M)$  is a volume form. We want to construct an orientation  $\Lambda$  on  $M$ , that is  $\Lambda_x$  on  $T_x M$  for all  $x \in M$ . Given an oriented basis  $v_1, \dots, v_n$  of  $T_x M$  we say that it is **positively oriented** if  $\omega(x)(v_1, \dots, v_n) > 0$ . For all  $x \in M$ , we define the orientation  $\Lambda_x$  on  $T_x M$  by considering the class of positively oriented ordered basis of  $T_x M$  which is compatible with the choice of an atlas on  $M$ . Take any atlas  $\{(U_\alpha, \phi_\alpha)\}$ , where  $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ . On  $U_\alpha$ ,

$$\omega = g_\alpha \phi_\alpha^* dx_1 \wedge \dots \wedge dx_n.$$

Since  $\omega \neq 0$ ,  $g_\alpha > 0$  or  $g_\alpha < 0$ . If  $g_\alpha < 0$  then switch  $x_1$  with  $x_2$ , so  $g_\alpha > 0$ . After this change of coordinates,  $(U_\alpha, \phi_\alpha)$  is positively oriented, so  $M$  is orientable.

$\Rightarrow$  Assume that  $M$  is orientable, that is there exists an atlas  $\{(U_\alpha, \phi_\alpha)\}$  of positively oriented charts. On  $U_\alpha$ , we consider

$$\omega_\alpha = \phi_\alpha^* dx_1 \wedge \dots \wedge dx_n.$$

Let  $f_\alpha$  be a partition of unity with respect to  $\{U_\alpha\}$ . Let  $\widetilde{\omega}_\alpha = f_\alpha \omega_\alpha \in \Omega^n(U_\alpha)$ . We may assume that  $\widetilde{\omega}_\alpha \in \Omega^n(M)$  by extending equal to zero outside  $U_\alpha$ . We define  $\omega = \sum_\alpha \widetilde{\omega}_\alpha \in \Omega^n(M)$ . For all  $\alpha$ , since  $\sum_\alpha f_\alpha = 1$  there exists  $\alpha$  such that  $\widetilde{\omega}_\alpha \neq 0$ , so  $\omega \neq 0$ . □

Let  $M$  be an orientable manifold of dimension  $n$ , and let  $\omega \in \Omega_c^n(M)$ . We want to define  $\int_M \omega$ . So far we defined for  $\omega$  such that  $\text{supp } \omega \subset U_\alpha$  where  $(U_\alpha, \phi_\alpha)$  is a chart.

**Definition 1.24.** Let  $\{(U_\alpha, \phi_\alpha)\}$  be a positively oriented atlas on  $M$ , and let  $f_\alpha$  be a partition of unity with respect to  $\{U_\alpha\}$ . Then  $\text{supp } f_\alpha \omega \subset U_\alpha$ , so let

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega.$$

**Remark 1.25.** Note that for each  $\alpha$ , we have that the support of  $f_\alpha \omega$  is contained in  $U_\alpha$  and therefore each term of the sum is well-defined as in (1). Indeed, we have

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* f_\alpha.$$

**Lemma 1.26.**  $\int_M \omega$  does not depend on  $\{(U_\alpha, \phi_\alpha)\}$  and  $f_\alpha$ .

*Proof.* Under the assumption that  $\text{supp } \omega \subset U_\alpha$  then we showed  $\int_{U_\alpha} \omega$  does not depend on  $(U_\alpha, \phi_\alpha)$ . Let  $\{(U_\alpha, \phi_\alpha)\}$  and  $\{(\overline{U}_\alpha, \overline{\phi}_\alpha)\}$  be two atlases with positively oriented charts, and let  $f_\alpha$  and  $\overline{f}_\alpha$  be two partitions of unity with respect to  $\{U_\alpha\}$  and  $\{\overline{U}_\alpha\}$  respectively. Then  $\sum_\alpha f_\alpha = \sum_\alpha \overline{f}_\alpha = 1$ , so  $\int_M f_\alpha \omega = \sum_\beta \int_M \overline{f}_\beta f_\alpha \omega$ . Thus

$$\int_M \omega = \sum_\alpha \int_M f_\alpha \omega = \sum_{\alpha, \beta} \int_M \overline{f}_\beta f_\alpha \omega = \sum_\beta \int_M \sum_\alpha f_\alpha \overline{f}_\beta \omega = \sum_\beta \int_M \overline{f}_\beta \omega.$$

□

**Proposition 1.27.** Let  $M$  and  $N$  be orientable manifolds of dimension  $n$ , and let  $\omega, \eta \in \Omega_c^n(M)$ .

1. *Linearity*

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

2. *Orientation reversal.* Let  $\overline{M}$  be the manifold  $M$  with opposite orientation  $\Lambda^- = \{\Lambda_x^- \mid x \in M\}$ , which is the orientation opposite than the one induced by  $M$  with orientation  $\Lambda$ . Then

$$\int_M \omega = - \int_{\overline{M}} \omega.$$

3. *Positivity.* Let  $\omega$  be the volume form on  $M$ . Then

$$\int_M \omega > 0.$$

4. *Diffeomorphism invariance.* Let  $F : N \rightarrow M$  be an orientation preserving diffeomorphism. Then

$$\int_M \omega = \int_N F^* \omega.$$

*Proof.*

1. Exercise. <sup>2</sup>

2. Exercise. <sup>3</sup>

3. Choose a positively oriented chart  $(U_\alpha, \phi_\alpha)$  on  $U_\alpha$ , so

$$\omega = g_\alpha \phi_\alpha^* dx_1 \wedge \cdots \wedge dx_n, \quad g_\alpha > 0.$$

Then  $\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega$  where  $f_\alpha$  is a partition of unity. For all  $x \in M$  there exists  $\alpha$  such that  $x \in U_\alpha$  and  $\int_{U_\alpha} f_\alpha \omega > 0$ , so  $\int_M \omega > 0$ .

4. Let  $(U_\alpha, \phi_\alpha)$  be a positively oriented atlas on  $M$ . Then  $(F^{-1}(U_\alpha), \phi_\alpha \circ F)$  is an atlas on  $N$  which is positively oriented. Let  $f_\alpha$  be a partition of unity with respect to  $\{U_\alpha\}$ . Then  $f_\alpha \circ F$  is a partition of the unity with respect to  $\{F^{-1}(U_\alpha)\}$ , so

$$\int_N F^* \omega = \sum_\alpha \int_N (f_\alpha \circ F) F^* \omega = \sum_\alpha \int_N F^* (f_\alpha \omega) = \sum_\alpha \int_M f_\alpha \omega = \int_M \omega.$$

□

<sup>2</sup>Exercise

<sup>3</sup>Exercise

## 1.7 Manifolds with boundary

Denote

$$\mathbb{R}_{\geq 0}^n = (\mathbb{R}_{\geq 0})^n, \quad \mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Let  $U \subset \mathbb{R}_+^n$  be open, and let  $F : U \rightarrow \mathbb{R}^m$  be a function. Then  $F$  is  $C^\infty$  if it can be extended to a  $C^\infty$ -function  $\tilde{F} : \tilde{U} \rightarrow \mathbb{R}^m$  where  $\tilde{U} \supset U$  and  $\tilde{U}$  is open.

**Definition 1.28.** A **manifold with boundary** of dimension  $n$  is a Hausdorff topological space  $M$  such that there exists an open covering  $\{U_\alpha\}$ , and for all  $\alpha$ , there exists a homeomorphism  $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$  such that for all  $\alpha$  and  $\beta$ ,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n \rightarrow \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n$$

is a diffeomorphism, so

$$\begin{array}{ccc} & U_\alpha \cap U_\beta & \\ \phi_\alpha \swarrow & & \searrow \phi_\beta \\ \mathbb{R}_+^n \supset \phi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow{\phi_\alpha \circ \phi_\beta^{-1}} & \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n \end{array}.$$

The **boundary** of  $M$  is

$$\partial M = \{x \in M \mid \exists \alpha, \phi_\alpha(x) \in \partial \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}\}.$$

Then  $(U_\alpha, \phi_\alpha)$  is called a **chart** and  $\{(U_\alpha, \phi_\alpha)\}$  is called an **atlas**.

**Remark 1.29.**

- $\partial M$  is closed in  $M$ .
- $\mathring{M} = M \setminus \partial M$  is a manifold of dimension  $n$ .

**Example 1.30.**

- $M = [0, 1]$  is a manifold with boundary  $\partial M = \{0, 1\}$ .
- The closed disc  $D = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$  is a manifold with boundary  $\partial D = S^{n-1}$ .
- $M = [0, 1] \times S^1$  is a manifold with boundary  $\partial M = S^1 \sqcup S^1$ .

**Remark 1.31.**

- We can define tangent spaces and differential forms exactly in the same way as usual manifolds.
- The definition of orientability is the same. If  $M$  is orientable, then  $\partial M$  is also orientable. As a convention, the positive orientation on the boundary of  $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}$  is given by  $(-1)^n dx_1 \wedge \dots \wedge dx_{n-1}$ . This induces a positive orientation on  $\partial M$ .
- Also partitions of unity for any open cover  $U_\alpha$  of  $M$  is defined the same way. If  $M$  is orientable, for any manifold with boundary, for all open covering  $U = \{U_\alpha\}$ , there exists a partition of unity  $f_\alpha$ . This implies that if  $\omega \in \Omega_c^n(M)$ , then  $\int_M \omega$  is defined the same way for manifolds.

## 1.8 Stokes' theorem

**Theorem 1.32** (Stokes). *For any manifold with boundary  $M$  of dimension  $n$ , and for any  $\omega \in \Omega_c^{n-1}(M)$  we have*

$$\int_M d\omega = \int_{\partial M} \omega \in \Omega_c^n(M).$$

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*Proof.* Let  $\{(U_\alpha, \phi_\alpha)\}$  be an atlas, and let  $f_\alpha : M \rightarrow \mathbb{R}$  be a partition of unity with respect to this cover. Then  $\sum_\alpha f_\alpha = 1$  on  $M$ , so

$$\int_M d\omega = \int_M d\left(\sum_\alpha f_\alpha \omega\right) = \sum_\alpha \int_M d(f_\alpha \omega) = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* d(f_\alpha \omega).$$

Proposition 1.16 implies that

$$(\phi_\alpha^{-1})^* d(f_\alpha \omega) = d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right).$$

Then  $(\phi_\alpha^{-1})^* (f_\alpha \omega)$  is an  $(n-1)$ -form on  $\phi_\alpha(U_\alpha)$ . In coordinates,

$$(\phi_\alpha^{-1})^* (f_\alpha \omega) = \sum_{j=1}^n \widetilde{f}_\alpha \omega_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

where  $\omega_j$  is a smooth function on  $\phi_\alpha(U_\alpha)$  and

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\phi_\alpha} & \phi_\alpha(U_\alpha) \\ f_\alpha \downarrow & \swarrow \widetilde{f}_\alpha & \\ [0, 1] & & \end{array}.$$

Then

$$\begin{aligned} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right) &= d\left(\sum_{j=1}^n \widetilde{f}_\alpha \omega_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n\right) \\ &= \sum_{j=1}^n \sum_{k=1}^n \frac{\partial}{\partial x_k} (\widetilde{f}_\alpha \omega_j) dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \sum_{j=1}^n \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) dx_1 \wedge \cdots \wedge dx_n, \end{aligned}$$

so

$$\sum_\alpha \int_{\phi_\alpha(U_\alpha)} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right) = \sum_\alpha \int_{\mathbb{R}_+^n} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right),$$

because  $\widetilde{f}_\alpha = 0$  outside  $\phi_\alpha(U_\alpha)$ . Thus

$$\begin{aligned} \int_M d\omega &= \sum_\alpha \int_{\mathbb{R}_+^n} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) dx_1 \wedge \cdots \wedge dx_n \\ &= \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) dx_n dx_{n-1} \cdots dx_1 \\ &= \sum_\alpha \sum_{j=1}^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} (-1)^{j-1} \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) \Big|_{x_n=0} dx_n dx_{n-1} \cdots \widehat{dx_j} \cdots dx_1 \\ &= \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^{n-1} (\widetilde{f}_\alpha \omega_j) \Big|_{x_n=0} dx_{n-1} \cdots dx_1, \end{aligned}$$

since  $(f_\alpha \omega_j)|_{x_n=0} = 0$  for  $j = 1, \dots, n-1$ , so

$$\int_M d\omega = \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^{n-1} (\widetilde{f}_\alpha \omega_j) \Big|_{x_n=0} dx_{n-1} \cdots dx_1 = \sum_\alpha \int_{\partial U_\alpha} f_\alpha|_{\partial U_\alpha} \omega = \int_{\partial M} \omega,$$

where  $\partial U_\alpha = U_\alpha \cap \partial M$ . □

## 1.9 Applications of Stokes' theorem

**Theorem 1.33** (Integration by parts). *Let  $M$  be an orientable  $n$ -dimensional manifold with boundary, let  $\omega \in \Omega_c^p(M)$ , let  $\eta \in \Omega_c^{n-p-1}(M)$ , and let  $p \in \{0, \dots, n-1\}$ . Then*

$$\int_{\partial M} \omega \wedge \eta = \int_M d\omega \wedge \eta + (-1)^p \int_M \omega \wedge d\eta.$$

*Proof.*

$$\int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta) = \int_M (d\omega \wedge \eta + (-1)^p \omega \wedge d\eta),$$

by Stokes and the Leibnitz rule.  $\square$

**Theorem 1.34** (Brouwer's fixed point theorem). *Let*

$$D = \{x \in \mathbb{R}^n \mid |x| \leq 1\},$$

*so*

$$\partial D = S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\},$$

*and let  $f : D \rightarrow D$  be a smooth morphism. Then  $f$  admits a fixed point, that is there exists  $x \in D$  such that  $f(x) = x$ .*

*Proof.* Assume that  $f(x) \neq x$  for all  $x \in D$ . For any  $x \in D$ , consider the ray starting from  $f(x)$  and passing through  $x$ . Let  $g(x)$  be the point where this ray intersects  $\partial D$  away from  $f(x)$ . Note that if  $x \in \partial D$  then  $g(x) = x$ . Then  $g : D \rightarrow \partial D$ . It is easy to check that  $g$  is smooth. Since  $\partial D = S^{n-1}$  is orientable by Proposition 1.23 there exists a volume form  $\omega \in \Omega^{n-1}(\partial D)$ , so  $\omega(x) \neq 0$ . Since  $\omega \in \Omega^{n-1}(\partial D)$ ,  $d\omega \in \Omega^n(\partial D)$ , which is an  $n$ -dimensional manifold, so  $d\omega = 0$ . Thus

$$0 < \int_{\partial D} \omega = \int_{\partial D} g^* \omega = \int_D dg^* \omega = \int_D g^* d\omega = 0,$$

by Stokes, a contradiction.  $\square$

**Example 1.35.** Recall any exact form is closed, since  $d^2 = 0$ . But the opposite is not always true. Let  $M = \mathbb{R}^2 \setminus \{0\}$ , and let

$$\omega = \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx \in \Omega^1(M).$$

Then  $\omega$  is closed, since

$$d\omega = \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right) dx \wedge dy - \frac{\partial}{\partial y} \left( \frac{y}{x^2 + y^2} \right) dy \wedge dx = 0.$$

We want to show that  $\omega$  is not exact. Assume that

$$\omega = df, \quad f \in \Omega^0(M) = \{C^\infty\text{-function}\}.$$

In particular  $\omega = df$  on  $S^1 \subset M$ . Let

$$\begin{aligned} \gamma & : [0, 2\pi] \longrightarrow S^1 \\ \theta & \longmapsto (\cos \theta, \sin \theta) \end{aligned}$$

Then

$$\int_{S^1} \omega = \int_0^{2\pi} \gamma^* \omega = \int_0^{2\pi} \left( \left( \frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} \right) \cos \theta d\theta - \left( \frac{\sin \theta}{\cos^2 \theta + \sin^2 \theta} \right) (-\sin \theta) d\theta \right) = \int_0^{2\pi} d\theta = 2\pi,$$

and

$$\int_{S^1} \omega = \int_{S^1} df = \int_{\partial S^1} f = \int_{\emptyset} f = 0,$$

so  $\omega$  is not exact.

**Theorem 1.36.** *Let  $M$  be an orientable manifold of dimension  $n$  without boundary, and let  $\omega \in \Omega_c^n(M)$ . Assume  $\omega$  is exact. Then*

$$\int_M \omega = 0.$$

*Proof.* Easy from Stokes. □

**Theorem 1.37.** *Let  $M$  be an orientable manifold of dimension  $n$  with boundary, and let  $\omega \in \Omega_c^{n-1}(M)$  be a closed form. Then*

$$\int_{\partial M} \omega = 0.$$

*Proof.* Easy from Stokes. □

**Definition 1.38.** Let  $M$  be an orientable manifold of dimension  $n$ , let  $\omega \in \Omega_c^k(M)$ , and let  $N \subset M$  be a submanifold of dimension  $k$ . We can define

$$\int_M \omega = \int_N i^* \omega,$$

where  $i : N \hookrightarrow M$  is the inclusion. We will denote

$$\omega|_N = i^* \omega \in \Omega_c^k(N).$$

**Theorem 1.39.** *Let  $M$  be an oriented manifold of dimension  $n$ , let  $\omega \in \Omega_c^k(M)$ , and let  $S \subset M$  be a compact orientable submanifold of dimension  $k$  such that  $\partial S = \emptyset$  and  $\int_S \omega \neq 0$ . Then*

- $\omega$  is not exact,
- $\omega|_S$  is not exact, and
- $S$  is not the boundary of an orientable manifold  $N \subset M$  of dimension  $k+1$ .

*Proof.* Exercise. <sup>4</sup> □

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<sup>4</sup>Exercise

## 2 De Rham cohomology

### 2.1 De Rham cohomology

**Definition 2.1.** Let  $M$  be a manifold of dimension  $n$ , and let  $p \geq 0$ . Then  $\omega_1, \omega_2 \in \Omega^p(M)$  are said to be **cohomologous** if  $\omega_1 - \omega_2 = d\eta$  where  $\eta \in \Omega^{p-1}(M)$ . In particular  $\omega \in \Omega^p(M)$  is cohomologous to zero if it is exact. Let

$$Z^p(M) = \text{Ker}(d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)) = \{\omega \in \Omega^p(M) \mid \omega \text{ is closed}\} \subset \Omega^p(M),$$

and let

$$B^p(M) = \text{Im}(d : \Omega^{p-1}(M) \rightarrow \Omega^p(M)) = \{\omega \in \Omega^p(M) \mid \omega \text{ is exact}\} \subset \Omega^p(M).$$

Then  $B^p(M) \subset Z^p(M)$  for all  $p \geq 0$ .

**Notation.** If  $p = 0$ , then  $B^0(M) = 0$ .

**Note.** If  $\omega_1, \omega_2 \in Z^p(M)$  then  $\omega_1 - \omega_2 \in B^p(M)$  if and only if  $\omega_1$  and  $\omega_2$  are cohomologous.

**Definition 2.2.** Denote the  $p$ -th de Rham cohomology group as

$$H^p(M) = Z^p(M) / B^p(M) = \{[\omega] \mid \omega \in Z^p(M)\}, \quad p \geq 0.$$

where

$$[\omega] = \{\omega' \in \Omega^p(M) \text{ cohomologous to } \omega\}$$

is the **de Rham class** of  $\omega$ .

**Remark.**  $H^p(M)$  is a vector space over  $\mathbb{R}$ .

**Definition 2.3.**  $b_p(M) = \dim H^p(M) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$  is the  $p$ -th **Betti number** of  $M$ .

**Proposition 2.4.** If  $M$  is connected then

$$H^0(M) = \mathbb{R},$$

that is  $b_0(M) = 1$ . More in general,  $b_0(M)$  is the number of connected components of  $M$ .

*Proof.* Assume  $M$  is connected. Then  $B^0(M) = 0$ , so

$$\begin{aligned} H^0(M) &= Z^0(M) = \{f \in \Omega^0(M) \text{ closed}\} \\ &= \left\{ f \in \Omega^0(M) \mid \text{locally } \forall x \in M, \frac{\partial}{\partial x_i} f(x) = 0 \right\} \\ &= \{f \in \Omega^0(M) \text{ locally constant}\} = \mathbb{R}. \end{aligned}$$

□

**Example.** Let  $M = S^1$ . Then  $H^0(M) = \mathbb{R}$ .

**Proposition 2.5.** Let  $M$  be a manifold of dimension  $n$ . Then

$$H^p(M) = 0, \quad p \geq n+1.$$

*Proof.* Recall  $\Omega^p(M) = 0$  if  $p \geq n+1$  because all alternating  $p$ -forms for  $p \geq n+1$  on an  $n$ -dimensional vector space are zero, so  $Z^p(M) = 0$ . Thus  $H^p(M) = 0$ . □

**Proposition 2.6.** Let  $M$  be a compact orientable manifold of dimension  $n$  without boundary. Then

$$H^n(M) \neq 0.$$

*Proof.*  $M$  is orientable, so there exists a volume form  $\omega \in \Omega^n(M) = \Omega_c^n(M)$ , by Proposition 1.23. Then  $\omega$  is closed, because  $d\omega$  is an  $(n+1)$ -form on  $M$ , so  $\omega \in Z^n(M)$ . We want to show that  $[\omega] \neq 0$  in  $H^n(M)$ . Assume  $[\omega] = 0$ , so  $\omega$  is exact. Thus  $\omega = d\eta$  where  $\eta$  is an  $(n-1)$ -form on  $M$ , so

$$0 < \int_M \omega = \int_M d\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0,$$

by Stokes, a contradiction. □

**Proposition 2.7.** *Let  $G : M \rightarrow N$  be a smooth morphism between manifolds. Then*

$$G^* : \Omega^p(N) \rightarrow \Omega^p(M), \quad p \geq 0$$

*takes closed forms of  $N$  to closed forms on  $M$  and exact forms of  $N$  to exact forms on  $M$ .*

*Proof.* Proposition 1.16 implies that  $G^*d = dG^*$ . If  $\omega$  is closed then  $dG^*\omega = G^*d\omega = G^*0 = 0$ , so  $G^*\omega$  is closed. If  $\omega = d\eta$  is exact then  $G^*\omega = dG^*\eta$  is also exact.  $\square$

Thus  $G^* : Z^p(N) \rightarrow Z^p(M)$  and  $G^* : B^p(N) \rightarrow B^p(M)$ , so there exists a linear map

$$\begin{array}{ccc} G^* & : & H^p(N) \longrightarrow H^p(M) \\ & & [\omega] \longmapsto [G^*\omega] \end{array}.$$

**Corollary 2.8.** *Let  $M$  and  $N$  be diffeomorphic manifolds. Then*

$$H^p(M) \cong H^p(N), \quad p \geq 0,$$

*that is  $H^p(M)$  is a diffeomorphic invariant.*

*Proof.* By Proposition 2.7 there exists  $F^* : H^p(N) \rightarrow H^p(M)$  and  $(F^{-1})^* : H^p(M) \rightarrow H^p(N)$ . By Proposition 1.8,

$$(F^{-1})^* F^* \omega = (F \circ F^{-1})^* \omega = \text{id}_N^* \omega = \omega, \quad \omega \in H^p(N)$$

so  $(F^{-1})^* \circ F^* = \text{id}_{H^p(N)}$ . Similarly  $F^* \circ (F^{-1})^* = \text{id}_{H^p(M)}$ , so  $F^*$  is an isomorphism.  $\square$

## 2.2 Homotopy invariance

**Definition 2.9.** Let  $M_0$  and  $M_1$  be manifolds, and let  $f_0, f_1 : M_0 \rightarrow M_1$  be smooth morphisms. Then  $f_0$  and  $f_1$  are **smoothly homotopic equivalent** if there exists a smooth morphism  $H : M_0 \times [0, 1] \rightarrow M_1$  such that  $H(x, 0) = f_0(x)$  and  $H(x, 1) = f_1(x)$  for all  $x \in M_0$ . A **homotopy** is a smooth morphism  $H : M_0 \times [0, 1] \rightarrow M_1$  where  $M_0$  and  $M_1$  are smooth manifolds.

**Notation 2.10.** Let  $f_t(x) = H(x, t)$ , so  $f_t : M_0 \rightarrow M_1$  is a smooth morphism. Then  $f_0$  and  $f_1$  are said to be homotopic equivalent, denoted by  $f_0 \sim f_1$ . Then  $\sim$  is an equivalence. <sup>5</sup>

**Definition 2.11.**  $M_0$  and  $M_1$  are **homotopy equivalent** if there exist smooth morphisms  $f : M_0 \rightarrow M_1$  and  $g : M_1 \rightarrow M_0$  such that  $f \circ g \sim \text{id}_{M_1}$  and  $g \circ f \sim \text{id}_{M_0}$ .

**Example 2.12.**

- Let  $M_0 = \mathbb{R}^n$  and  $M_1 = \{0\}$ . Then  $M_0$  and  $M_1$  are homotopy equivalent. Let

$$\begin{array}{ccc} f & : & M_0 \longrightarrow M_1 \\ x & \longmapsto & 0 \end{array}, \quad \begin{array}{ccc} g & : & M_1 \longrightarrow M_0 \\ 0 & \longmapsto & 0 \end{array}.$$

Then

$$\begin{array}{ccc} f \circ g & : & M_1 \longrightarrow M_1 \\ 0 & \longmapsto & 0 \end{array},$$

so  $f \circ g = \text{id}_{M_1}$ , and

$$\begin{array}{ccc} g \circ f & : & M_0 \longrightarrow M_0 \\ x & \longmapsto & 0 \end{array}.$$

We want to show that  $g \circ f \sim \text{id}_{M_0}$ . Define a smooth morphism

$$\begin{array}{ccc} H & : & M_0 \times [0, 1] \longrightarrow M_0 \\ (x, t) & \longmapsto & tx \end{array}.$$

Then  $H(x, 0) = 0 = (g \circ f)(x)$  for all  $x$ , and  $H(x, 1) = x = \text{id}_{M_0}(x)$  for all  $x$ , so  $g \circ f \sim \text{id}_{M_0}$ . More in general  $M \subset \mathbb{R}^n$  is called **convex** if for all  $x, y \in M$  the segment joining  $x$  to  $y$  is contained inside  $M$ . If  $M$  is convex then  $M$  is homotopy equivalent to  $M \times \{0\}$ .

<sup>5</sup>Exercise



- Let  $M_0 = \mathbb{R}^2 \setminus \{0\}$  and  $M_1 = S^1$ . Then  $M_0$  and  $M_1$  are homotopy equivalent. Let

$$\begin{aligned} f &: M_0 \longrightarrow M_1 \\ x &\longmapsto \frac{x}{|x|}, \end{aligned} \quad \begin{aligned} g &: M_1 \longrightarrow M_0 \\ x &\longmapsto x \end{aligned}.$$

Then

$$\begin{aligned} f \circ g &: M_1 \longrightarrow M_1 \\ x &\longmapsto x \end{aligned},$$

so  $f \circ g = \text{id}_{M_1}$ , and

$$\begin{aligned} g \circ f &: M_0 \longrightarrow M_0 \\ x &\longmapsto \frac{x}{|x|}. \end{aligned}$$

Let

$$\begin{aligned} H &: M_0 \times [0, 1] \longrightarrow M_0 \\ (x, t) &\longmapsto tx + (1-t) \frac{x}{|x|} \end{aligned}$$

be smooth. Then  $H(x, 0) = x/|x| = (g \circ f)(x)$  and  $H(x, 1) = x = \text{id}_{M_0}(x)$ , so  $g \circ f \sim \text{id}_{M_0}$ .

**Proposition 2.13.** *Let  $M$  and  $N$  be manifolds, and let  $H : M \times [0, 1] \rightarrow N$  be smooth. Denote*

$$\begin{aligned} f_t &: M \longrightarrow N \\ x &\longmapsto H(x, t), \end{aligned} \quad t \in [0, 1].$$

Then  $f_t^* : H^p(N) \rightarrow H^p(M)$  does not depend on  $t$  for all  $p \geq 0$ .

*Proof.* Let  $t_1, t_2 \in [0, 1]$  such that  $t_1 < t_2$ . The goal is  $f_{t_1}^*[\eta] = f_{t_2}^*[\eta]$  for all  $[\eta] \in H^p(N)$ . Let

$$\begin{aligned} i_k &: M \longrightarrow M \times [0, 1] \\ x &\longmapsto (x, t_k) \end{aligned}, \quad k = 1, 2.$$

Claim that for all  $p$  there exists a linear map  $h : \Omega^p(M \times [t_1, t_2]) \rightarrow \Omega^{p-1}(M)$  such that

$$d(h(\omega)) + h(d\omega) = i_2^*\omega - i_1^*\omega \in \Omega^p(M), \quad \omega \in \Omega^p(M \times [0, 1]). \quad (2)$$

Step 1. The claim implies the proposition. Let  $\eta \in \Omega^p(N)$  be closed, so  $d\eta = 0$ . Then  $H^*\eta$  is also closed, so let  $\omega = H^*\eta \in \Omega^p(M \times [t_1, t_2])$ . Apply  $h$ . Then  $d\omega = 0$ , so  $d(h(\omega)) = i_2^*\omega - i_1^*\omega$  is exact. Thus

$$f_{t_1}^*[\eta] = [f_{t_1}^*\eta] = [i_1^*H^*\eta] = [i_1^*\omega] = [i_2^*\omega] = [i_2^*H^*\eta] = [f_{t_2}^*\eta] = f_{t_2}^*[\eta],$$

so the proposition follows.

Step 2. The proof of the claim. Let  $\omega \in \Omega^p(M \times [t_1, t_2])$ . Then for all  $(x, t) \in M \times [t_1, t_2]$ ,  $\omega(x, t)$  is an alternating  $p$ -form on  $T_{(x,t)}(M \times [t_1, t_2])$ . We want an alternating  $(p-1)$ -form  $h(\omega)(x)$  on  $T_x M$ . Let  $v_1, \dots, v_{p-1} \in T_x M$ . Then

$$h(\omega)(x)(v_1, \dots, v_{p-1}) = \int_{t_1}^{t_2} \omega(x, t) \left( \frac{\partial}{\partial t}, v_1, \dots, v_{p-1} \right) dt$$

is a  $(p-1)$ -form on  $M$ , and  $\frac{\partial}{\partial t}$  is a global vector field. Check  $h$  is linear.<sup>6</sup> It is enough to prove (2) locally. Remark that exactness is not a local property. Fix local coordinates  $(x_1, \dots, x_n, t)$  around a point of  $M \times [0, 1]$ . Then

$$\omega = \sum_{|I|=p} \omega_I + \sum_{|J|=p-1} \omega_J, \quad \omega_I = g_I dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad \omega_J = g_J dx_{j_1} \wedge \dots \wedge dx_{j_{p-1}} \wedge dt,$$

where  $g_I$  and  $g_J$  are smooth functions. Any term of (2) is linear. So we just need to check (2) holds for  $\omega_I$  and  $\omega_J$ .

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<sup>6</sup>Exercise

$\omega_I$ . Let  $\omega = g(x, t) dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ . Then

$$d \left( h \left( \omega(x, t) \left( \frac{\partial}{\partial t}, v_1, \dots, v_{p-1} \right) \right) \right) = d(h(0)) = 0,$$

and

$$\begin{aligned} h(d\omega) &= h \left( \frac{\partial}{\partial t} g(x, t) dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} + \sum_{j=1}^n \frac{\partial}{\partial x_j} g(x, t) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \right) \\ &= \left( \int_{t_1}^{t_2} \frac{\partial}{\partial t} g(x, t) dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_p} + 0 \\ &= (g(x, t_2) - g(x, t_1)) dx_{i_1} \wedge \cdots \wedge dx_{i_p} = i_2^* \omega - i_1^* \omega, \end{aligned}$$

so (2) holds.

$\omega_J$ . Let  $\omega = g(x, t) dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \wedge dt$ . Then

$$\begin{aligned} d(h(\omega)) &= (-1)^{p-1} d \left( \left( \int_{t_1}^{t_2} g(x, t) dt \right) dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \right) \\ &= (-1)^{p-1} \sum_{j=1}^n \frac{\partial}{\partial x_j} \left( \int_{t_1}^{t_2} g(x, t) dt \right) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \\ &= (-1)^{p-1} \sum_{j=1}^n \left( \int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x, t) dt \right) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}}, \end{aligned}$$

and

$$\begin{aligned} h(d\omega) &= h \left( \sum_{j=1}^n \frac{\partial}{\partial x_j} g(x, t) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \wedge dt + 0 \right) \\ &= (-1)^{p-1} \sum_{j=1}^n \left( \int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x, t) dt \right) dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} = -d(h(\omega)), \end{aligned}$$

and  $i_2^* \omega = i_1^* \omega = 0$ , so (2) holds. □

**Corollary 2.14.** *Assume  $M$  and  $N$  are homotopy equivalent. Then there exist isomorphisms*

$$H^p(N) \rightarrow H^p(M), \quad p \geq 0.$$

*Proof.* There exist  $f : M \rightarrow N$  and  $g : N \rightarrow M$  such that  $g \circ f \sim \text{id}_M$  and  $f \circ g \sim \text{id}_N$ . By Proposition 2.13  $(g \circ f)^* : H^p(M) \rightarrow H^p(M)$  coincides with  $\text{id}_M^* = \text{id}_{H^p(M)}$ . Then  $f^* \circ g^* = (g \circ f)^* = \text{id}_{H^p(M)}$ . Similarly  $g^* \circ f^* = \text{id}_{H^p(N)}$ , so  $g^*$  and  $f^*$  are isomorphisms. □

**Definition 2.15.** Let  $M$  be a manifold. Then  $M$  is **smoothly contractible** if  $M$  is homotopy equivalent to a point.

**Example.**  $\mathbb{R}^n$  is contractible, by Example 2.12. If  $M \subset \mathbb{R}^n$  is convex then  $M$  is contractible.

**Theorem 2.16** (Poincaré lemma). *If  $M$  is a contractible manifold then*

$$H^p(M) = 0, \quad p \geq 1.$$

*Proof.* By previous Corollary 2.14, there exists an isomorphism  $H^p(M) \rightarrow H^p(\{\text{point}\})$ . Then  $\{\text{point}\}$  is a zero-dimensional manifold, so by Proposition 2.5,  $H^p(\{\text{point}\}) = 0$  for all  $p > 0$ . □

Thus  $H^p(\mathbb{R}^n) = 0$  for all  $p > 0$ , so  $\mathbb{R}^n$  is not diffeomorphic to any compact orientable manifold.