M4P54 Differential Topology

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Syllabus

Contents

0 Introduction

Lecture 1 Thursday 09/01/20

Differential topology is the study of the topology of a manifold using analysis. The topics are

- a review of differential forms,
- de Rham cohomology,
- Morse theory, and
- singular homology.

The following are references.

- J M Lee, Introduction to smooth manifolds, 2000
- L W Tu, Introduction to smooth manifolds, 2008
- J Milnor, Morse theory, 1960
- A Banyaga and D Hurtubise, Lectures on Morse homology, 2004

1 Differential forms on manifolds

1.1 Alternating p-forms on a vector space

Let V be a vector space over $\mathbb{R}$, and let $p \geq 0$. Then $V^p = V \times \cdots \times V$.

Definition 1.1. A multilinear map $\omega : V^p \to \mathbb{R}$ is called an **alternating p-form** if we have

$$\omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(p)}\right) = \epsilon(\sigma)\, \omega(v_1, \ldots, v_p), \qquad v_1, \ldots, v_p \in V \qquad \sigma \in \mathcal{S}_p,$$

where $\mathcal{S}_p$ is the group of permutations of p elements and $\epsilon(\sigma)$ is the signature of σ.

Recall that if m is the number of transpositions in a decomposition of σ, then $\epsilon(\sigma) = (-1)^m$, where a **transposition** is $(a_i a_j)$ for $a_i \neq a_j$.

Notation 1.2. $\Lambda^p V^* = \{\text{alternating } p\text{-forms } \omega \text{ on } V\}$ is called the **p-th exterior power** of V.

Check that it is a vector space. [1]

Example 1.3.

- $\Lambda^0 V^* = \mathbb{R}$.
- $\Lambda^1 V^* = V^* = \mathrm{Hom}(V, \mathbb{R})$, the **dual** of V.

Definition 1.4. Let $\omega_1 \in \Lambda^p V^*$ and $\omega_2 \in \Lambda^q V^*$. We define $\omega_1 \wedge \omega_2 \in \Lambda^{p+q} V^*$ the **exterior product** of ω_1 and ω_2 by

$$\omega_1 \wedge \omega_2 (v_1, \ldots, v_{p+q}) = \sum_{\sigma \in \mathcal{S}_{p,q}} \epsilon(\sigma)\, \omega_1\left(v_{\sigma(1)}, \ldots, v_{\sigma(p)}\right) \omega_2\left(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}\right), \qquad v_1, \ldots, v_{p+q} \in V,$$

where

$$\mathcal{S}_{p,q} = \{\sigma \in \mathcal{S}_{p+q} \mid \sigma(1) < \cdots < \sigma(p),\ \sigma(p+1) < \cdots < \sigma(p+q)\}.$$

Example 1.5.

- Assume $\omega_1, \omega_2 \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \omega_2 (v_1, v_2) = \omega_1(v_1)\, \omega_2(v_2) - \omega_1(v_2)\, \omega_2(v_1), \qquad v_1, v_2 \in V.$$

- Assume $\omega_1, \ldots, \omega_p \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \cdots \wedge \omega_p (v_1, \ldots, v_p) = \det\left(\omega_i(v_j)\right)_{i,j=1,\ldots,p}, \qquad v_1, \ldots, v_p \in V.$$

Proposition 1.6. *Let $\omega_i \in \Lambda^{p_i} V^*$ for $i = 1, 2, 3$.*

- *Associativity* $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$.
- *Distributivity* $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$, *assuming* $p_2 = p_3$.
- *Supercommutativity* $\omega_1 \wedge \omega_2 = (-1)^{p_1 \cdot p_2} \omega_2 \wedge \omega_1$.

Definition 1.7. Let $\Phi : V \to W$ be a linear map between vector spaces over $\mathbb{R}$. Let $\omega \in \Lambda^p W^*$. Then the **pull-back** $\Phi^*(\omega) \in \Lambda^p V^*$ of ω is an alternating p-form on V defined by

$$\Phi^*(\omega)(v_1, \ldots, v_p) = \omega(\Phi(v_1), \ldots, \Phi(v_p)), \qquad v_1, \ldots, v_p \in V.$$

[1] Exercise

Proposition 1.8. *Given $\Phi : V \to W$ a linear map,*

- *the pull-back*
$$\begin{array}{rccc} \Phi^* & : & \Lambda^p W^* & \longrightarrow & \Lambda^p V^* \\ & & \omega & \longmapsto & \Phi^*(\omega) \end{array}$$
is a linear map that preserves exterior products, that is
$$\Phi^*(\omega_1 \wedge \omega_2) = \Phi^*(\omega_1) \wedge \Phi^*(\omega_2), \qquad \omega_1 \in \Lambda^p W^*, \qquad \omega_2 \in \Lambda^q W^*,$$

- *if $\Psi : W \to Z$ is linear then*
$$(\Psi \circ \Phi)^*(\omega) = \Phi^*(\Psi^*(\omega)), \qquad \omega \in \Lambda^p Z^*,$$

- *assuming $V = W$ and $p = \dim V$, then*
$$\Phi^*(\omega) = (\det \Phi)\, \omega, \qquad \omega \in \Lambda^p V^*.$$

1.2 Differential forms on manifolds

Let M be a smooth manifold of dimension n, and let $x \in M$. Then the tangent space $\mathrm{T}_x M$ of M at x is a vector space of dimension n.

Notation 1.9. Let
$$\Lambda^p \mathrm{T}_x^* M = \Lambda^p (\mathrm{T}_x M)^*.$$
Consider the set
$$\Lambda^p \mathrm{T}^* M = \bigsqcup_{x \in M} \Lambda^p \mathrm{T}_x^* M,$$
the **p-th exterior bundle** on M. There exists a morphism $\pi : \Lambda^p \mathrm{T}^* M \to M$ such that for all $x \in M$, $\pi^{-1}(x) = \Lambda^p \mathrm{T}_x^* M$, so $\Lambda^p \mathrm{T}^* M$ is a vector bundle and it is a smooth manifold, and π is a smooth morphism.

Example 1.10.

- $\Lambda^0 \mathrm{T}^* M = M \times \mathbb{R}$.
- $\Lambda^1 \mathrm{T}^* M$ is the **cotangent bundle**, the dual of the tangent bundle.

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Definition 1.11. A **differential p-form** ω on M is a smooth section of π. That is, it is a smooth morphism $\omega : M \to \Lambda^p \mathrm{T}^* M$ such that $\pi \circ \omega = \mathrm{id}_M$.

Thus, $\omega(x) \in \Lambda^p \mathrm{T}_x^* M$.

Notation 1.12.
$$\Omega^p(M) = \{\text{differential } p\text{-forms } \omega \text{ on } M\}, \qquad \Omega^\bullet(M) = \bigoplus_p \Omega^p(M).$$

Example 1.13. $\Omega^0(M) \cong \{f : M \to \mathbb{R}\ \mathrm{C}^\infty\text{-function}\}$.

Exercise. If $n = \dim M$, then $\Omega^{n+1}(M) = 0$.

The algebra is the same as last week.

Definition 1.14. Let $\omega_1 \in \Omega^p(M)$ and $\omega_2 \in \Omega^q(M)$. Then $\omega_1 \wedge \omega_2 \in \Omega^{p+q}(M)$ is defined by
$$\omega_1 \wedge \omega_2(x) = \omega_1(x) \wedge \omega_2(x) \in \Lambda^{p+q} \mathrm{T}_x^* M, \qquad x \in M.$$

By Proposition 1.6, associativity, distributivity, and supercommutativity hold for $\Omega^p(M)$. Let $F : M \to N$ be a smooth morphism between manifolds. Then for all $x \in M$, the differential of F at x is the linear map
$$\mathrm{D}F_x : \mathrm{T}_x M \to \mathrm{T}_{F(x)} N.$$

Thus, for all $p \geq 0$, we have a natural map, called the **pull-back**,

$$\begin{array}{cccc} F_x^* & : & \Lambda^p \mathrm{T}_{F(x)}^* N & \longrightarrow & \Lambda^p \mathrm{T}_x^* M \\ & & \omega\left(v_1, \dots, v_p\right) & \longmapsto & \omega\left(\mathrm{D}F_x\left(v_1\right), \dots, \mathrm{D}F_x\left(v_p\right)\right) \end{array}, \qquad \omega \in \Lambda^p \mathrm{T}_{F(x)}^* N, \qquad v_1, \dots, v_p \in \mathrm{T}_x^* M.$$

Thus, we can define

$$\begin{array}{cccc} F^* & : & \Omega^p\left(N\right) & \longrightarrow & \Omega^p\left(M\right) \\ & & \omega\left(x\right) & \longmapsto & F^*\left(\omega\left(F\left(x\right)\right)\right) \end{array}, \qquad \omega \in \Omega^p\left(N\right).$$

By Proposition 1.8, the pull-back preserves the exterior product, so

$$F^*\left(\omega_1 \wedge \omega_2\right) = F^*\left(\omega_1\right) \wedge F^*\left(\omega_2\right).$$

If $G : N \to P$,

$$\left(G \circ F\right)^*\left(\omega\right) = F^*\left(G^*\left(\omega\right)\right).$$

1.3 Local description of p-forms

Let M be a manifold of dimension n, let $x_0 \in M$, let (U, ϕ) be a local chart around x_0, and let $(x_1, \dots, x_n)$ be local coordinates around x_0. A basis of $\mathrm{T}_{x_0} M$ is given by

$$\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}.$$

A basis of $\mathrm{T}_{x_0}^* M$ is given by

$$\left\{\mathrm{d}x_1, \dots, \mathrm{d}x_n\right\}, \qquad \mathrm{d}x_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}.$$

A basis of $\Lambda^p \mathrm{T}_{x_0}^* M$ is

$$\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad i_1 < \dots < i_p.$$

Thus, $\omega \in \Omega^p\left(M\right)$ is locally given by

$$\omega\left(x\right) = \sum_{|I|=p} f_I\left(x\right) \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad I = \left(i_1, \dots, i_p\right), \qquad i_1 < \dots < i_p,$$

where f_I is a C^∞-function on U for all I.

Example 1.15. Let $F : M \to N$ be a smooth morphism between manifolds of dimension n, and let $\omega \in \Omega^n\left(N\right)$. Locally,

$$\omega\left(y\right) = f\left(y\right) \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n, \qquad y \in N$$

for some $f \in \mathrm{C}^\infty$. Proposition 1.8 implies that

$$F^*\left(\omega\right)\left(x\right) = \left(f \circ F\right)\left(x\right) \det \mathrm{D}F_x \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n, \qquad x \in M,$$

where $y_i = p_i \circ F$ and $p_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th projection.

Let $f : M \to \mathbb{R}$ be a smooth function, so $f \in \Omega^0\left(M\right)$. Locally, the **differential** is

$$\begin{array}{cccc} \mathrm{d} & : & \Omega^0\left(M\right) & \longrightarrow & \Omega^1\left(M\right) \\ & & f & \longmapsto & \sum_{i=1}^n \frac{\partial f}{\partial x_i} \mathrm{d}x_i \end{array}.$$

Check that $\mathrm{d}f \in \Omega^1\left(M\right)$, so $\mathrm{d}f$ is a 1-form on M. Alternatively, $\mathrm{d}f = f^*\left(\mathrm{d}x\right)$ for $\mathrm{d}x$ a 1-form on $\mathbb{R}$, or $\mathrm{d}f\left(X\right) = X\left(f\right)$ for any vector field X on M. More in general, let $\omega \in \Omega^p\left(M\right)$. Locally,

$$\omega = \sum_{|I|=p} f_I \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad f_I \in \mathrm{C}^\infty,$$

so $\mathrm{d}\omega \in \Omega^{p+1}\left(M\right)$. Then the **de Rham differential** is

$$\begin{array}{cccc} \mathrm{d} & : & \Omega^p\left(M\right) & \longrightarrow & \Omega^{p+1}\left(M\right) \\ & & \omega & \longmapsto & \sum_{|I|=p} \mathrm{d}f_I \wedge \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p} \end{array}.$$

Proposition 1.16.

- *The Leibniz rule*

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \mathrm{d}\omega_2, \qquad w_1 \in \Omega^p(M), \qquad \omega_2 \in \Omega^q(M).$$

- $\mathrm{d}^2 = 0$, *that is*

$$\mathrm{d}(\mathrm{d}\omega) = 0, \qquad \omega \in \Omega^p(M).$$

- *Let $F : M \to N$ be a smooth morphism between manifolds. Then*

$$F^*(\mathrm{d}\omega) = \mathrm{d}(F^*(\omega)), \qquad \omega \in \Omega^p(M)$$

so

$$\begin{array}{ccc} \Omega^p(M) & \xrightarrow{\mathrm{d}} & \Omega^{p+1}(M) \\ {\scriptstyle F^*}\uparrow & & \uparrow{\scriptstyle F^*} \\ \Omega^p(N) & \xrightarrow[\mathrm{d}]{} & \Omega^{p+1}(N) \end{array}.$$

Lecture 3 Tuesday 14/01/20

Definition 1.17.

- $\omega \in \Omega^p(M)$ is **closed** if $\mathrm{d}\omega = 0$.
- $\omega \in \Omega^p(M)$ is **exact** if there exists $\omega' \in \Omega^{p-1}(M)$ such that $\mathrm{d}\omega' = \omega$.

ω is exact implies that ω is closed, since if $\omega = \mathrm{d}\omega'$ then $\mathrm{d}\omega = \mathrm{d}^2\omega' = 0$.

1.4 Integration on manifolds

Let M be a manifold of dimension n, let $F : M \to M$ be a smooth morphism, and let $\omega \in \Omega^n(M)$. Then

$$F^*(\omega)(x) = \det \mathrm{D}F_x \omega(F(x)).$$

Locally, assume $\omega = f \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$ for some coordinates $y_1, \ldots, y_n$ and $f \in \mathrm{C}^\infty$. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of M, and let $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$. Then

$$\begin{array}{ccccc} h_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} & : & \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n & \longrightarrow & \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \\ & & \omega(x) & \longmapsto & (f \circ h_{\alpha\beta})(x) \det(\mathrm{D}h_{\alpha\beta})_x \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \end{array}.$$

Let $D \subset \mathbb{R}^n$ be compact such that ∂D has zero measure, so D is a domain of integration, let $f : U \to \mathbb{R}$ be a C^∞-function where $U \subset \mathbb{R}^n$ is open such that $D \subset U$, and let $h : U \to h(U) \subset \mathbb{R}^n$ be a diffeomorphism. Then

$$\int_{h^{-1}(D)} f(y)\, \mathrm{d}y_1 \ldots \mathrm{d}y_n = \int_{h^{-1}(D)} f(y)\, \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n = \int_D (f \circ h)(x) |\det \mathrm{D}h_x|\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

Definition 1.18. Let us assume that $\omega = f(y) \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$ on U. We define

$$\int_D \omega = \int_D f(y)\, \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n, \qquad D \subset U.$$

Definition 1.19. Let $U \subset \mathbb{R}^n$ be an open set. We define the **support** of ω as

$$\operatorname{supp} \omega = \overline{\{x \in U \mid \omega(x) \neq 0\}}, \qquad \omega(x) \in \Lambda^p \mathrm{T}_x^* U.$$

Then ω has **compact support**, if $\operatorname{supp}\omega$ is compact.

Under this assumption, we can define

$$\int_U \omega = \int_D \omega \in \mathbb{R},$$

which is well-defined.

Fact. Under the same assumption, if $\phi : V \to U$ is a diffeomorphism, provided that $\det \mathrm{D}\phi_x > 0$, since $\det \mathrm{D}\phi_x \neq 0$ for all x, then

$$\int_U \omega = \int_V \phi^*(\omega).$$

The goal is to define $\int_M \omega$. Let V be a vector space over $\mathbb{R}$ of dimension n, and let $B = (b_1, \dots, b_n) \subset V$ and $B' = (b_1', \dots, b_n') \subset V$ be ordered bases of V. Then B and B' have the **same orientation** if $\det T > 0$ where

$$\begin{array}{cccc} T & : & V & \longrightarrow & V \\ & & b_i & \longmapsto & b_i' \end{array}$$

is a linear map. Let $\omega \in \Lambda^n V^*$ for $\omega \neq 0$. Then B and B' have the same orientation if and only if $\omega(b_1, \dots, b_n)$ has the same sign as $\omega(b_1', \dots, b_n')$, by Proposition 1.8. An **orientation** Λ of V is a set of all the ordered basis of V with the same orientation. Let $\phi : V \to W$ be an isomorphism of vector spaces with fixed orientations Λ_v and Λ_w respectively. We say that ϕ is **orientation preserving** if an ordered basis of V induces an ordered basis of W, so Λ_v induces Λ_w.

Example 1.20. Let $V = \mathbb{R}^n$, and let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. Then $e_1, \dots, e_n$ defines an orientation of V called **positive**.

Let M be a manifold. The idea is to find an orientation Λ_x of $\mathrm{T}_x M$ for all $x \in M$.

Special case. Let $M = U \subset \mathbb{R}^n$ be open. There exists a natural isomorphism $\phi_x : \mathrm{T}_x U \to \mathbb{R}^n$. Let Λ_x^+ be an orientation on $\mathrm{T}_x U$ such that ϕ_x is orientation preserving with respect to the positive orientation on $\mathbb{R}^n$. Let $\Lambda^+ = \{\Lambda_x^+\}$.

General case. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M. On U_α, we define the orientation so that $(\mathrm{D}\phi_\alpha)_x : \mathrm{T}_x U_\alpha \to \mathrm{T}_{\phi_\alpha(x)} \phi_\alpha(U) \subset \mathbb{R}^n$ is orientation preserving. This is called the positive orientation on the chart (U_α, ϕ_α). We define Λ on M, which is a collection of Λ^+ on $\mathrm{T}_x M$ for all $x \in M$. Then M is **orientable** if there exists an atlas with positive orientation charts. This coincides in assuming that $\det \mathrm{D}\left(\phi_\beta^{-1} \circ \phi_\alpha\right) > 0$ for all α and β.

For all $p \geq 0$, Lecture 4 Thursday 16/01/20

$$\Omega_{\mathrm{c}}^p(M) = \{\omega \in \Omega^p(M) \mid \operatorname{supp} M \text{ is compact}\}.$$

If M is compact $\Omega_{\mathrm{c}}^p(M) = \Omega^p(M)$. Let $\omega \in \Omega_{\mathrm{c}}^r(M)$. Assume $\operatorname{supp} \omega \subset U$ where (U, ϕ) is a chart of M, and $\phi : U \to \phi(U) \subset \mathbb{R}^n$. Assume also that (U, ϕ) is positively oriented. Let $\phi^{-1} : \phi(U) \to U$ such that $\left(\phi^{-1}\right)^*(\omega) \in \Omega_{\mathrm{c}}^n(\phi(U))$, that is $\operatorname{supp}\left(\phi^{-1}\right)^*(\omega) \subset \phi(U)$. We define

$$\int_M \omega = \int_{\phi(U)} \left(\phi^{-1}\right)^*(\omega).$$

We need to show that, under the assumptions above, $\int_M \omega$ does not depend on (U, ϕ). Let $\left(\overline{U}, \overline{\phi}\right)$ be also a positively oriented chart such that $\operatorname{supp} \omega \subset \overline{U}$. We want to show that

$$\int_{\phi(U)} \left(\phi^{-1}\right)^*(\omega) = \int_{\overline{\phi}(\overline{U})} \left(\overline{\phi}^{-1}\right)^*(\omega).$$

Let

$$\overline{\phi} \circ \phi^{-1} : \phi\left(U \cap \overline{U}\right) \to \overline{\phi}\left(U \cap \overline{U}\right),$$

so

$$\begin{tikzcd} & U \cap \overline{U} \arrow[dl, "\phi"'] \arrow[dr, "\overline{\phi}"] & \\ \mathbb{R}^n \supset \phi\left(U \cap \overline{U}\right) \arrow[rr, "\overline{\phi} \circ \phi^{-1}"'] & & \overline{\phi}\left(U \cap \overline{U}\right) \subset \mathbb{R}^n \end{tikzcd}.$$

Since both charts are positively oriented the determinant of the differential $\mathrm{D}\left(\overline{\phi} \circ \phi^{-1}\right)$ is positive. Then

$$
\begin{aligned}
\int_{\overline{\phi}(U)} \left(\overline{\phi}^{-1}\right)^* (\omega) &= \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\overline{\phi}^{-1}\right)^* (\omega) && \left(\overline{\phi}^{-1}\right)^* (\omega) = 0 \text{ outside } \overline{\phi}\left(U \cap \overline{U}\right) \\
&= \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\overline{\phi} \circ \phi^{-1}\right)^* \left(\overline{\phi}^{-1}\right)^* (\omega) && \det \mathrm{D}\left(\overline{\phi} \circ \phi^{-1}\right) > 0 \\
&= \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\phi^{-1}\right)^* \circ \overline{\phi}^* \circ \left(\overline{\phi}^{-1}\right)^* (\omega) && \text{pull-back} \\
&= \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\phi^{-1}\right)^* \circ \left(\overline{\phi}^{-1} \circ \overline{\phi}\right)^* (\omega) && \text{pull-back} \\
&= \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\phi^{-1}\right)^* (\omega) && \det \mathrm{D}\left(\overline{\phi} \circ \phi^{-1}\right) > 0 \\
&= \int_{\overline{\phi}(U)} \left(\phi^{-1}\right)^* (\omega) && \left(\phi^{-1}\right)^* (\omega) = 0 \text{ outside } \phi\left(U \cap \overline{U}\right).
\end{aligned}
$$

Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering. A **partition of unity** with respect to U is a collection of smooth functions $f_\alpha : M \to [0,1]$ such that

1. $\operatorname{supp} f_\alpha = \overline{\{x \in M \mid f_\alpha(x) = 0\}} \subset U_\alpha$ for all α,
2. $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$, and
3. for all $x \in M$, there exists $U \ni x$ open such that $\operatorname{supp} f_\alpha \cap U \neq \emptyset$ for only finitely many α.

Remark. 3 implies that 2 is a finite sum.

Example 1.21. Let

$$M = \mathrm{S}^1 = \left\{x \in \mathbb{R}^2 \;\middle|\; |x| = 1\right\}, \qquad U_1 = \mathrm{S}^1 \setminus \{(1,0)\}, \qquad U_2 = \mathrm{S}^1 \setminus \{(-1,0)\},$$

so $\{U_i\}$ is a cover. Let

$$f_1(\cos\theta, \sin\theta) = \frac{1}{2} - \frac{1}{2}\cos\theta, \qquad f_2(\cos\theta, \sin\theta) = \frac{1}{2} + \frac{1}{2}\cos\theta.$$

Then f_i is a partition of unity.

Theorem 1.22. *Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering of M. Then there exists a partition of unity f_α with respect to U.*

Proof. We omit the proof. □

Theorem 1.23. *Let M be a manifold, and let $n = \dim M$. Then M is orientable if and only if there exists $\omega \in \Omega^n(M)$ which is never vanishing on M.*

ω is called a **volume form** on M.

Proof.

$\Longleftarrow$ Assume $\omega \in \Omega^n(M)$ is a volume form. We want to define an orientation Λ_x of $\mathrm{T}_x M$ for all $x \in M$. Given an oriented basis $v_1, \dots, v_n$ of $\mathrm{T}_x M$ we say that it is positively oriented if $\omega(x)(v_1, \dots, v_n) > 0$. This defines Λ_x on $\mathrm{T}_x M$ which is compatible with the choice of an atlas on M. Indeed, the pull-back between two different charts is defined by the determinant, so it is orientation preserving, so M is orientable.

□