

M4P58 Modular Forms

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Autumn 2019

Syllabus

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0 Introduction

Lecture 1
Friday
04/10/19

The following are textbooks.

- Serre, A course in arithmetic, 1973
- J Shurman and F Diamond, A first course in modular forms, 2005

Let

$$f = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 = \sum_{n=1} b_n q^n = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \dots,$$

and let a_n be the number of solutions modulo n to the elliptic curve

$$E = \{(x, y) \in \mathbb{Z} \mid y^2 + y = x^3 - x^2 - 10x - 20\}.$$

- Modulo 2, there are $a_2 = 4$ solutions $(0, 0), (0, 1), (1, 0), (1, 1)$.
- Modulo 3, there are $a_3 = 4$ solutions $(1, 0), (1, -1), (-1, 0), (-1, -1)$.
- Modulo 5, there are $a_5 = 4$ solutions $(0, 0), (0, -1), (1, 0), (-1, -1)$.
- Modulo 7, there are $a_7 = 9$ solutions $(1, 3), (2, 2), (2, -3), (-1, 1), (-1, -2), (-2, 1), (-2, -2), (-3, 1), (-3, -2)$.

If $p \neq 11$, then

$$a_p - p = -b_p.$$

The following are some questions.

- What is the relationship between E and f ?
- Can we find similar relationships for other E ?
- How does one prove something like this?

Let

$$\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\} \subseteq \mathbb{C}.$$

Then \mathbb{H} has an action of

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

Modular forms are complex functions on \mathbb{H} with a high degree of symmetry. These functions are symmetric under the action of large discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$, in particular

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} \subseteq \mathrm{SL}_2(\mathbb{R}).$$

Why are these interesting to number theorists? Power series expansions often involve expressions of interest to number theorists. For example,

- Bernoulli numbers,
- divisor functions $\sigma_k(n) = \sum_{d|n} d^k$,
- number of points on elliptic curves, and
- traces of Galois representations.

1 Modular forms of level one

1.1 Modular functions and forms

1.1.1 Modular actions

Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

Then $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{C} \cup \{\infty\}$ by

$$\gamma \cdot z = \begin{cases} \frac{az+b}{cz+d} & z \neq -\frac{d}{c} \\ \infty & z = -\frac{d}{c} \end{cases} \quad \gamma \cdot \infty = \frac{a}{c}.$$

One checks that this gives a bijection from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$, where inverse is given by the inverse matrix

$$\gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and $\gamma \cdot (\gamma' \cdot z) = \gamma\gamma' \cdot z$. One obtains a left action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{C} \cup \{\infty\}$. An observation is

$$\mathrm{Im} \gamma z = \mathrm{Im} \frac{az+b}{cz+d} = \mathrm{Im} \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{\mathrm{Im}(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{(ad-bc)\mathrm{Im} z}{|cz+d|^2}.$$

In particular, if $\gamma \in \mathrm{SL}_2(\mathbb{R})$, then

$$\mathrm{Im} \gamma z = \frac{\mathrm{Im} z}{|cz+d|^2}.$$

So $\mathrm{SL}_2(\mathbb{R})$ preserves $\mathbb{H} \cup \{\infty\}$. More generally, if $\gamma \in \mathrm{GL}_2(\mathbb{R})$, then

$$\mathrm{Im} \gamma z = \frac{\det \gamma \mathrm{Im} z}{|cz+d|^2}.$$

So

$$\mathrm{GL}_2(\mathbb{R})_+ = \{\gamma \in \mathrm{GL}_2(\mathbb{R}) \mid \det \gamma > 0\}$$

preserves $\mathbb{H} \cup \{\infty\}$. Define

$$f|_{k,\gamma} : \mathbb{H} \longrightarrow \mathbb{C} \\ z \longmapsto \det \gamma^{k-1} f(\gamma z) (cz+d)^{-k}, \quad f : \mathbb{H} \rightarrow \mathbb{C}, \quad \gamma \in \mathrm{GL}_2(\mathbb{R})_+, \quad k \in \mathbb{Z},$$

where $\det \gamma^{k-1}$ is the fudge factor, which is one for $\gamma \in \mathrm{SL}_2(\mathbb{R})$, and $(cz+d)^{-k}$ is the twisted action on functions. Check that

$$f|_{k,\mathrm{id}} = f, \quad \left(f|_{k,\gamma}\right)|_{k,\gamma'} = f|_{k,\gamma'\gamma}.$$

This gives, for each k , a left action of $\mathrm{GL}_2(\mathbb{R})_+$ on functions $\mathbb{H} \rightarrow \mathbb{C}$, a **modular action of weight k** . A modular form of weight k will be a sufficiently nice function $f : \mathbb{H} \rightarrow \mathbb{C}$ such that $f|_{k,\gamma} = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. That is, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and all $z \in \mathbb{H}$,

$$f(\gamma z) (cz+d)^{-k} = f(z), \quad \implies \quad f(\gamma z) = f(z) (cz+d)^k,$$

the **modular transformation law of weight k** . The following are some observations.

- Let $k = 0$. Then constant functions satisfy $f(\gamma z) = f(z)$. It will turn out that all functions of weight zero are constant.
- Let k be odd, and $\gamma = -\mathrm{id}$. Then $\gamma z = z$ for all z and $cz+d = -1$, so $f(\gamma z) = f(z) (cz+d)^k$ gives $f(z) = f(z) (-1)^k$, so $f(z) = -f(z)$, so $f(z) = 0$ for all z . So no functions $f : \mathbb{H} \rightarrow \mathbb{C}$ satisfy the modular transformation law of weight k , for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, when k is odd.

Lecture 2
Friday
04/10/19

1.1.2 Review of complex analysis

Let $f : U \rightarrow \mathbb{C}$, for $U \subseteq \mathbb{C}$ open, and let $p \in U$.

Definition 1.1.1. f is **holomorphic** at p if

$$f'(p) = \lim_{\epsilon \rightarrow 0, \epsilon \in \mathbb{C}} \frac{f(p' + \epsilon) - f(p')}{\epsilon}$$

exists for all p' in a neighbourhood of p .

Proposition 1.1.2. f is holomorphic at p implies that f is continuous.

Proposition 1.1.3. f is holomorphic at p implies that f is infinitely differentiable at p , that is $f^{(n)}(p)$ exists for all $n \geq 0$. Moreover, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (z-p)^n = f(p) + f'(p)(z-p) + \frac{f''(p)}{2} (z-p)^2 + \dots,$$

for all z in a neighbourhood of p .

Corollary 1.1.4. If f is holomorphic and not identically zero on an open set U , then the zeroes of f are isolated on U .

More generally is the following.

Definition 1.1.5. f is **meromorphic** at p if there exists a neighbourhood U of p and $g, h : U \rightarrow \mathbb{C}$ holomorphic on U such that $f = g/h$ on $U \setminus \{p\}$. Such an f has a **Laurent series expansion** at p ,

$$f(z) = \sum_{i=-N}^{\infty} c_i (z-p)^i.$$

The smallest i such that $c_i \neq 0$ is denoted by $\text{ord}_p f$, the **order of vanishing** of f at p .

- If $\text{ord}_p f = -n$ for $n > 0$, we say f has a **pole of order n** .
- If $\text{ord}_p f = n$ for $n > 0$, we say f has a **zero of order n** .

Proposition 1.1.6.

- $\text{ord}_p fg = \text{ord}_p f + \text{ord}_p g$.
- $\text{ord}_p (f + g) \geq \min \{\text{ord}_p f, \text{ord}_p g\}$, with equality if $\text{ord}_p f \neq \text{ord}_p g$.

If f is holomorphic on $U \setminus \{p\}$ for U a neighbourhood of p , then f may or may not be meromorphic at p .

Example. $f(z) = e^{-1/z^2}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, but not meromorphic at zero.

Theorem 1.1.7. Let f be holomorphic on $U \setminus \{p\}$, and there exists $n > 0$ such that

$$\lim_{x \rightarrow p} (x-p)^n f(x)$$

exists. Then f is meromorphic on U , and $\text{ord}_p f \geq -n$.

1.1.3 Modular functions

Definition 1.1.8. $f : \mathbb{H} \rightarrow \mathbb{C}$ is a **weakly modular function of weight k** if

- f is meromorphic on \mathbb{H} , and
- f satisfies the modular transformation law of weight k .

Consider

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $\gamma z = z + 1$ and $cz + d = 1$. The modular transformation law gives $f(z + 1) = f(z)$. Let

$$D = \{q \mid |q| < 1\}.$$

Can define a function

$$\begin{aligned} g : D \setminus \{0\} &\longrightarrow \mathbb{C} \\ q &\longmapsto f\left(\frac{\log q}{2\pi i}\right), \end{aligned}$$

that is $f(z) = g(e^{2\pi iz})$ for $z \in \mathbb{H}$, where g is holomorphic or meromorphic on $\{z \mid 0 < |z| < 1\}$ if and only if f is holomorphic or meromorphic on \mathbb{H} .

Definition 1.1.9. $f : \mathbb{H} \rightarrow \mathbb{C}$ is a **modular form of weight k** if

1. f satisfies the modular transformation law of weight k ,
2. f is holomorphic on \mathbb{H} , and
3. f is holomorphic at ∞ , so the function $g : D \setminus \{0\} \rightarrow \mathbb{C}$, which is holomorphic on $D \setminus \{0\}$ by 2, extends to a holomorphic function on D .

Then $q \rightarrow 0$ in D if and only if $\text{Im } z \rightarrow +\infty$. Then 3 means $g(q)$ is bounded as $q \rightarrow 0$ so $f(z)$ is bounded as $\text{Im } z \rightarrow +\infty$. For f satisfying 3, $g : D \setminus \{0\} \rightarrow \mathbb{C}$ has a series expansion

$$g(q) = \sum_n a_n q^n = a_0 + a_1 q + \dots$$

in $q = e^{2\pi iz}$. We call this the **q -expansion** for f .

Definition 1.1.10. $f : \mathbb{H} \rightarrow \mathbb{C}$ is a **meromorphic modular form of weight k** if the same conditions 1 to 3 hold, but with holomorphic weakened to meromorphic.

Note. If f is only meromorphic at ∞ then a finite number of negative powers of q can appear.

Example.

- The **modular discriminant**

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

is a modular form of weight 12.

- The **j -invariant**

$$j(z) = \frac{1}{q} + 744 + 196844q + 21493760q^2 + \dots$$

is a meromorphic modular form of weight 0.

Lecture 3
Monday
07/10/19

1.1.4 Lattice functions

How can we construct modular forms?

Definition 1.1.11. A **lattice** in \mathbb{C} is an abelian subgroup of \mathbb{C} of the form $\mathbb{Z}w_1 + \mathbb{Z}w_2$, where $w_1, w_2 \in \mathbb{C}$ are \mathbb{R} -linearly independent. More generally if V is an \mathbb{R} -vector space, a **lattice** L in V is a discrete abelian subgroup of V that spans V over \mathbb{R} . For $L \subseteq \mathbb{C}$ a lattice and $\lambda \in \mathbb{C}^\times$, let

$$\lambda L = \{\lambda x \mid x \in L\} \subseteq \mathbb{C}.$$

We say that L and λL are **homothetic**. For $z \in \mathbb{H}$, let

$$L_{z,1} = \mathbb{Z} + \mathbb{Z}z = \{az + b \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}.$$

A question is when is $L_{z,1}$ homothetic to $L_{z',1}$, and what is a homothety factor?

- Suppose $L_{z,1} = \lambda L_{z',1}$. Then there exist a, b, c, d such that $\lambda z' = az + b$ and $\lambda = cz + d$, so

$$\begin{pmatrix} \lambda z' \\ \lambda \end{pmatrix} = \gamma \begin{pmatrix} z \\ 1 \end{pmatrix}. \quad (1)$$

On the other hand there exist a', b', c', d' such that $z = a'\lambda z' + b'\lambda$ and $1 = c'\lambda z' + d'\lambda$, so

$$\gamma' \begin{pmatrix} \lambda z' \\ \lambda \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix}. \quad (2)$$

(1) and (2) imply that

$$\gamma' \gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix},$$

so $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Moreover (1) implies that $z' = (az + b) / (cz + d)$.

- Conversely, if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $\gamma z = (az + b) / (cz + d)$, so

$$L_{\gamma z,1} = (cz + d)^{-1} L_{az+b, cz+d}.$$

But certainly $L_{az+b, cz+d} \subseteq L_{z,1}$. On the other hand if γ' is inverse to γ ,

$$\begin{pmatrix} z \\ 1 \end{pmatrix} = \gamma' \gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \gamma \begin{pmatrix} az + b \\ cz + d \end{pmatrix} = \begin{pmatrix} a'(az + b) + b'(cz + d) \\ c'(az + b) + d'(cz + d) \end{pmatrix},$$

so $z \in L_{az+b, cz+d}$ and $1 \in L_{az+b, cz+d}$. So $L_{az+b, cz+d} = L_{z,1}$, so $L_{\gamma z,1} = (cz + d)^{-1} L_{z,1}$.

Definition 1.1.12. A **lattice function of weight k** is a function $F : \{\text{lattices in } \mathbb{C}\} \rightarrow \mathbb{C}$ such that

$$F(\lambda L) = \lambda^{-k} F(L),$$

for all lattices L . Given such an F , can define

$$\begin{aligned} f &: \mathbb{H} \longrightarrow \mathbb{C} \\ z &\longmapsto F(L_{z,1}). \end{aligned}$$

If F has weight k , then

$$f(\gamma z) = F(L_{\gamma z,1}) = F((cz + d)^{-1} L_{z,1}) = (cz + d)^k F(L_{z,1}) = (cz + d)^k f(z).$$

1.2 Eisenstein series

Lecture 4
Friday
11/10/19

Definition 1.2.1. For $L \in \mathbb{C}$, define the **Eisenstein series**

$$G_k(L) = \sum_{w \in L, w \neq 0} \frac{1}{w^k}, \quad g_k(z) = G_k(L_{z,1}) = \sum_{\substack{m=-\infty \\ (m,n) \neq (0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k}.$$

Then

$$G_k(\lambda L) = \sum_{w' \in \lambda L, w' \neq 0} \frac{1}{w'^k} = \sum_{w \in L, w \neq 0} \frac{1}{(\lambda w)^k} = \lambda^{-k} G_k(L).$$

Corollary 1.2.2. g_k satisfies the modular transformation law of weight k .

The following are some questions.

- Does G_k , or g_k , converge?
- Is g_k holomorphic or meromorphic on \mathbb{H} ?
- Is g_k holomorphic at ∞ ?
- What is the q -expansion of g_k ?

1.2.1 Convergence and holomorphy on \mathbb{H}

Definition 1.2.3. Let $U \subseteq \mathbb{C}$ be open. A sequence of functions $f_n : U \rightarrow \mathbb{C}$ **converges uniformly on compact sets** to f if for all $C \subseteq U$ compact and all $\epsilon > 0$, there exists $N \in \mathbb{Z}$ such that for all $n > N$,

$$|f(z) - f_n(z)| < \epsilon, \quad z \in C.$$

Theorem 1.2.4. A uniform limit of holomorphic functions is holomorphic. If f_n converges to f uniformly on compact sets and f_n is holomorphic on U , then f is holomorphic on U .

Theorem 1.2.5. Let $k \geq 4$. The series $g_k(z)$ converges absolutely and uniformly on compact subsets of \mathbb{H} .

Proof. Let

$$P_{z,r} = \{az + b \mid a, b \in \mathbb{R}, \max(|a|, |b|) = r\} \subseteq \mathbb{C},$$

so $P_{z,r} = rP_{z,1}$, and there are $8r$ points on $P_{z,r} \cap L_{z,1}$. Then

$$g_k(z) = \sum_{r=1}^{\infty} \sum_{w \in L_{z,1} \cap P_{z,r}} \frac{1}{w^k}.$$

The function $z \mapsto |z|$ attains a non-zero minimum $\delta(z)$ on $P_{z,1}$, so on $P_{z,1}$, have $|z| > \delta(z)$, so $1/|z|^k < 1/\delta(z)^k$. On $P_{z,r}$, have $|z| > r\delta(z)$, so $1/|z|^k < 1/r^k \delta(z)^k$. Let $C \subseteq \mathbb{H}$ be compact. Then $z \mapsto \delta(z)$ is a continuous function on C and attains a minimum δ_C . For all $z \in C$ and all $w \in P_{z,r}$, get $|w| > r\delta_C$, so

$$\frac{1}{|w|^k} < \frac{1}{r^k \delta_C^k}.$$

Thus for $z \in C$, $g_k(z)$ is dominated by

$$\sum_{r=1}^{\infty} \frac{8r}{r^k \delta_C^k} = \frac{8}{\delta_C^k} \sum_{r=1}^{\infty} \frac{1}{r^{k-1}},$$

which converges absolutely for $k \geq 4$. □

Corollary 1.2.6. $g_k(z)$ is holomorphic on \mathbb{H} .

1.2.2 q -expansion and holomorphy at ∞

The idea is to understand series of the form

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k}.$$

Theorem 1.2.7. *A bounded holomorphic function on all of \mathbb{C} is constant.*

Lemma 1.2.8.

1.

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

2.

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Proof.

1. The right hand side converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$, so the right hand side is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$. Locally around $z = n$, the series looks like

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \cdots + \frac{1}{(z-n+1)^2} + \frac{1}{(z-n)^2} + \frac{1}{(z-n-1)^2} + \cdots = \frac{1}{(z-n)^2} + h_1(z),$$

where $h_1(z)$ is holomorphic in a neighbourhood of $z = n$. Similarly, the left hand side is meromorphic on \mathbb{C} , and the Laurent series near $z = n$ is

$$\frac{\pi^2}{\sin^2 \pi z} = \pi \left(\frac{1}{\pi^2 (z-n)^2} + \frac{1}{3} + \frac{1}{15} \pi^2 (z-n)^2 + \cdots \right) = \frac{1}{(z-n)^2} + h_2(z),$$

where $h_2(z)$ is a holomorphic function. So the difference

$$g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} - \frac{\pi^2}{\sin^2 \pi z}$$

is meromorphic on \mathbb{C} and holomorphic on $\mathbb{C} \setminus \mathbb{Z}$, and the Laurent expression around $z = n$ is

$$g(z) = \frac{1}{(z-n)^2} + h_1(z) - \left(\frac{1}{(z-n)^2} + h_2(z) \right) = h_1(z) - h_2(z),$$

so $g(z)$ is holomorphic at $z = n$ for all n . Consider $t \rightarrow \pm\infty$ for $z = a + it$. The right hand side is

$$R = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=a-N}^{a+N} \frac{1}{(z-n)^2} + \sum_{n=-\infty}^{a-N-1} \frac{1}{(z-n)^2} + \sum_{n=a+N+1}^{\infty} \frac{1}{(z-n)^2} = R_0 + R_- + R_+,$$

where R_0 has finitely many terms that converge to less than $\epsilon/2$ as $t \rightarrow \pm\infty$ and $R_- + R_+ < \epsilon/2$ for $N \gg 0$ independent of t , so $R < \epsilon$ converges to zero. Similarly, the left hand side is

$$\left| \frac{\pi^2}{\sin^2 \pi z} \right| = \left| \frac{2\pi^2}{e^{\pi i z} - e^{-\pi i z}} \right| \rightarrow 0,$$

so $\lim_{t \rightarrow \infty} g(a + it) = 0$. Moreover, $g(z+1) = g(z)$ for all z . Then

$$S = \{z \in \mathbb{C} \mid n-1 \leq \operatorname{Re} z \leq n, -N \leq \operatorname{Im} z \leq N\}, \quad n \in \mathbb{Z}$$

is compact, so $|g(z)|$ attains a maximum in S , so $g(z)$ is bounded in S . Since $g(z)$ is also bounded in $R_- + R_+$, $g(z)$ is bounded in \mathbb{C} , so g is constant. Since $\lim_{t \rightarrow \infty} g(a + it) = 0$, $g = 0$.

2. Check that the right hand side converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$, so the right hand side is meromorphic on $\mathbb{C} \setminus \mathbb{Z}$. Similarly, the left hand side is also meromorphic on $\mathbb{C} \setminus \mathbb{Z}$. Comparing derivatives,

$$-\frac{\pi^2}{\sin^2 \pi z} = -\frac{1}{z^2} - \sum_{n=1}^{\infty} \left(\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} \right),$$

so the difference is constant. Let $z = \frac{1}{2}$. The left hand side is $\pi \cot \pi/2 = 0$ and the right hand side is

$$\frac{2}{1} + \left(-\frac{2}{1} + \frac{2}{3} \right) + \left(-\frac{2}{3} + \frac{2}{5} \right) + \cdots \rightarrow 0, \quad n \rightarrow \infty,$$

so the difference is zero. □

Thus

$$\frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \pi \cot \pi z = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = \pi i \frac{q+1}{q-1} = \pi i - \frac{2\pi i}{1-q} = \pi i - 2\pi i \sum_{n=0}^{\infty} q^n.$$

Take $\frac{d^{k-1}}{dz^{k-1}}$. For $k \geq 2$ even, get

$$-(k-1)! \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = -(2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} q^n,$$

so

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n.$$

Collecting powers of q ,

$$\begin{aligned} g_k(z) &= \sum_{\substack{m=-\infty \\ (m,n) \neq (0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{n^k} + 2 \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{k-1} q^{nm} \\ &= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \end{aligned} \quad \begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} n^{-s} \\ \sigma_{k-1}(n) &= \sum_{d|n, d>0} d^{k-1}. \end{aligned}$$

Corollary 1.2.9. $g_k(z)$ is holomorphic at ∞ . In particular, g_k is a modular form of weight k .

1.2.3 Bernoulli numbers

Definition 1.2.10. The **Bernoulli numbers** b_k are defined by

$$\sum_{k=0}^{\infty} b_k \frac{x^k}{k!} = \frac{x}{e^x - 1},$$

a formal power series with rational coefficients.

Then

$$b_0 = 1, \quad b_1 = -\frac{1}{2}, \quad b_2 = \frac{1}{6}, \quad b_3 = 0, \quad b_4 = -\frac{1}{20}, \quad \dots, \quad b_{2k} \in \mathbb{Q}, \quad b_{2k+1} = 0, \quad \dots$$

Proposition 1.2.11. *For all even k ,*

$$\zeta(k) = -b_k \frac{(2\pi i)^k}{2k!}.$$

Proof. On one hand,

$$\pi z \cot \pi z = \pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1} = \pi i z + \sum_{k=0}^{\infty} b_k \frac{(2\pi i z)^k}{k!}.$$

On the other hand,

$$\begin{aligned} \pi \cot \pi z &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \frac{1}{z} - \frac{2z}{n^2} \sum_{n=1}^{\infty} \frac{1}{1 - z^2/n^2} \\ &= \frac{1}{z} - \sum_{n=1}^{\infty} \frac{2}{z} \sum_{k=1}^{\infty} \left(\frac{z^2}{n^2}\right)^k = \frac{1}{z} - \frac{2}{z} \sum_{k=1}^{\infty} z^{2k} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{1}{z} - \frac{2}{z} \sum_{k=1}^{\infty} \zeta(2k) z^{2k}, \end{aligned}$$

so

$$\pi i z + \sum_{k=0}^{\infty} b_k \frac{(2\pi i z)^k}{k!} = \pi z \cot \pi z = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k}.$$

Comparing,

$$b_{2k} \frac{(2\pi i)^{2k}}{(2k)!} = -2\zeta(2k),$$

get the desired formula. □

So

$$g_k(z) = \frac{-b_k (2\pi i)^k}{k!} + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Set the **normalised Eisenstein series**

$$E_k = \frac{g_k}{2\zeta(k)} = 1 - \frac{2k}{b_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Example.

$$\begin{aligned} E_4 &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, & E_6 &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \\ E_8 &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, & E_{12} &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n. \end{aligned}$$

An observation is if f is modular of weight k and g is modular of weight k' , then fg is modular of weight $k + k'$, and if $k = k'$, then $f + g$ is modular of weight k .

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Example. Important examples.

- The **modular discriminant**

$$\Delta(z) = \frac{E_4 - E_6^2}{1728} = q - 24q^2 + 252q^3 + \dots$$

is a modular form of weight 12.

- The **j-invariant**

$$j(z) = \frac{E_4^3}{\Delta} = \frac{1}{q} + 744 + 196844q + \dots$$

is a meromorphic modular form of weight 0.

1.3 Controlling modular forms

1.3.1 The fundamental domain

The idea is to control the action of $\mathrm{SL}_2(\mathbb{Z})$ on \mathbb{H} . If $f : \mathbb{H} \rightarrow \mathbb{C}$ satisfies $f(\gamma z) = (cz + d)^k f(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and if $D \subseteq \mathbb{H}$ such that D meets every $\mathrm{SL}_2(\mathbb{Z})$ -orbit in \mathbb{H} , then f is determined by its values on D .

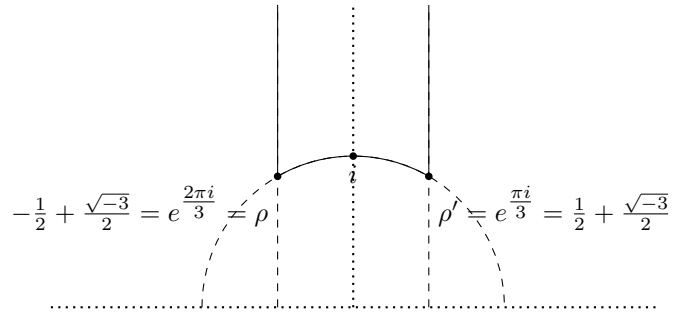
Definition 1.3.1. Let G be a group acting continuously on a complex analytic space X , such as $X = \mathbb{H}$. A subset $D \subseteq X$ is a **fundamental domain** for the action of G if

- D meets every G -orbit in X ,
- the subset $\{x \in D \mid \exists g \in G, gx \in D, gx \neq x\}$ has measure zero, and
- D is closed in X .

Define

$$\mathcal{D} = \{z \in \mathbb{H} \mid \tfrac{1}{2} \leq \operatorname{Re} z \leq \tfrac{1}{2}, |z| \geq 1\} \subseteq \mathbb{H},$$

so



Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : z \mapsto -\frac{1}{z}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : z \mapsto z + 1,$$

and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be the subgroup generated by S and T . We will see later that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Theorem 1.3.2.

1. For all $z \in \mathbb{H}$, there exists $\gamma \in \Gamma$ such that $\gamma z \in \mathcal{D}$.
2. Suppose $z, z' \in \mathcal{D}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma z = z'$. Then either
 - $z = z'$,
 - $\operatorname{Re} z = \pm \frac{1}{2}$ and $z' = z \mp 1$, or
 - $|z| = 1$ and $z' = -1/z$.

In particular, if $z \neq z'$, then z and z' are on the boundary of \mathcal{D} .

3. For $z \in \mathcal{D}$, let I_z be the stabiliser of z in $\mathrm{SL}_2(\mathbb{Z})$, that is

$$I_z = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma z = z\}.$$

Then $I_z = \{\pm \operatorname{id}\}$ unless

- $z = i$, where $I_z = \{\pm \operatorname{id}, \pm S\}$,
- $z = \rho$, where $I_z = \{\pm \operatorname{id}, \pm (ST), \pm (T^{-1}S)\}$, or
- $z = \rho'$, where $I_z = \{\pm \operatorname{id}, \pm (TS), \pm (ST^{-1})\}$.

Corollary 1.3.3. $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Proof. Fix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathring{\mathcal{D}}$ so $\mathrm{SL}_2(\mathbb{Z})z \cap \mathcal{D} = \{z\}$ and $I_z = \{\pm \operatorname{id}\}$. Consider γz . There exists $\gamma' \in \Gamma$ such that $\gamma'\gamma z \in \mathcal{D}$, so $\gamma'\gamma z = z$. So $\gamma'\gamma = \pm \operatorname{id}$, so $\gamma = \pm \gamma'^{-1}$. But $\gamma'^{-1} \in \Gamma$ and $-\operatorname{id} = S^2 \in \Gamma$, so $\gamma \in \Gamma$. \square

Proof of Theorem 1.3.2. Recall $\operatorname{Im} \gamma z = \operatorname{Im} z / |cz + d|^2$ for $\gamma \in \operatorname{SL}_2(\mathbb{Z})$.

1. As c and d vary, $\{cz + d\}$ forms a lattice in \mathbb{C} , so there exist only finitely many c and d such that $|cz + d| < 1$. So $\operatorname{Im} \gamma z$ attains a maximum as γ varies over Γ , so there exists $\gamma \in \Gamma$ such that $\operatorname{Im} \gamma z$ is maximal. There exists $n \in \mathbb{Z}$ such that $T^n \gamma z$ has real part between $-\frac{1}{2}$ and $\frac{1}{2}$. Consider $|T^n \gamma z|$. If this is less than one, then

$$\operatorname{Im} ST^n \gamma z = \operatorname{Im} \frac{-1}{T^n \gamma z} > \operatorname{Im} T^n \gamma z = \operatorname{Im} \gamma z.$$

Since $ST^n \gamma \in \Gamma$, this contradicts maximality so $|T^n \gamma z| \geq 1$, so $T^n \gamma z \in \mathcal{D}$.

□