M4P33 Algebraic Geometry

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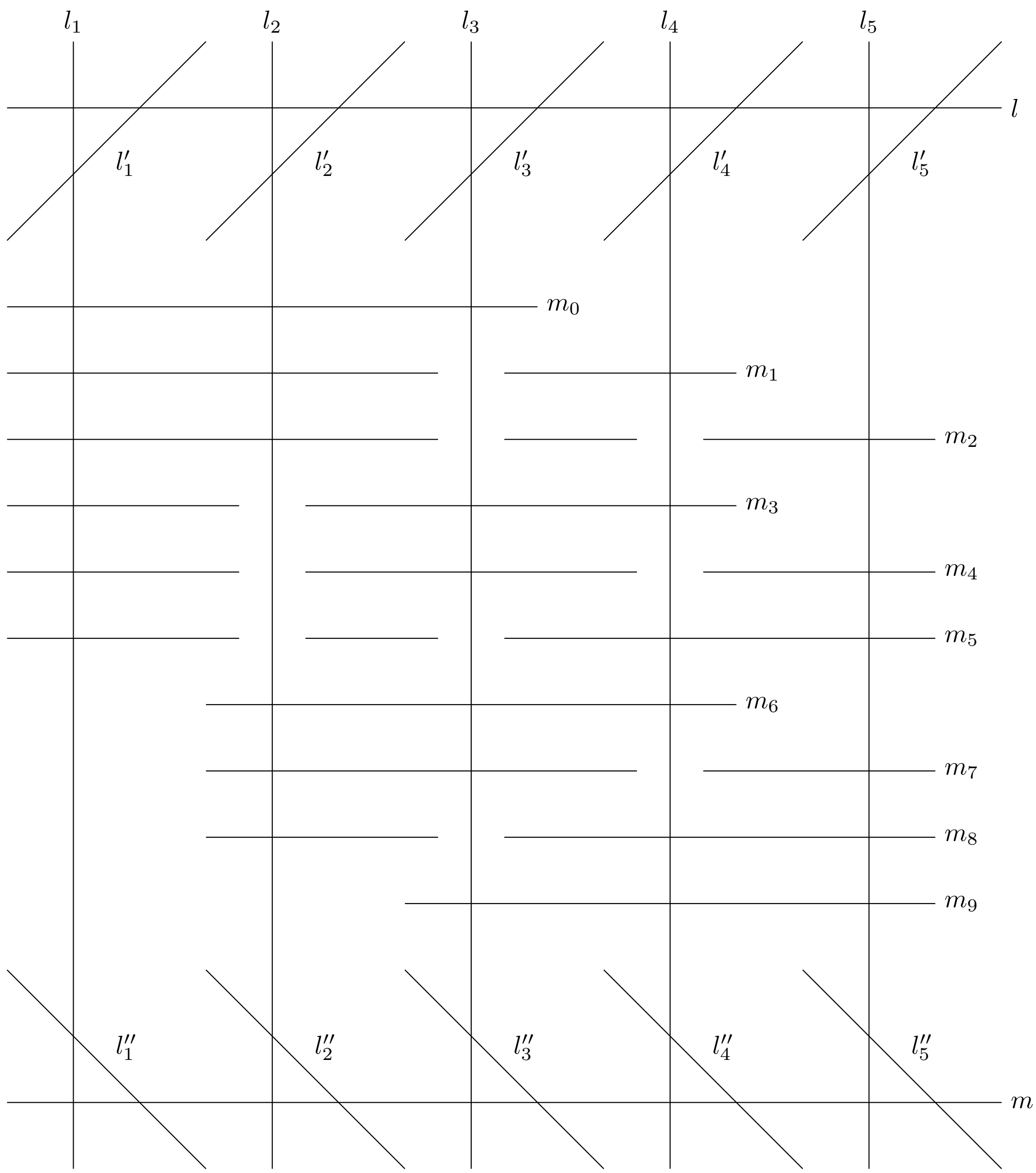

Syllabus

Affine varieties. Projective varieties. Morphisms. Rational maps. Nonsingular varieties. Intersections in projective space. The 27 lines on a cubic surface. Grassmannians. Divisors on curves. Elliptic curves.

Contents

0 Introduction

Lecture 1 Friday 11/01/19

I will not follow a particular book, but everything I am going to say will be contained in one of the following books.

- I Shafarevich, Basic algebraic geometry, 1974
- R Hartshorne, Algebraic geometry, 1977
- J Harris, Algebraic geometry: a first course, 1922
- M Reid, Undergraduate algebraic geometry, 1988

1 Affine varieties

Notation 1.1.

- R is a commutative ring with unity.
- K is a field.
- $K[x_1, \ldots, x_n]$ is the ring of polynomials in n variables.
- $\mathbb{A}^n$ is K^n as a set.

Definition 1.2. Let $S \subseteq K[x_1, \ldots, x_n]$ then

$$Z(S) = \{x \in \mathbb{A}^n \mid \forall f \in S,\ f(x) = 0\}$$

is called the **zero locus** of S. Subsets of $\mathbb{A}^n$ that are of this form are called **affine varieties**.

Remark 1.3. Some authors call **algebraic set** the object $Z(S)$. We will not follow this notation.

Example 1.4.

- Single points $p = (p_1, \ldots, p_n)$. $p = Z(S)$ where $S = \{x_1 - p_1, \ldots, x_n - p_n\}$.
- $\mathbb{A}^n = Z(0)$.
- $\emptyset = Z(1)$.
- Subspaces of $\mathbb{A}^n = K^n$.
- If $X = Z(f_1, \ldots, f_n) \subseteq \mathbb{A}^n$ and $Y = Z(g_1, \ldots, g_m) \subseteq \mathbb{A}^n$ are affine varieties then

  $$X \times Y = Z(f_1, \ldots, f_n, g_1, \ldots, g_m) \subseteq \mathbb{A}^{n+m}$$

  is a variety.

Remark 1.5. If $S \subseteq K[x_1, \ldots, x_n]$ and $I = \langle S \rangle$ then $Z(S) = Z(I)$.

Theorem 1.6 (Hilbert's basis theorem)**.** *If R is Noetherian then $R[x]$ is Noetherian.*

Corollary 1.7. *Every ideal in $K[x_1, \ldots, x_n]$ is finitely generated.*

Definition 1.8. Let $X \subseteq \mathbb{A}^n$ then

$$I(X) = \{f \in K[x_1, \ldots, x_n] \mid \forall x \in X,\ f(x) = 0\}.$$

Example 1.9. $I(p) = I((p_1, \ldots, p_n)) = \langle x_1 - p_1, \ldots, x_n - p_n \rangle$.

Goal is

$$\begin{array}{rcl} \{\text{affine varieties in } \mathbb{A}^n\} & \leftrightarrow & \{\text{ideals of } K[x_1, \ldots, x_n]\} \\ X & \mapsto & I(X) \\ Z(J) & \leftarrow\!\shortmid & J \end{array}.$$

$Z(I(X)) = X$ but $I(Z(J)) \supseteq J$.

Example 1.10. $J = \left\langle x^2 + 1 \right\rangle \subseteq \mathbb{R}[x] = I(\emptyset) = I\left(Z\left(x^2 + 1\right)\right)$.

Proposition 1.11.

- *If $X \subseteq Y$ then $I(Y) \subseteq I(X)$. If $I \subseteq J$ then $Z(J) \subseteq Z(I)$.*
- *$X \subseteq Z(I(X))$ and $S \subseteq I(Z(S))$.*
- *If X is affine then $Z(J(X)) = X$. If $X = Z(S)$ then take Z of $S \subseteq I(Z(S))$.*

Example 1.12. Let $J \subseteq \mathbb{C}[x]$. $J = \langle f \rangle$, where $f = (x - x_1)^{k_1} \dots (x - x_n)^{k_n}$.

Definition 1.13. Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal.

$$I \subseteq \sqrt{I} = \{f \in K[x_1, \dots, x_n] \mid \exists n \in \mathbb{N},\ f^n \in I\}.$$

If $\sqrt{I} = I$, we say I is a **radical ideal**. (Exercise: $\sqrt{I}$ is an ideal, $I \subseteq \sqrt{I}$, and $\sqrt{I} = \bigcap_{p \text{ prime}} p$)

Theorem 1.14 (Hilbert's Nullstellensatz)**.** *$I(Z(J)) = \sqrt{J}$. If $\sqrt{J} = J$ then*

$$\begin{array}{rcl} \{\textit{affine varieties}\} & \leftrightarrow & \{\textit{radical ideals}\} \\ X & \mapsto & I(X) \\ Z(J) & \leftarrow\!\shortmid & J \end{array}.$$

Lecture 2 Monday 14/01/19

Proposition 1.15.

1. *$Z(S) \cup Z(T) = Z(ST)$.*
2. *$\bigcap_i Z(S_i) = Z(\bigcup_i S_i)$.*
3. *$Z(0) = \mathbb{A}^n$ and $Z(1) = \emptyset$.*

Proof.

1. If $p \in Z(S) \cup Z(T)$, then $f(p) = 0$ for $f \in S$ or $f \in T$, so $f(x) = 0$ for $f \in ST$, where

$$ST = \left\{ \sum_{i \in I,\ I \text{ finite}} s_i t_i \right\} \subseteq S \cap T,$$

with equality if $S + T = R$. If $p \in Z(ST)$, there exists f such that $f(p) = 0$ for $f \in S$ or $f(p) = 0$ for $f \in T$, so $p \in Z(S) \cup Z(T)$.

□

Definition 1.16. The **Zariski topology** on $\mathbb{A}^n$ is the topology generated by closed sets of the form $Z(S)$. By the above proposition this is a topology.

Example 1.17. $\mathbb{A}^1$ is not Hausdorff.

Definition 1.18. A topological space X is **irreducible** if it cannot be expressed as a union $X = A \cup B$, where A and B are proper and closed subsets. $\emptyset$ is not considered irreducible.

Example 1.19. $\mathbb{A}^1$.

Example 1.20. Any non-empty open set of irreducible X is dense and irreducible. Suppose A is open then $X = A^c \cup \overline{A}$. Since X is irreducible then $A^c = X$, a contradiction, or $\overline{A} = X$. Suppose A is reducible. Let $A = (A \cap B) \cup (A \cap C)$, where B and C are closed. Then $X = A^c \cup (B \cup C)$. $A^c = X$ or $B \cup C = X$, which are contradictions.

Example 1.21. If A is irreducible then $\overline{A}$ is also irreducible. Suppose $\overline{A}$ is not irreducible. $\overline{A} = \left(\overline{A} \cap B\right) \cup \left(\overline{A} \cap C\right)$. Take $\bigcap A$, $A = (A \cap B) \cup (A \cap C)$, a contradiction.

Definition 1.22. An affine variety is **irreducible** if it is irreducible as a topological space.

Remark 1.23. A **quasi-affine variety** is an open set of an affine variety.

Proposition 1.24.

1. *$I(X \cup Y) = I(X) \cap I(Y)$.*
2. *$Z(I(X)) = \overline{X}$ for any $X \subseteq \mathbb{A}^n$.*

Proof.

1. If $f \in I(X \cup Y)$ then $f(p) = 0$ for all $p \in X \cup Y$, so $f \in I(X)$ and $f \in I(Y)$.
2. We know that $X \subseteq Z(I(X))$ hence $\overline{X} \subseteq Z(I(X))$. Now, let Y be a closed set containing X, that is $X \subseteq Y$. Then
$$I(Y) \subset I(X) \qquad \implies \qquad Z(I(X)) \subset Z(I(X)) = Y,$$
so any closed set containing Y contains $Z(I(X))$.

□

Proposition 1.25. *X is irreducible if and only if $I(X)$ is prime.*

Proof.

$\implies$ Let $f, g \in I(X)$.
$$X \subseteq Z(fg) = Z(f) \cup Z(g) \qquad \implies \qquad X = (X \cap Z(f)) \cup (X \cap Z(g)).$$
$Z(f) \subseteq X$, so $f \in I(X)$, or $Z(g) \subseteq X$, so $g \in I(X)$.

$\impliedby$ Exercise.

□

Example 1.26. $\mathbb{A}^n$.

Definition 1.27. If $X \subseteq \mathbb{A}^n$, the **coordinate ring** of X is
$$A(X) = \frac{A}{I(X)} = \frac{K[x_1, \dots, x_n]}{I(X)}.$$

Lecture 3 Tuesday 15/01/19

Example 1.28. Let $f \in K[x_1, \dots, x_n]$ be irreducible. If $n = 3$, $Z(f)$ is a surface. If $n = 2$, $Z(f)$ is a curve.

Example 1.29. Let $y - x^2 \in K[x, y]$. Then
$$\begin{array}{ccc} A(X) = \dfrac{K[x,y]}{\langle y - x^2 \rangle} \cong K\left[x, x^2\right] & \longrightarrow & K[x] \\ \sum_{i,j} a_{ij} x^i x^{2j} = \sum_{i,j} a_{ij} x^{2j+i} & \longmapsto & \sum_n b_n x^n \end{array}.$$

Example 1.30. Let $xy - 1 \in K[x, y]$. Then
$$A(X) = \frac{K[x,y]}{\langle xy - 1 \rangle} \cong K\left[x, \frac{1}{x}\right].$$

$A(X)$ cannot be $K[x]$.

Definition 1.31. A **Noetherian** topological space X is a topological space such that if
$$C_1 \supseteq C_2 \supseteq \dots$$
is a decreasing chain of closed sets then there is a k such that $C_k = C_{k+1} = \dots$.

Example 1.32. $\mathbb{A}^n$. Recall that if $A \subset B$ then $I(B) \subset I(A)$. So using the definition above,
$$I(C_1) \subseteq I(C_2) \subseteq \dots.$$
Since $K[x_1, \dots, x_n]$ is Noetherian then $I(C_i)$ stabilises. So $I(C_k) = I(C_{k+1}) = \dots$, but taking Z, we recover C_k so C_k stabilises as well.

Theorem 1.33. *If X is Noetherian then any non-empty closed subset can be expressed as a finite union of irreducible closed sets $X = Y_1 \cup \dots \cup Y_n$. Moreover, if we require that $Y_i \subseteq Y_j$ then this expression is unique.*

Proof. Let C be the collection of closed sets that do not satisfy that property. Let Y be a minimum closed inside C, in particular Y is reducible, so $Y = Y' \cup Y''$, for Y', Y'' closed. Hence $Y', Y'' \not\subset C$, so they can be expressed as a finite union of irreducibles, a contradiction. If $Y_i \not\subset Y_j$, then suppose

$$Y_1 \cup \dots \cup Y_n = X_1 \cup \dots \cup X_n.$$

Then $Y_1 \subset X_1 \cup X_n$, in particular $Y_1 = \bigcup_j (Y_1 \cap X_j)$, so there is a j such that $Y_1 \cap X_j = Y_1$, so $Y_1 \subset X_j$. We can assume $j = 1$ and repeat the same argument to find that $Y_1 = X_1$, so consider $\overline{Y \setminus Y_1} = Y_2 \cup \dots \cup Y_n$. But

$$Y_2 \cup \dots \cup Y_n = X_2 \cup \dots \cup X_n,$$

and the result follows by induction. □

Corollary 1.34. *Any affine variety in $\mathbb{A}^n$ can be expressed equally as a union of irreducible algebraic varieties.*

Definition 1.35. The **dimension** of a topological space is the supremum of n where

$$Y_0 \subset \dots \subset Y_n$$

is a sequence of irreducible closed sets.

Example 1.36. Dimension of $\mathbb{A}^1$ is one.

Definition 1.37. Let A be a ring and $\mathfrak{p}$ be a prime ideal, then the **height** of $\mathfrak{p}$ is the supremum of n where

$$\mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_n \subset \mathfrak{p},$$

where $\mathfrak{p}_i$ are prime. The **Krull dimension** of A is

$$\sup_{\mathfrak{p} \text{ prime}} \textit{height}(\mathfrak{p}).$$

Proposition 1.38. *If Y is affine then* $\dim(Y) = \dim(A(Y))$.

Proof. Let C be a closed and irreducible set $C \subset Y$, then $I(C) \supset I(Y)$, then $I(C)$ is prime. □

Proposition 1.39. *Let K be a field and B be an integral domain which is a finitely generated algebra, then*

- $\dim(B)$ *is the transcendence degree of $K(B)$ over K, and*
- *if $\mathfrak{p} \subseteq B$ is prime, then*

$$\textit{height}(\mathfrak{p}) + \dim\left(\frac{B}{\mathfrak{p}}\right) = \dim(B).$$

Proof. Atiyah Macdonald chapter 11. □

Proposition 1.40 (Krull Hauptidealsatz)**.** *Let A be a Noetherian ring and $f \in A$ not a zero divisor and not a unit. Then every prime ideal containing f has height one.*

Proof. Atiyah Macdonald page 122. □

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Proposition 1.41. *A Noetherian integral domain A is a UFD if and only if every prime ideal I of height one is principal.*

Theorem 1.42. *An irreducible variety $Y \subseteq \mathbb{A}^n$ has dimension $n-1$ if and only if $Y = Z(f)$ where f is an irreducible polynomial in $K[x_1, \dots, x_n]$.*

Proof.

$\Longrightarrow$ If Y has dimension $n-1$ then $I(Y)$ has height one, by the above proposition $I(Y) = \langle f \rangle$, so $Y = Z(f)$.

$\Longleftarrow$ Let $I = I(Y)$ then I is prime, by the Krull Hauptidealsatz we have that I has height one, so $\dim(Y) = n - 1$.

□

2 Projective varieties

Definition 2.1. The **projective space** $\mathbb{P}^n$ is defined as

$$\mathbb{P}^n = \frac{\mathbb{A}^{n+1} \setminus \{0\}}{\{x \sim \lambda x \mid \lambda \in K^*\}}.$$

A point in $\mathbb{P}^n$ is written as $[a_0 : \dots : a_n] = \overline{(a_0, \dots, a_n)}$.

Definition 2.2. A **graded ring** R is a ring together with a decomposition

$$R = \bigoplus_{d>0} R_d,$$

where R_d are abelian groups and $R_k \cdot R_t \subseteq R_{k+t}$.

Example 2.3. $K[x_0, \dots, x_n]$ is a graded ring, where R_d are monomials of degree d.

Notation 2.4. Let A be $K[x_0, \dots, x_n]$ without the grading and S be $K[x_0, \dots, x_n]$ as a graded ring.

Definition 2.5. An ideal $I \subseteq S$ is **homogeneous** if

$$I = \bigoplus_{d \geq 0} (I \cap S_d).$$

If $f = f_0 + \dots + f_d$, then $f_i \in I$.

Remark 2.6. I is homogeneous if and only if $I = \langle f_0, \dots, f_n \rangle$, where f_i are homogeneous.

Lemma 2.7. *If I, J are homogeneous then*

1. *$I + J$ is homogeneous,*
2. *IJ is homogeneous,*
3. *$I \cap J$ is homogeneous, and*
4. *$\sqrt{I}$ is homogeneous.*

Proof.

4. Let $f = f_0 + \dots + f_d \in \sqrt{I}$ then

$$f^n = (f_0 + \dots + f_d)^n = f_d^n + \dots \in I \qquad \implies \qquad f_d^n \in I \qquad \implies \qquad f_d \in \sqrt{I},$$

so $f - f_d \in \sqrt{I}$, by induction $f_i \in \sqrt{I}$.

□

Definition 2.8. If f is homogeneous of degree k then

$$f(\lambda \cdot x) = \lambda^k \cdot f(x),$$

in particular $f(x) = 0$ if and only if $f(\lambda \cdot x) = 0$, so it makes sense to define

$$Z(f) = \{x \in \mathbb{P}^n \mid f(x) = 0\}.$$

More generally, if $I \subseteq S$ is a homogeneous ideal then

$$Z(I) = \{x \in \mathbb{P}^n \mid f \in I \text{ homogeneous}, \ f(x) = 0\}.$$

Definition 2.9. A subset $X \subseteq \mathbb{P}^n$ is called a **projective variety** if $X = Z(T)$ for some homogeneous ideal T.

Proposition 2.10.

- $Z(S) \cup Z(T) = Z(ST)$.
- $\bigcap_\alpha Z(S_\alpha) = Z(\bigcup_\alpha S_\alpha)$.
- $Z(0) = \mathbb{P}^n$ *and* $Z(1) = \emptyset$.

Definition 2.11. We define the **Zariski topology** on $\mathbb{P}^n$ by taking closed sets to be $Z(T)$ for some T.

Definition 2.12.

- A projective variety is **irreducible** if it is an irreducible topological space.
- An open subset of a projective variety is called a **quasi-projective variety**.
- The **dimension** of a projective variety is its dimension as a topological space.
- If $T \subseteq S$ then
$$I(T) = \langle f \in S \mid f \text{ homogeneous}, \ \forall p \in T, \ f(p) = 0 \rangle .$$

Definition 2.13. If X is a projective variety the **homogeneous coordinate ring** is

$$S(X) = \frac{S}{I(X)}.$$

Definition 2.14. If $f \in S$ is linear and homogeneous, we call $Z(f)$ a **hyperplane**.

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21/01/19

Proposition 2.15.
$$\begin{array}{cccc} \phi_i & : & U_i = \mathbb{P}^n \setminus Z(x_i) & \longrightarrow & \mathbb{A}^n \\ & & [x_0 : \dots : x_n] & \longmapsto & \left(\dfrac{x_0}{x_i}, \dots, \dfrac{x_n}{x_i} \right) \end{array}$$

is a homeomorphism in the Zariski topology.

Proof. Let $\phi = \phi_0$ and $U = U_0$, let $C \subseteq \mathbb{A}^n$ be a closed set then we claim that $\phi^{-1}(C)$ is closed. Indeed, let $C = Z(S)$, then $\phi^{-1}(C) = Z(S') \cup U$ where

$$S' = \left\{ x_0^d \cdot f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right) \;\middle|\; f \in S \right\}.$$

Similarly, let $A \subseteq U$ is closed, we claim that $\phi(A)$ is closed. Let $\overline{A}$ be its closure in $\mathbb{P}^n$, then $\overline{A} = Z(B)$, so $\phi(A) = Z(B')$ where
$$B' = \{ f(1, x_1, \dots, x_n) \mid f \in B \} .$$

So we conclude that ϕ is a homeomorphism. ☐

Note. $\langle 1 \rangle = S$ and $\langle x_0, \dots, x_n \rangle \subsetneq S$ map to $\emptyset$ under Z. So in order to have a one-to-one correspondence we need the following.

- $Z(I) = \emptyset$ if and only if $\sqrt{I} \supseteq \langle x_0, \dots, x_n \rangle$. If we consider $Z(I)$ in $\mathbb{A}^{n+1}$, note that $x \in Z(I)$ if and only if $\lambda x \in Z(I)$. So $Z(I) = \emptyset$ if and only if $Z(I) \subseteq \{0\}$. So $\sqrt{I} \supseteq \langle x_0, \dots, x_n \rangle$.
- $I(Z(J)) = \sqrt{J}$ if $Z(J) \neq \emptyset$, since $I(Z(J)) = I(Z_a(J)) = \sqrt{J}$.

Corollary 2.16.

$$\left\{ \text{ projective varieties } \right\} \quad \leftrightsquigarrow \quad \left\{ \text{ homogeneous radical ideals not } \langle x_0, \dots, x_n \rangle \right\},$$

$$\left\{ \text{ irreducible projective varieties } \right\} \quad \leftrightsquigarrow \quad \left\{ \text{ homogeneous radical prime ideals } \right\}.$$

Example 2.17. $\mathbb{P}^n$ is irreducible.

Proposition 2.18.

- *$\mathbb{P}^n$ is Noetherian, that is satisfies the descending chain condition.*
- *Every projective variety can be written as a unique union of irreducible projective varieties. We call* ***irreducible components*** *the irreducible varieties in that decomposition.*

Theorem 2.19. *Let $Y \subseteq \mathbb{P}^n$ be an irreducible projective variety. Then*

$$\dim(S(Y)) = \dim(Y) + 1.$$

Proof. Let

$$\begin{array}{cccc} \phi_i & : & U = \mathbb{P}^n \setminus Z(x_i) & \longrightarrow & \mathbb{A}^n \\ & & [x_0 : \dots : x_n] & \longmapsto & \left(\dfrac{x_0}{x_i}, \dots, \dfrac{x_n}{x_i} \right) \end{array},$$

and $Y_i = \phi_i(Y \cap U_i)$. Let

$$\begin{array}{ccc} K[x_1, \dots, x_n] & \longrightarrow & \left(S(Y)_{x_i}\right)_0 \\ f(x_1, \dots, x_n) & \longmapsto & \dfrac{x_i^{\partial f} f\left(\dfrac{x_1}{x_i}, \dots, \dfrac{x_n}{x_i}\right)}{x_i^{\partial f}} \end{array},$$

then

$$A(Y_i) = \frac{K[x_1, \dots, x_n]}{I(Y_i)} \cong \left(S(Y)_{x_i}\right)_0,$$

moreover $S(Y)_{x_i} \cong A(Y_i)\left[x_i, x_i^{-1}\right]$. So

$$\dim(S(Y)) = \dim\left(S(Y)_{x_i}\right) = \dim\left(A(Y_i)\left[x_i, x_i^{-1}\right]\right) = tra\left(K(Y_i)(x_i)\right) = \dim(Y_i) + 1.$$

Therefore if $Y_i \neq \emptyset$, $\dim(Y_i) = \dim(S(Y)) - 1$ for all i, but since U_i cover Y we have $\dim(Y) = \max\{\dim(Y_i)\}$. (Exercise: if $\{U_n\}_n$ is a finite cover of a topological space Y then $\dim(Y) = \max\{\dim(Y_i)\}$) Since $\dim(Y_i)$ are the same if $Y_i \neq \emptyset$, we conclude that $\dim(Y) = \dim(Y_d)$ for some d. ◻

Lecture 6 Tuesday 22/01/19

Proposition 2.20. *Every Noetherian topological space is compact.*

Proof. Let X be a Noetherian topological space and let $\{U_n\}$ be a cover of X. So consider C, the collection of the union of finitely many open sets of $\{U_n\}$. Since X is Noetherian C has a maximum element, say $U_1 \cup \dots \cup U_n$. If $U_1 \cup \dots \cup U_n \subsetneq X$ then there is $x \in X$ not in the union, and we can find another $U_{\alpha_0} \ni x$. But then

$$U_1 \cup \dots \cup U_n \cup U_{\alpha_0} \supsetneq U_1 \cup \dots \cup U_n,$$

a contradiction. So $X = U_1 \cup \dots \cup U_n$. ◻

Corollary 2.21. *$\mathbb{P}^n$, $\mathbb{A}^n$, affine varieties, and projective varieties are all compact in the Zariski topology.*

Definition 2.22. A variety X is **complete** if for any other variety Y, the projection $X \times Y \to Y$ is closed.

Example 2.23. $\mathbb{P}^n$ is complete. $\mathbb{A}^n$ is not complete.

3 Morphisms

Definition 3.1. Suppose Y is a quasi-affine variety and $p \in Y$. We say that a function $f : Y \to \mathbb{A}^1$ is **regular** at p if there are $g, h \in K[x_1, \ldots, x_n]$ and $U \ni p$ such that $f = g/h$ in U with $h \neq 0$. A function is **regular** if it is regular for every $p \in Y$.

Example 3.2. Local is not global. Let $X = Z(x_1x_4 - x_2x_3) \subseteq \mathbb{A}^4$ and $U = X \setminus Z(x_2, x_4)$. Then

$$\begin{array}{cccc} \phi & : & U & \longrightarrow & \mathbb{A}^1 \\ & & (x_1, x_2, x_3, x_4) & \longmapsto & \begin{cases} \dfrac{x_1}{x_2} & x_2 \neq 0 \\ \dfrac{x_3}{x_4} & x_4 \neq 0 \end{cases} \end{array}$$

is a regular function.

Definition 3.3. Let Y be a quasi-projective variety, $f : Y \to \mathbb{A}^1$, and $p \in Y$. We say that f is **regular** at p if there are g, h homogeneous polynomials of the same degree and an open set $U \ni p$ such that $f = g/h$ on U and $h \neq 0$.

Lemma 3.4. *A regular function is continuous.*

Proof. It is enough to show that $f^{-1}(p)$ is closed. Since f is regular $f = g/h$ on some neighbourhood U, then

$$f^{-1}(p) \cap U = Z(g - ph) \cap U.$$

$\square$

Remark 3.5. If X is irreducible then $f = g$ on $U \subseteq X$, then $f = g$ on X. Because the set where $f - g = 0$ is closed and dense.

Definition 3.6. We will use the term **variety** to denote an affine, quasi-affine, projective, or quasi-projective variety.

Definition 3.7. A **morphism** is $f : X \to Y$ if f is continuous and for every $U \subseteq Y$ and every function $g : U \to \mathbb{A}^1$ the composition $g \circ f$ is regular.

Remark 3.8.

- Let $f : X \to Y$ and $g : Y \to Z$ then the composition $g \circ f$ of these two morphisms is the composition of f and g as functions.
- A morphism $f : X \to Y$ is an **isomorphism** if there is a morphism $g : Y \to X$ such that $f \circ g = id$ and $g \circ f = id$.

Definition 3.9. Let X be a variety. Denote the set of all regular functions of X by $\mathcal{O}(X)$. If $p \in X$ the **local ring** at $p \in X$ is

$$\mathcal{O}_p = \varinjlim_{U \ni p} (\mathcal{O}(U)).$$

An element of $\mathcal{O}_p$ is a pair (U, f), where $p \in U$ and f is regular at p, moreover $(U, f) \sim (V, g)$ if $f = g$ on $U \cap V$.

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Definition 3.10. Let Y be an irreducible variety, the **function field** $K(Y)$ of Y is the field whose elements are pairs (U, f) where U is open and f is regular on U, and

$$(U, f) + (V, g) = (U \cap V, f + g).$$

Remark 3.11.

- $K(Y)$ is indeed a field for if $(U, f) \neq 0$ then $U^{-1} = U \setminus Z(f)$, so $\left(U^{-1}, 1/f\right)$ is the inverse to (U, f).
- $K(Y)$ is the quotient field of $A(Y)$ or $S(Y)$.
- $\mathcal{O}(Y) \hookrightarrow \mathcal{O}_p \hookrightarrow K(Y)$ for all $p \in Y$.

Theorem 3.12. *If $Y \subseteq \mathbb{A}^n$ is an irreducible affine variety with coordinate ring $A(Y)$ then*

1. $\mathcal{O}(Y) = A(Y)$,
2. *for all $p \in Y$, if $\mathfrak{m}_p = \{f \in A(Y) \mid f(p) = 0\}$ then we have a one-to-one correspondence*
$$\{\text{ points of } Y\ \} \qquad \leftrightsquigarrow \qquad \{\text{ maximal ideals of } A(Y)\ \},$$
3. *for all $p \in Y$, $\mathcal{O}_p \cong A(Y)_{\mathfrak{m}_p}$ and $\dim(\mathcal{O}_p) = \dim(Y)$, and*
4. *$K(Y)$ is the quotient field of $A(Y)$.*

Proof.

1. Notice that there is a natural map $A \to \mathcal{O}(Y)$ with kernel $I(Y)$, so there is an injection $A(Y) \hookrightarrow \mathcal{O}(Y)$, that is
$$A(Y) \subseteq \mathcal{O}(Y) \subseteq \bigcap_{p \in Y} \mathcal{O}_p = \bigcap_{\mathfrak{m}_p} A(Y)_{\mathfrak{m}_p} = A(Y),$$
so $A(Y) = \mathcal{O}(Y)$.
2. We know that points of Y correspond to maximal ideals $\mathfrak{m}_p \supseteq I(Y)$. Taking the quotient, we get maximal ideals inside $A(Y)$.
3. There is a natural map $A(Y)_{\mathfrak{m}_p} \to \mathcal{O}_p$, which is injective by $\alpha : A(Y) \hookrightarrow \mathcal{O}(Y)$, and it is surjective by definition of $\mathcal{O}_p$. Moreover,
$$\dim(\mathcal{O}_p) = \dim(A_p)_{\mathfrak{m}_p} = \mathit{height}(\mathfrak{m}_p) = \dim(Y).$$
4. The quotient field of $A(Y)$ is the quotient field of $\mathcal{O}_p$ for all p, by 3, which is $K(Y)$ by definition.

$\square$

Theorem 3.13. *Let $Y \subseteq \mathbb{P}^n$ be irreducible and projective. Then*

1. $\mathcal{O}(Y) = K$,
2. *for all $p \in Y$, $\mathfrak{m}_p$ as before, $\mathcal{O}_p \cong \left(S(Y)_{\mathfrak{m}_p}\right)_0$, and*
3. $K(Y) \cong \left(S(Y)_{(0)}\right)_0$.

Proof. Recall that

$$\begin{array}{cccc} \phi_i & : & U_i = \mathbb{P}^n \setminus Z(x_i) & \longrightarrow & \mathbb{A}^n \\ & & [x_0 : \dots : x_n] & \longmapsto & \left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right) \end{array}$$

gives $\phi_i^* : A(Y_i) \cong \left(S(Y)_{x_i}\right)_0$ and $Y_i = \phi_i(Y \cap U_i)$.

1. $K \subseteq \mathcal{O}(Y)$. Take $f \in \mathcal{O}(Y)$, so f is regular at each Y_i, but $\mathcal{O}(Y_i) \cong A(Y_i)$, also by ϕ_i^*, $A(Y_i) \cong \left(S(Y)_{x_i}\right)_0$. Thus $f = g_i / x_i^{n_i}$, where $n_i = \deg(g_i)$, in particular $x_i^{n_i} f \in S(Y)_{n_i}$. Now, set $N \geq \sum_i n_i$, then $S(Y)_N \cdot f \subseteq S(Y)_N$, so we can iterate this process to obtain $S(Y)_N \cdot f^q \subseteq S(Y)_N$. In particular $x_0^N f \subset S$, hence $S(Y)[f]$ is contained in $x_0^{-N} S(Y)$. Therefore f is integral since $S(Y)[f]$ is finitely generated. There are $a_i \in S$ such that
$$f^k + a_1 f^{k-1} + \dots + a_k = 0.$$
Since f is homogeneous of degree zero we can take the constant terms of a_i and still have an equation, hence $a_i \in K$.
2. Let $p \in Y$, then $p \in Y_i$, by the previous theorem we know that $\mathcal{O}_p \cong A(Y_i)_{\mathfrak{m}_p}$. By ϕ_i^*, $\mathcal{O}_p \cong \left(\left(S(Y)_{x_i}\right)_{\mathfrak{m}_p}\right)_0$, but since $x_i \notin \mathfrak{m}_p$, hence $\mathcal{O}_p \cong \left(S(Y)_{\mathfrak{m}_p}\right)_0$.
3. Recall that the quotient field of Y is $K(Y) = K(Y_i)$, but $K(Y_i)$ is the quotient field of the coordinate ring $A(Y_i)$, by ϕ_i^*, this is $\left(S(Y)_{(0)}\right)_0$.

$\square$

Lecture 8 Monday 28/01/19

Proposition 3.14. *Let X be an irreducible variety and Y be an irreducible affine variety, then we have a bijection*
$$\alpha : Hom(X, Y) \xrightarrow{\sim} Hom(A(Y), \mathcal{O}(X)),$$
the set of morphisms from X to Y to the set of K-algebra homomorphisms.

Proof. Given a morphism $\phi : X \to Y$, by definition of morphism, ϕ takes regular functions at Y to regular functions at X. So if $f \in A(Y)$ then $\phi \circ f \in \mathcal{O}(X)$. Conversely, let $h : A(Y) \to \mathcal{O}(X)$ be a homomorphism of K-algebras. Recall that
$$A(Y) = \frac{A}{I(Y)} = \frac{k[x_1, \dots, x_n]}{I(Y)}.$$
Take $\overline{x_i} \in A(Y)$ and let $y_i = h(\overline{x_i}) \in \mathcal{O}(X)$ and define
$$\begin{array}{cccc} \psi & : & X & \longrightarrow & \mathbb{A}^n \\ & & p & \longmapsto & (y_1(p), \dots, y_n(p)) \end{array}.$$
We claim that $Im(\psi) \subseteq Y$, but since $Y = Z(I(Y))$, it is enough to show that if $f \in I(Y)$ then $f(\psi(p)) = 0$.
$$f(\psi(p)) = f(y_1(p), \dots, y_n(p)) = f(h(\overline{x_1}(p)), \dots, h(\overline{x_n}(p))) = h(f(x_1, \dots, x_n))(p) = 0.$$
$\square$

Lemma 3.15. *If X, Y are as before then $\psi : X \to Y$ is a morphism if and only if $\psi_i = x_i \circ \psi$ are regular functions.*

Proof. Suppose ψ_i are regular functions, then if p is a polynomial $p \circ \psi$ is regular, but since regular functions are quotients of polynomials, we conclude that $f \circ \psi$ is regular for any regular function f. $\square$

Corollary 3.16. *If X, Y are affine then $X \cong Y$ if and only if $A(X) \cong A(Y)$.*

Corollary 3.17. *The correspondence $X \mapsto A(X)$ induces an arrow reversing correspondence between the category of affine varieties and the category of K-integral domains.*

Lecture 9 Tuesday 29/01/19

Lecture 9 is a problem class.
Lecture 10 is a problem class.

Lecture 10 Friday 01/02/19

4 Rational maps

Lecture 11 Monday 04/02/19

Definition 4.1. Let X, Y be varieties. A **rational map** $f : X \dashrightarrow Y$ is a pair (U, f_U) where $U \subseteq X$ is open and f_U is a morphism on U and we identify $(U, f_U) \sim (V, g_V)$ if $f_U = g_V$ on $U \cap V$.

Lemma 4.2. *If X, Y are varieties and $\phi, \psi : X \to Y$ such that $\phi = \psi$ on $U \subseteq X$, then $\phi = \psi$ on X.*

Proof. We can assume that $Y \subseteq \mathbb{P}^n$ for some n, and hence we reduce to the case where $Y = \mathbb{P}^n$. So the product is $\phi \times \psi : X \to \mathbb{P}^n \times \mathbb{P}^n$. Let $\Delta \subseteq \mathbb{P}^n \times \mathbb{P}^n = Z(x_i y_j - x_j y_i)$. Since $\phi = \psi$ on U, $(\phi \times \psi)(U) \subseteq A$, so $(\phi \times \psi)\left(\overline{U}\right) = (\phi \times \psi)(X) \subseteq \Delta$. $\square$

Definition 4.3.

- A **dominant rational map** is a rational map $f : X \dashrightarrow Y$, such that $f_U(U)$ is dense for some, and hence all, (U, f_U).
- A **birational map** is a dominant rational map $f : X \dashrightarrow Y$ such that f admits an inverse $g : Y \dashrightarrow X$.

Theorem 4.4. *For any two varieties X, Y we have a correspondence*

$$\left\{ \text{ dominant rational maps } f : X \to Y \ \right\} \qquad \leftrightsquigarrow \qquad \left\{ \ K\text{-algebra homomorphisms } K(Y) \to K(X) \ \right\}.$$

Proof. Given a rational map $f : X \dashrightarrow Y$ and let $g \in K(Y)$. Let f_U be a representative of f then we have that if $(V, g) = g$, $g \circ f_U \in K(X)$. Since we can cover Y using affine varieties, we can assume Y is affine then $K(Y) = K(A(Y))$. If we start with a homomorphism $\theta : K(Y) \to K(X)$, let $y_1, \dots, y_n \in A(Y)$ be the generators of $A(Y)$, then $\theta(y_i) \in K(X)$. We can find U such that $\theta(y_i)$ are regular at U. Then this induces a map $A(Y) \to \mathcal{O}(U)$. But then we have a morphism $U \to Y$, and moreover this is the inverse of the map we defined previously. $\square$

Definition 4.5.

- A field extension L/K is **separably generated** if there is a transcendence basis $\{x_i\}$ for L/K such that L is a separable algebraic extension of $K(\{x_i\})$.
- Primitive element theorem. If L/K is finite and separable then $L/K(\alpha)$ for some $\alpha \in L$. If L is infinite and $\beta_1, \dots, \beta_n$ are generators for L/K then $\alpha = c_1\beta_1 + \dots + c_n\beta_n$ for $c_i \in K$.
- If K is perfect, any finitely generated extension L/K is separably generated.

Theorem 4.6. *Any variety X of dimension n is birational to a hypersurface $Y \subseteq \mathbb{P}^{n+1}$.*

Proof. Since $K(X) = K$ is finitely generated, by the theorem above it is separably generated. So we can find a transcendence basis $x_1, \dots, x_n \in K$ such that $K/k(x_1, \dots, x_n)$ is finite and separable. By the primitive element theorem, $K = k(x_1, \dots, x_n, y)$ for some y which is algebraic over $k(x_1, \dots, x_n)$, so y is the solution of a polynomial equation f in $k(x_1, \dots, x_n)$. In particular if we clear denominators we get a polynomial $f(x_1, \dots, x_n, y)$ in $\mathbb{A}^{n+1}$, by taking $Z(f)$ we get a hypersurface and taking its projective closure we get a hypersurface in $\mathbb{P}^n$. $\square$

Lecture 12 Tuesday 05/02/19

Corollary 4.7. *The following are equivalent.*

- *$F : X \dashrightarrow Y$ is birational.*
- *There exist U, V such that $F : U \to V$ is an isomorphism.*
- $K(Y) \cong K(X)$.

Definition 4.8. The **blow-up** of $\mathbb{A}^n$ at the origin 0, denoted by $\widetilde{\mathbb{A}^n}$, is

$$Z\left(x_i y_j - x_j y_i\right) \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

$$\begin{tikzcd} \widetilde{\mathbb{A}^n} \arrow[r, hook] \arrow[dr, "\pi"'] & \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[d, "(x,y)\mapsto x"] \\ & \mathbb{A}^n \end{tikzcd}.$$

Proposition 4.9.

1. *Let $P \in \mathbb{A}^n$, if $P \neq 0$ then $\pi^{-1}(P)$ is a single point, and $\widetilde{\mathbb{A}^n} \setminus \pi^{-1}(0) \cong \mathbb{A}^n \setminus \{0\}$.*
2. *$\pi^{-1}(0) \cong \mathbb{P}^{n-1}$.*
3. *Points of $\pi^{-1}(0)$ are in one-to-one correspondence with the set of lines through the origin.*
4. *$\widetilde{\mathbb{A}^n}$ is irreducible.*

Proof.

1. If $P \neq 0$ then $y_j = x_j y_i / x_i$ and this is true for every j, so this gives a unique point in $\mathbb{P}^{n-1}$.
2. Obvious.
3. A line through the origin is given by $x_i = t a_i$ for $t \neq 0$. Taking π^{-1} of this line we get $x_i = t a_i$ and $y_i = t a_i = a_i$. In other words if $x \neq 0$, $\pi^{-1}(X) = (X, [X])$.
4. $\widetilde{\mathbb{A}^n} \setminus \pi^{-1}(0) \cong \mathbb{A}^n \setminus \{0\}$ is dense and irreducible, by 3.

∎

Definition 4.10. If $Y \ni 0$ is a closed subvariety of $\mathbb{A}^n$ we define the **blow-up** of Y at 0 by

$$\widetilde{Y} = \overline{\pi^{-1}(Y \setminus \{0\})}.$$

More generally, we can blow-up any point by taking an affine change of coordinates. We also get a birational map $\pi : \widetilde{Y} \to Y$.

Example 4.11. Let $Y = Z\left(y^2 - x^2(x+1)\right)$. The equations of the blow-up are

$$\begin{cases} y^2 = x^2(x+1) \\ xu = yt \end{cases},$$

where $[t : u] \in \mathbb{P}^1$. Suppose $t \neq 0$.

$$\begin{cases} y^2 = x^2(x+1) \\ y = xu \end{cases} \implies (xu)^2 = x^2(x+1) \implies x^2\left(u^2 - x - 1\right) = 0.$$

Example 4.12. Let $y^2 = x^3$.

$$\begin{cases} y^2 = x^3 \\ y = xu \end{cases} \implies (xu)^2 = x^3 \implies x^2\left(u^2 - x\right) = 0.$$

5 Nonsingular varieties

Lecture 13 Friday 08/02/19

Definition 5.1. Let $Y \subseteq \mathbb{A}^n$ be an affine variety of dimension r, and suppose $I(Y) = \langle f_1, \dots, f_k \rangle$. Y is **nonsingular** at $P \in Y$ if $rank\left(\frac{\partial f_i(P)}{\partial x_j}\right) = n - r$. Y is **nonsingular** if it is nonsingular at every $P \in Y$.

Example 5.2. Let $x^2 = x^4 + y^4 \subseteq \mathbb{A}^2$, so $f = x^2 - x^4 - y^4$.

$$\frac{\partial f}{\partial x} = 2x - 4x^3 = 0 \qquad \implies \qquad x\left(1 - 2x^2\right) = 0 \qquad \implies \qquad x = 0 \text{ or } 2x^2 = 1,$$

$$\frac{\partial f}{\partial y} = -9y^3 = 0 \qquad \implies \qquad y = 0 \qquad \implies \qquad x^2 = x^4 \qquad \implies \qquad x = 0 \text{ or } x^2 = 1,$$

so $Sing(Y) = \{(0,0)\}$.

Example 5.3. Let $Y = Z(f) = Z\left(y^2 - x^3\right)$.

$$\frac{\partial f}{\partial x} = -3x^2 = 0, \qquad \frac{\partial f}{\partial y} = 2y = 0,$$

so $Sing(Y) = \{(0,0)\}$.

Definition 5.4. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, and residue field $A/\mathfrak{m} = K$. A is a **regular local ring** if $\dim_K\left(\mathfrak{m}/\mathfrak{m}^2\right) = \dim(A)$.

Note. $\left(\mathfrak{m}/\mathfrak{m}^2\right)^*$ is called the **Zariski-tangent space**.
Claim that $\mathfrak{m}/\mathfrak{m}^2$ is a K-vector space for $K = A/\mathfrak{m}$.

Theorem 5.5. *Let $Y \subseteq \mathbb{A}^n$ be an affine variety. Then Y is nonsingular at P if and only if $\mathcal{O}_P$ is a regular local ring.*

Proof. Let $P = (a_1, \dots, a_n) \in Y$ with corresponding maximal ideal $I_P = \langle x_1 - a_1, \dots, x_n - a_n \rangle$. We define a map

$$\begin{array}{cccc} \theta_P & : & A = K[x_1, \dots, x_n] & \longrightarrow & K^n \\ & & f & \longmapsto & \left(\dfrac{\partial f(P)}{\partial x_1}, \dots, \dfrac{\partial f(P)}{\partial x_n}\right) \end{array}.$$

Note that $\theta((x_i - a_i)(x_j - a_j)) = 0$, hence $\theta_P\left(I_P^2\right) = 0$, in particular we have an isomorphism $I_P/I_P^2 \cong K^n$. By the isomorphism, if $\alpha = I(Y) = \langle f_1, \dots, f_t \rangle$ then the rank of $\frac{\partial f_i(P)}{\partial x_j}$ corresponds to the dimension of α under the isomorphism, which is $\overline{\alpha}$ in I_P/I_P^2, $(\alpha + I_P)/I_P^2$. Now $\mathcal{O}_P = (A/\alpha)_{I_P}$. If $\mathfrak{m} = (I_P + \alpha)/\alpha$ then $\mathfrak{m}^2 = \left(I_P^2 + \alpha\right)/\alpha$, so $\mathfrak{m}/\mathfrak{m}^2 = I_P/\left(I_P^2 + \alpha\right)$. So

$$r = \dim\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right) = \dim\left(\frac{I_P}{I_P^2 + \alpha}\right) = \dim\left(\frac{I_P}{I_P^2}\right) - \dim\left(\frac{I_P^2 + \alpha}{I_P^2}\right) = n - rank\left(\frac{\partial f_i}{\partial x_j}\right).$$

So $\mathcal{O}_P$ is regular if and only if $rank\left(\frac{\partial f_i}{\partial x_j}\right) = n - r$. ☐

Definition 5.6. Let X be a variety. X is **nonsingular** at P if $\mathcal{O}_P$ is a regular local ring.

Theorem 5.7. *Let Y be a variety. Then $Sing(Y)$ is a proper and closed set. The set of nonsingular points of Y is open and dense.*

Proof. Prove that $Sing(Y)$ is closed, first. We know that the rank of the Jacobian is at most $n-r$, therefore the singular points occurs when the rank is less than $n-r$, which is to say that $Sing(Y)$ is given by the vanishing of the $(n-r) \times (n-r)$ minors of $\frac{\partial f_i}{\partial x_j}$ and $I(Y)$, hence is closed. To prove that it is proper $Sing(Y) \subsetneq Y$. ☐

Lecture 14 Monday 11/02/19

Lecture 14 is a problem class.
Lecture 15 is a problem class.

Lecture 15 Tuesday 12/02/19

6 Intersections in projective space

Lecture 16 Friday 15/02/19

Theorem 6.1. *Let $Y, Z \subseteq \mathbb{A}^n$ be varieties, with $\dim(Y) = r$ and $\dim(Z) = s$ then every irreducible component has dimension at least $r + s - n$.*

Proof. Suppose Z is a hypersurface. Then if $Y \subseteq Z$ the theorem holds, and if $Y \nsubseteq Z$ the theorem is true by homework 1. Let Z be general. Consider the diagonal in $\mathbb{A}^{2n}$ given by the image of the isomorphism $P \mapsto P \times P$, then $Y \cap Z$ corresponds to $(Y \times Z) \cap \Delta$. Recall that

$$\Delta = Z(x_1 - y_1) \cap \cdots \cap Z(x_n - y_n),$$

by the first case n times we have that each irreducible component has dimension

$$(r + s) - n - 2n = r + s - n.$$

□

Theorem 6.2. *Let $Y, Z \subseteq \mathbb{P}^n$ be varieties, where $\dim(Y) = r$ and $\dim(Z) = s$, then each irreducible component of $Y \cap Z$ has dimension at least $r + s - n$. Moreover, if $r + s - n \geq 0$ then $Y \cap Z \neq \emptyset$.*

Proof. Take the affine cone of Y and Z, $C(Y)$ and $C(Z)$, since $0 \in C(Y) \cap C(Z)$ we apply the previous theorem to get

$$(r + 1) + (s + 1) - (n + 1) = r + s - n + 1,$$

so therefore $Y \cap Z \neq \emptyset$. □

Definition 6.3. A **numerical polynomial** is a polynomial $f \in \mathbb{Q}[x]$ such that $f(n) \in \mathbb{Z}$ for $n \gg 0$, for n sufficiently large.

Theorem 6.4.

1. *If $f \in \mathbb{Q}[x]$ is a numerical polynomial then there are $c_0, \dots, c_r \in \mathbb{Z}$ such that*

$$f(x) = c_0 \binom{x}{r} + \cdots + c_r \binom{x}{0}.$$

2. *If for $n \gg 0$, $\Delta f = f(n+1) - f(n) = q$ and q is a numerical polynomial, then there exists p such that for $n \gg 0$, $p(n) = f(n)$.*

Proof.

1. By linear algebra we can find $c_0, \dots, c_r \in \mathbb{Q}$ such that

$$f(x) = c_0 \binom{x}{r} + \cdots + c_r \binom{x}{0},$$

then

$$\Delta f = c_0 \binom{x}{r-1} + \cdots + c_{r-1} \binom{x}{0}.$$

By induction on the degree of f we have that $c_0, \dots, c_{r-1} \in \mathbb{Z}$, but since $f(n) \in \mathbb{Z}$ for $n \gg 0$ then $c_r \in \mathbb{Z}$.

2. If

$$q = c_0 \binom{x}{r} + \cdots + c_r \binom{x}{0},$$

set

$$p = c_0 \binom{x}{r+1} + \cdots + c_r \binom{x}{1}.$$

$\Delta p = q$ gives $\Delta(f - p)(n) = 0$.

□

Definition 6.5.

- Let S be a graded ring. A **graded S-module** is a module M with a decomposition
$$M = \bigoplus_{d \in \mathbb{Z}} M_d,$$
such that $S_k \cdot M_d \subseteq M_{d+k}$.
- Let $l \in \mathbb{Z}$. The **twisted module** $M(l)$ is the graded S-module given by $M(l)_k = M_{l+k}$.
- $Ann(M) = \{x \in S \mid xM = 0\}$.

Theorem 6.6. *Let M be a finitely generated graded S-module. Then there is a filtration*
$$0 = M^0 \subseteq \cdots \subseteq M^r = M,$$
such that $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(l)$ for some $\mathfrak{p}_i$ prime ideals and $l_i \in \mathbb{Z}$, such that

- *prime $\mathfrak{p} \supseteq Ann(M)$ if and only if $\mathfrak{p} \subseteq \mathfrak{p}_i$, that is $\mathfrak{p}_i$ are minimal primes of M, and*
- *for each minimal prime $\mathfrak{p}$ of M the number of times $\mathfrak{p}$ appears in the set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ is $len_{S_\mathfrak{p}}(M_\mathfrak{p})$.*

Lecture 17 Monday 18/02/19

Definition 6.7. Let $\mathfrak{p}$ be a minimal prime of a graded S-module M. Then the **multiplicity** of M at $\mathfrak{p}$ is $len_{S_\mathfrak{p}}(M_\mathfrak{p})$.

Definition 6.8. Let M be a graded $S = K[x_1, \dots, x_n]$-module. The **Hilbert function** of M is $\phi_M(l) = \dim_K(M_l)$.

Theorem 6.9. *Let M be a graded $S = K[x_1, \dots, x_n]$-module. Then for $n \gg 0$, there is a unique polynomial $P_M \in \mathbb{Q}[x]$ such that $\phi_M(n) = P_M(n)$. P_M is called the* ***Hilbert polynomial****. It is a polynomial of degree* $\dim(Z(Ann(M)))$.

Proof. By the previous theorem, M has a filtration
$$0 = M^0 \subseteq \cdots \subseteq M^r = M,$$
such that M^i/M^{i-1} is of the form $(S/\mathfrak{p}_i)(l_i)$. Without loss of generality we can assume $M = S/\mathfrak{p}$, since l_i amounts to a translation $z \mapsto z + l_i$. If $\mathfrak{p} = \langle x_0, \dots, x_n \rangle$ then $S/\mathfrak{p} \cong K$, in particular $\phi_M(l_i) = 0$ if $l_i > 0$, but then take $P_M = 0$. We can assume $\dim(0) = -1$ and $\dim(\emptyset) = -1$. Suppose $\mathfrak{p} \neq \langle x_0, \dots, x_n \rangle$. Then there is $x_i \notin \mathfrak{p}$ and consider the short exact sequence
$$0 \to M \xrightarrow{x_i} M \to \frac{M}{x_i M} = M'' \to 0.$$
Taking Hilbert function we get that
$$\phi_{M''}(l) = \phi_M(l) - \phi_M(l-1) = \Delta\phi_M(l-1).$$
Note that $Ann(M'') = Ann(M) \cup \{x_i\}$, so $Z(Ann(M'')) = Z(\mathfrak{p}) \cap Z(x_i)$. Note that
$$\dim(Ann(M'')) = \dim(Z(\mathfrak{p})) - 1,$$
so we apply induction over $\dim(Ann(M))$. Thus $\phi_{M''}$ agrees with a polynomial $P_{M''}(n)$ for $n \gg 0$ but then $\Delta\phi_M = P_{M''}$ for $n \gg 0$, so ϕ_M agrees with a polynomial of degree
$$\dim(Ann(M'')) + 1 = \dim(Z(\mathfrak{p})).$$
∎

Definition 6.10. If $Y \subseteq \mathbb{P}^n$ of dimension r, the **Hilbert polynomial** of Y is the Hilbert polynomial of $S(Y)$. The degree of Y is $r!$ times the leading coefficient of P_Y.

Theorem 6.11.

1. *If* $Y \neq \emptyset$, *then* $\deg(Y) \in \mathbb{Z}_{>0}$.
2. $\deg(\mathbb{P}^n) = 1$.
3. *If* $Y = Y_1 \cup Y_2$ *with* $\dim(Y_i) = r$ *and* $\dim(Y_1 \cap Y_2) < r$ *then* $\deg(Y) = \deg(Y_1) + \deg(Y_2)$.
4. *If* H *is a hypersurface generated by* f *then* $\deg(H) = \deg(f)$.

Proof.

1. Obvious.
2. $$\phi_{\mathbb{P}^n}(z) = \binom{z+n}{n} = \frac{1}{n!}(z)\dots(n+1) = \frac{1}{n!}z^n + \dots.$$
3. Let $I = I(Y)$, $I_1 = I(Y_1)$, and $I_2 = I(Y_2)$. Consider the short exact sequence
$$0 \to \frac{S}{I} \to \frac{S}{I_1} \oplus \frac{S}{I_2} \to \frac{S}{I_1 + I_2} \to 0.$$
Taking Hilbert function,
$$\phi_{\frac{S}{I_1+I_2}} = \phi_{\frac{S}{I_1} \oplus \frac{S}{I_2}} - \phi_{\frac{S}{I}}.$$
Since $Z(I_1 + I_2) = Y_1 \cap Y_2$ and $\dim(Y_1 \cap Y_2) < r$ we have that $\phi_{S/I_1 \oplus S/I_2}$ and $\phi_{S/I}$ have the same leading coefficients, hence $\deg(Y) = \deg(Y_1) + \deg(Y_2)$.
4. Suppose $\deg(f) = d$ then consider the short exact sequence
$$0 \to S(-d) \xrightarrow{f} S \to \frac{S}{\langle f \rangle} \to 0.$$
Taking Hilbert functions,
$$\phi_{\frac{S}{\langle f \rangle}}(z) = \phi_S(z) - \phi_S(z-d) = \binom{z+n}{n} - \binom{z-d+n}{n} = \frac{d}{(n-1)!}z^{n-1} + \dots.$$

$\square$

Let $Y \subseteq \mathbb{P}^n$ be a projective variety and H a hypersurface then $Y \cap H = Z_1 \cup \dots \cup Z_k$, where each Z_j has dimension $r - 1 = \dim(Y) - 1$. Suppose $I(Z_j) = \mathfrak{p}_j$, then each $\mathfrak{p}_j$ is a minimal prime of $S/(I_Y + I_H)$, then the **intersection multiplicity** $i(Y, H; Z_j)$ is the multiplicity of $S/(I_Y + I_H)$ at $\mathfrak{p}_j$. Lecture 18 Tuesday 19/02/19

Theorem 6.12. *Let $Y \subseteq \mathbb{P}^n$ be a variety and H a hypersurface such that $Y \nsubseteq H$. If $Y \cap H = Z_1 \cup \dots \cup Z_k$ then*

$$\sum_{j=1}^{k} i(Y, H; Z_j) \deg(Z_j) = \deg(Y) \deg(H).$$

Corollary 6.13 (Bézout's theorem)**.** *If $Y, H \subseteq \mathbb{P}^2$ are curves and $Y \cap H = \{P_1, \dots, P_k\}$ then*

$$\sum_{j=1}^{k} i(Y, H; P_j) = \deg(Y) \deg(H).$$

Proof. Supppose H is generated by f, where $\deg(f) = d$, and let $I = I(Y)$.

$$0 \to \left(\frac{S}{I}\right)(-d) \xrightarrow{f} \frac{S}{I} \to \frac{S}{I + I_H} \to 0.$$

Taking Hilbert polynomials we get

$$\phi_{\frac{S}{I_1 + I_2}}(z) = \phi_{\frac{S}{I_Y}}(z) + \phi_{\frac{S}{I_Y}}(z - d).$$

Let $\deg(Y) = e$, then the right hand side is

$$\frac{e}{r!} z^r + \dots - \left(\frac{e}{r!}(z - d)^r + \dots\right) = \frac{de}{(r-1)!} z^{r-1} + \dots.$$

Now on the left hand side, by the structure theorem, there is a filtration

$$0 = M^0 \subseteq \dots \subseteq M^s = M,$$

where $M = S/(I_Y + I_H)$. Then

$$P_M = \sum_{i=1}^{s} P_i = \sum_{i=1}^{s} P_{\frac{M^i}{M^{i-1}}},$$

where each $M^i/M^{i-1} = (S/\mathfrak{p}_i)(l_i)$. Since we want to compare the leading coefficient from this with the one from the right hand side, we only care about the P_i's with degree $r - 1$. So the $\mathfrak{p}_j = I(Z_j)$ and the leading term is

$$\frac{\sum_{j=1}^{k} i(Y, H; Z_j) \deg(Z_j)}{(r-1)!} + \dots.$$

$\square$

7 The 27 lines on a cubic surface

Lecture 19 Friday 22/02/19

Theorem 7.1. *Let $S \subseteq \mathbb{P}^3$ be a nonsingular cubic surface given by a polynomial $f(x, y, z, t)$. Then S has exactly 27 lines.*

We start with a lemma.

Lemma 7.2.

1. *Given a point $p \in S$ then there are at most three lines through p. If there are two or three they must be spheres.*
2. *Every plane π intersect S in*
   - *an irreducible cubic,*
   - *a conic and a line, or*
   - *three distinct lines.*

Proof.

1. $l \subseteq S$ gives $T_p(l) = l \subseteq T_p(S)$, by 2, $T_p(S)$ intersect S in at most three lines.
2. We have to prove that there are no multiple lines in the intersection $S \cap \pi$. Changing coordinates if necessary, we can suppose $\pi = \{f = 0\}$ and $l = \{z = 0\}$ is the line in the intersection.

$$f = z^2 \cdot a(x, y, z, t) + t \cdot b(x, y, z, t).$$

   Claim that S is singular at $z = t = b = 0$.

$$Jac(f) = \begin{pmatrix} z^2 a_x + t b_x & z^2 a_y + t b_y & 2za + z^2 a_z + t b_z & z^2 a_t + b + t b_t \end{pmatrix}.$$

   Since S is smooth there are no multiple lines.

□

Lemma 7.3. *S has a line.*

Proof.

- Let $P \in S$ and consider $T_P(S)$. Then $T_P(S)$ intersects S in a plane cubic $C = S \cap T_P(S)$ which is singular at P. Otherwise we are done. Then C has to be a nodal or a cuspidal curve. So assume that C is a cuspidal curve, and change coordinates if necessary, assume that $P = [0 : 0 : 1 : 0]$ and $T_P(S) = \{t = 0\}$. So the equation of f has the shape

$$f = x^2 z - y^3 + gt,$$

  for some g of homogeneous degree two.

- We consider the point $P_\alpha = \left[1 : \alpha : \alpha^3 : 0\right] \in C \subset S$, consider the plane $x = 0$ and the line $P_\alpha Q$ in $\mathbb{P}^3$ passing through P_α and intersecting this plane $x = 0$ at $Q = (0, y, z, t)$. The line through P_α and Q is $\lambda P_\alpha + \mu Q$ and it lies inside S if

$$f(\lambda P_\alpha + \mu Q) = 0.$$

  After expanding this we have

$$P_\alpha Q \subset S \qquad \iff \qquad A(y, z, t) = B(y, z, t) = C(y, z, t) = 0,$$

  for A, B, C to be determined. There is a polynomial $R(\alpha)$ of degree 27 such that $R(\alpha) = 0$ if and only if $A = B = C$ have a common zero.

Lecture 20 Monday 25/02/19

- Let $f(x,y,z,t)$ be a polynomial, then the **polar form** of f is

$$f_1(x,y,z,t,x',y',z',t') = \frac{\partial f}{\partial x} \cdot x' + \frac{\partial f}{\partial y} \cdot y' + \frac{\partial f}{\partial z} \cdot z' + \frac{\partial f}{\partial t} \cdot t',$$

where $P = (x,y,z,t)$ and $Q = (x',y',z',t')$. Then

$$f(\lambda P + \mu Q) = \lambda^3 f(P) + \lambda^2 \mu f_1(P,Q) + \lambda \mu^2 f_1(Q,P) + \mu^3 f(Q).$$

The polar form of $f = x^2 z - y^3 + gt$ is

$$f_1 = 2xzx' - 3y^2 y' + x^2 z' + g(x,y,z,t)\, t' + t g_1,$$

where g_1 is the polar form of g. Recall $P_\alpha = \left(1, \alpha, \alpha^2, 0\right)$ and $Q = (0,y,z,t)$, so

$$\{f(\lambda P + \mu Q) = 0\} = PQ \subseteq S \qquad \iff \qquad f(P) = f_1(P,Q) = f_1(Q,P) = f(Q) = 0.$$

Thus

$$\begin{cases} A = z - 3\alpha^2 y + g\left(1, \alpha, \alpha^3, 0\right) t \\ B = -3\alpha y^2 + g_1\left(1, \alpha, \alpha^3, 0, 0, y, z, t\right) t \\ C = -y^3 + g(0,y,z,t)\, t \end{cases}.$$

- Note that

$$g\left(1, \alpha, \alpha^3, 0\right) = a^6 + \dots.$$

If $l = 0$,

$$z = 3\alpha^2 y + g(P)\, t = 3\alpha^2 y + \left[a^6\right] t.$$

Applying this to $B = 0$ we have

$$B = -3\alpha y^2 + g_1\left(1, \alpha, \alpha^3, 0, 0, y, 3\alpha^2 y - \left[a^6\right] t, t\right) t = b_0 y^2 + b_1 y t + b_2 t^2,$$

where

$$b_0 = -3\alpha, \qquad b_1 = 6\alpha^5 + \dots, \qquad b_2 = -2\alpha^9 + \dots.$$

Substituting z in C we get

$$C = c_0 y^3 + c_1 y^2 t + c_2 y t^2 + c_3 t^3,$$

where

$$c_0 = -1, \qquad c_1 = 9\alpha^4 + \dots, \qquad c_2 = -6\alpha^8 + \dots, \qquad c_3 = \alpha^{12} + \dots.$$

By Sylvester theorem B and C have a common zero if and only if

$$\det \begin{pmatrix} -3\alpha & 6\alpha^5 & -2\alpha^9 & & \\ & -3\alpha & 6\alpha^5 & -2\alpha^9 & \\ & & -3\alpha & 6\alpha^5 & -2\alpha^9 \\ -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} & \\ & -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} \end{pmatrix} = 0.$$

if and only if

$$\alpha^{27} \det \begin{pmatrix} -3 & 6 & -2 & & \\ & -3 & 6 & -2 & \\ & & -3 & 6 & -2 \\ -1 & 9 & -6 & 1 & \\ & -1 & 9 & -6 & 1 \end{pmatrix} = \alpha^{27} + \dots = 0.$$

This concludes the proof that S has a line because we know that the matrix has at least one root and for each root we get a value of α such that the line $P_\alpha Q \subseteq S$.

$\square$

Lecture 21 Tuesday 26/02/19

Proposition 7.4. *Let l be a line in S, then there are five pairs of lines (l_i, l_i') intersecting l such that*

- *$l \cup l_i \cup l_i'$ is coplanar, and*
- *$(l_i \cup l_i') \cap (l_j \cup l_j') = \emptyset$.*

Proof. Given any plane $\Pi \subseteq \mathbb{P}^3$, if Π contains a line l of S then $\Pi \cap S$ is l and a conic. l is given by $z = t = 0$.

$$f = Ax^2 + Bxy + Cy^2 + Dx + Ey + F, \qquad A, B, C, D, E, F \in K[z, t].$$

We want to prove that there are exactly five planes Π_i such that $f|_{\Pi_i}$ is a singular conic. The conic given by f is singular if and only if

$$\Delta = \det \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix} = 4ACF + BDE - AE^2 - B^2F - CD^2 = 0.$$

Δ is four times the usual determinant if *char* $(K) \neq 2$. Notice that Δ is a form of degree five in two variables z and t. We know that l, l_i, l_i' could be of two types.

1. $l : (t = 0)$, $l_1 : (x = 0)$, $l_1' = (y = 0)$.
2. $l : (t = 0)$, $l_1 : (x = 0)$, $l_1' = (x = t)$.

Assume we are in case 1. Suppose $z = 0$ is a solution, then we have to prove that z^2 is not a solution. Then the equation of f is

$$f = txy + gz.$$

So $B = t + az$, where $a \in K$, then $\Delta \equiv -t^2F \mod z^2$. If $F \neq 0$ then Δ is non-zero, but F is non-zero because F is nonsingular, thus there are no multiple roots. ☐

Corollary 7.5. *S has at least two distinct lines.*

Proof. Just take l_1 and l_2. ☐

Lemma 7.6. *If $l_1, \dots, l_4 \in \mathbb{P}^3$ are disjoint lines then*

- *either all four lines lie on a smooth quadric and they have an infinite number of transversals,*
- *or the four lines do not lie in any quadric and they have either one or two common transversals.*

Proof. Any three lines lie in a smooth quadric Q. ☐

Lemma 7.7.

- *Any line not the 17 lines intersect exactly three of the lines $l_1, \dots, l_5$.*
- *Conversely, given $ijk \subset \{1, 2, 3, 4, 5\}$ there is a line passing through l_i, l_j, l_k.*

8 Grassmannians

Lecture 22
Friday
01/03/19

Definition 8.1. Let V be a vector space of dimension n, then

$$G(k;n) = \{S \subseteq V \mid S \text{ subspace of dimension } k\}.$$

Remark 8.2. A point in $G(k;n)$ can be expressed as a basis $[v_1, \dots, v_k]$ for a k-dimensional space.

Theorem 8.3. *The map*

$$\begin{array}{rrcl} p & : \qquad G(k;n) & \longrightarrow & \mathbb{P}\left(\bigwedge^k (V)\right) \cong \mathbb{P}^{{}^nC_k - 1} \\ & [v_1, \dots, v_k] & \longmapsto & [v_1 \wedge \dots \wedge v_k] \end{array}$$

is an embedding. That is, image of p is closed.

Example 8.4. Claim that a line $L \subseteq \mathbb{P}^3$ gives a point in $G(2;4) \hookrightarrow \mathbb{P}^5$. $G(2;4)$ is a quadric in $\mathbb{P}^5$ given by $Z(xs - yt + zw)$.

Proof. Now we will see the coordinates of the map p. Given a vector space V of dimension n and a vector subspace $S \subseteq V$ of dimension k, then let $v_1, \dots, v_n$ be a basis for V, and $s_1, \dots, s_k$ be a basis for S, then the basis for S can be seen as a $k \times n$ matrix

$$M_S = \begin{pmatrix} s_{11} & \dots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{k1} & \dots & s_{kn} \end{pmatrix}.$$

If we change the basis for S then the matrix above gets multiplied by an invertible $k \times k$ matrix. Then this $k \times k$ matrix acts on the $k \times k$ minors of M_S. Suppose the first minor K_1 is non-zero then choose the inverse of that minor as a base change so that M_S will have the form

$$\begin{pmatrix} 1 & & & b_{11} & \dots & b_{1n-k} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & 1 & b_{k1} & \dots & b_{kn-k} \end{pmatrix}.$$

This gives a correspondence between matrices M_S with first non-zero minor and $\mathbb{A}^{k(n-k)}$. Therefore, the image of p has dimension $k(n-k)$. ☐

Similarly, we can define flag varieties. Given a vector space V and **flag**

$$0 \subseteq V_1 \subseteq \dots \subseteq V_n \subseteq V$$

of vector subspaces of dimension V_i, the **flag variety** denoted by $F(V)$ is the set of flags on V.

9 Divisors on curves

Definition 9.1. A **Weil divisor** is a formal finite sum

$$D = \sum_i a_i Y_i, \qquad a_i \in \mathbb{Z},$$

of algebraic subvarieties of codimension one.

Definition 9.2. More generally, an **algebraic cycle** is a formal sum of codimension p subvarieties

$$C = \sum_i a_i Y_i \subseteq X, \qquad a_i \in \mathbb{Z}.$$

By integrating algebraic cycles, we get a map from the space of p-cycles into the cohomology of the variety. **Hodge conjecture** states that this defines a bijection.

Example 9.3. In case $\dim(X) = 1$, then a divisor is just a sum of points with multiplicity. If $K = \mathbb{C}$ and

$$f = \frac{(z-1)(z-2)}{(z-3)(z-4)},$$

then $(f) = \overline{1} + \overline{2} - \overline{3} - \overline{4}$.

Definition 9.4. Let $K = \mathbb{C}$ and $\dim(X) = 1$. Let $D, V \subseteq X$. Then **linear equivalence** is $D \sim V$ if and only if $D - V = (f)$ for $f \in K(x)$.

Definition 9.5. The **class group** is divisors modulo $\sim$.

Definition 9.6. Let $X \subseteq Y$ be a subvariety, then

$$\mathcal{O}_{X,Y} = \{(U, f) \mid f \text{ regular at } U,\ U \cap Y \neq \emptyset\}.$$

Let f be a rational function, then

$$(f) = \sum_Y v_Y(f) Y,$$

where v_Y is the valuation associated to $\mathcal{O}_{X,Y}$.

Lecture 23
Monday
04/03/19

Definition 9.7. Let X be a smooth projective curve.

- A **divisor** D is a formal sum $K_1 p_1 + \cdots + K_n p_n$ of points, where $K_i \in \mathbb{Z}$,
- We say a divisor D is **effective** if $K_i \geq 0$.
- Given two divisors D, E, $D \geq E$ if and only if $D - E \geq 0$.
- The **degree** of D, denoted $\deg(D)$, is the sum $\sum_{i=1}^n K_i$.

Remark 9.8. Degree gives a map $\deg : Div \to \mathbb{Z}$. The set of all divisors on X has a natural group structure given by addition, we denote this group by $Div(X)$.

Notation 9.9. The subgroup of **degree zero divisors** is denoted by $Div^0(X)$.

Definition 9.10.

- For a non-zero homogeneous polynomial $f \in S(X)$ the **divisor** of f is
$$(f) = div(f) = \sum_{a \in V_X(f)} mult_a(f) \cdot a \in Div(f).$$
By Bézout's theorem, $\deg(div(f)) = \deg(X)\deg(f)$.
- If $Y \subseteq \mathbb{P}^2$ not containing X, then the **intersection** of X and Y is
$$X \cdot Y = \sum_{a \in X \cap Y} mult_a(X, Y) \cdot a.$$

Example 9.11. Let $X = Z(xz - y^2)$ and $Y = Z(z)$ then $X \cap Y = \{[1 : 0 : 0]\}$, so
$$X \cdot Y = 2 \cdot [1 : 0 : 0].$$

Lemma 9.12. *$mult_a(fg) = mult_a(f) + mult_a(g)$ gives $div(fg) = div(f) + div(g)$.*

Proof. Recall that $mult_a(f) = len(\mathcal{O}_a / \langle f \rangle) = \dim_K(\mathcal{O}_a / \langle f \rangle)$. Thus there is a short exact sequence
$$0 \to \frac{\mathcal{O}_a}{\langle f \rangle} \xrightarrow{g} \frac{\mathcal{O}_a}{\langle fg \rangle} \to \frac{\mathcal{O}_a}{\langle g \rangle} \to 0.$$
□

Definition 9.13. Let $f \in K^*(X)$, then if $f = g/h$ we define $mult_a(f) = mult_a(g) - mult_a(h)$.

If we take a different representation of f, say $f = g'/h'$, then $g/h = g'/h'$ gives $gh' = hg'$, so
$$\begin{cases} mult_a(gh') = mult_a(g) + mult_a(h') \\ mult_a(hg') = mult_a(h) + mult_a(g') \end{cases} \implies mult_a(g) - mult_a(h) = mult_a(g') - mult_a(h').$$

Analogously, we have
$$div(f) = \sum_{a \in Z(g) \cup Z(h)} mult_a(f)\, a = div(g) \cdot div(h).$$

Example 9.14. Let $f = xy/(x-y)^2$ on $\mathbb{P}^1$. Then
$$div(f) = [1 : 0] + [0 : 1] - 2[1 : 1].$$

Remark 9.15. Note that $\deg(div(f))$ is always zero because
$$\deg(div(f)) = \deg(div(g)) - \deg(div(h)) = (\deg(X))(\deg(g)) - (\deg(X))(\deg(h)) = 0.$$

Definition 9.16. A divisor on X is called principal if it is of the form $div(f)$ for some $f \in K^*(X)$. We denote the subgroup of all principal divisors by $Prin(X)$.

Definition 9.17. The quotient
$$Pic(X) = \frac{Div(X)}{Prin(X)}$$
is called the **Picard group** of X. Restricting to degree zero divisors, we get
$$Pic^0(X) = \frac{Div^0(X)}{Prin(X)},$$
where $Div^0(X)$ are the divisors of degree zero.

By the degree map $\deg : Div(X) \to \mathbb{Z}$ we have
$$\frac{Pic(X)}{Pic^0(X)} \cong \frac{Div(X)}{Div^0(X)} \cong \mathbb{Z}.$$

Example 9.18. Every degree zero divisor is principal. Suppose

$$D = K_1 [a_{1,0} : a_{1,1}] + \dots + K_n [a_{n,0} : a_{n,1}], \qquad \sum_{i=1}^{n} K_i = 0,$$

then set

$$f [x_0 : x_1] = \prod_{i=1}^{n} (a_{i,1} x_0 - a_{i,0} x_1)^{K_i}.$$

So $Pic^0 (\mathbb{P}^1) = \{0\}$ so $Pic (\mathbb{P}^1) = \mathbb{Z}$.

Lecture 24 Tuesday 05/03/19

Lemma 9.19 (Nakayama lemma)**.** *If R is local with maximal ideal $\mathfrak{m}$ and M is finitely generated then $M = \mathfrak{m} M$ gives $M = 0$.*

Corollary 9.20. *If R is local with maximal ideal $\mathfrak{m}$ then $\langle t_1, \dots, t_n \rangle = \mathfrak{m}$ if and only if $\langle \overline{t_1}, \dots, \overline{t_n} \rangle = \mathfrak{m}/\mathfrak{m}^2$.*

Proof. Let $N = \langle t_1, \dots, t_n \rangle \subseteq \mathfrak{m}$. Suppose $\langle \overline{t_1}, \dots, \overline{t_m} \rangle = \mathfrak{m}/\mathfrak{m}^2$. Then

$$N + \mathfrak{m}^2 = \mathfrak{m} + \mathfrak{m}^2 \qquad \implies \qquad \frac{N + \mathfrak{m}^2}{N} = \frac{\mathfrak{m} + \mathfrak{m}^2}{N} \qquad \implies \qquad \mathfrak{m} \left(\frac{\mathfrak{m}}{N} \right) = \frac{\mathfrak{m}}{N} \qquad \implies \qquad \frac{\mathfrak{m}}{N} = 0,$$

so $\langle t_1, \dots, t_n \rangle = \mathfrak{m}$. $\square$

Lemma 9.21. *Let $X \subseteq \mathbb{P}^2$ be a smooth curve, and $I_a \subseteq \mathcal{O}_a$ be the maximal ideal of the local ring $\mathcal{O}_a$.*

1. *I_a is principal, so $I_a = \langle \phi_a \rangle$ with $mult_a (\phi_a) = 1$.*
2. *Any non-zero $\phi \in \mathcal{O}_a$ can be written as $c \phi_a^m$, where $m = mult_a (\phi)$.*

Proof.

1. Since $\mathcal{O}_a$ is regular, $\dim (\mathfrak{m}/\mathfrak{m}^2) = 1$, in particular $\mathfrak{m} \neq \mathfrak{m}^2$ and we can find $\phi_a \in \mathfrak{m} \setminus \mathfrak{m}^2$, so $\langle \phi_a \rangle = \mathfrak{m}$. Thus any ideal has to be of the form $\langle \phi_a^k \rangle$ since $\langle \phi_a \rangle$ is maximal.
2. Take $\phi \in \mathcal{O}_a$ non-zero then $\langle \phi \rangle = \langle \phi_a^m \rangle$ gives $\phi = c \phi_a^m$, so

$$mult (\phi) = mult (c \phi_a^m) = mult (c) + mult (\phi_a^m) = m \cdot mult (\phi_a) = m.$$

$\square$

Lemma 9.22. *Let $X \subseteq \mathbb{P}^2$ be a smooth curve, and $a \in X$.*

1. *If $f, g \in S$, of same degree with $mult (X, f) \geq m$ and $mult (X, g) \geq m$ then*
   - *$mult_a (X, \lambda f + \mu g) \geq m$, and*
   - *there exist λ, μ such that $mult_a (X, \lambda f + \mu g) \geq m + 1$.*
2. *Let $Y \subseteq \mathbb{P}^2$ be another curve and $m = mult_a (X, Y)$. If $f \in S$ with $mult_a (X, f) \geq m$ then $mult_a (Y, f) \geq m$.*

Proof.

1. Write $f = u \phi_a^m$ and $g = v \phi_a^m$, so for any λ, μ we have $\lambda f + \mu g = (\lambda u + \mu v) \phi_a^m$ so $mult_a (\lambda f + \mu g) \geq m$, and we can find λ', μ' such that $\lambda' u + \mu' v = 0$ at a, so that $mult_a (\lambda f + \mu g) \geq m + 1$.
2. Let $I (X) = \langle g \rangle$, $I (Y) = \langle h \rangle$, and $k = mult_a (X, f) \geq m = mult_a (X, h)$. $f = u \phi_a^k$ and $h = v \phi_a^m$, so $\langle f \rangle \subset \langle h \rangle$. $\langle f, g \rangle \subset \langle g, h \rangle$, and we also have $\langle f, h \rangle \subset \langle g, h \rangle$, so $mult_a (f, h) \geq mult_a (g, h)$ gives $mult_a (Y, f) \geq mult_a (X, Y) = m$.

$\square$

Lemma 9.23. *Let $X \subset \mathbb{P}^2$ be smooth and $g, h \in S(X)$.*

1. *If $div(g) = div(h)$ then g, h are linearly dependent on $S(X)$.*
2. *If h is linear and $div(g) \geq div(h)$ then $h \mid g$ in $S(X)$.*

Proof.

1. By Bézout's theorem, $\deg(X) \cdot \deg(g) = \deg(X) \cdot \deg(h)$, so $\deg(g) = \deg(h)$. We know by the previous lemma that $mult_a(\lambda g + \mu h) \geq m_a$, and we can find $b \in X$ such that $mult_b(\lambda g + \mu h) \geq m_b + 1$. Summing up, we have
$$\sum_{a \in X} mult_a(\lambda g + \mu h) \geq d \deg(X) + 1,$$
but $\lambda g + \mu h$ has degree d, so this is a contradiction unless $\lambda g + \mu h = 0$, that is g, h are linearly dependent.
2. Exercise.

$\square$

Proposition 9.24. *Let $X \subseteq \mathbb{P}^2$ be a smooth cubic. Then for all distinct $a, b \in X$ we have $a - b \neq 0$, so there is no rational function ϕ such that $div(\phi) = a - b$.*

Proof. Assume that the result is false. Then there are $f, g \in S(X)$ of degree d such that

- there are points $a_1, \dots, a_{3d-1}$ and $a \neq b$ on X such that
$$div(g) = a_1 + \dots + a_{3d-1} + a, \qquad div(f) = a_1 + \dots + a_{3d-1} + b,$$
- among $a_1, \dots, a_{3d-1}$, there are at least $2d$ distinct points, since we can multiply f and g by a linear polynomial with distinct roots so the degree increases by one but the number of distinct points increases by three.

Pick a minimal d. If $d = 1$ then
$$div(g) = a_1 + a_2 + a, \qquad div(f) = a_1 + a_2 + b,$$
so $a = b = \Psi(a_1, a_2)$, a contradiction. So $d > 1$. Consider $(\lambda f + \mu g)$, so
$$div(\lambda f + \mu g) \geq a_1 + \dots + a_{3d-1}.$$
We can choose λ, μ such that
$$div(\lambda f + \mu g) \geq a_1 + \dots + a_{3d-1} + c,$$
for any given c. By Bézout's theorem,
$$div(\lambda f + \mu g) = a_1 + \dots + a_{3d-1} + c.$$
So we can choose a and b, so that $a = \Psi(a_1, a_2)$ and $b = \Psi(a_1, a_3)$.
$$div(f) = (a_1 + a_2 + \Psi(a_1, a_2)) + a_3 + \dots + a_{3d-1}, \qquad div(g) = (a_1 + a_3 + \Psi(a_1, a_3)) + a_2 + \dots + a_{3d-1}.$$
Set l, l' linear polynomials such that
$$div(l) = a_1 + a_2 + \Psi(a_1, a_2), \qquad div(l) = a_1 + a_3 + \Psi(a_1, a_3).$$
The quotient by $div(l)$ and $div(l')$ gives a polynomial whose divisor is
$$a_4 + \dots + a_{3d-1} + a_3, \qquad a_4 + \dots + a_{3d-1} + a_2,$$
but since we chose d minimum this gives a contradiction. $\square$

10 Elliptic curves

Definition 10.1. An **abelian variety** A is a smooth connected projective variety which has a group structure such that addition and taking inverse are regular functions. Lecture 25 Friday 08/03/19

Definition 10.2. An **elliptic curve** is a one-dimensional abelian variety.

Proposition 10.3. *Let X be an elliptic curve in $\mathbb{P}^2$, and fix $a_0 \in X$, then there is a bijection*

$$\begin{array}{cccccc} \Phi & : & X & \longrightarrow & Pic^0(X) = \dfrac{Div^0(X)}{Prin(X)} & . \\ & & a & \longmapsto & a - a_0 & \end{array}$$

Proof.

- Φ is injective by the last proposition.
- Φ is surjective. Suppose

  $$D = a_1 + \dots + a_m - b_1 - \dots - b_m.$$

  Consider the function l with $div(l) = a_1 + a_2 + \Psi(a_1, a_2)$, so

  $$D = D + div(l) = div(l) - \Psi(a_1, a_2) + a_3 + \dots.$$

  So we can assume $D = a_1 - b_1 = \Phi(\cdot)$.

  $$a_0 + a_1 + \Psi(a_0, a_1) - b_1 - \Psi(a_0, a_1) - \Psi(b_1, \Psi(a_0, a_1)) = 0,$$

  so

  $$D = a_1 - b_1 = \Psi(b_1, \Psi(a_0, a_1)) - a_0 = \Phi(\Psi(b_1, \Psi(a_0, a_1))).$$

  Thus Φ is surjective.

□

We know that $X \cong Pic^0(X)$, which is a group. But what is the expression for $g_1 + g_2$ for $g_1, g_2 \in X$? $\Phi(g_1 + g_2) = \Phi(g_1) + \Phi(g_2)$, so

$$\begin{aligned} g_1 + g_2 &= \Phi^{-1}(\Phi(g_1) + \Phi(g_2)) = \Phi^{-1}(g_1 - g_0 + g_2 - g_0) = \Phi^{-1}(\Psi(g_0, \Psi(g_1, g_2)) - g_0) \\ &= \Phi^{-1}(\Phi(\Psi(g_0, \Psi(g_1, g_2)))) = \left(\Phi^{-1} \circ \Phi\right)(\Psi(g_0, \Psi(g_1, g_2))) = \Psi(g_0, \Psi(g_1, g_2)). \end{aligned}$$

Let X be an elliptic curve and $a_0 \in X$, can consider the group law based on a_0. Then for $n \in \mathbb{Z}$, we can define $n \times a = a + \dots + a$.

Note. Given $a, b \in X$, the problem of finding whether or not there exists n such that $a = n \times b$ is extremely hard.

Lecture 26 is a problem class. Lecture 26 Monday 11/03/19
Lecture 27 is a problem class. Lecture 27 Tuesday 12/03/19
Lecture 28 is a problem class. Lecture 28 Friday 15/03/19
Lecture 29 is a problem class. Lecture 29 Monday 18/03/19
What's next in Lecture 30? Lecture 30 Tuesday 19/03/19