M4P54 Differential Topology

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Syllabus

Contents

0 Introduction

Lecture 1 Thursday 09/01/20

Differential topology is the study of the topology of a manifold using analysis. The topics are

- a review of differential forms,
- de Rham cohomology,
- Morse theory, and
- singular homology.

The following are references.

- J M Lee, Introduction to smooth manifolds, 2000
- L W Tu, Introduction to smooth manifolds, 2008
- J Milnor, Morse theory, 1960
- A Banyaga and D Hurtubise, Lectures on Morse homology, 2004

1 Differential forms on manifolds

1.1 Alternating p-forms on a vector space

Let V be a vector space over $\mathbb{R}$, and let $p \geq 0$. Then $V^p = V \times \cdots \times V$.

Definition 1.1. A multilinear map $\omega : V^p \to \mathbb{R}$ is called an **alternating p-form** if we have

$$\omega\left(v_{\sigma(1)}, \dots, v_{\sigma(p)}\right) = \epsilon(\sigma)\,\omega(v_1, \dots, v_p), \qquad v_1, \dots, v_p \in V \qquad \sigma \in \mathcal{S}_p,$$

where $\mathcal{S}_p$ is the group of permutations of p elements and $\epsilon(\sigma)$ is the signature of σ.

Recall that if m is the number of transpositions in a decomposition of σ, then $\epsilon(\sigma) = (-1)^m$, where a **transposition** is $(a_i a_j)$ for $a_i \neq a_j$.

Notation 1.2. $\Lambda^p V^* = \{\text{alternating } p\text{-forms } \omega \text{ on } V\}$ is called the **p-th exterior power** of V.

Check that it is a vector space. [1]

Example 1.3.

- $\Lambda^0 V^* = \mathbb{R}$.
- $\Lambda^1 V^* = V^* = \operatorname{Hom}(V, \mathbb{R})$, the **dual** of V.

Definition 1.4. Let $\omega_1 \in \Lambda^p V^*$ and $\omega_2 \in \Lambda^q V^*$. We define the **exterior product** $\omega_1 \wedge \omega_2 \in \Lambda^{p+q} V^*$ of ω_1 and ω_2 by

$$\omega_1 \wedge \omega_2(v_1, \dots, v_{p+q}) = \sum_{\sigma \in \mathcal{S}_{p,q}} \epsilon(\sigma)\,\omega_1\left(v_{\sigma(1)}, \dots, v_{\sigma(p)}\right)\omega_2\left(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}\right), \qquad v_1, \dots, v_{p+q} \in V,$$

where

$$\mathcal{S}_{p,q} = \{\sigma \in \mathcal{S}_{p+q} \mid \sigma(1) < \cdots < \sigma(p),\ \sigma(p+1) < \cdots < \sigma(p+q)\}.$$

Example 1.5.

- Assume $\omega_1, \omega_2 \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \omega_2(v_1, v_2) = \omega_1(v_1)\,\omega_2(v_2) - \omega_1(v_2)\,\omega_2(v_1), \qquad v_1, v_2 \in V.$$

- Assume $\omega_1, \dots, \omega_p \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \cdots \wedge \omega_p(v_1, \dots, v_p) = \det\left(\omega_i(v_j)\right)_{i,j=1,\dots,p}, \qquad v_1, \dots, v_p \in V.$$

Proposition 1.6. *Let $\omega_i \in \Lambda^{p_i} V^*$ for $i = 1, 2, 3$.*

- *Associativity* $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$.
- *Distributivity* $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$, *assuming* $p_2 = p_3$.
- *Supercommutativity* $\omega_1 \wedge \omega_2 = (-1)^{p_1 \cdot p_2}\,\omega_2 \wedge \omega_1$.

Definition 1.7. Let $\Phi : V \to W$ be a linear map between vector spaces over $\mathbb{R}$. Let $\omega \in \Lambda^p W^*$. Then the **pull-back** $\Phi^* \omega \in \Lambda^p V^*$ of ω is an alternating p-form on V defined by

$$\Phi^* \omega(v_1, \dots, v_p) = \omega(\Phi(v_1), \dots, \Phi(v_p)), \qquad v_1, \dots, v_p \in V.$$

[1] Exercise

Proposition 1.8. *Given $\Phi : V \to W$ a linear map,*

- *the pull-back*
$$\begin{array}{cccc} \Phi^* & : & \Lambda^p W^* & \longrightarrow & \Lambda^p V^* \\ & & \omega & \longmapsto & \Phi^* \omega \end{array}$$
is a linear map that preserves exterior products, that is
$$\Phi^* \left(\omega_1 \wedge \omega_2\right) = \Phi^* \omega_1 \wedge \Phi^* \omega_2, \qquad \omega_1 \in \Lambda^p W^*, \qquad \omega_2 \in \Lambda^q W^*,$$
- *if $\Psi : W \to Z$ is linear then*
$$\left(\Psi \circ \Phi\right)^* \omega = \Phi^* \Psi^* \omega, \qquad \omega \in \Lambda^p Z^*,$$
- *assuming $V = W$ and $p = \dim V$, then*
$$\Phi^* \omega = \left(\det \Phi\right) \omega, \qquad \omega \in \Lambda^p V^*.$$

1.2 Differential forms on manifolds

Let M be a smooth manifold of dimension n, and let $x \in M$. Then the tangent space $\mathrm{T}_x M$ of M at x is a vector space of dimension n.

Notation 1.9. Let
$$\Lambda^p \mathrm{T}_x^* M = \Lambda^p \left(\mathrm{T}_x M\right)^*.$$
Consider the set
$$\Lambda^p \mathrm{T}^* M = \bigsqcup_{x \in M} \Lambda^p \mathrm{T}_x^* M,$$
the **p-th exterior bundle** on M. There exists a morphism $\pi : \Lambda^p \mathrm{T}^* M \to M$ such that for all $x \in M$, $\pi^{-1}(x) = \Lambda^p \mathrm{T}_x^* M$, so $\Lambda^p \mathrm{T}^* M$ is a vector bundle and it is a smooth manifold, and π is a smooth morphism.

Example 1.10.

- $\Lambda^0 \mathrm{T}^* M = M \times \mathbb{R}$.
- $\Lambda^1 \mathrm{T}^* M$ is the **cotangent bundle**, the dual of the tangent bundle.

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Definition 1.11. A **differential p-form** ω on M is a smooth section of π, that is it is a smooth morphism $\omega : M \to \Lambda^p \mathrm{T}^* M$ such that $\pi \circ \omega = \mathrm{id}_M$.

Thus, $\omega(x) \in \Lambda^p \mathrm{T}_x^* M$.

Notation 1.12.
$$\Omega^p(M) = \left\{\text{differential } p\text{-forms } \omega \text{ on } M\right\}, \qquad \Omega^\bullet(M) = \bigoplus_p \Omega^p(M).$$

Example 1.13.
$$\Omega^0(M) \cong \left\{f : M \to \mathbb{R} \ \mathrm{C}^\infty\text{-function}\right\}.$$

Exercise. If $n = \dim M$, then $\Omega^{n+1}(M) = 0$.

The algebra is the same as last week.

Definition 1.14. Let $\omega_1 \in \Omega^p(M)$ and $\omega_2 \in \Omega^q(M)$. Then $\omega_1 \wedge \omega_2 \in \Omega^{p+q}(M)$ is defined by
$$\omega_1 \wedge \omega_2(x) = \omega_1(x) \wedge \omega_2(x) \in \Lambda^{p+q} \mathrm{T}_x^* M, \qquad x \in M.$$

By Proposition 1.6, associativity, distributivity, and supercommutativity hold for $\Omega^p(M)$. Let $F : M \to N$ be a smooth morphism between manifolds. Then for all $x \in M$, the differential of F at x is the linear map
$$\mathrm{D}F_x : \mathrm{T}_x M \to \mathrm{T}_{F(x)} N.$$

Thus, for all $p \geq 0$, we have a natural map, called the **pull-back**,

$$\begin{array}{cccc} F_x^* & : & \Lambda^p \mathrm{T}^*_{F(x)} N & \longrightarrow & \Lambda^p \mathrm{T}^*_x M \\ & & \omega\left(v_1, \ldots, v_p\right) & \longmapsto & \omega\left(\mathrm{D}F_x\left(v_1\right), \ldots, \mathrm{D}F_x\left(v_p\right)\right) \end{array}, \qquad \omega \in \Lambda^p \mathrm{T}^*_{F(x)} N, \qquad v_1, \ldots, v_p \in \mathrm{T}^*_x M.$$

Thus, we can define

$$\begin{array}{cccc} F^* & : & \Omega^p\left(N\right) & \longrightarrow & \Omega^p\left(M\right) \\ & & \omega\left(x\right) & \longmapsto & F^*\omega\left(F\left(x\right)\right) \end{array}, \qquad \omega \in \Omega^p\left(N\right).$$

By Proposition 1.8, the pull-back preserves the exterior product, so

$$F^*\left(\omega_1 \wedge \omega_2\right) = F^*\omega_1 \wedge F^*\omega_2.$$

If $G : N \to P$,

$$\left(G \circ F\right)^* \omega = F^* G^* \omega.$$

1.3 Local description of p-forms

Let M be a manifold of dimension n, let $x_0 \in M$, let (U, ϕ) be a local chart around x_0, and let $(x_1, \ldots, x_n)$ be local coordinates around x_0. A basis of $\mathrm{T}_{x_0} M$ is given by

$$\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}.$$

A basis of $\mathrm{T}^*_{x_0} M$ is given by

$$\left\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\right\}, \qquad \mathrm{d}x_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}.$$

A basis of $\Lambda^p \mathrm{T}^*_{x_0} M$ is

$$\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}, \qquad i_1 < \cdots < i_p.$$

Thus, $\omega \in \Omega^p\left(M\right)$ is locally given by

$$\omega\left(x\right) = \sum_{|I|=p} f_I\left(x\right) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}, \qquad I = \left(i_1, \ldots, i_p\right), \qquad i_1 < \cdots < i_p,$$

where f_I is a C^∞-function on U for all I.

Example 1.15. Let $F : M \to N$ be a smooth morphism between manifolds of dimension n, and let $\omega \in \Omega^n\left(N\right)$. Locally,

$$\omega\left(y\right) = f\left(y\right) \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n, \qquad y \in N,$$

for some $f \in \mathrm{C}^\infty$. By Proposition 1.8,

$$F^*\omega\left(x\right) = \left(f \circ F\right)\left(x\right) \det \mathrm{D}F_x \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n, \qquad x \in M,$$

where $y_i = \mathrm{p}_i \circ F$ and $\mathrm{p}_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th projection.

Let $f : M \to \mathbb{R}$ be a smooth function, so $f \in \Omega^0\left(M\right)$. Locally, the **differential** is

$$\begin{array}{cccc} \mathrm{d} & : & \Omega^0\left(M\right) & \longrightarrow & \Omega^1\left(M\right) \\ & & f & \longmapsto & \sum_{i=1}^n \frac{\partial f}{\partial x_i} \mathrm{d}x_i \end{array}.$$

Check that $\mathrm{d}f \in \Omega^1\left(M\right)$, so $\mathrm{d}f$ is a 1-form on M. Alternatively, $\mathrm{d}f = f^*\mathrm{d}x$ for $\mathrm{d}x$ a 1-form on $\mathbb{R}$, or $\mathrm{d}f\left(X\right) = X\left(f\right)$ for any vector field X on M. More in general, let $\omega \in \Omega^p\left(M\right)$. Locally,

$$\omega = \sum_{|I|=p} f_I \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}, \qquad f_I \in \mathrm{C}^\infty,$$

so $\mathrm{d}\omega \in \Omega^{p+1}\left(M\right)$. Then the **de Rham differential** is

$$\begin{array}{cccc} \mathrm{d} & : & \Omega^p\left(M\right) & \longrightarrow & \Omega^{p+1}\left(M\right) \\ & & \omega & \longmapsto & \sum_{|I|=p} \mathrm{d}f_I \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} \end{array}.$$

Proposition 1.16.

- *The Leibniz rule*
$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \mathrm{d}\omega_2, \qquad w_1 \in \Omega^p(M), \qquad \omega_2 \in \Omega^q(M).$$
- $\mathrm{d}^2 = 0$, *that is*
$$\mathrm{d}(\mathrm{d}\omega) = 0, \qquad \omega \in \Omega^p(M).$$
- *Let $F : M \to N$ be a smooth morphism between manifolds. Then*
$$F^* \mathrm{d}\omega = \mathrm{d}(F^*\omega), \qquad \omega \in \Omega^p(M),$$
so
$$\begin{tikzcd} \Omega^p(M) \arrow[r, "\mathrm{d}"] & \Omega^{p+1}(M) \\ \Omega^p(N) \arrow[u, "F^*"] \arrow[r, "\mathrm{d}"'] & \Omega^{p+1}(N) \arrow[u, "F^*"'] \end{tikzcd}.$$

Definition 1.17. Lecture 3 Tuesday 14/01/20

- $\omega \in \Omega^p(M)$ is **closed** if $\mathrm{d}\omega = 0$.
- $\omega \in \Omega^p(M)$ is **exact** if there exists $\omega' \in \Omega^{p-1}(M)$ such that $\mathrm{d}\omega' = \omega$.

ω is exact implies that ω is closed, since if $\omega = \mathrm{d}\omega'$ then $\mathrm{d}\omega = \mathrm{d}^2\omega' = 0$.

1.4 Integrations on manifolds

Let M be a manifold of dimension n, let $F : M \to M$ be a smooth morphism, and let $\omega \in \Omega^n(M)$. Then
$$F^*\omega(x) = \det \mathrm{D}F_x \omega(F(x)).$$
Locally, assume $\omega = f \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n$ for some coordinates $(y_1, \dots, y_n)$ and $f \in \mathrm{C}^\infty$. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of M, where $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$. Then
$$h_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \to \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n,$$
such that
$$h_{\alpha\beta}^*\omega(x) = (f \circ h_{\alpha\beta})(x) \det(\mathrm{D}h_{\alpha\beta})_x \, \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$
Let $D \subset \mathbb{R}^n$ be compact such that ∂D has zero measure, so D is a domain of integration, let $f : U \to \mathbb{R}$ be a C^∞-function where $U \subset \mathbb{R}^n$ is open such that $D \subset U$, and let $h : U \to h(U)$ be a diffeomorphism. Then
$$\int_{h^{-1}(D)} f(y) \, \mathrm{d}y_1 \dots \mathrm{d}y_n = \int_{h^{-1}(D)} f(y) \, \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n = \int_D (f \circ h)(x) |\det \mathrm{D}h_x| \, \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$
Let us assume that $\omega = f(y) \, \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n$ on U. We define
$$\int_D \omega = \int_D f(y) \, \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n, \qquad D \subset U.$$
Definition 1.18. Let $U \subset \mathbb{R}^n$ be an open set. We define the **support** of ω as
$$\operatorname{supp} \omega = \overline{\{x \in U \mid \omega(x) \neq 0\}}, \qquad \omega(x) \in \Lambda^p \mathrm{T}_x^* U.$$
Then ω has **compact support**, if $\operatorname{supp} \omega$ is compact.

Fact. Under this assumption, we can define
$$\int_U \omega = \int_D \omega \in \mathbb{R},$$
which is well-defined. Under the same assumption, if $\phi : V \to U$ is a diffeomorphism, provided that $\det \mathrm{D}\phi_x > 0$, since $\det \mathrm{D}\phi_x \neq 0$ for all x, then
$$\int_U \omega = \int_V \phi^* \omega.$$

1.5 Orientation

Let V be a vector space over $\mathbb{R}$ of dimension n, and let $B = (b_1, \dots, b_n) \subset V$ and $B' = (b'_1, \dots, b'_n) \subset V$ be ordered bases of V. Then B and B' have the **same orientation** if $\det T > 0$ where

$$\begin{array}{cccc} T & : & V & \longrightarrow & V \\ & & b_i & \longmapsto & b'_i \end{array}$$

is a linear map. Let $\omega \in \Lambda^n V^*$ for $\omega \neq 0$. Then B and B' have the same orientation if and only if $\omega(b_1, \dots, b_n)$ has the same sign as $\omega(b'_1, \dots, b'_n)$, by Proposition 1.8. An **orientation** Λ of V is a set of all the ordered basis of V with the same orientation. Let $\phi : V \to W$ be an isomorphism of vector spaces with fixed orientations Λ_v and Λ_w respectively. We say that ϕ is **orientation preserving** if an ordered basis of V induces an ordered basis of W, so Λ_v induces Λ_w. Let $V = \mathbb{R}^n$, and let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. Then $e_1, \dots, e_n$ defines an orientation of V called **positive**. Let M be a manifold. The idea is to find an orientation Λ_x of $\mathrm{T}_x M$ for all $x \in M$.

Special case. Let $M = U \subset \mathbb{R}^n$ be open. There exists a natural isomorphism $\phi_x : \mathrm{T}_x U \to \mathbb{R}^n$. Let Λ_x^+ be an orientation on $\mathrm{T}_x U$ such that ϕ_x is orientation preserving with respect to the positive orientation on $\mathbb{R}^n$. Let $\Lambda^+ = \{\Lambda_x^+\}$.

General case. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M. On U_α, we define the orientation so that $(\mathrm{D}\phi_\alpha)_x : \mathrm{T}_x U_\alpha \to \mathrm{T}_{\phi_\alpha(x)} \phi_\alpha(U) \subset \mathbb{R}^n$ is orientation preserving. This is called the positive orientation on the chart (U_α, ϕ_α). We define Λ on M, which is a collection of Λ^+ on $\mathrm{T}_x M$ for all $x \in M$. Then M is **orientable** if there exists an atlas with positive orientation charts. This coincides in assuming that $\det \mathrm{D}\left(\phi_\beta^{-1} \circ \phi_\alpha\right) > 0$ for all α and β.

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Notation 1.19. For all $p \geq 0$,

$$\Omega_c^p(M) = \{\omega \in \Omega^p(M) \mid \operatorname{supp} M \text{ is compact}\}.$$

If M is compact $\Omega_c^p(M) = \Omega^p(M)$. Let $\omega \in \Omega_c^r(M)$. Assume $\operatorname{supp} \omega \subset U$ where (U, ϕ) is a chart of M, and $\phi : U \to \phi(U) \subset \mathbb{R}^n$. Assume also that (U, ϕ) is positively oriented. Let $\phi^{-1} : \phi(U) \to U$ such that $\left(\phi^{-1}\right)^* \omega \in \Omega_c^n(\phi(U))$, that is $\operatorname{supp}\left(\phi^{-1}\right)^* \omega \subset \phi(U)$. We define

$$\int_M \omega = \int_{\phi(U)} \left(\phi^{-1}\right)^* \omega. \tag{1}$$

We need to show that, under the assumptions above, $\int_M \omega$ does not depend on (U, ϕ). Let $\left(\overline{U}, \overline{\phi}\right)$ be also a positively oriented chart such that $\operatorname{supp} \omega \subset \overline{U}$. We want to show that

$$\int_{\phi(U)} \left(\phi^{-1}\right)^* \omega = \int_{\overline{\phi}\left(\overline{U}\right)} \left(\overline{\phi}^{-1}\right)^* \omega.$$

Let $\overline{\phi} \circ \phi^{-1} : \phi\left(U \cap \overline{U}\right) \to \overline{\phi}\left(U \cap \overline{U}\right)$, so

$$\begin{tikzcd} & U \cap \overline{U} \arrow[dl, "\phi"'] \arrow[dr, "\overline{\phi}"] & \\ \mathbb{R}^n \supset \phi\left(U \cap \overline{U}\right) \arrow[rr, "\overline{\phi} \circ \phi^{-1}"'] & & \overline{\phi}\left(U \cap \overline{U}\right) \subset \mathbb{R}^n \end{tikzcd}.$$

Since both charts are positively oriented the determinant of the differential $\mathrm{D}\left(\overline{\phi} \circ \phi^{-1}\right)$ is positive, so

$$\begin{aligned} \int_{\overline{\phi}(U)} \left(\overline{\phi}^{-1}\right)^* \omega &= \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\overline{\phi}^{-1}\right)^* \omega = \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\overline{\phi} \circ \phi^{-1}\right)^* \left(\overline{\phi}^{-1}\right)^* \omega = \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\phi^{-1}\right)^* \overline{\phi}^* \left(\overline{\phi}^{-1}\right)^* \omega \\ &= \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\phi^{-1}\right)^* \left(\overline{\phi}^{-1} \circ \overline{\phi}\right)^* \omega = \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\phi^{-1}\right)^* \omega = \int_{\overline{\phi}(U)} \left(\phi^{-1}\right)^* \omega, \end{aligned}$$

by a property of the pull-back and since $\left(\overline{\phi}^{-1}\right)^* \omega = 0$ outside $\overline{\phi}\left(U \cap \overline{U}\right)$.

1.6 Partitions of unity

Definition 1.20. Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering. A **partition of unity** with respect to U is a collection of smooth functions $f_\alpha : M \to [0,1]$ such that

1. $\operatorname{supp} f_\alpha = \overline{\{x \in M \mid f_\alpha(x) = 0\}} \subset U_\alpha$ for all α,
2. $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$, and
3. for all $x \in M$, there exists $U \ni x$ open such that $\operatorname{supp} f_\alpha \cap U \neq \emptyset$ for only finitely many α.

Remark. 3 implies that 2 is a finite sum.

Example 1.21. Let

$$M = \mathrm{S}^1 = \left\{x \in \mathbb{R}^2 \;\middle|\; |x| = 1\right\}, \qquad U_1 = \mathrm{S}^1 \setminus \{(1,0)\}, \qquad U_2 = \mathrm{S}^1 \setminus \{(-1,0)\},$$

so $\{U_i\}$ is a cover. Let

$$f_1(\cos\theta, \sin\theta) = \frac{1}{2} - \frac{1}{2}\cos\theta, \qquad f_2(\cos\theta, \sin\theta) = \frac{1}{2} + \frac{1}{2}\cos\theta.$$

Then f_i is a partition of unity.

Proposition 1.22. *Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering of M. Then there exists a partition of unity f_α with respect to U.*

Proof. We omit the proof. □

Proposition 1.23. *Let M be a manifold, and let $n = \dim M$. Then M is orientable if and only if there exists $\omega \in \Omega^n(M)$ which is never vanishing on M, so $\omega(x) \neq 0$ for all $x \in M$.*

ω is called a **volume form** on M.

Proof.

$\Longleftarrow$ Assume $\omega \in \Omega^n(M)$ is a volume form. We want to construct an orientation Λ on M, that is Λ_x on $\mathrm{T}_x M$ for all $x \in M$. Given an oriented basis $v_1, \dots, v_n$ of $\mathrm{T}_x M$ we say that it is **positively oriented** if $\omega(x)(v_1, \dots, v_n) > 0$. For all $x \in M$, we define the orientation Λ_x on $\mathrm{T}_x M$ by considering the class of positively oriented ordered basis of $\mathrm{T}_x M$ which is compatible with the choice of an atlas on M. Take any atlas $\{(U_\alpha, \phi_\alpha)\}$, where $\phi_\alpha : U_\alpha \to \mathbb{R}^n$. On U_α,

$$\omega = g_\alpha \phi_\alpha^* \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$

Since $\omega \neq 0$, $g_\alpha > 0$ or $g_\alpha < 0$. If $g_\alpha < 0$ then switch x_1 with x_2, so $g_\alpha > 0$. After this change of coordinates, (U_α, ϕ_α) is positively oriented, so M is orientable. Lecture 5 Monday 20/01/20

$\Longrightarrow$ Assume that M is orientable, that is there exists an atlas $\{(U_\alpha, \phi_\alpha)\}$ of positively oriented charts. On U_α, we consider

$$\omega_\alpha = \phi_\alpha^* \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$

Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Let $\widetilde{\omega_\alpha} = f_\alpha \omega_\alpha \in \Omega^n(U_\alpha)$. We may assume that $\widetilde{\omega_\alpha} \in \Omega^n(M)$ by extending equal to zero outside U_α. We define $\omega = \sum_\alpha \widetilde{\omega_\alpha} \in \Omega^n(M)$. For all α, since $\sum_\alpha f_\alpha = 1$ there exists α such that $\widetilde{\omega_\alpha} \neq 0$, so $\omega \neq 0$.

□

Let M be an orientable manifold of dimension n, and let $\omega \in \Omega_{\mathrm{c}}^n(M)$. We want to define $\int_M \omega$. So far we defined for ω such that $\operatorname{supp} \omega \subset U_\alpha$ where (U_α, ϕ_α) is a chart.

Definition 1.24. Let $\{(U_\alpha, \phi_\alpha)\}$ be a positively oriented atlas on M, and let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $\operatorname{supp} f_\alpha \omega \subset U_\alpha$, so let

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega.$$

Remark 1.25. Note that for each α, we have that the support of $f_\alpha \omega$ is contained in U_α and therefore each term of the sum is well-defined as in (1). Indeed, we have

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} \left(\phi^{-1}\right)^* f_\alpha.$$

Lemma 1.26. *$\int_M \omega$ does not depend on $\{(U_\alpha, \phi_\alpha)\}$ and f_α.*

Proof. Under the assumption that $\operatorname{supp} \omega \subset U_\alpha$ then we showed $\int_{U_\alpha} \omega$ does not depend on (U_α, ϕ_α). Let $\{(U_\alpha, \phi_\alpha)\}$ and $\left\{\left(\overline{U_\alpha}, \overline{\phi_\alpha}\right)\right\}$ be two atlases with positively oriented charts, and let f_α and $\overline{f_\alpha}$ be two partitions of unity with respect to $\{U_\alpha\}$ and $\left\{\overline{U_\alpha}\right\}$ respectively. Then $\sum_\alpha f_\alpha = \sum_\alpha \overline{f_\alpha} = 1$, so $\int_M f_\alpha \omega = \sum_\beta \int_M \overline{f_\beta} f_\alpha \omega$. Thus

$$\int_M \omega = \sum_\alpha \int_M f_\alpha \omega = \sum_{\alpha,\beta} \int_M \overline{f_\beta} f_\alpha \omega = \sum_\beta \int_M \sum_\alpha f_\alpha \overline{f_\beta} \omega = \sum_\beta \int_M \overline{f_\beta} \omega.$$

□

Proposition 1.27. *Let M and N be orientable manifolds of dimension n, and let $\omega, \eta \in \Omega^n_c(M)$.*

1. *Linearity*

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

2. *Orientation reversal. Let $\overline{M}$ be the manifold M with opposite orientation $\Lambda^- = \{\Lambda^-_x \mid x \in M\}$, which is the orientation opposite than the one induced by M with orientation Λ. Then*

$$\int_M \omega = -\int_{\overline{M}} \omega.$$

3. *Positivity. Let ω be the volume form on M. Then*

$$\int_M \omega > 0.$$

4. *Diffeomorphism invariance. Let $F : N \to M$ be an orientation preserving diffeomorphism. Then*

$$\int_M \omega = \int_N F^* \omega.$$

Proof.

1. Exercise. [2]

2. Exercise. [3]

3. Choose a positively oriented chart (U_α, ϕ_α) on U_α, so

$$\omega = g_\alpha \phi_\alpha^* \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n, \qquad g_\alpha > 0.$$

   Then $\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega$ where f_α is a partition of unity. For all $x \in M$ there exists α such that $x \in U_\alpha$ and $\int_{U_\alpha} f_\alpha \omega > 0$, so $\int_M \omega > 0$.

4. Let (U_α, ϕ_α) be a positively oriented atlas on M. Then $\left(F^{-1}(U_\alpha), \phi_\alpha \circ F\right)$ is an atlas on N which is positively oriented. Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $f_\alpha \circ F$ is a partition of the unity with respect to $\left\{F^{-1}(U_\alpha)\right\}$, so

$$\int_N F^* \omega = \sum_\alpha \int_N (f_\alpha \circ F) F^* \omega = \sum_\alpha \int_N F^*(f_\alpha \omega) = \sum_\alpha \int_M f_\alpha \omega = \int_M \omega.$$

□

[2] Exercise
[3] Exercise

1.7 Manifolds with boundary

Denote
$$\mathbb{R}^n_{\geq 0} = (\mathbb{R}_{\geq 0})^n, \qquad \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$
Let $U \subset \mathbb{R}^n_+$ be open, and let $F : U \to \mathbb{R}^m$ be a function. Then F is C^∞ if it can be extended to a C^∞-function $\widetilde{F} : \widetilde{U} \to \mathbb{R}^m$ where $\widetilde{U} \supset U$ and $\widetilde{U}$ is open.

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Definition 1.28. A **manifold with boundary** of dimension n is a Hausdorff topological space M such that there exists an open covering $\{U_\alpha\}$, and for all α, there exists a homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ such that for all α and β,
$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+ \to \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+$$
is a diffeomorphism, so
$$\begin{tikzcd} & U_\alpha \cap U_\beta \arrow[dl, "\phi_\alpha"'] \arrow[dr, "\phi_\beta"] & \\ \mathbb{R}^n_+ \supset \phi_\alpha(U_\alpha \cap U_\beta) \arrow[rr, "\phi_\alpha \circ \phi_\beta^{-1}"'] & & \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+ \end{tikzcd}.$$
The **boundary** of M is
$$\partial M = \left\{x \in M \;\middle|\; \exists \alpha,\ \phi_\alpha(x) \in \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}\right\}.$$
Then (U_α, ϕ_α) is called a **chart** and $\{(U_\alpha, \phi_\alpha)\}$ is called an **atlas**.

Remark 1.29.

- ∂M is closed in M.
- $\mathring{M} = M \setminus \partial M$ is a manifold of dimension n.

Example 1.30.

- $M = [0, 1]$ is a manifold with boundary $\partial M = \{0, 1\}$.
- The closed disc $\mathrm{D} = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is a manifold with boundary $\partial \mathrm{D} = \mathrm{S}^{n-1}$.
- $M = [0, 1] \times \mathrm{S}^1$ is a manifold with boundary $\partial M = \mathrm{S}^1 \sqcup \mathrm{S}^1$.

Remark 1.31.

- We can define tangent spaces and differential forms exactly in the same way as usual manifolds.
- The definition of orientability is the same. If M is orientable, then ∂M is also orientable. As a convention, the positive orientation on the boundary of $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$ is given by $(-1)^n \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_{n-1}$. This induces a positive orientation on ∂M.
- Also partitions of unity for any open cover U_α of M is defined the same way. If M is orientable, for any manifold with boundary, for all open covering $U = \{U_\alpha\}$, there exists a partition of unity f_α. This implies that if $\omega \in \Omega^n_{\mathrm{c}}(M)$, then $\int_M \omega$ is defined the same way for manifolds.

1.8 Stokes' theorem

Theorem 1.32 (Stokes). *For any manifold with boundary M of dimension n, and for any $\omega \in \Omega^{n-1}_{\mathrm{c}}(M)$ we have*
$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \in \Omega^n_{\mathrm{c}}(M).$$

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas, and let $f_\alpha : M \to \mathbb{R}$ be a partition of unity with respect to this cover. Then $\sum_\alpha f_\alpha = 1$ on M, so

$$\int_M \mathrm{d}\omega = \int_M \mathrm{d}\left(\sum_\alpha f_\alpha \omega\right) = \sum_\alpha \int_M \mathrm{d}(f_\alpha \omega) = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} \left(\phi_\alpha^{-1}\right)^* \mathrm{d}\left(f_\alpha \omega\right).$$

By Proposition 1.16,

$$\left(\phi_\alpha^{-1}\right)^* \mathrm{d}\left(f_\alpha \omega\right) = \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right).$$

Then $\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)$ is an $(n-1)$-form on $\phi_\alpha\left(U_\alpha\right)$. In coordinates,

$$\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right) = \sum_{j=1}^n \widetilde{f_\alpha} \omega_j \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n,$$

where ω_j is a smooth function on $\phi_\alpha\left(U_\alpha\right)$ and

$$\begin{tikzcd} U_\alpha \arrow[r, "\widetilde{\phi_\alpha}"] \arrow[d, "f_\alpha"'] & \phi_\alpha\left(U_\alpha\right) \arrow[dl, "\widetilde{f_\alpha}"] \\ {[0,1]} & \end{tikzcd}.$$

Then

$$\begin{aligned}
\mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right) &= \mathrm{d}\left(\sum_{j=1}^n \widetilde{f_\alpha} \omega_j \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n\right) \\
&= \sum_{j=1}^n \sum_{k=1}^n \frac{\partial}{\partial x_k}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_k \wedge \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n \\
&= \sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_j \wedge \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n \\
&= \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n,
\end{aligned}$$

so

$$\sum_\alpha \int_{\phi_\alpha(U_\alpha)} \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right) = \sum_\alpha \int_{\mathbb{R}_+^n} \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right),$$

because $\widetilde{f_\alpha} = 0$ outside $\phi_\alpha\left(U_\alpha\right)$. Thus

$$\begin{aligned}
\int_M \mathrm{d}\omega &= \sum_\alpha \int_{\mathbb{R}_+^n} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n \\
&= \sum_\alpha \int_{-\infty}^\infty \dots \int_{-\infty}^\infty \int_0^\infty \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_n \, \mathrm{d}x_{n-1} \dots \mathrm{d}x_1 \\
&= \sum_\alpha \sum_{j=1}^n \int_{-\infty}^\infty \dots \widehat{\int_{-\infty}^\infty} \dots \int_{-\infty}^\infty \int_0^\infty (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right)\Big|_{x_n=0} \mathrm{d}x_n \, \mathrm{d}x_{n-1} \dots \widehat{\mathrm{d}x_j} \dots \mathrm{d}x_1 \\
&= \sum_\alpha \int_{-\infty}^\infty \dots \int_{-\infty}^\infty (-1)^{n-1} \left(\widetilde{f_\alpha} \omega_j\right)\Big|_{x_n=0} \mathrm{d}x_{n-1} \dots \mathrm{d}x_1,
\end{aligned}$$

since $\left(f_\alpha \omega_j\right)|_{x_n=0} = 0$ for $j = 1, \dots, n-1$, so

$$\int_M \mathrm{d}\omega = \sum_\alpha \int_{-\infty}^\infty \dots \int_{-\infty}^\infty (-1)^{n-1} \left(\widetilde{f_\alpha} \omega_j\right)\Big|_{x_n=0} \mathrm{d}x_{n-1} \dots \mathrm{d}x_1 = \sum_\alpha \int_{\partial U_\alpha} f_\alpha|_{\partial U_\alpha} \omega = \int_{\partial M} \omega,$$

where $\partial U_\alpha = U_\alpha \cap \partial M$. $\square$

1.9 Applications of Stokes' theorem

Lecture 7 Thursday 23/01/20

Theorem 1.33 (Integration by parts). *Let M be an orientable n-dimensional manifold with boundary, let $\omega \in \Omega_{\mathrm{c}}^p(M)$, let $\eta \in \Omega_{\mathrm{c}}^{n-p-1}(M)$, and let $p \in \{0, \dots, n-1\}$. Then*

$$\int_{\partial M} \omega \wedge \eta = \int_M \mathrm{d}\omega \wedge \eta + (-1)^p \int_M \omega \wedge \mathrm{d}\eta.$$

Proof.

$$\int_{\partial M} \omega \wedge \eta = \int_M \mathrm{d}(\omega \wedge \eta) = \int_M \left(\mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta\right),$$

by Stokes and the Leibniz rule. □

Theorem 1.34 (Brouwer's fixed point theorem). *Let*

$$D = \{x \in \mathbb{R}^n \mid |x| \le 1\},$$

so

$$\partial D = \mathrm{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\},$$

and let $f : D \to D$ be a smooth morphism. Then f admits a fixed point, that is there exists $x \in D$ such that $f(x) = x$.

Proof. Assume that $f(x) \ne x$ for all $x \in D$. For any $x \in D$, consider the ray starting from $f(x)$ and passing through x. Let $g(x)$ be the point where this ray intersects ∂D away from $f(x)$. Note that if $x \in \partial D$ then $g(x) = x$. Then $g : D \to \partial D$. It is easy to check that g is smooth. Since $\partial D = \mathrm{S}^{n-1}$ is orientable by Proposition 1.23 there exists a volume form $\omega \in \Omega^{n-1}(\partial D)$, so $\omega(x) \ne 0$. Since $\omega \in \Omega^{n-1}(\partial D)$, $\mathrm{d}\omega \in \Omega^n(\partial D)$, which is an n-dimensional manifold, so $\mathrm{d}\omega = 0$. Thus

$$0 < \int_{\partial D} \omega = \int_{\partial D} g^* \omega = \int_D \mathrm{d}g^* \omega = \int_D g^* \mathrm{d}\omega = 0,$$

by Stokes, a contradiction. □

Example 1.35. Recall any exact form is closed, since $\mathrm{d}^2 = 0$. But the opposite is not always true. Let $M = \mathbb{R}^2 \setminus \{0\}$, and let

$$\omega = \frac{x}{x^2 + y^2} \mathrm{d}y - \frac{y}{x^2 + y^2} \mathrm{d}x \in \Omega^1(M).$$

Then ω is closed, since

$$\mathrm{d}\omega = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) \mathrm{d}x \wedge \mathrm{d}y - \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) \mathrm{d}y \wedge \mathrm{d}x = 0.$$

We want to show that ω is not exact. Assume that

$$\omega = \mathrm{d}f, \qquad f \in \Omega^0(M) = \{\mathrm{C}^\infty\text{-function}\}.$$

In particular $\omega = \mathrm{d}f$ on $\mathrm{S}^1 \subset M$. Let

$$\begin{array}{rccc} \gamma & : & [0, 2\pi] & \longrightarrow & \mathrm{S}^1 \\ & & \theta & \longmapsto & (\cos\theta, \sin\theta) \end{array}.$$

Then

$$\int_{\mathrm{S}^1} \omega = \int_0^{2\pi} \gamma^* \omega = \int_0^{2\pi} \left(\left(\frac{\cos\theta}{\cos^2\theta + \sin^2\theta}\right) \cos\theta \mathrm{d}\theta - \left(\frac{\sin\theta}{\cos^2\theta + \sin^2\theta}\right)(-\sin\theta)\,\mathrm{d}\theta\right) = \int_0^{2\pi} \mathrm{d}\theta = 2\pi,$$

and

$$\int_{\mathrm{S}^1} \omega = \int_{\mathrm{S}^1} \mathrm{d}f = \int_{\partial \mathrm{S}^1} f = \int_{\emptyset} f = 0,$$

so ω is not exact.

Proposition 1.36. *Let M be an orientable manifold of dimension n without boundary, and let $\omega \in \Omega_{\mathrm{c}}^{n}(M)$. Assume ω is exact. Then*

$$\int_M \omega = 0.$$

Proof. Easy from Stokes. □

Proposition 1.37. *Let M be an orientable manifold of dimension n with boundary, and let $\omega \in \Omega_{\mathrm{c}}^{n-1}(M)$ be a closed form. Then*

$$\int_{\partial M} \omega = 0.$$

Proof. Easy from Stokes. □

Let M be an orientable manifold of dimension n, let $\omega \in \Omega_{\mathrm{c}}^{k}(M)$, and let $N \subset M$ be a submanifold of dimension k. We can define

$$\int_M \omega = \int_N i^* \omega,$$

where $i : N \hookrightarrow M$ is the inclusion. We will denote

$$\omega|_N = i^* \omega \in \Omega_{\mathrm{c}}^{k}(N).$$

Proposition 1.38. *Let M be an oriented manifold of dimension n, let $\omega \in \Omega_{\mathrm{c}}^{k}(M)$, and let $S \subset M$ be a compact orientable submanifold of dimension k such that $\partial S = \emptyset$ and $\int_S \omega \neq 0$. Then*

- *ω is not exact,*
- *$\omega|_S$ is not exact, and*
- *S is not the boundary of an orientable manifold $N \subset M$ of dimension $k+1$.*

Proof. Exercise. [4] □

[4] Exercise

2 De Rham cohomology

2.1 De Rham cohomology

Lecture 8 Monday 27/01/20

Definition 2.1. Let M be a manifold of dimension n, and let $p \ge 0$. Then $\omega_1, \omega_2 \in \Omega^p(M)$ are said to be **cohomologous** if $\omega_1 - \omega_2 = \mathrm{d}\eta$ where $\eta \in \Omega^{p-1}(M)$. In particular $\omega \in \Omega^p(M)$ is cohomologous to zero if it is exact. Let

$$\mathcal{Z}^p(M) = \mathrm{Ker}\left(\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)\right) = \{\omega \in \Omega^p(M) \mid \omega \text{ is closed}\} \subset \Omega^p(M),$$

and let

$$\mathcal{B}^p(M) = \mathrm{Im}\left(\mathrm{d} : \Omega^{p-1}(M) \to \Omega^p(M)\right) = \{\omega \in \Omega^p(M) \mid \omega \text{ is exact}\} \subset \Omega^p(M).$$

Then $\mathcal{B}^p(M) \subset \mathcal{Z}^p(M)$ for all $p \ge 0$.

Notation. If $p = 0$, then $\mathcal{B}^0(M) = 0$.

Note. If $\omega_1, \omega_2 \in \mathcal{Z}^p(M)$ then $\omega_1 - \omega_2 \in \mathcal{B}^p(M)$ if and only if ω_1 and ω_2 are cohomologous.

Definition 2.2. Denote the **p-th de Rham cohomology group** as

$$\mathrm{H}^p(M) = \mathcal{Z}^p(M) / \mathcal{B}^p(M) = \{[\omega] \mid \omega \in \mathcal{Z}^p(M)\}, \qquad p \ge 0.$$

where

$$[\omega] = \{\omega' \in \Omega^p(M) \text{ cohomologous to } \omega\}$$

is the **de Rham class** of ω.

Remark. $\mathrm{H}^p(M)$ is a vector space over $\mathbb{R}$.

Definition 2.3. $\mathrm{b}_p(M) = \dim \mathrm{H}^p(M) \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$ is the **p-th Betti number** of M.

Proposition 2.4. *If M is connected then*

$$\mathrm{H}^0(M) = \mathbb{R},$$

that is $\mathrm{b}_0(M) = 1$. More in general, $\mathrm{b}_0(M)$ is the number of connected components of M.

Proof. Assume M is connected. Then $\mathcal{B}^0(M) = 0$, so

$$\begin{aligned}\mathrm{H}^0(M) = \mathcal{Z}^0(M) &= \left\{f \in \Omega^0(M) \text{ closed}\right\} \\ &= \left\{f \in \Omega^0(M) \;\middle|\; \text{locally } \forall x \in M,\ \frac{\partial}{\partial x_i} f(x) = 0\right\} \\ &= \left\{f \in \Omega^0(M) \text{ locally constant}\right\} = \mathbb{R}.\end{aligned}$$

□

Example. Let $M = \mathrm{S}^1$. Then $\mathrm{H}^0(M) = \mathbb{R}$.

Proposition 2.5. *Let M be a manifold of dimension n. Then*

$$\mathrm{H}^p(M) = 0, \qquad p \ge n + 1.$$

Proof. Recall $\Omega^p(M) = 0$ if $p \ge n+1$ because all alternating p-forms for $p \ge n+1$ on an n-dimensional vector space are zero, so $\mathcal{Z}^p(M) = 0$. Thus $\mathrm{H}^p(M) = 0$. □

Proposition 2.6. *Let M be a compact orientable manifold of dimension n without boundary. Then*

$$\mathrm{H}^n(M) \ne 0.$$

Proof. M is orientable, so there exists a volume form $\omega \in \Omega^n(M) = \Omega^n_{\mathrm{c}}(M)$, by Proposition 1.23. Then ω is closed, because $\mathrm{d}\omega$ is an $(n+1)$-form on M, so $\omega \in \mathcal{Z}^n(M)$. We want to show that $[\omega] \ne 0$ in $\mathrm{H}^n(M)$. Assume $[\omega] = 0$, so ω is exact. Thus $\omega = \mathrm{d}\eta$ where η is an $(n-1)$-form on M, so

$$0 < \int_M \omega = \int_M \mathrm{d}\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0,$$

by Stokes, a contradiction. □

Proposition 2.7. *Let $G : M \to N$ be a smooth morphism between manifolds. Then*

$$G^* : \Omega^p(N) \to \Omega^p(M), \qquad p \geq 0$$

takes closed forms of N to closed forms on M and exact forms of N to exact forms on M.

Proof. By Proposition 1.16, $G^* \mathrm{d} = \mathrm{d} G^*$. If ω is closed then $\mathrm{d} G^* \omega = G^* \mathrm{d}\omega = G^* 0 = 0$, so $G^* \omega$ is closed. If $\omega = \mathrm{d}\eta$ is exact then $G^* \omega = \mathrm{d} G^* \eta$ is also exact. ☐

Thus $G^* : \mathcal{Z}^p(N) \to \mathcal{Z}^p(M)$ and $G^* : \mathcal{B}^p(N) \to \mathcal{B}^p(M)$, so there exists a linear map

$$\begin{array}{cccc} G^* & : & \mathrm{H}^p(N) & \longrightarrow & \mathrm{H}^p(M) \\ & & [\omega] & \longmapsto & [G^* \omega] \end{array}.$$

Corollary 2.8. *Let M and N be diffeomorphic manifolds. Then*

$$\mathrm{H}^p(M) \cong \mathrm{H}^p(N), \qquad p \geq 0,$$

that is $\mathrm{H}^p(M)$ is a diffeomorphic invariant.

Proof. By Proposition 2.7 there exists $F^* : \mathrm{H}^p(N) \to \mathrm{H}^p(M)$ and $\left(F^{-1}\right)^* : \mathrm{H}^p(M) \to \mathrm{H}^p(N)$. By Proposition 1.8,

$$\left(F^{-1}\right)^* F^* \omega = \left(F \circ F^{-1}\right)^* \omega = \mathrm{id}_N^* \omega = \omega, \qquad \omega \in \mathrm{H}^p(N)$$

so $\left(F^{-1}\right)^* \circ F^* = \mathrm{id}_{\mathrm{H}^p(N)}$. Similarly $F^* \circ \left(F^{-1}\right)^* = \mathrm{id}_{\mathrm{H}^p(M)}$, so F^* is an isomorphism. ☐

2.2 Homotopy invariance

Definition 2.9. Let M_0 and M_1 be manifolds, and let $f_0, f_1 : M_0 \to M_1$ be smooth morphisms. Then f_0 and f_1 are **smoothly homotopic equivalent** if there exists a smooth morphism $H : M_0 \times [0,1] \to M_1$ such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$ for all $x \in M_0$. A **homotopy** is a smooth morphism $H : M_0 \times [0,1] \to M_1$ where M_0 and M_1 are smooth manifolds.

Lecture 9 Tuesday 28/01/20

Notation 2.10. Let $f_t(x) = H(x,t)$, so $f_t : M_0 \to M_1$ is a smooth morphism. Then f_0 and f_1 are said to be homotopic equivalent, denoted by $f_0 \sim f_1$. Then $\sim$ is an equivalence. [5]

Definition 2.11. M_0 and M_1 are **homotopy equivalent** if there exist smooth morphisms $f : M_0 \to M_1$ and $g : M_1 \to M_0$ such that $f \circ g \sim \mathrm{id}_{M_1}$ and $g \circ f \sim \mathrm{id}_{M_0}$.

Example 2.12.

- Let $M_0 = \mathbb{R}^n$ and $M_1 = \{0\}$. Then M_0 and M_1 are homotopy equivalent. Let

$$\begin{array}{ccccc} f & : & M_0 & \longrightarrow & M_1 \\ & & x & \longmapsto & 0 \end{array}, \qquad \begin{array}{ccccc} g & : & M_1 & \longrightarrow & M_0 \\ & & 0 & \longmapsto & 0 \end{array}.$$

Then

$$\begin{array}{ccccc} f \circ g & : & M_1 & \longrightarrow & M_1 \\ & & 0 & \longmapsto & 0 \end{array},$$

so $f \circ g = \mathrm{id}_{M_1}$, and

$$\begin{array}{ccccc} g \circ f & : & M_0 & \longrightarrow & M_0 \\ & & x & \longmapsto & 0 \end{array}.$$

We want to show that $g \circ f \sim \mathrm{id}_{M_0}$. Define a smooth morphism

$$\begin{array}{ccccc} H & : & M_0 \times [0,1] & \longrightarrow & M_0 \\ & & (x,t) & \longmapsto & tx \end{array}.$$

Then $H(x,0) = 0 = (g \circ f)(x)$ for all x, and $H(x,1) = x = \mathrm{id}_{M_0}(x)$ for all x, so $g \circ f \sim \mathrm{id}_{M_0}$. More in general $M \subset \mathbb{R}^n$ is called **convex** if for all $x, y \in M$ the segment joining x to y is contained inside M. If M is convex then M is homotopy equivalent to $M \times \{0\}$.

[5] Exercise

- Let $M_0 = \mathbb{R}^2 \setminus \{0\}$ and $M_1 = \mathrm{S}^1$. Then M_0 and M_1 are homotopy equivalent. Let
$$\begin{array}{cccc} f & : & M_0 & \longrightarrow & M_1 \\ & & x & \longmapsto & \dfrac{x}{|x|} \end{array}, \qquad \begin{array}{cccc} g & : & M_1 & \longrightarrow & M_0 \\ & & x & \longmapsto & x \end{array}.$$
Then
$$\begin{array}{cccc} f \circ g & : & M_1 & \longrightarrow & M_1 \\ & & x & \longmapsto & x \end{array},$$
so $f \circ g = \mathrm{id}_{M_1}$, and
$$\begin{array}{cccc} g \circ f & : & M_0 & \longrightarrow & M_0 \\ & & x & \longmapsto & \dfrac{x}{|x|} \end{array}.$$
Let
$$\begin{array}{cccc} H & : & M_0 \times [0,1] & \longrightarrow & M_0 \\ & & (x,t) & \longmapsto & tx + (1-t)\dfrac{x}{|x|} \end{array}$$
be smooth. Then $H(x,0) = x/|x| = (g \circ f)(x)$ and $H(x,1) = x = \mathrm{id}_{M_0}(x)$, so $g \circ f \sim \mathrm{id}_{M_0}$.

Proposition 2.13. *Let M and N be manifolds, and let $H : M \times [0,1] \to N$ be smooth. Denote*
$$\begin{array}{cccc} f_t & : & M & \longrightarrow & N \\ & & x & \longmapsto & H(x,t) \end{array}, \qquad t \in [0,1].$$
Then $f_t^* : \mathrm{H}^p(N) \to \mathrm{H}^p(M)$ does not depend on t for all $p \ge 0$.

Proof. Let $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$. The goal is $f_{t_1}^*[\eta] = f_{t_2}^*[\eta]$ for all $[\eta] \in \mathrm{H}^p(N)$. Let
$$\begin{array}{cccc} i_k & : & M & \longrightarrow & M \times [0,1] \\ & & x & \longmapsto & (x,t_k) \end{array}, \qquad k = 1,2.$$
Claim that for all p there exists a linear map $h : \Omega^p(M \times [t_1,t_2]) \to \Omega^{p-1}(M)$ such that
$$\mathrm{d}(h(\omega)) + h(\mathrm{d}\omega) = i_2^*\omega - i_1^*\omega \in \Omega^p(M), \qquad \omega \in \Omega^p(M \times [0,1]). \tag{2}$$

Step 1. The claim implies the proposition. Let $\eta \in \Omega^p(N)$ be closed, so $\mathrm{d}\eta = 0$. Then $H^*\eta$ is also closed, so let $\omega = H^*\eta \in \Omega^p(M \times [t_1,t_2])$. Apply h. Then $\mathrm{d}\omega = 0$, so $\mathrm{d}(h(\omega)) = i_2^*\omega - i_1^*\omega$ is exact. Thus
$$f_{t_1}^*[\eta] = \left[f_{t_1}^*\eta\right] = [i_1^*H^*\eta] = [i_1^*\omega] = [i_2^*\omega] = [i_2^*H^*\eta] = \left[f_{t_2}^*\eta\right] = f_{t_2}^*[\eta],$$
so the proposition follows.

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Step 2. The proof of the claim. Let $\omega \in \Omega^p(M \times [t_1,t_2])$. Then for all $(x,t) \in M \times [t_1,t_2]$, $\omega(x,t)$ is an alternating p-form on $\mathrm{T}_{(x,t)}(M \times [t_1,t_2])$. We want an alternating $(p-1)$-form $h(\omega)(x)$ on T_xM. Let $v_1, \dots, v_{p-1} \in \mathrm{T}_xM$. Then
$$h(\omega)(x)(v_1,\dots,v_{p-1}) = \int_{t_1}^{t_2} \omega(x,t)\left(\frac{\partial}{\partial t}, v_1, \dots, v_{p-1}\right)\mathrm{d}t$$
is a $(p-1)$-form on M, and $\frac{\partial}{\partial t}$ is a global vector field. Check h is linear. [6] It is enough to prove (2) locally. Remark that exactness is not a local property. Fix local coordinates $(x_1,\dots,x_n,t)$ around a point of $M \times [0,1]$. Then
$$\omega = \sum_{|I|=p} \omega_I + \sum_{|J|=p-1} \omega_J, \qquad \omega_I = g_I \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad \omega_J = g_J \mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}} \wedge \mathrm{d}t,$$
where g_I and g_J are smooth functions. Any term of (2) is linear. So we just need to check (2) holds for ω_I and ω_J.

[6] Exercise

ω_I. Let $\omega = g(x,t)\,\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}$. Then

$$\mathrm{d}\left(h\left(\omega(x,t)\left(\frac{\partial}{\partial t}, v_1, \dots, v_{p-1}\right)\right)\right) = \mathrm{d}(h(0)) = 0,$$

and

$$\begin{aligned} h(\mathrm{d}\omega) &= h\left(\frac{\partial}{\partial t} g(x,t)\,\mathrm{d}t \wedge \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p} + \sum_{j=1}^n \frac{\partial}{\partial x_j} g(x,t)\,\mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}\right) \\ &= \left(\int_{t_1}^{t_2} \frac{\partial}{\partial t} g(x,t)\ \mathrm{d}t\right)\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p} + 0 \\ &= (g(x,t_2) - g(x,t_1))\,\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p} = i_2^*\omega - i_1^*\omega, \end{aligned}$$

so (2) holds.

ω_J. Let $\omega = g(x,t)\,\mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}} \wedge \mathrm{d}t$. Then

$$\begin{aligned} \mathrm{d}(h(\omega)) &= (-1)^{p-1}\,\mathrm{d}\left(\left(\int_{t_1}^{t_2} g(x,t)\ \mathrm{d}t\right)\mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}}\right) \\ &= (-1)^{p-1} \sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\int_{t_1}^{t_2} g(x,t)\ \mathrm{d}t\right)\mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}} \\ &= (-1)^{p-1} \sum_{j=1}^n \left(\int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x,t)\ \mathrm{d}t\right)\mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}}, \end{aligned}$$

and

$$\begin{aligned} h(\mathrm{d}\omega) &= h\left(\sum_{j=1}^n \frac{\partial}{\partial x_j} g(x,t)\,\mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}} \wedge \mathrm{d}t + 0\right) \\ &= (-1)^{p-1} \sum_{j=1}^n \left(\int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x,t)\ \mathrm{d}t\right)\mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}} = -\mathrm{d}(h(\omega)), \end{aligned}$$

and $i_2^*\omega = i_1^*\omega = 0$, so (2) holds.

□

Corollary 2.14. *Assume M and N are homotopy equivalent. Then there exist isomorphisms*

$$\mathrm{H}^p(N) \to \mathrm{H}^p(M), \qquad p \ge 0.$$

Proof. There exist $f : M \to N$ and $g : N \to M$ such that $g \circ f \sim \mathrm{id}_M$ and $f \circ g \sim \mathrm{id}_N$. By Proposition 2.13 $(g \circ f)^* : \mathrm{H}^p(M) \to \mathrm{H}^p(M)$ coincides with $\mathrm{id}_M^* = \mathrm{id}_{\mathrm{H}^p(M)}$. Then $f^* \circ g^* = (g \circ f)^* = \mathrm{id}_{\mathrm{H}^p(M)}$. Similarly $g^* \circ f^* = \mathrm{id}_{\mathrm{H}^p(N)}$, so g^* and f^* are isomorphisms. □

Definition 2.15. Let M be a manifold. Then M is **smoothly contractible** if M is homotopy equivalent to a point.

Example. $\mathbb{R}^n$ is contractible, by Example 2.12. If $M \subset \mathbb{R}^n$ is convex then M is contractible.

Theorem 2.16 (Poincaré lemma)**.** *If M is a contractible manifold then*

$$\mathrm{H}^p(M) = 0, \qquad p \ge 1.$$

Proof. By previous Corollary 2.14, there exists an isomorphism $\mathrm{H}^p(M) \to \mathrm{H}^p(\{\text{point}\})$. Then $\{\text{point}\}$ is a zero-dimensional manifold, so by Proposition 2.5, $\mathrm{H}^p(\{\text{point}\}) = 0$ for all $p > 0$. □

Thus $\mathrm{H}^p(\mathbb{R}^n) = 0$ for all $p > 0$, so $\mathbb{R}^n$ is not diffeomorphic to any compact orientable manifold. Lecture 11 Monday 03/02/20

Proposition 2.17. *Let M be a manifold, and let $\omega \in \Omega^p(M)$ be a closed p-form for $p > 0$. Then for all $x \in X$, there exists a neighbourhood $U \ni x$ such that ω is exact on U, that is there exists $\eta \in \Omega^{p-1}(U)$ such that $\omega = \mathrm{d}\eta$ on U.*

Proof. Let (U, ϕ) be a chart around x. I may assume that $V = \phi(U)$ is a ball in $\mathbb{R}^n$. Then U is diffeomorphic to $B = \{z \mid |z - z_0| < r\}$ for some $z_0 \in \mathbb{R}^n$ and $r > 0$, so $\mathrm{H}^p(U) \cong \mathrm{H}^p(B)$ for all $p \geq 0$. Since B is contractible, $\mathrm{H}^p(B) = 0$ for all $p > 0$. The restriction of ω on U gives a class $[\omega] \in \mathrm{H}^p(U) = 0$, so ω is cohomologous to zero on U. Thus ω is exact on U. ☐

Definition 2.18. Let M be a manifold, let $\gamma : [0,1] \to M$ be a continuous or smooth path, and let $x = \gamma(0)$ and $y = \gamma(1)$. A **homotopy of paths** from x to y is a map

$$\begin{array}{cccccc} F & : & [0,1] \times [0,1] & \longrightarrow & M \\ & & (0,t) & \longmapsto & x \\ & & (1,t) & \longmapsto & y \end{array}.$$

Proposition 2.19. *Let γ_0 and γ_1 be homotopic paths on a manifold M, and let $\omega \in \Omega^1(M)$ be closed. Then*

$$\int_0^1 \gamma_0^*\omega = \int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

Proof. Lee's introduction to smooth manifolds. The idea is

$$\int_{\gamma_0} \omega - \int_{\gamma_1} \omega = \int_{\gamma_0 \cup \gamma_1} \omega = \int_{\mathrm{Im}\, F} \omega = 0,$$

by Stokes' theorem. ☐

Recall that M is **simply connected**, so $\pi_1(M) = 0$, if any path γ from x to x is homotopic equivalent to a point.

Proposition 2.20. *Let M be a simply connected orientable manifold. Then*

$$\mathrm{H}^1(M) = 0.$$

Proof. Let $\omega \in \Omega^1(M)$ be a closed form. Then claim that ω is exact if and only if $\int_\gamma \omega = 0$ for all loops γ, that is paths from x to x.

- The proof of the claim. Assume that $\omega = \mathrm{d}f$ is exact for $f \in \Omega^0(M)$. By Proposition 2.19,

  $$\int_\gamma \omega = \int_{\text{trivial loop}} \omega = 0.$$

  Assume that $\int_\gamma \omega = 0$ for all loops γ. Fix x. Let

  $$f(y) = \int_x^y \omega.$$

  Since $\int_{\gamma_1 \cup \gamma_2} \omega = 0$, f is well-defined, that is it does not depend on the choice of the path. Then $\mathrm{d}f = \omega$. This can be checked locally, that is in an open set of $\mathbb{R}^n$. Here it follows from the fundamental theorem of calculus.

- The claim implies the proposition. Being simply connected, any loop inside M is homotopic equivalent to the trivial loop. For all loops γ and for all closed ω, $\int_\gamma \omega = 0$ by Proposition 2.19, so ω is exact. Thus $[\omega] = 0$ in $\mathrm{H}^1(M)$.

☐

2.3 Some homological algebra

Let $C^\bullet$ be a sequence of vector spaces, that is C^k is a vector space for $k \in \mathbb{Z}$.

Definition 2.21. $(C^\bullet, d^\bullet)$ is a **cochain complex** if $C^\bullet$ is a sequence of vector spaces and $d^\bullet$ is a sequence of linear maps $d^k : C^k \to C^{k+1}$ such that the composition $d^{k+1} \circ d^k : C^k \to C^{k+1} \to C^{k+2}$ is zero for all k. Then $d^\bullet$ is the **differential**.

Definition 2.22. The elements of

$$\mathcal{Z}^k(C^\bullet, d^\bullet) = \operatorname{Ker}\left(d^k : C^k \to C^{k+1}\right) \subset C^k$$

are called **cocycles**. The elements of

$$\mathcal{B}^k(C^\bullet, d^\bullet) = \operatorname{Im}\left(d^k : C^{k-1} \to C^k\right) \subset C^k$$

are called **coboundaries**. Then $d^{k-1} \circ d^k = 0$, so $\mathcal{B}^k \subset \mathcal{Z}^k$. The quotients

$$\mathrm{H}^k(C^\bullet, d^\bullet) = \mathcal{Z}^k(C^\bullet, d^\bullet) / \mathcal{B}^k(C^\bullet, d^\bullet)$$

are the **k-th cohomology groups** of $(C^\bullet, d^\bullet)$.

Definition 2.23. Let $(C^\bullet, d^\bullet)$ and $(D^\bullet, d^\bullet)$ be two cochain complexes. A map $f : (C^\bullet, d^\bullet) \to (D^\bullet, d^\bullet)$ is a sequence of linear maps $f^k : C^k \to D^k$ such that $f^{k+1} \circ d^k = d^k \circ f^k$ for all k, so

$$\begin{tikzcd}
\dots \arrow[r] & C^k \arrow[r, "d^k"] \arrow[d, "f^k"] & C^{k+1} \arrow[r, "d^{k+1}"] \arrow[d, "f^{k+1}"] & C^{k+2} \arrow[r] \arrow[d, "f^{k+2}"] & \dots \\
\dots \arrow[r] & D^k \arrow[r, "d^k"] & D^{k+1} \arrow[r, "d^{k+1}"] & D^{k+2} \arrow[r] & \dots
\end{tikzcd}.$$

Proposition 2.24. *Let $f : (C^\bullet, d^\bullet) \to (D^\bullet, d^\bullet)$ be a map between cochain complexes. Then there exists a natural induced map*

$$f^k : \mathrm{H}^k(C^\bullet, d^\bullet) \to \mathrm{H}^k(D^\bullet, d^\bullet).$$

Proof. Let $[\omega] \in \mathrm{H}^k(C^\bullet, d^\bullet) = \mathcal{Z}^k(C^\bullet, d^\bullet) / \mathcal{B}^k(C^\bullet, d^\bullet)$ for $\omega \in \mathcal{Z}^k(C^\bullet, d^\bullet)$, that is $d^k(\omega) = 0$. I want to check that $f^k(\omega) \in \mathcal{Z}^k(D^\bullet, d^\bullet)$. By definition of maps, $d^k\left(f^k(\omega)\right) = f^{k+1}\left(d^k(\omega)\right) = 0$, so there is a map

$$\mathcal{Z}^k(C^\bullet, d^\bullet) \to \mathcal{Z}^k(D^\bullet, d^\bullet).$$

Now I need to check that if $\omega \in \mathcal{B}^k(C^\bullet, d^\bullet)$ then $f^k(\omega) \in \mathcal{B}^k(D^\bullet, d^\bullet)$. [7] $\square$

Lecture 12 Tuesday 04/02/20

Definition 2.25. A sequence of linear maps

$$C^1 \xrightarrow{f^1} C^2 \to \dots \to C^{k-1} \xrightarrow{f^{k-1}} C^k$$

between vector spaces is said to be **exact** if for all i, $\operatorname{Ker} f^i = \operatorname{Im} f^{i-1}$.

Example 2.26.

- A sequence
$$0 \to C^1 \xrightarrow{f^1} C^2$$
is exact if and only if f^1 is injective.
- A sequence
$$C^1 \xrightarrow{f^1} C^2 \to 0$$
is exact if and only if f^1 is surjective.
- An exact sequence
$$0 \to C^1 \xrightarrow{f^1} C^2 \xrightarrow{f^2} C^3 \to 0$$
is called a **short exact sequence**. In particular f^1 is injective and f^2 is surjective.

[7] Exercise

- Any long exact sequence
$$C^1 \xrightarrow{f^1} C^2 \to \dots \to C^{q-1} \xrightarrow{f^{q-1}} C^q$$
can be split into short exact sequences
$$\begin{tikzcd}[column sep=small]
\dots \arrow[r] & C^{k-1} \arrow[rr, "f^{k-1}"] \arrow[dr] & & C^k \arrow[rr, "f^k"] \arrow[dr] & & C^{k+1} \arrow[r] & \dots \\
& & \operatorname{Im} f^{k-1} = \operatorname{Ker} f^k \arrow[ur] \arrow[dr] & & \operatorname{Im} f^k = \operatorname{Ker} f^{k+1} \arrow[ur] \arrow[dr] & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}, \qquad k = 2, \dots, q-1.$$

Lemma 2.27 (Snake lemma)**.** *Consider the commutative diagram*
$$\begin{tikzcd}
& C^1 \arrow[r, "f^1"] \arrow[d, "\alpha_1"] & C^2 \arrow[r, "f^2"] \arrow[d, "\alpha_2"] & C^3 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & D^1 \arrow[r, "g^1"] & D^2 \arrow[r, "g^2"] & D^3 &
\end{tikzcd},$$
such that both the horizontal lines are exact sequences. Then there exists a long exact sequence
$$\operatorname{Ker} \alpha_1 \to \operatorname{Ker} \alpha_2 \to \operatorname{Ker} \alpha_3 \xrightarrow{\delta} \operatorname{Coker} \alpha_1 \to \operatorname{Coker} \alpha_2 \to \operatorname{Coker} \alpha_3.$$
If
$$\begin{tikzcd}
0 \arrow[r] & C^1 \arrow[r, "f^1"] \arrow[d, "\alpha_1"] & C^2 \arrow[r, "f^2"] \arrow[d, "\alpha_2"] & C^3 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & D^1 \arrow[r, "g^1"] & D^2 \arrow[r, "g^2"] & D^3 \arrow[r] & 0
\end{tikzcd},$$
then
$$0 \to \operatorname{Ker} \alpha_1 \to \operatorname{Ker} \alpha_2 \to \operatorname{Ker} \alpha_3 \xrightarrow{\delta} \operatorname{Coker} \alpha_1 \to \operatorname{Coker} \alpha_2 \to \operatorname{Coker} \alpha_3 \to 0.$$

Proof. We are going to construct $\delta : \operatorname{Ker} \alpha_3 \to \operatorname{Coker} \alpha_1$. Let $x \in \operatorname{Ker} \alpha_3$. There exists $y \in C^2$ such that $f^2(y) = x$ because f^2 is surjective. Let $z = \alpha_2(y)$ then
$$g^2(z) = g^2(\alpha_2(y)) = \alpha_3\left(f^2(y)\right) = \alpha_3(x) = 0,$$
since $x \in \operatorname{Ker} \alpha_3$. Then $z \in \operatorname{Ker} g^2 = \operatorname{Im} g^1$, so there exists $w \in D^1$ such that $z = g^1(w)$. The idea is
$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& \operatorname{Ker} \alpha_1 \arrow[r] \arrow[d] & \operatorname{Ker} \alpha_2 \arrow[r] \arrow[d] & \operatorname{Ker} \alpha_3 \arrow[d] & \\
& C^1 \arrow[r, "f^1"] \arrow[d, "\alpha_1"] & y \in C^2 \arrow[r, "f^2"] \arrow[d, "\alpha_2"] & x \in C^3 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & w \in D^1 \arrow[r, "g^1"'] \arrow[d] & z \in D^2 \arrow[r, "g^2"'] \arrow[d] & D^3 \arrow[d] & \\
& \operatorname{Coker} \alpha_1 \arrow[r] \arrow[d] & \operatorname{Coker} \alpha_2 \arrow[r] \arrow[d] & \operatorname{Coker} \alpha_3 \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}.$$

Define $\delta(x) = [w] \in \operatorname{Coker} \alpha^1 = D^1 / \operatorname{Im} \alpha^1$. Need to check that δ is well-defined, so $[w]$ does not depend on our choice of w and y. The rest is an exercise. [8] $\square$

[8] Exercise

2.4 The Mayer-Vietoris sequence

The idea is given a manifold M, we may write $M = U \cup V$ with open U and V so that $H^i(U)$, $H^i(V)$, and $H^i(U \cap V)$ are easy to compute, so this will give us $H^i(M)$. Let M be a manifold, and let U and V be open such that $M = U \cup V$. Assume $U \cap V \neq \emptyset$. Let

$$i_U : U \to M, \qquad i_V : V \to M, \qquad j_U : U \cap V \to U, \qquad j_V : U \cap V \to V$$

be inclusions, and let $i_U^*, i_V^*, j_U^*, j_V^*$ be pull-backs.

Proposition 2.28. *For all p there exist short exact sequences*

$$0 \to \Omega^p(M) \xrightarrow{f} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{g} \Omega^p(U \cap V) \to 0,$$

where $f = (i_U^*, i_V^*)$ and $g = j_V^* - j_U^*$. More precisely, if $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^p(V)$ then $g(\omega_1, \omega_2) = j_V^*\omega_2 - j_U^*\omega_1$.

Proof.

- f is injective. Assume $\omega \in \Omega^p(M)$ such that $f(\omega) = 0$, so $i_U^*\omega = i_V^*\omega = 0$. Since $M = U \cup V$ then $\omega = 0$ on M, so f is injective.
- $\operatorname{Im} f = \operatorname{Ker} g$. Let $f(\omega) \in \operatorname{Im} f$, so $f(\omega) = (i_U^*\omega, i_V^*\omega)$. Then $g(f(\omega)) = j_V^* i_V^*\omega - j_U^* i_U^*\omega = l^*\omega - l^*\omega = 0$, where

$$\begin{tikzcd} & U \arrow[dr, "i_U"] & \\ U \cap V \arrow[ur, "j_U"] \arrow[rr, "l"] \arrow[dr, "j_V"'] & & M \\ & V \arrow[ur, "i_V"'] & \end{tikzcd},$$

  so $\operatorname{Im} f \subset \operatorname{Ker} g$. Now let $(\omega_1, \omega_2) \in \operatorname{Ker} g$, so $j_V^*\omega_2 = j_U^*\omega_1$ for $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^p(V)$. The restriction of ω_2 on $U \cap V$ coincides with the restriction of ω_1 on $U \cap V$. Then define

$$\omega = \begin{cases} \omega_1 & \text{on } U \\ \omega_2 & \text{on } V \end{cases}.$$

  Then $f(\omega) = (\omega_1, \omega_2)$, so $\operatorname{Ker} g \subset \operatorname{Im} f$.
- g is surjective. Let $\eta \in \Omega^p(U \cap V)$, and let $\{f_U, f_V\}$ be a partition of unity with respect to $\{U, V\}$. Then $\operatorname{supp} f_U \subset U$ and $f_U + f_V = 1$. Let $\eta_1 \in \Omega^p(U)$ be defined by

$$\eta_1 = \begin{cases} f_V \cdot \eta & \text{on } U \cap V \\ 0 & \text{outside } \operatorname{supp} f_V \end{cases},$$

  and let $\eta_2 \in \Omega^p(V)$ be defined by

$$\eta_2 = \begin{cases} f_U \cdot \eta & \text{on } U \cap V \\ 0 & \text{outside } \operatorname{supp} f_U \end{cases}.$$

  Then $g(-\eta_2, \eta_1) = \eta_1|_{U \cap V} + \eta_2|_{U \cap V} = (f_U + f_V) \cdot \eta = \eta$, so $\eta \in \operatorname{Im} g$.

$\square$

Lecture 13 Thursday 06/02/20

Theorem 2.29 (Mayer-Vietoris)**.** *Let M be a manifold, and let U and V be open in M such that $M = U \cup V$ and $U \cap V \neq \emptyset$. Then for all $p \geq 0$ there exists a linear $\delta : \mathrm{H}^p(U \cap V) \to \mathrm{H}^{p+1}(M)$ such that*

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}^p(M) \arrow[r, "(i_U^*, i_V^*)"] & \mathrm{H}^p(U) \oplus \mathrm{H}^p(V) \arrow[r, "j_V^* - j_U^*"] & \mathrm{H}^p(U \cap V) \arrow[dll, "\delta"] \\ & \mathrm{H}^{p+1}(M) \arrow[r, "(i_U^*, i_V^*)"] & \mathrm{H}^{p+1}(U) \oplus \mathrm{H}^{p+1}(V) \arrow[r, "j_V^* - j_U^*"] & \mathrm{H}^{p+1}(U \cap V) \arrow[r] & \dots \end{tikzcd}$$

is exact.

Example 2.30. Let $M = \mathrm{S}^1$, let $N = (0,1)$ and $S = (0,-1)$, and let $U = M \setminus \{N\}$ and $V = M \setminus \{S\}$, so $M = U \cup V$ and $U \cap V = M \setminus \{N, S\}$. Then

$$\mathrm{H}^p(U) \cong \mathrm{H}^p(V) \cong \mathrm{H}^p((0,1)) \cong \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \qquad (0,1) \subset \mathbb{R},$$

and

$$\mathrm{H}^p(U \cap V) = \mathrm{H}^p(U \setminus \{S\}) = \mathrm{H}^p\left(\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)\right) = \begin{cases} \mathbb{R}^2 & p = 0 \\ 0 & p > 0 \end{cases}, \qquad \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right) \subset \mathbb{R},$$

so

$$\begin{array}{ccccccccccccc} 0 \to & \mathrm{H}^0(M) & \to & \mathrm{H}^0(U) \oplus \mathrm{H}^0(V) & \xrightarrow{\phi} & \mathrm{H}^0(U \cap V) & \xrightarrow{\delta} & \mathrm{H}^1(M) & \to & \mathrm{H}^1(U) \oplus \mathrm{H}^1(V) & \to & \mathrm{H}^1(U \cap V) & \to \dots \\ & \wr\!\!\parallel & & \wr\!\!\parallel & & \wr\!\!\parallel & & & & \wr\!\!\parallel & & \wr\!\!\parallel & \\ & \mathbb{R} & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R}^2 & & & & 0 \oplus 0 & & 0 & \end{array}.$$

Then $\operatorname{Im} \phi = \mathbb{R} \subset \mathrm{H}^0(U \cap V) = \mathbb{R}^2$. Thus

$$\mathrm{H}^1(M) = \operatorname{Coker} \phi = \mathbb{R}^2 / \operatorname{Im} \phi \cong \mathbb{R}.$$

Remark 2.31. Let

$$0 \to C^1 \to \dots \to C^k \to 0$$

be an exact sequence. Then

$$\sum_k (-1)^k \dim C^k = 0.$$[9]

In our $M = \mathrm{S}^1$ case $1 - 2 + 2 - \dim \mathrm{H}^1(M) = 0$, so $\dim \mathrm{H}^1(M) = 1$. Thus $\mathrm{H}^1(M) \cong \mathbb{R}$.

Example 2.32. Let $M = \mathrm{S}^n \subset \mathbb{R}^{n+1}$ be the n-dimensional sphere. Then

$$\mathrm{H}^p(M) = \begin{cases} \mathbb{R} & p = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

By induction on n.

$n = 1$. Ok.

$n > 1$. Let $U = M \setminus \{N\}$ and $V = M \setminus \{S\}$, so $U \cap V \neq \emptyset$ and $U \cup V = M$. Then

$$U \cong V \cong \mathbb{R}^n, \qquad U \cap V = V \setminus \{N\} \cong \mathbb{R}^n \setminus \{0\} \cong \mathrm{S}^{n-1},$$

so

$$\begin{array}{ccccccccccc} 0 \longrightarrow & \mathrm{H}^0(M) & \longrightarrow & \mathrm{H}^0(U) \oplus \mathrm{H}^0(V) & \longrightarrow & \mathrm{H}^0(U \cap V) & \xrightarrow{\delta} & \mathrm{H}^1(M) & \longrightarrow & \mathrm{H}^1(U) \oplus \mathrm{H}^1(V) & \longrightarrow \dots \\ & \wr\!\!\parallel & & \wr\!\!\parallel & & \wr\!\!\parallel & & & & \wr\!\!\parallel & \\ & \mathbb{R} & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R} & & & & 0 \oplus 0 & \end{array}.$$

Then $1 - 2 + 1 - \dim \mathrm{H}^1(M) = 0$, so $\dim \mathrm{H}^1(M) = 0$. Thus $\mathrm{H}^1(M) = 0$. Then for $p > 0$

$$\begin{array}{ccccccccc} \dots \longrightarrow & \mathrm{H}^p(U) \oplus \mathrm{H}^p(V) & \longrightarrow & \mathrm{H}^p(U \cap V) & \xrightarrow{\delta} & \mathrm{H}^{p+1}(M) & \longrightarrow & \mathrm{H}^{p+1}(U) \oplus \mathrm{H}^{p+1}(V) & \longrightarrow \dots \\ & \wr\!\!\parallel & & & & & & \wr\!\!\parallel & \\ & 0 \oplus 0 & & & & & & 0 \oplus 0 & \end{array}$$

is exact, so $\mathrm{H}^p(U \cap V) \cong \mathrm{H}^{p+1}(M)$. By induction

$$\mathrm{H}^p(U \cap V) = \mathrm{H}^{p+1}(M) = \begin{cases} \mathbb{R} & p = n - 1 \\ 0 & \text{otherwise} \end{cases}.$$

[9] Exercise

Proof of Theorem 2.29. By Proposition 2.28 for all p

$$\begin{tikzcd} 0 \arrow[r] & \Omega^p(M) \arrow[r] \arrow[d, "\mathrm{d}_M^p"] & \Omega^p(U) \oplus \Omega^p(V) \arrow[r] \arrow[d, "\left(\mathrm{d}_U^p, \mathrm{d}_V^p\right)"] & \Omega^p(U \cap V) \arrow[r] \arrow[d, "\mathrm{d}_{U \cap V}^p"] & 0 \\ 0 \arrow[r] & \Omega^{p+1}(M) \arrow[r] & \Omega^{p+1}(U) \oplus \Omega^{p+1}(V) \arrow[r] & \Omega^{p+1}(U \cap V) \arrow[r] & 0 \end{tikzcd}$$

are exact. Recall d commutes with the pull-back. By the strong snake lemma,

$$\begin{tikzcd} & \operatorname{Coker} \mathrm{d}_M^{p-1} \arrow[r] \arrow[d, "\partial_M^p = \mathrm{d}_M^p"] & \operatorname{Coker}\left(\mathrm{d}_U^{p-1}, \mathrm{d}_V^{p-1}\right) \arrow[r] \arrow[d, "\left(\partial_U^p, \partial_V^p\right) = \left(\mathrm{d}_U^p, \mathrm{d}_V^p\right)"] & \operatorname{Coker} \mathrm{d}_{U \cap V}^{p-1} \arrow[r] \arrow[d, "\partial_{U \cap V}^p = \mathrm{d}_{U \cap V}^p"] & 0 \\ 0 \arrow[r] & \operatorname{Ker} \mathrm{d}_M^{p+1} \arrow[r] & \operatorname{Ker}\left(\mathrm{d}_U^{p+1}, \mathrm{d}_V^{p+1}\right) \arrow[r] & \operatorname{Ker} \mathrm{d}_{U \cap V}^{p+1} & \end{tikzcd},$$

which is well-defined, since $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$. By the weak snake lemma again,

$$\operatorname{Ker} \partial_M^p \to \operatorname{Ker}\left(\partial_U^p, \partial_V^p\right) \to \operatorname{Ker} \partial_{U \cap V}^p \xrightarrow{\delta} \operatorname{Coker} \partial_M^p \to \operatorname{Coker}\left(\partial_U^p, \partial_V^p\right) \to \operatorname{Coker} \partial_{U \cap V}^p.$$

Then $\operatorname{Coker} \mathrm{d}_M^{p-1} = \Omega^p(M) / \operatorname{Im} \mathrm{d}_M^{p-1}$. There exists

$$\mathrm{H}^p(M) = \operatorname{Ker} \mathrm{d}_M^p / \operatorname{Im} \mathrm{d}_M^{p-1} \xrightarrow{\sim} \operatorname{Ker}\left(\Omega^p(M) / \operatorname{Im} \mathrm{d}_M^{p-1} \to \operatorname{Ker} \mathrm{d}_M^{p+1}\right) = \operatorname{Ker} \partial_M^p.$$

Similarly, $\operatorname{Ker}\left(\partial_U^p, \partial_V^p\right) \cong \mathrm{H}^p(U) \oplus \mathrm{H}^p(V)$ and $\operatorname{Ker} \partial_{U \cap V}^p \cong \mathrm{H}^p(U \cap V)$. There exists

$$\mathrm{H}^{p+1}(M) = \operatorname{Ker} \mathrm{d}_M^{p+1} / \operatorname{Im} \mathrm{d}_M^p \xrightarrow{\sim} \operatorname{Coker}\left(\Omega^p(M) / \operatorname{Im} \mathrm{d}_M^{p-1} \to \operatorname{Ker} \mathrm{d}_M^{p+1}\right) = \operatorname{Coker} \partial_M^p.$$

Similarly, $\operatorname{Coker}\left(\partial_U^p, \partial_V^p\right) \cong \mathrm{H}^{p+1}(U) \oplus \mathrm{H}^{p+1}(V)$ and $\operatorname{Coker} \partial_{U \cap V}^p \cong \mathrm{H}^{p+1}(U \cap V)$. ☐

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Example 2.33. Let $\mathbb{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$ be the torus. Then

$$\mathrm{H}^p\left(\mathbb{T}^2\right) = \begin{cases} \mathbb{R} & p = 0, 2 \\ \mathbb{R} \oplus \mathbb{R} & p = 1 \end{cases}.$$

We leave the proof as an exercise. [10]

Definition 2.34. Let M be a manifold, and let $U = \{U_i\}$ be an open cover of M. Then U is said to be **good** if for all $I = (i_1, \dots, i_p)$, $U_{i_1} \cap \dots \cap U_{i_p}$ is either $\emptyset$ or contractible.

Lemma 2.35. *Let M be a connected manifold which admits a finite good cover. Then for all $p \geq 0$, $\mathrm{H}^p(M)$ is a finite dimensional vector space.*

Exercise. Find a counterexample without assuming there exists a finite good cover.

Proof. Let U be a finite good cover. Define $k = \# U$. By induction on k.

$k = 1$. $M = U_1$ is contractible, so

$$\mathrm{H}^p(M) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & \text{otherwise} \end{cases}.$$

$k > 1$. Assume ok for covers with at most $k-1$ elements. Let $U = \bigcup_{i=1}^{k-1} U_i$ and $V = U_k$. Then $U \cup V = M$ and $U \cap V \neq \emptyset$, so Mayer-Vietoris holds. By induction $\mathrm{H}^p(U)$ and $\mathrm{H}^p(V)$ are finite dimensional, since $\mathrm{H}^p(U)$ is covered by $k-1$ of U_i and $\mathrm{H}^p(V)$ is contractible. Then $U \cap V = \bigcup_{i=1}^{k-1} (U_i \cap U_k)$, and $\{U_i \cap U_k\}$ is a good cover of $U \cap V$ with $k-1$ elements. [11] By induction $\mathrm{H}^p(U \cap V)$ is finite dimensional. Thus $\mathrm{H}^p(M)$ is also finite dimensional.

☐

[10] Exercise
[11] Exercise

Fact. Any manifold admits a good cover.

Theorem 2.36. *Let M be a compact connected manifold. Then $\mathrm{H}^p(M)$ is finite dimensional.*

Proof. Follows from the fact and Lemma 2.35. $\square$

2.5 Compactly supported de Rham cohomology

Let M be a manifold, and let $\omega \in \Omega_c^p(M)$. Then $\mathrm{d}\omega \in \Omega_c^{p+1}(M)$ and $\mathrm{d}^2 = 0$, so

$$\Omega_c^p(M) \xrightarrow{\mathrm{d}} \Omega_c^{p+1}(M) \xrightarrow{\mathrm{d}} \dots .$$

Definition 2.37. The p**-th compactly supported de Rham cohomology group** is

$$\mathrm{H}_c^p(M) = \mathcal{Z}_c^p(M) / \mathcal{B}_c^p(M) = \mathrm{Ker}\left(\mathrm{d} : \Omega_c^p(M) \to \Omega_c^{p+1}(M)\right) / \mathrm{Im}\left(\mathrm{d} : \Omega_c^{p-1}(M) \to \Omega_c^p(M)\right).$$

Example. If M is compact, then

$$\mathrm{H}_c^p(M) = \mathrm{H}^p(M), \qquad p \ge 0.$$

Proposition 2.38. *Let M be a non-compact connected manifold. Then*

$$\mathrm{H}_c^0(M) = 0.$$

Recall if M is connected $\mathrm{H}^0(M) = \mathbb{R}$, since $\mathrm{H}^0(M) = \{f \text{ constant on } M\}$.

Proof. $\mathrm{H}_c^0(M) = \{f \text{ constant on } M \text{ and with compact support}\}$. Since M is non-compact, if $f \in \Omega_c^0(M)$, then $\mathrm{supp}\, f \subsetneq M$. Thus there exists $x \in M$ such that $f(x) = 0$, so $f \equiv 0$, since f is constant. $\square$

Remark 2.39. Let $f : M \to N$ be a smooth morphism between manifolds, and let $\omega \in \Omega_c^p(N) \subset \Omega^p(N)$. Then $f^*\omega \in \Omega^p(M)$, and $\mathrm{supp}\, f^*\omega \subset f^{-1}(\mathrm{supp}\,\omega)$, which is not compact in general, so $f^*\omega \notin \Omega_c^p(M)$ in general. If f is **proper**, that is $f^{-1}(K)$ is compact for all compact subsets $K \subset N$, then $f^* : \Omega_c^p(N) \to \Omega_c^p(M)$ is well-defined.

Exercise. If f is a diffeomorphism then f^* induces an isomorphism $\mathrm{H}_c^p(N) \to \mathrm{H}_c^p(M)$.

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Definition 2.40. Let M_0 and M_1 be manifolds without boundary, and let $f_i : M_0 \to M_1$ be smooth morphisms for $i = 0, 1$. Then f_0 and f_1 are **properly smoothly homotopic** if there exists a smooth $H : M_0 \times [0,1] \to M$ such that $H(\cdot, i) = f_i(\cdot)$ for $i = 0, 1$ and H is proper. Then M_0 and M_1 are **properly smoothly homotopically equivalent** if there exist smooth morphisms $f : M_0 \to M_1$ and $g : M_1 \to M_0$ such that $f \circ g \sim \mathrm{id}_{M_1}$ and $g \circ f \sim \mathrm{id}_{M_0}$, where the equivalences are properly homotopic.

Notation. $f_t(\cdot) = H(\cdot, t) : M_0 \to M_1$.

Remark 2.41. To say that H is proper is not the same as saying f_t is proper for all t.

Exercise. Find H such that f_t is proper but H is not. A hint is to let $M_0 = M_1 = \mathbb{R}$ and $H : \mathbb{R} \times [0,1] \to \mathbb{R}$ such that $f_t^{-1}(0)$ is bounded for all t but $H^{-1}(0)$ is not.

Proposition 2.42. *If M_0 and M_1 are properly homotopically equivalent then*

$$\mathrm{H}_c^p(M_0) \cong \mathrm{H}_c^p(M_1).$$

Let M be a manifold, and let $i : U \hookrightarrow M$ be an open set. Then there exist linear **push-forwards**

$$i_* : \Omega_c^p(U) \to \Omega_c^p(M), \qquad p \ge 0.$$

Let $\omega \in \Omega_c^p(U)$. Then $\omega = 0$ outside U. We can define

$$i_*\omega = \begin{cases} \omega & \text{on } U \\ 0 & \text{on } M \setminus U \end{cases}.$$

If $j : V \hookrightarrow U$ and $i : U \hookrightarrow M$, then $(i \circ j)_* = i_* \circ j_*$.

Lemma 2.43. *Let M be a manifold, and let $i : U \hookrightarrow M$ be an immersion such that U is open. Then for all $p \ge 0$, $i_* : \Omega_c^p(U) \to \Omega_c^p(M)$ commutes with d, that is*

$$\mathrm{d}(i_* \omega) = i_* \mathrm{d}\omega, \qquad \omega \in \Omega_c^p(U).$$

In particular if ω is closed then $i_*\omega$ is closed, and if ω is exact then $i_*\omega$ is exact.

Proof.

$$\mathrm{d}(i_*\omega) = \begin{cases} \mathrm{d}\omega & \text{on } U \\ 0 & \text{on } M \setminus U \end{cases} = i_* \mathrm{d}\omega.$$

Let ω be closed, so $\mathrm{d}\omega = 0$. Then $\mathrm{d}(i_*\omega) = i_*\mathrm{d}\omega = 0$, so $i_*\omega$ is closed. Similarly for exactness. □

Let $U \hookrightarrow M$ be as before. Then there exist

$$i_* : \mathrm{H}_c^p(U) \to \mathrm{H}_c^p(M), \qquad p \ge 0.$$

Proposition 2.44 (Punctured manifolds)**.** *Let M be a manifold of dimension n, let $x \in M$, and let $i : M \setminus \{x\} \hookrightarrow M$. Then*

- *for all $p \ge 2$, $i_* : \mathrm{H}_c^p(M \setminus \{x\}) \to \mathrm{H}_c^p(M)$ is an isomorphism.*
- *for all $p \ge 1$, if M is compact $i_* : \mathrm{H}_c^p(M \setminus \{x\}) \to \mathrm{H}_c^p(M) = \mathrm{H}^p(M)$ is an isomorphism.*

Proof.

- Injectivity.

$p \ge 2$. Let $\omega \in \Omega_c^p(M \setminus \{x\})$ be closed such that $i_*[\omega] = 0$, so $[i_*\omega] = 0$ in $\mathrm{H}_c^p(M)$. The goal is $[\omega] = 0$. There exists $\eta \in \Omega_c^{p-1}(M)$ such that $i_*\omega = \mathrm{d}\eta$. By Poincaré lemma there exists $U \subset M$ containing x such that $\mathrm{H}^q(U) = 0$ for all $q \ge 1$. Then $i_*\omega = 0$ in a neighbourhood of x because $\operatorname{supp}\omega \subset M \setminus \{x\}$, so $\mathrm{d}\eta = 0$ in a neighbourhood of x. By taking U smaller we can assume η is closed. Since $p \ge 2$, $[\eta] \in \mathrm{H}^{p-1}(U) = 0$, so η is exact. Then there exists $\sigma \in \Omega^{p-2}(U)$ such that $\eta = \mathrm{d}\sigma$ on U. Let $(U, M \setminus \{x\})$ be an open cover of M, let $\left(f_U, f_{M\setminus\{x\}}\right)$ be a partition of unity, and let $\eta' = \eta - \mathrm{d}(i_*(f_U\sigma))$. On a neighbourhood of x, $\eta' = 0$ because $i_*(f_U\sigma) = \sigma$, so $\operatorname{supp}\eta' \subset M \setminus \{x\}$. Thus $\eta' \in \Omega_c^{p-1}(M \setminus \{x\})$ and $\omega = \mathrm{d}\eta'$, so $[\omega] = 0$.

$p = 1$. The same proof. Let $\omega \in \Omega_c^1(M \setminus \{x\})$ be closed such that $[i_*\omega] = 0$. There exists $\eta \in \Omega_c^0(M)$ such that $i_*\omega = \mathrm{d}\eta$. By taking an open set $U \subset M$ such that $x \in U$, we may assume $\mathrm{d}\eta = 0$, so $\eta = c$ is constant on U. Let $\eta' = \eta - c$. Then $\eta' = 0$ on U. If M is compact then $\eta' \in \Omega_c^0(M \setminus \{x\})$. Thus $\omega = \mathrm{d}\eta'$, so $[\omega] = 0$.

- Surjectivity.

$p \ge 1$. Let $[\omega] \in \Omega_c^p(M)$ such that ω is closed. By Poincaré lemma there exists an open $U \ni x$ such that ω is exact, so there exists $\sigma \in \Omega^{p-1}(U)$ such that $\omega = \mathrm{d}\sigma$. Let $\left(f_U, f_{M\setminus\{x\}}\right)$ be a partition of unity as before, and let $\omega' = \omega - \mathrm{d}(i_*(f_U\sigma))$. Then $\omega' = 0$ in a neighbourhood of x and $[\omega'] = [\omega]$, and $\omega'|_{M\setminus\{x\}} \in \Omega_c^p(M \setminus \{x\})$. Thus $\left[i_*\ \omega'|_{M\setminus\{x\}}\right] = [\omega'] = [\omega]$.

□

Exercise. Compute $\mathrm{H}_c^1\left(\mathbb{R}^2 \setminus \{0\}\right)$ by hands.

Example 2.45.

$$\mathrm{H}_c^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}.$$

Recall $\mathbb{R}^n \cong \mathrm{S}^n \setminus \{x\}$ for $x \in \mathrm{S}^n$. By Proposition 2.44, by $M = \mathrm{S}^n$,

$$\mathrm{H}_c^p(\mathbb{R}^n) = \mathrm{H}_c^p(\mathrm{S}^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}, \qquad p \ge 1,$$

and $\mathrm{H}_c^0(\mathbb{R}^n) = 0$.

Let M be a manifold such that $M = U \cup V$ for open U and V such that $U \cap V \neq \emptyset$, and let Lecture 16 Thursday 13/02/20

$$\begin{tikzcd} & U \arrow[dr, hook, "i_U"] & \\ U \cap V \arrow[ur, hook, "j_U"] \arrow[dr, hook, "j_V"'] & & M \\ & V \arrow[ur, hook, "i_V"'] & \end{tikzcd}, \qquad \begin{tikzcd} & \Omega^p(U) \arrow[dr, "i_{U*}"] & \\ \Omega^p(U \cap V) \arrow[ur, "j_{U*}"] \arrow[dr, "j_{V*}"'] & & \Omega^p(M) \\ & \Omega^p(V) \arrow[ur, "i_{V*}"'] & \end{tikzcd}, \qquad p \geq 0.$$

Proposition 2.46. *We have a short exact sequence*

$$0 \leftarrow \Omega^p(M) \xleftarrow{i} \Omega^p(U) \oplus \Omega^p(V) \xleftarrow{j} \Omega^p(U \cap V) \leftarrow 0,$$

where $i = i_{U*} + i_{V*}$ and $j = (-j_{U*}, j_{V*})$.

Proof.

- j is injective. Let $\omega \in \Omega^p(U \cap V)$ such that $j(\omega) = 0$, so $j_{U*}\omega = j_{V*}\omega = 0$. Then $\omega = 0$, so j is injective.
- $\operatorname{Ker} i = \operatorname{Im} j$. Let $\omega \in \Omega^p(U \cap V)$. Then $i(j(\omega)) = i(-j_{U*}\omega, j_{V*}\omega) = -i_{U*}j_{U*}\omega + i_{V*}j_{V*}\omega = 0$, so $\operatorname{Ker} i \supset \operatorname{Im} j$. Let $(\omega_1, \omega_2) \in \operatorname{Ker} i$. Then $i_{U*}\omega_1 + i_{V*}\omega_2 = 0$, so $i_{V*}\omega_1 = -i_{V*}\omega_2$, so $\operatorname{supp}\omega_1 \subset U \cap V$ and $\operatorname{supp}\omega_2 \subset U \cap V$, so there exists $\eta \in \Omega^p(U \cap V)$ such that $j_{U*}\eta = -\omega_1$ and $j_{V*}\eta = \omega_2$, so $(\omega_1, \omega_2) = j(\eta)$, so $\operatorname{Ker} i \subset \operatorname{Im} j$.
- i is surjective. Let $\omega \in \Omega_c^p(M)$, and let $\{f_U, f_V\}$ be a partition of unity with respect to $\{U, V\}$. Define $\omega_U = f_U \cdot \omega|_U \in \Omega_c^p(U)$ and $\omega_V = f_V \cdot \omega|_V \in \Omega_c^p(V)$. Then $i(\omega_U, \omega_V) = i_{U*}\omega_U + i_{V*}\omega_V = f_U \cdot \omega + f_V \cdot \omega = (f_U + f_V) \cdot \omega = \omega$.

$\square$

Thus for all p we get

$$\begin{tikzcd} 0 \arrow[r] & \Omega_c^p(U \cap V) \arrow[r] \arrow[d, "\mathrm{d}"] & \Omega_c^p(U) \oplus \Omega_c^p(V) \arrow[r] \arrow[d, "(\mathrm{d},\mathrm{d})"] & \Omega_c^p(M) \arrow[r] \arrow[d, "\mathrm{d}"] & 0 \\ 0 \arrow[r] & \Omega_c^{p+1}(U \cap V) \arrow[r] & \Omega_c^{p+1}(U) \oplus \Omega_c^{p+1}(V) \arrow[r] & \Omega_c^{p+1}(M) \arrow[r] & 0 \end{tikzcd}.$$

Theorem 2.47. *There exists $\delta : \mathrm{H}_c^p(M) \to \mathrm{H}_c^{p+1}(U \cap V)$ such that*

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}_c^p(U \cap V) \arrow[r] & \mathrm{H}_c^p(U) \oplus \mathrm{H}_c^p(V) \arrow[r] & \mathrm{H}_c^p(M) \arrow[dll, "\delta"] & \\ & \mathrm{H}_c^{p+1}(U \cap V) \arrow[r] & \mathrm{H}_c^{p+1}(U) \oplus \mathrm{H}_c^{p+1}(V) \arrow[r] & \mathrm{H}_c^{p+1}(M) \arrow[r] & \dots \end{tikzcd}.$$

Proof. Same proof as Mayer-Vietoris for $\mathrm{H}^p(M)$. $\square$

2.6 Poincaré duality

Let M be an orientable manifold. Then $\mathrm{H}^p(M) \cong \mathrm{H}_c^{n-p}(M)^*$, the dual of $\mathrm{H}_c^{n-p}(M)$.

Proposition 2.48. *Let M be a manifold. Then the bilinear map*

$$\begin{array}{cccc} \cup & : & \mathrm{H}^p(M) \times \mathrm{H}^q(M) & \longrightarrow & \mathrm{H}^{p+q}(M) \\ & & ([\omega], [\eta]) & \longmapsto & [\omega \wedge \eta] \end{array}$$

is well-defined, and

$$[\omega] \cup [\eta] = (-1)^{p \cdot q} [\eta] \cup [\omega].$$

Proof. Follows from the Leibniz rule and Proposition 1.6. $\square$

Lemma 2.49. *Let M be oriented without boundary of dimension n. Then there exists a linear map*

$$\begin{array}{cccc} \mathrm{I}_M & : & \mathrm{H}_{\mathrm{c}}^n(M) & \longrightarrow & \mathbb{R} \\ & & [\omega] & \longmapsto & \int_M \omega \end{array},$$

and I_M is surjective.

I_M is called **integration**.

Proof. Let $\omega \in \Omega_{\mathrm{c}}^n(M)$ such that $[\omega] = 0$, so ω is exact. By Stokes $\int_M \omega = 0$, so I_M is well-defined and it is linear. It is enough to show there exists closed $\omega \in \Omega_{\mathrm{c}}^n(M)$ such that $\int_M \omega \neq 0$. Take a volume form ω_0, which exists because M is oriented. Take $f \in \mathrm{C}^\infty(M)$ for $f \geq 0$ and with compact support. Let $\omega = f \cdot \omega_0 \in \Omega_{\mathrm{c}}^n(M)$. Then ω is closed because $\Omega_{\mathrm{c}}^{n+1}(M) = 0$ and $\int_M \omega = \int_M (f \cdot \omega_0) > 0$, by definition of volume forms. ☐

Example 2.50. Let $M = \mathrm{S}^n$, and let $\omega \in \Omega_{\mathrm{c}}^n(M)$ such that $\int_M \omega = 0$. We want to show that ω is exact. Since M is compact, $\mathrm{H}_{\mathrm{c}}^n(M) = \mathrm{H}^n(M) = \mathbb{R}$. By Lemma 2.49 $\mathrm{I}_M : \mathrm{H}_{\mathrm{c}}^n(M) \to \mathbb{R}$ is surjective, and $\mathrm{H}_{\mathrm{c}}^n(M) = \mathbb{R}$, so I_M is injective. Since $\int_M \omega = 0$, $\mathrm{I}_M([\omega]) = 0$, so $[\omega] = 0$. Thus ω is exact.

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Let M be a connected manifold of dimension n. If $\omega_2 \in \mathrm{H}_{\mathrm{c}}^q(M)$ then $[\omega_1 \wedge \omega_2] \in \mathrm{H}_{\mathrm{c}}^{p+q}(M)$. Then

$$\cup : \mathrm{H}^p(M) \times \mathrm{H}_{\mathrm{c}}^q(M) \to \mathrm{H}_{\mathrm{c}}^{p+q}(M).$$

Let M be an oriented manifold without boundary of dimension n. Then

$$\begin{array}{cccc} \mathrm{I}_M & : & \mathrm{H}_{\mathrm{c}}^n(M) & \longrightarrow & \mathbb{R} \\ & & [\omega] & \longmapsto & \int_M \omega \end{array}.$$

Choose $q = n - p$. Then

$$\mathrm{I}_M \circ \cup : \mathrm{H}^p(M) \times \mathrm{H}_{\mathrm{c}}^{n-p}(M) \to \mathrm{H}_{\mathrm{c}}^n(M) \to \mathbb{R}.$$

Recall that if $\phi : V \times W \to \mathbb{R}$ is bilinear, then there exists

$$\begin{array}{ccc} V & \longrightarrow & W^* = \mathrm{Hom}(W, \mathbb{R}) \\ v & \longmapsto & \phi_v \end{array}, \qquad \begin{array}{cccc} \phi_v & : & W & \longrightarrow & \mathbb{R} \\ & & w & \longmapsto & \phi(v, w) \end{array}.$$

Thus, we get

$$\mathrm{H}^p(M) \to \mathrm{H}_{\mathrm{c}}^{n-p}(M)^*.$$

Poincaré duality says that this is an isomorphism.

Example. Assume M is compact and oriented. Then $\mathrm{H}^p(M) \xrightarrow{\sim} \mathrm{H}^{n-p}(M)$, so $\mathrm{b}^p(M) = \mathrm{b}^{n-p}(M)$.

Example 2.51. Let $U \subset \mathbb{R}^n$ be an open subset diffeomorphic to $\mathbb{R}^n$. Then

$$\mathrm{H}^p(U) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \qquad \mathrm{H}_{\mathrm{c}}^p(U) = \begin{cases} 0 & p < n \\ \mathbb{R} & p = n \end{cases}.$$

We want to show that Poincaré duality holds. We just need to check that Poincaré duality holds for $p = 0$. It is enough to show that $\phi : \mathrm{H}^0(U) \to \mathrm{H}_{\mathrm{c}}^n(U)^*$ is injective, that is there exists ω such that $\phi(\omega) \neq 0$. Given $\omega \in \mathrm{H}^0(U)$,

$$\begin{array}{cccc} \phi(\omega) & : & \mathrm{H}_{\mathrm{c}}^n(U) & \longrightarrow & \mathbb{R} \\ & & \eta & \longmapsto & \int_U \eta \wedge \omega \end{array}.$$

Then $\omega = c$ is a constant function on U, so

$$\begin{array}{cccc} \phi(\omega) & : & \mathrm{H}_{\mathrm{c}}^n(U) & \longrightarrow & \mathbb{R} \\ & & \eta & \longmapsto & \int_U c\omega \end{array}.$$

If $c \neq 0$ there exists η such that this map is not zero, so $\phi(\omega) \neq 0$. Thus ϕ is an isomorphism.

We will prove the following.

Theorem 2.52 (Poincaré duality)**.** *Assume that M is an oriented manifold, without boundary, such that there exists a finite open cover $U = \{U_i\}$ such that $U_{i_1} \cap \dots \cap U_{i_q}$ is $\emptyset$ or diffeomorphic to $\mathbb{R}^n$. Then*

$$\mu_M : \mathrm{H}^p(M) \xrightarrow{\sim} \mathrm{H}_{\mathrm{c}}^{n-p}(M)^*, \qquad p \ge 0, \qquad n = \dim M$$

is an isomorphism.

Any compact manifold M admits such a cover.

Lemma 2.53. *Let*

$$C^1 \xrightarrow{f^1} C^2 \xrightarrow{f^2} C^3$$

be exact, where C^i are vector spaces of finite dimension. Then there exists

$$\left(C^3\right)^* \xrightarrow{\left(f^2\right)^*} \left(C^2\right)^* \xrightarrow{\left(f^1\right)^*} \left(C^1\right)^*,$$

which is also exact, where $\left(f^1\right)^* \phi = \phi \circ f^1$ and $\left(f^2\right)^* \phi = \phi \circ f^2$.

Proof. By assumption $\operatorname{Ker} f^2 = \operatorname{Im} f^1$. We want to prove $\operatorname{Ker}\left(f^1\right)^* = \operatorname{Im}\left(f^2\right)^*$.

- Let $\phi \in \operatorname{Im}\left(f^2\right)^*$. Then there exists $\psi \in \left(C^3\right)^*$ such that $\left(f^2\right)^* \psi = \phi$, so $\psi \circ f^2 = \phi$, so $0 = \psi \circ f^2 \circ f^1 = \phi \circ f^1 = \left(f^1\right)^* \phi$, so $\phi \in \operatorname{Ker}\left(f^1\right)^*$.
- Let $\phi \in \operatorname{Ker}\left(f^1\right)^*$. Then $\phi \circ f^1 = 0$, so $\operatorname{Ker} f^2 = \operatorname{Im} f^1 \subset \operatorname{Ker} \phi$, so there exists $\overline{\phi} : C^2 / \operatorname{Ker} f^2 \to \mathbb{R}$, so there exists $\psi : C^3 \to \mathbb{R}$ extending $\overline{\phi}$ such that $\psi \circ f^2 = \phi$, so $\left(f^2\right)^* \psi = \phi$, so $\phi \in \operatorname{Im}\left(f^2\right)^*$.

$\square$

Lemma 2.54 (Five lemma)**.** *Let*

$$\begin{array}{ccccccccc}
C^1 & \xrightarrow{f^1} & C^2 & \xrightarrow{f^2} & C^3 & \xrightarrow{f^3} & C^4 & \xrightarrow{f^4} & C^5 \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\
D^1 & \xrightarrow{g^1} & D^2 & \xrightarrow{g^2} & D^3 & \xrightarrow{g^3} & D^4 & \xrightarrow{g^4} & D^5
\end{array},$$

such that the horizontal lines are exact. Suppose

- *α_1 is surjective,*
- *α_5 is injective, and*
- *α_2 and α_4 are isomorphisms.*

Then α_3 is an isomorphism.

Proof. Let $x \in C^3$ such that $\alpha_3(x) = 0$, so if $y = f^3(x)$ then $\alpha_4(y) = 0$. Since α_4 is an isomorphism, $y = 0$. Then $x \in \operatorname{Ker} f^3 = \operatorname{Im} f^2$, so there exists $z \in C^2$ such that $f^2(z) = x$. Let $w = \alpha_2(z)$ then $g^2(w) = 0$, so $w \in \operatorname{Ker} g^2 = \operatorname{Im} g^1$. Then there exists $t \in D^1$ such that $g^1(t) = w$. Since α_1 is surjective there exists $s \in C^1$ such that $\alpha_1(s) = t$, so

$$\begin{array}{ccccccccc}
s \in C^1 & \xrightarrow{f^1} & z \in C^2 & \xrightarrow{f^2} & x \in C^3 & \xrightarrow{f^3} & y \in C^4 & \xrightarrow{f^4} & C^5 \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\
t \in D^1 & \xrightarrow{g^1} & w \in D^2 & \xrightarrow{g^2} & 0 \in D^3 & \xrightarrow{g^3} & 0 \in D^4 & \xrightarrow{g^4} & D^5
\end{array}.$$

We want to show that $f^1(s) = z$, and $\alpha_2\left(f^1(s)\right) = g^1\left(\alpha_1(s)\right) = g^1(t) = w = \alpha_2(z)$, so $f^1(s) = z$, since α_2 is injective. Thus $x = f^2(z) = f^2\left(f^1(s)\right) = 0$, so α_3 is injective. Show that α_3 is surjective. [12] $\square$

[12] Exercise

Proof of Theorem 2.52. Let $N = \#\mathcal{U}$. We proceed by induction on N. Then $N = 1$ is ok, so let $N > 1$. Let $U = \bigcup_{i=1}^{N-1} U_i$ and $V = U_N$, so $M = U \cup V$. Both U and V, and $U \cap V$, satisfy Poincaré duality by induction. The idea is to use classical Mayer-Vietoris and compact support Mayer-Vietoris, and the five lemma. By Mayer-Vietoris, Lecture 18 Tuesday 18/02/20

$$\mathrm{H}^{p-1}(U) \oplus \mathrm{H}^{p-1}(V) \xrightarrow{g} \mathrm{H}^{p-1}(U \cap V) \xrightarrow{\delta} \mathrm{H}^p(M) \xrightarrow{f} \mathrm{H}^p(U) \oplus \mathrm{H}^p(V) \to \dots,$$

where $f = (i_U^*, i_V^*)$ and $g = j_V^* - j_U^*$. By compact support Mayer-Vietoris,

$$\dots \to \mathrm{H}_\mathrm{c}^{n-p}(U) \oplus \mathrm{H}_\mathrm{c}^{n-p}(V) \xrightarrow{i} \mathrm{H}_\mathrm{c}^{n-p}(M) \xrightarrow{\delta_\mathrm{c}} \mathrm{H}_\mathrm{c}^{n-(p-1)}(M) \xrightarrow{j} \mathrm{H}_\mathrm{c}^{n-(p-1)}(U) \oplus \mathrm{H}_\mathrm{c}^{n-(p-1)}(V),$$

where $j = (-j_{U*}, j_{V*})$ and $i = i_{U*} + i_{V*}$. Taking the dual, by Lemma 2.53,

$$\mathrm{H}_\mathrm{c}^{n-(p-1)}(U)^* \oplus \mathrm{H}_\mathrm{c}^{n-(p-1)}(V)^* \xrightarrow{j^*} \mathrm{H}_\mathrm{c}^{n-(p-1)}(U \cap V)^* \xrightarrow{\delta_\mathrm{c}^*} \mathrm{H}_\mathrm{c}^{n-p}(M)^* \xrightarrow{i^*} \mathrm{H}_\mathrm{c}^{n-p}(U)^* \oplus \mathrm{H}_\mathrm{c}^{n-p}(V)^* \to \dots.$$

We get a diagram

$$\begin{tikzcd}
\mathrm{H}^{p-1}(U) \oplus \mathrm{H}^{p-1}(V) \arrow[r, "g"] \arrow[d, "n_{p-1} \cdot \mu_U \oplus \mu_V"] & \mathrm{H}^{p-1}(U \cap V) \arrow[r, "\delta"] \arrow[d, "n_{p-1} \cdot \mu_{U \cap V}"] & \mathrm{H}^p(M) \arrow[r, "f"] \arrow[d, "n_p \cdot \mu_M"] & \mathrm{H}^p(U) \oplus \mathrm{H}^p(V) \arrow[r] \arrow[d, "n_p \cdot \mu_U \oplus \mu_V"] & \dots \\
\mathrm{H}_\mathrm{c}^{n-(p-1)}(U)^* \oplus \mathrm{H}_\mathrm{c}^{n-(p-1)}(V)^* \arrow[r, "j^*"] & \mathrm{H}_\mathrm{c}^{n-(p-1)}(U \cap V)^* \arrow[r, "\delta_\mathrm{c}^*"] & \mathrm{H}_\mathrm{c}^{n-p}(M)^* \arrow[r, "i^*"] & \mathrm{H}_\mathrm{c}^{n-p}(U)^* \oplus \mathrm{H}_\mathrm{c}^{n-p}(V)^* \arrow[r] & \dots
\end{tikzcd},$$

where $n_0 = 1$ and $n_p = (-1)^{p-1} n_{p-1}$. The goal is to show that μ_M is an isomorphism. The idea is by the five lemma, it is enough to show that

1. all the other vertical arrows are isomorphisms, and
2. the diagram is commutative.

We know 1 is ok by induction on N. We need to show 2.

- The first square. We want to show that $\mu_{U \cap V} \circ g = j^* \circ (\mu_U \oplus \mu_V)$. Let $\omega_U \in \Omega^{p-1}(U)$ and $\omega_V \in \Omega^{p-1}(V)$ be closed forms. We want to show

$$\mu_{U \cap V}(g([\omega_U], [\omega_V])) = j^*(\mu_U([\omega_U]), \mu_V([\omega_V])),$$

  in $\mathrm{H}_\mathrm{c}^{n-(p-1)}(U \cap V)^*$, that is we want to show that on any element of $\mathrm{H}_\mathrm{c}^{n-(p-1)}(U \cap V)$ they coincide. Let $\eta \in \Omega_\mathrm{c}^{n-(p-1)}(U \cap V)$. Recall $g = j_V^* - j_U^*$. Then

$$\int_{U \cap V} g(\omega_U, \omega_V) \wedge \eta = -\int_U \omega_U \wedge j_{U*}\eta + \int_V \omega_V \wedge j_{V*}\eta,$$

  since $g(\omega_U, \omega_V) = j_V^*\omega_V - j_U^*\omega_U$. Lecture 19 Thursday 20/02/20

- The second square. We want an explicit construction of δ and δ_c. Let $\omega \in \Omega^p(M)$ be a closed form, and let $\{f_U, f_V\}$ be a partition of the unity with respect to $\{U, V\}$. Define

$$\omega_U = f_U \cdot \omega|_U \in \Omega_\mathrm{c}^p(U), \qquad \omega_V = f_V \cdot \omega|_V \in \Omega_\mathrm{c}^p(V),$$

  so $(\omega_U, \omega_V) \in \Omega_\mathrm{c}^p(U) \oplus \Omega_\mathrm{c}^p(V)$. Recall $i = i_{U*} + i_{V*}$. Then

$$i(\omega_U, \omega_V) = i_{U*}\omega_U + i_{V*}\omega_V = \omega_U + \omega_V = f_U \cdot \omega + f_V \cdot \omega = \omega.$$

  If ω is closed, then $i(\mathrm{d}\omega_U, \mathrm{d}\omega_V) = \mathrm{d}(i_{U*}\omega_U) + \mathrm{d}(i_{V*}\omega_V) = 0$, so $(\mathrm{d}\omega_U, \mathrm{d}\omega_V) \in \operatorname{Ker} i = \operatorname{Im} j \subset \Omega_\mathrm{c}^{p+1}(U) \oplus \Omega_\mathrm{c}^{p+1}(V)$. Since j is injective there exists a unique $\delta_\mathrm{c}(\omega) \in \Omega_\mathrm{c}^{p+1}(U \cap V)$ such that $j(\delta_\mathrm{c}(\omega)) = (\mathrm{d}\omega_U, \mathrm{d}\omega_V)$. Since $f_U + f_V = 1$, $\mathrm{d}f_U + \mathrm{d}f_V = 0$, so $\mathrm{d}f_U = -\mathrm{d}f_V$. Then

$$j(\delta_\mathrm{c}(\omega)) = (\mathrm{d}\omega_U, \mathrm{d}\omega_V) = (\mathrm{d}f_U \wedge \omega|_U, \mathrm{d}f_V \wedge \omega|_V) = (-\mathrm{d}f_V \wedge \omega|_U, \mathrm{d}f_V \wedge \omega|_V) = j(\mathrm{d}f_V \wedge \omega|_{U \cap V}).$$

Since j is injective, $\delta_{\mathrm{c}}(\omega) = \mathrm{d}f_V \wedge \omega|_{U\cap V}$, so $\delta_{\mathrm{c}} : \Omega_{\mathrm{c}}^p(M) \to \Omega_{\mathrm{c}}^{p+1}$. Let η be a form on M. Since $\delta_{\mathrm{c}}(\mathrm{d}\eta) = \mathrm{d}f_V \wedge \mathrm{d}\eta|_{U\cap V} = -\mathrm{d}\delta_{\mathrm{c}}(\eta)$, δ_{c} maps closed forms to closed forms and exact forms to exact forms, so

$$\begin{array}{cccc} \delta_{\mathrm{c}} & : & \mathrm{H}_{\mathrm{c}}^p(M) & \longrightarrow & \mathrm{H}_{\mathrm{c}}^{p+1}(U\cap V) \\ & & \omega & \longmapsto & \mathrm{d}f_V \wedge \omega|_{U\cap V} \end{array}.$$

By construction, it makes the long exact sequence exact. Similarly

$$\begin{array}{cccc} \delta & : & \mathrm{H}^p(U\cap V) & \longrightarrow & \mathrm{H}^{p+1}(M) \\ & & \omega & \longmapsto & \begin{cases} \mathrm{d}f_V \wedge \omega & \text{on } U\cap V \\ 0 & \text{otherwise} \end{cases} \end{array}.$$

Now we check that the second square is commutative, that is

$$n_{p-1}\cdot\mu_M(\delta([\omega_1])) = n_p\cdot\delta_{\mathrm{c}}^*(\mu_{U\cap V}([\omega_1])), \qquad \omega_1 \in \Omega^{p-1}(U\cap V).$$

That is,

$$n_{p-1}\int_M \delta(\omega_1)\wedge\omega_2 = n_p\int_{U\cap V}\omega_1\wedge\delta_{\mathrm{c}}(\omega_2), \qquad \omega_2 \in \Omega_{\mathrm{c}}^{n-p}(M).$$

Then for all $\omega_2 \in \Omega_{\mathrm{c}}^{n-p}(M)$,

$$n_{p-1}\int_M \delta(\omega_1)\wedge\omega_2 = n_{p-1}\int_{U\cap V}\mathrm{d}f_V\wedge\omega_1\wedge\omega_2 = n_p\int_{U\cap V}\omega_1\wedge\mathrm{d}f_V\wedge\omega_2 = n_p\int_{U\cap V}\omega_1\wedge\delta_{\mathrm{c}}(\omega_2).$$

- The third square. To check $(\mu_U\oplus\mu_V)\circ f = i^*\circ\mu_M$, so

$$(\mu_U\oplus\mu_V)(f([\omega])) = i^*(\mu_M([\omega])), \qquad \omega\in\Omega^p(M),$$

in $\mathrm{H}_{\mathrm{c}}^{n-p}(U)^*\oplus\mathrm{H}_{\mathrm{c}}^{n-p}(V)^*$. Let $\eta_U\in\Omega_{\mathrm{c}}^{n-p}(U)$ and $\eta_V\in\Omega_{\mathrm{c}}^{n-p}(V)$. Then

$$\int_U \omega|_U\wedge\eta_U + \int_V \omega|_V\wedge\eta_V = \int_M \omega\wedge i(\eta_U,\eta_V).$$

□

The following is an easy consequence.

Corollary 2.55. *Let M be an oriented compact connected manifold of dimension n. Then*

$$\mathrm{H}^n(M) = \mathbb{R},$$

and

$$\mathrm{H}_{\mathrm{c}}^p(M\setminus\{x\}) = \mathrm{H}^p(M), \qquad x\in M, \qquad 1\le p<n.$$

Definition 2.56. The **Euler characteristic** of M is

$$\chi(M) = \sum_{p=0}^n (-1)^p \dim\mathrm{H}^p(M).$$

Corollary 2.57. *If M is a compact oriented manifold of odd dimension then $\chi(M) = 0$.*

Proof. By Poincaré duality, $\dim\mathrm{H}^i(M) = \dim\mathrm{H}^{n-i}(M)$. □

2.7 Degree of a morphism

Let M and N be connected oriented manifolds of dimension n, and let $f : M\to N$ be a proper smooth morphism. Then $f^* : \mathrm{H}_{\mathrm{c}}^n(N)\cong\mathbb{R}\to\mathrm{H}_{\mathrm{c}}^n(M)\cong\mathbb{R}$ by Poincaré duality and connectedness, so $f(x) = c\cdot x$ for some $\deg f = c\in\mathbb{R}$. For any $\omega\in\Omega_{\mathrm{c}}^n(M)$,

$$\int_M f^*\omega = \deg f\cdot\int_M \omega.$$

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Proposition 2.58. *Let M, N, P be connected oriented manifolds of dimension n.*

- *If $f : M \to N$ and $g : N \to P$ are smooth and proper morphisms then*

$$\deg (g \circ f) = \deg f \cdot \deg g$$

- *If f is a diffeomorphism then*

$$\deg g = \begin{cases} 1 & f \text{ is orientation preserving} \\ -1 & \text{otherwise} \end{cases}.$$

- *If $f, g : M \to N$ are smooth proper and properly homotopic equivalent then*

$$\deg f = \deg g.$$

Theorem 2.59 (Mapping degree theorem)**.** *Let $f : M \to N$ be a proper smooth morphism between connected oriented manifolds of dimension n. Then $\deg f \in \mathbb{Z}$.*

Definition 2.60. Let $f : M \to N$ be a smooth morphism. Then $y \in N$ is **regular** if for all $x \in f^{-1}(y)$, $\mathrm{D}f_x$ has maximal rank.

Theorem 2.61 (Preimage theorem)**.** *Let $f : M \to N$ be a smooth morphism, and let $y \in N$ be a regular value. Then $f^{-1}(y)$ is a manifold of dimension $\dim M - \operatorname{rk} \mathrm{D}f_x$ where $x \in f^{-1}(y)$.*

Theorem 2.62 (Implicit function theorem)**.** *Let $f : M \to N$ be a smooth morphism, and let $x \in M$ be such that $\mathrm{D}f_x$ is an isomorphism. Then there exists an open $x \in U \subset M$ such that $f|_U : U \to f(U)$ is an isomorphism.*

Theorem 2.63 (Sard's theorem)**.** *Let $f : M \to N$ be smooth. Then if $Z \subset N$ is the set of regular values of f then $Z \cap f(M)$ is dense in $f(M)$.*

Proof of Theorem 2.59. Recall $\dim M = \dim N$, and if $\omega \in \Omega_{\mathrm{c}}^n(N)$ and if $\mathrm{D}f_x$ is of rank less than n for all x, then $\deg f = 0$. We are done. In particular we may assume there exists x such that $\mathrm{D}f_x$ has rank equal to n. Let $y = f(x)$. By Sard's theorem, we may assume that for all $x \in f^{-1}(y)$, $\mathrm{D}f_x$ has rank n. By the preimage theorem $f^{-1}(y)$ is a manifold of dimension zero, so

$$f^{-1}(y) = \{x_1, \dots, x_k\}$$

is a finite set, because f is proper. By the implicit function theorem, for all i there exists an open set $U_i \ni x$ such that $f|_U$ is a diffeomorphism and $f(U_i) = U$. Let $\omega \in \Omega_{\mathrm{c}}^n(N)$ be such that $\int_U \omega = 1$ and $\operatorname{supp} \omega \subset U$. Since $f|_{U_i}$ is a diffeomorphism

$$\int_{U_i} f|_{U_i}^* \omega = \operatorname{sgn}(\det \mathrm{D}f_x) \int_U \omega,$$

and $f|_{U_i}^* \omega$ has support in U_i. Since $\operatorname{supp} f^* \omega \subset \bigcup_i U_i$,

$$\int_M f^* \omega = \sum_{i=1}^k \int_{U_i} f^* \omega = \sum_{i=1}^k \operatorname{sgn}(\det \mathrm{D}f_x) \int_U \omega = \sum_{i=1}^k \operatorname{sgn}(\det \mathrm{D}f_x) \int_M \omega,$$

so $\deg f = \sum_{i=1}^k \operatorname{sgn}(\det \mathrm{D}f_{x_i}) \in \mathbb{Z}$, which does not depend on y, if y is a regular point. $\square$

As an immediate consequence of the proof of the theorem, we get the following.

Corollary 2.64. *Let $f : M \to N$ be a proper smooth morphism between connected oriented manifolds of dimension n. Assume that f is not surjective. Then $\deg f = 0$.*

Example 2.65. Let $M = \mathrm{S}^n = N$, and let

$$\begin{array}{cccccc} f & : & M & \longrightarrow & N \\ & & x & \longmapsto & -x \end{array}$$

be the antipodal map. Claim that $\deg f = (-1)^{n+1}$. Let $i : \mathrm{S}^n \hookrightarrow \mathbb{R}^{n+1}$, let

$$\widetilde{\omega} = x_1 \mathrm{d}x_2 \wedge \dots \wedge \mathrm{d}x_{n+1} \in \Omega^n\left(\mathbb{R}^{n+1}\right),$$

and let $\omega = i_* \widetilde{\omega} \in \Omega^n(\mathrm{S}^n)$. By Stokes and $\mathrm{S}^n = \partial \mathrm{D}^{n+1}$,

$$\int_{\mathrm{S}^n} \omega = \int_{\mathrm{S}^n} i^* \widetilde{\omega} = \int_{\mathrm{D}^{n+1}} \mathrm{d}\widetilde{\omega} = \int_{\mathrm{D}^{n+1}} \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_{n+1} \neq 0,$$

so f can be extended to

$$\begin{array}{cccccc} \widetilde{f} & : & \mathbb{R}^{n+1} & \longrightarrow & \mathbb{R}^{n+1} \\ & & x & \longmapsto & -x \end{array}.$$

Then $\widetilde{f} \circ i = i \circ f$ and $\widetilde{f}^* \widetilde{\omega} = (-1)^{n+1} \widetilde{\omega}$, so

$$f^* \omega = f^* i^* \widetilde{\omega} = (i \circ f)^* \widetilde{\omega} = \left(\widetilde{f} \circ i\right)^* \widetilde{\omega} = i^* \widetilde{f}^* \widetilde{\omega} = (-1)^{n+1} i^* \widetilde{\omega} = (-1)^{n+1} \omega.$$

Thus

$$(-1)^{n+1} \int_{\mathrm{S}^n} \omega = \int_{\mathrm{S}^n} f^* \omega = \deg f \int_{\mathrm{S}^n} \omega,$$

so $\deg f = (-1)^{n+1}$.

3 Morse theory

Definition 3.1. Let M be a manifold of dimension n, and let $f : M \to \mathbb{R}$ be smooth. A **critical point** of f is a point $x \in M$ such that $\mathrm{D}f_x = 0$, that is if $x_1, \dots, x_n$ are local coordinates at x, then

$$\frac{\partial}{\partial x_i} f(x) = 0, \qquad i = 1, \dots, n.$$

For such x, we define the Hessian of f to be

$$\mathrm{H}_f = \left(\frac{\partial^2}{\partial x_i \partial x_j} f(x) \right).$$

Then x is called **non-degenerate** if $\det \mathrm{H}_f(x) \neq 0$. A function such that every critical point is non-degenerate is called a **Morse function**.

Fact. By Sard's theorem most of the functions satisfy this property.