M4P55 Commutative Algebra

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Autumn 2019

Syllabus

Contents

0 Introduction

Lecture 1
Thursday
03/10/19

The prerequisites are

- groups,
- rings,
- fields, and
- a solid linear algebra.

This course is good for

- algebraic geometry, and
- algebraic number theory.

The following are books.

- M Reid, Undergraduate commutative algebra, 1995
- M F Atiyah and I G Macdonald, Introduction to commutative algebra, 1969

The following is the structure of the course.

- Generalities on rings, such as ideals, and examples.
- Localisation of rings between a ring R and the fraction field K of R, such as $\mathbb{Z}$ and $\mathbb{Q}$.
- Finiteness conditions of Noetherian rings and Artinian rings.
- Integral closure and normal rings, such as $\mathbb{Z}[i] \subset \mathbb{Q}(i)$ and $\mathbb{Z}\left[\sqrt{-3}\right] \subset \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right] \subset \mathbb{Q}\left(\sqrt{-3}\right)$.
- Discrete valuation rings.
- Completion of rings with topology.

1 Rings and ideals

Definition 1.1. A **commutative ring** is a set $(A, +, \cdot, 0, 1)$ such that

1. $(A, +, 0)$ is an abelian group,
2. for all $x, y, z \in A$,
   - $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
   - $x \cdot y = y \cdot x$,
   - $x \cdot (y + z) = x \cdot y + x \cdot z$, and
3. for all $x \in A$, $x \cdot 1 = 1 \cdot x = x$.

Remark 1.2.

- One is uniquely determined by 3, since $1' = 1' \cdot 1 = 1$.
- If $1 = 0$, then $0 = x \cdot 0 = x \cdot 1 = x$, since

$$x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0,$$

so $x \cdot 0 = 0$. So every element is zero. Hence $R = \{0\}$.

Definition 1.3. A **homomorphism of rings** $f : A \to B$ is a map such that for all $x, y \in A$,

$$f(x + y) = f(x) + f(y), \qquad f(xy) = f(x) f(y), \qquad f(1) = 1.$$

Example. If $A \subset B$ is closed under $+$ and $\cdot$, and $1 \in A$, then

$$\begin{array}{ccc} A & \longrightarrow & B \\ x & \longmapsto & x \end{array}$$

is a homomorphism.

Remark 1.4.

- A composition of homomorphisms is a homomorphism.
- An **isomorphism** is a bijective homomorphism.

Definition 1.5. A subset I of a ring A is an **ideal** if I is a subgroup of the additive group $(A, +)$ which is closed under multiplication by elements of A, so $xI \subset I$ for any $x \in A$. Sometimes this is written as $I \triangleleft A$. In this case the **quotient group** A/I is naturally a ring, where $(x + I)(y + I)$ is defined as $xy + I$.

Proposition 1.6. *Let I be an ideal of a commutative ring A. Then there is a natural bijection between the ideals $J \subset A$ such that $I \subset J$ and the ideals of A/I.*

Proof. Let

$$\begin{array}{ccc} A & \longrightarrow & A/I \\ x & \longmapsto & x + I \end{array}$$

be the natural surjective map. Send J to its image under this map. $\square$

Definition 1.7. If $f : A \to B$ is a homomorphism, then

$$\operatorname{Ker} f = \{x \in A \mid f(x) = 0\}$$

is an ideal in A, and

$$\operatorname{Im} f = f(A) \cong A / \operatorname{Ker} f \subset B.$$

2 Polynomials and formal power series

Lecture 2 Tuesday 08/10/19

Definition 2.1. Let R be a ring. The **polynomial ring** with coefficients in R is

$$R[x] = \{a_0 + \dots + a_n x^n \mid a_i \in R,\ n \in \mathbb{Z}_{\geq 0}\}.$$

The addition is coefficient-wise, and the multiplication is given by the formula

$$\left(\sum_{i \geq 0} a_i x^i\right) \left(\sum_{j \geq 0} b_j x^j\right) = \sum_{i \geq 0} \left(\sum_{j+k=i,\ j \geq 0,\ k \geq 0} a_j b_k\right) x^i,$$

where all but finitely many coefficients are zero. Define

$$R[x_1, \dots, x_n] = R[x_1] \dots [x_n] = \left\{ \sum_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} x_1^{i_1} \dots x_n^{i_n} \;\middle|\; a_{i_1, \dots, i_n} \in R \right\},$$

where all but finitely many coefficients $a_{i_1, \dots, i_n}$ are equal to zero.

Definition 2.2. The **ring of formal power series** with coefficients in R is

$$R[[t]] = \{a_0 + a_1 t + \dots \mid a_i \in R\}.$$

The addition is coefficient-wise, and the multiplication is given by the formula

$$\left(\sum_{i \geq 0} a_i t^i\right) \left(\sum_{j \geq 0} b_j t^j\right) = \sum_{i \geq 0} \left(\sum_{j+k=i,\ j \geq 0,\ k \geq 0} a_j b_k\right) x^i.$$

Define

$$R[[t_1, \dots, t_n]] = R[[t_1]] \dots [[t_n]].$$

In $R[[t]]$ many products equal one unlike in $R[t]$, for example $(1-t)(1+t+\dots) = 1$.

3 Zero-divisors, nilpotents, units

Definition 3.1. Let A be a ring. An element $x \in A$ is a **zero-divisor** if $x \neq 0$ but $xy = 0$ for some $y \neq 0$ in A. A ring without zero-divisors is called an **integral domain**. An element $x \in A$ is **nilpotent** if $x^n = 0$ for some $n \in \mathbb{Z}_{>0}$. A **unit** $x \in A$ is an element such that $xy = 1$ for some $y \in A$. The units of A form a group under multiplication, denoted by A^*, or $A^\times$.

Definition 3.2. Let $x \in A$. Then the set

$$\langle x \rangle = \{xy \mid y \in A\}$$

is an ideal. Such ideals are called **principal ideals**.

Remark. $x \in A^*$ if and only if $\langle x \rangle = A$, and R is a field if and only if $R^* = R \setminus \{0\}$.

Proposition 3.3. *Let A be a non-zero ring. Then the following are equivalent.*

1. *A is a field.*
2. *There are no ideals in A other than $\langle 0 \rangle$ and A.*
3. *Every non-zero homomorphism $f : A \to B$ is injective.*

Proof.

$1 \implies 2$. Clear.

$2 \implies 3$. $\operatorname{Ker} f \subset A$ is an ideal. Since $f \neq 0$, $\operatorname{Ker} f \neq A$. Hence $\operatorname{Ker} f = 0$.

$3 \implies 1$. Take any $x \neq 0$ in A. Look at $\langle x \rangle$. Define $B = A / \langle x \rangle$. Then take $f : A \to B$ to be the natural surjective map. If f is not identically zero, we get a contradiction with 3.

∎

4 Prime ideals and maximal ideals

Definition 4.1. An ideal $I \subset A$ is called **prime** if $I \neq A$ and if whenever $xy \in I$, then $x \in I$ or $y \in I$. An ideal $J \subset A$ is called **maximal** if there is no ideal J' such that $J \subsetneq J' \subsetneq A$.

Lemma 4.2. *An ideal $I \subset A$ is prime if and only if A/I is an integral domain.*

Proof. Obvious. □

Lemma 4.3. *An ideal $J \subset A$ is maximal if and only if A/J is a field.*

Proof. Obvious. □

Definition 4.4. The set of prime ideals of A is called the **spectrum** of A and is denoted by $\operatorname{Spec} A$.

Proposition 4.5. *If $f : A \to B$ is a ring homomorphism and $I \subset B$ is a prime ideal, then $f^{-1}(I)$ is a prime ideal of A.*

Proof. It is easy to see that $f^{-1}(I)$ is an ideal in A. Suppose $xy \in f^{-1}(I)$ for some $x, y \in A$. Then $f(x) f(y) = f(xy) \in I$. Since I is prime, $f(x) \in I$ or $f(y) \in I$, so $x \in f^{-1}(I)$ or $y \in f^{-1}(I)$. □

So we get a canonical map

$$\begin{array}{cccc} f^* & : & \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\ & & I \subset B & \longmapsto & f^{-1}(I) \subset A \end{array}.$$

Lecture 3 Wednesday 09/10/19

Remark 4.6. If $f : A \to B$ is a ring homomorphism, then $f^{-1}(\mathfrak{p})$, where $\mathfrak{p} \subset B$ is a prime ideal, is a prime ideal. But this is false for maximal ideals. Let $A = \mathbb{Z}$, let $B = \mathbb{Q}$, and let $f(x) = x$. Then $\langle 0 \rangle \subset \mathbb{Q}$ is a maximal ideal and $f^{-1}(\langle 0 \rangle) = \langle 0 \rangle \subset \mathbb{Z}$ is not a maximal ideal. For example, $\langle 0 \rangle \subsetneq \langle 2 \rangle \subsetneq \mathbb{Z}$.

Theorem 4.7. *Let A be a non-zero ring. Then A has at least one maximal ideal. In particular,* $\operatorname{Spec} A$ *is not empty.*

The proof is based on Zorn's lemma. Let S be a set. Then a **partial order** is a binary relation $\le$ such that

- $x \le x$ for all $x \in S$,
- $x \le y \le z$ implies that $x \le z$, and
- $x \le y$ and $y \le x$ imply that $x = y$,

where not all pairs are comparable. A **chain** $T \subset S$ is a subset in which every two elements are comparable.

Lemma 4.8 (Zorn)**.** *Suppose that S is a partially ordered set such that every chain $T \subset S$ has an upper bound, that is an element $t \in S$ such that $x \le t$ for all $x \in T$. Then S has a maximal element, that is there exists $s \in S$ such that if $x \in S$ and $x \ge s$, then $x = s$.*

Zorn's lemma is equivalent to the axiom of choice.

Proof of Theorem 4.7. Let Σ be the set of all ideals of A which are not equal to A. Then $\langle 0 \rangle \in \Sigma$, so $\Sigma \neq \emptyset$. Equip Σ with partial order given by inclusion. Enough to check the assumption of Zorn's lemma. Suppose T is a chain of ideals, so it is a collection of ideals J_i for $i \in T$. Consider instead

$$I = \bigcup_{i \in T} J_i.$$

Claim that T is a chain implies that I is an ideal. Then $x \in I$ implies that $x \in J_i$ for some i. Take any $x, y \in I$. Then $x \in J_i$ and $y \in J_k$ for some $i, k \in T$, so T is a chain, hence $i \le k$ or $k \le i$, so $J_i \subset J_k$ or $J_k \subset J_i$. Without loss of generality assume $J_i \subset J_k$. Then $x, y \in J_k$, so $x + y \in J_k \subset I$. Clearly, I is an upper bound. □

Corollary 4.9. *Any ideal of A is contained in a maximal ideal of A.*

Proof. If $I \subset A$ is an ideal, apply Theorem 4.7 to A/I. $\square$

Corollary 4.10. *Any non-unit of A is contained in a maximal ideal.*

Proof. Apply Corollary 4.9 to $\langle a \rangle$. $\square$

Example. The maximal ideals of $\mathbb{Z}$ are $\langle p \rangle$, where p is prime.

Definition 4.11. A ring A is **local** if A has exactly one maximal ideal.

Example. Any field is a local ring. If k is a field, then $k[[t]]$ is a local ring.

Lemma 4.12 (Prime avoidance)**.** *Let A be a ring and let $\mathfrak{p} \subset A$ be a prime ideal. Suppose that $I_1, \dots, I_n$ are ideals in A such that $\bigcap_{j=1}^n I_j \subset \mathfrak{p}$. Then $I_j \subset \mathfrak{p}$ for some j. If, moreover, $\bigcap_{j=1}^k I_j = \mathfrak{p}$, then $I_j = \mathfrak{p}$ for some j.*

Proof. Suppose that I_j is not a subset of $\mathfrak{p}$ for any j. Then there exists $x_j \in I_j$ such that $x_j \notin \mathfrak{p}$. Hence

$$x_1, \dots, x_n \in I_1 \dots I_n \subset \bigcap_{j=1}^n I_j \subset \mathfrak{p},$$

so $x_1 (x_2 \dots x_n) \in \mathfrak{p}$. Then $x_1 \notin \mathfrak{p}$ implies that $x_2 \dots x_n \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime we get a contradiction. For the second claim, we know that some $I_j \subset \mathfrak{p}$. But $\mathfrak{p} = \bigcap_{j=1}^k I_j \subset I_k$ for all k. Hence $\mathfrak{p} = I_j$. $\square$

5 Nilradical and the Jacobson radical

Lecture 4 Thursday 10/10/19

Proposition 5.1. *The set $\mathcal{N}(A)$ consisting of all nilpotents of the ring A and zero is an ideal. Then $\mathcal{N}(A)$ is called the **nilradical** of A. The quotient $A/\mathcal{N}(A)$ has no nilpotents.*

Proof. Suppose $x \in A$ is nilpotent, so $x^n = 0$. For any $a \in A$, $(ax)^n = a^n x^n = 0$. Let x and y be nilpotents. Say $x^n = y^m = 0$. Then

$$(x+y)^{n+m} = \sum_{i,j \geq 0,\ i+j=n+m} a_{ij} x^i y^j, \qquad a_{ij} \in A.$$

Clearly, either $i \geq n$ or $j \geq m$. Then $a_{ij} x^i y^j = 0$. Therefore, $(x+y)^{n+m} = 0$, hence $x + y \in \mathcal{N}(A)$. If $x + \mathcal{N}(A)$ is nilpotent in $A/\mathcal{N}(A)$, then $x^n + \mathcal{N}(A) = \mathcal{N}(A)$ is the trivial coset. Hence $x^n \in \mathcal{N}(A)$. Thus $(x^n)^m = 0$ for some m. $\square$

Definition 5.2. A ring A such that $\mathcal{N}(A) = 0$ is called a **reduced ring**.

Proposition 5.3. *$\mathcal{N}(A)$ is the intersection of all prime ideals of A.*

Proof.

- $\subset$ Let I be the intersection of all prime ideals of A. Let $f \in A$ be such that $f^n = 0$. Take any prime ideal $\mathfrak{p} \subset A$. We know that $f^n = 0 \in \mathfrak{p}$. Then $f(f \dots f) \in \mathfrak{p}$ and $\mathfrak{p}$ prime implies that $f \in \mathfrak{p}$, so $f \in I$.
- $\supset$ Let us prove the converse. Suppose f is not nilpotent, so $f^n \neq 0$ for all $n \geq 1$. We will show that there exists a prime ideal $\mathfrak{p} \subset A$ that does not contain f. Let us consider all ideals of A that do not contain f^m, where $m \in \mathbb{Z}_{>0}$. Let Σ be the set of ideals $J \subset A$ such that

  $$J \cap \{f^m \mid m \geq 1\} = \emptyset.$$

  The zero ideal $\langle 0 \rangle$ is in Σ. So $\Sigma \neq \emptyset$. Equip Σ with a partial order given by inclusion. Applying Zorn's lemma we obtain that Σ contains a maximal element. Call it $\mathfrak{p}$. By construction, $\mathfrak{p} \cap \{f^m \mid m \geq 1\} = \emptyset$, so $f \notin \mathfrak{p}$. It remains to prove that $\mathfrak{p}$ is prime. Enough to prove that if $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$, then $xy \notin \mathfrak{p}$. Consider the ideal $\mathfrak{p} + \langle x \rangle \supsetneq \mathfrak{p}$. Since $\mathfrak{p}$ is maximal in Σ, thus $\mathfrak{p} + \langle x \rangle$ is not in Σ. By definition of Σ there exists $n \geq 1$ such that $f^n \in \mathfrak{p} + \langle x \rangle$. Similarly, there exists $m \geq 1$ such that $f^m \in \mathfrak{p} + \langle y \rangle$. Then $(\mathfrak{p} + \langle x \rangle)(\mathfrak{p} + \langle y \rangle) \subset \mathfrak{p} + \langle xy \rangle$. In particular, $f^{n+m} = f^n f^m \in \mathfrak{p} + \langle xy \rangle$. If $xy \in \mathfrak{p}$, then $f^{n+m} \in \mathfrak{p}$, which is not possible. Therefore, $xy \notin \mathfrak{p}$. So $\mathfrak{p}$ is a prime ideal that does not contain f.

$\square$

Definition 5.4. The **Jacobson radical** $\mathcal{J}(A)$ is the intersection of all maximal ideals of A.

Proposition 5.5. *$x \in \mathcal{J}(A)$ if and only if $1 - xy \in A^*$ for all $y \in A$.*

Proof.

$\implies$ Let $x \in \mathcal{J}(A)$. Suppose there exists $y \in A$ such that $1 - xy$ is not a unit. By Corollary 4.10 every non-unit is contained in a maximal ideal. Say $M \subset A$ is a maximal ideal and $1 - xy \in M$. But $x \in \mathcal{J}(A) \subset M$. Then $1 = (1 - xy) + xy \in M$, but then $M \neq A$. A contradiction.

$\impliedby$ Given $x \in A$ such that $1 - xy \in A^*$ for all $y \in A$, we must have $x \in \mathcal{J}(A)$. If $x \notin \mathcal{J}(A)$, then there exists a maximal ideal $M \subset A$ such that $x \notin M$. Then $M + \langle x \rangle = A \ni 1$. Thus $1 = m + xy$, where $y \in A$. But by assumption $1 - xy \in A^*$, so $m \in A^*$. But then $M = A$. A contradiction.

$\square$

Definition 5.6. Let I be an ideal of A. The **radical** of I is the set

$$\operatorname{rad} I = \{x \in A \mid \exists n \geq 1, \ x^n \in I\}.$$

Proposition 5.7. *The radical of I is the intersection of all prime ideals of A that contain I.*

Proof. Apply Proposition 5.3 to A/I. $\square$

Lecture 5 Tuesday 15/10/19

Definition 5.8. Let I be an indexing set. For each $i \in I$ we are given a ring R_i. Consider the product set $\prod_{i \in I} R_i$. This is $(x_i)_{i \in I}$ for $x_i \in R_i$. Define

$$0 = (0)_{i \in I} \in \prod_{i \in I} R_i, \qquad 1 = (1)_{i \in I} \in \prod_{i \in I} R_i.$$

Define addition and multiplication coordinate-wise, so

$$(a_i)_{i \in I} + (b_i)_{i \in I} = (a_i + b_i)_{i \in I}, \qquad (a_i)_{i \in I} \cdot (b_i)_{i \in I} = (a_i \cdot b_i)_{i \in I}, \qquad (a_i)_{i \in I}, (b_i)_{i \in I} \in \prod_{i \in I} R_i.$$

Then $\prod_{i \in I} R_i$ is a ring, the **product of rings**.

A warning is if I has at least two elements, then $\prod_{i \in I} R_i$ has zero-divisors.

Example. $R_1 \times R_2$ has $(1, 0) \cdot (0, 1) = (0, 0) = 0$.

If $h_i : R \to R_i$ is a ring homomorphism for $i \in I$, then $(h_i)_{i \in I}$ is a ring homomorphism $R \to \prod_{i \in I} R_i$.

Remark 5.9. Let $\mathfrak{p}_i$ for $i \in I$ be all prime ideals of R. Let $h_i : R \to R/\mathfrak{p}_i$. Then

$$h = (h_i)_{i \in I} : R \to \prod_{i \in I} R/\mathfrak{p}_i$$

is a homomorphism, and

$$\operatorname{Ker} h = \bigcap_{i \in I} \operatorname{Ker} h_i = \bigcap_{i \in I} \mathfrak{p}_i = \mathcal{N}(R).$$

So there is an injective map

$$R/\mathcal{N}(R) \hookrightarrow \prod_{i \in I} R/\mathfrak{p}_i,$$

a product of integral domains. Now take $f_j : R \to R/M_j$, so if we take the indexing set J to be the set of all maximal ideals of R, then we obtain an injective map

$$R/\mathcal{J}(R) \hookrightarrow \prod_{j \in J} R/M_j,$$

a product of fields.

6 Localisation of rings

Example. Fix a prime p. Then

$$\mathbb{Z} \subset \left\{ \frac{m}{p^k} \;\middle|\; m \in \mathbb{Z},\ k \in \mathbb{Z}_{\geq 0} \right\} \subset \mathbb{Q}.$$

Definition 6.1. A subset S of a ring A is called a **multiplicative set** if $1 \in S$ and $0 \notin S$, and S is closed under multiplication.

Example 6.2.

- Let $a \in A$ be a non-nilpotent. Then $\{1, a, \dots\}$ is a multiplicative set.
- Let $\mathfrak{p} \subsetneq A$ be a prime ideal. Then $A \setminus \mathfrak{p}$ is a multiplicative set. Indeed, if $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$ then $xy \notin \mathfrak{p}$ by the definition of a prime ideal.
- If we have a family $\mathfrak{p}_i$ for $i \in I$ of prime ideals, then $A \setminus \bigcup_{i \in I} \mathfrak{p}_i$ is a multiplicative set.
- A^* is a multiplicative set.
- All non-zero-divisors in A form a multiplicative set.
- Let $I \subsetneq A$ be an ideal. Then $1 + I = \{1 + x \mid x \in I\}$ is a multiplicative set.

Definition 6.3. Consider $A \times S$ and the equivalence relation on $A \times S$ defined as

$$(a, s) \sim (b, t) \qquad \iff \qquad \exists u \in S,\ u\,(at - bs) = 0.$$

Check that this is indeed an equivalence relation. [1] The following is some notation.

- The equivalence class of (a, s) is written as a/s. For example, if $t \in S$, then $a/s = at/st$.
- The set of equivalence classes is denoted by $S^{-1}A$.

Define

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \qquad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}, \qquad a, b \in A, \qquad s, t \in S.$$

Need to check that these operations are well-defined. [2] Define $\frac{0}{1}$ as the zero of $S^{-1}A$, and $\frac{1}{1}$ as the one of $S^{-1}A$. Then $S^{-1}A$ is a ring, the **localisation of** A **with respect to** S.

Lemma 6.4. *There is a ring homomorphism*

$$\begin{array}{rcccc} f & : & A & \longrightarrow & S^{-1}A \\ & & x & \longmapsto & \dfrac{x}{1} \end{array}.$$

This f is injective if and only if S has no zero-divisors.

Proof. If S contains a zero-divisor, say u, then there exists $a \in A$ for $a \neq 0$ such that $ua = 0$. Then

$$f\,(a) = \frac{a}{1} = \frac{au}{u} = \frac{0}{u} = 0.$$

So $\operatorname{Ker} f$ contains a, hence f is not injective. If f has no zero-divisors, then $ua = u\,(a - 0) \neq 0$ if $a \neq 0$ and any $u \in S$. Hence $f\,(a) \neq 0$. ☐

If A is an integral domain, then $\operatorname{Ker} f = 0$. So $A \hookrightarrow S^{-1}A$. Lecture 6 Thursday 16/10/19

[1] Exercise

[2] Exercise

Example. Let $R = \mathbb{Z}$.

- If $S = \{1, a, \dots\}$, then
$$S^{-1}\mathbb{Z} = \left\{ \frac{n}{a^m} \;\middle|\; n \in \mathbb{Z},\ m \in \mathbb{Z}_{\geq 0} \right\}.$$
- If $S = \mathbb{Z} \setminus p\mathbb{Z}$, then
$$S^{-1}\mathbb{Z} = \left\{ \frac{n}{m} \;\middle|\; p \nmid m \right\}.$$
- If $S = \mathbb{Z} \setminus \bigcup_{p_i \text{ prime}} p_i\mathbb{Z}$, then
$$S^{-1}\mathbb{Z} = \left\{ \frac{n}{m} \;\middle|\; p_i \nmid m \right\}.$$
- If $S = \mathbb{Z}^* = \{\pm 1\}$, then $S^{-1}\mathbb{Z} = \mathbb{Z}$.
- If $S = \{\text{all non-zero elements}\}$, then $S^{-1}\mathbb{Z} = \mathbb{Q}$.
- If $S = \{1 + I \mid I \subset \mathbb{Z} \text{ ideal}\} = \{1 + nk \mid k \in \mathbb{Z}\}$, then
$$S^{-1}\mathbb{Z} = \left\{ \frac{m}{1 + nk} \;\middle|\; m, k \in \mathbb{Z} \right\},$$
where n is fixed.

Example. Let $R = k[x]$, where k is a field.

- If $S = k[x]^* = k^*$, then $S^{-1}k[x] = k[x]$.
- If $S = \{\text{all non-zero elements}\}$, then
$$S^{-1}k[x] = k(x) = \left\{ \frac{f(x)}{g(x)} \;\middle|\; g(x) \text{ arbitrary non-zero polynomial} \right\}.$$

Example 6.5. Let k be a field, and let $A = k[x, y] / \langle xy \rangle$. Note that A has zero-divisors, since $xy = 0$ in A, but $x \neq 0$ in A and $y \neq 0$ in A. Then $S = \{1, x, \dots\}$ is a multiplicative set, since $x^n \neq 0$ in A for $n = 1, 2, \dots$, because no power of the polynomial x is in $\langle xy \rangle$. What is $S^{-1}A$? Let $f : A \to S^{-1}A$. Then $a \in \operatorname{Ker} f$ if and only if $a/1 = 0/1$, if and only if $u \cdot (a \cdot 1 - 0 \cdot 1) = 0$ for some $u \in S$, if and only if $ua = 0$. Let $a \neq 0$. Then $u = 1$ is not interesting. Take $u = x$ and $a = y$, then $xy = 0$, hence $y \in \operatorname{Ker} f$. Then f is a homomorphism, hence $\operatorname{Ker} f$ is an ideal. So $\langle y \rangle = yA \subset \operatorname{Ker} f$. In general,
$$a = \sum_{i,j \geq 0} a_{ij} x^i y^j \equiv a_{00} + \sum_{i \geq 1} a_{i0} x^i + \sum_{j \geq 1} a_{0j} y^j \mod \langle xy \rangle.$$
Then $\operatorname{Ker} f = yA = \langle y \rangle$, since $\sum_{j \geq 1} a_{0j} y^j$ goes to zero, since it is annihilated by x, and $x^n \sum_{i \geq 0} a_i x^i$ is never zero in A. Thus $f(A) = k[x]$, and
$$S^{-1}A = \left\{ \frac{f(x)}{x^n} \;\middle|\; f(x) \in k[x],\ n \geq 0 \right\} = k\left[x, x^{-1}\right] = \left\{ \sum_{i \in \mathbb{Z},\ a_i = 0 \text{ for almost all } i} a_i x^i \;\middle|\; a_i \in k \right\}.$$

Lemma 6.6 (Universal property of localisation)**.** *Let A be a ring, and $S \subset A$ a multiplicative set. Let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in B for all $s \in S$. Then there exists a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$ where $f : A \to S^{-1}A$ is the canonical map, so* Lecture 7 Thursday 17/10/19

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "g"] & \\ S^{-1}A \arrow[r, "\exists! h"'] & B \end{tikzcd}.$$

Proof. Define

$$\begin{array}{rcccl} h & : & S^{-1}A & \longrightarrow & B \\ & & \dfrac{a}{s} & \longmapsto & \dfrac{g(a)}{g(s)}, \qquad a \in A, \qquad s \in S. \end{array}$$

This is well-defined, that is if $a/s = b/t$ then $g(a)\,g(s)^{-1} = g(b)\,g(t)^{-1}$. [3] This is a ring homomorphism. [4] Now easy to check that

$$(h \circ f)(a) = h\left(\frac{a}{1}\right) = \frac{g(a)}{g(1)} = \frac{g(a)}{1} = g(a), \qquad a \in A.$$

Moreover, if $h' : S^{-1}A \to B$ and $g = h' \circ f$ then for all $a \in A$ we have $(h' \circ f)(a) = g(a)$. Since h' is a ring homomorphism, for all $s \in S$, $h'(1/s) = 1/h'(s/1) = 1/g(s)$. Hence

$$h'\left(\frac{a}{s}\right) = h'\left(\frac{a}{1}\right) h'\left(\frac{1}{s}\right) = \frac{h'(f(a))}{h'(f(s))} = \frac{g(a)}{g(s)} = h\left(\frac{a}{s}\right).$$

□

For all ideal $I \subseteq A$, set

$$S^{-1}I = \left\{ \frac{i}{s} \in S^{-1}A \;\middle|\; i \in I,\ s \in S \right\},$$

the ideal of $S^{-1}A$ generated by $f(I)$.

Proposition 6.7. *Let $S \subset A$ be a multiplicative subset, and let $I_1, \dots, I_n$ be ideals of A. Then*

1. $S^{-1}(I_1 + \dots + I_n) = S^{-1}I_1 + \dots + S^{-1}I_n$,
2. $S^{-1}(I_1 \cdot \dots \cdot I_n) = S^{-1}I_1 \cdot \dots \cdot S^{-1}I_n$,
3. $S^{-1}\left(\bigcap_{i=1}^n I_i\right) = \bigcap_{j=1}^n S^{-1}I_j$, *and*
4. $S^{-1}(\operatorname{rad} I) = \operatorname{rad} S^{-1}I$ *for every ideal I.*

Proof. Exercise. [5] □

There is a map

$$\{\text{ideals } I \text{ of } A\} \to \left\{\text{ideals } S^{-1}I \text{ of } S^{-1}A\right\}.$$

Proposition 6.8. *Every ideal of $S^{-1}A$ is of the form $S^{-1}I$ for some ideal $I \subseteq A$.*

Proof. Let J be any ideal of $S^{-1}A$. Define $I = f^{-1}A$. Know I is an ideal of A. Claim that $J = S^{-1}I$. Say $a/s \in J$. Since J is an ideal, $s(a/s) \in J$, so $a/1 \in J$, so $a \in I$. Hence $a/s \in S^{-1}I$. So $J \subseteq S^{-1}I$. Conversely, $f(I) = f\left(f^{-1}(J)\right) \subseteq J$. Thus $S^{-1}I \subseteq J$. □

Theorem 6.9. *The only prime ideals of $S^{-1}A$ are of the form $S^{-1}\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal of A such that $\mathfrak{p} \cap S = \emptyset$. Hence there is a bijection*

$$\{\ \textit{prime ideals of } S^{-1}A\ \} \qquad \leftrightsquigarrow \qquad \{\ \textit{prime ideals of } A \textit{ that do not intersect } S\ \}.$$

Proof. Prove $S^{-1}\mathfrak{p}$ is prime if $\mathfrak{p}$ is prime and $\mathfrak{p} \cap S = \emptyset$. Say $a/s \cdot b/t \in S^{-1}\mathfrak{p}$ for $a/s, b/t \in S^{-1}A$. This implies $v(abu - cst) = 0$ for some $u, v \in S$ and $c \in \mathfrak{p}$. Hence $abuv = cstv \in \mathfrak{p}$, so $ab \in \mathfrak{p}$, as u and v are units, so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Hence $S^{-1}\mathfrak{p}$ is prime. Next note that $f^{-1}\left(S^{-1}\mathfrak{p}\right) = \mathfrak{p}$, assuming $\mathfrak{p} \cap S = \emptyset$. For if $a \in A$ lies in $S^{-1}\mathfrak{p}$ then by definition there exists $s \in S$ such that $sa \in \mathfrak{p}$. Then s is a unit and so $a \in \mathfrak{p}$. Hence $\mathfrak{p}$ is uniquely determined by $S^{-1}\mathfrak{p}$. Now let $\mathfrak{q}$ be an arbitrary prime ideal of $S^{-1}A$. Then certainly $\mathfrak{q} = S^{-1}I$ for $I = f^{-1}(\mathfrak{q})$. But the preimage of a prime ideal is prime. So I is prime. Moreover, $I \cap S = \emptyset$ as no $s \in S$ is in $\mathfrak{q}$, since $\mathfrak{q}$ is prime, so $\mathfrak{q}$ contains no units. □

[3] Exercise
[4] Exercise
[5] Exercise

7 Spec R as a topological space

Lecture 8 Tuesday 22/10/19

A set X with a collection $\mathcal{U}$ of subsets $U \subset X$ is called a **topological space** if the following properties hold.

1. $\mathcal{U}$ contains $\emptyset$ and X.
2. If U and U' are in $\mathcal{U}$, then $U \cap U'$ is in $\mathcal{U}$.
3. If U_i are in $\mathcal{U}$, where i is an element of an indexing set S, then $\bigcup_{i \in S} U_i$ is in $\mathcal{U}$.

Then the elements of $\mathcal{U}$ are called **open subsets** of X. The following is an equivalent definition. A set X with a family $\mathcal{V}$ of subsets $V \subset X$ is called a **topological space** if the following properties hold.

1. $\mathcal{V}$ contains $\emptyset$ and X.
2. If V and V' are in $\mathcal{V}$, then $V \cup V'$ is in $\mathcal{V}$.
3. If V_i are in $\mathcal{V}$, where i is an element of an indexing set S, then $\bigcap_{i \in S} V_i$ is in $\mathcal{V}$.

Then the elements of $\mathcal{U}$ are called **closed subsets** of X. For the equivalence, if U is in $\mathcal{U}$, then define the closed subsets as $X \setminus U$ for U in $\mathcal{U}$, and vice versa. Let R be a ring with unity. Let $I \subset R$ be an ideal. Let V_I be the set of all prime ideals in R that contain I. Define $\mathrm{U}_I = \operatorname{Spec} R \setminus \mathrm{V}_I$.

Proposition 7.1. *The collection of subsets* $\mathrm{V}_I \subset \operatorname{Spec} R$*, for all ideals* $I \subset R$*, satisfies* 1, 2, 3 *of closed subsets, hence defines a topology on* $\operatorname{Spec} R$*.*

Proof.

1. If $I = 0$ is the zero ideal, then $\mathrm{V}_0 = \operatorname{Spec} R$, all prime ideals of R. If $I = R$, then no prime ideals of R contain R, so $\mathrm{V}_R = \emptyset$, so 1 holds.
2. It is enough to check that $\mathrm{V}_I \cup \mathrm{V}_J = \mathrm{V}_{IJ} = \mathrm{V}_{I \cap J}$. Note that $IJ \subset I \cap J$. An element of V_I is a prime ideal $\mathfrak{p} \supset I$, so $\mathfrak{p} \supset IJ$. Conversely, let $\mathfrak{p}$ be a prime ideal such that $IJ \subset \mathfrak{p}$. Claim that $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$. Suppose not. Then there exists $x \in I$ such that $x \notin \mathfrak{p}$ and there exists $y \in J$ such that $y \notin \mathfrak{p}$. Then $xy \in IJ \subset \mathfrak{p}$. This contradicts the definition of prime ideals. So the claim is proved. Thus 2 holds.
3. J_i for $i \in S$ is a collection of ideals. Claim that $\bigcap_{i \in S} \mathrm{V}_{J_i} = \mathrm{V}_J$, where $J = \sum_{i \in S} J_i$ is the smallest ideal of R containing all J_i for $i \in S$. The elements of J are finite sums, where each summand is in some J_i. If $\mathfrak{p} \supset J_i$ for $i \in S$, then $\mathfrak{p} \supset J$. Conversely, if $\mathfrak{p} \supset J \supset J_i$, then $\mathfrak{p} \supset J_i$ for all $i \in S$.

$\square$

Recall that if $f : A \to B$ is a homomorphism of rings, then $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ sends any prime ideal $\mathfrak{p} \subset B$ to the inverse image $f^{-1}(\mathfrak{p})$, which is a prime ideal in A. This breaks down for maximal ideals.

Example. Take $f : \mathbb{Z} \to \mathbb{Q}$, then $f^{-1}(0) = 0$, which is not maximal in $\mathbb{Z}$.

A map of topological spaces is **continuous** if the inverse image of any open set is open. Equivalently, the inverse images of closed sets are closed.

Proposition 7.2. f^* *is a continuous map.*

Proof. Let I be an ideal in A. We need to show that $(f^*)^{-1}(\mathrm{V}_I) = \mathrm{V}_J$ for some ideal J in B. Let J be the smallest ideal in B containing $f(I)$.

- $\subset$ Fix $\mathfrak{p}$ in V_I, a prime ideal in A such that $\mathfrak{p} \supset I$. The elements of the left hand side that are mapped to $\mathfrak{p}$ by f^* are the prime ideals $\mathfrak{q} \subset B$ such that $\mathfrak{p} = f^{-1}(\mathfrak{q})$. We have $I \subset \mathfrak{p}$, so $f(I) \subset f(\mathfrak{p}) \subset \mathfrak{q}$, so $J \subset \mathfrak{q}$, by definition of J.
- $\supset$ Take any prime ideal $\mathfrak{q} \subset B$ such that $J \subset \mathfrak{q}$. We have $I \subset f^{-1}(f(I)) \subset f^{-1}(J) \subset f^{-1}(\mathfrak{q})$, so $f^{-1}(\mathfrak{q})$ is a prime ideal in A containing I. This ideal is exactly $f^*(\mathfrak{q})$, so $f^*(\mathfrak{q})$ is in V_I. Since $\mathfrak{q} \in (f^*)^{-1}(f^*(\mathfrak{q})) \subset (f^*)^{-1}(\mathrm{V}_I)$, so we are done.

$\square$

The following are particular cases. Lecture 9 Wednesday 23/10/19

- Assume f is surjective. Then $B \cong A/\operatorname{Ker} f$. Then

$$\begin{array}{rcl} \{\text{prime ideals in } B\} & \longrightarrow & \{\text{prime ideals in } A \text{ containing } \operatorname{Ker} f\} \\ \mathfrak{p} \subset B & \longmapsto & f^{-1}(\mathfrak{p}) \end{array}.$$

  So in this case f^* is injective and its image is $\mathrm{V}_{\operatorname{Ker} f}$.

- Let S be a multiplicative set in A. Let $f : A \to S^{-1}A$ be the associated canonical map. By Theorem 6.9 the prime ideals of $S^{-1}A$ are $S^{-1}\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal in A such that $\mathfrak{p} \cap S = \emptyset$. Thus $f^* : \operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ is injective and its image consists of $\mathfrak{p} \subset A$ such that $\mathfrak{p} \cap S = \emptyset$.

Example.

- Let k be a field. Then $\operatorname{Spec} k$ is one point.

- Let $R = k[x]$, an integral domain. This is a PID, so every ideal is $\langle p(x) \rangle$, where $p(x) \in k[x]$ is monic. Then $\langle p(x) \rangle$ is prime if and only if $p(x)$ is irreducible, so

$$\operatorname{Spec} k[x] = \{\langle 0 \rangle\} \cup \{\langle p(x) \rangle \mid p(x) \text{ is monic and irreducible}\}.$$

  In particular, if k is algebraically closed, such as $k = \mathbb{C}$, then

$$\operatorname{Spec} k[x] = \{\langle 0 \rangle\} \cup \{\langle x - a \rangle \mid a \in k\}.$$

- Let $R = \mathbb{Z}$, a PID. Then

$$\operatorname{Spec} \mathbb{Z} = \{\langle 0 \rangle\} \cup \{\langle p \rangle \mid p \text{ is a prime number}\}.$$

- Let $R = \mathbb{Z}[i]$ be the Gaussian integers, a PID. The tautological map $f : \mathbb{Z} \to \mathbb{Z}[i]$ gives rise to $f^* : \operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}$. Take a usual prime p and decompose p into a product of primes in $\mathbb{Z}[i]$.

  - $2 = (1+i)(1-i) = -i(1+i)^2$, where $1+i$ is a prime in $\mathbb{Z}[i]$.
  - If $p \equiv 1 \mod 4$, then $p = (a+bi)(a-bi)$. In this case $a+bi$ and $a-bi$ are not associated primes.
  - If $p \equiv 3 \mod 4$, then p stays prime in $\mathbb{Z}[i]$.

  Then

$$\begin{array}{ccll} \operatorname{Spec} \mathbb{Z}[i] & \longrightarrow & \operatorname{Spec} \mathbb{Z} & \\ \langle 0 \rangle & \longmapsto & \langle 0 \rangle & \\ \langle 1+i \rangle & \longmapsto & \langle 2 \rangle & \text{ramified} \\ \langle 3 \rangle & \longmapsto & \langle 3 \rangle & \text{inert} \\ \langle 1+2i \rangle, \langle 1-2i \rangle & \longmapsto & \langle 5 \rangle & \text{split} \end{array}.$$

- Let R be an integral domain and let k be the fraction field of R, so $f : R \hookrightarrow k$. Then $\operatorname{Spec} k = \{\langle 0 \rangle\}$ and $f^* : \operatorname{Spec} k \to \operatorname{Spec} R$.

- Let k be a field, so $f : k \hookrightarrow k[x]$. Then $f^* : \operatorname{Spec} k[x] \to \operatorname{Spec} k$. If $\mathfrak{p} \subset k[x]$, then $\mathfrak{p} \cap k = \{\langle 0 \rangle\}$, otherwise if $\mathfrak{p}$ contains a unit of $k[x]$ then $\mathfrak{p} = k[x]$. A contradiction.

Usually, every point of a topological space is a closed subset. But this is not always true. Recall that if Y is a subset of a topological space X, then the **closure** of Y is the smallest closed subset of X containing Y. It is the same as the intersection of all closed subsets containing Y. Claim that if $\mathfrak{p} \subseteq R$ is a prime ideal, then the closure of $\mathfrak{p}$ is $\mathrm{V}_{\mathfrak{p}}$. Any closed subset of $\operatorname{Spec} R$ containing $\mathfrak{p}$ is V_J, where $J \subset \mathfrak{p}$. This V_J visibly contains $\mathrm{V}_{\mathfrak{p}}$. Hence $\mathrm{V}_{\mathfrak{p}}$ is the intersection of all such V_J.

Example. In $\operatorname{Spec} \mathbb{Z}$, the point $\langle p \rangle$ is closed, because $\mathrm{V}_{\langle p \rangle} = \{\langle p \rangle\}$. The point $\langle 0 \rangle$ is not closed, as $\mathrm{V}_{\langle 0 \rangle} = \operatorname{Spec} \mathbb{Z}$. The closure of $\langle 0 \rangle$ is all of $\operatorname{Spec} \mathbb{Z}$.

Example. Let $R = k[[t]] = \{a_0 + a_1 t + \dots \mid a_i \in k\}$, a local ring. Its unique maximal ideal is $\langle t \rangle$. This is also a unique non-zero prime ideal. [6] All ideals are $\langle 0 \rangle$ and $\langle t^n \rangle$. Then $\operatorname{Spec} k[[t]] = \{\langle 0 \rangle, \langle t \rangle\}$. Similarly, $\langle 0 \rangle$ is not a closed point, since its closure is $\operatorname{Spec} k[[t]]$, and $\langle t \rangle$ is a closed point.

[6] Exercise

8 Determinants

Lecture 10 Thursday 24/10/19

Let R be a commutative ring with unity. Let A be a matrix $A = (a_{ij})_{i,j=1}^{n}$ for $a_{ij} \in R$. Then

$$\det A = \sum_{\pi \in \mathcal{S}_n} \operatorname{sgn} \pi \cdot a_{1\pi(1)} \cdot \dots \cdot a_{n\pi(n)} \in R,$$

where $\operatorname{sgn} : \mathcal{S}_n \to \{\pm 1\}$. Let

$$\mathrm{M}_{ij} = \det (A \text{ without } j\text{-th column and } i\text{-th row}) \in R.$$

Then

$$(-1)^{j+1} a_{i1} \mathrm{M}_{j1} + \dots + (-1)^{j+n} a_{in} \mathrm{M}_{jn} = \begin{cases} \det A & i = j \\ 0 & i \neq j \end{cases}.$$

Define the **adjoint matrix** of A as the $n \times n$ matrix $A^\vee$ with entries $(A^\vee)_{ij} = (-1)^{i+j} \mathrm{M}_{ji}$, so

$$A^\vee = \left((-1)^{i+j} \mathrm{M}_{ij}\right)^{\mathsf{T}}.$$

Then $A \cdot A^\vee = A^\vee \cdot A = \det A \cdot \mathrm{I}_n$, where I_n is the identity matrix.

9 Modules

Definition 9.1. Let A be a commutative ring with unity. An **A-module** M is an abelian group with an additional structure $A \times M \to M$ such that

$$\lambda (x + y) = \lambda x + \lambda y, \qquad (\mu + \lambda) x = \mu x + \lambda x, \qquad \mu (\lambda x) = (\mu \lambda) x, \qquad 1x = x, \qquad \lambda, \mu \in R, \qquad x, y \in M.$$

Example 9.2.

- If R is a field, then an R-module is the same as a vector space.
- If $R = \mathbb{Z}$, then an R-module is the same as an abelian group. Remark that if G is an abelian group then $n \cdot g = g + \dots + g$.
- If R is any ring, then subgroups of R that are R-modules are the same as ideals.
- If k is a field, then $k[x]$-modules are vector spaces V over k equipped with a linear transformation $L : V \to V$. Here x acts on V as L.

Definition 9.3. If M and N are R-modules, then a **homomorphism of R-modules** $f : M \to N$ is a homomorphism of abelian groups such that $f(rx) = rf(x)$ for all $x \in M$ and $r \in R$.

Definition 9.4. Let $\operatorname{Hom}_R(M, N)$ be the set of R-module homomorphisms $M \to N$.

This is an abelian group. Moreover, it is an R-module. If $r \in R$ and $f \in \operatorname{Hom}_R(M, N)$ then $r \cdot f$ sends $x \in M$ to $rf(x) \in N$. Warning that if R is not commutative $\operatorname{Hom}_R(M, N)$ is just an abelian group.

Definition 9.5. Let M and N be submodules of an R-module. Define

$$(N : M) = \{r \in R \mid rM \subset N\}.$$

This is an ideal in R.

Example. The **annihilator** of M is

$$(0 : M) = \{r \in R \mid rM = 0\} = \operatorname{Ann} M.$$

Definition 9.6. An R-module M is **finitely generated** if there are elements $x_1, \dots, x_n \in M$ such that for any $m \in M$ there are $r_1, \dots, r_n \in R$ such that $m = r_1 x_1 + \dots + r_n x_n$.

Example. There is a **free** finitely generated module

$$R^{\oplus n} = \{(t_1, \dots, t_n) \mid t_i \in R\},$$

with coordinate-wise addition and multiplication.

Remark. Any finitely generated R-module is a quotient of a free finitely generated R-module. Indeed, define

$$\begin{array}{cccc} f_i & : & R^{\oplus n} & \longrightarrow & M \\ & & (t_1, \dots, t_n) & \longmapsto & t_1 x_1 + \dots + t_n x_n \end{array}.$$

Comment that JM is the smallest submodule of M containing all elements rm for $r \in J$ and $m \in M$, so

$$JM = \{\text{finite sums } r_1 m_1 + \dots + r_k m_k\} \subset M.$$

Lemma 9.7. *Let A be a ring. Let M be a finitely generated A-module. Let $J \subset A$ be an ideal such that $JM = M$. Then there is an $a \in J$ such that $(1 - a) M = 0$.*

Proof. If $M = 0$, then it is fine. Suppose $M \neq 0$ and $m_1, \dots, m_n$ are generators of M. Then $m_i \in M = JM$, so

$$m_1 = x_{11} m_1 + \dots + x_{1n} m_n, \qquad \dots, \qquad m_n = x_{n1} m_1 + \dots + x_{nn} m_n,$$

for $x_{ij} \in J$. Define $X = (x_{ij})_{i,j=1}^n$. Then

$$\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = X \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \qquad \Longleftrightarrow \qquad (\mathrm{I}_n - X) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

Consider the adjoint matrix $(\mathrm{I}_n - X)^\vee$. Then

$$(\mathrm{I}_n - X)^\vee (\mathrm{I}_n - X) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0 \qquad \Longleftrightarrow \qquad \det (\mathrm{I}_n - X) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0.$$

We have $\det (\mathrm{I}_n - X) \in A$. Then $\det (\mathrm{I}_n - X)$ is a product of diagonal entries $\prod_{i=1}^n (1 - x_{ii})$, plus other terms but every non-diagonal term contains at least one factor in J, so is in J. Finally, $\det (\mathrm{I}_n - X) = 1 - a$, where $a \in J$. Now, $(1 - a) m_i = 0$ for $i = 1, \dots, n$. Hence $(1 - a) M = 0$. ☐

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Remark. If M is not finitely generated then this is false, such as $A = \mathbb{Z}$ and $M = \mathbb{Q}$. If p is a prime, then $p\mathbb{Q} = \mathbb{Q}$. So for $J = \langle p \rangle$ we have $JM = M$. But no non-zero integer annihilates $\mathbb{Q}$, since $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.

Corollary 9.8. *Let R be a ring and let M be a finitely generated R-module. If $f : M \to M$ is a surjective R-module endomorphism, then f is an isomorphism.*

Proof. Define $A = R[t]$. Let us equip M with the structure of an A-module. Define $t \cdot m = f(m)$ for $m \in M$. This makes sense because $f(rx) = rf(x)$ for all $r \in R$. Then M is finitely generated also as an A-module. If $f(M) = M$, then $tM = M$. Take $J = \langle t \rangle \subset A$. By Lemma 9.7 there exists $a \in \langle t \rangle$ such that $(1 - a) M = 0$. Take $v \in M$ such that $f(v) = 0$. Then $tv = 0$, so $av = 0$. Since $(1 - a) v = 0$, we conclude $v = 0$. ☐

Theorem 9.9 (Nakayama's lemma)**.** *Let A be a ring and let $J \subset A$ be an ideal contained in the Jacobson radical $\mathcal{J}(A)$. If M is a finitely generated A-module such that $JM = M$, then $M = 0$.*

Proof. Lemma 9.7 implies that there exists $a \in J$ such that $(1 - a) M = 0$. But $a \in \mathcal{J}(A)$, so $1 - a$ is a unit in A. Then there exists $u \in A$ such that $u(1 - a) = 1$. Hence $M = u(1 - a) M = 0$. ☐

Corollary 9.10. *Let A be a ring and J an ideal contained in the Jacobson radical of A. Suppose M is an A-module, and $N \subset M$ is a submodule such that M/N is a finitely generated A-module. Then $M = N + JM$ implies $M = N$.*

Proof. Apply Nakayama's lemma to M/N. Indeed, we have $M/N = J(M/N)$, so $M/N = 0$. ☐

Recall a ring is local when it has a unique maximal ideal. The quotient is called the **residue field**.

Example. For k a field, $k[[t]] \supset \langle t \rangle$ and $k[[t_1, \dots, t_n]] \supset \langle t_1, \dots, t_n \rangle$ are local rings. [7]

Theorem 9.11. *Let R be a local ring with maximal ideal J and residue field $k = R/J$. Let M be a finitely generated R-module.*

1. *M/JM is a finite-dimensional vector space over k.*
2. *Let $v_1, \dots, v_n$ be a basis of M/JM as a vector space over k. Choose $\widetilde{v_1}, \dots, \widetilde{v_n} \in M$ to be representatives of $v_1, \dots, v_n$ respectively. That is, $v_i = \widetilde{v_i} + JM$. Then $\widetilde{v_1}, \dots, \widetilde{v_n}$ generate M as an R-module. Moreover, this is a minimal set of generators of M. That is, no proper subset generates M.*
3. *All minimal sets of generators of M are obtained in this way. In particular, all such sets have n elements, where $n = \dim_k M/JM$.*

Proof. J is the Jacobson radical of A.

1. Any quotient of a finitely generated R-module is a finitely generated R-module. Hence M/JM is a finitely generated R-module. But if $x \in J$ then $x \cdot M/JM = 0$. So R acts on M/JM via the quotient $k = R/J$. One says that the action of R descends to an action of k. Thus M/JM is a k-module, which is finitely generated. In other words, M/JM is a finite-dimensional k-vector space.
2. Consider
$$N = R\widetilde{v_1} + \dots R\widetilde{v_n} = \{r_1\widetilde{v_1} + \dots + r_n\widetilde{v_n} \mid r_i \in R\} \subset M.$$
Then M/JM is generated by $v_1, \dots, v_n$, hence $M = N + JM$, since $M/JM = N/JN$. By Corollary 9.10 we have $M = N$. If a proper subset of $\widetilde{v_1}, \dots, \widetilde{v_n}$ generates M, then a proper subset of $v_1, \dots, v_n$ generates an n-dimensional vector space. A contradiction.

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3. Suppose $m_1, \dots, m_n$ is any minimal generating set of the R-module M. Consider $\overline{m_1}, \dots, \overline{m_n} \in M/JM$. Then $\overline{m_1}, \dots, \overline{m_n}$ span the vector space M/JM. If this is not a basis, then M/JM is spanned by a proper subset of $\overline{m_1}, \dots, \overline{m_n}$. In particular, a basis is a proper subset. By part 2 a proper subset of $m_1, \dots, m_n$ generates M. This contradicts the minimality of $m_1, \dots, m_n$.

∎

The moral of the story is any finitely generated module M over a local ring R has a minimal set of generators, where $m_1, \dots, m_n$ is a minimal set of generators of M if and only if $\overline{m_1}, \dots, \overline{m_n}$ is a basis of the k-vector space M/JM, and n is well-defined.

10 Localisation of modules

Let A be a ring with a multiplicative set $S \subset A$.

Definition 10.1. Let M be an A-module. Consider the set $M \times S$. Equip it with a relation $\sim$ such that
$$(m, s) \sim (n, t) \qquad \iff \qquad \exists u \in S, \ u(mt - ns) = 0.$$
This is an equivalence relation.

- Define $S^{-1}M$ as the set of equivalence classes.
- The equivalence class of (m, s) is written as m/s.

Turn $S^{-1}M$ into a $S^{-1}A$-module as follows. Let $\frac{0}{1}, \frac{1}{1} \in S^{-1}M$, and
$$\frac{m}{s} + \frac{b}{t} = \frac{mt + bs}{st}, \qquad \frac{a}{s} \cdot \frac{m}{t} = \frac{am}{st}, \qquad a \in A, \qquad m \in M, \qquad s \in S, \qquad t \in S.$$
This is the **localisation of M with respect to S**.

[7] Exercise

Now let us consider a particular kind of multiplicative set.

Definition 10.2. Let $\mathfrak{p} \subset A$ be a prime ideal. Let $S = A \setminus \mathfrak{p}$. This is a multiplicative set. Then the **localisation** $S^{-1}A$ **of** A **at** $\mathfrak{p}$ is written as $A_{\mathfrak{p}}$.

Theorem 10.3. *Let $\mathfrak{p} \subset A$ be a prime ideal. Then $A_{\mathfrak{p}}$ is a local ring with unique maximal ideal*

$$\mathfrak{p}A_{\mathfrak{p}} = \left\{ \frac{x}{y} \;\middle|\; x \in \mathfrak{p},\ y \notin \mathfrak{p} \right\}.$$

Remark. In general, a ring R with an ideal J is a local ring with maximal ideal J if and only if $R^* = R \setminus J$. Indeed, if $J \subset R$ is a maximal ideal, then for any $x \in R \setminus J$, $J + xR$ contains one. This forces x to be a unit. Conversely, if $R^* = R \setminus J$ then J is maximal and is a unique maximal ideal.

Proof. Suppose $a/s \in A_{\mathfrak{p}}^*$. Then $a/s \cdot b/t = 1/1$ for some $b \in A$ and $t \in A \setminus \mathfrak{p}$. By definition $u\,(ab - st) = 0$ for $u \in A \setminus \mathfrak{p}$, so $uab = ust \notin \mathfrak{p}$, since all factors are in $S = A \setminus \mathfrak{p}$. Therefore, $a \notin \mathfrak{p}$, hence $a/s \notin \mathfrak{p}A_{\mathfrak{p}}$. Conversely, if $a/s \notin \mathfrak{p}A_{\mathfrak{p}}$ for $s \notin \mathfrak{p}$, then $a \notin \mathfrak{p}$. Thus a/s is a unit in $A_{\mathfrak{p}}$ because $a/s \cdot s/a = 1$. ☐

Example 10.4. Let $R = \mathbb{Z}$ and $\mathfrak{p} = \langle p \rangle$. Then

$$p\mathbb{Z}_{\langle p \rangle} = \left\{ \frac{x}{y} \;\middle|\; p \mid x,\ p \nmid y \right\} \subset \left\{ \frac{x}{y} \;\middle|\; x \in \mathbb{Z},\ p \nmid y \right\} = \mathbb{Z}_{\langle p \rangle}$$

is the unique maximal ideal.

Proposition 10.5. *Let M be an A-module. Consider $M_{\mathfrak{p}} = (A \setminus \mathfrak{p})^{-1} M$, where $\mathfrak{p} \subset A$ is a maximal ideal. Then $M = 0$ if and only if $M_{\mathfrak{p}} = 0$ for any maximal ideal $\mathfrak{p}$.*

Proof.

$\implies$ Obvious.

$\impliedby$ Assume $M \neq 0$, so there exists $x \in M$ such that $x \neq 0$. Define

$$I = \operatorname{Ann} x = \{a \in A \mid ax = 0\},$$

so $1 \notin I$ since $x \neq 0$. Choose a maximal ideal $\mathfrak{p}$ containing I. If $M_{\mathfrak{p}} = 0$, then $x/1 = 0$. We know that $x \in \operatorname{Ker}(M \to M_{\mathfrak{p}})$ if and only if $ux = 0$ for some $u \in A \setminus \mathfrak{p}$. A contradiction, since $I \subset \mathfrak{p}$.

☐

The following is a corollary. Let M be a finitely generated A-module. Then $m_1, \dots, m_n$ generate M if and only if $m_1, \dots, m_n$ generate the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ for any maximal ideal $\mathfrak{p} \subset A$. By Theorem 9.11 applied to $A_{\mathfrak{p}}$, this is if and only if the images $\overline{m_1}, \dots, \overline{m_n}$ in $M/\mathfrak{p}M \cong M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ generate the $k(\mathfrak{p})$-vector space for every maximal ideal $\mathfrak{p} \subset A$, where $k(\mathfrak{p}) = A/\mathfrak{p}$.

Corollary 10.6. *Assume A is an integral domain with field of fractions K. In this case A is a subring of K. For any prime ideal $\mathfrak{p} \subset A$ the local ring $A_{\mathfrak{p}}$ is also a subring of K. Then*

$$A = \bigcap_{\text{all prime ideals } \mathfrak{p} \subset A} A_{\mathfrak{p}},$$

as subsets of K.

Proof. Clearly, $A \subset A_{\mathfrak{p}}$, so the left hand side is in the right hand side. Let us prove that if $x \in K$ is contained in each $A_{\mathfrak{p}}$, then $x \in A$. Consider

$$I = \{a \in A \mid ax \in A\}.$$

Visibly, I is an ideal in A. We are given that $x = m/s$, where $m \in A$ and $s \in A \setminus \mathfrak{p}$. Hence $s \in I$. So I contains an element not in $\mathfrak{p}$ for every $\mathfrak{p}$. Then $I = A$, because otherwise I is contained in some maximal ideal but maximal ideals are prime. Hence $1 \in I$, so $x \in A$. ☐

Lecture 13 is a problem class. Lecture 13 Thursday 31/10/19

Lecture 14 is a test. Lecture 14 Tuesday 05/11/19

11 Chain conditions

Lecture 15 Wednesday 06/11/19

Lemma 11.1. *Let Σ be a partially ordered set. The following are equivalent.*

- *Every maximal non-empty subset of Σ has a maximal element, so no element of the subset is bigger.*
- *Every ascending chain of elements of Σ is stationary, so there exists $i_0 \in I$ such that $a_{i_0} = a_i$ for all $i > i_0$.*

Proof.

$\implies$ Take a maximal element of the chain, say a_{i_0}. Then for any $i \ge i_0$ we have $a_i = a_{i_0}$.

$\impliedby$ Suppose $S \subset \Sigma$ has no maximal element. Then choose any element in S, say a_1. This is not maximal, so can choose $a_2 \in S$ such that $a_1 < a_2$. Keep doing this, get an infinite chain which is not stationary, because $a_i \ne a_j$ for all $i \ne j$.

$\square$

Definition 11.2. Let A be a ring and let M be an A-module. Then M is called **Noetherian** if any ascending chain of submodules of M is stationary. In other words, if $M_1 \subset M_2 \subset \dots \subset M$ are A-submodules, then there exists n such that $M_n = M_{n+1} = \dots$. Then M is called **Artinian** if any descending chain of submodules of M is stationary. The ring A is **Noetherian**, or **Artinian**, if such is the A-module A.

Proposition 11.3. *Let A be a ring and let M be an A-module. The following are equivalent.*

- *M is Noetherian.*
- *Every A-submodule of M is finitely generated.*

In particular, A is a Noetherian ring if and only if every ideal in A is finitely generated.

Proof.

$\implies$ Suppose that $N \subset M$ is a submodule which is not finitely generated. Let $N_1 = 0$. Since N is not finitely generated we can find $0 \ne x \in N$ such that $N_2 = Ax$ is the submodule generated by x, where $N \ne N_2$. So we continue. If $0 = N_1 \subsetneq \dots \subsetneq N_m$ are constructed, then $N_m \ne N$, so there exists $y \in N$ such that $y \notin N_m$. Define $N_{m+1} = N_m + Ay$, the smallest module containing N_m and y. Since N is not finitely generated, this chain is not stationary.

$\impliedby$ Let $M_1 \subset M_2 \subset \dots \subset M$. Must prove that this chain is stationary. Define

$$N = \bigcup_{i \in I} M_i.$$

This is a submodule of M. We know that $N = Rx_1 + \dots + Rx_n$ where $x_1, \dots, x_n \in N$. Then x_k is contained in some M_{i_k}. Suppose that $i_0 = \max\{i_1, \dots, i_n\}$. Then $x_{i_1}, \dots, x_{i_n} \in M_{i_0}$, since $M_{i_1} \subset M_{i_0}, \dots, M_{i_k} \subset M_{i_0}$. But now we see that $M_{i_0} \supset N$. Since $M_{i_0} \subset N$, we must have $N = M_{i_0}$. Hence $M_{i_0} = M_{i_0+1} = \dots$.

$\square$

Proposition 11.4. *Suppose M is an A-module. Let $N \subset M$ be a submodule. Then M is Noetherian if and only if N and M/N are Noetherian, and M is Artinian if and only if N and M/N are Artinian.*

Proof. The Noetherian case.

$\implies$ Suppose M is Noetherian. Ascending chains of submodules of N are ascending chains of submodules of M, so must be stationary. Let $f : M \to N$ be the canonical map. If $L_1 \subset L_2 \subset \dots$ is a chain of submodules of M/N, then $f^{-1}(L_1) \subset f^{-1}(L_2) \subset \dots$ is a chain of submodules of M. This is stationary. Since $f\left(f^{-1}(L_i)\right) = L_i$, the original chain of L_i's is stationary.

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$\Longleftarrow$ Now assume that N and M/N are Noetherian. We need to prove that an ascending chain $M_1 \subset M_2 \subset \dots$ of submodules of M is stationary. Then $N \cap M_1 \subset N \cap M_2 \subset \dots$ is a chain of submodules of N. Similarly, $M_1/N \cap M_1 \subset M_2/N \cap M_2 \subset \dots$. Indeed, $M_1 \to M_2$ is clearly injective, and $\operatorname{Ker}(M_1 \to M_2/N \cap M_2) = N \cap M_1$. Therefore, $M_1/N \cap M_1$ injectively maps to $M_2/N \cap M_2$. Then

$$\begin{tikzcd} M_1/M_1 \cap N \arrow[r, hook] & M_2/M_2 \cap N \arrow[r, hook] & \dots \arrow[r, hook] & M/N \\ M_1 \arrow[u, two heads] \arrow[r, hook] & M_2 \arrow[u, two heads] \arrow[r, hook] & \dots \arrow[r, hook] & M \arrow[u, two heads] \\ M_1 \cap N \arrow[u, hook] \arrow[r, hook] & M_2 \cap N \arrow[u, hook] \arrow[r, hook] & \dots \arrow[r, hook] & M \arrow[u, hook] \end{tikzcd}.$$

If F and G are submodules of H, then we have a natural map

$$\begin{array}{ccc} F & \longrightarrow & (F+G)/G \\ x & \longmapsto & x+G \end{array}.$$

The kernel of this map is $F \cap G$. The map $F \to (F+G)/G$ is surjective. So we have a canonical isomorphism $F/F \cap G \xrightarrow{\sim} (F+G)/G$. Apply this to $F = M_i$, $G = N$, and $H = M$. Then

$$\begin{tikzcd} (M_1+N)/N \arrow[r, hook] & (M_2+N)/N \arrow[r, hook] & \dots \arrow[r, hook] & M/N \\ M_1/M_1 \cap N \arrow[u, "\sim"] \arrow[r, hook] & M_2/M_2 \cap N \arrow[u, "\sim"] \arrow[r, hook] & \dots \arrow[r, hook] & M/N \arrow[u, "\sim"] \end{tikzcd}.$$

There exists $a \in \mathbb{N}$ such that $M_i \cap N = M_a \cap N$ for all $i \geq a$. There exists $b \in \mathbb{N}$ such that $(M_i+N)/N = (M_b+N)/N$ for all $i \geq b$. Define $c = \max\{a, b\}$. Then

$$\begin{tikzcd} (M_c+N)/N \arrow[r, "\sim"] & (M_i+N)/N \\ y \in M_c \arrow[u, two heads] \arrow[r, hook] & M_i \ni x \arrow[u, two heads] \\ M_c \cap N \arrow[u, hook] \arrow[r, "\sim"] & M_i \cap N \arrow[u, hook] \end{tikzcd}.$$

Claim that $M_i = M_c$ for all $i \geq c$. It remains to show that any $x \in M_i$ is in fact in M_c. Since the top arrow is an isomorphism, and $M_c \to (M_c+N)/N$ is surjective, we can find $y \in M_c$ whose image in $(M_i+N)/N$ is equal to the image of x. Then $x - y \in M_i$ goes to zero in $(M_i+N)/N$. Thus $x - y \in M_i \cap N$. Hence $x - y \in M_c \cap N \subset M_c$. Hence $x = (x-y) + y \in M_c$. Therefore, $M_c = M_i$.

□

Corollary 11.5. *Let A be a Noetherian ring and let M be a finitely generated A-module. Then M is Noetherian. Similarly, if A is Artinian, then any finitely generated A-module is Artinian.*

Proof. Recall that any finitely generated A-module is a quotient of a free module $A^{\oplus n} = A \oplus \dots \oplus A$. Proposition 11.4 implies that since A is a submodule of $A^{\oplus 2}$ via $x \mapsto (x, 0)$, and the quotient is isomorphic to A, that $A^{\oplus 2}$ is Noetherian. Hence $A^{\oplus n}$ is Noetherian. Applying Proposition 11.4 to the surjective map $A^{\oplus n} \to M$ we prove that M is Noetherian. □

Corollary 11.6. *Let M be an A-module. If $0 = M_0 \subset \dots \subset M_n = M$ are A-submodules such that M_{i+1}/M_i is a Noetherian A-module, then M is also Noetherian. The same statement is true for Artinian modules.*

Proof. Apply Proposition 11.4. Then M_1/M_0 is Noetherian and M_2/M_1 is Noetherian implies that M_2 is Noetherian, etc. □

Lemma 11.7. *Let A be a Noetherian ring. Let $S \subset A$ be a multiplicative set. Then $S^{-1}A$ is Noetherian.*

Proof. By Lemma 11.1 it is enough to prove that any non-empty set of ideals of $S^{-1}A$ has a maximal element. So take J a non-empty set of ideals of $S^{-1}A$. Let $f : A \to S^{-1}A$ be the map $f(a) = a/1$. Consider $\left\{ f^{-1}(I) \mid I \in J \right\}$. This is a set of ideals of A. It has a maximal element, say I_0, since A is Noetherian. Then $I_0 = S^{-1}f(I_0)$ is a maximal element of J. □

12 Primary decomposition

Definition 12.1. An ideal $I \subsetneq R$ is called **primary** if for all $x, y \in R$ such that $xy \in I$ we have either $x \in I$ or $y^n \in I$ for some $n \geq 1$. Equivalently, every zero-divisor in R/I is a nilpotent element of R/I.

Example. If $R = \mathbb{Z}$ and p a prime number then $\langle p^n \rangle$ is a primary ideal.

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Proposition 12.2. *If* $\operatorname{rad} I$ *is a maximal ideal, then* I *is primary. In particular, any power of a maximal ideal is primary.*

Proof. Recall $\operatorname{rad} I$ is the intersection of all prime ideals containing I. In particular, if $\operatorname{rad} I$ is a maximal ideal, then it is a unique prime ideal containing I. Then R/I has a unique prime ideal $\operatorname{rad} I/I$, so R/I is a local ring. Hence $\mathcal{N}(R/I) = \mathcal{J}(R/I) = \operatorname{rad} I/I$. Clearly, $(R/I) \setminus (\operatorname{rad} I/I) = (R/I)^*$. Thus any element of R/I is either a unit, or a nilpotent element. Hence I is primary. If $M \subset R$ is a maximal ideal, then $\operatorname{rad} M^n = M$. $\square$

Proposition 12.3. *Let* $I \subset R$ *be a primary ideal. Then* $\operatorname{rad} I$ *is a prime ideal. This is the smallest prime ideal of* R *that contains* I*.*

Remark.

$$\{\text{ideals } I \subset R \mid \operatorname{rad} I \text{ is a maximal ideal}\} \subset \{\text{primary ideals}\} \subset \{\text{ideals } I \subset R \mid \operatorname{rad} I \text{ is a prime ideal}\}.$$

Proof. Suppose $xy \in \operatorname{rad} I$, so $x^m y^m = (xy)^m \in I$, but $x \notin \operatorname{rad} I$, so $x^m \notin I$. So in R/I we have $x^m y^m = 0$ and $x^m \neq 0$. Since I is primary, every zero-divisor in R/I is nilpotent. Hence $(y^m)^n = 0$ for some $n \geq 1$. But then in R we have $y^{mn} \in I$, so $y \in \operatorname{rad} I$. This proves that $\operatorname{rad} I$ is prime. Recall that $\operatorname{rad} I$ is the intersection of all prime ideals containing I. If $\operatorname{rad} I$ is already a prime ideal, it is the smallest ideal containing I. $\square$

A **primary decomposition** of an ideal $I \subset R$ is the representation

$$I = \bigcap_{m=1} J_m,$$

where $J_1, \dots, J_m$ are primary ideals of R. The aim is that any ideal in a Noetherian ring has a primary decomposition.

Example. Let $R = \mathbb{Z}$. Then $n = \prod_{i=1}^m p_i^{a_i}$, where p_i's are prime numbers, and $a_i \geq 1$, so

$$\langle n \rangle = \prod_{i=1}^m \langle p_i^{a_i} \rangle = \bigcap_{i=1}^m \langle p_i^{a_i} \rangle.$$

Clearly, $\langle p_i \rangle$ are maximal ideals of $\mathbb{Z}$. So, $\langle p_i^{a_i} \rangle$ are primary ideals of $\mathbb{Z}$.

Definition 12.4. Let $I \subsetneq R$ be an ideal. Then I is called **irreducible** if for any ideals J and K of R such that $I = J \cap K$ we have $I = J$ or $I = K$. In other words, I is irreducible if $I \neq J \cap K$, where $I \subsetneq J$ and $I \subsetneq K$.

Proposition 12.5.

1. *Any prime ideal is irreducible.*
2. *In a Noetherian ring, any irreducible ideal is primary.*

Exercise.

$$\{\text{prime ideals}\} \subset \{\text{irreducible ideals}\} \subset \{\text{primary ideals}\}.$$

Show that these are strict in general.

Proof.

1. Suppose $\mathfrak{p} \subset R$ is a prime ideal such that $\mathfrak{p} = J \cap K$, and $\mathfrak{p} \neq J$ and $\mathfrak{p} \neq K$. Let $x \in J \setminus \mathfrak{p}$ and $y \in K \setminus \mathfrak{p}$. Then $xy \in JK \subset J \cap K = \mathfrak{p}$. This is a contradiction, since $\mathfrak{p}$ is prime.

2. Let I be an irreducible ideal of a Noetherian ring R. Consider R/I. Suppose $x, y \in R/I$ such that $xy = 0$ and $x \neq 0$. The task is to show that $y^n = 0$ for some $n \geq 1$. Since R is Noetherian, R/I is Noetherian. Consider
$$\operatorname{Ann} y^m = \{\alpha \in R/I \mid \alpha y^m = 0\}.$$
Then $\operatorname{Ann} y \subset \operatorname{Ann} y^2 \subset \cdots \subset R/I$. There exists $n \geq 1$ such that $\operatorname{Ann} y^n = \operatorname{Ann} y^{n+i}$, for all $i \geq 0$. Claim that $\langle x \rangle \cap \langle y^n \rangle = \langle 0 \rangle$. Suppose $0 \neq a \in \langle x \rangle \cap \langle y^n \rangle$. Then $ay = 0$ and also $a = by^n$ for some $b \in R/I$. Then $0 = ay = by^{n+1}$. This says that $b \in \operatorname{Ann} y^{n+1} = \operatorname{Ann} y^n$. Hence $by^n = 0$, so $a = 0$, a contradiction. But the ideal $I \subset R$ is irreducible, hence the ideal $\langle 0 \rangle \subset R/I$ is irreducible. We know that $\langle x \rangle \neq 0$. Thus $\langle y^n \rangle = \langle 0 \rangle$, so $y^n = 0$. This finishes the proof.

$\square$

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Theorem 12.6 (Noether)**.** *Every ideal in a Noetherian ring has a primary decomposition.*

Proof. We shall in fact prove that every ideal is a finite intersection of irreducible ideals. Suppose this does not hold for a Noetherian ring R. Let Σ be the set of proper ideals of R that are not finite intersections of irreducible ideals. Assume $\Sigma \neq \emptyset$. In a Noetherian ring every non-empty set of ideals has a maximal element. Take a maximal element of Σ. This is an ideal $I \subsetneq R$. Then I is not a finite intersection of irreducible ideals, in particular I is not irreducible. Thus $I = J \cap K$, where J and K are ideals of R, and $J \supsetneq I$ and $K \supsetneq I$. Since I is a maximal element of Σ, we can write $J = \bigcap_{m=1}^n J_m$ and $K = \bigcap_{s=1}^r K_s$, where each J_m and each K_s is irreducible. Hence
$$I = \left(\bigcap_{m=1}^n J_m\right) \cap \left(\bigcap_{s=1}^r K_s\right)$$
is a finite intersection of irreducible ideals. This is a contradiction. This shows that $\Sigma = \emptyset$. $\square$

Lemma 12.7. *Let $I_1, \ldots, I_n$ be primary ideals in R such that $\operatorname{rad} I_1 = \cdots = \operatorname{rad} I_n$. Then $\bigcap_{j=1}^n I_j$ is also a primary ideal and*
$$\operatorname{rad} \bigcap_{j=1}^n I_j = \operatorname{rad} I_1 = \cdots = \operatorname{rad} I_n.$$

Proof. Let $\mathfrak{p} = \operatorname{rad} I_j$ for $j = 1, \ldots, n$, and let $I = \bigcap_{j=1}^n I_j$. Suppose $x, y \in R$ such that $xy \in I$, but $x \notin I$. Hence $x \notin I_j$ for some j. We have $xy \in I_j$ but $x \notin I_j$ thus $y \in \operatorname{rad} I_j$, since I_j is primary. So $y \in \mathfrak{p}$. Then
$$\operatorname{rad} I = \operatorname{rad} \bigcap_{j=1}^n I_j = \bigcap_{j=1}^n \operatorname{rad} I_j = \mathfrak{p},$$
by problem sheet 2 question 2 (b). Hence $y \in \operatorname{rad} I$. This shows that I is primary. Moreover, $\operatorname{rad} I = \mathfrak{p}$. $\square$

Lemma 12.8. *Let I be a primary ideal of R such that $\operatorname{rad} I$ is a prime ideal $\mathfrak{p}$. We say that I is a* ***$\mathfrak{p}$-primary ideal****. Then*
$$(I : \langle x \rangle) = \begin{cases} R & x \in I \\ \text{a } \mathfrak{p}\text{-primary ideal} & x \notin I \end{cases}.$$

Proof. $x \in I$ implies that $1 \in (I : \langle x \rangle)$. Hence $\langle I : \langle x \rangle \rangle = R$. Now assume $x \notin I$. Then
$$(I : \langle x \rangle) = \{y \in R \mid xy \in I\}.$$
Since I is primary, this implies $y^n \in I$ and $y \in \operatorname{rad} I = \mathfrak{p}$. So $I \subset (I : \langle x \rangle) \subset \mathfrak{p}$, so $\mathfrak{p} = \operatorname{rad} I \subset \operatorname{rad}(I : \langle x \rangle) \subset \mathfrak{p}$, so $\operatorname{rad}(I : \langle x \rangle) = \mathfrak{p}$. It remains to show that $(I : \langle x \rangle)$ is primary. Assume $yz \in (I : \langle x \rangle)$ whereas $y \notin \operatorname{rad}(I : \langle x \rangle) = \mathfrak{p}$. We must show that $z \in (I : \langle x \rangle)$. Then $yz \in (I : \langle x \rangle)$ implies that $y(xz) = xyz \in I$. Since I is primary and $y \notin \mathfrak{p} = \operatorname{rad} I$, no power of y is contained in I, therefore $xz \in I$, so $z \in (I : \langle x \rangle)$. $\square$

Call a primary decomposition $I = \bigcap_{j=1}^{k} I_j$ **minimal** if

- $\operatorname{rad} I_j \ne \operatorname{rad} I_k$ for $j \ne k$, and
- for every $j = 1, \dots, n$, $\bigcap_{k=1,\ k \ne j}^{n} I_k \subset I_j$.

Can achieve this by Lemma 12.7.

Theorem 12.9 (First uniqueness theorem)**.** *Let $I = \bigcap_{j=1}^{n} I_j$ be a minimal primary decomposition. Write $\mathfrak{p}_j = \operatorname{rad} I_j$ for $j = 1, \dots, n$. Then the ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are precisely the prime ideals of R of the form $\operatorname{rad}(I : \langle x \rangle)$, where $x \in R$. In particular, $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ do not depend on the primary decomposition chosen.*

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Proof. Take any $x \in R$. Then

$$(I : \langle x \rangle) = \left(\bigcap_{j=1}^{k} I_j : \langle x \rangle \right) = \left\{ y \in R \;\middle|\; xy \in \bigcap_{j=1}^{k} I_j \right\} = \bigcap_{j=1}^{k} \{ y \in R \mid xy \in I_j \} = \bigcap_{j=1}^{k} (I_j : \langle x \rangle).$$

Take the radicals of these ideals. Problem sheet 2 question 2 (b) says that the radical of an intersection is the intersection of their radicals, so $\operatorname{rad}(I : \langle x \rangle) = \bigcap_{j=1}^{k} \operatorname{rad}(I_j : \langle x \rangle)$. Note that by Lemma 12.8

$$\operatorname{rad}(I_j : \langle x \rangle) = \begin{cases} R & x \in I_j \\ \mathfrak{p}_j & x \notin I_j \end{cases},$$

so $\operatorname{rad}(I : \langle x \rangle) = \bigcap_{x \notin I_j} \mathfrak{p}_j$. So we recover all of $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ and nothing else. Lemma 4.12 says that $\mathfrak{p} = \bigcap_{i=1}^{m} J_i$ is prime implies that $\mathfrak{p}$ is one of the J_i's. Hence if $\operatorname{rad}(I : \langle x \rangle)$ is a prime ideal, then it is one of $\mathfrak{p}_j = \operatorname{rad}(I_j : \langle x \rangle)$ for $x \notin I_j$. ☐

Remark. These prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are called the **associated primes** of I.

Example.

- Let $R = \mathbb{Z}$. Then

$$\{\text{prime ideals}\} = \{\langle 0 \rangle\} \cup \{\text{maximal ideals}\} = \{\langle 0 \rangle\} \cup \{\langle p \rangle \mid p \text{ prime}\},$$
$$\{\text{primary ideals}\} = \{\text{irreducible ideals}\} = \{\langle 0 \rangle\} \cup \{\langle p^n \rangle \mid p \text{ prime}\}.$$

  For example, $\langle 4 \rangle \subsetneq \langle 2 \rangle \cap \langle 2 \rangle \subsetneq \mathbb{Z}$ is irreducible.

- Let $R = k[x]$. Then

$$\{\text{prime ideals}\} = \{\langle 0 \rangle\} \cup \{\text{maximal ideals}\} = \{\langle 0 \rangle\} \cup \{\langle p(x) \rangle \mid p(x) \text{ monic irreducible polynomial}\},$$
$$\{\text{primary ideals}\} = \{\text{irreducible ideals}\} = \{\langle 0 \rangle\} \cup \{\langle p(x)^n \rangle \mid p(x) \text{ monic irreducible polynomial}\}.$$

- Let $R = k[x, y]$. Then $\langle x \rangle$ is prime, since $k[x, y] / \langle x \rangle \cong k[y]$ is an integral domain, and $\langle x, y \rangle$ is maximal, since $k[x, y] / \langle x, y \rangle \cong k$ is a field.
  - $\langle x, y^2 \rangle$ is not prime, since $k[x, y] / \langle x, y^2 \rangle \cong k \oplus ky$ is not an integral domain, where $y^2 = 0$. Then $\operatorname{rad} \langle x, y^2 \rangle = \langle x, y \rangle$, so Proposition 12.2 implies that $\langle x, y^2 \rangle$ is primary.
  - $\langle xy \rangle$ is not prime, since $x^n, y^n \notin \langle xy \rangle$ for all $n \ge 1$ and $xy \in \langle xy \rangle$, and $k[x, y] / \langle xy \rangle$ has zero-divisors which are not nilpotent, so $\langle xy \rangle$ is also not primary. Then $\langle xy \rangle = \langle x \rangle = \langle x \rangle \langle y \rangle = \langle x \rangle \cap \langle y \rangle$ is a primary decomposition, where $\langle x \rangle$ and $\langle y \rangle$ are prime, hence primary.
  - $\langle x^a y^b \rangle = \langle x^a \rangle \cap \langle x^b \rangle$ for $a, b \ge 1$ is a primary decomposition, since $\langle x^a \rangle$ and $\langle y^b \rangle$ are primary. For example, $\operatorname{rad} \langle x^a \rangle = \langle x \rangle$, since $k[x, y] / \langle x^a \rangle \cong k[y] \oplus \dots \oplus k[y] x^{a-1}$ has no non-nilpotent zero-divisors.
  - $\langle x^2, xy^2 \rangle = \langle x \rangle \langle x, y^2 \rangle$ for $a, b \ge 1$ is not primary, since y gives a zero-divisor in $k[x, y] / \langle x^2, xy^2 \rangle$ which is not nilpotent. Find a primary decomposition. [8]
  - $\langle x^2, xy, y^2 \rangle = \langle x, y \rangle^2$, so it is primary but not irreducible, since $\langle x^2, xy, y^2 \rangle = \langle x^2, y \rangle \cap \langle x, y^2 \rangle$.

[8] Exercise

13 Artinian rings and modules

Lecture 20 Tuesday 19/11/19

Definition 13.1. Let A be a ring and let M be an A-module. Then M is a **simple** A-module if the only proper submodule of M is zero. A **composition series** is a descending chain of submodules $M = M_0 \supsetneq \dots \supsetneq M_n = 0$ such that M_i/M_{i+1} is a simple A-module for $i = 0, \dots, n-1$.

Proposition 13.2. *The following are equivalent.*

- *M is both Noetherian and Artinian.*
- *M has a composition series.*

Proof.

$\implies$ Look at all proper submodules of M. Since M is Noetherian, this set has a maximal element. Call it M_1. It is also Noetherian, so continue and build a descending chain. Since M_1 is maximal, M/M_1 is simple. All M_i/M_{i+1} are simple. Since M is Artinian, this chain is stationary, so $M_n = 0$ for some n.

$\impliedby$ Corollary 11.6 says that if M_i/M_{i+1} is both Noetherian and Artinian, then so is M. A simple module is both Noetherian and Artinian.

□

Proposition 13.3. *If M has a composition series of length n, then any composition series of M has length n.*

Proof. Let us denote by $\mathrm{l}(M)$ the smallest length of a composition series of M.

Step 1. For a proper submodule $N \subsetneq M$ we have $\mathrm{l}(N) < \mathrm{l}(M)$. Indeed, let (M_i) be a composition series of length $\mathrm{l}(M)$. Define $N_i = N \cap M_i$, so

$$\begin{array}{ccccccc} M = M_0 & \supset & \dots & \supset & M_{n-1} & \supset & M_n = 0 \\ \cup & & & & \cup & & \\ N = N_0 & \supset & \dots & \supset & N_{n-1} & \supset & N_n = 0 \end{array}.$$

Then $\mathrm{Ker}\,(N_i \to M_i/M_{i+1}) = N_{i+1}$, so $N_i/N_{i+1} \subset M_i/M_{i+1}$, which is simple. After eliminating repetitions we get a composition of length at most $\mathrm{l}(M)$. If the length is exactly $\mathrm{l}(M)$, then $N_{n-1} = M_{n-1}$, $N_{n-2} = M_{n-2}$, etc, and finally $N = M$.

Step 2. Any proper chain of submodules of M has length at most $\mathrm{l}(M)$. Passing to a proper submodule decreases $\mathrm{l}(M)$ at least by one. So the chain contains no more than $\mathrm{l}(M)$ terms.

Step 3. So consider any composition series of M. By step 2, it has length at most $\mathrm{l}(M)$. By minimality of $\mathrm{l}(M)$, it has length equal to $\mathrm{l}(M)$.

□

Define the **length** of a Noetherian and Artinian module M to be $\mathrm{l}(M)$, the length of any composition series.

Exercise. Any chain of submodules of M can be made into a composition series by inserting some submodules.

Proposition 13.4. *Let M be a Noetherian and Artinian module. If $N \subset M$ is a submodule, then*

$$\mathrm{l}(M) = \mathrm{l}(N) + \mathrm{l}(M/N).$$

Proof. Exercise. [9] □

[9] Exercise

Example 13.5. Suppose R is a k-algebra, that is k is a field contained in R and R is a vector space over k. For example, $R = k$ or $R = k[x_1, \dots, x_n]/I$, where I is an ideal in $k[x_1, \dots, x_n]$. Lecture 21 Wednesday 20/11/19

- If $\dim_k R < \infty$, then R is an Artinian ring. Indeed, ideals of R are vector subspaces, so any chain of ideals has finite length. Hence R is both Artinian and Noetherian.
- If R is a finite set, then R is Artinian and Noetherian.
- Let $k[[x]] = \left\{ \sum_{i \geq 0} a_i x_i \;\middle|\; a_i \in k \right\}$. Then $\langle x \rangle \supsetneq \langle x^2 \rangle \supsetneq \dots$ is an infinite descending chain. So $k[[x]]$ is not Artinian. Similarly, $k[x]$ is not Artinian.

Remark. Hilbert's basis theorem says that if R is Noetherian, then so is $R[x]$. The analogue of this does not hold for Artinian rings.

Lemma 13.6. *An Artinian integral domain is a field.*

Proof. Take $x \neq 0$ in an Artinian ring A. Consider $\langle x \rangle \supset \langle x^2 \rangle \supset \dots$, which is stationary, so $\langle x^n \rangle = \langle x^{n+1} \rangle$ for some $n \geq 0$. Therefore, $x^n = ax^{n+1}$, so $x^n(ax - 1) = 0$. Then $x^n \neq 0$, so $ax = 1$, so x is invertible, that is $x \in A^*$. $\square$

Corollary 13.7. *In an Artinian ring every prime ideal is maximal.*

Corollary 13.8. *In an Artinian ring the nilradical is the same as the Jacobson radical.*

Lemma 13.9. *An Artinian ring has only finitely many maximal ideals.*

Proof. Assume this is false, so $M_1, M_2, \dots$ are pairwise different maximal ideals. Then $M_1 \supset M_1 \cap M_2 \supset \dots$ is stationary, so for some $n \geq 1$ we have $M_1 \cap \dots \cap M_n = (M_1 \cap \dots \cap M_n) \cap M_{n+1}$, so $M_1 \cap \dots \cap M_n \subset M_{n+1}$. By the prime avoidance lemma, M_{n+1} contains some M_i, where $i \in \{1, \dots, n\}$. These are maximal ideals, hence $M_i = M_{n+1}$. This is a contradiction. $\square$

Definition. An ideal $I \subset R$ is called **nilpotent** if $I^n = 0$ for some $n \geq 1$.

Lemma 13.10. *If A is an Artinian or Noetherian ring, then the nilradical $\mathcal{N}(A)$ is a nilpotent ideal, that is $\mathcal{N}(A)^m = 0$ for some m.*

Proof.

- Assume A is Artinian. Consider $\mathcal{N}(A) \supset \mathcal{N}(A)^2 \supset \dots$. There exists $n \geq 1$ such that $\mathcal{N}(A)^n = \mathcal{N}(A)^{n+1}$. Claim that $\mathcal{N}(A)^n = 0$. Assume $\mathcal{N}(A)^n \neq 0$. Consider all ideals $I \subset A$ such that $I\mathcal{N}(A)^n \neq 0$. This set is non-empty. It contains $\mathcal{N}(A)$, since $\mathcal{N}(A)^{n+1} = \mathcal{N}(A)^n = 0$. Then A is Artinian, so this set has a minimal element. Call it I. So we have $I\mathcal{N}(A)^n \neq 0$ hence $x\mathcal{N}(A)^n \neq 0$ for some $x \in I$. We have $\langle x \rangle \mathcal{N}(A)^n \neq 0$ where $\langle x \rangle \subset I$. Then $\langle x \rangle$ belongs to our set, so by minimality of I we must have $I = \langle x \rangle$. We have $0 \neq x\mathcal{N}(A)^n = x\mathcal{N}(A)^n \mathcal{N}(A)^n$, since $\mathcal{N}(A)^n = \mathcal{N}(A)^m$, for any $m \geq n$. So $x\mathcal{N}(A)^n$ is an ideal in our set. Then $x \in I$ so $x\mathcal{N}(A)^n \subset I$. By minimality of I we must have $x\mathcal{N}(A)^n = I$. Therefore $\langle x \rangle \mathcal{N}(A)^n = I = \langle x \rangle$, so x can be written as xy for some $y \in \mathcal{N}(A)^n \subset \mathcal{N}(A)$. Thus $y^r = 0$ for some $r \geq 1$. Then $x = \dots = xy^r = 0$. This says that $I = \langle x \rangle = 0$. This is a contradiction because $I\mathcal{N}(A)^n \neq 0$.
- Now assume A is Noetherian. Then every ideal is finitely generated, in particular $\mathcal{N}(A) = \langle x_1, \dots, x_n \rangle$. Since each x_i is nilpotent there exists $m \geq 1$ such that $x_i^m = 0$ for all i. Then any product of mn elements of $\mathcal{N}(A)$ is $\sum_{a_1 + \dots + a_n = mn} cx_1^{a_1} \dots x_n^{a_n}$. So there exists i, for $1 \leq i \leq n$, such that $a_i \geq m$. Hence $x_1^{a_1} \dots x_n^{a_n} = 0$, so $\mathcal{N}(A)^{mn} = 0$.

$\square$

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Corollary 13.11. *Every ideal in a Noetherian or Artinian ring contains some power of its radical.*

Proof. If A is Noetherian, then A/I is also Noetherian. We have $\operatorname{rad} I/I = \mathcal{N}(A/I)$. By Lemma 13.10 there exists $m \geq 1$ such that $\mathcal{N}(A/I)^m = 0$. Then $\operatorname{rad} I^m \subset I$. The same proof works in the Artinian case. $\square$

Lemma 13.12. *Let V be a vector space over a field k. The following are equivalent.*

- $\dim_k V < \infty$.
- *V is a Noetherian k-module.*
- *V is an Artinian k-module.*

Proof. Obvious. $\square$

Lemma 13.13. *Let A be a ring. Let $I_1, \dots, I_n$ be maximal ideals of A, possibly with repetitions. Suppose $I_1 \dots I_n = 0$. Then A is Noetherian if and only if A is Artinian.*

Proof.

$\implies$ Define $M_r = I_1 \dots I_r$ for $r = 1, \dots, n$, so $A \supset M_1 = I_1 \supset \dots \supset M_n = \prod_{r=1}^n I_r = 0$. Then A is a Noetherian A-module, so every subquotient module is also Noetherian. In particular, M_i/M_{i+1} is a Noetherian A-module. Since $I_{r+1}M_r = M_{r+1}$, I_{r+1} acts trivially on M_r/M_{r+1}. Therefore, M_r/M_{r+1} is naturally an A/I_{r+1}-module. But A/I_{r+1} is a field. Call it k. The A-submodules of M_r/M_{r+1} are the same as the k-submodules of M_r/M_{r+1}. By Lemma 13.12 M_r/M_{r+1} is an Artinian k-module hence M_r/M_{r+1} is an Artinian A-module. Now Proposition 11.4 implies that A is Artinian.

$\impliedby$ Similar.

$\square$

Definition 13.14. The **Krull dimension** of a ring A is the supremum of all $n \ge 0$ such that A has an ascending chain of prime ideals $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_n$. If the supremum does not exist, the dimension is infinite. It is denoted $\dim A$.

Example 13.15.

- Any field has dimension zero.
- If A is Artinian, then every prime ideal is maximal, hence $\dim A = 0$.
- Since $\langle 0 \rangle \subsetneq \langle p \rangle$ for p a prime number, $\dim \mathbb{Z} = 1$.
- If k is a field, then $\dim k[t] = 1$.
- If k is a field, then $\dim k[[t]] = 1$.
- If k is a field, then $\dim k[t_1, \dots, t_n] = n$.

Theorem 13.16. *A ring is Artinian if and only if it is Noetherian and has dimension zero.*

Proof.

$\implies$ Suppose A is Artinian. Corollary 13.7 says that every prime ideal is maximal, hence $\dim A = 0$. Lemma 13.9 says that A has only finitely many maximal ideals, say $I_1, \dots, I_n$. Then $I_1 \dots I_n \subset I_1 \cap \dots \cap I_n = \mathcal{N}(A)$. Lemma 13.10 says $\mathcal{N}(A)^m = 0$ for some $m \ge 1$. Hence $I_1^m \dots I_n^m = 0$. Lemma 13.13 now implies that A is Noetherian.

$\impliedby$ Now assume that A is Noetherian and $\dim A = 0$. By Emmy Noether's theorem the ideal $\langle 0 \rangle$ has a primary decomposition, that is $\langle 0 \rangle = J_1 \cap \dots \cap J_n$, where J_i's are primary. Recall $\operatorname{rad} J_i$ is a prime ideal of A. Since $\dim A = 0$, this ideal is maximal. The associated primes of $\langle 0 \rangle$ are maximal ideals. By Corollary 13.11 each J_i contains a power of $\operatorname{rad} J_i$. Therefore, the product of these powers of these maximal ideals is contained in $\prod_{i=1}^n J_i \subset \bigcap_{i=1}^n J_i = \langle 0 \rangle$. Now Lemma 13.13 implies that A is Artinian.

$\square$

Theorem 13.17. *Any Artinian ring is a product of finitely many local Artinian rings.*

Definition 13.18. Two ideals $I, J \subset R$ are **coprime** if $I + J = R$.

Example. Two distinct maximal ideals are coprime.

Suppose $I_1, \dots, I_n$ are ideals of R. Define

$$\begin{array}{rcrcl} \phi & : & R & \longrightarrow & \prod_{j=1}^n R/I_j \\ & & x & \longmapsto & (x + I_1, \dots, x + I_n) \end{array} .$$

Lemma 13.19 (Chinese remainder theorem)**.**

- *If I_r and I_s are coprime whenever $r \ne s$, then $\prod_{j=1}^n I_j = \bigcap_{j=1}^n I_j$.*
- *ϕ is surjective if and only if I_r and I_s are coprime for $r \ne s$.*
- *ϕ is injective if and only if $\bigcap_{j=1}^n I_j = \langle 0 \rangle$.*

Proof. See problem sheet 4 question 2. □

Lecture 23 is a test. Lecture 23 Tuesday 26/11/19