M4P58 Modular Forms

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Syllabus

Contents

0 Introduction

Lecture 1
Friday
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The following are textbooks.

- Serre, A course in arithmetic, 1973
- J Shurman and F Diamond, A first course in modular forms, 2005

Let

$$f = q \prod_{n=1}^{\infty} (1 - q^n)^2 \left(1 - q^{11n}\right)^2 = \sum_{n=1} b_n q^n = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \dots,$$

and let a_n be the number of solutions modulo n to the elliptic curve

$$E = \left\{(x, y) \in \mathbb{Z} \mid y^2 + y = x^3 - x^2 - 10x - 20\right\}.$$

- Modulo 2, there are $a_2 = 4$ solutions $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$.
- Modulo 3, there are $a_3 = 4$ solutions $(1,0)$, $(1,-1)$, $(-1,0)$, $(-1,-1)$.
- Modulo 5, there are $a_5 = 4$ solutions $(0,0)$, $(0,-1)$, $(1,0)$, $(-1,-1)$.
- Modulo 7, there are $a_7 = 9$ solutions $(1,3)$, $(2,2)$, $(2,-3)$, $(-1,1)$, $(-1,-2)$, $(-2,1)$, $(-2,-2)$, $(-3,1)$, $(-3,-2)$.

If $p \ne 11$, then

$$a_p - p = -b_p.$$

The following are some questions.

- What is the relationship between E and f?
- Can we find similar relationships for other E?
- How does one prove something like this?

Let

$$\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R},\ y > 0\} \subseteq \mathbb{C}.$$

Then $\mathbb{H}$ has an action of

$$\mathrm{SL}_2(\mathbb{R}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{R},\ ad - bc = 1\right\}.$$

Modular forms are complex functions on $\mathbb{H}$ with a high degree of symmetry. These functions are symmetric under the action of large discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$, in particular

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{Z},\ ad - bc = 1\right\} \subseteq \mathrm{SL}_2(\mathbb{R}).$$

Why are these interesting to number theorists? Power series expansions often involve expressions of interest to number theorists. For example,

- Bernoulli numbers,
- divisor functions $\sigma_k(n) = \sum_{d \mid n} d^k$,
- number of points on elliptic curves, and
- traces of Galois representations.

1 Modular forms of level one

1.1 Modular functions and forms

1.1.1 Modular actions

Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}.$$

Then $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{C} \cup \{\infty\}$ by

$$\gamma \cdot z = \begin{cases} \dfrac{az+b}{cz+d} & z \ne -\dfrac{d}{c} \\ \infty & z = -\dfrac{d}{c} \end{cases} \qquad \gamma \cdot \infty = \frac{a}{c}.$$

One checks that this gives a bijection from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$, where inverse is given by the inverse matrix

$$\gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and $\gamma \cdot (\gamma' \cdot z) = \gamma\gamma' \cdot z$. One obtains a left action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{C} \cup \{\infty\}$. An observation is

$$\operatorname{Im} \gamma z = \operatorname{Im} \frac{az+b}{cz+d} = \operatorname{Im} \frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = \frac{\operatorname{Im}(az+b)(c\overline{z}+d)}{|cz+d|^2} = \frac{(ad-bc)\operatorname{Im} z}{|cz+d|^2}.$$

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In particular, if $\gamma \in \mathrm{SL}_2(\mathbb{R})$, then

$$\operatorname{Im} \gamma z = \frac{\operatorname{Im} z}{|cz+d|^2}.$$

So $\mathrm{SL}_2(\mathbb{R})$ preserves $\mathbb{H} \cup \{\infty\}$. More generally, if $\gamma \in \mathrm{GL}_2(\mathbb{R})$, then

$$\operatorname{Im} \gamma z = \frac{\det \gamma \operatorname{Im} z}{|cz+d|^2}.$$

So

$$\mathrm{GL}_2(\mathbb{R})_+ = \{\gamma \in \mathrm{GL}_2(\mathbb{R}) \mid \det \gamma > 0\}$$

preserves $\mathbb{H} \cup \{\infty\}$. Define

$$\begin{array}{rccc} f|_{k,\gamma} & : & \mathbb{H} & \longrightarrow & \mathbb{C} \\ & & z & \longmapsto & \det \gamma^{k-1} f(\gamma z)(cz+d)^{-k} \end{array}, \qquad f : \mathbb{H} \to \mathbb{C}, \qquad \gamma \in \mathrm{GL}_2(\mathbb{R})_+, \qquad k \in \mathbb{Z},$$

where $\det \gamma^{k-1}$ is the fudge factor, which is one for $\gamma \in \mathrm{SL}_2(\mathbb{R})$, and $(cz+d)^{-k}$ is the twisted action on functions. Check that

$$f|_{k,\mathrm{id}} = f, \qquad \left(f|_{k,\gamma} \right)\Big|_{k,\gamma'} = f|_{k,\gamma'\gamma}.$$

This gives, for each k, a left action of $\mathrm{GL}_2(\mathbb{R})_+$ on functions $\mathbb{H} \to \mathbb{C}$, a **modular action of weight** k. A modular form of weight k will be a sufficiently nice function $f : \mathbb{H} \to \mathbb{C}$ such that $f|_{k,\gamma} = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. That is, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and all $z \in \mathbb{H}$,

$$f(\gamma z)(cz+d)^{-k} = f(z), \qquad \implies \qquad f(\gamma z) = f(z)(cz+d)^k,$$

the **modular transformation law of weight** k. The following are some observations.

- Let $k = 0$. Then constant functions satisfy $f(\gamma z) = f(z)$. It will turn out that all functions of weight zero are constant.
- Let k be odd, and $\gamma = -\mathrm{id}$. Then $\gamma z = z$ for all z and $cz + d = -1$, so $f(\gamma z) = f(z)(cz+d)^k$ gives $f(z) = f(z)(-1)^k$, so $f(z) = -f(z)$, so $f(z) = 0$ for all z. So no functions $f : \mathbb{H} \to \mathbb{C}$ satisfy the modular transformation law of weight k, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, when k is odd.

1.1.2 Review of complex analysis

Let $f : U \to \mathbb{C}$, for $U \subseteq \mathbb{C}$ open, and let $p \in U$.

Definition 1.1.1. f is **holomorphic** at p if

$$f'(p') = \lim_{\epsilon \to 0,\ \epsilon \in \mathbb{C}} \frac{f(p' + \epsilon) - f(p')}{\epsilon}$$

exists for all p' in a neighbourhood of p.

Proposition 1.1.2. *f is holomorphic at p implies that f is continuous.*

Proposition 1.1.3. *f is holomorphic at p implies that f is infinitely differentiable at p, that is $f^{(n)}(p)$ exists for all $n \geq 0$. Moreover, we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n = f(p) + f'(p)(z-p) + \frac{f'(p)}{2}(z-p)^2 + \dots,$$

for all z in a neighbourhood of p.

Corollary 1.1.4. *If f is holomorphic and not identically zero on an open set U, then the zeroes of f are isolated on U.*

More generally is the following.

Definition 1.1.5. f is **meromorphic** at p if there exists a neighbourhood U of p and $g, h : U \to \mathbb{C}$ holomorphic on U such that $f = g/h$ on $U \setminus \{p\}$. Such an f has a **Laurent series expansion** at p,

$$f(z) = \sum_{i=-N}^{\infty} c_i (z-p)^i.$$

The smallest i such that $c_i \neq 0$ is denoted by $\operatorname{ord}_p f$, the **order of vanishing** of f at p.

- If $\operatorname{ord}_p f = -n$ for $n > 0$, we say f has a **pole of order** n.
- If $\operatorname{ord}_p f = n$ for $n > 0$, we say f has a **zero of order** n.

Proposition 1.1.6.

- $\operatorname{ord}_p fg = \operatorname{ord}_p f + \operatorname{ord}_p g$.
- $\operatorname{ord}_p (f+g) \geq \min\{\operatorname{ord}_p f, \operatorname{ord}_p g\}$, *with equality if* $\operatorname{ord}_p f \neq \operatorname{ord}_p g$.

If f is holomorphic on $U \setminus \{p\}$ for U a neighbourhood of p, then f may or may not be meromorphic at p.

Example. $f(z) = e^{-1/z^2}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, but not meromorphic at zero.

Theorem 1.1.7. *Let f be holomorphic on $U \setminus \{p\}$, and there exists $n > 0$ such that*

$$\lim_{x \to p} (x-p)^n f(x)$$

exists. Then f is meromorphic on U, and $\operatorname{ord}_p f \geq -n$.

1.1.3 Modular functions

Definition 1.1.8. $f : \mathbb{H} \to \mathbb{C}$ is a **weakly modular function of weight** k if

- f is meromorphic on $\mathbb{H}$, and
- f satisfies the modular transformation law of weight k.

Consider

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $\gamma z = z + 1$ and $cz + d = 1$. The modular transformation law gives $f(z+1) = f(z)$. Let

$$\mathrm{D} = \{q \mid |q| < 1\}.$$

Can define a function

$$\begin{array}{rcccc} g & : & \mathrm{D} \setminus \{0\} & \longrightarrow & \mathbb{C} \\ & & q & \longmapsto & f\left(\dfrac{\log q}{2\pi i}\right) \end{array},$$

that is $f(z) = g\left(e^{2\pi i z}\right)$ for $z \in \mathbb{H}$, where g is holomorphic or meromorphic on $\{z \mid 0 < |z| < 1\}$ if and only if f is holomorphic or meromorphic on $\mathbb{H}$.

Definition 1.1.9. $f : \mathbb{H} \to \mathbb{C}$ is a **modular form of weight** k if

1. f satisfies the modular transformation law of weight k,
2. f is holomorphic on $\mathbb{H}$, and
3. f is holomorphic at ∞, so the function $g : \mathrm{D} \setminus \{0\} \to \mathbb{C}$, which is holomorphic on $\mathrm{D} \setminus \{0\}$ by 2, extends to a holomorphic function on D.

Then $q \to 0$ in D if and only if $\mathrm{Im}\, z \to +\infty$. Then 3 means $g(q)$ is bounded as $q \to 0$ so $f(z)$ is bounded as $\mathrm{Im}\, z \to +\infty$. For f satisfying 3, $g : \mathrm{D} \setminus \{0\} \to \mathbb{C}$ has a series expansion

$$g(q) = \sum_n a_n q^n = a_0 + a_1 q + \dots$$

in $q = e^{2\pi i z}$. We call this the q**-expansion** for f.

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Definition 1.1.10. $f : \mathbb{H} \to \mathbb{C}$ is a **meromorphic modular form of weight** k if the same conditions 1 to 3 hold, but with holomorphic weakened to meromorphic.

Note. If f is only meromorphic at ∞ then a finite number of negative powers of q can appear.

Example.

- The **modular discriminant**

  $$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

  is a modular form of weight 12.

- The j**-invariant**

  $$\mathrm{j}(z) = \frac{1}{q} + 744 + 196844q + 21493760q^2 + \dots$$

  is a meromorphic modular form of weight 0.

1.1.4 Lattice functions

How can we construct modular forms?

Definition 1.1.11. A **lattice** in $\mathbb{C}$ is an abelian subgroup of $\mathbb{C}$ of the form $\mathbb{Z}w_1 + \mathbb{Z}w_2$, where $w_1, w_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent. More generally if V is an $\mathbb{R}$-vector space, a **lattice** L in V is a discrete abelian subgroup of V that spans V over $\mathbb{R}$. For $L \subseteq \mathbb{C}$ a lattice and $\lambda \in \mathbb{C}^\times$, let

$$\lambda L = \{\lambda x \mid x \in L\} \subseteq \mathbb{C}.$$

We say that L and λL are **homothetic**. For $z \in \mathbb{H}$, let

$$\mathrm{L}_{z,1} = \mathbb{Z} + \mathbb{Z}z = \{az + b \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}.$$

A question is when is $\mathrm{L}_{z,1}$ homothetic to $\mathrm{L}_{z',1}$, and what is a homothety factor?

- Suppose $\mathrm{L}_{z,1} = \lambda \mathrm{L}_{z',1}$. Then there exist a, b, c, d such that $\lambda z' = az + b$ and $\lambda = cz + d$, so

$$\begin{pmatrix} \lambda z' \\ \lambda \end{pmatrix} = \gamma \begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{1}$$

  On the other hand there exist a', b', c', d' such that $z = a'\lambda z' + b'\lambda$ and $1 = c'\lambda z' + d'\lambda$, so

$$\gamma' \begin{pmatrix} \lambda z' \\ \lambda \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{2}$$

  (1) and (2) imply that

$$\gamma'\gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix},$$

  so $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Moreover (1) implies that $z' = (az + b) / (cz + d)$.

- Conversely, if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $\gamma z = (az + b) / (cz + d)$, so

$$\mathrm{L}_{\gamma z,1} = (cz + d)^{-1} \mathrm{L}_{az+b,cz+d}.$$

  But certainly $\mathrm{L}_{az+b,cz+d} \subseteq \mathrm{L}_{z,1}$. On the other hand if γ' is inverse to γ,

$$\begin{pmatrix} z \\ 1 \end{pmatrix} = \gamma'\gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \gamma \begin{pmatrix} az + b \\ cz + d \end{pmatrix} = \begin{pmatrix} a'(az + b) + b'(cz + d) \\ c'(az + b) + d'(cz + d) \end{pmatrix},$$

  so $z \in \mathrm{L}_{az+b,cz+d}$ and $1 \in \mathrm{L}_{az+b,cz+d}$. So $\mathrm{L}_{az+b,cz+d} = \mathrm{L}_{z,1}$, so $\mathrm{L}_{\gamma z,1} = (cz + d)^{-1} \mathrm{L}_{z,1}$.

Definition 1.1.12. A **lattice function of weight** k is a function $F : \{\text{lattices in } \mathbb{C}\} \to \mathbb{C}$ such that

$$F(\lambda L) = \lambda^{-k} F(L),$$

for all lattices L. Given such an F, can define

$$\begin{array}{cccc} f & : & \mathbb{H} & \longrightarrow & \mathbb{C} \\ & & z & \longmapsto & F(\mathrm{L}_{z,1}) \end{array}.$$

If F has weight k, then

$$f(\gamma z) = F(\mathrm{L}_{\gamma z,1}) = F\left((cz + d)^{-1} \mathrm{L}_{z,1}\right) = (cz + d)^k F(\mathrm{L}_{z,1}) = (cz + d)^k f(z).$$

1.2 Eisenstein series

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Definition 1.2.1. For $L \in \mathbb{C}$, define the **Eisenstein series**

$$\mathrm{G}_k(L) = \sum_{w \in L,\ w \neq 0} \frac{1}{w^k}, \qquad \mathrm{g}_k(z) = \mathrm{G}_k(\mathrm{L}_{z,1}) = \sum_{\substack{m=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k}.$$

Then

$$\mathrm{G}_k(\lambda L) = \sum_{w' \in \lambda L,\ w' \neq 0} \frac{1}{w'^k} = \sum_{w \in L,\ w \neq 0} \frac{1}{(\lambda w)^k} = \lambda^{-k} \mathrm{G}_k(L).$$

Corollary 1.2.2. *g_k satisfies the modular transformation law of weight k.*

The following are some questions.

- Does G_k, or g_k, converge?
- Is g_k holomorphic or meromorphic on $\mathbb{H}$?
- Is g_k holomorphic at ∞?
- What is the q-expansion of g_k?

1.2.1 Convergence and holomorphy on $\mathbb{H}$

Definition 1.2.3. Let $U \subseteq \mathbb{C}$ be open. A sequence of functions $f_n : U \to \mathbb{C}$ **converges uniformly on compact sets** to f if for all $C \subseteq U$ compact and all $\epsilon > 0$, there exists $N \in \mathbb{Z}$ such that for all $n > N$,

$$|f(z) - f_n(z)| < \epsilon, \qquad z \in C.$$

Theorem 1.2.4. *A uniform limit of holomorphic functions is holomorphic. If f_n converges to f uniformly on compact sets and f_n is holomorphic on U, then f is holomorphic on U.*

Theorem 1.2.5. *Let $k \geq 4$. The series $\mathrm{g}_k(z)$ converges absolutely and uniformly on compact subsets of $\mathbb{H}$.*

Proof. Let

$$P_{z,r} = \{az + b \mid a, b \in \mathbb{R},\ \max(|a|, |b|) = r\} \subseteq \mathbb{C},$$

so $P_{z,r} = rP_{z,1}$, and there are $8r$ points on $P_{z,r} \cap \mathrm{L}_{z,1}$. Then

$$\mathrm{g}_k(z) = \sum_{r=1}^{\infty} \sum_{w \in \mathrm{L}_{z,1} \cap P_{z,r}} \frac{1}{w^k}.$$

The function $z \mapsto |z|$ attains a non-zero minimum $\delta(z)$ on $P_{z,1}$, so on $P_{z,1}$, have $|z| > \delta(z)$, so $1/|z|^k < 1/\delta(z)^k$. On $P_{z,r}$, have $|z| > r\delta(z)$, so $1/|z|^k < 1/r^k\delta(z)^k$. Let $C \subseteq \mathbb{H}$ be compact. Then $z \mapsto \delta(z)$ is a continuous function on C and attains a minimum δ_C. For all $z \in C$ and all $w \in P_{z,r}$, get $|w| > r\delta_C$, so

$$\frac{1}{|w|^k} < \frac{1}{r^k \delta_C^k}.$$

Thus for $z \in C$, $\mathrm{g}_k(z)$ is dominated by

$$\sum_{r=1}^{\infty} \frac{8r}{r^k \delta_C^k} = \frac{8}{\delta_C^k} \sum_{r=1}^{\infty} \frac{1}{r^{k-1}},$$

which converges absolutely for $k \geq 4$. ∎

Corollary 1.2.6. *$\mathrm{g}_k(z)$ is holomorphic on $\mathbb{H}$.*

1.2.2 q-expansion and holomorphy at ∞

The idea is to understand series of the form

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k}.$$

Theorem 1.2.7. *A bounded holomorphic function on all of $\mathbb{C}$ is constant.*

Lemma 1.2.8.

1.
$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

2.
$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}.$$

Proof.

1. The right hand side converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$, so the right hand side is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$. Locally around $z = n$, the series looks like

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \dots + \frac{1}{(z-n+1)^2} + \frac{1}{(z-n)^2} + \frac{1}{(z-n-1)^2} + \dots = \frac{1}{(z-n)^2} + h_1(z),$$

where $h_1(z)$ is holomorphic in a neighbourhood of $z = n$. Similarly, the left hand side is meromorphic on $\mathbb{C}$, and the Laurent series near $z = n$ is

$$\frac{\pi^2}{\sin^2 \pi z} = \pi \left(\frac{1}{\pi^2 (z-n)^2} + \frac{1}{3} + \frac{1}{15} \pi^2 (z-n)^2 + \dots \right) = \frac{1}{(z-n)^2} + h_2(z),$$

where $h_2(z)$ is a holomorphic function. So the difference

$$g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} - \frac{\pi^2}{\sin^2 \pi z}$$

is meromorphic on $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \mathbb{Z}$, and the Laurent expression around $z = n$ is

$$g(z) = \frac{1}{(z-n)^2} + h_1(z) - \left(\frac{1}{(z-n)^2} + h_2(z) \right) = h_1(z) - h_2(z),$$

so $g(z)$ is holomorphic at $z = n$ for all n. Consider $t \to \pm\infty$ for $z = a + it$. The right hand side is

$$R = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=a-N}^{a+N} \frac{1}{(z-n)^2} + \sum_{n=-\infty}^{a-N-1} \frac{1}{(z-n)^2} + \sum_{n=a+N+1}^{\infty} \frac{1}{(z-n)^2} = R_0 + R_- + R_+,$$

where R_0 has finitely many terms that converge to less than $\epsilon/2$ as $t \to \pm\infty$ and $R_- + R_+ < \epsilon/2$ for $N \gg 0$ independent of t, so $R < \epsilon$ converges to zero. Similarly, the left hand side is

$$\left| \frac{\pi^2}{\sin^2 \pi z} \right| = \left| \frac{2\pi^2}{e^{\pi i z} - e^{-\pi i z}} \right| \to 0,$$

so $\lim_{t \to \infty} g(a+it) = 0$. Moreover, $g(z+1) = g(z)$ for all z. Then

$$S = \{ z \in \mathbb{C} \mid n-1 \le \operatorname{Re} z \le n, \ -N \le \operatorname{Im} z \le N \}, \qquad n \in \mathbb{Z}$$

is compact, so $|g(z)|$ attains a maximum in S, so $g(z)$ is bounded in S. Since $g(z)$ is also bounded in $R_- + R_+$, $g(z)$ is bounded in $\mathbb{C}$, so g is constant. Since $\lim_{t \to \infty} g(a+it) = 0$, $g = 0$.

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2. Check that the right hand side converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$, so the right hand side is meromorphic on $\mathbb{C} \setminus \mathbb{Z}$. Similarly, the left hand side is also meromorphic on $\mathbb{C} \setminus \mathbb{Z}$. Comparing derivatives,
$$-\frac{\pi^2}{\sin^2 \pi z} = -\frac{1}{z^2} - \sum_{n=1}^{\infty} \left(\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} \right),$$
so the difference is constant. Let $z = \frac{1}{2}$. The left hand side is $\pi \cot \pi/2 = 0$ and the right hand side is
$$\frac{2}{1} + \left(-\frac{2}{1} + \frac{2}{3} \right) + \left(-\frac{2}{3} + \frac{2}{5} \right) + \dots \to 0, \qquad n \to \infty,$$
so the difference is zero.

$\square$

Thus
$$\frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \pi \cot \pi z = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = \pi i \frac{q+1}{q-1} = \pi i - \frac{2\pi i}{1-q} = \pi i - 2\pi i \sum_{n=0}^{\infty} q^n.$$
Take $\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}$. For $k \ge 2$ even, get
$$-(k-1)! \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = -(2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} q^n,$$
so
$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n.$$
Collecting powers of q,
$$\begin{aligned} \mathrm{g}_k(z) &= \sum_{\substack{m=-\infty \\ (m,n) \ne (0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{n^k} + 2 \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{k-1} q^{nm} && \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \\ &= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n && \sigma_{k-1}(n) = \sum_{d \mid n,\ d>0} d^{k-1}. \end{aligned}$$

Corollary 1.2.9. *$\mathrm{g}_k(z)$ is holomorphic at ∞. In particular, g_k is a modular form of weight k.*

1.2.3 Bernoulli numbers

Definition 1.2.10. The **Bernoulli numbers** b_k are defined by
$$\sum_{k=0}^{\infty} b_k \frac{x^k}{k!} = \frac{x}{e^x - 1},$$
a formal power series with rational coefficients.

Then
$$b_0 = 1, \qquad b_1 = -\tfrac{1}{2}, \qquad b_2 = \tfrac{1}{6}, \qquad b_3 = 0, \qquad b_4 = -\tfrac{1}{20}, \qquad \dots, \qquad b_{2k} \in \mathbb{Q}, \qquad b_{2k+1} = 0, \qquad \dots.$$

Proposition 1.2.11. *For all even k,*

$$\zeta(k) = -b_k \frac{(2\pi i)^k}{2k!}.$$

Proof. On one hand,

$$\pi z \cot \pi z = \pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1} = \pi i z + \sum_{k=0}^{\infty} b_k \frac{(2\pi i z)^k}{k!}.$$

On the other hand,

$$\begin{aligned}\pi \cot \pi z &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \frac{1}{z} - \frac{2z}{n^2} \sum_{n=1}^{\infty} \frac{1}{1 - z^2/n^2} \\ &= \frac{1}{z} - \sum_{n=1}^{\infty} \frac{2}{z} \sum_{k=1}^{\infty} \left(\frac{z^2}{n^2}\right)^k = \frac{1}{z} - \frac{2}{z} \sum_{k=1}^{\infty} z^{2k} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{1}{z} - \frac{2}{z} \sum_{k=1}^{\infty} \zeta(2k) z^{2k},\end{aligned}$$

so

$$\pi i z + \sum_{k=0}^{\infty} b_k \frac{(2\pi i z)^k}{k!} = \pi z \cot \pi z = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k}.$$

Comparing,

$$b_{2k} \frac{(2\pi i)^{2k}}{(2k)!} = -2\zeta(2k),$$

get the desired formula. □

So

$$\mathrm{g}_k(z) = \frac{-b_k (2\pi i)^k}{k!} + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Set the **normalised Eisenstein series**

$$\mathrm{E}_k = \frac{\mathrm{g}_k}{2\zeta(k)} = 1 - \frac{2k}{b_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Example.

$$\mathrm{E}_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \qquad \mathrm{E}_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

$$\mathrm{E}_8 = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, \qquad \mathrm{E}_{12} = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n.$$

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An observation is if f is modular of weight k and g is modular of weight k', then fg is modular of weight $k + k'$, and if $k = k'$, then $f + g$ is modular of weight k.

Example. Important examples.

- The **modular discriminant**

  $$\Delta(z) = \frac{\mathrm{E}_4 - \mathrm{E}_6^2}{1728} = q - 24q^2 + 252q^3 + \dots$$

  is a modular form of weight 12.

- The j**-invariant**

  $$\mathrm{j}(z) = \frac{\mathrm{E}_4^3}{\Delta} = \frac{1}{q} + 744 + 196844q + \dots$$

  is a meromorphic modular form of weight 0.

1.3 Controlling modular forms

1.3.1 The fundamental domain

The idea is to control the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. If $f : \mathbb{H} \to \mathbb{C}$ satisfies $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and if $D \subseteq \mathbb{H}$ such that D meets every $\mathrm{SL}_2(\mathbb{Z})$-orbit in $\mathbb{H}$, then f is determined by its values on D.

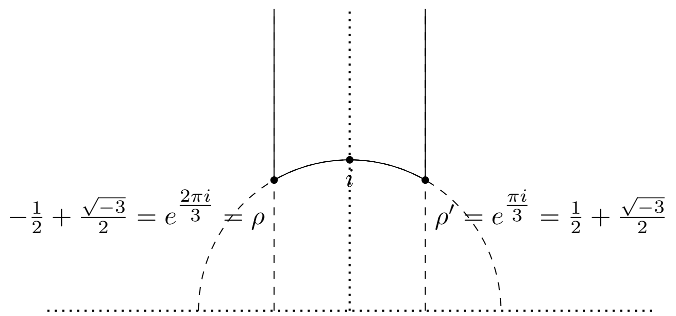
Definition 1.3.1. Let G be a group acting continuously on a complex analytic space X, such as $X = \mathbb{H}$. A subset $D \subseteq X$ is a **fundamental domain** for the action of G if

- D meets every G-orbit in X,
- the subset $\{x \in D \mid \exists g \in G,\ gx \in D,\ gx \neq x\}$ has measure zero, and
- D is closed in X.

Define

$$\mathcal{D} = \left\{ z \in \mathbb{H} \;\middle|\; \tfrac{1}{2} \le \mathrm{Re}\, z \le \tfrac{1}{2},\ |z| \ge 1 \right\} \subseteq \mathbb{H},$$

so



Let

$$\mathrm{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : z \mapsto -\frac{1}{z}, \qquad \mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : z \mapsto z+1,$$

and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be the subgroup generated by S and T. We will see later that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Theorem 1.3.2.

1. *For all $z \in \mathbb{H}$, there exists $\gamma \in \Gamma$ such that $\gamma z \in \mathcal{D}$.*
2. *Suppose $z, z' \in \mathcal{D}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma z = z'$. Then either*
   - $z = z'$,
   - $\mathrm{Re}\, z = \pm\frac{1}{2}$ *and* $z' = z \mp 1$, *or*
   - $|z| = 1$ *and* $z' = -1/z$.

   In particular, if $z \neq z'$, then z and z' are on the boundary of $\mathcal{D}$.
3. *For $z \in \mathcal{D}$, let I_z be the stabiliser of z in $\mathrm{SL}_2(\mathbb{Z})$, that is*

   $$\mathrm{I}_z = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma z = z\}.$$

   Then $\mathrm{I}_z = \{\pm \mathrm{id}\}$ unless
   - $z = i$, *where* $\mathrm{I}_z = \{\pm \mathrm{id}, \pm \mathrm{S}\}$,
   - $z = \rho$, *where* $\mathrm{I}_z = \left\{\pm \mathrm{id}, \pm(\mathrm{ST}), \pm\left(\mathrm{T}^{-1}\mathrm{S}\right)\right\}$, *or*
   - $z = \rho'$, *where* $\mathrm{I}_z = \left\{\pm \mathrm{id}, \pm(\mathrm{TS}), \pm\left(\mathrm{ST}^{-1}\right)\right\}$.

Corollary 1.3.3. $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Proof. Fix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathring{\mathcal{D}}$ so $\mathrm{SL}_2(\mathbb{Z})\, z \cap \mathcal{D} = \{z\}$ and $\mathrm{I}_z = \{\pm \mathrm{id}\}$. Consider γz. There exists $\gamma' \in \Gamma$ such that $\gamma' \gamma z \in \mathcal{D}$, so $\gamma' \gamma z = z$. So $\gamma' \gamma = \pm \mathrm{id}$, so $\gamma = \pm \gamma'^{-1}$. But $\gamma'^{-1} \in \Gamma$ and $-\mathrm{id} = \mathrm{S}^2 \in \Gamma$, so $\gamma \in \Gamma$. ∎

Proof of Theorem 1.3.2. Recall $\mathrm{Im}\,\gamma z = \mathrm{Im}\,z/|cz+d|^2$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

1. As c and d vary, $\{cz+d\}$ forms a lattice in $\mathbb{C}$, so there exist only finitely many c and d such that $|cz+d| < 1$. So $\mathrm{Im}\,\gamma z$ attains a maximum as γ varies over Γ, so there exists $\gamma \in \Gamma$ such that $\mathrm{Im}\,\gamma z$ is maximal. There exists $n \in \mathbb{Z}$ such that $\mathrm{T}^n\gamma z$ has real part between $-\frac{1}{2}$ and $\frac{1}{2}$. Consider $|\mathrm{T}^n\gamma z|$. If this is less than one, then
$$\mathrm{Im}\,\mathrm{ST}^n\gamma z = \mathrm{Im}\,\frac{-1}{\mathrm{T}^n\gamma z} > \mathrm{Im}\,\mathrm{T}^n\gamma z = \mathrm{Im}\,\gamma z.$$
Since $\mathrm{ST}^n\gamma \in \Gamma$, this contradicts maximality so $|\mathrm{T}^n\gamma z| \ge 1$, so $\mathrm{T}^n\gamma z \in \mathcal{D}$. Lecture 7 Friday 18/10/19

2, 3. Let $z, z' \in \mathcal{D}$ such that $\gamma z = z'$. Without loss of generality $\mathrm{Im}\,z' \ge \mathrm{Im}\,z$, so $|cz+d| \le 1$. Note that $|cz+d| \ge \mathrm{Im}\,(cz+d) \ge \frac{\sqrt{3}}{2}c$, so $c = -1, 0, 1$. Note that can replace γ with $-\gamma$ if convenient.

$c = 0$. Then $ad = 1$, so can assume $a = d = 1$, so $\gamma z = z + b$. Since $z, z+b \in \mathcal{D}$, $b = \pm 1$ and $\mathrm{Re}\,z = \mp\frac{1}{2}$.

$c = 1$. Have $|z+d| \le 1$ and $|z| \ge 1$, so $d = -1, 0, 1$.

- $d = 0$. Then $|z| = 1$, and $\gamma z = (az-1)/z = a - 1/z$. The only possibilities are
  - $a = 0$ and $\gamma = \mathrm{S}$,
  - $a = 1$ and $\gamma = \mathrm{TS}$, so $z = \rho'$, or
  - $a = -1$ and $\gamma = \mathrm{T}^{-1}\mathrm{S}$, so $z = \rho$.
- $d = 1$. Then $z = \rho$, and $\gamma z = ((b+1)z+b)/(z+1) = b + 1 - 1/(z+1)$, so $b = 0$ or $b = -1$.
- $d = -1$. Then $z = \rho'$ is similar.

$c = -1$. Similar.

□

1.3.2 Further review of complex analysis

Recall that on any compact set, a meromorphic function has only finitely many zeroes and poles. If $f(z) = g\left(e^{2\pi iz}\right)$ is meromorphic at infinity and g is meromorphic on $\mathrm{D} = |q| < 1$, zeroes and poles of g are discrete with respect to q, and $\mathrm{Im}\,z \gg 0$ if and only if $|q| < \epsilon$.

Definition 1.3.4. Let $U \subseteq \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$ be meromorphic on U. If f has a pole at p, can write
$$f(z) = \sum_{n=\mathrm{ord}_p f<0}^{\infty} a_n (z-p)^n.$$
The coefficient a_{-1} is called the **residue** $\mathrm{Res}_p\, f$ of f at p.

Theorem 1.3.5 (Residue theorem)**.** *Let V be a region in $\mathbb{C}$ whose boundary ∂V is a simple closed curve. Then*
$$\frac{1}{2\pi}\int_{\partial V} f(z)\,\mathrm{d}z = \sum_{p\in V\ \text{pole of}\ f} \mathrm{Res}_p\, f.$$

Definition 1.3.6. Let f be meromorphic on $U \subseteq \mathbb{C}$ open. Then the **logarithmic derivative** $\mathrm{d}\log f$ is the function f'/f.

If $f(z) = c_n(z-p)^n + c_{n+1}(z-p)^{n+1} + \dots$, then if $n \ne 0$, then the leading term of f' is $nc_n(z-p)^{n-1}$ and the leading term of f is $c_n(z-p)^n$, so the leading term of f'/f is $n(z-p)^{-1}$. If $n = 0$, then f'/f is holomorphic. So f'/f is meromorphic with simple poles precisely at the points where $\mathrm{ord}_p\, f \ne 0$, and $\mathrm{Res}_p\, f'/f$ at such p is $\mathrm{ord}_p\, f$.

Theorem 1.3.7 (Argument principle)**.**
$$\frac{1}{2\pi i}\int_{\partial V}\frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{p\in V}\mathrm{ord}_p\, f.$$

1.3.3 Controlling modular forms

Theorem 1.3.8. *Let f be a non-zero meromorphic modular form of weight k. Then*

$$\operatorname{ord}_\infty f + \frac{\operatorname{ord}_\rho f}{3} + \frac{\operatorname{ord}_i f}{2} + \sum_{p \in \mathbb{H}/\operatorname{SL}_2(\mathbb{Z}),\ p \nsim \{i,\rho\}} \operatorname{ord}_p f = \frac{k}{12}.$$

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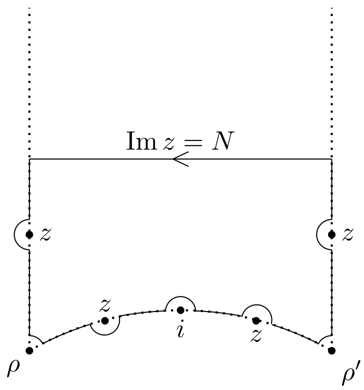
Proof. Consider the closed curve $C_{N,\epsilon}$,



,

where the z's are zeroes or poles of f, and the circles are of radius ϵ. Consider

$$\frac{1}{2\pi i} \int_{C_{N,\epsilon}} \frac{f'(z)}{f(z)}\, \mathrm{d}z = \sum_{p \in \mathbb{H}/\operatorname{SL}_2(\mathbb{Z}),\ p \nsim \{i,\rho\}} \operatorname{ord}_p f, \qquad \epsilon \to 0.$$

So it suffices to show

$$\lim_{\epsilon \to 0,\ N \to \infty} \frac{1}{2\pi i} \int_{C_{N,\epsilon}} \frac{f'(z)}{f(z)}\, \mathrm{d}z = -\operatorname{ord}_\infty f - \frac{\operatorname{ord}_\rho f}{3} - \frac{\operatorname{ord}_i f}{2} + \frac{k}{12}.$$

The vertical parts of the boundary cancel. The integral over the circular part of $\partial\mathcal{D}$ approaches

$$\frac{1}{2\pi i} \int_\rho^i \frac{f'(z)}{f(z)}\, \mathrm{d}z + \frac{1}{2\pi i} \int_i^{\rho'} \frac{f'(z)}{f(z)}\, \mathrm{d}z = \frac{1}{2\pi i} \left(\int_\rho^i \frac{f'(z)}{f(z)}\, \mathrm{d}z - \int_\rho^i \frac{f'(-1/z)}{f(-1/z)}\, \mathrm{d}z \right)$$

Since $f(-1/z) = z^k f(z)$,

$$\mathrm{d}\left(z^k f(z)\right) = \left(k z^{k-1} f(z) + z^k f'(z)\right) \mathrm{d}z,$$

so

$$\frac{1}{2\pi i} \int_\rho^i \frac{f'(z)}{f(z)}\, \mathrm{d}z + \frac{1}{2\pi i} \int_i^{\rho'} \frac{f'(z)}{f(z)}\, \mathrm{d}z = \frac{1}{2\pi i} \int_\rho^i \frac{f'(z)}{f(z)} - \frac{k z^{k-1} f(z) + z^k f'(z)}{z^k f(z)}\, \mathrm{d}z = -\frac{1}{2\pi i} \int_\rho^i \frac{k}{z}\, \mathrm{d}z = \frac{k}{12}.$$

Since $\mathrm{d}q = 2\pi i q \mathrm{d}z$, the top part is

$$\frac{1}{2\pi i} \int_{\frac{1}{2}+iN}^{\frac{1}{2}-iN} \frac{f'(z)}{f(z)}\, \mathrm{d}z = -\frac{1}{2\pi i} \int_{\text{circle of radius } \epsilon} \frac{g'(q)}{g(q)}\, \mathrm{d}q = -\operatorname{ord}_\infty f.$$

Near i, $f'/f = \operatorname{ord}_i f\,(z-i)^{-1} + h(z)$, where $h(z)$ is holomorphic and $h(z) \to 0$ as $\epsilon \to 0$. Then the circle $C_{\epsilon,i}$ of radius ϵ centered at i is

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C_{\epsilon,i}} \frac{f'(z)}{f(z)}\, \mathrm{d}z = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\text{arc of half circle centered at } i} \frac{\operatorname{ord}_i f}{z-i}\, \mathrm{d}z = -\frac{\operatorname{ord}_i f}{2}.$$

Similarly, at ρ and ρ', get that the circles $C_{\epsilon,\rho}$ and $C_{\epsilon,\rho'}$ of radius ϵ centered at ρ and ρ' are

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C_{\epsilon,\rho}} \frac{f'(z)}{f(z)}\, \mathrm{d}z = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C_{\epsilon,\rho'}} \frac{f'(z)}{f(z)}\, \mathrm{d}z = -\frac{\operatorname{ord}_\rho f}{6},$$

which gives $-\operatorname{ord}_\rho f/3$. □

1.3.4 Holomorphic modular forms

Let
$$\mathrm{M}_k = \{\text{holomorphic modular forms of weight } k\},$$
and let
$$\mathrm{S}_k = \{\text{cusp forms of weight } k\} = \{f \in \mathrm{M}_k \mid \operatorname{ord}_\infty f > 0\} \subseteq \mathrm{M}_k.$$

Corollary 1.3.9.

- $\mathrm{M}_k = 0$ *if* $k < 0$, $k = 2$, *or* k *odd.*
- M_0 *are constants.*
- $\mathrm{M}_4 = \mathbb{C}\mathrm{E}_4$, *where* $\operatorname{ord}_\rho \mathrm{E}_4 = 1$ *and no other zeroes.*
- $\mathrm{M}_6 = \mathbb{C}\mathrm{E}_6$, *where* $\operatorname{ord}_i \mathrm{E}_6 = 1$ *and no other zeroes.*
- $\mathrm{M}_8 = \mathbb{C}\mathrm{E}_8$, *where* $\operatorname{ord}_\rho \mathrm{E}_8 = 2$ *and no other zeroes.*
- $\mathrm{M}_{10} = \mathbb{C}\mathrm{E}_{10}$, *where* $\operatorname{ord}_\rho \mathrm{E}_{10} = \operatorname{ord}_i \mathrm{E}_{10} = 1$ *and no other zeroes.*
- $\mathrm{M}_{12} = \mathbb{C}\mathrm{E}_{12} \oplus \mathbb{C}\Delta$, *where* $\operatorname{ord}_\infty \Delta = 1$ *and no other zeroes.*

Corollary 1.3.10. $\Delta : \mathrm{M}_k \to \mathrm{S}_{k+12}$ *is an isomorphism. On the other hand,*
$$\mathrm{M}_k \cong \mathbb{C}\mathrm{E}_k \oplus \mathrm{S}_k, \qquad k \geq 4 \text{ even},$$
so
$$\mathrm{M}_k \cong \mathbb{C}\mathrm{E}_k \oplus \cdots \oplus \mathbb{C}\mathrm{E}_{k-12r}\Delta^r, \qquad k - 12r \in \{0, 4, 6, 8, 10, 14\}.$$
So for $k \geq 4$, *the set*
$$\begin{cases} \mathrm{E}_k, \dots, \mathrm{E}_{k-12\lfloor k/12 \rfloor}\Delta^{\lfloor k/12 \rfloor} & k \not\equiv 2 \mod 12 \\ \mathrm{E}_k, \dots, \mathrm{E}_{14}\Delta^{\lfloor k/12 \rfloor - 1} & k \equiv 2 \mod 12 \end{cases}$$
is a basis for M_k.

Corollary 1.3.11. $\mathrm{E}_4^2 = \mathrm{E}_8$ *and* $\mathrm{E}_4\mathrm{E}_6 = \mathrm{E}_{10}$.

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A variant is to write $k = 4n + 6m$ with $m = 0, 1$ and $n \geq 0$, for $k \geq 4$. Then $\mathrm{M}_k = \mathbb{C}\mathrm{E}_4^n\mathrm{E}_6^m \oplus \mathrm{S}_k$ gives a basis
$$\mathrm{E}_4^n\mathrm{E}_6^m, \dots, \mathrm{E}_4^{n-3\lfloor n/3 \rfloor}\mathrm{E}_6^m\Delta^{\lfloor n/3 \rfloor}$$
for M_k. Since $\Delta = \left(\mathrm{E}_4^3 - \mathrm{E}_6^2\right)/1728$, we see every modular form of weight k is a polynomial in E_4 and E_6, and
$$\Delta \in q + q^2\mathbb{Z}[[q]], \qquad \mathrm{E}_4^n\mathrm{E}_6^m \in 1 + q\mathbb{Z}[[q]], \qquad \mathrm{E}_4^{n-3}\mathrm{E}_6^m\Delta \in q + q^2\mathbb{Z}[[q]], \qquad \dots$$
have integer coefficients. The upshot is if the q-expansion of f has integer coefficients, then f is an integer combination of
$$\mathrm{E}_4^n\mathrm{E}_6^m, \dots, \mathrm{E}_4^{n-3\lfloor n/3 \rfloor}\mathrm{E}_6^m\Delta^{\lfloor n/3 \rfloor}.$$

Notation. $\mathrm{M}_k(\mathbb{Z}) \subseteq \mathrm{M}_k$ consists of modular forms with integer q-expansions.

Theorem 1.3.12. $\mathrm{M}_k(\mathbb{Z})$ *spans* M_k, *and* $f \in \mathrm{M}_k$ *lies in* $\mathrm{M}_k(\mathbb{Z})$ *if and only if* f *is an integral polynomial in* $\mathrm{E}_4, \mathrm{E}_6, \Delta$.

Definition 1.3.13. A **graded ring** is a ring R, together with a direct sum decomposition, as abelian groups,
$$R = \bigoplus_{i \in \mathbb{Z}} R_i,$$
such that $R_i \cdot R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}$.

Example.

- $R = \mathbb{C}[X, Y]$, where R_i are polynomials homogeneous of degree i.
- $R = \bigoplus_{k \in \mathbb{Z}} \mathrm{M}_k$.

Let $\mathbb{C}[X,Y]$ be graded with $\deg X = 4$ and $\deg Y = 6$. Have a homomorphism of graded rings

$$\begin{array}{ccc} \mathbb{C}[X,Y] & \longrightarrow & \bigoplus_{k\in\mathbb{Z}} \mathrm{M}_k \\ (X,Y) & \longmapsto & (\mathrm{E}_4, \mathrm{E}_6) \end{array}.$$

Theorem 1.3.14. *This is an isomorphism of graded rings.*

Proof. This map is surjective, since every $f \in \mathrm{M}_k$ is a polynomial in E_4 and E_6. Remains to show this map is injective. Suppose not. There exists $P(X,Y)$, homogeneous of degree k, such that $P(\mathrm{E}_4,\mathrm{E}_6) = 0$. Write $k = 4n + 6m$ with $m = 0, 1$. If $P = c_0 X^n Y^n + \dots + c_r X^{n-3r} Y^{m+2r}$ where $r = \lfloor n/3 \rfloor$, then

$$c_0 \mathrm{E}_4^n \mathrm{E}_6^n + \dots + c_r \mathrm{E}_4^{n-3r} \mathrm{E}_6^{m+2r} = 0.$$

Dividing by $\mathrm{E}_4^{n-3r}\mathrm{E}_6^{m+2r}$, get $Q\left(\mathrm{E}_4^3/\mathrm{E}_6^2\right) = 0$ where $Q(X) = c_0 X^r + \dots + c_r$. Since the roots of Q are discrete, and $\mathrm{E}_4^3/\mathrm{E}_6^2$ is non-constant, this is impossible. $\square$

1.3.5 Meromorphic modular forms

Note. The meromorphic modular forms of weight zero form a field. For example, $\mathrm{j}(z) = \mathrm{E}_4^3/\Delta = 1728\mathrm{E}_4^3/\left(\mathrm{E}_4^3 - \mathrm{E}_6^2\right)$ is a non-constant meromorphic modular form, with a pole of order one at infinity, a zero of order three at ρ, and no other zeroes or poles.

Theorem 1.3.15. j *gives a bijection between* $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ *and* $\mathbb{C}$.

Proof. Given $\lambda \in \mathbb{C}$, want $z \in \mathbb{H}$ such that $\mathrm{j}(z) = \lambda$. Consider $g = j - \lambda$. This is meromorphic of weight zero. There is a pole at infinity, and no other poles, and

$$\mathrm{ord}_\infty g + \frac{\mathrm{ord}_\rho g}{3} + \frac{\mathrm{ord}_i g}{2} + \sum_{p \in \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}),\ p \nsim \{i,\rho\}} \mathrm{ord}_p g = 0.$$

The only possibilities are

- g has a zero at ρ of order three, and no other zeroes,
- g has a zero at i of order two, and no other zeroes, or
- g has a simple zero somewhere else, and no others.

In each case, the zero of g is a unique $\mathrm{SL}_2(\mathbb{Z})$-orbit on which $\mathrm{j}(z) = \lambda$. So j is bijective. $\square$