

# M4P57 Complex Manifolds

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Spring 2020

**Syllabus**

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# 1 Introduction

Lecture 1  
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The following are references.

- O Biquard and A Höring, Kähler geometry and Hodge theory, 2008.
- J P Demailly, Complex analytic and differential geometry, 2012.
- C Voisin, Hodge theory and complex algebraic geometry, 2002.
- R O Wells, Differential analysis on complex manifolds, 1973.
- A Gathmann, Algebraic geometry, 2002
- P Griffiths and J Harris, Principles of algebraic geometry, 1978.

Complex manifolds are manifolds over  $\mathbb{C}^n$ .

**Example 1.1.**  $\mathbb{C}^1$  is a complex manifold. Any open  $U \subset \mathbb{C}^n$  is a complex manifold.

**Example 1.2.** The sphere  $S^2 \subset \mathbb{R}^3$  is a complex manifold by

$$S^2 \cong \mathbb{C} \cup \{\infty\} = \mathbb{P}_{\mathbb{C}}^1 = \mathbb{CP}^1.$$

More in general  $\mathbb{P}_{\mathbb{C}}^n$  is a complex manifold for all  $n$ .

**Example 1.3.** The torus

$$S^1 \times S^1 = \mathbb{R}^2 / \mathbb{Z}^2 = \mathbb{C} / \mathbb{Z}^2$$

is a complex manifold. More in general a  $2n$ -dimensional torus  $\mathbb{C}^n / \Lambda$  for a lattice  $\Lambda \cong \mathbb{Z}^{2n}$  is a complex manifold.

**Example 1.4.** Compact Riemannian surfaces of genus  $g > 1$ , called **hyperbolics**, are all complex manifolds.

**Example 1.5.** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic. The graph of  $f$ ,

$$\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{C}\} \subset \mathbb{C} \times \mathbb{C},$$

is a complex manifold. From  $\Gamma_f$  we can recover  $f$ , by

$$f(x) = q(p^{-1}(x) \cap \Gamma_f),$$

where  $p, q : \mathbb{C}^2 \rightarrow \mathbb{C}$  are the projections to the first and second factors. This allows us to define  $f^{-1}$ . Assume  $f$  is bijective. Define

$$\tau : \begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ (x, y) & \longmapsto & (y, x) \end{array}.$$

Define

$$\Gamma_{f^{-1}} = \tau(\Gamma_f).$$

Then  $f^{-1}$  is the function induced by  $\Gamma_{f^{-1}}$ . This makes sense even if  $f$  is not bijective. Then we get a multivalued function, such as  $\log z$  as the inverse of  $\exp z$ .

**Example 1.6.** Generalising Example 1.5, we can consider two complex manifolds  $M$  and  $N$  and we can consider holomorphisms  $f : M \rightarrow N$ . Given  $M$ ,

$$\text{Aut } M = \{f : M \rightarrow M \text{ holomorphic bijective and } f^{-1} \text{ holomorphic}\}.$$

If  $M = \mathbb{C}$ , there are lots of  $C^\infty$ -functions  $\mathbb{C} \rightarrow \mathbb{C}$  but the automorphisms of  $\mathbb{C}$  are just affine linear maps. If  $M = \mathbb{C}/\mathbb{Z}^2$ , then  $\text{Aut } M$  is interesting.

**Example 1.7.** Algebraic geometry is the zeroes of polynomials. That is, fix  $m$ , and take polynomials  $f_1, \dots, f_k$  in  $m$  variables. Define

$$M = \{(x_1, \dots, x_m) \in \mathbb{C}^m \mid f_1(x_1, \dots, x_m) = \dots = f_k(x_1, \dots, x_m) = 0\}.$$

Then  $M$  is called an **algebraic variety**. If  $M$  is smooth then  $M$  is a complex manifold. Fix  $m$ , take homogeneous polynomials  $F_1, \dots, F_k$  in  $m + 1$  variables, where  $F$  is **homogeneous** if it is the sum of monomials of the same degree. Consider

$$N = \{(x_0, \dots, x_m) \in \mathbb{P}_{\mathbb{C}}^m \mid F_1(x_0, \dots, x_m) = \dots = F_k(x_0, \dots, x_m) = 0\}.$$

Then  $N$  is called a **projective variety**. If  $N$  is smooth then  $N$  is a complex manifold.

The idea is if  $M$  is a differentiable manifold, then  $M$  contains lots of submanifolds  $N$ . This is not true for complex manifolds. There exist complex manifolds without any proper complex submanifolds, which is not true for projective varieties. The following are questions.

- What can we say about the topology of complex manifolds? For example, what is  $\pi_1(M)$ ? What is the cohomology of  $M$ ?
- Assume that  $M$  and  $N$  are complex manifolds which are diffeomorphic. Are they also isomorphic, so there exists a biholomorphism  $M \rightarrow N$ ?

What is next?

- Hodge decomposition theorem. Understand the cohomology of  $M$  by using the complex structure.
- Kodaira embedding theorem. Understand when a compact complex manifold is projective.

**Note.** If  $M \subset \mathbb{P}_{\mathbb{C}}^m$  is a compact complex manifold then  $M$  is projective.

**Example.** Let  $M = \Gamma_{\exp}$  for  $\exp : \mathbb{C} \rightarrow \mathbb{C}$ . Then  $M \subset \mathbb{C}^2$  but it is not algebraic.

## 2 Local theory

### 2.1 Holomorphic functions in several variables

**Notation 2.1.** Given  $z_0 \in \mathbb{C}$  and  $r > 0$ , the **disc** is

$$D(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$

and  $\partial D(z_0, r)$  is the boundary of  $D(z_0, r)$ .

**Definition 2.2.** Let  $U \subset \mathbb{C}$ , and let  $f : U \rightarrow \mathbb{C}$  be a function. Then  $f$  is **holomorphic at**  $z_0 \in U$  if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

**Theorem 2.3** (Cauchy). *Let  $U \subset \mathbb{C}$  be open, let  $f$  be holomorphic on  $U$ , and let  $z_0 \in U$ . Assume that if  $D = D(z_0, r) \subset U$  then  $\overline{D} \subset U$ . Then*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz.$$

**Notation 2.4.** Fix  $z_0 = (z_{01}, \dots, z_{0n}) \in \mathbb{C}^n$  and  $R = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$ . Then the **polydisc** is

$$D(z_0, R) = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i - z_{0i}| < r_i \text{ for each } i\},$$

where  $R$  is the **polyradius**.

**Definition 2.5.** Let  $U \subset \mathbb{C}^n$  be open, let  $f : U \rightarrow \mathbb{C}$  be a continuous function, and let  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ . Then  $f$  is **holomorphic at**  $z$ , if assuming that  $D = D(z, R) \subset U$  for some  $R = (r_1, \dots, r_n)$  then

$$f(z_1, \dots, z_{i-1}, \cdot, z_{i+1}, \dots, z_n) : D(z_i, r_i) \rightarrow \mathbb{C}$$

is holomorphic for all  $i$ .

**Example 2.6.** Any convergent power series in  $n$ -variables is holomorphic.

The opposite is also true.

**Theorem 2.7** (Cauchy). *Let  $U \subset \mathbb{C}^n$  be an open set, let  $f : U \rightarrow \mathbb{C}$  be holomorphic, and let  $z = (z_1, \dots, z_n) \in U$ . Assume that if  $D = D(z, R)$  for some  $R = (r_1, \dots, r_n)$  then  $\overline{D} \subset U$ . If  $z' = (z'_1, \dots, z'_n) \in D$  then*

$$f(z') = \frac{1}{(2\pi i)^n} \int_{\partial D(z_1, r_1)} \cdots \int_{\partial D(z_n, r_n)} \frac{f(z)}{(z - z'_1) \cdots (z - z'_n)} dz_n \cdots dz_1.$$

*Proof.* Use induction on  $n$  and Cauchy theorem at each step. □

**Corollary 2.8.** *Let  $U \subset \mathbb{C}^n$  be open, let  $f : U \rightarrow \mathbb{C}$  be holomorphic, and let  $z = (z_1, \dots, z_n) \in U$ . Then there exists  $D = D(z, R) \subset U$  for some  $R = (r_1, \dots, r_n)$  and there exists*

$$p(w) = \sum_{m_1, \dots, m_n \geq 0} a_{m_1, \dots, m_n} (w_1 - z_1)^{m_1} \cdots (w_n - z_n)^{m_n},$$

such that  $p$  is convergent on  $D$  and  $f(w) = p(w)$  inside  $D$ .

*Proof.* The idea is to use Theorem 2.7 and  $1/(1-w) = \sum_{k \geq 0} w^k$ . □

**Definition 2.9.** Let  $U \subset \mathbb{C}^n$  be open. Then  $f : U \rightarrow \mathbb{C}^m$  is **holomorphic** if  $f_i = p_i \circ f$  is holomorphic for any  $i = 1, \dots, m$  where  $p_i : \mathbb{C}^m \rightarrow \mathbb{C}$  is the  $i$ -th projection, so  $f = (f_1, \dots, f_m)$ .

**Fact.** If  $f : U \rightarrow \mathbb{C}^m$  is holomorphic and  $g : V \rightarrow \mathbb{C}^p$  is holomorphic where  $V \supset f(U)$  then  $g \circ f$  is holomorphic.

**Definition 2.10.** Let  $U \subset \mathbb{C}^n$  be open. A holomorphic function  $f : U \rightarrow \mathbb{C}^m$  is **biholomorphic at**  $z_0 \in U$  if there exists an open neighbourhood  $V \subset U$  of  $z_0$  such that  $f : V \rightarrow f(V)$  is bijective and  $f^{-1} : f(V) \rightarrow V$  is holomorphic. Then  $f$  is **biholomorphic** if  $f$  is bijective and  $f$  is biholomorphic at any point.

**Note.**  $f(V)$  is automatically open in  $\mathbb{C}^m$  if  $m = n$ .

**Example 2.11.** Let  $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be linear such that  $\det \Phi \neq 0$ . Then  $\Phi$  is a biholomorphism.

**Example 2.12.** Let  $U = \mathbb{C} \setminus \{0\}$  and

$$\begin{array}{ccc} f & : & U \longrightarrow U \\ z & \longmapsto & z^2 \end{array}.$$

Check that  $f$  is biholomorphic at any point of  $U$  but  $f$  is not biholomorphic.

**Remark.**  $\mathbb{C}^n \cong \mathbb{R}^{2n}$  and  $\mathbb{C}^m \cong \mathbb{R}^{2m}$ . Then a holomorphic  $f : U \subset \mathbb{C}^n \rightarrow \mathbb{C}^m$  is also a diffeomorphism  $U \subset \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2m}$ .

**Theorem 2.13** (Hartogs). *Let  $n \geq 2$ , let  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  and  $\delta = (\delta_1, \dots, \delta_n)$  such that  $\epsilon_i > \delta_i > 0$ , let  $U = D(0, \epsilon) \setminus \overline{D(0, \delta)}$ , and let  $f : U \rightarrow \mathbb{C}^m$  be holomorphic. Then there exists a holomorphic  $\bar{f} : D(0, \epsilon) \rightarrow \mathbb{C}^m$  such that  $\bar{f}|_U = f$ .*

**Example.** Hartogs theorem is false for  $n = 1$ . If  $f(z) = 1/z$ , for all  $\epsilon > \delta > 0$ , then  $f$  cannot be extended.

## 2.2 Cauchy formula in one variable

Let  $\omega = x + iy \in \mathbb{C}$  for  $x, y \in \mathbb{R}$ , and let  $f : U \rightarrow \mathbb{C}$  be  $C^\infty$  for some  $U \subset \mathbb{C}$ . Recall that

$$\frac{\partial f}{\partial \omega} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f, \quad \frac{\partial f}{\partial \bar{\omega}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f.$$

Recall that  $f$  is holomorphic if and only if  $\frac{\partial f}{\partial \bar{\omega}} = 0$  on  $U$ . More in general, let  $U \subset \mathbb{C}^n$  be open, let  $z_i = x_i + iy_i$ , and let  $f : U \rightarrow \mathbb{C}$  be a  $C^\infty$ -function. Then  $f$  is holomorphic if and only if  $\frac{\partial f}{\partial \bar{z}_i} = 0$  for all  $i = 1, \dots, n$ . Let  $\omega \in \mathbb{C}$ . Since  $dx \wedge dy = -dy \wedge dx$ , let

$$dA = \frac{i}{2} d\omega \wedge d\bar{\omega} = \frac{i}{2} (dx + idy) \wedge (dx - idy) = dx \wedge dy,$$

which is the Lebesgue measure on  $\mathbb{R}^2 \cong \mathbb{C}$ .

**Proposition 2.14.** *Let  $f : U \rightarrow \mathbb{C}$  for  $U \subset \mathbb{C}$  be a  $C^\infty$ -function, and let  $D = D(z, r)$  such that  $\bar{D} \subset U$ . Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f}{\omega - z} d\omega - \frac{1}{\pi} \int_D \frac{1}{\omega - z} \frac{\partial f}{\partial \bar{\omega}} dA.$$

*Proof.* Assume  $z = 0$ . Recall that  $f(\omega) = 1/\omega$  is locally integrable around zero, so

$$\int_D \frac{1}{\omega} \frac{\partial f}{\partial \bar{\omega}} dA = \lim_{\epsilon \rightarrow 0} \int_{D \setminus D(0, \epsilon)} \frac{1}{\omega} \frac{\partial f}{\partial \bar{\omega}} dA.$$

Away from zero

$$\begin{aligned} d\left(\frac{f}{\omega} d\omega\right) &= \frac{1}{\omega} df \wedge d\omega + f d\left(\frac{1}{\omega}\right) \wedge d\omega = \frac{1}{\omega} \left( \frac{\partial f}{\partial \omega} d\omega + \frac{\partial f}{\partial \bar{\omega}} d\bar{\omega} \right) \wedge d\omega + f \frac{\partial}{\partial \omega} \left( \frac{1}{\omega} \right) d\omega \wedge d\omega \\ &= \frac{1}{\omega} \frac{\partial f}{\partial \bar{\omega}} d\bar{\omega} \wedge d\omega = \frac{2i}{\omega} \frac{\partial f}{\partial \bar{\omega}} dA. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{\pi} \int_D \frac{1}{\omega} \frac{\partial f}{\partial \bar{\omega}} dA &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{D \setminus D(0, \epsilon)} \frac{1}{\omega} \frac{\partial f}{\partial \bar{\omega}} dA \\ &= \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \int_{D \setminus D(0, \epsilon)} d\left(\frac{f}{\omega} d\omega\right) & \frac{1}{\omega} \frac{\partial f}{\partial \bar{\omega}} dA &= \frac{1}{2i} d\left(\frac{f}{\omega} d\omega\right) \\ &= \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \left( \int_{\partial D} \frac{f}{\omega} d\omega - \int_{\partial D(0, \epsilon)} \frac{f}{\omega} d\omega \right) & \text{Stokes' theorem} \\ &= \frac{1}{2\pi i} \left( \int_{\partial D} \frac{f}{\omega} d\omega - 2\pi i f(0) \right) & \lim_{\epsilon \rightarrow 0} \int_{\partial D(0, \epsilon)} \frac{1}{\omega} d\omega &= 2\pi i. \end{aligned}$$

□

If  $f$  is holomorphic, then  $\frac{\partial f}{\partial \bar{\omega}} = 0$ , which implies Theorem 2.3.

### 2.3 Rank theorem

Let  $U \subset \mathbb{C}^n$  be open, and let  $f : U \rightarrow \mathbb{C}^m$  be holomorphic. Then the **Jacobian** is

$$J_f = \left( \frac{\partial f_j}{\partial z_i}(z) \right),$$

where  $f_j = p_j \circ f$  and  $p_j : \mathbb{C}^m \rightarrow \mathbb{C}$  is the  $j$ -th projection.

**Exercise.** Show that the real Jacobian, which is  $2n \times 2n$ , has non-negative determinants.

**Theorem 2.15** (Rank theorem). *Let  $z \in U$  such that  $r = \text{rk } J_f(z')$  is constant around  $z$ . Then there exist open  $z \in V \subset U \subset \mathbb{C}^n$  and  $f(z) \in W \subset f(U) \subset \mathbb{C}^m$  such that  $\phi : D(0, 1)^n \rightarrow V$  and  $\psi : D(0, 1)^m \rightarrow W$  are biholomorphisms such that*

$$\eta = \psi^{-1} \circ f \circ \phi : \begin{array}{ccc} D(0, 1)^n & \longrightarrow & D(0, 1)^m \\ (z_1, \dots, z_n) & \longmapsto & (z_1, \dots, z_r, 0, \dots, 0) \end{array},$$

so

$$\begin{array}{ccccc} \mathbb{C}^n \supset U & \supset & V \ni z & \xrightarrow{f} & f(z) \in W \subset f(U) \subset \mathbb{C}^m \\ & & \uparrow \phi & & \uparrow \psi \\ & & D(0, 1)^n & \xrightarrow{\eta} & D(0, 1)^m \end{array}.$$

**Theorem 2.16** (Inverse function theorem). *Let  $f : U \rightarrow \mathbb{C}^n$  be holomorphic for  $U \subset \mathbb{C}^n$ , and let  $z \in U$  such that  $\det J_f(z) \neq 0$ . Then  $f$  is a biholomorphism at  $z$ .*

*Proof.*  $\det J_f(z) \neq 0$  if and only if  $\text{rk } J_f(z) = n$ , so  $\text{rk } J_f(z') = n$  around  $z$ , since  $\det J_f(z)$  is a continuous function. Let  $\phi$  and  $\psi$  as in the theorem. Then  $\eta = \psi^{-1} \circ f \circ \phi = \text{id}$ , so on  $V$ ,  $f = \psi \circ \phi^{-1}$  is a composition of biholomorphisms, which is a biholomorphism.  $\square$

**Remark 2.17.** Let  $f : U \rightarrow \mathbb{C}^n$  for  $U \subset \mathbb{C}^n$ . Then  $\det J_f(z)$  is a holomorphism, so

$$Z = \{z \in U \mid \det J_f(z) = 0\}$$

is closed.

### 2.4 Holomorphic differential forms

Let  $U \subset \mathbb{C}^n$  be open.

**Definition 2.18.** A **holomorphic vector field** on  $U$  is the expression

$$X = \sum_i a_i \frac{\partial}{\partial z_i},$$

where  $a_i$  are holomorphic functions on  $U$ .

For all  $x \in U$ , the **tangent space** is

$$T_x U = \left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle \cong \mathbb{C}^n.$$

If  $x \in U$ , then  $X(x) \in T_x U$ .

**Notation 2.19.**

$$H^0(U, \mathcal{O}_U) = \{\text{holomorphic functions } f : U \rightarrow \mathbb{C}\}, \quad H^0(U, T_U) = \{\text{holomorphic vector fields on } U\}.$$

**Remark.**  $R = H^0(U, \mathcal{O}_U)$  is a ring and  $M = H^0(U, T_U)$  is a module over  $R$ . That is, if  $X \in H^0(U, T_U)$  and  $f \in H^0(U, \mathcal{O}_U)$ , then  $fX \in H^0(U, T_U)$ .

**Definition 2.20.** Let  $R$  be a ring and  $M$  be an  $R$ -module for  $p \geq 1$ . The  $p$ -th exterior power  $\Lambda^p M$  of  $M$  is the  $R$ -module  $M^{\otimes p}$  with the relations

$$m_1 \otimes \cdots \otimes m_p - \epsilon(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(p)}, \quad m_1, \dots, m_p \in M, \quad \sigma \in \mathcal{S}_p,$$

where  $\epsilon(\sigma) = (-1)^m$  is the signature of  $\sigma$  and  $m$  is the number of transpositions defining  $\sigma$ . Then  $M^* = \text{Hom}_R(M, R)$  is the **dual** of  $M$  as an  $R$ -module.

Let  $R = H^0(U, \mathcal{O}_U)$  and  $M = H^0(U, T_U)$ .

**Definition 2.21.** Let  $p > 0$ . We define a **holomorphic  $p$ -form**, as an element of

$$H^0(U, \Omega_U^p) = \Lambda^p M^*.$$

If  $p = 0$ , by convention a **holomorphic 0-form** is just an element in  $R$ .

Let  $z_1, \dots, z_n$  be coordinates for  $U$ . Recall  $\eta \in M$  is given by  $\eta = \sum_i a_i \frac{\partial}{\partial z_i}$  for holomorphic functions  $a_i \in R$ . Then  $\omega \in M^*$  is given by the expression

$$\sum_i b_i dz_i, \quad b_i \in R, \quad dz_i \left( \frac{\partial}{\partial z_j} \right) = \delta_{ij}.$$

More in general  $\omega \in H^0(U, \Omega_U^p)$  is given by

$$\omega = \sum_{|I|=p} f_I dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad f_I \in R, \quad I = (i_1, \dots, i_p), \quad i_1 < \cdots < i_p,$$

where  $dz_{i_1}, \dots, dz_{i_p}$  is an  $R$ -basis of  $H^0(U, \Omega_U^p)$ .

**Example.**

$$H^0(U, \Omega_U^p) \cong \Lambda^p H^0(U, \Omega_U^1)$$

is an isomorphism as  $R$ -modules. This is not true for complex manifolds in general.

The **exterior product** is

$$\begin{aligned} H^0(U, \Omega_U^p) \otimes H^0(U, \Omega_U^q) &\longrightarrow H^0(U, \Omega_U^{p+q}) \\ \omega_1 \otimes \omega_2 &\longmapsto \omega_1 \wedge \omega_2 \end{aligned},$$

where we just need to define

$$\omega_1 \wedge \omega_2 = f dz_{i_1} \wedge dz_{i_p} \otimes g dz_{j_1} \wedge dz_{j_q} = f g dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_q},$$

by linearity. Then  $\omega_1 \wedge \omega_2 = 0$  if  $\{i_1, \dots, i_p\} \cap \{j_1, \dots, j_q\} \neq \emptyset$ , since  $dz_i \wedge dz_i = 0$ .

**Exercise.** Check that this definition coincides with the definition in M4P54.

The **exterior derivative** is

$$\begin{aligned} d : H^0(U, \Omega_U^p) &\longrightarrow H^0(U, \Omega_U^{p+1}) \\ f dz_{i_1} \wedge \cdots \wedge dz_{i_p} &\longmapsto \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p}. \end{aligned}$$

By definition  $d$  is  $\mathbb{C}$ -linear, but not  $R$ -linear. That is,

$$d(a\omega_1 + b\omega_2) = ad\omega_1 + bd\omega_2, \quad \omega_1, \omega_2 \in H^0(U, \Omega_U^p), \quad a, b \in \mathbb{C}.$$

**Theorem 2.22.** Let  $U \subset \mathbb{C}^n$  be open. Then

- the Leibnitz rule

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2, \quad \omega_1 \in H^0(U, \Omega_U^p), \quad \omega_2 \in H^0(U, \Omega_U^q),$$

- $d^2 = 0$ , that is

$$d(d\omega) = 0, \quad \omega \in H^0(U, \Omega_U^p).$$

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**Definition 2.23.** Let  $f : U \subset \mathbb{C}^n \rightarrow \mathbb{C}^m$  be holomorphic, let  $f_i = p_i \circ f : U \rightarrow \mathbb{C}$  where  $p_i : \mathbb{C}^m \rightarrow \mathbb{C}$  is the  $i$ -th projection, and let  $f(U) \subset V \subset \mathbb{C}^m$  be open. Then if

$$\omega = h dz_{i_1} \wedge \cdots \wedge dz_{i_p} \in H^0(V, \Omega_V^p), \quad h \in H^0(U, \mathcal{O}_U),$$

then we can define the **pull-back** of  $\omega$ ,

$$f^* \omega = h \circ f df_{i_1} \wedge \cdots \wedge df_{i_p} \in H^0(U, \Omega_U^p),$$

since  $f_i \in H^0(V, \mathcal{O}_V) = H^0(V, \Omega_V^0)$  implies that  $df_i \in H^0(V, \Omega_V^1)$ , so

$$\begin{array}{ccc} U & \xrightarrow{f} & f(U) \subset V \\ & \searrow h \circ f \in H^0(U, \mathcal{O}_U) & \downarrow h \\ & & \mathbb{C} \end{array} .$$

This is linear over  $\mathbb{C}$  and over  $H^0(U, \mathcal{O}_U)$ .

**Proposition 2.24.** Let  $U \subset \mathbb{C}^n$ ,  $V \subset \mathbb{C}^m$ , and  $W \subset \mathbb{C}^{m'}$  be open, let  $f : U \rightarrow \mathbb{C}^m$  and  $g : V \rightarrow \mathbb{C}^{m'}$  be holomorphic such that  $V \supset f(U)$  and  $W \supset g(V)$ , and let  $\omega \in H^0(V, \Omega_V^p)$  and  $\eta \in H^0(W, \Omega_W^q)$ . Then

- $f^*(\omega + \eta) = f^*\omega + f^*\eta$  if  $p = q$ ,
- $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$ ,
- $df^*\omega = f^*d\omega$ , and
- $f^*g^*\omega = (g \circ f)^*\omega$ .

Let  $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ , and let  $z_i = x_i + iy_i$  for  $i = 1, \dots, n$  and  $x_i, y_i \in \mathbb{R}$ . Then

$$dz_i = dx_i + idy_i,$$

so any holomorphic form is a differentiable form on  $\mathbb{R}^{2n}$ . A  $(p, q)$ -**form** is a differentiable  $(p + q)$ -form of the expression

$$\omega = \sum_{|I|=p, |J|=q} f_{I,J} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}, \quad d\bar{z}_j = dx_j - idy_j, \quad f_{I,J} : U \rightarrow \mathbb{C} \cong \mathbb{R}^2 \in C^\infty.$$

We denote

$$C^\infty(U, \Omega_U^{p,q}) = \{\text{differentiable } (p + q)\text{-forms on } U\}.$$

If  $\omega$  is a  $(p, q)$ -form, then the **conjugate**  $\bar{\omega}$  of  $\omega$  is the  $(q, p)$ -form defined by

$$\bar{\omega} = \sum_{|I|=p, |J|=q} \overline{f_{I,J}} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_q}.$$

### 3 Complex manifolds

#### 3.1 Objects

**Definition 3.1.** A **complex manifold** of dimension  $n$  is a connected Hausdorff topological space  $X$ , with a countable open cover  $\{U_\alpha\}$  of  $X$  such that for all  $\alpha$ , there exists  $\phi_\alpha : U_\alpha \rightarrow \mathbb{C}^n$  such that  $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$  is a homeomorphism and

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta)$$

is a biholomorphism for each  $\alpha$  and  $\beta$ , so

$$\begin{array}{ccc} & U_\alpha \cap U_\beta & \\ \phi_\alpha \swarrow & & \searrow \phi_\beta \\ \mathbb{C}^n \supset \phi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow{\phi_\alpha \circ \phi_\beta^{-1}} & \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n \end{array} .$$

The pair  $(U_\alpha, \phi_\alpha)$  is called a **holomorphic chart**. The set  $\{(U_\alpha, \phi_\alpha)\}$  is called a **holomorphic atlas** or a **complex structure**.

Recall  $X$  is Hausdorff if for all  $x, y \in X$  there exist  $U$  and  $V$  open in  $X$  such that  $U \cap V = \emptyset$  and  $x \in U$  and  $y \in V$ .

**Example 3.2.**

- If  $U \subset \mathbb{C}^n$  is an open set then  $U$  is a complex manifold. More in general if  $X$  is a complex manifold and  $U \subset X$  is open then  $U$  is a complex manifold. Let  $\{(U_\alpha, \phi_\alpha)\}$  be a complex structure on  $M$ . Then

$$\{(\overline{U_\alpha}, \overline{\phi_\alpha})\} = \{(U_\alpha \cap U, \phi_\alpha|_{\overline{U_\alpha}})\}$$

is a complex structure of  $M$ .

- If  $X$  and  $Y$  are complex manifolds, then  $X \times Y$  is a complex manifold.

**Example 3.3.** The projective space  $\mathbb{P}_{\mathbb{C}}^n$  or  $\mathbb{CP}^n$ . Let  $V^* = \mathbb{C}^{n+1} \setminus \{0\}$ , with coordinates  $(z_0, \dots, z_n)$ . Define an equivalence on  $V^*$  as

$$v_1 \sim v_2 \iff \exists \lambda \in \mathbb{C}, v_1 = \lambda v_2.$$

Check that  $\sim$  is an equivalence. Consider the Euclidean topology on  $V^*$ . Then there exists an induced topology on  $X = V^*/\sim = \{[v] \mid v \in V^*\}$ , with quotient map

$$\begin{array}{ccc} q & : & V^* \longrightarrow X \\ & & v \longmapsto [v] \end{array} .$$

Given  $v = (z_0, \dots, z_n) \in V^*$  we denote  $[v] = [z_0, \dots, z_n]$  such that  $z_i \neq 0$  for some  $i$ . Two elements  $[x_0, \dots, x_n]$  and  $[y_0, \dots, y_n]$  of  $X$  define the same point if and only if there exists  $\lambda$  such that  $x_i = \lambda y_i$  for all  $i$ . Let

$$V_i = \{(z_0, \dots, z_n) \in V^* \mid z_i \neq 0\},$$

which is open in  $V^*$ , and let  $U_i = q(V_i)$ , which is open in  $X$ , such that  $\{U_i\}$  is a cover of  $X$ , that is  $\bigcup_i U_i = X$ . Let

$$H_i = \{(z_0, \dots, z_n) \in V^* \mid z_i = 1\}.$$

Then there exists a homeomorphism

$$\begin{array}{ccc} r_i & : & H_i \longrightarrow \mathbb{C}^n \\ & & (z_0, \dots, z_n) \longmapsto [z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_n] \end{array} ,$$

and let

$$\begin{array}{ccc} q_i = q|_{H_i} & : & H_i \subset V^* \longrightarrow U_i \subset X \\ & & (z_0, \dots, z_n) \longmapsto [z_0, \dots, z_n] \end{array}$$

be also a homeomorphism.

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- $q_i$  is surjective. Take  $[x_0, \dots, x_n] \in U_i$ . Then  $x_i \neq 0$ , so choose  $\lambda = 1/x_i$ . Then

$$[x_0, \dots, x_n] = \left[ \frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right] = q(z_0, \dots, z_n), \quad z_j = \frac{x_j}{x_i},$$

and in particular  $z_i = 1$ , so there exists  $(z_0, \dots, z_n) \in H_i$  such that  $q_i(z_0, \dots, z_n) = [x_0, \dots, x_n]$ .

- $q_i$  is injective.<sup>1</sup>

For all  $i$ ,  $q_i^{-1} : U_i \rightarrow H_i$  and  $r_i : H_i \rightarrow \mathbb{C}^n$  are homeomorphisms, so  $\phi_i = r_i \circ q_i^{-1} : U_i \rightarrow \mathbb{C}^n$  is also a homeomorphism. We want to show that  $(U_i, \phi_i)$  define a holomorphic atlas, so

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$$

is a biholomorphism. Consider the case  $j = 0$  and  $i = 1$ . Then  $\phi_0(U_0 \cap U_1) = \{(x_1, \dots, x_n) \mid x_1 \neq 0\}$ , so

$$\begin{aligned} \phi_1 \circ \phi_0^{-1} : \phi_0(U_0 \cap U_1) &\longrightarrow \phi_1(U_0 \cap U_1) \\ (x_1, \dots, x_n) &\longmapsto \left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right) \end{aligned}$$

is a biholomorphism. Thus  $X$  is a compact complex manifold. If  $n = 1$ , then  $\mathbb{P}_{\mathbb{C}}^1 \cong \mathbb{S}^2$ .

**Example 3.4.** The complex torus. Let

$$\begin{aligned} \Lambda = \mathbb{Z}^{2n} &\longrightarrow \mathbb{C}^n \\ (a_1, \dots, a_n, b_1, \dots, b_n) &\longmapsto (a_1 + ib_1, \dots, a_n + ib_n) \end{aligned}$$

Define an equivalence on  $\mathbb{C}^n$  by

$$v_1 \sim v_2 \iff v_1 - v_2 \in \Lambda.$$

Then  $X = \mathbb{C}^n / \sim$  with quotient map  $q : \mathbb{C}^n \rightarrow X$  is Hausdorff and compact. Topologically  $X \cong [0, 1]^{2n} / \sim$ . For each  $x \in \mathbb{C}^n$ , we can find an open set  $x \in U \subset \mathbb{C}^n$  such that  $q|_U : U \rightarrow X$  is a homeomorphism. The idea is if  $x \in (0, 1)^{2n}$  then we can take  $U = (0, 1)^{2n}$ . If not, there exists a translation of  $\mathbb{C}^n \rightarrow \mathbb{C}^n$  such that the property holds. We define

$$\phi_V = q|_U^{-1} : V \subset \mathbb{C}^n / \Lambda \rightarrow U \subset \mathbb{C}^n, \quad V = q(U).$$

Show that  $(V, \phi_V)$  define a complex structure on  $X$ .<sup>2</sup> This is also a compact complex manifold. More in general  $\mathbb{C}^n / \Lambda$  where  $\Lambda \cong \mathbb{Z}^{2n}$  is a lattice is a compact complex manifold.

## 3.2 Morphisms

**Definition 3.5.** Let  $f : X \rightarrow Y$  be a continuous morphism between complex manifolds. Then  $f$  is **holomorphic** if there exists a complex structure  $\{(U_\alpha, \phi_\alpha)\}$  on  $Y$  and for all  $y \in Y$  there exists a holomorphic chart  $(V_\alpha, \psi_\alpha)$  such that  $x \in V_\alpha$  and  $f(V_\alpha) \subset U_\alpha$  around any point  $x$  of  $f^{-1}(y)$  and  $\phi_\alpha \circ f \circ \psi_\alpha^{-1}$  is holomorphic, so

$$\begin{array}{ccc} X \supset V_\alpha & \xrightarrow{f} & U_\alpha \subset Y \\ \psi_\alpha \downarrow & & \downarrow \phi_\alpha \\ \psi_\alpha(V_\alpha) & \xrightarrow{\tilde{f}} & \phi_\alpha(U_\alpha) \end{array}$$

Then  $J_f = J_{\tilde{f}}$ , and a **holomorphic function on  $X$**  is a holomorphic function  $f : X \rightarrow \mathbb{C}$ .

**Exercise 3.6.** If  $X$  is a compact complex manifold then any holomorphic function  $f : X \rightarrow \mathbb{C}$  is constant.

<sup>1</sup>Exercise

<sup>2</sup>Exercise

**Definition 3.7.** Let  $f : X \rightarrow Y$  be a holomorphic function between complex manifolds. Then  $f$  is

- a **submersion** if  $\dim Y \geq \dim X = r$  and  $\text{rk } J_f = r$  at any point,
- an **immersion** if  $r = \dim X \leq \dim Y$  and  $\text{rk } J_f = r$  at any point, and
- an **embedding** if it is an immersion and  $f : X \rightarrow f(X)$  is a homeomorphism.

**Example 3.8.** Let  $f_2, \dots, f_n : \mathbb{C} \rightarrow \mathbb{C}$  be holomorphic, and let

$$\begin{aligned} f &: \mathbb{C} \longrightarrow \mathbb{C}^n \\ z &\longmapsto (z, f_2(z), \dots, f_n(z)) \end{aligned}$$

Then  $f$  is an embedding.

**Example 3.9.** Let  $X = \mathbb{C}^2/\Lambda$  for  $\Lambda = \mathbb{Z}^4 \subset \mathbb{C}^2$ , and let  $q : \mathbb{C}^2 \rightarrow X$ . Fix  $\lambda \in \mathbb{C}$ . Let

$$\begin{aligned} f &: \mathbb{C} \longrightarrow \mathbb{C}^2 \\ z &\longmapsto (z, \lambda z) \end{aligned}$$

Then  $\tilde{f} = q \circ f : \mathbb{C} \rightarrow X$  is an immersion.

- If  $\lambda = 0$  or  $\lambda = \frac{1}{2}$ , then  $\tilde{f}(\mathbb{C})$  is a closed submanifold.
- If  $\lambda$  is general then  $\tilde{f}(\mathbb{C})$  is dense inside  $X$ , so it is not closed. Thus it is not a complex submanifold of  $X$ .

**Definition 3.10.** Let  $i : X \rightarrow Y$  be an embedding of complex manifolds. If  $i(X) \subset Y$  is closed then  $i(X)$  is called a **complex submanifold** of  $Y$ . The **codimension** of  $X$  in  $Y$  is  $\dim Y - \dim X$ .

**Theorem 3.11.**

1. Let  $i : X \rightarrow Y$  be a submanifold of codimension  $k$ , and let  $n = \dim X$ . Then for all  $x \in X$ , there exists an open neighbourhood  $x \in U \subset Y$  and a submersion  $f : U \rightarrow D(0, 1)^k \subset \mathbb{C}^k$  such that  $X \cap U = f^{-1}(0)$ .
2. If  $X \subset Y$  is a closed subset such that for all  $x \in X$  there exists  $U \ni x$  open in  $Y$  and a submersion  $f : U \rightarrow D(0, 1)^k$  such that  $X \cap U = f^{-1}(0)$ , then  $X$  is a complex submanifold.

*Proof.*

1. We can assume that if there exists a holomorphic chart  $(U, \psi)$  on  $Y$  such that  $x \in U$  and if  $V = i^{-1}(U)$  then there exists  $\phi : V \rightarrow \mathbb{C}^n$  such that  $(V, \phi)$  is a holomorphic chart on  $X$ . After possibly shrinking  $U$  smaller, by the rank theorem, there exist biholomorphic  $a : \psi(U) \rightarrow D(0, 1)^{n+k}$  and  $b : \phi(U) \rightarrow D(0, 1)^n$  such that the induced morphism is given by

$$\begin{aligned} D(0, 1)^n &\longrightarrow D(0, 1)^{n+k} \\ (z_1, \dots, z_n) &\longmapsto (z_1, \dots, z_n, 0, \dots, 0) \end{aligned}$$

Let

$$\begin{aligned} c &: D(0, 1)^{n+k} \longrightarrow D(0, 1)^k \\ (z_1, \dots, z_{n+k}) &\longmapsto (z_{n+1}, \dots, z_{n+k}) \end{aligned}$$

so

$$\begin{array}{ccccc} Y & \supset & U & \xrightarrow{\phi} & \phi(U) & \xrightarrow{b} & D(0, 1)^n \subset \mathbb{C}^n \\ \uparrow i & & \uparrow i & & & & \downarrow \\ X & \supset & V & \xrightarrow{\psi} & \psi(U) & \xrightarrow{a} & D(0, 1)^{n+k} \subset \mathbb{C}^{n+k} \end{array} \quad \begin{array}{c} \curvearrowright \\ c \end{array}$$

Then  $f$  is the composition  $c \circ a \circ \psi : U \rightarrow D(0, 1)^n$ .

2. Let  $\{(U_\alpha, \phi_\alpha)\}$  be a complex structure on  $Y$ , and let  $V_\alpha = X \cap U_\alpha$  and  $\psi_\alpha = \phi_\alpha|_{V_\alpha}$ . The goal is to show that  $\{(V_\alpha, \psi_\alpha)\}$  defines a complex structure on  $X$ . By assumption,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k} \rightarrow \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k}$$

is biholomorphic. Let  $U' = \phi_\beta(U)$ , let  $X' = \phi_\beta(X \cap U)$ , and let  $f' = f \circ \phi_\beta^{-1}$ , so

$$\begin{array}{ccccccc} & & & \phi_\alpha(U) & \subset & \phi_\alpha(U_\alpha \cap U_\beta) & \subset \mathbb{C}^{n+k} \\ & & & \nearrow \phi_\alpha & & \uparrow \phi_\alpha \circ \phi_\beta^{-1} & \\ Y & \supset & U_\alpha \cap U_\beta & \supset & U & \xrightarrow{\phi_\beta} & U' \subset \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k} \\ \uparrow i & & \cup & & \cup & \searrow f & \\ X & \supset & X \cap U_\alpha \cap U_\beta & \supset & X \cap U & \xrightarrow{f'} & D(0,1)^k \subset \mathbb{C}^k \\ & & & & \cup & & \\ & & & & X' & & \end{array}$$

Then  $f'^{-1}(0) = \phi_\beta(X \cap U_\alpha \cap U_\beta)$  and  $f'$  is also a submersion. By the rank theorem, we may assume that  $U' = D(0,1)^{n+k}$  and  $f'(z_1, \dots, z_{n+k}) = (z_1, \dots, z_k)$ , so  $\phi_\beta(X' \cap U_\alpha \cap U_\beta) = f'^{-1}(0)$ . Thus

$$(\psi_\alpha \circ \psi_\beta^{-1})(z_1, \dots, z_n) = (\phi_\alpha \circ \phi_\beta^{-1})(z_1, \dots, z_n, 0, \dots, 0)$$

is also a biholomorphism. □

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**Example 3.12.** Let  $U \subset \mathbb{C}^n$  be open, let  $k \leq n$ , let  $f_1, \dots, f_k : U \rightarrow \mathbb{C}$ , and let

$$V = \{x \in \mathbb{C}^n \mid f_1(x) = \dots = f_k(x) = 0\}.$$

Assume that  $\left(\frac{\partial f_i}{\partial z_j}\right)$  has maximal rank  $k$  at any point of  $U$ . Then  $V$  is a complex submanifold of  $U$ . The idea is if  $f = (f_1, \dots, f_k) : U \rightarrow \mathbb{C}^k$ , then  $f$  is a submersion around any point of  $V$ , and use the previous Theorem 3.11.

**Example 3.13.** Let  $f : X \rightarrow Y$  be a holomorphism between complex manifolds, and let  $W \subset X$  be a submanifold. Then  $f|_W : W \rightarrow Y$  is holomorphic.

**Exercise 3.14.** Let  $X = \mathbb{C}^n$ . Show that all the compact submanifolds of  $X$  are zero-dimensional, that is points.

**Exercise 3.15.** Let  $X$  and  $Y$  be compact manifolds. Recall that  $X \times Y$  is also a complex manifold. Assume  $f : X \rightarrow Y$ , so

$$\Gamma_f = \{(x, f(x)) \mid x \in X\} \subset X \times Y.$$

Show that  $\Gamma_f$  is a complex submanifold.

**Example 3.16.** Let  $n, m > 0$ , and let

$$\text{Mat}_{n,m} \mathbb{C} = \{(n \times m)\text{-matrices}\} \cong \mathbb{C}^{n \cdot m}.$$

Then  $\text{Mat}_{n,m} \mathbb{C}$  is a complex manifold. Let

$$\text{GL}_n \mathbb{C} = \{(n \times n)\text{-matrices } A \mid A \text{ invertible}\}.$$

Then  $\text{GL}_n \mathbb{C}$  is a complex manifold, open in  $\text{Mat}_{n,n} \mathbb{C}$ .

**Example 3.17.** Projective manifolds. Let  $R = \mathbb{C}[x_0, \dots, x_n]$  be the ring of polynomials, and let  $X = \mathbb{P}_{\mathbb{C}}^n$  be the complex projective space. Then  $f \in R$  is homogeneous of degree  $d$  if  $f(\lambda x) = \lambda^d f(x)$ . Let  $q : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}_{\mathbb{C}}^n$ , let  $F_1, \dots, F_k$  be homogeneous polynomials in  $R$ , and let

$$V = \{F_1 = \dots = F_k = 0\} \subset \mathbb{C}^{n+1} \setminus \{0\}, \quad W = q(V) \subset \mathbb{P}_{\mathbb{C}}^n,$$

so  $q^{-1}(W) = V$ , because  $F_i$  are homogeneous. Since  $V$  is closed in  $\mathbb{C}^{n+1} \setminus \{0\}$ ,  $W$  is closed in  $\mathbb{P}_{\mathbb{C}}^n$ . Claim that if  $V$  is a submanifold of  $\mathbb{C}^{n+1} \setminus \{0\}$  then  $W$  is a compact submanifold of  $\mathbb{P}_{\mathbb{C}}^n$ . If  $\{U_i\}$  is the open covering given by

$$U_i = \{[x_0, \dots, x_n] \mid x_i \neq 0\},$$

then it is enough to show that  $W \cap U_i$  is a complex submanifold of  $U_i$  for all  $i$ . Assume  $i = n$ . Let  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ . Then  $q(x) = \mathbb{C}^*$  for all  $x \in X$  but  $\mathbb{P}_{\mathbb{C}}^n \times \mathbb{C}^* \neq \mathbb{C}^{n+1} \setminus \{0\}$ . We want to show there exists a biholomorphism

$$\begin{aligned} \phi_n : \quad U_n \times \mathbb{C}^* &\longrightarrow q^{-1}(U_n) = \{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} \mid x_n \neq 0\} \\ ([x_0, \dots, x_n], t) &\longmapsto \left( \frac{tx_0}{x_n}, \dots, \frac{tx_{n-1}}{x_n}, t \right), \end{aligned}$$

such that

$$\begin{aligned} \phi_n^{-1} : \quad q^{-1}(U_n) &\longrightarrow U_n \times \mathbb{C}^* \\ (y_0, \dots, y_n) &\longmapsto (q(y_0, \dots, y_n), y_n) = ([y_0, \dots, y_n], y_n). \end{aligned}$$

From this, it follows that  $V \cap q^{-1}(U_n) \cong (W \cap U_n) \times \mathbb{C}^*$ , so the claim follows.

**Example 3.18.** Plane curves. Let  $X = \mathbb{P}_{\mathbb{C}}^2$ , let  $F \in R[x_0, x_1, x_2]$  be homogeneous of degree  $d$ , and let  $W = \{F = 0\} \subset \mathbb{P}_{\mathbb{C}}^2$ . Then  $W$  is a compact complex submanifold if and only if for all  $x \in \mathbb{P}_{\mathbb{C}}^2$ ,  $\partial_{x_i} F(x) \neq 0$  for some  $i$ .

$d = 1$ .  $W$  is the projective line, so  $F = ax_0 + bx_1 + cx_2$  for  $a, b, c$  not all zero. Then  $W$  is a complex submanifold. There exists a biholomorphism  $\mathbb{P}_{\mathbb{C}}^1 \rightarrow W$ .

$d = 2$ .  $W$  is a conic, so  $F$  is a degree two polynomial. Then  $F = x_0x_1$  does not define a manifold. If  $F = x_0x_1 - x_2^2$ , then  $W$  is a complex submanifold of  $X$ . There exists

$$\begin{aligned} \mathbb{P}_{\mathbb{C}}^1 &\longrightarrow W \subset X \\ [t_0, t_1] &\longmapsto [t_0^2, t_1^2, t_0t_1]. \end{aligned}$$

Check that it is a biholomorphism. <sup>3</sup> This is true for any  $f$  of degree two such that  $W$  is a complex submanifold.

$d \geq 3$ . If  $W$  is a complex submanifold then we will show that  $W$  is not biholomorphic to  $\mathbb{P}_{\mathbb{C}}^1$ .

**Definition 3.19.** Let  $X$  be a complex manifold of dimension  $n$ , and let  $x \in X$ . Then there exists a chart  $(U, \phi)$  around  $x$  such that  $\phi(U) \subset \mathbb{C}^n$ . The **holomorphic tangent space**  $T_x X$  of  $X$  at  $x$ , is the vector space over  $\mathbb{C}$  generated by

$$\left( \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right).$$

Let  $X$  be a real manifold. The **real tangent space**  $T_x^{\mathbb{R}} X$  is the vector space over  $\mathbb{R}$  defined by

$$\left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right),$$

where  $(x_1, \dots, x_n, y_1, \dots, y_n)$  are coordinates of  $\mathbb{R}^{2n}$ . The **complex tangent space**  $T_x^{\mathbb{R}} X$  is the vector space over  $\mathbb{C}$  generated by

$$\left( \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right),$$

a  $2n$ -dimensional vector space over  $\mathbb{C}$ . Then  $T_x^{\mathbb{C}} X = T_x^{\mathbb{R}} X \otimes_{\mathbb{R}} \mathbb{C}$ .

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<sup>3</sup>Exercise

### 3.3 Holomorphic forms on complex manifolds

**Definition 3.20.** Let  $X$  be a complex manifold of dimension  $n$ . Let  $\{(U_\alpha, \phi_\alpha)\}$  be a complex structure on  $X$ . A **holomorphic  $p$ -form** on  $X$  is the data  $\omega_\alpha$ , the  $p$ -forms on  $\phi_\alpha(U_\alpha) \subset \mathbb{C}^n$  such that if

$$h_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta),$$

then  $h_{\alpha\beta}^* \omega_\beta = \omega_\alpha$  for all  $\alpha$  and  $\beta$ .

**Notation 3.21.**

$$\begin{aligned}\Omega_x^p(X) &= H^0(X, \Omega_x^p) = \{\text{holomorphic } p\text{-forms on } X\}, \\ \mathcal{O}_x(X) &= H^0(X, \mathcal{O}_x) = \{\text{holomorphic functions on } X\}.\end{aligned}$$

$R = \mathcal{O}_x(X)$  is a ring and  $M = \Omega_x^p(X)$  is an  $R$ -module.

**Lemma 3.22.** Let  $f : X \rightarrow Y$  be holomorphic. Then  $f^* : \Omega^p(Y) \rightarrow \Omega^p(X)$ .

*Proof.* Let  $\{(U_\alpha, \phi_\alpha)\}$  be a complex structure on  $Y$ . We can write  $f^{-1}(U_\alpha) = \bigcup_{\alpha,\beta} V_{\alpha,\beta}$  where  $\{(V_{\alpha,\beta}, \psi_{\alpha,\beta})\}$  is a complex structure on  $X$ , so

$$\mathbb{C}^n \xleftarrow{\psi_{\alpha,\beta}} V_{\alpha,\beta} \xrightarrow{f|_{V_{\alpha,\beta}}} U_\alpha \xrightarrow{\phi_\alpha} \mathbb{C}^n.$$

Assume  $\omega$  is defined by  $\omega_\alpha$  on  $\phi_\alpha(U_\alpha)$ . Let

$$\omega_{\alpha,\beta} = \left( \left( \psi_{\alpha,\beta}^{-1} \right)^* \circ f^* \circ \phi_\alpha^* \right) (\omega_\alpha)$$

be a  $p$ -form on  $\psi_{\alpha,\beta}(V_{\alpha,\beta})$ . Check that  $\omega_{\alpha,\beta}$  are compatible with respect to the atlas on  $X$ .<sup>4</sup> □

As in the local case, we can define

$$\begin{array}{ccc} \Omega_x^p(X) \otimes \Omega_x^q(X) & \longrightarrow & \Omega_x^{p+q}(X) \\ \omega_1 \otimes \omega_2 & \longmapsto & \omega_1 \wedge \omega_2 \end{array}.$$

Similarly there exists  $d : \Omega_x^p(X) \rightarrow \Omega_x^{p+1}(X)$ .

### 3.4 Holomorphic vector bundles

**Definition 3.23.** Let  $X$  be a complex manifold. A **holomorphic vector bundle**  $E$  of rank  $r$  on  $X$  is

- a complex manifold  $E$ ,
- a holomorphism  $\pi : E \rightarrow X$ , and
- an open covering  $U_\alpha$  of  $X$ ,

such that there exists a biholomorphism  $\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{C}^r$  such that if  $p_\alpha : U_\alpha \times \mathbb{C}^r \rightarrow U_\alpha$  is the projection then  $\pi|_{\pi^{-1}(U_\alpha)} = p_\alpha \circ \psi_\alpha$ , so

$$\begin{array}{ccc} E & \supset & \pi^{-1}(U_\alpha) \xrightarrow{\psi_\alpha} U_\alpha \times \mathbb{C}^r \\ \pi \downarrow & & \downarrow \pi \swarrow p_\alpha \\ X & \supset & U_\alpha \end{array}.$$

For any  $x \in X$ , there exists  $\alpha$  such that  $x \in U_\alpha$ , so

$$\begin{array}{ccc} \pi^{-1}(x) & \xrightarrow{\psi_\alpha} & \{x\} \times \mathbb{C}^r \\ \pi \downarrow & & \swarrow p_\alpha \\ x & & \end{array}.$$

<sup>4</sup>Exercise

Then  $E(x) = \pi^{-1}(x)$  is a vector space of rank  $r$  over  $\mathbb{C}$ . Let  $U_\alpha \ni x \in U_\beta$ . There exists a biholomorphism

$$\mathbb{C}^r \cong p_\alpha^{-1}(x) \rightarrow p_\beta^{-1}(x) \cong \mathbb{C}^r,$$

because they are both biholomorphic to  $\pi^{-1}(x)$ , so  $g_{\alpha\beta}(x) \in \mathrm{GL}_r \mathbb{C}$  because all the biholomorphisms from  $\mathbb{C}^r \rightarrow \mathbb{C}^r$  are linear. The holomorphisms  $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathrm{GL}_r \mathbb{C}$  are called **transition functions**. Then

$$\begin{array}{ccc} p_\alpha^{-1}(x) & \xrightarrow{\mathrm{id}} & p_\alpha^{-1}(x) \\ & \searrow & \nearrow \\ & p_\beta^{-1}(x) & \end{array} \quad \Rightarrow \quad (g_{\alpha\beta} \circ g_{\beta\alpha})(x) = x, \quad x \in U_\alpha \cap U_\beta,$$

and

$$\begin{array}{ccc} p_\alpha^{-1}(x) & \xrightarrow{g_{\alpha\gamma}} & p_\gamma^{-1}(x) \\ & \searrow & \nearrow \\ & p_\beta^{-1}(x) & \end{array} \quad \Rightarrow \quad (g_{\alpha\beta} \circ g_{\beta\gamma})(x) = g_{\alpha\gamma}(x), \quad x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

**Definition 3.24.** Let  $X$  be a complex manifold, and let  $E$  and  $F$  be vector bundles on  $X$  of rank  $r$  and  $s$  respectively, with  $\pi : E \rightarrow X$  and  $\pi' : F \rightarrow X$ . A **holomorphic map**  $f : E \rightarrow F$  is a holomorphic function  $E \rightarrow F$  such that  $\pi = \pi' \circ f$  and such that the rank of the induced linear map  $E(x) \rightarrow F(x)$  is independent of  $x \in X$ , so

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \pi \searrow & & \swarrow \pi' \\ & X & \end{array}, \quad \begin{array}{ccc} E(x) = \pi^{-1}(x) & \xrightarrow{f} & \pi'^{-1}(x) = F(x) \\ \pi \searrow & & \swarrow \pi' \\ & x & \end{array}.$$

**Example 3.25.**  $\pi : E = X \times \mathbb{C}^r \rightarrow X$  is a vector bundle of rank  $r$ , called **trivial**.

**Definition 3.26.** A vector bundle of rank one is called a **line bundle**.

**Definition 3.27.** Let  $\pi : E \rightarrow X$  and  $\pi'^{-1} : F \rightarrow X$  be vector bundles on  $X$  of rank  $r$  and  $s$  respectively.

- The **direct sum**  $E \oplus F$  is a  $(r+s)$ -vector bundle such that  $(E \oplus F)(x) = E(x) \oplus F(x)$  for all  $x \in X$ . The idea is to take an open cover which trivialises both  $E$  and  $F$ . Find the transition function of  $E \oplus F$ .<sup>5</sup>
- The **tensor product**  $E \otimes F$  is the  $(r \cdot s)$ -vector bundle such that  $(E \otimes F)(x) = E(x) \otimes F(x)$  for all  $x \in X$ .
- The  **$p$ -th exterior power** of  $E$  is the vector bundle  $\Lambda^p E$  such that  $(\Lambda^p E)(x) = \Lambda^p(E(x))$ . If  $p = r = \mathrm{rk} E$  then  $\det E = \Lambda^r E$  is a line bundle on  $X$ .
- The **dual** of  $E$  is the rank  $r$  vector bundle  $E^*$  such that  $E^*(x) = (E(x))^*$ , the dual  $\mathrm{Hom}(E(x), \mathbb{C})$  of  $E(x)$ .
- Let  $f : E \rightarrow F$  be holomorphic maps. Then the **kernel**  $\mathrm{Ker} f$  is a vector bundle such that  $(\mathrm{Ker} f)(x) = \mathrm{Ker} f(x) \subset E(x)$ . The **cokernel**  $\mathrm{Coker} f$  is a vector bundle such that  $(\mathrm{Coker} f)(x) = \mathrm{Coker} f(x)$ .

**Example 3.28.** Let  $X = \mathbb{P}_{\mathbb{C}}^1$ , and let

$$\mathcal{O}(-1) = \{(x, v) \mid x = [x_0, \dots, x_n] \in \mathbb{P}_{\mathbb{C}}^n, v = \mu(x_0, \dots, x_n), \mu \in \mathbb{C}\} \subset \mathbb{P}_{\mathbb{C}}^n \times \mathbb{C}^{n+1}.$$

Then  $\pi = p_1 : \mathcal{O}(-1) \rightarrow \mathbb{P}_{\mathbb{C}}^n$ , so

$$\pi^{-1}([x_0, \dots, x_n]) = \{v = \mu(x_0, \dots, x_n) \mid \mu \in \mathbb{C}\} \cong \mathbb{C}^1.$$

<sup>5</sup>Exercise



Let  $\{U_i\}$  be an open covering of  $X$  given by  $U_i = \{[x_0, \dots, x_n] \mid x_i \neq 0\}$ . We define

$$\begin{aligned} \psi_i : \quad & \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C} \\ & ([x_0, \dots, x_n], [v_0, \dots, v_n]) \longmapsto ([x_0, \dots, x_n], v_i) \end{aligned}$$

Then  $\psi_i$  is a biholomorphism, so  $\mathcal{O}(-1)$  is a complex manifold and  $\mathcal{O}(-1)$  is a line bundle. The **tautological line bundle**  $\mathcal{O}(1)$  is the dual of  $\mathcal{O}(-1)$ . Let

$$\mathcal{O}(k) = \begin{cases} X \times \mathbb{C} & k = 0 \\ \mathcal{O}(1)^{\otimes k} & k > 0, \\ \mathcal{O}(-1)^{\otimes k} & k < 0 \end{cases}$$

so  $\mathcal{O}(k) = \mathcal{O}(-k)^*$ .<sup>6</sup> On  $\mathbb{P}_{\mathbb{C}}^n$  these are the only line bundles. That is, if  $\mathcal{L}$  is a line bundle on  $\mathbb{P}_{\mathbb{C}}^1$ , there exists  $k \in \mathbb{Z}$  such that  $\mathcal{L} \cong \mathcal{O}(k)$ . Let  $X = \mathbb{P}_{\mathbb{C}}^1$ , and let  $E$  be a line bundle of rank  $r$  on  $X$ . Then

$$E \cong \bigoplus_{i=1}^r \mathcal{O}(a_i), \quad a_1, \dots, a_r \in \mathbb{Z}.$$

This is false for  $X = \mathbb{P}_{\mathbb{C}}^n$ , with  $n \geq 2$ .

**Definition 3.29.** Let  $f : Y \rightarrow X$  be a holomorphism between complex manifolds, and let  $E$  be a vector bundle of rank  $r$  on  $X$ . Then there exists a vector bundle  $f^*E$  of rank  $r$  on  $Y$  defined by

$$f^*E = \{(y, v) \in Y \times E \mid f(y) = \pi(v)\},$$

the **fibre product** of  $E$  and  $Y$  over  $X$ , such that

$$\begin{array}{ccc} f^*E & \xrightarrow{f'} & E \\ \pi' \downarrow & & \downarrow \pi \\ Y & \xrightarrow{f} & X \end{array}$$

Let  $U = \{U_i\}$  be an open cover of  $X$  which trivialises  $E$ , so

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\psi_i} & U_i \times \mathbb{C}^r \\ & \searrow \pi \quad \swarrow p_1 & \\ & U_i & \end{array},$$

and let  $U' = \{f^{-1}(U_i)\}$  be an open covering of  $Y$ , so

$$\begin{array}{ccccccc} \pi'^{-1}(f^{-1}(U_i)) & \xrightarrow{f'} & \pi^{-1}(U_i) & \xrightarrow{\psi_i} & U_i \times \mathbb{C}^r & \xrightarrow{p_2} & \mathbb{C}^r \\ \pi' \downarrow & & \downarrow \pi & & & & \\ f^{-1}(U_i) & \xrightarrow{f} & U_i & & & & \end{array}.$$

Then

$$\pi'^{-1}(f^{-1}(U_i)) = \{(y, v) \in f^{-1}(U_i) \times \pi^{-1}(U_i) \mid f(y) = \pi(v)\}.$$

Let

$$\begin{aligned} \pi'^{-1}(f^{-1}(U_i)) & \longrightarrow f^{-1}(U_i) \times \mathbb{C}^r \\ (y, v) & \longmapsto (y, p_2(\psi_i(v))) \end{aligned}$$

Then  $f^*E$  is a vector bundle, and

$$f^*E(y) = \pi'^{-1}(y) = E(f(y)), \quad y \in Y.$$

**Notation 3.30.** Let  $f : Y \rightarrow X$  be a morphism, and let  $E$  be a vector bundle on  $X$ . Then  $f^*E = E|_Y$ , mostly used if  $f : Y \hookrightarrow X$ .

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<sup>6</sup>Exercise

**Definition 3.31.** Let  $E$  be a holomorphic vector bundle on a complex manifold  $X$ , and let  $\pi : E \rightarrow X$ . A **section** of  $E$  is a holomorphic function  $s : X \rightarrow E$  such that  $\pi \circ s = \text{id}_X$ .

**Example 3.32.** Let  $E = X \times \mathbb{C}^r$  be the trivial vector bundle of rank  $r$ . Fix  $v \in \mathbb{C}^r$ . Then

$$\begin{aligned} s_v &: X \longrightarrow E \\ x &\longmapsto (x, v) \end{aligned}$$

is a section of  $E$ . If  $v_1, \dots, v_r$  is a basis of  $\mathbb{C}^r$  then  $s_{v_1}, \dots, s_{v_r}$  have the property that  $s_{v_1}(x), \dots, s_{v_r}(x)$  forms a basis of  $E(x)$ . Vice versa, assume  $E$  is a vector bundle on  $X$  of rank  $r$  such that there exist sections  $s_1, \dots, s_r$  of  $E$  such that for all  $x \in X$ ,  $s_1(x), \dots, s_r(x)$  is a basis of  $E(x)$ . Then  $E \cong X \times \mathbb{C}^r$ , since

$$\begin{aligned} X \times \mathbb{C}^r &\longrightarrow E \\ (x, (v_1, \dots, v_r)) &\longmapsto \sum_i v_i s_i(x) \end{aligned}$$

is a biholomorphism. Then  $s_1, \dots, s_r$  is called a **holomorphic frame** for  $E$ . Recall that for all  $E \rightarrow X$  and for all  $x \in X$  there exists open  $U \ni x$  such that  $E|_U$  is trivial, so there exists a frame on  $U$  for  $E|_U$ . This is called a **local frame** around  $x$ .

**Example 3.33.** Let  $X$  be a complex manifold of dimension  $n$ , and let  $z_1, \dots, z_n$  be coordinates on  $\mathbb{C}^n$ . There exists an atlas  $\{(U_\alpha, \phi_\alpha)\}$  for  $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{C}^n$ . For all  $x \in U_\alpha$ ,  $T_x U_\alpha \rightarrow T_{\phi_\alpha(x)} V_\alpha$ , and  $T_{\phi_\alpha(x)} V_\alpha = \left\langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right\rangle$  is a frame of  $T_{V_\alpha}$ . Let

$$T_X = \bigcup_{x \in X} T_x X,$$

and let  $\pi^{-1} : T_X \rightarrow X$  such that  $\pi^{-1}(x) = T_x X$ . Then  $T_X$  is a holomorphic vector bundle of rank  $n$  called the **tangent bundle**, where  $U = \{U_\alpha\}$  and

$$\psi_\alpha : \pi^{-1}(U_\alpha) = T_X|_{U_\alpha} \rightarrow T_{\mathbb{C}^n}|_{V_\alpha} \cong V_\alpha \times \mathbb{C}^r \rightarrow U_\alpha \times \mathbb{C}^r$$

defines the trivialisation. Let

$$\Omega_X^1 = T_X^*$$

be the **cotangent bundle** of  $X$ , and let

$$\Omega_X^p = \Lambda^p \Omega_X^1, \quad p \geq 1.$$

A holomorphic  $p$ -form on  $X$  is a section of  $\Omega_X^p$ .<sup>7</sup>

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<sup>7</sup>Exercise