M3P21 Geometry II: Algebraic Topology

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Contents

0 Some underlying geometric notions

0.1 Introduction

Lecture 1 Friday 11/01/19

Combines topological spaces with algebraic objects, groups.

- How to show that a torus is not homeomorphic to a sphere?
- How to show that $\mathbb{R}^n \ncong \mathbb{R}^m$ if $n \neq m$?

Content is fundamental groups and homology. We will follow chapter one and two from

- A Hatcher, Algebraic topology, 2002

Prerequisites are the following.

- Point set topology. Topological spaces, continuous maps, product and quotient topologies, Hausdorff spaces, etc.
- Basic group theory. Normal subgroups and quotients, isomorphism theorems, free groups, presentation of groups, etc.

0.2 Homotopy

Let X, Y be topological spaces and $I = [0, 1]$.

Definition 0.1. A **homotopy** is a continuous map $F : X \times I \to Y$. For every $t \in I$ we obtain a continuous map

$$\begin{array}{cccc} f_t : & X & \to & Y \\ & x & \mapsto & f_t(x) = F(x, t) \end{array}.$$

Definition 0.2. Two continuous maps $f_0, f_1 : X \to Y$ are **homotopic** if there exists a homotopy $F : X \times I \to Y$ such that

$$f_0(x) = F(x, 0), \qquad f_1(x) = F(x, 1),$$

for all $x \in X$. We write $f_0 \cong f_1$.

(Exercise: this is an equivalence relation)

Definition 0.3. Let $A \subseteq X$ be a subspace. A retraction of X onto A is a continuous map $r : X \to A$ such that

- $r(X) = A$, and
- $r|_A = id_A$.

Example 0.4. If $X \neq \emptyset$, $p \in X$, then X retracts to p by the constant map $X \to \{p\}$.

Definition 0.5. A **deformation retraction** of X onto $A \subseteq X$ is a retraction that is homotopic to the identity. That is, there is a continuous map

$$\begin{array}{cccc} F : & X \times I & \to & A \\ & (x, t) & \mapsto & f_t(x) \end{array},$$

such that $f_0 = id_X$ and $f_1 : X \to A$ is the deformation retraction.

Example 0.6. In the closed n-dimensional disk

$$D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\},$$

deformation retracts to $(0, \ldots, 0) \in \mathbb{R}^n$. $f_t(x) = t \cdot x$. $t = 1$ gives $f_1 = id_{D^n}$ and $t = 0$ gives $f_0 : D^n \to (0, \ldots, 0)$.

Example 0.7. Let S^n be the n-sphere,

$$S^n = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

The cylinder $S^n \times I$ deformation retracts to $S^n \times \{0\}$, by defining $f_t(x, r) = (x, t \cdot r)$.

An observation is if X is a topological space, $f : X \to \{p\}$, and $p \in X$ is a deformation retraction of X to p, then X is path connected. Indeed, if $F : X \to I \to X$ is a homotopy from id_X to f and $x \in X$ is a point, then this gives a path

$$\begin{array}{ccc} I & \to & X \\ t & \mapsto & F(x, t) \end{array}$$

that connects x to p. This implies that not all retractions are deformation retractions. For example, take a space that is not path connected and retract it to a point.

Definition 0.8. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \to X$ such that $fg \cong id_Y$ and $gf = id_X$. If there exists a homotopy equivalence between X and Y, X and Y are **homotopy equivalent** or they have the same **homotopy type**.

Lemma 0.9. *A deformation retraction $f : X \to A$ is a homotopy equivalence.*

Proof. Let $i : A \hookrightarrow X$ be the inclusion map. Then $fi = id_A$ and $if = f \cong id_X$ by definition. □

Example 0.10. The disk with two holes is equivalent to ∞.

Example 0.11. $\mathbb{R}^n$ deformation retracts to a point, by $f_t(x) = t \cdot x$.

Definition 0.12.

- X is **contractible** if it is homotopy equivalent to a point.
- A contractible map is **nullhomotopic** if it is homotopy equivalent to a constant map.

0.3 Cell complexes

Example 0.13. Torus $S^1 \times S^1$ is the union of a point, two open intervals, and the open disk $Int(D^2)$.

These are called **cells**. Can think of disks D^n glued together.

Lecture 2
Tuesday
15/01/19

Definition 0.14. A **CW-complex**, or **cell complex**, is a topological space X such that there exists a decomposition $X = \cup_{n \in \mathbb{N}} X^n$, where the X^n are constructed inductively in the following way.

- X^n is a discrete set.
- For each $n \geq 0$ there is an collection of closed n disks $\{D^n_\alpha\}$ together with continuous maps $\phi_\alpha : \partial D^n_\alpha \to X^{n-1}$ such that
$$X^n = \frac{X^{n-1} \sqcup \bigsqcup_\alpha D^n_\alpha}{\sim},$$
where $x \sim \phi_\alpha(x)$ for all $x \in \partial D^n_\alpha$ for all α.
- A subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n.

Remark 0.15.

- As a set,
$$X^n = X^{n-1} \sqcup \bigsqcup_\alpha e^n_\alpha,$$
where each e^n_α is homeomorphic to an open n-disk. These e^n_α are called the n-**cells** of X.
- If $X = X^m$ for some m, then X is called **finite dimensional**. The minimal m such that $X = X^m$ is the dimension of X.

Example 0.16.

- $[0, 1]$ is a CW-complex.
- $\mathbb{R}$ is a CW-complex.
- S^1 is a CW-complex.
- A graph is a CW-complex.
- $S^n = D^{n-1}/\partial D^n$ is a CW-complex.
- Can also decompose S^2 into one 0-cell, one 1-cell, and two 2-cells.
- The torus $S^1 \times S^1$ is one 0-cell, two 1-cells, and one 2-cell.
- The Möbius strip is two 0-cells, three 1-cells, and one 2-cell.
- The Klein bottle is one 0-cell, two 1-cells, and one 2-cell.

Definition 0.17. If X is a CW-complex with finitely many cells the **Euler characteristic** $\chi(X)$ of X is the number of even cells minus the number of odd cells.

A fact is that $\chi(X)$ does not depend of the choice of cells decomposition.

Example 0.18.

- $$\chi\left(S^1\right) = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}.$$
- $\chi\left(S^1 \times S^1\right) = 0$.

This is the generalisation of the following observation by Leonhard Euler. Let P be a convex polyhedron, where

- V is the number of vertices of P,
- E is the number of edges of P, and
- F is the nubmer of faces of P.

Then $V - E + F = 2$.

1 The fundamental group

Let X be a topological space. A path is a continuous map $f : I \to X$, where $I = [0, 1]$.

Definition 1.1. Two paths f_0, f_1 are **homotopic** if there exists a homotopy between f_0 and f_1 preserving the endpoints, that is a continuous map

$$\begin{array}{rccc} F : & I \times I & \to & X \\ & (s, t) & \mapsto & f_t(s) \end{array},$$

such that

$$f_t(0) = f_0(0), \qquad f_t(1) = f_0(1),$$

for all $t \in I$, and

$$F(s, 0) = f_0(s), \qquad F(s, 1) = f_1(s).$$

Example 1.2. Let $X \subseteq \mathbb{R}^n$ be a convex set. Then all the paths in X are homotopic if they have the same endpoints.

Proof. If $f_0, f_1 : I \to X$ are two paths such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$. Define $f_t(s) = (1-t) f_0(s) + t f_1(s)$. ☐

Lemma 1.3. *Being homotopic is an equivalence relation on the set of paths with fixed endpoints. We will write $f_0 \sim f_1$ for two homotopic paths f_0 and f_1.*

Proof.

- f is homotopic to f.
- If f_0 is homotopic to f_1 by a homotopy f_t, then f_1 is homotopic to f_0 by the homotopy f_{1-t}.
- If f_0 is homotopic to f_1 by a homotopy f_t and $f_1 = g_0$ is homotopic to g_1 by a homotopy g_t, then f_0 is homotopic to g_1 by the homotopy
$$h_t = \begin{cases} f_{2t} & 0 \le t \le \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}.$$
Then
$$\begin{array}{rccc} H: & I \times I & \to & X \\ & (s,t) & \mapsto & h_t(s) \end{array}$$
is continuous because its restriction to the closed subsets $I \times [0,1/2]$ and $I \times [1/2,1]$ is continuous. If the restriction to two closed subsets is continuous then the restriction to the union of these subsets is continuous.

☐

Lecture 3 Wednesday 16/01/19

Let X be a topological space and $I = [0,1]$. If $f : I \to X$ is a path, $[f]$ is the class of all paths on X homotopic to f.

Definition 1.4. Let $f, g : I \to X$ be two paths such that $f(1) = g(0)$. The **product path** $f \cdot g$ is the path
$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}.$$

A convention is that whenever we write $f \cdot g$ we implicitly assume $f(1) = g(0)$.

Lemma 1.5. *Let f_0, f_1, g_0, g_1 be paths on X such that $f_1 \cong f_0$ and $g_0 \cong g_1$. Then $f_0 \cdot g_0 \cong f_1 \cdot g_1$.*

Proof.
$$\begin{array}{ccc} I \times I & \to & X \\ (s,t) & \mapsto & (f_t \cdot g_t)(s) \end{array}$$
is a homotopy between $f_0 \cdot g_0$ and $f_1 \cdot g_1$. ☐

Remark 1.6. **Reparametrisation** is a continuous $\phi : [0,1] \to [0,1]$ such that $\phi(0) = 0$ and $\phi(1) = 1$. If $f : I \to X$ is a path, then $f \cdot \phi \cong f$

Proof. Define $\phi_t(s) = (1-t)\phi(s) + ts$, then $f \cdot \phi_t$ is a homotopy between $f \cdot \phi$ and f. ☐

For $x \in X$, let
$$\begin{array}{rccc} c_x : & I & \to & X \\ & s & \mapsto & x \end{array}$$
be the **constant path** at x. For a path $f : I \to X$, define
$$\begin{array}{rccc} f^{-1} : & I & \to & X \\ & s & \mapsto & f(1-s) \end{array}.$$

Lemma 1.7. *Let $f, g, h : I \to X$ be paths. Then*

1. $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$,
2. $f \cdot c_{f(1)} \cong f$ *and* $c_{f(0)} \cdot f \cong f$, *and*
3. $f \cdot f^{-1} \cong c_{f(0)}$ *and* $f^{-1} \cdot f \cong c_{f(1)}$.

Proof.

1. $((f \cdot g) \cdot h) \cdot \phi = f \cdot (g \cdot h)$, where
$$\phi(s) = \begin{cases} \frac{s}{2} & s \in \left[0, \frac{1}{2}\right] \\ s - \frac{1}{4} & s \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ 2s - 1 & s \in \left[\frac{3}{4}, 1\right] \end{cases},$$
so $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$ by reparametrisation.
2. Again reparametrisation.
3. Define
$$H(s,t) = \begin{cases} f(\max\{1-2s, t\}) & s \in \left[0, \frac{1}{2}\right] \\ f(\max\{2s-1, t\}) & s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$
H is continuous, $H(s,0) = f^{-1} \cdot f$, and $H(s,1) = c_{f(1)}$.

$\square$

Definition 1.8. A **loop** with **basepoint** $x_0 \in X$ is a path $f : I \to X$ such that $f(0) = f(1) = x_0$.

Definition 1.9. Denote by $\Pi_1(X, x_0)$ the set of homotopy classes $[f]$ of loops $f : I \to X$ with basepoint x_0.

Proposition 1.10. $\Pi_1(X, x_0)$ *is a group with product* $[f][g] = [f \cdot g]$ *and neutral element* $c_{x_0} : I \to X$, *the constant path at* x_0.

Proof. Follows directly from Lemma 1.5 and Lemma 1.7. $\square$

Definition 1.11. $\Pi_1(X, x_0)$ is the **fundamental group** of X at x_0.

Example 1.12. Let $X \subseteq \mathbb{R}^n$ be a convex set and $x_0 \in X$. Then $\Pi_1(X, x_0) = 0$.

Proof. X is convex gives all loops are homotopic to each other. $\square$

Assume $x_0, x_1 \in X$ such that x_0 and x_1 are in the same path component of X. Let $h : I \to X$ be a path such that $h(0) = x_0$ and $h(1) = x_1$. Define

$$\begin{array}{rccc} \beta_h : & \Pi_1(X, x_1) & \to & \Pi_1(X, x_0) \\ & [f] & \mapsto & \left[h \cdot f \cdot h^{-1}\right] \end{array}.$$

This is well-defined by Lemma 1.5.

Proposition 1.13. $\beta_h : \Pi_1(X, x_1) \to \Pi_1(X, x_0)$ *is an isomorphism.*

Proof. It is a homomorphism.

$$\beta_h[f \cdot g] = \left[h \cdot f \cdot g \cdot h^{-1}\right] = \left[h \cdot f \cdot h^{-1}\right]\left[h \cdot g \cdot h^{-1}\right] = \beta_h[f] \cdot \beta_h[g],$$

and $\beta_h[c_{x_1}] = [c_{x_1}]$. It is bijective with $\beta_h^{-1} = \beta_{h^{-1}}$. $\square$

If X is path connected, we often write $\Pi_1(X)$ instead of $\Pi_1(X, x_0)$.

Definition 1.14. X is **simply connected** if it is path connected and $\Pi_1(X) = 0$.

Proposition 1.15. *X is simply connected if and only if there exists a unique homotopy class of paths between any two points of X.*

Proof.

$\implies$ There exists a path between any two points. Let f, g be two paths from x_0 to x_1 for $x_0, x_1 \in X$. $f \cdot g^{-1} \cong g \cdot g^{-1}$ gives $f \cong f \cdot g^{-1} \cdot g \cong g \cdot g^{-1} \cdot g \cong g$.

$\Longleftarrow$ Let X be path connected. $x_1 = x_0$ gives that all loops at x_0 are homotopic to each other, so $\Pi_1(X) = 0$.

$\square$

1.1 The fundamental group of the circle

Goal is to show that $\Pi_1(S^1) \cong \mathbb{Z}$. Lecture 4 Friday 18/01/19

Definition 1.16. A **covering space** of a space X is a space $\widetilde{X}$ and a continuous map $p : \widetilde{X} \to X$ such that for each $x \in X$ there is an open $U \subseteq X$ such that

- $p^{-1}(U) = \bigcup_{j \in J} \widetilde{U_j}$, where $\widetilde{U_j} \subseteq \widetilde{X}$ is open,
- $\widetilde{U_i} \cap \widetilde{U_j} = \emptyset$ if $i \neq j$, and
- $p|_{\widetilde{U_j}} \to U$ is a homeomorphism for all $j \in J$.

Such a U is called **evenly covered**. The $\widetilde{U_j}$ are called **sheets**.

Example 1.17. $\begin{array}{cccc} p : & \mathbb{R} & \to & S^1 \\ & s & \mapsto & (\cos(2\pi s, \sin(2\pi s))) \end{array}$.

Definition 1.18. Let $p : \widetilde{X} \to X$ be a covering space. A **lift** of a continuous map $f : Y \to X$ is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ such that $p \cdot \widetilde{f} = f$.

Proposition 1.19. *Let $p : \widetilde{X} \to X$ be a covering space and $f : Y \to X$ be a continuous map. If there are two lifts $\widetilde{f_1}, \widetilde{f_2} : Y \to \widetilde{X}$ of f such that $\widetilde{f_1}(y) = \widetilde{f_2}(y)$ for some $y \in Y$ and if Y is connected, then $\widetilde{f_1} = \widetilde{f_2}$.*

This is the **unique lifting property**.

Proof. Let $y \in Y$ and $U \subseteq X$ be an evenly convered of $f(y)$, so $p^{-1}(U) = \bigcup_j \widetilde{U_j}$. Let $\widetilde{U_1}$ be the sheet such that $\widetilde{f_1}(y) \in \widetilde{U_1}$, and let $\widetilde{U_2}$ be the sheet such that $\widetilde{f_2}(y) \in \widetilde{U_2}$. Let $N \subseteq Y$ be open and $y \in N$ such that $\widetilde{f_1}(N) \subseteq \widetilde{U_1}$ and $\widetilde{f_2}(N) \subseteq \widetilde{U_2}$. We have $p \cdot \widetilde{f_1} = p \cdot \widetilde{f_2}$.

$$\widetilde{f_1}(y) = \widetilde{f_2}(y) \qquad \iff \qquad \widetilde{U_1} = \widetilde{U_2} \qquad \iff \qquad \widetilde{f_1 |_N} = \widetilde{f_2 |_N}.$$

Let $A = \left\{ y \in Y \mid \widetilde{f_1}(y) = \widetilde{f_2}(y) \right\}$, so A is open, $Y \setminus A$ is open, and $A \neq \emptyset$. Thus $A = Y$. $\square$

Proposition 1.20. *Let $p : \widetilde{X} \to X$ be a covering space and $F : Y \times I \to X$ be a continuous map such that there exists a lift $\widetilde{f_0} : Y \times \{0\} \to \widetilde{X}$ of $F|_{Y \times \{0\}}$. Then there is a unique lift $\widetilde{F} : Y \times I \to \widetilde{X}$ of F such that $\widetilde{F}|_{Y \times \{0\}} = \widetilde{f_0}$.*

This is the **homotopy lifting property**.

Proof. Let $y_0 \in Y$ and $t \in I$. There are open $y_0 \in N_t \subseteq Y$ and $t \in (a_t, b_t) \subseteq I$ such that $F(N_t \times (a_t, b_t)) \subseteq U \times X$, where U is open in X and evenly covered. Compactness of I gives that there exist $0 = t_0 < \dots < t_m = 1$ and there exists $y_0 \in N \subseteq Y$ open such that $F(N \times [t_i, t_{i+1}]) \subseteq U_i \subseteq X$, where U_i is open in X and evenly covered. We inductively construct a lift $\widetilde{F}|_{N \times I}$ of $F|_{N \times I}$. $\widetilde{F}|_{N \times [0,0]} = \widetilde{f_0}|_{N \times [0,0]}$ exists. Assume the lift has been constructed on $N \times [0, t_i]$. Let $\widetilde{U_i} \subseteq \widetilde{X}$ be such that $p|_{\widetilde{U_i}} : \widetilde{U_i} \to U_i$ such that $\widetilde{F}(F)(y_0, t_i) \subseteq \widetilde{U_i}$. After shrinking N, may assume $\widetilde{F}(N \times \{t_i\}) \subseteq \widetilde{U_i}$. Define $\widetilde{F}$ on $N \times [t_i, t_{i+1}]$ to be composition of F with the homeomorphism $p_i^{-1} : U_i \to \widetilde{U_i}$. After finitely many steps we obtain a lift $\widetilde{F} : N \times I \to \widetilde{X}$, where $y_0 \in N \subseteq Y$ is open, so for each $y \in Y$ there is a neighbourhood $N_y \subseteq Y$ such that $F \mid_{N \times I}: N \times I \to X$ lifts. For all $y \in Y$, $\{y\} \times I$ is connected and can be lifted, so by Proposition 1.19 the lift of $N \times I$ is unique. Thus there is a unique lift $\widetilde{F} : Y \times I \to \widetilde{X}$. ☐

Corollary 1.21. *Let $f : I \to X$ be a path, $f(0) = x_0$, and $p : \widetilde{X} \to X$ be a covering space. For each $\widetilde{x_0} \in p^{-1}(x_0)$, there is a unique lift $\widetilde{f} : I \to \widetilde{X}$ such that $\widetilde{f}(0) = \widetilde{x_0}$.*

Proof. Proposition 1.20 for Y a point. ☐

Theorem 1.22. *Let $x_0 = (1,0) \in S^1$. $\Pi_1(S^1, x_0)$ is the infinite cyclic group generated by the homotopy class of the loop*

$$\begin{array}{rccc} \omega : & I & \to & S^1 \\ & s & \mapsto & (\cos(2\pi s), \sin(2\pi s)) \end{array}.$$

Remark 1.23.

- $[\omega]^n = [\omega_n]$, where $\omega_1(s) = (\cos(2\pi n s), \sin(2\pi n s))$.
- $$\begin{array}{rccc} p : & \mathbb{R} & \to & S^1 \\ & s & \mapsto & \cos(2\pi s), \sin(2\pi s) \end{array}$$

  is a covering space. ω_n lifts to $\widetilde{\omega_n} : I \to \mathbb{R}$ by $\widetilde{\omega_n}(s) = ns$, $\widetilde{\omega_n}(0) = 0$, and $\widetilde{\omega_n}(1) = n$.

Proof of Theorem 1.22.

- If $f : I \to S^1$ be a loop at $x_0 = (1,0)$, then $[f] = [\omega_n]$ for some n.
- If $[\omega_n] = [\omega_m]$, then $n = m$.

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