

M4P54 Differential Topology

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Syllabus

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0 Introduction

Differential topology is the study of the topology of a manifold using analysis. The topics are

- a review of differential forms,
- de Rham cohomology,
- Morse theory, and
- singular homology.

The following are references.

- J M Lee, Introduction to smooth manifolds, 2000
- L W Tu, Introduction to smooth manifolds, 2008
- J Milnor, Morse theory, 1960
- A Banyaga and D Hurtubise, Lectures on Morse homology, 2004

Lecture 1
Thursday
09/01/20

1 Differential forms on manifolds

1.1 Alternating p -forms on a vector space

Let V be a vector space over \mathbb{R} , and let $p \geq 0$. Then $V^p = V \times \cdots \times V$.

Definition 1.1. A multilinear map $\omega : V^p \rightarrow \mathbb{R}$ is called an **alternating p -form** if we have

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(p)}) = \epsilon(\sigma) \omega(v_1, \dots, v_p), \quad v_1, \dots, v_p \in V \quad \sigma \in \mathcal{S}_p,$$

where \mathcal{S}_p is the group of permutations of p elements and $\epsilon(\sigma)$ is the signature of σ .

Recall that if m is the number of transpositions in a decomposition of σ , then $\epsilon(\sigma) = (-1)^m$, where a **transposition** is $(a_i a_j)$ for $a_i \neq a_j$.

Notation 1.2. $\Lambda^p V^* = \{\text{alternating } p\text{-forms } \omega \text{ on } V\}$ is called the **p -th exterior power** of V .

Check that it is a vector space.¹

Example 1.3.

- $\Lambda^0 V^* = \mathbb{R}$.
- $\Lambda^1 V^* = V^* = \text{Hom}(V, \mathbb{R})$, the **dual** of V .

Definition 1.4. Let $\omega_1 \in \Lambda^p V^*$ and $\omega_2 \in \Lambda^q V^*$. We define the **exterior product** $\omega_1 \wedge \omega_2 \in \Lambda^{p+q} V^*$ of ω_1 and ω_2 by

$$\omega_1 \wedge \omega_2(v_1, \dots, v_{p+q}) = \sum_{\sigma \in \mathcal{S}_{p,q}} \epsilon(\sigma) \omega_1(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \omega_2(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}), \quad v_1, \dots, v_{p+q} \in V,$$

where

$$\mathcal{S}_{p,q} = \{\sigma \in \mathcal{S}_{p+q} \mid \sigma(1) < \cdots < \sigma(p), \sigma(p+1) < \cdots < \sigma(p+q)\}.$$

Example 1.5.

- Assume $\omega_1, \omega_2 \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \omega_2(v_1, v_2) = \omega_1(v_1) \omega_2(v_2) - \omega_1(v_2) \omega_2(v_1), \quad v_1, v_2 \in V.$$

- Assume $\omega_1, \dots, \omega_p \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \cdots \wedge \omega_p(v_1, \dots, v_p) = \det(\omega_i(v_j))_{i,j=1,\dots,p}, \quad v_1, \dots, v_p \in V.$$

Proposition 1.6. Let $\omega_i \in \Lambda^{p_i} V^*$ for $i = 1, 2, 3$.

- *Associativity* $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$.
- *Distributivity* $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$, assuming $p_2 = p_3$.
- *Supercommutativity* $\omega_1 \wedge \omega_2 = (-1)^{p_1 p_2} \omega_2 \wedge \omega_1$.

Definition 1.7. Let $\Phi : V \rightarrow W$ be a linear map between vector spaces over \mathbb{R} . Let $\omega \in \Lambda^p W^*$. Then the **pull-back** $\Phi^* \omega \in \Lambda^p V^*$ of ω is an alternating p -form on V defined by

$$\Phi^* \omega(v_1, \dots, v_p) = \omega(\Phi(v_1), \dots, \Phi(v_p)), \quad v_1, \dots, v_p \in V.$$

¹Exercise

Proposition 1.8. Given $\Phi : V \rightarrow W$ a linear map,

- the pull-back

$$\begin{aligned} \Phi^* &: \Lambda^p W^* \longrightarrow \Lambda^p V^* \\ \omega &\longmapsto \Phi^* \omega \end{aligned}$$

is a linear map that preserves exterior products, that is

$$\Phi^* (\omega_1 \wedge \omega_2) = \Phi^* \omega_1 \wedge \Phi^* \omega_2, \quad \omega_1 \in \Lambda^p W^*, \quad \omega_2 \in \Lambda^q W^*,$$

- if $\Psi : W \rightarrow Z$ is linear then

$$(\Psi \circ \Phi)^* \omega = \Phi^* \Psi^* \omega, \quad \omega \in \Lambda^p Z^*,$$

- assuming $V = W$ and $p = \dim V$, then

$$\Phi^* \omega = (\det \Phi) \omega, \quad \omega \in \Lambda^p V^*.$$

1.2 Differential forms on manifolds

Let M be a smooth manifold of dimension n , and let $x \in M$. Then the tangent space $T_x M$ of M at x is a vector space of dimension n .

Notation 1.9. Let

$$\Lambda^p T_x^* M = \Lambda^p (T_x M)^*.$$

Consider the set

$$\Lambda^p T^* M = \bigsqcup_{x \in M} \Lambda^p T_x^* M,$$

the p -th exterior bundle on M . There exists a morphism $\pi : \Lambda^p T^* M \rightarrow M$ such that for all $x \in M$, $\pi^{-1}(x) = \Lambda^p T_x^* M$, so $\Lambda^p T^* M$ is a vector bundle and it is a smooth manifold, and π is a smooth morphism.

Example 1.10.

- $\Lambda^0 T^* M = M \times \mathbb{R}$.
- $\Lambda^1 T^* M$ is the **cotangent bundle**, the dual of the tangent bundle.

Definition 1.11. A **differential p -form** ω on M is a smooth section of π . That is, it is a smooth morphism $\omega : M \rightarrow \Lambda^p T^* M$ such that $\pi \circ \omega = \text{id}_M$.

Thus, $\omega(x) \in \Lambda^p T_x^* M$.

Notation 1.12.

$$\Omega^p(M) = \{\text{differential } p\text{-forms } \omega \text{ on } M\}, \quad \Omega^\bullet(M) = \bigoplus_p \Omega^p(M).$$

Example 1.13.

$$\Omega^0(M) \cong \{f : M \rightarrow \mathbb{R} \text{ } C^\infty\text{-function}\}.$$

Exercise. If $n = \dim M$, then $\Omega^{n+1}(M) = 0$.

The algebra is the same as last week.

Definition 1.14. Let $\omega_1 \in \Omega^p(M)$ and $\omega_2 \in \Omega^q(M)$. Then $\omega_1 \wedge \omega_2 \in \Omega^{p+q}(M)$ is defined by

$$\omega_1 \wedge \omega_2(x) = \omega_1(x) \wedge \omega_2(x) \in \Lambda^{p+q} T_x^* M, \quad x \in M.$$

By Proposition 1.6, associativity, distributivity, and supercommutativity hold for $\Omega^p(M)$. Let $F : M \rightarrow N$ be a smooth morphism between manifolds. Then for all $x \in M$, the differential of F at x is the linear map

$$DF_x : T_x M \rightarrow T_{F(x)} N.$$

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Thus, for all $p \geq 0$, we have a natural map, called the **pull-back**,

$$\begin{aligned} F_x^* : \Lambda^p T_{F(x)}^* N &\longrightarrow \Lambda^p T_x^* M \\ \omega(v_1, \dots, v_p) &\longmapsto \omega(DF_x(v_1), \dots, DF_x(v_p)) \end{aligned}, \quad \omega \in \Lambda^p T_{F(x)}^* N, \quad v_1, \dots, v_p \in T_x^* M.$$

Thus, we can define

$$\begin{aligned} F^* : \Omega^p(N) &\longrightarrow \Omega^p(M) \\ \omega(x) &\longmapsto F^* \omega(F(x)) \end{aligned}, \quad \omega \in \Omega^p(N).$$

By Proposition 1.8, the pull-back preserves the exterior product, so

$$F^*(\omega_1 \wedge \omega_2) = F^* \omega_1 \wedge F^* \omega_2.$$

If $G : N \rightarrow P$,

$$(G \circ F)^* \omega = F^* G^* \omega.$$

1.3 Local description of p -forms

Let M be a manifold of dimension n , let $x_0 \in M$, let (U, ϕ) be a local chart around x_0 , and let (x_1, \dots, x_n) be local coordinates around x_0 . A basis of $T_{x_0} M$ is given by

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}.$$

A basis of $T_{x_0}^* M$ is given by

$$\{dx_1, \dots, dx_n\}, \quad dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij}.$$

A basis of $\Lambda^p T_{x_0}^* M$ is

$$dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad i_1 < \dots < i_p.$$

Thus, $\omega \in \Omega^p(M)$ is locally given by

$$\omega(x) = \sum_{|I|=p} f_I(x) dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad I = (i_1, \dots, i_p), \quad i_1 < \dots < i_p,$$

where f_I is a C^∞ -function on U for all I .

Example 1.15. Let $F : M \rightarrow N$ be a smooth morphism between manifolds of dimension n , and let $\omega \in \Omega^n(N)$. Locally,

$$\omega(y) = f(y) dy_1 \wedge \dots \wedge dy_n, \quad y \in N,$$

for some $f \in C^\infty$. Proposition 1.8 implies that

$$F^* \omega(x) = (f \circ F)(x) \det DF_x dx_1 \wedge \dots \wedge dx_n, \quad x \in M,$$

where $y_i = p_i \circ F$ and $p_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is the i -th projection.

Let $f : M \rightarrow \mathbb{R}$ be a smooth function, so $f \in \Omega^0(M)$. Locally, the **differential** is

$$\begin{aligned} d : \Omega^0(M) &\longrightarrow \Omega^1(M) \\ f &\longmapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i. \end{aligned}$$

Check that $df \in \Omega^1(M)$, so df is a 1-form on M . Alternatively, $df = f^* dx$ for dx a 1-form on \mathbb{R} , or $df(X) = X(f)$ for any vector field X on M . More in general, let $\omega \in \Omega^p(M)$. Locally,

$$\omega = \sum_{|I|=p} f_I dx_{i_1} \wedge \dots \wedge dx_{i_p}, \quad f_I \in C^\infty,$$

so $d\omega \in \Omega^{p+1}(M)$. Then the **de Rham differential** is

$$\begin{aligned} d : \Omega^p(M) &\longrightarrow \Omega^{p+1}(M) \\ \omega &\longmapsto \sum_{|I|=p} df_I \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}. \end{aligned}$$

Proposition 1.16.

- *The Leibnitz rule*

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2, \quad \omega_1 \in \Omega^p(M), \quad \omega_2 \in \Omega^q(M).$$

- $d^2 = 0$, *that is*

$$d(d\omega) = 0, \quad \omega \in \Omega^p(M).$$

- *Let $F : M \rightarrow N$ be a smooth morphism between manifolds. Then*

$$F^*d\omega = d(F^*\omega), \quad \omega \in \Omega^p(M)$$

so

$$\begin{array}{ccc} \Omega^p(M) & \xrightarrow{d} & \Omega^{p+1}(M) \\ F^* \uparrow & & \uparrow F^* \\ \Omega^p(N) & \xrightarrow{d} & \Omega^{p+1}(N) \end{array}.$$

Definition 1.17.

- $\omega \in \Omega^p(M)$ is **closed** if $d\omega = 0$.
- $\omega \in \Omega^p(M)$ is **exact** if there exists $\omega' \in \Omega^{p-1}(M)$ such that $d\omega' = \omega$.

ω is exact implies that ω is closed, since if $\omega = d\omega'$ then $d\omega = d^2\omega' = 0$.

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1.4 Integrations on manifolds

Let M be a manifold of dimension n , let $F : M \rightarrow M$ be a smooth morphism, and let $\omega \in \Omega^n(M)$. Then

$$F^*\omega(x) = \det DF_x \omega(F(x)).$$

Locally, assume $\omega = f dy_1 \wedge \cdots \wedge dy_n$ for some coordinates y_1, \dots, y_n and $f \in C^\infty$. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of M , where $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^n$. Then

$$h_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \rightarrow \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n,$$

such that

$$h_{\alpha\beta}^* \omega(x) = (f \circ h_{\alpha\beta})(x) \det(Dh_{\alpha\beta})_x dx_1 \wedge \cdots \wedge dx_n.$$

Let $D \subset \mathbb{R}^n$ be compact such that ∂D has zero measure, so D is a domain of integration, let $f : U \rightarrow \mathbb{R}$ be a C^∞ -function where $U \subset \mathbb{R}^n$ is open such that $D \subset U$, and let $h : U \rightarrow h(U)$ be a diffeomorphism. Then

$$\int_{h^{-1}(D)} f(y) dy_1 \cdots dy_n = \int_{h^{-1}(D)} f(y) dy_1 \wedge \cdots \wedge dy_n = \int_D (f \circ h)(x) |\det Dh_x| dx_1 \wedge \cdots \wedge dx_n.$$

Let us assume that $\omega = f(y) dy_1 \wedge \cdots \wedge dy_n$ on U . We define

$$\int_D \omega = \int_D f(y) dy_1 \wedge \cdots \wedge dy_n, \quad D \subset U.$$

Definition 1.18. Let $U \subset \mathbb{R}^n$ be an open set. We define the **support** of ω as

$$\text{supp } \omega = \overline{\{x \in U \mid \omega(x) \neq 0\}}, \quad \omega(x) \in \Lambda^p T_x^* U.$$

Then ω has **compact support**, if $\text{supp } \omega$ is compact.

Fact. Under this assumption, we can define

$$\int_U \omega = \int_D \omega \in \mathbb{R},$$

which is well-defined. Under the same assumption, if $\phi : V \rightarrow U$ is a diffeomorphism, provided that $\det D\phi_x > 0$, since $\det D\phi_x \neq 0$ for all x , then

$$\int_U \omega = \int_V \phi^* \omega.$$

1.5 Orientation

Let V be a vector space over \mathbb{R} of dimension n , and let $B = (b_1, \dots, b_n) \subset V$ and $B' = (b'_1, \dots, b'_n) \subset V$ be ordered bases of V . Then B and B' have the **same orientation** if $\det T > 0$ where

$$\begin{array}{ccc} T & : & V \longrightarrow V \\ & & b_i \longmapsto b'_i \end{array}$$

is a linear map. Let $\omega \in \Lambda^n V^*$ for $\omega \neq 0$. Then B and B' have the same orientation if and only if $\omega(b_1, \dots, b_n)$ has the same sign as $\omega(b'_1, \dots, b'_n)$, by Proposition 1.8. An **orientation** Λ of V is a set of all the ordered basis of V with the same orientation. Let $\phi : V \rightarrow W$ be an isomorphism of vector spaces with fixed orientations Λ_v and Λ_w respectively. We say that ϕ is **orientation preserving** if an ordered basis of V induces an ordered basis of W , so Λ_v induces Λ_w . Let $V = \mathbb{R}^n$, and let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. Then e_1, \dots, e_n defines an orientation of V called **positive**. Let M be a manifold. The idea is to find an orientation Λ_x of $T_x M$ for all $x \in M$.

Special case. Let $M = U \subset \mathbb{R}^n$ be open. There exists a natural isomorphism $\phi_x : T_x U \rightarrow \mathbb{R}^n$. Let Λ_x^+ be an orientation on $T_x U$ such that ϕ_x is orientation preserving with respect to the positive orientation on \mathbb{R}^n . Let $\Lambda^+ = \{\Lambda_x^+\}$.

General case. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M . On U_α , we define the orientation so that $(D\phi_\alpha)_x : T_x U_\alpha \rightarrow T_{\phi_\alpha(x)} \phi_\alpha(U) \subset \mathbb{R}^n$ is orientation preserving. This is called the positive orientation on the chart (U_α, ϕ_α) . We define Λ on M , which is a collection of Λ^+ on $T_x M$ for all $x \in M$. Then M is **orientable** if there exists an atlas with positive orientation charts. This coincides in assuming that $\det D(\phi_\beta^{-1} \circ \phi_\alpha) > 0$ for all α and β .

Notation 1.19. For all $p \geq 0$,

$$\Omega_c^p(M) = \{\omega \in \Omega^p(M) \mid \text{supp } \omega \text{ is compact}\}.$$

If M is compact $\Omega_c^p(M) = \Omega^p(M)$. Let $\omega \in \Omega_c^p(M)$. Assume $\text{supp } \omega \subset U$ where (U, ϕ) is a chart of M , and $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$. Assume also that (U, ϕ) is positively oriented. Let $\phi^{-1} : \phi(U) \rightarrow U$ such that $(\phi^{-1})^* \omega \in \Omega_c^p(\phi(U))$, that is $\text{supp } (\phi^{-1})^* \omega \subset \phi(U)$. We define

$$\int_M \omega = \int_{\phi(U)} (\phi^{-1})^* \omega. \quad (1)$$

We need to show that, under the assumptions above, $\int_M \omega$ does not depend on (U, ϕ) . Let $(\bar{U}, \bar{\phi})$ be also a positively oriented chart such that $\text{supp } \omega \subset \bar{U}$. We want to show that

$$\int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\bar{\phi}(\bar{U})} (\bar{\phi}^{-1})^* \omega.$$

Let $\bar{\phi} \circ \phi^{-1} : \phi(U \cap \bar{U}) \rightarrow \bar{\phi}(U \cap \bar{U})$, so

$$\begin{array}{ccc} & U \cap \bar{U} & \\ \phi \swarrow & & \searrow \bar{\phi} \\ \mathbb{R}^n \supset \phi(U \cap \bar{U}) & \xrightarrow{\bar{\phi} \circ \phi^{-1}} & \bar{\phi}(U \cap \bar{U}) \subset \mathbb{R}^n \end{array}.$$

Since both charts are positively oriented the determinant of the differential $D(\bar{\phi} \circ \phi^{-1})$ is positive, so

$$\begin{aligned} \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi}^{-1})^* \omega &= \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi}^{-1})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\bar{\phi} \circ \phi^{-1})^* (\bar{\phi}^{-1})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* \bar{\phi}^* (\bar{\phi}^{-1})^* \omega \\ &= \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* (\bar{\phi}^{-1} \circ \bar{\phi})^* \omega = \int_{\bar{\phi}(U \cap \bar{U})} (\phi^{-1})^* \omega = \int_{\phi(U)} (\phi^{-1})^* \omega, \end{aligned}$$

by a property of the pull-back and since $(\bar{\phi}^{-1})^* \omega = 0$ outside $\bar{\phi}(U \cap \bar{U})$.

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1.6 Partitions of unity

Definition 1.20. Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering. A **partition of unity** with respect to U is a collection of smooth functions $f_\alpha : M \rightarrow [0, 1]$ such that

1. $\text{supp } f_\alpha = \overline{\{x \in M \mid f_\alpha(x) > 0\}} \subset U_\alpha$ for all α ,
2. $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$, and
3. for all $x \in M$, there exists $U \ni x$ open such that $\text{supp } f_\alpha \cap U \neq \emptyset$ for only finitely many α .

Remark. 3 implies that 2 is a finite sum.

Example 1.21. Let

$$M = S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}, \quad U_1 = S^1 \setminus \{(1, 0)\}, \quad U_2 = S^1 \setminus \{(-1, 0)\},$$

so $\{U_i\}$ is a cover. Let

$$f_1(\cos \theta, \sin \theta) = \frac{1}{2} - \frac{1}{2} \cos \theta, \quad f_2(\cos \theta, \sin \theta) = \frac{1}{2} + \frac{1}{2} \cos \theta.$$

Then f_i is a partition of unity.

Theorem 1.22. Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering of M . Then there exists a partition of unity f_α with respect to U .

Proof. We omit the proof. □

Proposition 1.23. Let M be a manifold, and let $n = \dim M$. Then M is orientable if and only if there exists $\omega \in \Omega^n(M)$ which is never vanishing on M , so $\omega(x) \neq 0$ for all $x \in M$.

ω is called a **volume form** on M .

Proof.

\Leftarrow Assume $\omega \in \Omega^n(M)$ is a volume form. We want to construct an orientation Λ on M , that is Λ_x on $T_x M$ for all $x \in M$. Given an oriented basis v_1, \dots, v_n of $T_x M$ we say that it is **positively oriented** if $\omega(x)(v_1, \dots, v_n) > 0$. For all $x \in M$, we define the orientation Λ_x on $T_x M$ by considering the class of positively oriented ordered basis of $T_x M$ which is compatible with the choice of an atlas on M . Take any atlas $\{(U_\alpha, \phi_\alpha)\}$, where $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$. On U_α ,

$$\omega = g_\alpha \phi_\alpha^* dx_1 \wedge \dots \wedge dx_n.$$

Since $\omega \neq 0$, $g_\alpha > 0$ or $g_\alpha < 0$. If $g_\alpha < 0$ then switch x_1 with x_2 , so $g_\alpha > 0$. After this change of coordinates, (U_α, ϕ_α) is positively oriented, so M is orientable.

\Rightarrow Assume that M is orientable, that is there exists an atlas $\{(U_\alpha, \phi_\alpha)\}$ of positively oriented charts. On U_α , we consider

$$\omega_\alpha = \phi_\alpha^* dx_1 \wedge \dots \wedge dx_n.$$

Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Let $\widetilde{\omega}_\alpha = f_\alpha \omega_\alpha \in \Omega^n(U_\alpha)$. We may assume that $\widetilde{\omega}_\alpha \in \Omega^n(M)$ by extending equal to zero outside U_α . We define $\omega = \sum_\alpha \widetilde{\omega}_\alpha \in \Omega^n(M)$. For all α , since $\sum_\alpha f_\alpha = 1$ there exists α such that $\widetilde{\omega}_\alpha \neq 0$, so $\omega \neq 0$. □

Let M be an orientable manifold of dimension n , and let $\omega \in \Omega_c^n(M)$. We want to define $\int_M \omega$. So far we defined for ω such that $\text{supp } \omega \subset U_\alpha$ where (U_α, ϕ_α) is a chart.

Definition 1.24. Let $\{(U_\alpha, \phi_\alpha)\}$ be a positively oriented atlas on M , and let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $\text{supp } f_\alpha \omega \subset U_\alpha$, so let

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega.$$

Remark 1.25. Note that for each α , we have that the support of $f_\alpha \omega$ is contained in U_α and therefore each term of the sum is well-defined as in (1). Indeed, we have

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* f_\alpha.$$

Lemma 1.26. $\int_M \omega$ does not depend on $\{(U_\alpha, \phi_\alpha)\}$ and f_α .

Proof. Under the assumption that $\text{supp } \omega \subset U_\alpha$ then we showed $\int_{U_\alpha} \omega$ does not depend on (U_α, ϕ_α) . Let $\{(U_\alpha, \phi_\alpha)\}$ and $\{(\overline{U}_\alpha, \overline{\phi}_\alpha)\}$ be two atlases with positively oriented charts, and let f_α and \overline{f}_α be two partitions of unity with respect to $\{U_\alpha\}$ and $\{\overline{U}_\alpha\}$ respectively. Then $\sum_\alpha f_\alpha = \sum_\alpha \overline{f}_\alpha = 1$, so $\int_M f_\alpha \omega = \sum_\beta \int_M \overline{f}_\beta f_\alpha \omega$. Thus

$$\int_M \omega = \sum_\alpha \int_M f_\alpha \omega = \sum_{\alpha, \beta} \int_M \overline{f}_\beta f_\alpha \omega = \sum_\beta \int_M \sum_\alpha f_\alpha \overline{f}_\beta \omega = \sum_\beta \int_M \overline{f}_\beta \omega.$$

□

Proposition 1.27. Let M and N be orientable manifolds of dimension n , and let $\omega, \eta \in \Omega_c^n(M)$.

1. *Linearity*

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

2. *Orientation reversal.* Let \overline{M} be the manifold M with opposite orientation $\Lambda^- = \{\Lambda_x^- \mid x \in M\}$, which is the orientation opposite than the one induced by M with orientation Λ . Then

$$\int_M \omega = - \int_{\overline{M}} \omega.$$

3. *Positivity.* Let ω be the volume form on M . Then

$$\int_M \omega > 0.$$

4. *Diffeomorphism invariance.* Let $F : N \rightarrow M$ be an orientation preserving diffeomorphism. Then

$$\int_M \omega = \int_N F^* \omega.$$

Proof.

1. Exercise. ²

2. Exercise. ³

3. Choose a positively oriented chart (U_α, ϕ_α) on U_α , so

$$\omega = g_\alpha \phi_\alpha^* dx_1 \wedge \cdots \wedge dx_n, \quad g_\alpha > 0.$$

Then $\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega$ where f_α is a partition of unity. For all $x \in M$ there exists α such that $x \in U_\alpha$ and $\int_{U_\alpha} f_\alpha \omega > 0$, so $\int_M \omega > 0$.

4. Let (U_α, ϕ_α) be a positively oriented atlas on M . Then $(F^{-1}(U_\alpha), \phi_\alpha \circ F)$ is an atlas on N which is positively oriented. Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $f_\alpha \circ F$ is a partition of the unity with respect to $\{F^{-1}(U_\alpha)\}$, so

$$\int_N F^*(\omega) = \sum_\alpha \int_N (f_\alpha \circ F) F^* \omega = \sum_\alpha \int_N F^*(f_\alpha \omega) = \sum_\alpha \int_M f_\alpha \omega = \int_M \omega.$$

□

²Exercise

³Exercise

1.7 Manifolds with boundary

Denote

$$\mathbb{R}_{\geq 0}^n = (\mathbb{R}_{\geq 0})^n, \quad \mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Let $U \subset \mathbb{R}_+^n$ be open, and let $F : U \rightarrow \mathbb{R}^m$ be a function. Then F is C^∞ if it can be extended to a C^∞ -function $\tilde{F} : \tilde{U} \rightarrow \mathbb{R}^m$ where $\tilde{U} \supset U$ and \tilde{U} is open.

Definition 1.28. A **manifold with boundary** of dimension n is a Hausdorff topological space M such that there exists an open covering $\{U_\alpha\}$, and for all α , there exists a homeomorphism $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ such that for all α and β ,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n \rightarrow \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n$$

is a diffeomorphism, so

$$\begin{array}{ccc} & U_\alpha \cap U_\beta & \\ \phi_\alpha \swarrow & & \searrow \phi_\beta \\ \mathbb{R}_+^n \supset \phi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow{\phi_\alpha \circ \phi_\beta^{-1}} & \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}_+^n \end{array}.$$

The **boundary** of M is

$$\partial M = \{x \in M \mid \exists \alpha, \phi_\alpha(x) \in \partial \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}\}.$$

Then (U_α, ϕ_α) is called a **chart** and $\{(U_\alpha, \phi_\alpha)\}$ is called an **atlas**.

Remark 1.29.

- ∂M is closed in M .
- $\mathring{M} = M \setminus \partial M$ is a manifold of dimension n .

Example 1.30.

- $M = [0, 1]$ is a manifold with boundary $\partial M = \{0, 1\}$.
- The closed disc $D = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is a manifold with boundary $\partial D = S^{n-1}$.
- $M = [0, 1] \times S^1$ is a manifold with boundary $\partial M = S^1 \sqcup S^1$.

Remark 1.31.

- We can define tangent spaces and differential forms exactly in the same way as usual manifolds.
- The definition of orientability is the same. If M is orientable, then ∂M is also orientable. As a convention, the positive orientation on the boundary of $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\}$ is given by $(-1)^n dx_1 \wedge \dots \wedge dx_{n-1}$. This induces a positive orientation on ∂M .
- Also partitions of unity for any open cover U_α of M is defined the same way. If M is orientable, for any manifold with boundary, for all open covering $U = \{U_\alpha\}$, there exists a partition of unity f_α . This implies that if $\omega \in \Omega_c^n(M)$, then $\int_M \omega$ is defined the same way for manifolds.

1.8 Stokes' theorem

Theorem 1.32 (Stokes). *For any manifold with boundary M of dimension n , and for any $\omega \in \Omega_c^{n-1}(M)$ we have*

$$\int_M d\omega = \int_{\partial M} \omega \in \Omega_c^n(M).$$

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Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas, and let $f_\alpha : M \rightarrow \mathbb{R}$ be a partition of unity with respect to this cover. Then $\sum_\alpha f_\alpha = 1$ on M , so

$$\int_M d\omega = \int_M d\left(\sum_\alpha f_\alpha \omega\right) = \sum_\alpha \int_M d(f_\alpha \omega) = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* d(f_\alpha \omega).$$

Proposition 1.16 implies that

$$(\phi_\alpha^{-1})^* d(f_\alpha \omega) = d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right).$$

Then $(\phi_\alpha^{-1})^* (f_\alpha \omega)$ is an $(n-1)$ -form on $\phi_\alpha(U_\alpha)$. In coordinates,

$$(\phi_\alpha^{-1})^* (f_\alpha \omega) = \sum_{j=1}^n \widetilde{f}_\alpha \omega_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

where ω_j is a smooth function on $\phi_\alpha(U_\alpha)$ and

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\phi_\alpha} & \phi_\alpha(U_\alpha) \\ f_\alpha \downarrow & \swarrow \widetilde{f}_\alpha & \\ [0, 1] & & \end{array}.$$

Then

$$\begin{aligned} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right) &= d\left(\sum_{j=1}^n \widetilde{f}_\alpha \omega_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n\right) \\ &= \sum_{j=1}^n \sum_{k=1}^n \frac{\partial}{\partial x_k} (\widetilde{f}_\alpha \omega_j) dx_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \sum_{j=1}^n \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \\ &= \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) dx_1 \wedge \cdots \wedge dx_n, \end{aligned}$$

so

$$\sum_\alpha \int_{\phi_\alpha(U_\alpha)} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right) = \sum_\alpha \int_{\mathbb{R}_+^n} d\left((\phi_\alpha^{-1})^* (f_\alpha \omega)\right),$$

because $\widetilde{f}_\alpha = 0$ outside $\phi_\alpha(U_\alpha)$. Thus

$$\begin{aligned} \int_M d\omega &= \sum_\alpha \int_{\mathbb{R}_+^n} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) dx_1 \wedge \cdots \wedge dx_n \\ &= \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) dx_n dx_{n-1} \cdots dx_1 \\ &= \sum_\alpha \sum_{j=1}^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} (-1)^{j-1} \frac{\partial}{\partial x_j} (\widetilde{f}_\alpha \omega_j) \Big|_{x_n=0} dx_n dx_{n-1} \cdots \widehat{dx_j} \cdots dx_1 \\ &= \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^{n-1} (\widetilde{f}_\alpha \omega_j) \Big|_{x_n=0} dx_{n-1} \cdots dx_1, \end{aligned}$$

since $(f_\alpha \omega_j)|_{x_n=0} = 0$ for $j = 1, \dots, n-1$, so

$$\int_M d\omega = \sum_\alpha \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (-1)^{n-1} (\widetilde{f}_\alpha \omega_j) \Big|_{x_n=0} dx_{n-1} \cdots dx_1 = \sum_\alpha \int_{\partial U_\alpha} f_\alpha|_{\partial U_\alpha} \omega = \int_{\partial M} \omega,$$

where $\partial U_\alpha = U_\alpha \cap \partial M$. □

1.9 Applications of Stokes' theorem

Theorem 1.33 (Integration by parts). *Let M be an orientable n -dimensional manifold with boundary, let $\omega \in \Omega_c^p(M)$, let $\eta \in \Omega_c^{n-p-1}(M)$, and let $p \in \{0, \dots, n-1\}$. Then*

$$\int_{\partial M} \omega \wedge \eta = \int_M d\omega \wedge \eta + (-1)^p \int_M \omega \wedge d\eta.$$

Proof.

$$\int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta) = \int_M (d\omega \wedge \eta + (-1)^p \omega \wedge d\eta),$$

by Stokes and the Leibnitz rule. \square

Theorem 1.34 (Brouwer's fixed point theorem). *Let*

$$D = \{x \in \mathbb{R}^n \mid |x| \leq 1\},$$

so

$$\partial D = S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\},$$

and let $f : D \rightarrow D$ be a smooth morphism. Then f admits a fixed point, that is there exists $x \in D$ such that $f(x) = x$.

Proof. Assume that $f(x) \neq x$ for all $x \in D$. For any $x \in D$, consider the ray starting from $f(x)$ and passing through x . Let $g(x)$ be the point where this ray intersects ∂D away from $f(x)$. Note that if $x \in \partial D$ then $g(x) = x$. Then $g : D \rightarrow \partial D$. It is easy to check that g is smooth. Since $\partial D = S^{n-1}$ is orientable by Proposition 1.23 there exists a volume form $\omega \in \Omega^{n-1}(\partial D)$, so $\omega(x) \neq 0$. Since $\omega \in \Omega^{n-1}(\partial D)$, $d\omega \in \Omega^n(\partial D)$, which is an n -dimensional manifold, so $d\omega = 0$. Thus

$$0 < \int_{\partial D} \omega = \int_{\partial D} g^* \omega = \int_D dg^* \omega = \int_D g^* d\omega = 0,$$

by Stokes, a contradiction. \square

Example 1.35. Recall any exact form is closed, since $d^2 = 0$. But the opposite is not always true. Let $M = \mathbb{R}^2 \setminus \{0\}$, and let

$$\omega = \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx \in \Omega^1(M).$$

Then ω is closed, since

$$d\omega = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) dx \wedge dy - \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) dy \wedge dx = 0.$$

We want to show that ω is not exact. Assume that

$$\omega = df, \quad f \in \Omega^0(M) = \{C^\infty\text{-function}\}.$$

In particular $\omega = df$ on $S^1 \subset M$. Let

$$\begin{aligned} \gamma &: [0, 2\pi] \longrightarrow S^1 \\ \theta &\longmapsto (\cos \theta, \sin \theta). \end{aligned}$$

Then

$$\int_{S^1} \omega = \int_0^{2\pi} \gamma^* \omega = \int_0^{2\pi} \left(\left(\frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} \right) \cos \theta d\theta - \left(\frac{\sin \theta}{\cos^2 \theta + \sin^2 \theta} \right) (-\sin \theta) d\theta \right) = \int_0^{2\pi} d\theta = 2\pi,$$

and

$$\int_{S^1} \omega = \int_{S^1} df = \int_{\partial S^1} f = \int_{\emptyset} f = 0,$$

so ω is not exact.

Theorem 1.36. *Let M be an orientable manifold of dimension n without boundary, and let $\omega \in \Omega_c^n(M)$. Assume ω is exact. Then*

$$\int_M \omega = 0.$$

Proof. Easy from Stokes. □

Theorem 1.37. *Let M be an orientable manifold of dimension n with boundary, and let $\omega \in \Omega_c^{n-1}(M)$ be a closed form. Then*

$$\int_{\partial M} \omega = 0.$$

Proof. Easy from Stokes. □

Definition 1.38. Let M be an orientable manifold of dimension n , let $\omega \in \Omega_c^k(M)$, and let $N \subset M$ be a submanifold of dimension k . We can define

$$\int_M \omega = \int_N i^* \omega,$$

where $i : N \hookrightarrow M$ is the inclusion. We will denote

$$\omega|_N = i^* \omega \in \Omega_c^k(N).$$

Theorem 1.39. *Let M be an oriented manifold of dimension n , let $\omega \in \Omega_c^k(M)$, and let $S \subset M$ be a compact orientable submanifold of dimension k such that $\partial S = \emptyset$ and $\int_S \omega \neq 0$. Then*

- ω is not exact,
- $\omega|_S$ is not exact, and
- S is not the boundary of an orientable manifold $N \subset M$ of dimension $k+1$.

Proof. Exercise. ⁴ □

⁴Exercise

2 De Rham cohomology

2.1 De Rham cohomology

Let M be a manifold of dimension n , and let $p \geq 0$. Then $\omega_1, \omega_2 \in \Omega^p(M)$ are said to be **cohomologous** if $\omega_1 - \omega_2 = d\eta$ where $\eta \in \Omega^{p-1}(M)$. In particular $\omega \in \Omega^p(M)$ is cohomologous to zero if it is exact. Let

$$Z^p(M) = \text{Ker}(d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)) = \{\omega \in \Omega^p(M) \mid \omega \text{ is closed}\} \subset \Omega^p(M),$$

and let

$$B^p(M) = \text{Im}(d : \Omega^{p-1}(M) \rightarrow \Omega^p(M)) = \{\omega \in \Omega^p(M) \mid \omega \text{ is exact}\} \subset \Omega^p(M).$$

Then $B^p(M) \subset Z^p(M)$ for all $p \geq 0$.

Notation 2.1. If $p = 0$, then $B^0(M) = 0$.

Note. If $\omega_1, \omega_2 \in Z^p(M)$ then $\omega_1 - \omega_2 \in B^p(M)$ if and only if ω_1 and ω_2 are cohomologous.

Denote the p -th de Rham cohomology group as

$$H^p(M) = Z^p(M) / B^p(M) = \{[\omega] \mid \omega \in Z^p(M)\}, \quad p \geq 0.$$

where

$$[\omega] = \{\omega' \in \Omega^p(M) \text{ cohomologous to } \omega\}$$

is the **de Rham class** of ω .

Remark 2.2. $H^p(M)$ is a vector space over \mathbb{R} .

Definition 2.3. $b_p(M) = \dim H^p(M) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is the p -th **Betti number** of M .

Proposition 2.4. If M is connected then

$$H^0(M) = \mathbb{R},$$

that is $b_0(M) = 1$. More in general, $b_0(M)$ is the number of connected components of M .

Proof. Assume M is connected. Then $B^0(M) = 0$, so

$$\begin{aligned} H^0(M) &= Z^0(M) = \{f \in \Omega^0(M) \text{ closed}\} \\ &= \left\{ f \in \Omega^0(M) \mid \text{locally } \forall x \in M, \frac{\partial}{\partial x_i} f(x) = 0 \right\} \\ &= \{f \in \Omega^0(M) \text{ is locally constant}\} = \mathbb{R}. \end{aligned}$$

□

Example. Let $M = S^1$. Then $H^0(M) = \mathbb{R}$.

Proposition 2.5. Let M be a manifold of dimension n . Then

$$H^p(M) = 0, \quad p \geq n + 1.$$

Proof. Recall $\Omega^p(M) = 0$ if $p \geq n + 1$ because all alternating p -forms for $p \geq n + 1$ on an n -dimensional vector space are zero, so $Z^p(M) = 0$. Thus $H^p(M) = 0$. □

Proposition 2.6. Let M be a compact orientable manifold of dimension n without boundary. Then

$$H^n(M) \neq 0.$$

Proof. M is orientable, so there exists a volume form $\omega \in \Omega^n(M) = \Omega_c^n(M)$, by Proposition 1.23. Then ω is closed, because $d\omega$ is an $(n+1)$ -form on M , so $\omega \in Z^n(M)$. We want to show that $[\omega] \neq 0$ in $H^n(M)$. Assume $[\omega] = 0$, so ω is exact. Thus $\omega = d\eta$ where η is an $(n-1)$ -form on M , so

$$0 < \int_M \omega = \int_M d\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0,$$

by Stokes, a contradiction. □

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Proposition 2.7. *Let $G : M \rightarrow N$ be a smooth morphism between manifolds. Then for all $p \geq 0$, $G^* : \Omega^p(N) \rightarrow \Omega^p(M)$ takes closed forms of N to closed forms on M and exact forms of N to exact forms on M .*

Proof. Proposition 1.16 implies that $G^*d = dG^*$. If ω is closed then $dG^*\omega = G^*d\omega = G^*0 = 0$, so $G^*\omega$ is closed. If $\omega = d\eta$ is exact then $G^*\omega = dG^*\eta$ is also exact. \square

Thus $G^* : Z^p(N) \rightarrow Z^p(M)$ and $G^* : B^p(N) \rightarrow B^p(M)$, so there exists a linear map

$$\begin{array}{ccc} G^* & : & H^p(N) \longrightarrow H^p(M) \\ & & [\omega] \longmapsto [G^*\omega] \end{array}.$$

Proposition 2.8. *Let M and N be diffeomorphic manifolds. Then for all $p \geq 0$, $H^p(M) \cong H^p(N)$, that is $H^p(M)$ is a diffeomorphic invariant.*

Proof. By Proposition 2.7 there exists $F^* : H^p(N) \rightarrow H^p(M)$ and $(F^{-1})^* : H^p(M) \rightarrow H^p(N)$. By Proposition 1.8, for all $\omega \in H^p(N)$,

$$(F^{-1})^* F^* \omega = (F \circ F^{-1})^* \omega = \text{id}_N^* \omega = \omega,$$

so $(F^{-1})^* \circ F^* = \text{id}_{H^p(N)}$. Similarly $F^* \circ (F^{-1})^* = \text{id}_{H^p(M)}$, so F^* is an isomorphism. \square

Definition 2.9. Let M_0 and M_1 be manifolds, and let $f_0, f_1 : M_0 \rightarrow M_1$ be smooth morphisms. Then f_0 and f_1 are smoothly **homotopic equivalent** if there exists a smooth morphism $H : M_0 \times [0, 1] \rightarrow M_1$ such that $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$ for all $x \in M_0$.