M4P54 Differential Topology

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Spring 2020

Syllabus

Contents

0 Introduction

Lecture 1
Thursday
09/01/20

Differential topology is the study of the topology of a manifold using analysis. The topics are

- a review of differential forms,
- de Rham cohomology,
- Morse theory, and
- singular homology.

The following are references.

- J M Lee, Introduction to smooth manifolds, 2000
- L W Tu, Introduction to smooth manifolds, 2008
- J Milnor, Morse theory, 1960
- A Banyaga and D Hurtubise, Lectures on Morse homology, 2004

1 Differential forms on manifolds

1.1 Alternating p-forms on a vector space

Let V be a vector space over $\mathbb{R}$, and let $p \geq 0$. Then $V^p = V \times \cdots \times V$.

Definition 1.1. A multilinear map $\omega : V^p \to \mathbb{R}$ is called an **alternating p-form** if we have

$$\omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(p)}\right) = \epsilon(\sigma)\, \omega(v_1, \ldots, v_p), \qquad v_1, \ldots, v_p \in V \qquad \sigma \in \mathcal{S}_p,$$

where $\mathcal{S}_p$ is the group of permutations of p elements and $\epsilon(\sigma)$ is the signature of σ.

Recall that if m is the number of transpositions in a decomposition of σ, then $\epsilon(\sigma) = (-1)^m$, where a **transposition** is $(a_i a_j)$ for $a_i \neq a_j$.

Notation 1.2. $\Lambda^p V^* = \{\text{alternating } p\text{-forms } \omega \text{ on } V\}$ is called the **p-th exterior power** of V.

Check that it is a vector space. [1]

Example 1.3.

- $\Lambda^0 V^* = \mathbb{R}$.
- $\Lambda^1 V^* = V^* = \mathrm{Hom}(V, \mathbb{R})$, the **dual** of V.

Definition 1.4. Let $\omega_1 \in \Lambda^p V^*$ and $\omega_2 \in \Lambda^q V^*$. We define the **exterior product** $\omega_1 \wedge \omega_2 \in \Lambda^{p+q} V^*$ of ω_1 and ω_2 by

$$\omega_1 \wedge \omega_2 (v_1, \ldots, v_{p+q}) = \sum_{\sigma \in \mathcal{S}_{p,q}} \epsilon(\sigma)\, \omega_1\left(v_{\sigma(1)}, \ldots, v_{\sigma(p)}\right) \omega_2\left(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}\right), \qquad v_1, \ldots, v_{p+q} \in V,$$

where

$$\mathcal{S}_{p,q} = \{\sigma \in \mathcal{S}_{p+q} \mid \sigma(1) < \cdots < \sigma(p),\ \sigma(p+1) < \cdots < \sigma(p+q)\}.$$

Example 1.5.

- Assume $\omega_1, \omega_2 \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \omega_2 (v_1, v_2) = \omega_1(v_1)\, \omega_2(v_2) - \omega_1(v_2)\, \omega_2(v_1), \qquad v_1, v_2 \in V.$$

- Assume $\omega_1, \ldots, \omega_p \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \cdots \wedge \omega_p (v_1, \ldots, v_p) = \det\left(\omega_i(v_j)\right)_{i,j=1,\ldots,p}, \qquad v_1, \ldots, v_p \in V.$$

Proposition 1.6. *Let $\omega_i \in \Lambda^{p_i} V^*$ for $i = 1, 2, 3$.*

- *Associativity* $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$.
- *Distributivity* $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$, *assuming* $p_2 = p_3$.
- *Supercommutativity* $\omega_1 \wedge \omega_2 = (-1)^{p_1 \cdot p_2}\, \omega_2 \wedge \omega_1$.

Definition 1.7. Let $\Phi : V \to W$ be a linear map between vector spaces over $\mathbb{R}$. Let $\omega \in \Lambda^p W^*$. Then the **pull-back** $\Phi^* \omega \in \Lambda^p V^*$ of ω is an alternating p-form on V defined by

$$\Phi^* \omega(v_1, \ldots, v_p) = \omega\left(\Phi(v_1), \ldots, \Phi(v_p)\right), \qquad v_1, \ldots, v_p \in V.$$

[1] Exercise

Proposition 1.8. *Given $\Phi : V \to W$ a linear map,*

- *the pull-back*
$$\begin{array}{rccc} \Phi^* & : & \Lambda^p W^* & \longrightarrow & \Lambda^p V^* \\ & & \omega & \longmapsto & \Phi^* \omega \end{array}$$
is a linear map that preserves exterior products, that is
$$\Phi^* \left(\omega_1 \wedge \omega_2\right) = \Phi^* \omega_1 \wedge \Phi^* \omega_2, \qquad \omega_1 \in \Lambda^p W^*, \qquad \omega_2 \in \Lambda^q W^*,$$

- *if $\Psi : W \to Z$ is linear then*
$$\left(\Psi \circ \Phi\right)^* \omega = \Phi^* \Psi^* \omega, \qquad \omega \in \Lambda^p Z^*,$$

- *assuming $V = W$ and $p = \dim V$, then*
$$\Phi^* \omega = \left(\det \Phi\right) \omega, \qquad \omega \in \Lambda^p V^*.$$

1.2 Differential forms on manifolds

Let M be a smooth manifold of dimension n, and let $x \in M$. Then the tangent space $\mathrm{T}_x M$ of M at x is a vector space of dimension n.

Notation 1.9. Let
$$\Lambda^p \mathrm{T}_x^* M = \Lambda^p \left(\mathrm{T}_x M\right)^*.$$
Consider the set
$$\Lambda^p \mathrm{T}^* M = \bigsqcup_{x \in M} \Lambda^p \mathrm{T}_x^* M,$$
the **p-th exterior bundle** on M. There exists a morphism $\pi : \Lambda^p \mathrm{T}^* M \to M$ such that for all $x \in M$, $\pi^{-1}\left(x\right) = \Lambda^p \mathrm{T}_x^* M$, so $\Lambda^p \mathrm{T}^* M$ is a vector bundle and it is a smooth manifold, and π is a smooth morphism.

Example 1.10.

- $\Lambda^0 \mathrm{T}^* M = M \times \mathbb{R}$.
- $\Lambda^1 \mathrm{T}^* M$ is the **cotangent bundle**, the dual of the tangent bundle.

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Definition 1.11. A **differential p-form** ω on M is a smooth section of π. That is, it is a smooth morphism $\omega : M \to \Lambda^p \mathrm{T}^* M$ such that $\pi \circ \omega = \mathrm{id}_M$.

Thus, $\omega\left(x\right) \in \Lambda^p \mathrm{T}_x^* M$.

Notation 1.12.
$$\Omega^p\left(M\right) = \left\{\text{differential } p\text{-forms } \omega \text{ on } M\right\}, \qquad \Omega^\bullet\left(M\right) = \bigoplus_p \Omega^p\left(M\right).$$

Example 1.13.
$$\Omega^0\left(M\right) \cong \left\{f : M \to \mathbb{R} \ \mathrm{C}^\infty\text{-function}\right\}.$$

Exercise. If $n = \dim M$, then $\Omega^{n+1}\left(M\right) = 0$.

The algebra is the same as last week.

Definition 1.14. Let $\omega_1 \in \Omega^p\left(M\right)$ and $\omega_2 \in \Omega^q\left(M\right)$. Then $\omega_1 \wedge \omega_2 \in \Omega^{p+q}\left(M\right)$ is defined by
$$\omega_1 \wedge \omega_2\left(x\right) = \omega_1\left(x\right) \wedge \omega_2\left(x\right) \in \Lambda^{p+q} \mathrm{T}_x^* M, \qquad x \in M.$$

By Proposition 1.6, associativity, distributivity, and supercommutativity hold for $\Omega^p\left(M\right)$. Let $F : M \to N$ be a smooth morphism between manifolds. Then for all $x \in M$, the differential of F at x is the linear map
$$\mathrm{D}F_x : \mathrm{T}_x M \to \mathrm{T}_{F(x)} N.$$

Thus, for all $p \geq 0$, we have a natural map, called the **pull-back**,

$$\begin{array}{cccc} F_x^* & : & \Lambda^p \mathrm{T}_{F(x)}^* N & \longrightarrow & \Lambda^p \mathrm{T}_x^* M \\ & & \omega\left(v_1, \dots, v_p\right) & \longmapsto & \omega\left(\mathrm{D}F_x\left(v_1\right), \dots, \mathrm{D}F_x\left(v_p\right)\right) \end{array}, \qquad \omega \in \Lambda^p \mathrm{T}_{F(x)}^* N, \qquad v_1, \dots, v_p \in \mathrm{T}_x^* M.$$

Thus, we can define

$$\begin{array}{cccc} F^* & : & \Omega^p\left(N\right) & \longrightarrow & \Omega^p\left(M\right) \\ & & \omega\left(x\right) & \longmapsto & F^*\omega\left(F\left(x\right)\right) \end{array}, \qquad \omega \in \Omega^p\left(N\right).$$

By Proposition 1.8, the pull-back preserves the exterior product, so

$$F^*\left(\omega_1 \wedge \omega_2\right) = F^*\omega_1 \wedge F^*\omega_2.$$

If $G : N \to P$,

$$\left(G \circ F\right)^* \omega = F^* G^* \omega.$$

1.3 Local description of p-forms

Let M be a manifold of dimension n, let $x_0 \in M$, let (U, ϕ) be a local chart around x_0, and let $(x_1, \dots, x_n)$ be local coordinates around x_0. A basis of $\mathrm{T}_{x_0} M$ is given by

$$\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}.$$

A basis of $\mathrm{T}_{x_0}^* M$ is given by

$$\left\{\mathrm{d}x_1, \dots, \mathrm{d}x_n\right\}, \qquad \mathrm{d}x_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}.$$

A basis of $\Lambda^p \mathrm{T}_{x_0}^* M$ is

$$\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad i_1 < \dots < i_p.$$

Thus, $\omega \in \Omega^p\left(M\right)$ is locally given by

$$\omega\left(x\right) = \sum_{|I|=p} f_I\left(x\right) \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad I = \left(i_1, \dots, i_p\right), \qquad i_1 < \dots < i_p,$$

where f_I is a C^∞-function on U for all I.

Example 1.15. Let $F : M \to N$ be a smooth morphism between manifolds of dimension n, and let $\omega \in \Omega^n\left(N\right)$. Locally,

$$\omega\left(y\right) = f\left(y\right) \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n, \qquad y \in N,$$

for some $f \in \mathrm{C}^\infty$. Proposition 1.8 implies that

$$F^*\omega\left(x\right) = \left(f \circ F\right)\left(x\right) \det \mathrm{D}F_x \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n, \qquad x \in M,$$

where $y_i = \mathrm{p}_i \circ F$ and $\mathrm{p}_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th projection.

Let $f : M \to \mathbb{R}$ be a smooth function, so $f \in \Omega^0\left(M\right)$. Locally, the **differential** is

$$\begin{array}{cccc} \mathrm{d} & : & \Omega^0\left(M\right) & \longrightarrow & \Omega^1\left(M\right) \\ & & f & \longmapsto & \sum_{i=1}^n \frac{\partial f}{\partial x_i} \mathrm{d}x_i \end{array}.$$

Check that $\mathrm{d}f \in \Omega^1\left(M\right)$, so $\mathrm{d}f$ is a 1-form on M. Alternatively, $\mathrm{d}f = f^*\mathrm{d}x$ for $\mathrm{d}x$ a 1-form on $\mathbb{R}$, or $\mathrm{d}f\left(X\right) = X\left(f\right)$ for any vector field X on M. More in general, let $\omega \in \Omega^p\left(M\right)$. Locally,

$$\omega = \sum_{|I|=p} f_I \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad f_I \in \mathrm{C}^\infty,$$

so $\mathrm{d}\omega \in \Omega^{p+1}\left(M\right)$. Then the **de Rham differential** is

$$\begin{array}{cccc} \mathrm{d} & : & \Omega^p\left(M\right) & \longrightarrow & \Omega^{p+1}\left(M\right) \\ & & \omega & \longmapsto & \sum_{|I|=p} \mathrm{d}f_I \wedge \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p} \end{array}.$$

Proposition 1.16.

- *The Leibniz rule*
$$\mathrm{d}\left(\omega_1 \wedge \omega_2\right) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \mathrm{d}\omega_2, \qquad w_1 \in \Omega^p(M), \qquad \omega_2 \in \Omega^q(M).$$
- $\mathrm{d}^2 = 0$, *that is*
$$\mathrm{d}\left(\mathrm{d}\omega\right) = 0, \qquad \omega \in \Omega^p(M).$$
- *Let $F : M \to N$ be a smooth morphism between manifolds. Then*
$$F^* \mathrm{d}\omega = \mathrm{d}\left(F^*\omega\right), \qquad \omega \in \Omega^p(M),$$
so
$$\begin{array}{ccc} \Omega^p(M) & \xrightarrow{\mathrm{d}} & \Omega^{p+1}(M) \\ {\scriptstyle F^*}\uparrow & & \uparrow{\scriptstyle F^*} \\ \Omega^p(N) & \xrightarrow[\mathrm{d}]{} & \Omega^{p+1}(N) \end{array}.$$

Definition 1.17. Lecture 3 Tuesday 14/01/20

- $\omega \in \Omega^p(M)$ is **closed** if $\mathrm{d}\omega = 0$.
- $\omega \in \Omega^p(M)$ is **exact** if there exists $\omega' \in \Omega^{p-1}(M)$ such that $\mathrm{d}\omega' = \omega$.

ω is exact implies that ω is closed, since if $\omega = \mathrm{d}\omega'$ then $\mathrm{d}\omega = \mathrm{d}^2\omega' = 0$.

1.4 Integrations on manifolds

Let M be a manifold of dimension n, let $F : M \to M$ be a smooth morphism, and let $\omega \in \Omega^n(M)$. Then
$$F^*\omega(x) = \det \mathrm{D}F_x \omega(F(x)).$$
Locally, assume $\omega = f \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n$ for some coordinates $(y_1, \dots, y_n)$ and $f \in \mathrm{C}^\infty$. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of M, where $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$. Then
$$h_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha\left(U_\alpha \cap U_\beta\right) \subset \mathbb{R}^n \to \phi_\beta\left(U_\alpha \cap U_\beta\right) \subset \mathbb{R}^n,$$
such that
$$h_{\alpha\beta}^*\omega(x) = \left(f \circ h_{\alpha\beta}\right)(x) \det\left(\mathrm{D}h_{\alpha\beta}\right)_x \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$
Let $D \subset \mathbb{R}^n$ be compact such that ∂D has zero measure, so D is a domain of integration, let $f : U \to \mathbb{R}$ be a C^∞-function where $U \subset \mathbb{R}^n$ is open such that $D \subset U$, and let $h : U \to h(U)$ be a diffeomorphism. Then
$$\int_{h^{-1}(D)} f(y)\ \mathrm{d}y_1 \dots \mathrm{d}y_n = \int_{h^{-1}(D)} f(y)\ \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n = \int_D (f \circ h)(x) |\det \mathrm{D}h_x|\ \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$
Let us assume that $\omega = f(y) \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n$ on U. We define
$$\int_D \omega = \int_D f(y)\ \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n, \qquad D \subset U.$$
Definition 1.18. Let $U \subset \mathbb{R}^n$ be an open set. We define the **support** of ω as
$$\operatorname{supp} \omega = \overline{\{x \in U \mid \omega(x) \neq 0\}}, \qquad \omega(x) \in \Lambda^p \mathrm{T}_x^* U.$$
Then ω has **compact support**, if $\operatorname{supp} \omega$ is compact.

Fact. Under this assumption, we can define
$$\int_U \omega = \int_D \omega \in \mathbb{R},$$
which is well-defined. Under the same assumption, if $\phi : V \to U$ is a diffeomorphism, provided that $\det \mathrm{D}\phi_x > 0$, since $\det \mathrm{D}\phi_x \neq 0$ for all x, then
$$\int_U \omega = \int_V \phi^*\omega.$$

1.5 Orientation

Let V be a vector space over $\mathbb{R}$ of dimension n, and let $B = (b_1, \dots, b_n) \subset V$ and $B' = (b'_1, \dots, b'_n) \subset V$ be ordered bases of V. Then B and B' have the **same orientation** if $\det T > 0$ where

$$\begin{array}{ccccc} T & : & V & \longrightarrow & V \\ & & b_i & \longmapsto & b'_i \end{array}$$

is a linear map. Let $\omega \in \Lambda^n V^*$ for $\omega \neq 0$. Then B and B' have the same orientation if and only if $\omega(b_1, \dots, b_n)$ has the same sign as $\omega(b'_1, \dots, b'_n)$, by Proposition 1.8. An **orientation** Λ of V is a set of all the ordered basis of V with the same orientation. Let $\phi : V \to W$ be an isomorphism of vector spaces with fixed orientations Λ_v and Λ_w respectively. We say that ϕ is **orientation preserving** if an ordered basis of V induces an ordered basis of W, so Λ_v induces Λ_w. Let $V = \mathbb{R}^n$, and let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. Then $e_1, \dots, e_n$ defines an orientation of V called **positive**. Let M be a manifold. The idea is to find an orientation Λ_x of $\mathrm{T}_x M$ for all $x \in M$.

Special case. Let $M = U \subset \mathbb{R}^n$ be open. There exists a natural isomorphism $\phi_x : \mathrm{T}_x U \to \mathbb{R}^n$. Let Λ_x^+ be an orientation on $\mathrm{T}_x U$ such that ϕ_x is orientation preserving with respect to the positive orientation on $\mathbb{R}^n$. Let $\Lambda^+ = \{\Lambda_x^+\}$.

General case. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M. On U_α, we define the orientation so that $(\mathrm{D}\phi_\alpha)_x : \mathrm{T}_x U_\alpha \to \mathrm{T}_{\phi_\alpha(x)} \phi_\alpha(U) \subset \mathbb{R}^n$ is orientation preserving. This is called the positive orientation on the chart (U_α, ϕ_α). We define Λ on M, which is a collection of Λ^+ on $\mathrm{T}_x M$ for all $x \in M$. Then M is **orientable** if there exists an atlas with positive orientation charts. This coincides in assuming that $\det \mathrm{D}\left(\phi_\beta^{-1} \circ \phi_\alpha\right) > 0$ for all α and β.

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Notation 1.19. For all $p \geq 0$,

$$\Omega_{\mathrm{c}}^p(M) = \{\omega \in \Omega^p(M) \mid \operatorname{supp} M \text{ is compact}\}.$$

If M is compact $\Omega_{\mathrm{c}}^p(M) = \Omega^p(M)$. Let $\omega \in \Omega_{\mathrm{c}}^r(M)$. Assume $\operatorname{supp} \omega \subset U$ where (U, ϕ) is a chart of M, and $\phi : U \to \phi(U) \subset \mathbb{R}^n$. Assume also that (U, ϕ) is positively oriented. Let $\phi^{-1} : \phi(U) \to U$ such that $\left(\phi^{-1}\right)^* \omega \in \Omega_{\mathrm{c}}^n(\phi(U))$, that is $\operatorname{supp}\left(\phi^{-1}\right)^* \omega \subset \phi(U)$. We define

$$\int_M \omega = \int_{\phi(U)} \left(\phi^{-1}\right)^* \omega. \tag{1}$$

We need to show that, under the assumptions above, $\int_M \omega$ does not depend on (U, ϕ). Let $\left(\overline{U}, \overline{\phi}\right)$ be also a positively oriented chart such that $\operatorname{supp} \omega \subset \overline{U}$. We want to show that

$$\int_{\phi(U)} \left(\phi^{-1}\right)^* \omega = \int_{\overline{\phi}\left(\overline{U}\right)} \left(\overline{\phi}^{-1}\right)^* \omega.$$

Let $\overline{\phi} \circ \phi^{-1} : \phi\left(U \cap \overline{U}\right) \to \overline{\phi}\left(U \cap \overline{U}\right)$, so

$$\begin{tikzcd} & U \cap \overline{U} \arrow[dl, "\phi"'] \arrow[dr, "\overline{\phi}"] & \\ \mathbb{R}^n \supset \phi\left(U \cap \overline{U}\right) \arrow[rr, "\overline{\phi} \circ \phi^{-1}"'] & & \overline{\phi}\left(U \cap \overline{U}\right) \subset \mathbb{R}^n \end{tikzcd}.$$

Since both charts are positively oriented the determinant of the differential $\mathrm{D}\left(\overline{\phi} \circ \phi^{-1}\right)$ is positive, so

$$\begin{aligned} \int_{\overline{\phi}(U)} \left(\overline{\phi}^{-1}\right)^* \omega &= \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\overline{\phi}^{-1}\right)^* \omega = \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\overline{\phi} \circ \phi^{-1}\right)^* \left(\overline{\phi}^{-1}\right)^* \omega = \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\phi^{-1}\right)^* \overline{\phi}^* \left(\overline{\phi}^{-1}\right)^* \omega \\ &= \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\phi^{-1}\right)^* \left(\overline{\phi}^{-1} \circ \overline{\phi}\right)^* \omega = \int_{\overline{\phi}\left(U \cap \overline{U}\right)} \left(\phi^{-1}\right)^* \omega = \int_{\overline{\phi}(U)} \left(\phi^{-1}\right)^* \omega, \end{aligned}$$

by a property of the pull-back and since $\left(\overline{\phi}^{-1}\right)^* \omega = 0$ outside $\overline{\phi}\left(U \cap \overline{U}\right)$.

1.6 Partitions of unity

Definition 1.20. Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering. A **partition of unity** with respect to U is a collection of smooth functions $f_\alpha : M \to [0,1]$ such that

1. $\operatorname{supp} f_\alpha = \overline{\{x \in M \mid f_\alpha(x) = 0\}} \subset U_\alpha$ for all α,
2. $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$, and
3. for all $x \in M$, there exists $U \ni x$ open such that $\operatorname{supp} f_\alpha \cap U \neq \emptyset$ for only finitely many α.

Remark. 3 implies that 2 is a finite sum.

Example 1.21. Let

$$M = \mathrm{S}^1 = \left\{x \in \mathbb{R}^2 \;\middle|\; |x| = 1\right\}, \qquad U_1 = \mathrm{S}^1 \setminus \{(1,0)\}, \qquad U_2 = \mathrm{S}^1 \setminus \{(-1,0)\},$$

so $\{U_i\}$ is a cover. Let

$$f_1(\cos\theta, \sin\theta) = \frac{1}{2} - \frac{1}{2}\cos\theta, \qquad f_2(\cos\theta, \sin\theta) = \frac{1}{2} + \frac{1}{2}\cos\theta.$$

Then f_i is a partition of unity.

Proposition 1.22. *Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering of M. Then there exists a partition of unity f_α with respect to U.*

Proof. We omit the proof. ◻

Proposition 1.23. *Let M be a manifold, and let $n = \dim M$. Then M is orientable if and only if there exists $\omega \in \Omega^n(M)$ which is never vanishing on M, so $\omega(x) \neq 0$ for all $x \in M$.*

ω is called a **volume form** on M.

Proof.

$\Longleftarrow$ Assume $\omega \in \Omega^n(M)$ is a volume form. We want to construct an orientation Λ on M, that is Λ_x on $\mathrm{T}_x M$ for all $x \in M$. Given an oriented basis $v_1, \dots, v_n$ of $\mathrm{T}_x M$ we say that it is **positively oriented** if $\omega(x)(v_1, \dots, v_n) > 0$. For all $x \in M$, we define the orientation Λ_x on $\mathrm{T}_x M$ by considering the class of positively oriented ordered basis of $\mathrm{T}_x M$ which is compatible with the choice of an atlas on M. Take any atlas $\{(U_\alpha, \phi_\alpha)\}$, where $\phi_\alpha : U_\alpha \to \mathbb{R}^n$. On U_α,

$$\omega = g_\alpha \phi_\alpha^* \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$

Since $\omega \neq 0$, $g_\alpha > 0$ or $g_\alpha < 0$. If $g_\alpha < 0$ then switch x_1 with x_2, so $g_\alpha > 0$. After this change of coordinates, (U_α, ϕ_α) is positively oriented, so M is orientable.

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$\Longrightarrow$ Assume that M is orientable, that is there exists an atlas $\{(U_\alpha, \phi_\alpha)\}$ of positively oriented charts. On U_α, we consider

$$\omega_\alpha = \phi_\alpha^* \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$

Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Let $\widetilde{\omega_\alpha} = f_\alpha \omega_\alpha \in \Omega^n(U_\alpha)$. We may assume that $\widetilde{\omega_\alpha} \in \Omega^n(M)$ by extending equal to zero outside U_α. We define $\omega = \sum_\alpha \widetilde{\omega_\alpha} \in \Omega^n(M)$. For all α, since $\sum_\alpha f_\alpha = 1$ there exists α such that $\widetilde{\omega_\alpha} \neq 0$, so $\omega \neq 0$.

◻

Let M be an orientable manifold of dimension n, and let $\omega \in \Omega_{\mathrm{c}}^n(M)$. We want to define $\int_M \omega$. So far we defined for ω such that $\operatorname{supp} \omega \subset U_\alpha$ where (U_α, ϕ_α) is a chart.

Definition 1.24. Let $\{(U_\alpha, \phi_\alpha)\}$ be a positively oriented atlas on M, and let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $\operatorname{supp} f_\alpha \omega \subset U_\alpha$, so let

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega.$$

Remark 1.25. Note that for each α, we have that the support of $f_\alpha \omega$ is contained in U_α and therefore each term of the sum is well-defined as in (1). Indeed, we have

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} \left(\phi^{-1}\right)^* f_\alpha.$$

Lemma 1.26. *$\int_M \omega$ does not depend on $\{(U_\alpha, \phi_\alpha)\}$ and f_α.*

Proof. Under the assumption that $\operatorname{supp} \omega \subset U_\alpha$ then we showed $\int_{U_\alpha} \omega$ does not depend on (U_α, ϕ_α). Let $\{(U_\alpha, \phi_\alpha)\}$ and $\left\{\left(\overline{U_\alpha}, \overline{\phi_\alpha}\right)\right\}$ be two atlases with positively oriented charts, and let f_α and $\overline{f_\alpha}$ be two partitions of unity with respect to $\{U_\alpha\}$ and $\left\{\overline{U_\alpha}\right\}$ respectively. Then $\sum_\alpha f_\alpha = \sum_\alpha \overline{f_\alpha} = 1$, so $\int_M f_\alpha \omega = \sum_\beta \int_M \overline{f_\beta} f_\alpha \omega$. Thus

$$\int_M \omega = \sum_\alpha \int_M f_\alpha \omega = \sum_{\alpha,\beta} \int_M \overline{f_\beta} f_\alpha \omega = \sum_\beta \int_M \sum_\alpha f_\alpha \overline{f_\beta} \omega = \sum_\beta \int_M \overline{f_\beta} \omega.$$

□

Proposition 1.27. *Let M and N be orientable manifolds of dimension n, and let $\omega, \eta \in \Omega_c^n(M)$.*

1. *Linearity*

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

2. *Orientation reversal. Let $\overline{M}$ be the manifold M with opposite orientation $\Lambda^- = \{\Lambda_x^- \mid x \in M\}$, which is the orientation opposite than the one induced by M with orientation Λ. Then*

$$\int_M \omega = -\int_{\overline{M}} \omega.$$

3. *Positivity. Let ω be the volume form on M. Then*

$$\int_M \omega > 0.$$

4. *Diffeomorphism invariance. Let $F : N \to M$ be an orientation preserving diffeomorphism. Then*

$$\int_M \omega = \int_N F^* \omega.$$

Proof.

1. Exercise. [2]

2. Exercise. [3]

3. Choose a positively oriented chart (U_α, ϕ_α) on U_α, so

$$\omega = g_\alpha \phi_\alpha^* \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n, \qquad g_\alpha > 0.$$

   Then $\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega$ where f_α is a partition of unity. For all $x \in M$ there exists α such that $x \in U_\alpha$ and $\int_{U_\alpha} f_\alpha \omega > 0$, so $\int_M \omega > 0$.

4. Let (U_α, ϕ_α) be a positively oriented atlas on M. Then $\left(F^{-1}(U_\alpha), \phi_\alpha \circ F\right)$ is an atlas on N which is positively oriented. Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $f_\alpha \circ F$ is a partition of the unity with respect to $\left\{F^{-1}(U_\alpha)\right\}$, so

$$\int_N F^* \omega = \sum_\alpha \int_N (f_\alpha \circ F) F^* \omega = \sum_\alpha \int_N F^*(f_\alpha \omega) = \sum_\alpha \int_M f_\alpha \omega = \int_M \omega.$$

□

[2] Exercise
[3] Exercise

1.7 Manifolds with boundary

Denote

$$\mathbb{R}^n_{\geq 0} = (\mathbb{R}_{\geq 0})^n, \qquad \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Let $U \subset \mathbb{R}^n_+$ be open, and let $F : U \to \mathbb{R}^m$ be a function. Then F is C^∞ if it can be extended to a C^∞-function $\widetilde{F} : \widetilde{U} \to \mathbb{R}^m$ where $\widetilde{U} \supset U$ and $\widetilde{U}$ is open.

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Definition 1.28. A **manifold with boundary** of dimension n is a Hausdorff topological space M such that there exists an open covering $\{U_\alpha\}$, and for all α, there exists a homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ such that for all α and β,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta (U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+ \to \phi_\alpha (U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+$$

is a diffeomorphism, so

$$\begin{tikzcd} & U_\alpha \cap U_\beta \arrow[dl, "\phi_\alpha"'] \arrow[dr, "\phi_\beta"] & \\ \mathbb{R}^n_+ \supset \phi_\alpha (U_\alpha \cap U_\beta) \arrow[rr, "\phi_\alpha \circ \phi_\beta^{-1}"'] & & \phi_\beta (U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+ \end{tikzcd}.$$

The **boundary** of M is

$$\partial M = \left\{x \in M \;\middle|\; \exists \alpha,\ \phi_\alpha (x) \in \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}\right\}.$$

Then (U_α, ϕ_α) is called a **chart** and $\{(U_\alpha, \phi_\alpha)\}$ is called an **atlas**.

Remark 1.29.

- ∂M is closed in M.
- $\mathring{M} = M \setminus \partial M$ is a manifold of dimension n.

Example 1.30.

- $M = [0, 1]$ is a manifold with boundary $\partial M = \{0, 1\}$.
- The closed disc $\mathrm{D} = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is a manifold with boundary $\partial \mathrm{D} = \mathrm{S}^{n-1}$.
- $M = [0, 1] \times \mathrm{S}^1$ is a manifold with boundary $\partial M = \mathrm{S}^1 \sqcup \mathrm{S}^1$.

Remark 1.31.

- We can define tangent spaces and differential forms exactly in the same way as usual manifolds.
- The definition of orientability is the same. If M is orientable, then ∂M is also orientable. As a convention, the positive orientation on the boundary of $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$ is given by $(-1)^n \, \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_{n-1}$. This induces a positive orientation on ∂M.
- Also partitions of unity for any open cover U_α of M is defined the same way. If M is orientable, for any manifold with boundary, for all open covering $U = \{U_\alpha\}$, there exists a partition of unity f_α. This implies that if $\omega \in \Omega^n_{\mathrm{c}} (M)$, then $\int_M \omega$ is defined the same way for manifolds.

1.8 Stokes' theorem

Theorem 1.32 (Stokes)**.** *For any manifold with boundary M of dimension n, and for any $\omega \in \Omega^{n-1}_{\mathrm{c}} (M)$ we have*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \in \Omega^n_{\mathrm{c}} (M).$$

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas, and let $f_\alpha : M \to \mathbb{R}$ be a partition of unity with respect to this cover. Then $\sum_\alpha f_\alpha = 1$ on M, so

$$\int_M \mathrm{d}\omega = \int_M \mathrm{d}\left(\sum_\alpha f_\alpha \omega\right) = \sum_\alpha \int_M \mathrm{d}(f_\alpha \omega) = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} \left(\phi_\alpha^{-1}\right)^* \mathrm{d}\left(f_\alpha \omega\right).$$

Proposition 1.16 implies that

$$\left(\phi_\alpha^{-1}\right)^* \mathrm{d}\left(f_\alpha \omega\right) = \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right).$$

Then $\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)$ is an $(n-1)$-form on $\phi_\alpha\left(U_\alpha\right)$. In coordinates,

$$\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right) = \sum_{j=1}^n \widetilde{f_\alpha} \omega_j \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n,$$

where ω_j is a smooth function on $\phi_\alpha\left(U_\alpha\right)$ and

$$\begin{tikzcd} U_\alpha \arrow[r, "\widetilde{\phi_\alpha}"] \arrow[d, "f_\alpha"'] & \phi_\alpha\left(U_\alpha\right) \arrow[dl, "\widetilde{f_\alpha}"] \\ {[0,1]} & \end{tikzcd}.$$

Then

$$\begin{aligned}
\mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right) &= \mathrm{d}\left(\sum_{j=1}^n \widetilde{f_\alpha} \omega_j \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n\right) \\
&= \sum_{j=1}^n \sum_{k=1}^n \frac{\partial}{\partial x_k}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_k \wedge \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n \\
&= \sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_j \wedge \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n \\
&= \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n,
\end{aligned}$$

so

$$\sum_\alpha \int_{\phi_\alpha(U_\alpha)} \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right) = \sum_\alpha \int_{\mathbb{R}_+^n} \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right),$$

because $\widetilde{f_\alpha} = 0$ outside $\phi_\alpha\left(U_\alpha\right)$. Thus

$$\begin{aligned}
\int_M \mathrm{d}\omega &= \sum_\alpha \int_{\mathbb{R}_+^n} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n \\
&= \sum_\alpha \int_{-\infty}^\infty \dots \int_{-\infty}^\infty \int_0^\infty \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j}\left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_n \, \mathrm{d}x_{n-1} \dots \mathrm{d}x_1 \\
&= \sum_\alpha \sum_{j=1}^n \int_{-\infty}^\infty \dots \widehat{\int_{-\infty}^\infty} \dots \int_{-\infty}^\infty \int_0^\infty (-1)^{j-1} \frac{\partial}{\partial x_j}\left.\left(\widetilde{f_\alpha} \omega_j\right)\right|_{x_n=0} \mathrm{d}x_n \, \mathrm{d}x_{n-1} \dots \widehat{\mathrm{d}x_j} \dots \mathrm{d}x_1 \\
&= \sum_\alpha \int_{-\infty}^\infty \dots \int_{-\infty}^\infty (-1)^{n-1} \left.\left(\widetilde{f_\alpha} \omega_j\right)\right|_{x_n=0} \mathrm{d}x_{n-1} \dots \mathrm{d}x_1,
\end{aligned}$$

since $\left.\left(f_\alpha \omega_j\right)\right|_{x_n=0} = 0$ for $j = 1, \dots, n-1$, so

$$\int_M \mathrm{d}\omega = \sum_\alpha \int_{-\infty}^\infty \dots \int_{-\infty}^\infty (-1)^{n-1} \left.\left(\widetilde{f_\alpha} \omega_j\right)\right|_{x_n=0} \mathrm{d}x_{n-1} \dots \mathrm{d}x_1 = \sum_\alpha \int_{\partial U_\alpha} \left. f_\alpha\right|_{\partial U_\alpha} \omega = \int_{\partial M} \omega,$$

where $\partial U_\alpha = U_\alpha \cap \partial M$. ∎

1.9 Applications of Stokes' theorem

Lecture 7 Thursday 23/01/20

Theorem 1.33 (Integration by parts). *Let M be an orientable n-dimensional manifold with boundary, let $\omega \in \Omega_{\mathrm{c}}^p(M)$, let $\eta \in \Omega_{\mathrm{c}}^{n-p-1}(M)$, and let $p \in \{0, \dots, n-1\}$. Then*

$$\int_{\partial M} \omega \wedge \eta = \int_M \mathrm{d}\omega \wedge \eta + (-1)^p \int_M \omega \wedge \mathrm{d}\eta.$$

Proof.

$$\int_{\partial M} \omega \wedge \eta = \int_M \mathrm{d}(\omega \wedge \eta) = \int_M \left(\mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta\right),$$

by Stokes and the Leibnitz rule. ☐

Theorem 1.34 (Brouwer's fixed point theorem). *Let*

$$D = \{x \in \mathbb{R}^n \mid |x| \le 1\},$$

so

$$\partial D = \mathrm{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\},$$

and let $f : D \to D$ be a smooth morphism. Then f admits a fixed point, that is there exists $x \in D$ such that $f(x) = x$.

Proof. Assume that $f(x) \ne x$ for all $x \in D$. For any $x \in D$, consider the ray starting from $f(x)$ and passing through x. Let $g(x)$ be the point where this ray intersects ∂D away from $f(x)$. Note that if $x \in \partial D$ then $g(x) = x$. Then $g : D \to \partial D$. It is easy to check that g is smooth. Since $\partial D = \mathrm{S}^{n-1}$ is orientable by Proposition 1.23 there exists a volume form $\omega \in \Omega^{n-1}(\partial D)$, so $\omega(x) \ne 0$. Since $\omega \in \Omega^{n-1}(\partial D)$, $\mathrm{d}\omega \in \Omega^n(\partial D)$, which is an n-dimensional manifold, so $\mathrm{d}\omega = 0$. Thus

$$0 < \int_{\partial D} \omega = \int_{\partial D} g^*\omega = \int_D \mathrm{d}g^*\omega = \int_D g^*\mathrm{d}\omega = 0,$$

by Stokes, a contradiction. ☐

Example 1.35. Recall any exact form is closed, since $\mathrm{d}^2 = 0$. But the opposite is not always true. Let $M = \mathbb{R}^2 \setminus \{0\}$, and let

$$\omega = \frac{x}{x^2 + y^2}\mathrm{d}y - \frac{y}{x^2 + y^2}\mathrm{d}x \in \Omega^1(M).$$

Then ω is closed, since

$$\mathrm{d}\omega = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right)\mathrm{d}x \wedge \mathrm{d}y - \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right)\mathrm{d}y \wedge \mathrm{d}x = 0.$$

We want to show that ω is not exact. Assume that

$$\omega = \mathrm{d}f, \qquad f \in \Omega^0(M) = \{\mathrm{C}^\infty\text{-function}\}.$$

In particular $\omega = \mathrm{d}f$ on $\mathrm{S}^1 \subset M$. Let

$$\begin{array}{cccc} \gamma & : & [0, 2\pi] & \longrightarrow & \mathrm{S}^1 \\ & & \theta & \longmapsto & (\cos\theta, \sin\theta) \end{array}.$$

Then

$$\int_{\mathrm{S}^1} \omega = \int_0^{2\pi} \gamma^*\omega = \int_0^{2\pi} \left(\left(\frac{\cos\theta}{\cos^2\theta + \sin^2\theta}\right)\cos\theta\mathrm{d}\theta - \left(\frac{\sin\theta}{\cos^2\theta + \sin^2\theta}\right)(-\sin\theta)\,\mathrm{d}\theta\right) = \int_0^{2\pi} \mathrm{d}\theta = 2\pi,$$

and

$$\int_{\mathrm{S}^1} \omega = \int_{\mathrm{S}^1} \mathrm{d}f = \int_{\partial \mathrm{S}^1} f = \int_{\emptyset} f = 0,$$

so ω is not exact.

Proposition 1.36. *Let M be an orientable manifold of dimension n without boundary, and let $\omega \in \Omega_{\mathrm{c}}^{n}(M)$. Assume ω is exact. Then*

$$\int_M \omega = 0.$$

Proof. Easy from Stokes. ☐

Proposition 1.37. *Let M be an orientable manifold of dimension n with boundary, and let $\omega \in \Omega_{\mathrm{c}}^{n-1}(M)$ be a closed form. Then*

$$\int_{\partial M} \omega = 0.$$

Proof. Easy from Stokes. ☐

Let M be an orientable manifold of dimension n, let $\omega \in \Omega_{\mathrm{c}}^{k}(M)$, and let $N \subset M$ be a submanifold of dimension k. We can define

$$\int_M \omega = \int_N i^* \omega,$$

where $i : N \hookrightarrow M$ is the inclusion. We will denote

$$\omega|_N = i^* \omega \in \Omega_{\mathrm{c}}^{k}(N).$$

Proposition 1.38. *Let M be an oriented manifold of dimension n, let $\omega \in \Omega_{\mathrm{c}}^{k}(M)$, and let $S \subset M$ be a compact orientable submanifold of dimension k such that $\partial S = \emptyset$ and $\int_S \omega \neq 0$. Then*

- *ω is not exact,*
- *$\omega|_S$ is not exact, and*
- *S is not the boundary of an orientable manifold $N \subset M$ of dimension $k+1$.*

Proof. Exercise. [4] ☐

[4] Exercise

2 De Rham cohomology

2.1 De Rham cohomology

Lecture 8 Monday 27/01/20

Definition 2.1. Let M be a manifold of dimension n, and let $p \ge 0$. Then $\omega_1, \omega_2 \in \Omega^p(M)$ are said to be **cohomologous** if $\omega_1 - \omega_2 = \mathrm{d}\eta$ where $\eta \in \Omega^{p-1}(M)$. In particular $\omega \in \Omega^p(M)$ is cohomologous to zero if it is exact. Let

$$\mathcal{Z}^p(M) = \operatorname{Ker}\left(\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)\right) = \{\omega \in \Omega^p(M) \mid \omega \text{ is closed}\} \subset \Omega^p(M),$$

and let

$$\mathcal{B}^p(M) = \operatorname{Im}\left(\mathrm{d} : \Omega^{p-1}(M) \to \Omega^p(M)\right) = \{\omega \in \Omega^p(M) \mid \omega \text{ is exact}\} \subset \Omega^p(M).$$

Then $\mathcal{B}^p(M) \subset \mathcal{Z}^p(M)$ for all $p \ge 0$.

Notation. If $p = 0$, then $\mathcal{B}^0(M) = 0$.

Note. If $\omega_1, \omega_2 \in \mathcal{Z}^p(M)$ then $\omega_1 - \omega_2 \in \mathcal{B}^p(M)$ if and only if ω_1 and ω_2 are cohomologous.

Definition 2.2. Denote the **p-th de Rham cohomology group** as

$$\mathrm{H}^p(M) = \mathcal{Z}^p(M) / \mathcal{B}^p(M) = \{[\omega] \mid \omega \in \mathcal{Z}^p(M)\}, \qquad p \ge 0.$$

where

$$[\omega] = \{\omega' \in \Omega^p(M) \text{ cohomologous to } \omega\}$$

is the **de Rham class** of ω.

Remark. $\mathrm{H}^p(M)$ is a vector space over $\mathbb{R}$.

Definition 2.3. $\mathrm{b}_p(M) = \dim \mathrm{H}^p(M) \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$ is the **p-th Betti number** of M.

Proposition 2.4. *If M is connected then*

$$\mathrm{H}^0(M) = \mathbb{R},$$

that is $\mathrm{b}_0(M) = 1$. More in general, $\mathrm{b}_0(M)$ is the number of connected components of M.

Proof. Assume M is connected. Then $\mathcal{B}^0(M) = 0$, so

$$\begin{aligned} \mathrm{H}^0(M) = \mathcal{Z}^0(M) &= \left\{f \in \Omega^0(M) \text{ closed}\right\} \\ &= \left\{f \in \Omega^0(M) \;\middle|\; \text{locally } \forall x \in M,\ \frac{\partial}{\partial x_i} f(x) = 0\right\} \\ &= \left\{f \in \Omega^0(M) \text{ locally constant}\right\} = \mathbb{R}. \end{aligned}$$

□

Example. Let $M = \mathrm{S}^1$. Then $\mathrm{H}^0(M) = \mathbb{R}$.

Proposition 2.5. *Let M be a manifold of dimension n. Then*

$$\mathrm{H}^p(M) = 0, \qquad p \ge n+1.$$

Proof. Recall $\Omega^p(M) = 0$ if $p \ge n+1$ because all alternating p-forms for $p \ge n+1$ on an n-dimensional vector space are zero, so $\mathcal{Z}^p(M) = 0$. Thus $\mathrm{H}^p(M) = 0$. □

Proposition 2.6. *Let M be a compact orientable manifold of dimension n without boundary. Then*

$$\mathrm{H}^n(M) \ne 0.$$

Proof. M is orientable, so there exists a volume form $\omega \in \Omega^n(M) = \Omega^n_{\mathrm{c}}(M)$, by Proposition 1.23. Then ω is closed, because $\mathrm{d}\omega$ is an $(n+1)$-form on M, so $\omega \in \mathcal{Z}^n(M)$. We want to show that $[\omega] \ne 0$ in $\mathrm{H}^n(M)$. Assume $[\omega] = 0$, so ω is exact. Thus $\omega = \mathrm{d}\eta$ where η is an $(n-1)$-form on M, so

$$0 < \int_M \omega = \int_M \mathrm{d}\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0,$$

by Stokes, a contradiction. □

Proposition 2.7. *Let $G : M \to N$ be a smooth morphism between manifolds. Then*

$$G^* : \Omega^p(N) \to \Omega^p(M), \qquad p \ge 0$$

takes closed forms of N to closed forms on M and exact forms of N to exact forms on M.

Proof. Proposition 1.16 implies that $G^*\mathrm{d} = \mathrm{d}G^*$. If ω is closed then $\mathrm{d}G^*\omega = G^*\mathrm{d}\omega = G^*0 = 0$, so $G^*\omega$ is closed. If $\omega = \mathrm{d}\eta$ is exact then $G^*\omega = \mathrm{d}G^*\eta$ is also exact. □

Thus $G^* : \mathcal{Z}^p(N) \to \mathcal{Z}^p(M)$ and $G^* : \mathcal{B}^p(N) \to \mathcal{B}^p(M)$, so there exists a linear map

$$\begin{array}{rccc} G^* & : & \mathrm{H}^p(N) & \longrightarrow & \mathrm{H}^p(M) \\ & & [\omega] & \longmapsto & [G^*\omega] \end{array}.$$

Corollary 2.8. *Let M and N be diffeomorphic manifolds. Then*

$$\mathrm{H}^p(M) \cong \mathrm{H}^p(N), \qquad p \ge 0,$$

that is $\mathrm{H}^p(M)$ is a diffeomorphic invariant.

Proof. By Proposition 2.7 there exists $F^* : \mathrm{H}^p(N) \to \mathrm{H}^p(M)$ and $\left(F^{-1}\right)^* : \mathrm{H}^p(M) \to \mathrm{H}^p(N)$. By Proposition 1.8,

$$\left(F^{-1}\right)^* F^*\omega = \left(F \circ F^{-1}\right)^* \omega = \mathrm{id}_N^* \omega = \omega, \qquad \omega \in \mathrm{H}^p(N)$$

so $\left(F^{-1}\right)^* \circ F^* = \mathrm{id}_{\mathrm{H}^p(N)}$. Similarly $F^* \circ \left(F^{-1}\right)^* = \mathrm{id}_{\mathrm{H}^p(M)}$, so F^* is an isomorphism. □

2.2 Homotopy invariance

Definition 2.9. Let M_0 and M_1 be manifolds, and let $f_0, f_1 : M_0 \to M_1$ be smooth morphisms. Then f_0 and f_1 are **smoothly homotopic equivalent** if there exists a smooth morphism $H : M_0 \times [0,1] \to M_1$ such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$ for all $x \in M_0$. A **homotopy** is a smooth morphism $H : M_0 \times [0,1] \to M_1$ where M_0 and M_1 are smooth manifolds.

Lecture 9 Tuesday 28/01/20

Notation 2.10. Let $f_t(x) = H(x,t)$, so $f_t : M_0 \to M_1$ is a smooth morphism. Then f_0 and f_1 are said to be homotopic equivalent, denoted by $f_0 \sim f_1$. Then $\sim$ is an equivalence. [5]

Definition 2.11. M_0 and M_1 are **homotopy equivalent** if there exist smooth morphisms $f : M_0 \to M_1$ and $g : M_1 \to M_0$ such that $f \circ g \sim \mathrm{id}_{M_1}$ and $g \circ f \sim \mathrm{id}_{M_0}$.

Example 2.12.

- Let $M_0 = \mathbb{R}^n$ and $M_1 = \{0\}$. Then M_0 and M_1 are homotopy equivalent. Let

$$\begin{array}{rcccc} f & : & M_0 & \longrightarrow & M_1 \\ & & x & \longmapsto & 0 \end{array}, \qquad \begin{array}{rcccc} g & : & M_1 & \longrightarrow & M_0 \\ & & 0 & \longmapsto & 0 \end{array}.$$

  Then

$$\begin{array}{rcccc} f \circ g & : & M_1 & \longrightarrow & M_1 \\ & & 0 & \longmapsto & 0 \end{array},$$

  so $f \circ g = \mathrm{id}_{M_1}$, and

$$\begin{array}{rcccc} g \circ f & : & M_0 & \longrightarrow & M_0 \\ & & x & \longmapsto & 0 \end{array}.$$

  We want to show that $g \circ f \sim \mathrm{id}_{M_0}$. Define a smooth morphism

$$\begin{array}{rcccc} H & : & M_0 \times [0,1] & \longrightarrow & M_0 \\ & & (x,t) & \longmapsto & tx \end{array}.$$

  Then $H(x,0) = 0 = (g \circ f)(x)$ for all x, and $H(x,1) = x = \mathrm{id}_{M_0}(x)$ for all x, so $g \circ f \sim \mathrm{id}_{M_0}$. More in general $M \subset \mathbb{R}^n$ is called **convex** if for all $x, y \in M$ the segment joining x to y is contained inside M. If M is convex then M is homotopy equivalent to $M \times \{0\}$.

[5] Exercise

- Let $M_0 = \mathbb{R}^2 \setminus \{0\}$ and $M_1 = \mathrm{S}^1$. Then M_0 and M_1 are homotopy equivalent. Let

$$f : \begin{array}{ccc} M_0 & \longrightarrow & M_1 \\ x & \longmapsto & \dfrac{x}{|x|} \end{array}, \qquad g : \begin{array}{ccc} M_1 & \longrightarrow & M_0 \\ x & \longmapsto & x \end{array}.$$

Then

$$f \circ g : \begin{array}{ccc} M_1 & \longrightarrow & M_1 \\ x & \longmapsto & x \end{array},$$

so $f \circ g = \mathrm{id}_{M_1}$, and

$$g \circ f : \begin{array}{ccc} M_0 & \longrightarrow & M_0 \\ x & \longmapsto & \dfrac{x}{|x|} \end{array}.$$

Let

$$H : \begin{array}{ccc} M_0 \times [0,1] & \longrightarrow & M_0 \\ (x,t) & \longmapsto & tx + (1-t)\dfrac{x}{|x|} \end{array}$$

be smooth. Then $H(x,0) = x/|x| = (g \circ f)(x)$ and $H(x,1) = x = \mathrm{id}_{M_0}(x)$, so $g \circ f \sim \mathrm{id}_{M_0}$.

Proposition 2.13. *Let M and N be manifolds, and let $H : M \times [0,1] \to N$ be smooth. Denote*

$$f_t : \begin{array}{ccc} M & \longrightarrow & N \\ x & \longmapsto & H(x,t) \end{array}, \qquad t \in [0,1].$$

Then $f_t^* : \mathrm{H}^p(N) \to \mathrm{H}^p(M)$ does not depend on t for all $p \ge 0$.

Proof. Let $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$. The goal is $f_{t_1}^*[\eta] = f_{t_2}^*[\eta]$ for all $[\eta] \in \mathrm{H}^p(N)$. Let

$$i_k : \begin{array}{ccc} M & \longrightarrow & M \times [0,1] \\ x & \longmapsto & (x, t_k) \end{array}, \qquad k = 1, 2.$$

Claim that for all p there exists a linear map $h : \Omega^p(M \times [t_1,t_2]) \to \Omega^{p-1}(M)$ such that

$$\mathrm{d}(h(\omega)) + h(\mathrm{d}\omega) = i_2^*\omega - i_1^*\omega \in \Omega^p(M), \qquad \omega \in \Omega^p(M \times [0,1]). \tag{2}$$

Step 1. The claim implies the proposition. Let $\eta \in \Omega^p(N)$ be closed, so $\mathrm{d}\eta = 0$. Then $H^*\eta$ is also closed, so let $\omega = H^*\eta \in \Omega^p(M \times [t_1,t_2])$. Apply h. Then $\mathrm{d}\omega = 0$, so $\mathrm{d}(h(\omega)) = i_2^*\omega - i_1^*\omega$ is exact. Thus

$$f_{t_1}^*[\eta] = \left[f_{t_1}^*\eta\right] = [i_1^* H^*\eta] = [i_1^*\omega] = [i_2^*\omega] = [i_2^* H^*\eta] = \left[f_{t_2}^*\eta\right] = f_{t_2}^*[\eta],$$

so the proposition follows.

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Step 2. The proof of the claim. Let $\omega \in \Omega^p(M \times [t_1,t_2])$. Then for all $(x,t) \in M \times [t_1,t_2]$, $\omega(x,t)$ is an alternating p-form on $\mathrm{T}_{(x,t)}(M \times [t_1,t_2])$. We want an alternating $(p-1)$-form $h(\omega)(x)$ on $\mathrm{T}_x M$. Let $v_1, \dots, v_{p-1} \in \mathrm{T}_x M$. Then

$$h(\omega)(x)(v_1, \dots, v_{p-1}) = \int_{t_1}^{t_2} \omega(x,t)\left(\frac{\partial}{\partial t}, v_1, \dots, v_{p-1}\right) \mathrm{d}t$$

is a $(p-1)$-form on M, and $\frac{\partial}{\partial t}$ is a global vector field. Check h is linear. [6] It is enough to prove (2) locally. Remark that exactness is not a local property. Fix local coordinates $(x_1, \dots, x_n, t)$ around a point of $M \times [0,1]$. Then

$$\omega = \sum_{|I|=p} \omega_I + \sum_{|J|=p-1} \omega_J, \qquad \omega_I = g_I \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad \omega_J = g_J \mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}} \wedge \mathrm{d}t,$$

where g_I and g_J are smooth functions. Any term of (2) is linear. So we just need to check (2) holds for ω_I and ω_J.

[6] Exercise

ω_I. Let $\omega = g(x,t)\,\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}$. Then

$$\mathrm{d}\left(h\left(\omega(x,t)\left(\frac{\partial}{\partial t}, v_1, \dots, v_{p-1}\right)\right)\right) = \mathrm{d}(h(0)) = 0,$$

and

$$\begin{aligned} h(\mathrm{d}\omega) &= h\left(\frac{\partial}{\partial t} g(x,t)\,\mathrm{d}t \wedge \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p} + \sum_{j=1}^n \frac{\partial}{\partial x_j} g(x,t)\,\mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}\right) \\ &= \left(\int_{t_1}^{t_2} \frac{\partial}{\partial t} g(x,t)\ \mathrm{d}t\right)\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p} + 0 \\ &= (g(x,t_2) - g(x,t_1))\,\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p} = i_2^*\omega - i_1^*\omega, \end{aligned}$$

so (2) holds.

ω_J. Let $\omega = g(x,t)\,\mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}} \wedge \mathrm{d}t$. Then

$$\begin{aligned} \mathrm{d}(h(\omega)) &= (-1)^{p-1}\,\mathrm{d}\left(\left(\int_{t_1}^{t_2} g(x,t)\ \mathrm{d}t\right)\mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}}\right) \\ &= (-1)^{p-1} \sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\int_{t_1}^{t_2} g(x,t)\ \mathrm{d}t\right)\mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}} \\ &= (-1)^{p-1} \sum_{j=1}^n \left(\int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x,t)\ \mathrm{d}t\right)\mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}}, \end{aligned}$$

and

$$\begin{aligned} h(\mathrm{d}\omega) &= h\left(\sum_{j=1}^n \frac{\partial}{\partial x_j} g(x,t)\,\mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}} \wedge \mathrm{d}t + 0\right) \\ &= (-1)^{p-1} \sum_{j=1}^n \left(\int_{t_1}^{t_2} \frac{\partial}{\partial x_j} g(x,t)\ \mathrm{d}t\right)\mathrm{d}x_j \wedge \mathrm{d}x_{j_1} \wedge \dots \wedge \mathrm{d}x_{j_{p-1}} = -\mathrm{d}(h(\omega)), \end{aligned}$$

and $i_2^*\omega = i_1^*\omega = 0$, so (2) holds.

$\square$

Corollary 2.14. *Assume M and N are homotopy equivalent. Then there exist isomorphisms*

$$\mathrm{H}^p(N) \to \mathrm{H}^p(M), \qquad p \ge 0.$$

Proof. There exist $f : M \to N$ and $g : N \to M$ such that $g \circ f \sim \mathrm{id}_M$ and $f \circ g \sim \mathrm{id}_N$. By Proposition 2.13 $(g \circ f)^* : \mathrm{H}^p(M) \to \mathrm{H}^p(M)$ coincides with $\mathrm{id}_M^* = \mathrm{id}_{\mathrm{H}^p(M)}$. Then $f^* \circ g^* = (g \circ f)^* = \mathrm{id}_{\mathrm{H}^p(M)}$. Similarly $g^* \circ f^* = \mathrm{id}_{\mathrm{H}^p(N)}$, so g^* and f^* are isomorphisms. $\square$

Definition 2.15. Let M be a manifold. Then M is **smoothly contractible** if M is homotopy equivalent to a point.

Example. $\mathbb{R}^n$ is contractible, by Example 2.12. If $M \subset \mathbb{R}^n$ is convex then M is contractible.

Theorem 2.16 (Poincaré lemma). *If M is a contractible manifold then*

$$\mathrm{H}^p(M) = 0, \qquad p \ge 1.$$

Proof. By previous Corollary 2.14, there exists an isomorphism $\mathrm{H}^p(M) \to \mathrm{H}^p(\{\text{point}\})$. Then $\{\text{point}\}$ is a zero-dimensional manifold, so by Proposition 2.5, $\mathrm{H}^p(\{\text{point}\}) = 0$ for all $p > 0$. $\square$

Thus $\mathrm{H}^p(\mathbb{R}^n) = 0$ for all $p > 0$, so $\mathbb{R}^n$ is not diffeomorphic to any compact orientable manifold. Lecture 11 Monday 03/02/20

Proposition 2.17. *Let M be a manifold, and let $\omega \in \Omega^p(M)$ be a closed p-form for $p > 0$. Then for all $x \in X$, there exists a neighbourhood $U \ni x$ such that ω is exact on U, that is there exists $\eta \in \Omega^{p-1}(U)$ such that $\omega = \mathrm{d}\eta$ on U.*

Proof. Let (U, ϕ) be a chart around x. I may assume that $V = \phi(U)$ is a ball in $\mathbb{R}^n$. Then U is diffeomorphic to $B = \{z \mid |z - z_0| < r\}$ for some $z_0 \in \mathbb{R}^n$ and $r > 0$, so $\mathrm{H}^p(U) \cong \mathrm{H}^p(B)$ for all $p \geq 0$. Since B is contractible, $\mathrm{H}^p(B) = 0$ for all $p > 0$. The restriction of ω on U gives a class $[\omega] \in \mathrm{H}^p(U) = 0$, so ω is cohomologous to zero on U. Thus ω is exact on U. ◻

Definition 2.18. Let M be a manifold, let $\gamma : [0,1] \to M$ be a continuous or smooth path, and let $x = \gamma(0)$ and $y = \gamma(1)$. A **homotopy of paths** from x to y is a map

$$\begin{array}{cccccc} F & : & [0,1] \times [0,1] & \longrightarrow & M & \\ & & (0,t) & \longmapsto & x & . \\ & & (1,t) & \longmapsto & y & \end{array}$$

Proposition 2.19. *Let γ_0 and γ_1 be homotopic paths on a manifold M, and let $\omega \in \Omega^1(M)$ be closed. Then*

$$\int_0^1 \gamma_0^* \omega = \int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

Proof. Lee's introduction to smooth manifolds. The idea is

$$\int_{\gamma_0} \omega - \int_{\gamma_1} \omega = \int_{\gamma_0 \cup \gamma_1} \omega = \int_{\operatorname{Im} F} \omega = 0,$$

by Stokes' theorem. ◻

Recall that M is **simply connected**, so $\pi_1(M) = 0$, if any path γ from x to x is homotopic equivalent to a point.

Proposition 2.20. *Let M be a simply connected orientable manifold. Then*

$$\mathrm{H}^1(M) = 0.$$

Proof. Let $\omega \in \Omega^1(M)$ be a closed form. Then claim that ω is exact if and only if $\int_\gamma \omega = 0$ for all loops γ, that is paths from x to x.

- The proof of the claim. Assume that $\omega = \mathrm{d}f$ is exact for $f \in \Omega^0(M)$. By Proposition 2.19,

  $$\int_\gamma \omega = \int_{\text{trivial loop}} \omega = 0.$$

  Assume that $\int_\gamma \omega = 0$ for all loops γ. Fix x. Let

  $$f(y) = \int_x^y \omega.$$

  Since $\int_{\gamma_1 \cup \gamma_2} \omega = 0$, f is well-defined, that is it does not depend on the choice of the path. Then $\mathrm{d}f = \omega$. This can be checked locally, that is in an open set of $\mathbb{R}^n$. Here it follows from the fundamental theorem of calculus.

- The claim implies the proposition. Being simply connected, any loop inside M is homotopic equivalent to the trivial loop. For all loops γ and for all closed ω, $\int_\gamma \omega = 0$ by Proposition 2.19, so ω is exact. Thus $[\omega] = 0$ in $\mathrm{H}^1(M)$.

◻

2.3 Some homological algebra

Let $C^\bullet$ be a sequence of vector spaces, that is C^k is a vector space for $k \in \mathbb{Z}$.

Definition 2.21. $(C^\bullet, d^\bullet)$ is a **cochain complex** if $C^\bullet$ is a sequence of vector spaces and $d^\bullet$ is a sequence of linear maps $d^k : C^k \to C^{k+1}$ such that the composition $d^{k+1} \circ d^k : C^k \to C^{k+1} \to C^{k+2}$ is zero for all k. Then $d^\bullet$ is the **differential**.

Definition 2.22. The elements of

$$\mathcal{Z}^k (C^\bullet, d^\bullet) = \mathrm{Ker}\left(d^k : C^k \to C^{k+1}\right) \subset C^k$$

are called **cocycles**. The elements of

$$\mathcal{B}^k (C^\bullet, d^\bullet) = \mathrm{Im}\left(d^k : C^{k-1} \to C^k\right) \subset C^k$$

are called **coboundaries**. Then $d^{k-1} \circ d^k = 0$, so $\mathcal{B}^k \subset \mathcal{Z}^k$. The quotients

$$\mathrm{H}^k (C^\bullet, d^\bullet) = \mathcal{Z}^k (C^\bullet, d^\bullet) / \mathcal{B}^k (C^\bullet, d^\bullet)$$

are the k**-th cohomology groups** of $(C^\bullet, d^\bullet)$.

Definition 2.23. Let $(C^\bullet, d^\bullet)$ and $(D^\bullet, d^\bullet)$ be two cochain complexes. A map $f : (C^\bullet, d^\bullet) \to (D^\bullet, d^\bullet)$ is a sequence of linear maps $f^k : C^k \to D^k$ such that $f^{k+1} \circ d^k = d^k \circ f^k$ for all k, so

$$\begin{tikzcd} \dots \arrow[r] & C^k \arrow[r, "d^k"] \arrow[d, "f^k"] & C^{k+1} \arrow[r, "d^{k+1}"] \arrow[d, "f^{k+1}"] & C^{k+2} \arrow[r] \arrow[d, "f^{k+2}"] & \dots \\ \dots \arrow[r] & D^k \arrow[r, "d^k"] & D^{k+1} \arrow[r, "d^{k+1}"] & D^{k+2} \arrow[r] & \dots \end{tikzcd} .$$

Proposition 2.24. *Let $f : (C^\bullet, d^\bullet) \to (D^\bullet, d^\bullet)$ be a map between cochain complexes. Then there exists a natural induced map*

$$f^k : \mathrm{H}^k (C^\bullet, d^\bullet) \to \mathrm{H}^k (D^\bullet, d^\bullet).$$

Proof. Let $[\omega] \in \mathrm{H}^k (C^\bullet, d^\bullet) = \mathcal{Z}^k (C^\bullet, d^\bullet) / \mathcal{B}^k (C^\bullet, d^\bullet)$ for $\omega \in \mathcal{Z}^k (C^\bullet, d^\bullet)$, that is $d^k (\omega) = 0$. I want to check that $f^k (\omega) \in \mathcal{Z}^k (D^\bullet, d^\bullet)$. By definition of maps, $d^k \left(f^k (\omega)\right) = f^{k+1} \left(d^k (\omega)\right) = 0$, so there is a map

$$\mathcal{Z}^k (C^\bullet, d^\bullet) \to \mathcal{Z}^k (D^\bullet, d^\bullet).$$

Now I need to check that if $\omega \in \mathcal{B}^k (C^\bullet, d^\bullet)$ then $f^k (\omega) \in \mathcal{B}^k (D^\bullet, d^\bullet)$. [7] $\square$

Lecture 12 Tuesday 04/02/20

Definition 2.25. A sequence of linear maps

$$C^1 \xrightarrow{f^1} C^2 \to \dots \to C^{k-1} \xrightarrow{f^{k-1}} C^k$$

between vector spaces is said to be **exact** if for all i, $\mathrm{Ker}\, f^i = \mathrm{Im}\, f^{i-1}$.

Example 2.26.

- A sequence
$$0 \to C^1 \xrightarrow{f^1} C^2$$
is exact if and only if f^1 is injective.
- A sequence
$$C^1 \xrightarrow{f^1} C^2 \to 0$$
is exact if and only if f^1 is surjective.
- An exact sequence
$$0 \to C^1 \xrightarrow{f^1} C^2 \xrightarrow{f^2} C^3 \to 0$$
is called a **short exact sequence**. In particular f^1 is injective and f^2 is surjective.

[7] Exercise

- Any long exact sequence
$$C^1 \xrightarrow{f^1} C^2 \to \dots \to C^{q-1} \xrightarrow{f^{q-1}} C^q$$
can be split into short exact sequences
$$\begin{tikzcd}
\dots \arrow[r] & C^{k-1} \arrow[rr, "f^{k-1}"] \arrow[dr] & & C^k \arrow[rr, "f^k"] \arrow[dr] & & C^{k+1} \arrow[r] & \dots \\
& & \operatorname{Im} f^{k-1} = \operatorname{Ker} f^k \arrow[ur] \arrow[dr] & & \operatorname{Im} f^k = \operatorname{Ker} f^{k+1} \arrow[ur] \arrow[dr] & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}, \qquad k = 2, \dots, q-1.$$

Lemma 2.27 (Snake lemma)**.** *Consider the commutative diagram*
$$\begin{tikzcd}
& C^1 \arrow[r, "f^1"] \arrow[d, "\alpha_1"] & C^2 \arrow[r, "f^2"] \arrow[d, "\alpha_2"] & C^3 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & D^1 \arrow[r, "g^1"] & D^2 \arrow[r, "g^2"] & D^3 &
\end{tikzcd},$$
such that both the horizontal lines are exact sequences. Then there exists a long exact sequence
$$\operatorname{Ker} \alpha_1 \to \operatorname{Ker} \alpha_2 \to \operatorname{Ker} \alpha_3 \xrightarrow{\delta} \operatorname{Coker} \alpha_1 \to \operatorname{Coker} \alpha_2 \to \operatorname{Coker} \alpha_3.$$
If
$$\begin{tikzcd}
0 \arrow[r] & C^1 \arrow[r, "f^1"] \arrow[d, "\alpha_1"] & C^2 \arrow[r, "f^2"] \arrow[d, "\alpha_2"] & C^3 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & D^1 \arrow[r, "g^1"] & D^2 \arrow[r, "g^2"] & D^3 \arrow[r] & 0
\end{tikzcd},$$
then
$$0 \to \operatorname{Ker} \alpha_1 \to \operatorname{Ker} \alpha_2 \to \operatorname{Ker} \alpha_3 \xrightarrow{\delta} \operatorname{Coker} \alpha_1 \to \operatorname{Coker} \alpha_2 \to \operatorname{Coker} \alpha_3 \to 0.$$

Proof. We are going to construct $\delta : \operatorname{Ker} \alpha_3 \to \operatorname{Coker} \alpha_1$. Let $x \in \operatorname{Ker} \alpha_3$. There exists $y \in C^2$ such that $f^2(y) = x$ because f^2 is surjective. Let $z = \alpha_2(y)$ then
$$g^2(z) = g^2(\alpha_2(y)) = \alpha_3\left(f^2(y)\right) = \alpha_3(x) = 0,$$
since $x \in \operatorname{Ker} \alpha_3$. Then $z \in \operatorname{Ker} g^2 = \operatorname{Im} g^1$, so there exists $w \in D^1$ such that $z = g^1(w)$. The idea is
$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& \operatorname{Ker} \alpha_1 \arrow[r] \arrow[d] & \operatorname{Ker} \alpha_2 \arrow[r] \arrow[d] & \operatorname{Ker} \alpha_3 \arrow[d] \arrow[ddll, bend left, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- (\tikztotarget.west)}] & \\
& C^1 \arrow[r, "f^1"] \arrow[d, "\alpha_1"] & y \in C^2 \arrow[r, "f^2"] \arrow[d, "\alpha_2"] & x \in C^3 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & w \in D^1 \arrow[r, "g^1"'] \arrow[d] & z \in D^2 \arrow[r, "g^2"'] \arrow[d] & D^3 \arrow[d] & \\
& \operatorname{Coker} \alpha_1 \arrow[r] \arrow[d] & \operatorname{Coker} \alpha_2 \arrow[r] \arrow[d] & \operatorname{Coker} \alpha_3 \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}.$$
Define $\delta(x) = [w] \in \operatorname{Coker} \alpha^1 = D^1 / \operatorname{Im} \alpha^1$. Need to check that δ is well-defined, so $[w]$ does not depend on our choice of w and y. The rest is an exercise. [8] ☐

[8] Exercise

2.4 The Mayer-Vietoris sequence

The idea is given a manifold M, we may write $M = U \cup V$ with open U and V so that $H^i(U)$, $H^i(V)$, and $H^i(U \cap V)$ are easy to compute, so this will give us $H^i(M)$. Let M be a manifold, and let U and V be open such that $M = U \cup V$. Assume $U \cap V \neq \emptyset$. Let

$$i_U : U \to M, \qquad i_V : V \to M, \qquad j_U : U \cap V \to U, \qquad j_V : U \cap V \to V$$

be inclusions, and let $i_U^*, i_V^*, j_U^*, j_V^*$ be pull-backs.

Proposition 2.28. *For all p there exist short exact sequences*

$$0 \to \Omega^p(M) \xrightarrow{f} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{g} \Omega^p(U \cap V) \to 0,$$

where $f = (i_U^*, i_V^*)$ and $g = j_V^* - j_U^*$. More precisely, if $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^p(V)$ then $g(\omega_1, \omega_2) = j_V^*\omega_2 - j_U^*\omega_1$.

Proof.

- f is injective. Assume $\omega \in \Omega^p(M)$ such that $f(\omega) = 0$, so $i_U^*\omega = i_V^*\omega = 0$. Since $M = U \cup V$ then $\omega = 0$ on M, so f is injective.
- $\mathrm{Im}\, f = \mathrm{Ker}\, g$. Let $f(\omega) \in \mathrm{Im}\, f$, so $f(\omega) = (i_U^*\omega, i_V^*\omega)$. Then $g(f(\omega)) = j_V^* i_V^*\omega - j_U^* i_U^*\omega = l^*\omega - l^*\omega = 0$, where

$$\begin{tikzcd} & U \arrow[dr, "i_U"] & \\ U \cap V \arrow[ur, "j_U"] \arrow[rr, "l"] \arrow[dr, "j_V"'] & & M \\ & V \arrow[ur, "i_V"'] & \end{tikzcd},$$

  so $\mathrm{Im}\, f \subset \mathrm{Ker}\, g$. Now let $(\omega_1, \omega_2) \in \mathrm{Ker}\, g$, so $j_V^*\omega_2 = j_U^*\omega_1$ for $\omega_1 \in \Omega^p(U)$ and $\omega_2 \in \Omega^p(V)$. The restriction of ω_2 on $U \cap V$ coincides with the restriction of ω_1 on $U \cap V$. Then define

$$\omega = \begin{cases} \omega_1 & \text{on } U \\ \omega_2 & \text{on } V \end{cases}.$$

  Then $f(\omega) = (\omega_1, \omega_2)$, so $\mathrm{Ker}\, g \subset \mathrm{Im}\, f$.
- g is surjective. Let $\eta \in \Omega^p(U \cap V)$, and let $\{f_U, f_V\}$ be a partition of unity with respect to $\{U, V\}$. Then $\mathrm{supp}\, f_U \subset U$ and $f_U + f_V = 1$. Let $\eta_1 \in \Omega^p(U)$ be defined by

$$\eta_1 = \begin{cases} f_V \cdot \eta & \text{on } U \cap V \\ 0 & \text{outside } \mathrm{supp}\, f_V \end{cases},$$

  and let $\eta_2 \in \Omega^p(V)$ be defined by

$$\eta_2 = \begin{cases} f_U \cdot \eta & \text{on } U \cap V \\ 0 & \text{outside } \mathrm{supp}\, f_U \end{cases}.$$

  Then $g(-\eta_2, \eta_1) = \eta_1|_{U \cap V} + \eta_2|_{U \cap V} = (f_U + f_V) \cdot \eta = \eta$, so $\eta \in \mathrm{Im}\, g$.

$\square$

Lecture 13 Thursday 06/02/20

Theorem 2.29 (Mayer-Vietoris). *Let M be a manifold, and let U and V be open in M such that $M = U \cup V$ and $U \cap V \neq \emptyset$. Then for all $p \geq 0$ there exists a linear $\delta : \mathrm{H}^p(U \cap V) \to \mathrm{H}^{p+1}(M)$ such that*

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}^p(M) \arrow[r, "(i_U^*, i_V^*)"] & \mathrm{H}^p(U) \oplus \mathrm{H}^p(V) \arrow[r, "j_V^* - j_U^*"] & \mathrm{H}^p(U \cap V) \arrow[dll, "\delta"'] \\ & \mathrm{H}^{p+1}(M) \arrow[r, "(i_U^*, i_V^*)"] & \mathrm{H}^{p+1}(U) \oplus \mathrm{H}^{p+1}(V) \arrow[r, "j_V^* - j_U^*"] & \mathrm{H}^{p+1}(U \cap V) \arrow[r] & \dots \end{tikzcd}$$

is exact.

Example 2.30. Let $M = \mathrm{S}^1$, let $N = (0,1)$ and $S = (0,-1)$, and let $U = M \setminus \{N\}$ and $V = M \setminus \{S\}$, so $M = U \cup V$ and $U \cap V = M \setminus \{N, S\}$. Then

$$\mathrm{H}^p(U) \cong \mathrm{H}^p(V) \cong \mathrm{H}^p((0,1)) \cong \begin{cases} \mathbb{R} & p = 0 \\ 0 & p > 0 \end{cases}, \qquad (0,1) \subset \mathbb{R},$$

and

$$\mathrm{H}^p(U \cap V) = \mathrm{H}^p(U \setminus \{S\}) = \mathrm{H}^p\left(\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)\right) = \begin{cases} \mathbb{R}^2 & p = 0 \\ 0 & p > 0 \end{cases}, \qquad \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right) \subset \mathbb{R},$$

so

$$\begin{array}{ccccccccccccc} 0 \to & \mathrm{H}^0(M) & \to & \mathrm{H}^0(U) \oplus \mathrm{H}^0(V) & \xrightarrow{\phi} & \mathrm{H}^0(U \cap V) & \xrightarrow{\delta} & \mathrm{H}^1(M) & \to & \mathrm{H}^1(U) \oplus \mathrm{H}^1(V) & \to & \mathrm{H}^1(U \cap V) & \to \dots \\ & \wr\| & & \wr\| & & \wr\| & & & & \wr\| & & \wr\| & \\ & \mathbb{R} & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R}^2 & & & & 0 \oplus 0 & & 0 & \end{array}.$$

Then $\operatorname{Im} \phi = \mathbb{R} \subset \mathrm{H}^0(U \cap V) = \mathbb{R}^2$. Thus

$$\mathrm{H}^1(M) = \operatorname{Coker} \phi = \mathbb{R}^2 / \operatorname{Im} \phi \cong \mathbb{R}.$$

Remark 2.31. Let

$$0 \to C^1 \to \dots \to C^k \to 0$$

be an exact sequence. Then

$$\sum_k (-1)^k \dim C^k = 0.$$ [9]

In our $M = \mathrm{S}^1$ case $1 - 2 + 2 - \dim \mathrm{H}^1(M) = 0$, so $\dim \mathrm{H}^1(M) = 1$. Thus $\mathrm{H}^1(M) \cong \mathbb{R}$.

Example 2.32. Let $M = \mathrm{S}^n \subset \mathbb{R}^{n+1}$ be the n-dimensional sphere. Then

$$\mathrm{H}^p(M) = \begin{cases} \mathbb{R} & p = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

By induction on n.

$n = 1$. Ok.

$n > 1$. Let $U = M \setminus \{N\}$ and $V = M \setminus \{S\}$, so $U \cap V \neq \emptyset$ and $U \cup V = M$. Then

$$U \cong V \cong \mathbb{R}^n, \qquad U \cap V = V \setminus \{N\} \cong \mathbb{R}^n \setminus \{0\} \cong \mathrm{S}^{n-1},$$

so

$$\begin{array}{ccccccccccc} 0 \longrightarrow & \mathrm{H}^0(M) & \longrightarrow & \mathrm{H}^0(U) \oplus \mathrm{H}^0(V) & \longrightarrow & \mathrm{H}^0(U \cap V) & \xrightarrow{\delta} & \mathrm{H}^1(M) & \longrightarrow & \mathrm{H}^1(U) \oplus \mathrm{H}^1(V) & \longrightarrow \dots \\ & \wr\| & & \wr\| & & \wr\| & & & & \wr\| & \\ & \mathbb{R} & & \mathbb{R} \oplus \mathbb{R} & & \mathbb{R} & & & & 0 \oplus 0 & \end{array}.$$

Then $1 - 2 + 1 - \dim \mathrm{H}^1(M) = 0$, so $\dim \mathrm{H}^1(M) = 0$. Thus $\mathrm{H}^1(M) = 0$. Then for $p > 0$

$$\begin{array}{ccccccccc} \dots \longrightarrow & \mathrm{H}^p(U) \oplus \mathrm{H}^p(V) & \longrightarrow & \mathrm{H}^p(U \cap V) & \xrightarrow{\delta} & \mathrm{H}^{p+1}(M) & \longrightarrow & \mathrm{H}^{p+1}(U) \oplus \mathrm{H}^{p+1}(V) & \longrightarrow \dots \\ & \wr\| & & & & & & \wr\| & \\ & 0 \oplus 0 & & & & & & 0 \oplus 0 & \end{array}$$

is exact, so $\mathrm{H}^p(U \cap V) \cong \mathrm{H}^{p+1}(M)$. By induction

$$\mathrm{H}^p(U \cap V) = \mathrm{H}^{p+1}(M) = \begin{cases} \mathbb{R} & p = n - 1 \\ 0 & \text{otherwise} \end{cases}.$$

[9] Exercise

Proof of Theorem 2.29. By Proposition 2.28 for all p

$$\begin{tikzcd} 0 \arrow[r] & \Omega^p(M) \arrow[r] \arrow[d, "\mathrm{d}_M^p"] & \Omega^p(U) \oplus \Omega^p(V) \arrow[r] \arrow[d, "(\mathrm{d}_U^p, \mathrm{d}_V^p)"] & \Omega^p(U \cap V) \arrow[r] \arrow[d, "\mathrm{d}_{U \cap V}^p"] & 0 \\ 0 \arrow[r] & \Omega^{p+1}(M) \arrow[r] & \Omega^{p+1}(U) \oplus \Omega^{p+1}(V) \arrow[r] & \Omega^{p+1}(U \cap V) \arrow[r] & 0 \end{tikzcd}$$

are exact. Recall d commutes with the pull-back. By the strong snake lemma,

$$\begin{tikzcd} & \operatorname{Coker} \mathrm{d}_M^{p-1} \arrow[r] \arrow[d, "\partial_M^p = \mathrm{d}_M^p"] & \operatorname{Coker} \left(\mathrm{d}_U^{p-1}, \mathrm{d}_V^{p-1}\right) \arrow[r] \arrow[d, "(\partial_U^p, \partial_V^p) = (\mathrm{d}_U^p, \mathrm{d}_V^p)"] & \operatorname{Coker} \mathrm{d}_{U \cap V}^{p-1} \arrow[r] \arrow[d, "\partial_{U \cap V}^p = \mathrm{d}_{U \cap V}^p"] & 0 \\ 0 \arrow[r] & \operatorname{Ker} \mathrm{d}_M^{p+1} \arrow[r] & \operatorname{Ker} \left(\mathrm{d}_U^{p+1}, \mathrm{d}_V^{p+1}\right) \arrow[r] & \operatorname{Ker} \mathrm{d}_{U \cap V}^{p+1} & \end{tikzcd},$$

which is well-defined, since $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$. By the weak snake lemma again,

$$\operatorname{Ker} \partial_M^p \to \operatorname{Ker} (\partial_U^p, \partial_V^p) \to \operatorname{Ker} \partial_{U \cap V}^p \xrightarrow{\delta} \operatorname{Coker} \partial_M^p \to \operatorname{Coker} (\partial_U^p, \partial_V^p) \to \operatorname{Coker} \partial_{U \cap V}^p.$$

Then $\operatorname{Coker} \mathrm{d}_M^{p-1} = \Omega^p(M) / \operatorname{Im} \mathrm{d}_M^{p-1}$. There exists

$$\mathrm{H}^p(M) = \operatorname{Ker} \mathrm{d}_M^p / \operatorname{Im} \mathrm{d}_M^{p-1} \xrightarrow{\sim} \operatorname{Ker} \left(\Omega^p(M) / \operatorname{Im} \mathrm{d}_M^{p-1} \to \operatorname{Ker} \mathrm{d}_M^{p+1}\right) = \operatorname{Ker} \partial_M^p.$$

Similarly, $\operatorname{Ker} (\partial_U^p, \partial_V^p) \cong \mathrm{H}^p(U) \oplus \mathrm{H}^p(V)$ and $\operatorname{Ker} \partial_{U \cap V}^p \cong \mathrm{H}^p(U \cap V)$. There exists

$$\mathrm{H}^{p+1}(M) = \operatorname{Ker} \mathrm{d}_M^{p+1} / \operatorname{Im} \mathrm{d}_M^p \xrightarrow{\sim} \operatorname{Coker} \left(\Omega^p(M) / \operatorname{Im} \mathrm{d}_M^{p-1} \to \operatorname{Ker} \mathrm{d}_M^{p+1}\right) = \operatorname{Coker} \partial_M^p.$$

Similarly, $\operatorname{Coker} (\partial_U^p, \partial_V^p) \cong \mathrm{H}^{p+1}(U) \oplus \mathrm{H}^{p+1}(V)$ and $\operatorname{Coker} \partial_{U \cap V}^p \cong \mathrm{H}^{p+1}(U \cap V)$. □

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Example 2.33. Let $\mathbb{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$ be the torus. Then

$$\mathrm{H}^p\left(\mathbb{T}^2\right) = \begin{cases} \mathbb{R} & p = 0, 2 \\ \mathbb{R} \oplus \mathbb{R} & p = 1 \end{cases}.$$

We leave the proof as an exercise. [10]

Definition 2.34. Let M be a manifold, and let $U = \{U_i\}$ be an open cover of M. Then U is said to be **good** if for all $I = (i_1, \dots, i_p)$, $U_{i_1} \cap \dots \cap U_{i_p}$ is either $\emptyset$ or contractible.

Lemma 2.35. *Let M be a connected manifold which admits a finite good cover. Then for all $p \geq 0$, $\mathrm{H}^p(M)$ is a finite dimensional vector space.*

Exercise. Find a counterexample without assuming there exists a finite good cover.

Proof. Let U be a finite good cover. Define $k = \#U$. By induction on k.

$k = 1$. $M = U_1$ is contractible, so

$$\mathrm{H}^p(M) = \begin{cases} \mathbb{R} & p = 0 \\ 0 & \text{otherwise} \end{cases}.$$

$k > 1$. Assume ok for covers with at most $k-1$ elements. Let $U = \bigcup_{i=1}^{k-1} U_i$ and $V = U_k$. Then $U \cup V = M$ and $U \cap V \neq \emptyset$, so Mayer-Vietoris holds. By induction $\mathrm{H}^p(U)$ and $\mathrm{H}^p(V)$ are finite dimensional, since $\mathrm{H}^p(U)$ is covered by $k-1$ of U_i and $\mathrm{H}^p(V)$ is contractible. Then $U \cap V = \bigcup_{i=1}^{k-1} (U_i \cap U_k)$, and $\{U_i \cap U_k\}$ is a good cover of $U \cap V$ with $k-1$ elements. [11] By induction $\mathrm{H}^p(U \cap V)$ is finite dimensional. Thus $\mathrm{H}^p(M)$ is also finite dimensional.

□

[10] Exercise

[11] Exercise

Fact. Any manifold admits a good cover.

Theorem 2.36. *Let M be a compact connected manifold. Then $\mathrm{H}^p(M)$ is finite dimensional.*

Proof. Follows from the fact and Lemma 2.35. ☐

2.5 Compactly supported de Rham cohomology

Let M be a manifold, and let $\omega \in \Omega_c^p(M)$. Then $d\omega \in \Omega_c^{p+1}(M)$ and $d^2 = 0$, so

$$\Omega_c^p(M) \xrightarrow{d} \Omega_c^{p+1}(M) \xrightarrow{d} \dots.$$

Let

$$\mathrm{H}_c^p(M) = \mathcal{Z}_c^p(M) / \mathcal{B}_c^p(M) = \mathrm{Ker}\left(d : \Omega_c^p(M) \to \Omega_c^{p+1}(M)\right) / \mathrm{Im}\left(d : \Omega_c^{p-1}(M) \to \Omega_c^p(M)\right).$$

Example. If M is compact, then

$$\mathrm{H}_c^p(M) = \mathrm{H}^p(M), \qquad p \ge 0.$$

Lemma 2.37. *Let M be a non-compact connected manifold. Then*

$$\mathrm{H}_c^0(M) = 0.$$

Recall if M is connected $\mathrm{H}^0(M) = \mathbb{R}$.

Proof. $\mathrm{H}^0(M) = \{f \text{ constant on } M\}$ and $\mathrm{H}_c^0(M) = \{f \text{ constant on } M \text{ and with compact support}\}$. Since M is non-compact, if $f \in \Omega_c^0(M)$, then $\mathrm{supp}\, f \subsetneq M$. Thus there exists $x \in M$ such that $f(x) = 0$, so $f \equiv 0$, since f is constant. ☐

Let $f : M \to N$ be a smooth morphism between manifolds, and let $\omega \in \Omega_c^p(N) \subset \Omega^p(N)$. Then $f^*\omega \in \Omega^p(M)$, and $\mathrm{supp}\, f^*\omega \subset f^{-1}(\mathrm{supp}\,\omega)$, which is not compact in general, so $f^*\omega \notin \Omega_c^p(M)$ in general.

Definition 2.38. f is called **proper** if $f^{-1}(K)$ is compact for all compact subsets $K \subset N$.

If f is proper, then $f^* : \Omega_c^p(N) \to \Omega_c^p(M)$ is well-defined.

Exercise. If f is a diffeomorphism then f^* induces an isomorphism $\mathrm{H}_c^p(N) \to \mathrm{H}_c^p(M)$.

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Definition 2.39. Let M_0 and M_1 be manifolds without boundary, and let $f_i : M_0 \to M_1$ be smooth morphisms for $i = 0, 1$. Then f_0 and f_1 are **smoothly properly homotopic** if there exists a smooth $H : M_0 \times [0,1] \to M$ such that $H(\cdot, i) = f_i(\cdot)$ for $i = 0, 1$ and H is proper.

Notation. $f_t(\cdot) = H(\cdot, t) : M_0 \to M_1$.

Remark. To say that H is proper is not the same as saying f_t is proper for all t.

Exercise. Find H such that f_t is proper but H is not. A hint is to let $M_0 = M_1 = \mathbb{R}$ and $H : \mathbb{R} \times [0,1] \to \mathbb{R}$ such that $f_t^{-1}(0)$ is bounded for all t but $H^{-1}(0)$ is not.

Definition 2.40. M_0 and M_1 are **properly smoothly homotopically equivalent** if there exist smooth morphisms $f : M_0 \to M_1$ and $g : M_1 \to M_0$ such that $f \circ g \sim \mathrm{id}_{M_1}$ and $g \circ f \sim \mathrm{id}_{M_0}$, where the equivalences are properly homotopic.

Proposition 2.41. *If M_0 and M_1 are properly homotopically equivalent then*

$$\mathrm{H}_c^p(M_0) \cong \mathrm{H}_c^p(M_1).$$

Let M be a manifold, and let $i : U \hookrightarrow M$ be an open set. Then there exist linear **push-forwards**

$$i_* : \Omega_c^p(U) \to \Omega_c^p(M), \qquad p \ge 0.$$

Let $\omega \in \Omega_c^p(U)$. Then $\omega = 0$ outside $K \subset U$. We can define

$$i_*\omega = \begin{cases} \omega & \text{on } U \\ 0 & \text{on } M \setminus U \end{cases}.$$

If $j : V \hookrightarrow U$ and $i : U \hookrightarrow M$, then $(i \circ j)_* = i_* \circ j_*$.

Lemma 2.42. *Let M be a manifold, and let $i : U \hookrightarrow M$ be an immersion such that U is open. Then for all $p \ge 0$, $i_* : \Omega_c^p(U) \to \Omega_c^p(M)$ commutes with* d*, that is*

$$\mathrm{d}(i_*\omega) = i_*\mathrm{d}\omega, \qquad \omega \in \Omega_c^p(U).$$

In particular if ω is closed then $i_*\omega$ is closed, and if ω is exact then $i_*\omega$ is exact.

Proof.

$$\mathrm{d}(i_*\omega) = \begin{cases} \mathrm{d}\omega & \text{on } U \\ 0 & \text{otherwise} \end{cases} = i_*\mathrm{d}\omega.$$

Let ω be closed, so $\mathrm{d}\omega = 0$. Then $\mathrm{d}(i_*\omega) = i_*\mathrm{d}\omega = 0$, so $i_*\omega$ is closed. Similarly for exactness. □

Let $U \hookrightarrow M$ be as before. Then there exist

$$i_* : \mathrm{H}_c^p(U) \to \mathrm{H}_c^p(M), \qquad p \ge 0.$$

Theorem 2.43 (Punctured manifolds)**.** *Let M be a manifold of dimension n, let $x \in M$, and let $i : M \setminus \{x\} \hookrightarrow M$. Then*

- *for all $p \ge 2$, $i_* : \mathrm{H}_c^p(M \setminus \{x\}) \to \mathrm{H}_c^p(M)$ is an isomorphism.*
- *for all $p \ge 1$, if M is compact $i_* : \mathrm{H}_c^p(M \setminus \{x\}) \to \mathrm{H}_c^p(M) = \mathrm{H}^p(M)$ is an isomorphism.*

Proof.

- Injectivity.

$p \ge 2$. Let $\omega \in \Omega_c^p(M \setminus \{x\})$ be closed such that $i_*[\omega] = 0$, so $[i_*\omega] = 0$ in $\mathrm{H}_c^p(M)$. The goal is $[\omega] = 0$. There exists $\eta \in \Omega_c^{p-1}(M)$ such that $i_*\omega = \mathrm{d}\eta$. By Poincaré lemma there exists $U \subset M$ containing x such that $\mathrm{H}^q(U) = 0$ for all $q \ge 1$. Then $i_*\omega = 0$ in a neighbourhood of x because $\operatorname{supp}\omega \subset M \setminus \{x\}$, so $\mathrm{d}\eta = 0$ in a neighbourhood of x. By taking U smaller we can assume η is closed. Since $p \ge 2$, $[\eta] \in \mathrm{H}^{p-1}(U) = 0$, so η is exact. Then there exists $\sigma \in \Omega^{p-2}(U)$ such that $\eta = \mathrm{d}\sigma$ on U. Let $(U, M \setminus \{x\})$ be an open cover of M, let $\left(f_U, f_{M \setminus \{x\}}\right)$ be a partition of unity, and let $\eta' = \eta - \mathrm{d}(i_*(f_U\sigma))$. On a neighbourhood of x, $\eta' = 0$ because $i_*(f_U\sigma) = \sigma$, so $\operatorname{supp}\eta' \subset M \setminus \{x\}$. Thus $\eta' \in \Omega_c^{p-1}(M \setminus \{x\})$ and $\omega = \mathrm{d}\eta'$, so $[\omega] = 0$.

$p = 1$. The same proof. Let $\omega \in \Omega_c^1(M \setminus \{x\})$ be closed such that $[i_*\omega] = 0$. There exists $\eta \in \Omega_c^0(M)$ such that $i_*\omega = \mathrm{d}\eta$. By taking an open set $U \subset M$ such that $x \in U$, we may assume $\mathrm{d}\eta = 0$. Then η is constant on U, so $\eta = c$. Let $\eta' = \eta - c$. Then $\eta' = 0$ on U. If M is compact then $\eta' \in \Omega_c^0(M \setminus \{x\})$. Thus $\omega = \mathrm{d}\eta'$, so $[\omega] = 0$.

- Surjectivity.

$p \ge 1$. Let $[\omega] \in \Omega_c^p(M)$ such that ω is closed. By Poincaré lemma there exists open $U \ni x$ such that ω is exact, so there exists $\sigma \in \Omega^{p-1}(U)$ such that $\omega = \mathrm{d}\sigma$. Let $\left(f_U, f_{M \setminus \{x\}}\right)$ be a partition of unity as before, and let $\omega' = \omega - \mathrm{d}(i_*(f_U\sigma))$. Then $\omega' = 0$ in a neighbourhood of x and $[\omega'] = [\omega]$, and $\omega'|_{M \setminus \{x\}} \in \Omega_c^p(M \setminus \{x\})$. Thus $\left[i_* \, \omega'|_{M \setminus \{x\}}\right] = [\omega'] = [\omega]$.

□

Exercise. Compute $\mathrm{H}_c^1\left(\mathbb{R}^2 \setminus \{0\}\right)$ by hands.

Example 2.44.

$$\mathrm{H}_c^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}.$$

Recall $\mathbb{R}^n \cong \mathrm{S}^n \setminus \{x\}$ for $x \in \mathrm{S}^n$. By Theorem 2.43, by $M = \mathrm{S}^n$,

$$\mathrm{H}_c^p(\mathbb{R}^n) = \mathrm{H}_c^p(\mathrm{S}^n) = \begin{cases} \mathbb{R} & p = n \\ 0 & \text{otherwise} \end{cases}, \qquad p \ge 1,$$

and $\mathrm{H}_c^0(\mathbb{R}^n) = 0$.

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Let M be a manifold such that $M = U \cup V$ for open U and V such that $U \cap V \neq \emptyset$, and let

$$\begin{tikzcd} & U \arrow[dr, hook, "i_U"] & \\ U \cap V \arrow[ur, hook, "j_U"] \arrow[dr, hook, "j_V"'] & & M \\ & V \arrow[ur, hook, "i_V"'] & \end{tikzcd}, \qquad \begin{tikzcd} & \Omega^p(U) \arrow[dr, "i_{U*}"] & \\ \Omega^p(U \cap V) \arrow[ur, "j_{U*}"] \arrow[dr, "j_{V*}"'] & & \Omega^p(M) \\ & \Omega^p(V) \arrow[ur, "i_{V*}"'] & \end{tikzcd}, \qquad p \geq 0.$$

Proposition 2.45. *We have a short exact sequence*

$$0 \leftarrow \Omega^p(M) \overset{i}{\leftarrow} \Omega^p(U) \oplus \Omega^p(V) \overset{j}{\leftarrow} \Omega^p(U \cap V) \leftarrow 0,$$

where $i = i_{U*} + i_{V*}$ and $j = (j_{U*}, -j_{V*})$.

Proof.

- j is injective. Let $\omega \in \Omega^p(U \cap V)$ such that $j(\omega) = 0$, so $j_{U*}\omega = j_{V*}\omega = 0$. Then $\omega = 0$, so j is injective.
- $\operatorname{Ker} i = \operatorname{Im} j$. Let $\omega \in \Omega^p(U \cap V)$. Then $i(j(\omega)) = i(j_{U*}\omega, -j_{V*}\omega) = i_{U*}j_{V*}\omega - i_{U*}j_{V*}\omega = 0$, so $\operatorname{Ker} i \supset \operatorname{Im} j$. Let $(\omega_1, \omega_2) \in \operatorname{Ker} i$. Then $i_{U*}\omega_1 + i_{V*}\omega_2 = 0$, so $i_{V*}\omega_1 = -i_{V*}\omega_2$, so $\operatorname{supp}\omega_1 \subset U \cap V$ and $\operatorname{supp}\omega_2 \subset U \cap V$, so there exists $\eta \in \Omega^p(U \cap V)$ such that $j_{U*}\eta = \omega_1$ and $j_{V*}\eta = -\omega_2$, so $(\omega_1, \omega_2) = j(\eta)$, so $\operatorname{Ker} i \subset \operatorname{Im} j$.
- i is surjective. Let $\omega \in \Omega^p_{\mathrm{c}}(M)$, and let $\{f_U, f_V\}$ be a partition of unity with respect to $\{U, V\}$. Define $\omega_U = f_U \cdot \omega|_U \in \Omega^p_{\mathrm{c}}(U)$ and $\omega_V = f_V \cdot \omega|_V \in \Omega^p_{\mathrm{c}}(V)$. Then $i(\omega_U, \omega_V) = i_{U*}\omega_U + i_{V*}\omega_V = f_U \cdot \omega + f_V \cdot \omega = (f_U + f_V) \cdot \omega = \omega$.

$\square$

Thus for all p we get

$$\begin{tikzcd} 0 \arrow[r] & \Omega^p_{\mathrm{c}}(U \cap V) \arrow[r] \arrow[d, "\mathrm{d}"] & \Omega^p_{\mathrm{c}}(U) \oplus \Omega^p_{\mathrm{c}}(V) \arrow[r] \arrow[d, "(\mathrm{d},\mathrm{d})"] & \Omega^p_{\mathrm{c}}(M) \arrow[r] \arrow[d, "\mathrm{d}"] & 0 \\ 0 \arrow[r] & \Omega^{p+1}_{\mathrm{c}}(U \cap V) \arrow[r] & \Omega^{p+1}_{\mathrm{c}}(U) \oplus \Omega^{p+1}_{\mathrm{c}}(V) \arrow[r] & \Omega^{p+1}_{\mathrm{c}}(M) \arrow[r] & 0 \end{tikzcd}.$$

Theorem 2.46. *There exists $\delta : \mathrm{H}^p_{\mathrm{c}}(M) \to \mathrm{H}^{p+1}_{\mathrm{c}}(U \cap V)$ such that*

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{H}^p_{\mathrm{c}}(U \cap V) \arrow[r] & \mathrm{H}^p_{\mathrm{c}}(U) \oplus \mathrm{H}^p_{\mathrm{c}}(V) \arrow[r] & \mathrm{H}^p_{\mathrm{c}}(M) \arrow[dll, "\delta"] \\ & \mathrm{H}^{p+1}_{\mathrm{c}}(U \cap V) \arrow[r] & \mathrm{H}^{p+1}_{\mathrm{c}}(U) \oplus \mathrm{H}^{p+1}_{\mathrm{c}}(V) \arrow[r] & \mathrm{H}^{p+1}_{\mathrm{c}}(M) \arrow[r] & \dots \end{tikzcd}.$$

Proof. Same proof as Mayer-Vietoris for $\mathrm{H}^p(M)$. $\square$

2.6 Poincaré duality

Let M be an orientable manifold. Then

$$\mathrm{H}^p(M) \cong \mathrm{H}^{n-p}_{\mathrm{c}}(M)^*.$$

Proposition 2.47. *Let M be a manifold. Then the bilinear map*

$$\begin{array}{cccc} \cup & : & \mathrm{H}^p(M) \times \mathrm{H}^q(M) & \longrightarrow & \mathrm{H}^{p+q}(M) \\ & & ([\omega], [\eta]) & \longmapsto & [\omega \wedge \eta] \end{array}$$

is well-defined, and if $[\omega] \cup [\eta] = [\omega \wedge \eta]$ then $[\omega] \cup [\eta] = (-1)^{p \cdot q} [\eta] \wedge [\omega]$.

Proof. Follows from the Leibnitz rule and Proposition 1.6. $\square$

Lemma 2.48. *Let M be oriented without boundary of dimension n. Then there exists a linear map*

$$\begin{array}{rccc} \mathrm{I}_M & : & \mathrm{H}^n_{\mathrm{c}}(M) & \longrightarrow & \mathbb{R} \\ & & [\omega] & \longmapsto & \int_M \omega \end{array},$$

and I_M is surjective.

Then I_M is called **integration**.

Proof. Let $\omega \in \Omega^n_{\mathrm{c}}(M)$ such that $[\omega] = 0$, so ω is exact. By Stokes $\int_M \omega = 0$, so I_M is well-defined and it is linear. It is enough to show there exists closed $\omega \in \Omega^n_{\mathrm{c}}(M)$ such that $\int_M \omega \neq 0$. Take a volume form ω_0, which exists because M is oriented. Take $f \in \mathrm{C}^\infty(M)$ for $f \geq 0$ and with compact support. Let $f \cdot \omega_0 \in \Omega^n_{\mathrm{c}}(M)$. Then ω is closed because $\Omega^{n+1}_{\mathrm{c}}(M) = 0$ and $\int_M \omega = \int_M (f \cdot \omega_0) > 0$, by definition of volume forms. ☐

Example 2.49. Let $M = \mathrm{S}^n$, and let $\omega \in \Omega^n_{\mathrm{c}}(M)$ such that $\int_M \omega = 0$. We want to show that ω is exact. Since M is compact, $\mathrm{H}^n_{\mathrm{c}}(M) = \mathrm{H}^n(M) = \mathbb{R}$. By Lemma 2.48 $\mathrm{I}_M : \mathrm{H}^n_{\mathrm{c}}(M) \to \mathbb{R}$ is surjective, and $\mathrm{H}^n_{\mathrm{c}}(M) = \mathbb{R}$, so I_M is injective. Since $\int_M \omega = 0$, $\mathrm{I}_M([\omega]) = 0$, so $[\omega] = 0$. Thus ω is exact.