M3P21 Geometry II: Algebraic Topology

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Contents

0 Some underlying geometric notions

0.1 Introduction

Lecture 1 Friday 11/01/19

Combines topological spaces with algebraic objects, groups.

- How to show that a torus is not homeomorphic to a sphere?
- How to show that $\mathbb{R}^n \ncong \mathbb{R}^m$ if $n \neq m$?

Content is fundamental groups and homology. We will follow chapter one and two from

- A Hatcher, Algebraic topology, 2002

Prerequisites are the following.

- Point set topology. Topological spaces, continuous maps, product and quotient topologies, Hausdorff spaces, etc.
- Basic group theory. Normal subgroups and quotients, isomorphism theorems, free groups, presentation of groups, etc.

0.2 Homotopy

Let X, Y be topological spaces and $I = [0, 1]$.

Definition 0.1. A **homotopy** is a continuous map $F : X \times I \to Y$. For every $t \in I$ we obtain a continuous map

$$\begin{array}{rccl} f_t : & X & \to & Y \\ & x & \mapsto & f_t(x) = F(x,t) \end{array}.$$

Definition 0.2. Two continuous maps $f_0, f_1 : X \to Y$ are **homotopic** if there exists a homotopy $F : X \times I \to Y$ such that

$$f_0(x) = F(x,0), \qquad f_1(x) = F(x,1),$$

for all $x \in X$. We write $f_0 \cong f_1$.

(Exercise: this is an equivalence relation)

Definition 0.3. Let $A \subseteq X$ be a subspace. A **retraction** of X onto A is a continuous map $r : X \to A$ such that

- $r(X) = A$, and
- $r\mid_A = id_A$.

Example 0.4. If $X \neq \emptyset$, $p \in X$, then X retracts to p by the constant map $X \to \{p\}$.

Definition 0.5. A **deformation retraction** of X onto $A \subseteq X$ is a retraction that is homotopic to the identity. That is, there is a continuous map

$$\begin{array}{rccc} F : & X \times I & \to & A \\ & (x,t) & \mapsto & f_t(x) \end{array},$$

such that $f_0 = id_X$ and $f_1 : X \to A$ is the deformation retraction.

Example 0.6. The closed n-dimensional n**-disc**

$$D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$

deformation retracts to $(0, \dots, 0) \in \mathbb{R}^n$. Let $f_t(x) = t \cdot x$. $t = 1$ gives $f_1 = id_{D^n}$ and $t = 0$ gives $f_0 : D^n \to (0, \dots, 0)$.

Example 0.7. Let S^n be the n**-sphere**,

$$\partial D^{n+1} = S^n = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

The cylinder $S^n \times I$ deformation retracts to $S^n \times \{0\}$, by defining $f_t(x, r) = (x, t \cdot r)$.

An observation is if X is a topological space, and $f : X \to \{p\}$ for $p \in X$ is a deformation retraction of X to p, then X is path connected. Indeed, if $F : X \times I \to X$ is a homotopy from id_X to f and $x \in X$ is a point, then this gives a path

$$\begin{array}{ccc} I & \to & X \\ t & \mapsto & F(x, t) \end{array}$$

that connects x to p. This implies that not all retractions are deformation retractions.

Example 0.8. A retraction that is not a deformation retraction. Take a space that is not path connected and retract it to a point. Let $X = \{0, 1\}$ with discrete topology. $x \mapsto 0$ is a retraction, but not a deformation retraction because X is not path connected.

Definition 0.9. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \to X$ such that $fg \cong id_Y$ and $gf \cong id_X$. If there exists a homotopy equivalence between X and Y, X and Y are **homotopy equivalent** or they have the same **homotopy type**.

Lemma 0.10. *A deformation retraction $f : X \to A$ is a homotopy equivalence.*

Proof. Let $i : A \hookrightarrow X$ be the inclusion map. Then $fi = id_A$ and $if = f \cong id_X$ by definition. □

Example 0.11. The disc with two holes is equivalent to ∞.

Example 0.12. $\mathbb{R}^n$ deformation retracts to a point, by $f_t(x) = t \cdot x$.

Definition 0.13.

- X is **contractible** if it is homotopy equivalent to a point.
- A continuous map is **nullhomotopic** if it is homotopy equivalent to a constant map.

0.3 Cell complexes

Example 0.14. The torus $S^1 \times S^1$ is the union of a point, two open intervals, and the open disc $Int(D^2)$.

These are called **cells**. Can think of discs D^n glued together.

Lecture 2 Tuesday 15/01/19

Definition 0.15. A **CW-complex**, or **cell complex**, is a topological space X such that there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X^n,$$

where the X^n are constructed inductively in the following way.

- X^n is a discrete set.
- For each $n \geq 0$ there is an collection of closed n-discs $\{D^n_\alpha\}$ together with continuous maps

  $$\phi_\alpha : \partial D^n_\alpha \to X^{n-1},$$

  such that

  $$X^n = \frac{X^{n-1} \sqcup \bigsqcup_\alpha D^n_\alpha}{\sim},$$

  where $x \sim \phi_\alpha(x)$ for all $x \in \partial D^n_\alpha$ for all α.
- A subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n.

Remark 0.16.

- As a set,
$$X^n = X^{n-1} \sqcup \bigsqcup_{\alpha} e_{\alpha}^n,$$
where each e_{α}^n is homeomorphic to an open n-disc. These e_{α}^n are called the **n-cells** of X.
- If $X = X^m$ for some m, then X is called **finite dimensional**. The minimal m such that $X = X^m$ is the **dimension** of X.

Example 0.17.

- $[0, 1]$ is a CW-complex.
- $\mathbb{R}$ is a CW-complex.
- S^1 is a CW-complex.
- A graph is a CW-complex.
- $S^n = D^n / \partial D^n$ is a CW-complex. See worksheet 1.

Can also decompose CW-complexes.

- The sphere S^2 is one 0-cell, one 1-cell, and two 2-cells.
- The torus $S^1 \times S^1$ is one 0-cell, two 1-cells, and one 2-cell.
- The Möbius strip is two 0-cells, three 1-cells, and one 2-cell.
- The Klein bottle is one 0-cell, two 1-cells, and one 2-cell.

Definition 0.18. If X is a CW-complex with finitely many cells the **Euler characteristic** $\chi(X)$ of X is the number of even cells minus the number of odd cells.

A fact is that $\chi(X)$ does not depend of the choice of cells decomposition.

Example 0.19.

- $\chi(S^n) = 0$ if n is odd and $\chi(S^n) = 2$ if n is even.
- $\chi\left(S^1 \times S^1\right) = 0$.

This is the generalisation of the following observation by Leonhard Euler. Let P be a convex polyhedron, where

- V is the number of vertices of P,
- E is the number of edges of P, and
- F is the number of faces of P.

Then $V - E + F = 2$.

Example 0.20. A topological space that is not a CW-complex. $X = \{0, 1\}$ with trivial topology does not contain any closed points. A fact is that CW-complexes are always Hausdorff.

1 The fundamental group

1.1 Paths and homotopy

Let X be a topological space. A **path** is a continuous map $f : I \to X$, where $I = [0, 1]$.

Definition 1.1. Two paths f_0, f_1 are **homotopic** if there exists a homotopy between f_0 and f_1 preserving the endpoints, that is a continuous map

$$\begin{array}{cccc} F : & I \times I & \to & X \\ & (s,t) & \mapsto & f_t(s) \end{array},$$

such that

$$f_t(0) = f_0(0), \qquad f_t(1) = f_0(1),$$

for all $t \in I$, and

$$F(s,0) = f_0(s), \qquad F(s,1) = f_1(s),$$

for all $s \in I$.

Example 1.2. Let $X \subseteq \mathbb{R}^n$ be a convex set. Then all the paths in X are homotopic if they have the same endpoints.

Proof. Let $f_0, f_1 : I \to X$ be two paths such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$. Define

$$f_t(s) = (1-t) f_0(s) + t f_1(s).$$

□

Lemma 1.3. *Being homotopic is an equivalence relation on the set of paths with fixed endpoints. We will write $f_0 \cong f_1$ for two homotopic paths f_0 and f_1.*

Proof.

- f is homotopic to f.
- If f_0 is homotopic to f_1 by a homotopy f_t, then f_1 is homotopic to f_0 by the homotopy f_{1-t}.
- If f_0 is homotopic to f_1 by a homotopy f_t and $f_1 = g_0$ is homotopic to g_1 by a homotopy g_t, then f_0 is homotopic to g_1 by the homotopy

$$h_t = \begin{cases} f_{2t} & 0 \le t \le \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}.$$

  Then

$$\begin{array}{cccc} H : & I \times I & \to & X \\ & (s,t) & \mapsto & h_t(s) \end{array}$$

  is continuous because its restriction to the closed subsets $I \times [0, 1/2]$ and $I \times [1/2, 1]$ is continuous, since if the restriction to two closed subsets is continuous then the restriction to the union of these subsets is continuous.

□

Let X be a topological space and $I = [0, 1]$. If $f : I \to X$ is a path, $[f]$ is the class of all paths on X homotopic to f. Lecture 3 Wednesday 16/01/19

Definition 1.4. Let $f, g : I \to X$ be two paths such that $f(1) = g(0)$. The **product path** $f \cdot g$ is the path

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}.$$

A convention is that whenever we write $f \cdot g$ we implicitly assume $f(1) = g(0)$.

Lemma 1.5. *Let f_0, f_1, g_0, g_1 be paths on X such that $f_1 \cong f_0$ and $g_0 \cong g_1$. Then $f_0 \cdot g_0 \cong f_1 \cdot g_1$.*

Proof.

$$\begin{array}{ccc} I \times I & \to & X \\ (s,t) & \mapsto & (f_t \cdot g_t)(s) \end{array}$$

is a homotopy between $f_0 \cdot g_0$ and $f_1 \cdot g_1$. □

Remark 1.6. Let $\phi : [0,1] \to [0,1]$ be continuous such that $\phi(0) = 0$ and $\phi(1) = 1$. If $f : I \to X$ is a path, then $f \circ \phi \cong f$. This is a **reparametrisation**.

Proof. Define

$$\phi_t(s) = (1-t)\phi(s) + ts,$$

then $f \circ \phi_t$ is a homotopy between $f \circ \phi$ and f. □

For $x \in X$, let the **constant path** at x be

$$c_x : \begin{array}{ccc} I & \to & X \\ s & \mapsto & x \end{array}.$$

For a path $f : I \to X$, define

$$f^{-1} : \begin{array}{ccc} I & \to & X \\ s & \mapsto & f(1-s) \end{array}.$$

Lemma 1.7. *Let $f, g, h : I \to X$ be paths. Then*

1. *$(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$,*
2. *$f \cdot c_{f(1)} \cong f$ and $c_{f(0)} \cdot f \cong f$, and*
3. *$f \cdot f^{-1} \cong c_{f(0)}$ and $f^{-1} \cdot f \cong c_{f(1)}$.*

Proof.

1. $((f \cdot g) \cdot h) \circ \phi = f \cdot (g \cdot h)$, where

$$\phi(s) = \begin{cases} \frac{s}{2} & s \in \left[0, \frac{1}{2}\right] \\ s - \frac{1}{4} & s \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ 2s - 1 & s \in \left[\frac{3}{4}, 1\right] \end{cases},$$

so $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$ by reparametrisation.

2. Again reparametrisation, by

$$\psi(s) = \begin{cases} 2s & s \in \left[0, \frac{1}{2}\right] \\ 1 & s \in \left[\frac{1}{2}, 1\right] \end{cases}, \qquad \chi(s) = \begin{cases} 0 & s \in \left[0, \frac{1}{2}\right] \\ 2s - 1 & s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

3. Define

$$H(s,t) = \begin{cases} f(\max\{1-2s, t\}) & s \in \left[0, \frac{1}{2}\right] \\ f(\max\{2s-1, t\}) & s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

H is continuous, and

$$H(s,0) = f^{-1} \cdot f, \qquad H(s,1) = c_{f(1)}.$$

The inverse is similar.

□

Definition 1.8. A **loop** with **basepoint** $x_0 \in X$ is a path $f : I \to X$ such that $f(0) = f(1) = x_0$.

Definition 1.9. Denote by $\pi_1(X, x_0)$ the set of homotopy classes $[f]$ of loops $f : I \to X$ with basepoint x_0.

Proposition 1.10. *$\pi_1(X, x_0)$ is a group with product $[f][g] = [f \cdot g]$ and neutral element $c_{x_0} : I \to X$, the constant path at x_0.*

Proof. Follows directly from Lemma 1.5 and Lemma 1.7. ☐

Definition 1.11. $\pi_1(X, x_0)$ is the **fundamental group** of X at x_0.

Example 1.12. Let $X \subseteq \mathbb{R}^n$ be a convex set and $x_0 \in X$. Then $\pi_1(X, x_0) = 0$.

Proof. X is convex gives that all loops are homotopic to each other. ☐

Example 1.13.

- The fundamental group of a space X with the trivial topology is trivial, since X is simply connected, because all maps $f : I \to X$ are continuous, so path connected and all paths are homotopic.
- The fundamental group of a space X with the discrete topology is trivial, since $f : I \to X$ continuous gives f constant.

Assume $x_0, x_1 \in X$ such that x_0 and x_1 are in the same path component of X. Let $h : I \to X$ be a path such that $h(0) = x_0$ and $h(1) = x_1$. Define

$$\begin{array}{cccc} \beta_h : & \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ & [f] & \mapsto & \left[h \cdot f \cdot h^{-1}\right] \end{array}.$$

This is well-defined by Lemma 1.5.

Proposition 1.14. *$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is an isomorphism.*

Proof. It is a homomorphism.

$$\beta_h[f \cdot g] = \left[h \cdot f \cdot g \cdot h^{-1}\right] = \left[h \cdot f \cdot h^{-1}\right]\left[h \cdot g \cdot h^{-1}\right] = \beta_h[f] \cdot \beta_h[g],$$

and $\beta_h[c_{x_1}] = [c_{x_1}]$. It is bijective with $(\beta_h)^{-1} = \beta_{h^{-1}}$. ☐

If X is path connected, we often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Definition 1.15. X is **simply connected** if it is path connected and $\pi_1(X) = 0$.

Proposition 1.16. *X is simply connected if and only if there exists a unique homotopy class of paths between any two points of X.*

Proof.

$\implies$ There exists a path between any two points. Let f, g be two paths from x_0 to x_1 for $x_0, x_1 \in X$. $f \cdot g^{-1} \cong g \cdot g^{-1}$ gives $f \cong f \cdot g^{-1} \cdot g \cong g \cdot g^{-1} \cdot g \cong g$.

$\impliedby$ X is path connected. $x_1 = x_0$ gives that all loops at x_0 are homotopic to each other, so $\pi_1(X) = 0$.

☐

1.2 The fundamental group of the circle

Goal is to show that $\pi_1\left(S^1\right) \cong \mathbb{Z}$. Lecture 4 Friday 18/01/19

Definition 1.17. A **covering space** of a space X is a space $\widetilde{X}$ and a continuous map $p : \widetilde{X} \to X$ such that for each $x \in X$ there is an open $x \in U \subseteq X$ such that

- $p^{-1}(U) = \bigcup_{j \in J} \widetilde{U_j}$, where $\widetilde{U_j} \subseteq \widetilde{X}$ is open,
- $\widetilde{U_i} \cap \widetilde{U_j} = \emptyset$ if $i \neq j$, and
- $p\mid_{\widetilde{U_j}} : \widetilde{U_j} \to U_i$ is a homeomorphism for all $j \in J$.

Such a U is called **evenly covered**. The $\widetilde{U_j}$ are called **sheets**.

Example 1.18.

$$\begin{array}{rccc} p : & \mathbb{R} & \to & S^1 \\ & s & \mapsto & (\cos(2\pi s, \sin(2\pi s))) \end{array}$$

Definition 1.19. Let $p : \widetilde{X} \to X$ be a covering space. A **lift** of a continuous map $f : Y \to X$ is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ such that $p \circ \widetilde{f} = f$, so

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}.$$

Proposition 1.20 (Unique lifting property)**.** *Let $p : \widetilde{X} \to X$ be a covering space and $f : Y \to X$ be a continuous map. If there are two lifts $\widetilde{f_1}, \widetilde{f_2} : Y \to \widetilde{X}$ of f such that $\widetilde{f_1}(y) = \widetilde{f_2}(y)$ for some $y \in Y$ and if Y is connected, then $\widetilde{f_1} = \widetilde{f_2}$.*

Proof. Let $y \in Y$ and $U \subseteq X$ be an evenly covered neighbourhood of $f(y)$. Then

$$p^{-1}(U) = \bigcup_j \widetilde{U_j}.$$

Let $\widetilde{U_1}$ be the sheet such that $\widetilde{f_1}(y) \in \widetilde{U_1}$, and let $\widetilde{U_2}$ be the sheet such that $\widetilde{f_2}(y) \in \widetilde{U_2}$. Let $N \subseteq Y$ be open and $y \in N$ such that $\widetilde{f_1}(N) \subseteq \widetilde{U_1}$ and $\widetilde{f_2}(N) \subseteq \widetilde{U_2}$. We have $p \circ \widetilde{f_1} = p \circ \widetilde{f_2}$.

$$\widetilde{f_1}(y) = \widetilde{f_2}(y) \qquad \iff \qquad \widetilde{U_1} = \widetilde{U_2} \qquad \iff \qquad \widetilde{f_1}\mid_N = \widetilde{f_2}\mid_N .$$

Let

$$A = \left\{ y \in Y \mid \widetilde{f_1}(y) = \widetilde{f_2}(y) \right\},$$

so A is open and $Y \setminus A$ is open. Thus $A \neq \emptyset$ gives $A = Y$. □

Proposition 1.21 (Homotopy lifting property)**.** *Let $p : \widetilde{X} \to X$ be a covering space and $F : Y \times I \to X$ be a continuous map such that there exists a lift $\widetilde{f_0} : Y \times \{0\} \to \widetilde{X}$ of $F\mid_{Y \times \{0\}}$. Then there is a unique lift $\widetilde{F} : Y \times I \to \widetilde{X}$ of F such that $\widetilde{F}\mid_{Y \times \{0\}} = \widetilde{f_0}$.*

Proof. Let $y_0 \in Y$ and $t \in I$. There are open $y_0 \in N_t \subseteq Y$ and $t \in (a_t, b_t) \subseteq I$ such that $F(N_t \times (a_t, b_t)) \subseteq U \subseteq X$, where $U \subseteq X$ is open and evenly covered. Compactness of I gives that there exist

$$0 = t_0 < \dots < t_m = 1,$$

and there exists $y_0 \in N \subseteq Y$ open such that $F(N \times [t_i, t_{i+1}]) \subseteq U_i \subseteq X$, where $U_i \subseteq X$ is open and evenly covered. We inductively construct a lift $\widetilde{F}\mid_{N \times I}$ of $F\mid_{N \times I}$.

- $\widetilde{F}\mid_{N\times[0,0]}=\widetilde{f_0}\mid_{N\times[0,0]}$ exists.
- Assume the lift has been constructed on $N\times[0,t_i]$. Let $\widetilde{U_i}\subseteq\widetilde{X}$ be such that $p\mid_{\widetilde{U_i}}:\widetilde{U_i}\to U_i$ such that $\widetilde{F}(y_0,t_i)\subseteq\widetilde{U_i}$. After shrinking N, may assume $\widetilde{F}(N\times\{t_i\})\subseteq\widetilde{U_i}$. Define $\widetilde{F}$ on $N\times[t_i,t_{i+1}]$ to be composition of F with the homeomorphism $p^{-1}:U_i\to\widetilde{U_i}$.

After finitely many steps we obtain a lift $\widetilde{F}:N\times I\to\widetilde{X}$, where $y_0\in N\subseteq Y$ is open, so for each $y\in Y$ there is a neighbourhood $N_y\subseteq Y$ such that $F\mid_{N_y\times I}:N_y\times I\to X$ lifts. For all $y\in Y$, $\{y\}\times I$ is connected and can be lifted, so Proposition 1.20 gives that the lift of $N\times I$ is unique. Thus there is a unique lift $\widetilde{F}:Y\times I\to\widetilde{X}$. ◻

Example 1.22. Let X be a topological space and A be discrete. Then $p:X\times A\to X$ is a covering space. This is the **trivial covering**. (Exercise: show the unique lifting property and the homotopy lifting property for the trivial covering)

Corollary 1.23. *Let $f:I\to X$ be a path, $f(0)=x_0$, and $p:\widetilde{X}\to X$ be a covering space. For each $\widetilde{x_0}\in p^{-1}(x_0)$, there is a unique lift $\widetilde{f}:I\to\widetilde{X}$ such that $\widetilde{f}(0)=\widetilde{x_0}$.*

Proof. Proposition 1.21 for Y a point. ◻

Theorem 1.24. *Let $x_0=(1,0)\in S^1$. $\pi_1(S^1,x_0)$ is the infinite cyclic group generated by the homotopy class of the loop*

$$\begin{array}{rccc}\omega: & I & \to & S^1\\ & s & \mapsto & (\cos(2\pi s),\sin(2\pi s))\end{array}.$$

Remark 1.25.

- $[\omega]^n=[\omega_n]$, where
$$\omega_n(s)=(\cos(2\pi ns),\sin(2\pi ns)).$$
- $$\begin{array}{rccc}p: & \mathbb{R} & \to & S^1\\ & s & \mapsto & (\cos(2\pi s),\sin(2\pi s))\end{array}$$
  is a covering space.
- ω_n lifts to
$$\begin{array}{rccc}\widetilde{\omega_n}: & I & \to & \mathbb{R}\\ & s & \mapsto & ns\end{array},$$
  such that $\widetilde{\omega_n}(0)=0$ and $\widetilde{\omega_n}(1)=n$.

Proof of Theorem 1.24.

- If $f:I\to S^1$ be a loop at x_0, then the homotopy lifting property gives that there exists a lift $\widetilde{f}:I\to\mathbb{R}$ such that $\widetilde{f}(0)=0$. Since $p\left(\widetilde{f}(1)\right)=f(1)=x_0$, then $\widetilde{f}(1)=n$ for some $n\in\mathbb{Z}$. $\widetilde{\omega_n}:I\to\mathbb{R}$ is another path such that $\widetilde{\omega_n}(0)=0$ and $\widetilde{\omega_n}(1)=n$, so $\widetilde{f}\cong\widetilde{\omega_n}$. Let $F:I\times I\to\mathbb{R}$ be a homotopy equivalence between $\widetilde{f}$ and $\widetilde{\omega_n}$. Then $p\circ F:I\times I\to S^1$ gives a homotopy between $p\circ\widetilde{f}=f$ and $p\circ\widetilde{\omega_n}=\omega_n$.
- Let $m,n\in\mathbb{Z}$ and assume $\omega_m\cong\omega_n$. Let $F:I\times I\to S^1$ be a homotopy.
$$F(0,t)=\omega_m(t),\qquad F(1,t)=\omega_n(t),\qquad F(s,0)=F(s,1)=x_0,$$
  for all $s,t\in I$. The unique lifting property gives that $\widetilde{\omega_n},\widetilde{\omega_m}:I\to\mathbb{R}$ are unique lifts such that $\widetilde{\omega_n}(0)=0=\widetilde{\omega_m}(0)$. The homotopy lifting property gives that F lifts uniquely to a homotopy $\widetilde{F}:I\times I\to\mathbb{R}$ between $\widetilde{\omega_n}$ and $\widetilde{\omega_m}$, and $\widetilde{F}(s,1)\in\mathbb{Z}$ for all $s\in I$. Thus $\widetilde{F}(s,1)=n=m$, so $\omega_m\cong\omega_n$ if and only if $n=m$.

$\square$ Lecture 5 Tuesday 22/01/19

Lecture 5 is a problem class.

Lecture 6 Wednesday 23/01/19

Theorem 1.26. *Every non-constant polynomial $p \in \mathbb{C}[z]$ has a root in $\mathbb{C}$.*

Proof. May assume

$$p(z) = z^n + a_1 z^{n-1} + \dots + a_n.$$

Assume p has no roots in $\mathbb{C}$. For each $r \in \mathbb{R}_{\geq 0}$ we obtain a loop

$$\begin{array}{rccc} f_r : & I & \to & \mathbb{C} \\ & s & \mapsto & \dfrac{p\left(re^{2\pi i s}\right)/p(r)}{\left|p\left(re^{2\pi i s}\right)/p(r)\right|} \end{array},$$

so $|f_r(s)| = 1$. $f_r(0) = 1$ and $f_r(1) = 1$, so f_r is a loop based at 1. f_0 is the constant loop at 1. $f_r(s)$ depends continuously on r, so $f_r \simeq f_0$ for all $r \in \mathbb{R}_{\geq 0}$ and $[f_r] = [f_0] = 0 \in \pi_1\left(S^1\right)$. Fix $r \in \mathbb{R}_{\geq 0}$ such that $r > 1$ and $r > |a_1| + \dots + |a_n|$. For $|z| = r$ we have

$$|z^n| > (|a_1| + \dots + |a_n|)\left|z^{n-1}\right| \geq \left|a_1 z^{n-1}\right| + \dots + |a_n| \geq \left|a_1 z^{n-1} + \dots + a_n\right|.$$

Hence, for $0 \leq t \leq 1$ the polynomial $p_t(z) = z^n + t\left(a_1 z^{n-1} + \dots + a_n\right)$ has no root z with $|z| = r$. Define

$$F_r(t, s) = \frac{p_t\left(re^{2\pi i s}\right)/p_t(r)}{\left|p_t\left(re^{2\pi i s}\right)/p_t(r)\right|}.$$

$F_r(0, s) = \omega_n(s)$ and $F_r(1, s) = f_r(s)$, so $[\omega_n] = [f_r] = 0 \in \pi_1\left(S^1\right)$. Theorem 1.24 gives that $n = 0$, so p is constant. $\square$

See Hatcher Theorem 1.9 and Theorem 1.10 for more applications.

Proposition 1.27. *Let X, Y be topological spaces, $x_0 \in X$, and $y_0 \in Y$. Then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. A map

$$\begin{array}{rccc} f : & Z & \to & X \times Y \\ & z & \mapsto & (g(z), h(z)) \end{array}$$

is continuous if and only if $g : Z \to X$ and $h : Z \to Y$ are continuous. For $Z = I$,

$$\{\text{ loops in } X \times Y \text{ based at } (x_0, y_0)\} \quad \leftrightsquigarrow \quad \{\text{ loops in } X \text{ based at } x_0\} \times \{\text{ loops in } Y \text{ based at } y_0\}.$$

Two loops

$$\begin{array}{rccc} f_1 : & I & \to & X \times Y \\ & s & \mapsto & (g_1(s), h_1(s)) \end{array}, \qquad \begin{array}{rccc} f_2 : & I & \to & X \times Y \\ & s & \mapsto & (g_2(s), h_2(s)) \end{array}$$

are homotopic if and only if $g_1 \simeq g_2$ and $h_1 \simeq h_2$, so there is a bijection

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

$f_1 \cdot f_2 = (g_1 \cdot g_2, h_1 \cdot h_2)$ and the constant loop is mapped to the constant loop, so this is also a group isomorphism. $\square$

Example 1.28. The torus $S^1 \times S^1$ has

$$\pi_1\left(S^1 \times S^1\right) \cong \pi_1\left(S^1\right) \times \pi_1\left(S^1\right) \cong \mathbb{Z}^2.$$

1.3 Induced homomorphisms

Let X, Y be topological spaces, $x_0 \in X$, and $\phi : X \to Y$. An observation is that ϕ induces a homomorphism

$$\begin{array}{rccc} \phi_* : & \pi_1(X, x_0) & \to & \pi_1(Y, \phi(x_0)) \\ & [f] & \mapsto & [\phi \circ f] \end{array}.$$

ϕ_* is well-defined, since if f_t is a homotopy between the loops f_0 and f_1 based at x_0, then $\phi \circ f_t$ is a homotopy of loops between $\phi \circ f_0$ and $\phi \circ f_1$. Moreover,

$$\phi \circ (f \cdot g) = (\phi \circ f) \cdot (\phi \circ g),$$

and ϕ maps the constant path at x_0 to the constant path at $\phi(x_0)$, so ϕ is a homomorphism.

Proposition 1.29.

1. *Let $\psi : X \to Y$ and $\phi : Y \to Z$ be continuous maps between topological spaces, $x_0 \in X$, and*

$$\psi_* : \pi_1(X, x_0) \to \pi_1(Y, \psi(x_0)), \qquad \phi_* : \pi_1(Y, \psi(x_0)) \to \pi_1(Z, (\phi \cdot \psi)(x_0)),$$

$$(\phi \circ \psi)_* : \pi_1(X, x_0) \to \pi_1(Z, (\phi \cdot \psi)(x_0)).$$

   Then $(\phi \circ \psi)_* = \phi_* \circ \psi_*$.

2. *Let $id_X : X \to X$ be the identity then*

$$(id_X)_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

   is the identity.

Proof.

1. Let $f : I \to X$ be a loop at x_0, then

$$(\phi \circ \psi)_*([f]) = [(\phi \circ \psi) \circ f] = [\phi \circ (\psi \circ f)] = \phi_*([\psi \circ f]) = \phi_*(\psi_*([f])) = (\phi_* \circ \psi_*)([f]).$$

2. $(id_X)_*([f]) = [id_X \circ f] = [f]$.

□

These two observations yield in particular that if $\phi : X \to Y$ is a homeomorphism with inverse $\psi : Y \to X$, then

$$\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$$

is an isomorphism with inverse ψ_*.

Proposition 1.30. *If $n \geq 2$, then $\pi_1(S^n) = 0$.*

A Quotient topology

Recall that if X is a set with equivalence relation $\sim$, there is a quotient set $X/\sim$. The quotient map

$$\begin{array}{rccc} \pi : & X & \to & \frac{X}{\sim} \\ & x & \mapsto & [x] \end{array}$$

is characterised by the following universal property. For every map $g : X \to Y$ such that

$$a \sim b \qquad \implies \qquad g(a) = g(b),$$

there exists a unique $f : X/\sim \to Y$ such that $g = f \cdot \pi$, so

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ \frac{X}{\sim} \arrow[r, "\exists! f"'] & Y \end{tikzcd} .$$

Let X be a topological space and $\sim$ be an equivalence relation on X. We define a topology on $X/\sim$ by

$$U \subseteq \frac{X}{\sim} \text{ open} \qquad \iff \qquad \pi^{-1}(U) \text{ open}.$$

Remark A.1.

- This is the largest topology on $X/\sim$ such that π is continuous. Exercise 1 states that if Z is a topological space and $f : X/\sim \to Z$ is a map, then f is continuous if and only if $f \circ \pi : X \to Z$ is continuous. This implies that the topological quotient $\pi : X \to X/\sim$ is characterised by the following universal property. For any topological space Z and a continuous $g : X \to Z$ such that

$$a \sim b \qquad \implies \qquad g(a) = g(b),$$

  there exists a unique continuous map $f : X/\sim \to Z$ such that $gf \cdot \pi$, so

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ \frac{X}{\sim} \arrow[r, "\exists! f"'] & Z \end{tikzcd} .$$

- The quotient map is in general not open. For example, if $\pi : [0, 1] \to S^1$, then $[0, 1] \subset [0, 1)$ is open but $\pi([0, 1)) \subseteq S^1$ is not open.

- If X is Hausdorff, in general $X/\sim$ is not Hausdorff.

- If $\sim$ is the trivial relation, then $\pi : X \to X/\sim$ is a homeomorphism. Exercise 3 states that if X, Y are topological spaces, X is compact, Y is Hausdorff, and $\pi : X \to Y$ is surjective and continuous, then π is a quotient, that is there exists $\sim$ on X and $\pi : X \to Y \cong X/\sim$ is a quotient map.

- In particular, if $\pi : X \to Y$ is bijective, then π is a homeomorphism. Exercise $4, 5, 6$ states that if f is continuous and surjective, $f(\partial D^n)$ is a point, and f is a bijection on $D^n \setminus \partial D^n$, then

$$\begin{tikzcd} D^n \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ \frac{D^n}{\partial D^n} \arrow[r, "\sim"'] & S^1 \end{tikzcd} .$$