

M4P32 Number Theory: Elliptic Curves

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0 Introduction

0.1 Outline

1. Introduction
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The theme of the course will be studying polynomial equations over \mathbb{Z} or \mathbb{Q} .

Example. For $x^7 + y^7 + z^7 = 1$, what are the solutions with $x, y, z \in \mathbb{Q}$? Answer is hard. However $x^2 - 1 = 0$ is easy. In fact any equation in one variable is easy to solve over \mathbb{Q} . For example, $3x^5 - 9x^3 + x^2 + 148/81 = 0$ iff $243x^5 - 729x^3 + 81x^2 + 148 = 0$. Letting $x = r/s$ for $(r, s) = 1$ and $s \geq 1$, such as $r = -2$ and $s = 5$ in $-2/5$, gives $243r^5 - 729r^3s^2 + 81r^2s^3 + 148s^5 = 0$. So $r^2 \mid 148s^5$, but $(r, s) = 1$, so $r^2 \mid 148$. Similarly $s^2 \mid 243$. Now check the finitely many possibilities to get $x = 2/3$ as the only solution in \mathbb{Q} .

If k is a field, such as $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_p, \mathbb{Q}_p$, we write $k[x_1, \dots, x_n]$ for the polynomial ring in n variables x_1, \dots, x_n . A **monomial** is an expression $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ for $\alpha_i \in \mathbb{Z}_{\geq 0}$. The **degree** of $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is $\alpha_1 + \dots + \alpha_n$. An element of $k[x_1, \dots, x_n]$ is just a finite sum of multiples of monomials with coefficients in k . Its degree is the largest degree of any monomial occurring in it. For example, $5x_{10}^3 + x_2x_3^{10} - (2/3)x_7^7$ has degree 11. Typically we will be looking at the case of two variables x_1, x_2 , which we will usually call x, y .

Theorem 0.1 (Falting's theorem). The general equation in two variables of degree greater than four over \mathbb{Q} is known to only have finitely many solutions in \mathbb{Q} .

Example. $x^4 + y^4 = 17$ only has $(\pm 1, \pm 2)$ and $(\pm 2, \pm 1)$ as solutions, which is proved only in 2001. $(x^{100} + 5y^{100} - 7)(x - y) = 0$ is not a general equation and has infinitely many solutions.

Example. $x^2 + y^2 = -1$ has no solutions in \mathbb{Q} . $x^2 + y^2 = 0$ has finitely many solutions in \mathbb{Q} ($x = y = 0$). $x^2 + y^2 = 1$ has infinitely many solutions in \mathbb{Q} . Let $x = a/c$ and $y = b/c$ for $a, b, c \in \mathbb{Z}$ gives Pythagorean triples $a^2 + b^2 = c^2$, such as $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, etc.

Example. $x^2 + y^2 = 3$ has solutions in \mathbb{R} but no solutions in \mathbb{Q} . Equivalently, $a^2 + b^2 = 3c^2$ has no solutions with $a, b, c \in \mathbb{Z}$ except $a = b = c = 0$. Suppose we have a solution. Then $a^2 + b^2 \equiv 0 \pmod{3}$. A fact is that if $n \in \mathbb{Z}$, then $n^2 \equiv 0 \pmod{3}$ if $3 \mid n$, or $n^2 \equiv 1 \pmod{3}$ if $3 \nmid n$. So $a \equiv b \equiv 0 \pmod{3}$, or $3 \mid a, b$. Write $a = 2A$ and $b = 3B$. Then $(3A)^2 + (3B)^2 = 3c^2$ iff $9(A^2 + B^2) = 3c^2$, iff $3(A^2 + B^2) = c^2$. So $3 \mid c^2$ and $3 \mid c$. Write $c = 3C$. Then $3(A^2 + B^2) = (3C)^2$ iff $3(A^2 + B^2) = 9C^2$, iff $A^2 + B^2 = 3C^2$. Thus $a = b = c = 0$.

$x^2 + y^2 = -1$ has an obstruction in \mathbb{R} . $x^2 + y^2 = 3$ has an obstruction in \mathbb{Q}_3 . Hasse principle will tell us that for general equations of degree two in x, y , there are either infinitely many solutions in \mathbb{Q} or no solutions, and furthermore either no solutions in \mathbb{R} or no solutions in \mathbb{Q}_p for some prime p .

We study plane conics and plane cubics. Plane refers to 2 variables x, y . Conic refers to degree two while cubic refers to degree three.

Example. $x^2 + 2y^2 = 6$ has a rational solution $(2, 1)$. Drawing lines at $(2, 1)$ with rational slope $y - 1 = m(x - 2)$ will get all rational solutions by intersecting with $x^2 + 2y^2 = 6$. Then $x^2 + 2(m(x - 2) + 1)^2 = 6$ gives $(2m^2 + 1)x^2 + 4m(1 - 2m)x + 2(2m^2 - 1)^2 = 0$. The sum of the roots of this equation is $4m(2m - 1) / (2m^2 + 1)$. Since $x = 2$ is a root, the other root is

$$x_0 = \frac{4m(2m - 1)}{2m^2 + 1} - 2 = \frac{4m^2 - 4m - 2}{2m^2 + 1}, \quad y_0 = m(x_0 - 2) + 1 = m\left(\frac{-4m - 4}{2m^2 + 1}\right) + 1 = \frac{-2m^2 - 4m + 1}{2m^2 + 1}.$$

If $m \in \mathbb{Q}$, $(x_0, y_0) \in \mathbb{Q}^2$ and conversely, which is easy. For example, $m = 1$ gives $(x_0, y_0) = (-2/3, -5/3)$ and $m \rightarrow \infty$ gives $(x_0, y_0) \rightarrow (2, -1)$. Exercises for $x^2 + y^2 = 1$, $x^2 + y^2 = 0$, $xy = 0$, $x^2 - y^2 = 0$.

0.2 References

1. J W S Cassels, Lectures on elliptic curves, 1991
2. J H Silverman, The arithmetic of elliptic curves, 1986
3. J H Silverman and J Tate, 1992

1 The p -adic numbers

Definition 1.1. A **norm** on a field k is a function $|\cdot| : k \rightarrow \mathbb{R}$ such that:

1. $|x| \geq 0$ with equality iff $x = 0$,
2. $|xy| = |x| |y|$, and
3. $|x + y| \leq |x| + |y|$.

Note. $|1| = |-1| = 1$ and $|-x| = |-1| |x| = |x|$.

Example. Let $k = \mathbb{Q}$ or \mathbb{R} and $|x|$ is the absolute value of x , that is $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

Example. Discrete norm $|x| = 0$ if $x = 0$ and $|x| = 1$ if $x \neq 0$.

Remark 1.2. Any norm on k defines a metric by $d(x, y) = |x - y|$. In particular, a norm determines a topology.

Example. The discrete norm determines the discrete topology.

Definition 1.3. Let p be a prime number. Let $a/b \in \mathbb{Q}$ with $(a, b) = 1$. Let p^n be the biggest power of p dividing a/b , so $a/b = p^n (c/d)$ with $(c, p) = (d, p) = 1$. Then the **p -adic norm** is $|a/b|_p = p^{-n}$.

Example. $|1/6|_5 = 1$, $|1/6|_3 = 3$. $p^n \rightarrow 0$ into the p -adic topology as $n \rightarrow \infty$.

Lemma 1.4. $|\cdot|_p$ is a norm on \mathbb{Q} .

Proof.

1. Trivial.
2. Unique factorisation.
3. Prove $3' : |x + y|_p \leq \max(|x|_p, |y|_p)$. Without loss of generality, $x, y, x + y \neq 0$. Multiplying x, y by any power of p does not affect the truth of $3'$, so without loss of generality $x, y \in \mathbb{Z}$. Then we have to show that $p^r \mid x$ and $p^r \mid x + y$, which is obvious.

□

$3'$ is the **ultrametric inequality**.

Definition 1.5. If $|\cdot|$ satisfies $3'$, we say that $|\cdot|$ is **nonarchimedean**. Otherwise $|\cdot|_p$ is **archimedean**.

Say that two norms $|\cdot|_1, |\cdot|_2$ on a field k are equivalent if $|\cdot|_1 = |\cdot|_2^\alpha$ for some $\alpha > 0$. If two norms are equivalent, they define the same topology. **Ostrowski's theorem** states that up to equivalence, the only norms on \mathbb{Q} are the usual archimedean norm, the discrete norm, and the p -adic norms.

Lemma 1.6. If $|\cdot|$ is nonarchimedean and $|x| \neq |y|$, then $|x + y| = \max(|x|, |y|)$.

Proof. Without loss of generality, $|x| > |y|$. Write $x = (x + y) + (-y)$, so $|x| \leq \max(|x + y|, |-y|)$. So equality holds in all inequalities, so $|x| = |x + y|$. □

Example. Think about $D(a, r) = \{x \mid |x - a| < r\}$ and $D(b, r) = \{x \mid |x - b| < r\}$. What are the possibilities for $D(a, r) \cap D(b, r)$?

Example. TODO Exercise: directly prove Lemma 1.6 from the definition.

Let k be a field, let $|\cdot|$ be any norm on k . $d(x, y) = |x - y|$ is a metric, so k is a metric space with metric d . We have the usual definitions for sequences (x_n) . Say that x_n **converges** to x if for all $\epsilon > 0$, there exists N such that $n \geq N$ gives $|x_n - x| < \epsilon$. Say that (x_n) is **Cauchy** if $\epsilon > 0$, there exists N such that $m, n \geq N$ gives $|x_n - x_m| < \epsilon$. Say that (x_n) is **convergent** if x_n converges to x for some $x \in k$. Write $x_n \rightarrow x$. If (x_n) is convergent, then (x_n) is Cauchy. Say that k is **complete** with respect to $|\cdot|$ if all Cauchy sequences converge.

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Example. Let $k = \mathbb{Q}$ with the usual archimedean norm $|\cdot|$. $1, 1.4, 1.41, 1.414, \dots \rightarrow \sqrt{2} \notin \mathbb{Q}$.

Example. Let $k = \mathbb{Q}$ with $|\cdot|_2$. $3, 33, 333, 3333, \dots$ is a Cauchy sequence. In fact, if $m > n$, $|x_m - x_n|_2 = 2^{-n}$. So if $m, n \geq N$, $|x_m - x_n| \leq 2^{-N}$. $x_n = (10^n - 1)/3$ so $x_n + 1/3 = 10^n/3 = 2^n(5^n/3)$. $|x_n + 1/3|_2 = 2^{-n}$, so $x_n + 1/3 \rightarrow 0$.

Example. Let $k = \mathbb{Q}$ with $|\cdot|_5$. $|5^{2^n}|_5 = 5^{-2^n} \rightarrow 0$ as $n \rightarrow \infty$.

Example. Let $k = \mathbb{Q}$ with $|\cdot|_2$. (5^{2^n}) is $5, 25, 625, \dots \rightarrow 1$. Want to show that $5^{2^n} - 1 \rightarrow 0$, so that bigger and bigger powers of two divide $5^{2^n} - 1$. Use the lemma that $t \equiv 1 \pmod{2^k}$ gives $t^2 \equiv 1 \pmod{2^{k+1}}$.

Example. Let $k = \mathbb{Q}$ with $|\cdot|_7$. 5^{2^n} is not Cauchy. In fact $\pmod{7}$ it looks like $5, 4, 2, 4, 2, \dots$, so $|x_n - x_{n+1}|_7 = 1$.

Example. An example of a Cauchy sequence in \mathbb{Q} for some $|\cdot|_p$ which does not converge. Want to find a Cauchy sequence (x_n) with $x_n^2 \rightarrow 7$, so write down a sequence (x_n) such that $|x_n^2 - 7|_3 \rightarrow 0$. If $x_1 = 1$, then $x_1^2 \equiv 1 \pmod{3}$. If $x_2 = 4$, then $x_2^2 \equiv 7 \pmod{9}$. Find some $n \in \mathbb{N}$ such that $x_3 = 4 + 9n$ and $x_3^2 \equiv 7 \pmod{27}$. $x_3^2 = (4 + 9n)^2 = 16 + 8(9n) + (9n)^2 \equiv 16 + 8(9n) \pmod{27}$. So want $9 + 8(9n) \equiv 0 \pmod{27}$, so $1 + 8n \equiv 0 \pmod{3}$, such as $n = 1$. So $x_3 = 13$. Similar $x_4 = 13 + 27n$ and find some $n \in \mathbb{N}$.

Example. Show that for all primes p there is a Cauchy sequence (x_n) in \mathbb{Q} for $|\cdot|_p$ with the property that $x_n^2 \rightarrow t$ for some integer t which is not a perfect square, such as $t = 1 - p$ if $p > 2$ and $t = -7$ if $p = 2$. (TODO Exercise)

There is a general notion of completing a field with respect to a norm. The result is a field K containing k , for which all Cauchy sequences in k converge, and such that K is minimal with this property.

Example. Let $k = \mathbb{Q}$ with the archimedean norm $|\cdot|$, then $K = \mathbb{R}$.

Let k be a field and $|\cdot|$ be a norm on k . Let R be the ring of Cauchy sequences in k , so elements of R are (x_n) for $x_n \in k$, where $(x_n) + (y_n) = (x_n + y_n)$ and $(x_n)(y_n) = (x_n y_n)$.

Let $I = \{(x_n) \in R \mid x_n \rightarrow 0 \text{ as } n \rightarrow \infty\}$. Claim that I is an ideal in R . $(x_n), (y_n) \in I$ then $(x_n + y_n) \in I$ because $x_n \rightarrow 0$ and $y_n \rightarrow 0$ gives $x_n + y_n \rightarrow 0$. If $(x_n) \in R$, $(y_n) \in I$ then (x_n) is bounded, so $x_n y_n \rightarrow 0$.

Define the completion of k to be $\hat{k} = R/I$. Claim that \hat{k} is a field, so I is a maximal ideal of R . Need to show that if $(x_n) \in R$ with $x_n \not\rightarrow 0$, then there exists $(y_n) \in R$ such that $(x_n y_n) \in 1 + I$. $x_n \not\rightarrow 0$ gives ϵ, N such that $n \geq N$ gives $|x_n| \geq \epsilon > 0$. Set $y_n = 0$ if $n < N$ and $y_n = 1/x_n$ if $n \geq N$. Show that $(y_n) \in R$ (TODO Exercise). Then $(x_n y_n) = 0$ if $n < N$ and $(x_n y_n) = 1$ if $n \geq N$, so $x_n y_n \rightarrow 1$ as $n \rightarrow \infty$, as required.

k is a subfield of \hat{k} by $x \mapsto (x)$. Check that \hat{k} has a natural norm extending $|\cdot|$ on k , and \hat{k} is complete with respect to this norm, so $|(x_n)| = \lim_{n \rightarrow \infty} |x_n|$. (TODO Exercise)

Prove that if $|\cdot|$ on k is a nonarchimedean norm, then so is the induced norm on \hat{k} . (TODO Exercise) Furthermore, the sequence $|x_n|$ is eventually constant for any Cauchy sequence $(x_n) \in R \setminus I$. (TODO Exercise)

Note. For $|\cdot|_p$ on \mathbb{Q} this means that $|\cdot|$ is taking values in $p^{\mathbb{Z}}$.

Definition 1.7. The p -adic numbers \mathbb{Q}_p is the completion of \mathbb{Q} with respect to the p -adic norm. The p -adic integers \mathbb{Z}_p is the closed unit disc in \mathbb{Q}_p , so $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p = 1\}$.

Prove that \mathbb{Z}_p is a subring of \mathbb{Q}_p . Use that $|\cdot|_p$ is nonarchimedean. (TODO Exercise) Also $\mathbb{Q} \cap \mathbb{Z}_p = \{r/s \mid (r, s) = 1, p \nmid s\}$. In particular $\mathbb{Z} \subset \mathbb{Z}_p$. (TODO Exercise)

Prove that \mathbb{Q} is dense in \mathbb{Q}_p . Indeed k is dense in \hat{k} . (TODO Exercise) A less obvious fact is that \mathbb{Z} is dense in \mathbb{Z}_p .

Definition 1.8. Let k be a field with a norm $|\cdot|$, and (x_n) be any sequence. Then $\sum_{n=1}^{\infty} x_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n$, if this limit exists.

Lemma 1.9. If k is nonarchimedean and x_1, \dots, x_r satisfy $|x_i| \leq R$ for some R , then $|\sum_{n=1}^r x_i| \leq R$.

Proof. Induction with ultrametric inequality. □

Corollary 1.10. If k is nonarchimedean then (x_n) is Cauchy iff $x_n - x_{n+1} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Forward direction is obvious. Conversely, $m > n$ gives $x_m - x_n = (x_m - x_{m-1}) + \dots + (x_{n+1} - x_n)$. If $|x_{i+1} - x_i| \leq \epsilon$ for all $i \geq N$ then $|x_m - x_n| \leq \epsilon$ for all $m, n \geq N$ by Lemma 1.9. \square

Lemma 1.11. If k is complete and nonarchimedean then $\sum_{n=1}^{\infty} x_n$ converges iff $x_n \rightarrow 0$ as $n \rightarrow \infty$. If $|x_n|^{n-1} \leq R$ for all n and $x_n \rightarrow 0$ then $|\sum_{n=1}^{\infty} x_n| \leq R$.

Proof. Apply Lemma 1.9 and Corollary 1.10 to the sequence $(\sum_{i=1}^n x_i)$. \square

Lemma 1.12. If $a_n \in \mathbb{Z}$ for all $n \geq 0$, then $\sum_{n=0}^{\infty} a_n p^n$ converges in \mathbb{Q}_p . If $a_n = 0$ for $n < T$ and $p \nmid a_T$ then $|\sum_{n=0}^{\infty} a_n p^n| = p^{-T}$.

Proof. $|a_n p^n|_p \leq |p^n|_p = p^{-n} \rightarrow 0$ as $n \rightarrow \infty$, so $\sum_{n=0}^N a_n p^n$ converges by Lemma 1.11. If $N \geq T$ then $|\sum_{n=0}^N a_n p^n| = p^{-T}$. (TODO: Exercise, use lemma 1.6) \square

Proposition 1.13.

1. If $a_n \in \{0, \dots, p-1\}$, then $\sum_{n=0}^{\infty} a_n p^n \in \mathbb{Z}_p$. Further more if $a_n, b_n \in \{0, \dots, p-1\}$ and $\sum_{n=0}^{\infty} a_n p^n = \sum_{n=0}^{\infty} b_n p^n$ then $a_n = b_n$ for all n .
2. If $x \in \mathbb{Z}_p$ then there is a unique sequence (a_n) as in 1 with $x = \sum_{n=0}^{\infty} a_n p^n$.

Proof.

1. Convergence is Lemma 1.12. If $(a_n) \neq (b_n)$ let T be minimal such that $a_T \neq b_T$. Then

$$\left| \sum_{n=0}^{\infty} a_n p^n - \sum_{n=0}^{\infty} b_n p^n \right| = \left| \sum_{n=0}^{\infty} (a_n - b_n) p^n \right| = p^{-T}$$

by Lemma 1.12, so $\sum_{n=0}^{\infty} a_n p^n \neq \sum_{n=0}^{\infty} b_n p^n$.

2. Let $x \in \mathbb{Z}_p$. Since \mathbb{Q} is dense in \mathbb{Q}_p by construction, there exists $r/s \in \mathbb{Q}$, $(r, s) = 1$ with $|x - r/s|_p < 1$. Since $|x|_p \leq 1$, $|r/s|_p \leq 1$, so $p \nmid s$. So there exists $\gamma \in \mathbb{Z}$ with $|\gamma - r/s|_p < 1$, so $s\gamma \equiv r \pmod{p}$. Then $|\gamma - x|_p < 1$. Now choose $a_0 \in \{0, \dots, p-1\}$ with $\gamma \equiv a_0 \pmod{p}$. Then $|a_0 - x|_p < 1$. So $|(a_0 - x)/p|_p \leq 1$, and repeating the process, there exists $a_1 \in \{0, \dots, p-1\}$ with $|(a_0 - x)/p - a_1|_p < 1$, and so on. $|x - a_0 - a_1 p|_p \leq 1/p^2$, etc. \square

Corollary 1.14. Every element of \mathbb{Q}_p has a unique expression as $a = \sum_{n \geq -T} a_n p^n$ for $a_n \in \{0, \dots, p-1\}$, $a_{-T} \neq 0$, and $|a|_p = p^T$.

Proof. Given $a \in \mathbb{Q}_p$, let T be such that $|a|_p = p^T$. Apply Proposition 1.13 to $p^T a$. \square

Corollary 1.15. \mathbb{Z} is dense in \mathbb{Z}_p .

Proof. If $a \in \mathbb{Z}_p$, write $a = \sum_{n=0}^{\infty} a_n p^n$. Then $\sum_{n=0}^N a_n p^n$ is a sequence in \mathbb{Z} which converges to a . \square

For each $n \geq 1$ there is a ring homomorphism $\mathbb{Z}_p \rightarrow \mathbb{Z}/p^n \mathbb{Z}$ that sends $\sum_{i=0}^{\infty} a_i p^i \mapsto \sum_{i=0}^n a_i p^i$. The kernel of this map is $p^n \mathbb{Z}_p$, so $\mathbb{Z}_p / p^n \mathbb{Z}_p \cong \mathbb{Z} / p^n \mathbb{Z}$.

Note. In fact, $\mathbb{Z}_p \cong \lim_{\leftarrow n} \mathbb{Z} / p^n \mathbb{Z}$.

$p\mathbb{Z}_p$ is a maximal ideal of \mathbb{Z}_p because $\mathbb{Z}_p / p\mathbb{Z}_p = \mathbb{Z} / p\mathbb{Z}$ is a field. If $a \in \mathbb{Z}_p$, $a \notin p\mathbb{Z}_p$, then $a \in \mathbb{Z}_p^*$. Write $a = a_0 + pa_1 + \dots$ for $a_0 \in \{1, \dots, p-1\}$.

Example. $a = a_0 + pA$. Show that a has an inverse using the formula for the sum of a geometric progression. (TODO Exercise) For example $(1-p)^{-1} = 1 + p + \dots$

Alternatively, $a \in \mathbb{Z}_p$ is in $p\mathbb{Z}_p$ iff $|a|_p < 1$. So if $a \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$, then $|a|_p = 1$. Then $|1/a|_p = 1/|a|_p = 1$, so $1/a \in \mathbb{Z}_p$. $1/a$ exists because \mathbb{Q}_p is a field. So $p\mathbb{Z}_p$ is the unique maximal ideal of \mathbb{Z}_p , so \mathbb{Z}_p is a local ring. $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$ is the group of units. Terminology: $a \in \mathbb{Q}_p$ is a **unit** iff $a \in \mathbb{Z}_p^*$, so iff $|a|_p = 1$.

Lemma 1.16. $a \in \mathbb{Z}_p^*$ iff $|a|_p = 1$.

Proof. TODO - bring above paragraph down here. □

Corollary 1.17. Every element of $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$ is uniquely of the form $p^n u$ for $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^*$.

Proof. Set $|a|_p = p^{-n}$, then $|p^{-n}a|_p = 1$, so $u = p^{-n}a$, so $a = up^n$. □

Aim is to solve the equation $x^2 - 7 = 0$ in \mathbb{Z}_3 starting with a solution to $x^2 \equiv 7 \pmod{3}$. We did this by solving $x^2 \equiv 7 \pmod{3^n}$ for $n = 1, 2, \dots$. Write $f(x) = x^2 - 7$. Then $|f(1)|_3 = 1/3$, $|f(4)|_3 = 1/9$, $|f(13)|_3 = 1/27$, etc. So starting with $x_0 = 1$, we construct a sequence (x_n) with $|x_{n+1} - x_n| \leq 1/3^{n+1}$ and $|f(x_{n+1})|_3 \leq 1/3^n$. So in particular (x_n) is Cauchy, and so converges to some $x \in \mathbb{Q}_3$ with $f(x) = 0$. Newton-Raphson method over \mathbb{R} works over \mathbb{Q}_p , and this is called **Hensel's Lemma**.

Definition 1.18. If R is a ring and $f(X) \in R[X]$ is a polynomial, say $f(x) = \sum_{n=0}^m a_n x^n$, then $f'(x) = \sum_{n=0}^m n a_n x^{n-1}$.

In addition, for any x, h we have $f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \dots$. Why is $\frac{1}{2}f''(x)$ defined? If $f(x) = \sum_{n=0}^\infty a_n x^n$, $\frac{1}{2}f''(x) = \sum_{n=2}^\infty \frac{1}{2}n(n-1)a_n x^{n-2}$, where $n(n-1)$ is even.

Theorem 1.19 (Hensel's Lemma). Let k be a field which is complete with respect to a nonarchimedean norm $|\cdot|$. Let $R = \{x \in k \mid |x| \leq 1\}$, such as $k = \mathbb{Q}_p$, $R = \mathbb{Z}_p$, $|\cdot| = |\cdot|_p$. Let $f(x) \in R[x]$, and $t_0 \in R$ with $|f(t_0)| > |f'(t_0)|^2$. Then there is a unique $t \in R$ with $f(t) = 0$ and $|t - t_0| < |f'(t_0)|$. Moreover, $|f'(t)| = |f'(t_0)|$, and $|t - t_0| = |f(t_0)| / |f'(t_0)| < 1$.

Example. Let $k = \mathbb{Q}_3$ and $f(x) = x^2 - 7$. Then $f'(x) = 2x$. $t_0 = 1$, so $|f(t_0)|_3 = |-6|_3 = 1/3$. $|f'(t_0)|_3 = |2|_3 = 1$. So there is a unique $t \in \mathbb{Z}_3$ with $t^2 = 7$ and $|t - 1|_3 < 1$, so $t \equiv 1 \pmod{3}$.

Harder part of the proof of Hensel's Lemma is existence, which is done by constructing a sequence t_0, t_1, \dots as above. For uniqueness, suppose $s \neq t$ and $f(s) = f(t) = 0$, and $|s - t_0| < |f'(t_0)|$ and $|t - t_0| < |f'(t_0)|$, so $|s - t| \leq \max(|s - t_0|, |t - t_0|) < |f'(t_0)|$. Taylor expansion gives $f(s) = f(t) + (s - t)f'(t) + \dots$. So there exists $x \in R$ with $(s - t)f'(t) = (s - t)^2 = x$. So $f'(t) = (s - t)x$. $|f'(t)| = |s - t||x| \leq |s - t| < |f'(t_0)|$. So $|f'(t)| < |f'(t_0)|$. On the other hand it is easy to show that $|f'(t)| = |f'(t_0)|$, a contradiction. See handout for the full proof of Hensel's Lemma and some remarks about in what sense it is best possible.

What are the squares in \mathbb{Q}_p^* ? We know that we can write any element of \mathbb{Q}_p^* uniquely as $p^n u$ for u a unit, that is $|u|_p = 1$. Suppose that $p^n u = (p^r v)^2$ for $r \in \mathbb{Z}$ and v a unit. Then $p^n = p^{2r}$ and $u = v^2$, that is $n = 2r$ and $u = v^2$, that is the squares in \mathbb{Q}_p^* are exactly the $p^{2r}u$, where $u \in \mathbb{Z}_p^*$ is a square. If $p > 2$, we claim that u is a square iff u is a square modulo p , that is u is a quadratic residue. If $u = t^2$, then $u \equiv t^2 \pmod{p}$. Conversely, suppose that $u \equiv t_0^2 \pmod{p}$. Now use Hensel's lemma with $f(X) = X^2 - u$ and t_0 as above. $|f(t_0)|_p = |t_0^2 - u|_p \leq |p|_p = 1/p$. $|f'(t_0)|_p = |2t_0|_p = |t_0|_p = 1$. If $p \nmid t_0$ then $p \nmid u$ because $u \equiv t_0^2 \pmod{p}$. Hensel's lemma gives a unique $t \in \mathbb{Z}_p$ such that $t^2 = u$ and $|t - t_0| < 1$, that is $t \equiv t_0 \pmod{p}$. In this case, the equation $X^2 = u$ has exactly two solutions, namely t and $-t$. In any field, a polynomial of degree d has at most d roots. For $p = 2$, $u \in \mathbb{Z}_2^*$ is a square iff there exists t_0 such that $u \equiv t_0^2 \pmod{8}$, that is $u \equiv 1 \pmod{8}$, since $(2m+1)^2 = 4m(m+1) + 1 \equiv 1 \pmod{8}$.

(TODO missing)

2 Basic algebraic geometry

We want to consider curves in the plane given by the equations $f(x, y) = 0$. We want to understand the points of intersection of two such curves.

Example. If C has degree two and g has degree one, we expect to get in general two points of intersection.

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More generally, if f has degree a and g has degree b , can hope to get ab points of intersection. This can fail to be true in several different ways.

1. $y = x^2$ and $y = 0$, that is $f = y - x^2$, $g = y$, meet only at $(0,0)$. Solution is to count this point with multiplicity two. In this course, we will explain exactly when the multiplicity is one.
2. For $x^2 - y^2 = 0$ and $x^3 - y^3 = 0$, any point with $x = y$ is on the intersection. Solution is to demand that f and g are coprime, that is they have no common factor, unlike $x^2 - y^2 = (x - y)(x + y)$ and $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.
3. $y = x^2$ and $y = 2$ intersect at $(\pm\sqrt{2}, \sqrt{2})$. But if for example we are working over \mathbb{Q} , these points do not exist. Solution is to only work over fields which are algebraically closed, such as \mathbb{C} .
4. $y = x + 1$ and $y = x$ meet at infinity in projective space.

Definition 2.1. A field K is **algebraically closed** if every polynomial over K of degree at least one factors as a product of linear factors.

Example. $K = \mathbb{C}$ is algebraically closed. $K = \mathbb{Q}, \mathbb{R}$ is not algebraically closed. Any finite field is not algebraically closed.

A basic fact is if k is a field, there exists an algebraically closed field K with $k \subset K$, and in fact there exists a smallest possible choice, the **algebraic closure** of k .

Definition 2.2. If $f \in k[x_1, \dots, x_n]$ of degree d , then the **homogenisation** of f is the polynomial $F \in k[X_1, \dots, X_{n+1}]$ obtained as follows. If

$$f = \sum_{\underline{i}=(i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n} a_{\underline{i}} x_1^{i_1} \dots x_n^{i_n} \quad \implies \quad F = \sum_{\underline{i}} a_{\underline{i}} X_1^{i_1} \dots X_n^{i_n} X_{n+1}^{d-(i_1+\dots+i_n)}.$$

Given $G \in k[X_1, \dots, X_{n+1}]$ homogeneous of degree d , that is every monomial in G has degree exactly d , then for each $1 \leq i \leq n+1$, the i -th **dehomogenisation** is given by setting $X_i = 1$ and writing x_j for X_j .

Example. Let $G = X_1^2 X_2^2 + X_3^4 + X_2 X_3^2$, so $n = 2$. First dehomogenisation is $x_2^2 + x_3^4 + x_2 x_3^2$. Second dehomogenisation is $x_1^2 + x_3^4 + x_3^2$. Third dehomogenisation is $x_1^2 x_2^2 + 1 + x_2$.

If $n = 2$, usually write x, y and X, Y, Z for our variables.

Example. If $f = y^2 + x^3 + x + 1$, $F = Y^2 Z + X^3 + X Z^2 + Z^3$. Dehomogenisations of F are $y^2 z + 1 + z^2 + z^3$, $z + x^3 + x z^2 + z^3$, and $y^2 + x^3 + x + 1$.

Let k be a field, and define an equivalence relation on k^{n+1} by $(a_1, \dots, a_{n+1}) \sim (b_1, \dots, b_{n+1})$ iff there exists $\lambda \in k^*$ such that $b_i = \lambda a_i$ for $1 \leq i \leq n+1$.

Definition 2.3. For each $n \geq 0$, $\mathbb{P}^n(k)$, the **n -dimensional projective space over k** , is defined to be

$$k^{n+1} \setminus \{(0, \dots, 0)\} / \sim.$$

Write $[a_1 : \dots : a_{n+1}]$ for the equivalence class of (a_1, \dots, a_{n+1}) in $\mathbb{P}^n(k)$. This makes sense as long as some $a_i \neq 0$.

Let us see why we can think of $\mathbb{P}^n(k)$ as being k^n together with some points at infinity.

Example. $\mathbb{P}^1(k)$ should be k together with a point infinity. Any point of $\mathbb{P}^1(k)$ is of the form $[a_1 : a_2]$ with a_1, a_2 not both zero. If $a_2 \neq 0$, then $[a_1 : a_2] = [a_1/a_2 : 1]$. On the other hand, if $[a : 1] = [b : 1]$, then there exists $\lambda \in k^*$ such that $(a, 1) = \lambda(b, 1) = (b\lambda, \lambda)$, so $\lambda = 1$ and $a = b$. So there is a bijection between k and the points $[a_1 : a_2]$ with $a_2 \neq 0$, given by $k \rightarrow \mathbb{P}^1(k)$ by $a \mapsto [a : 1]$. The remaining points of $\mathbb{P}^1(k)$ are those with $a_2 = 0$. Then $[a_1 : 0] = [1 : 0]$, so we have exactly one point in $\mathbb{P}^1(k) \setminus k$, the point at infinity.

For each $1 \leq i \leq n+1$, let $\phi_i : k^n \rightarrow \mathbb{P}^n(k)$ be the map putting one in the i -th coordinate, that is $(a_1, \dots, a_n) \mapsto [a_1 : \dots : a_{i-1} : 1 : a_i : \dots : a_n]$.

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Example. $\phi_{n+1}((a_1, \dots, a_n)) = [a_1 : \dots : a_n : 1]$.

Lemma 2.4. Each map ϕ_i is injective.

Proof. If $[a_i : \dots : a_{i-1} : 1 : a_i : \dots : a_n] = [b_1 : \dots : b_{i-1} : 1 : b_i : \dots : b_n]$ then $(a_1, \dots, a_{i-1}, 1, a_i, \dots, a_n) = \lambda(b_1, \dots, b_{i-1}, 1, b_i, \dots, b_n)$. Looking at i -th coordinate, $\lambda = 1$, so $a_j = b_j$ for all j . \square

Usually think of k^n as lining inside $\mathbb{P}^n(k)$ via ϕ_{n+1} . Think of the complement as the points at infinity, that is the points $[x_1 : \dots : x_n : 0]$. Given a polynomial $F \in k[X_1, \dots, X_{n+1}]$, it never makes sense to ask what the value of F at $[a_1 : \dots : a_{n+1}] \in \mathbb{P}^n(k)$ is, because $[a_1 : \dots : a_{n+1}] = [\lambda a_1 : \dots : \lambda a_{n+1}]$. On the other hand, if F is homogeneous of degree d , then $F(\lambda a_1, \dots, \lambda a_{n+1}) = \lambda^d F(a_1, \dots, a_{n+1})$, so it does make sense to ask whether or not $F([a_1 : \dots : a_{n+1}]) = 0$. The **graph** of $F(X, Y, Z)$ homogeneous are the points P of $\mathbb{P}^2(k)$ where $F(P) = 0$. If F is the homogenisation of f , then the graph of F is the graph of f together with points at infinity. (TODO Exercise: $f(a, b) = 0$ iff $F(a, b, 1) = 0$)

Example. f is a line, say $f = y - mx - c$, so $F = Y - mX - cZ$. What are the points at infinity? If $Z = 0$ and $Y = mX$, that is $[1 : m : 0]$, the line $y = mx + c$ has one point at infinity, namely $[1 : m : 0]$. If $X = 0$, points at infinity are $Z = 0$, $[0 : Y : 0] = [0 : 1 : 0]$. So two lines meet at infinity iff they are parallel.

Proof of the following theorem is in Fulton's Algebraic Curves.

Theorem 2.5 (Bézout's theorem). Let F, G be homogeneous polynomials in $k[X, Y, Z]$ of degrees a, b respectively. If F and G have no common factors, and k is algebraically closed, then the graphs of $F = 0$ and $G = 0$ meet at ab points in $\mathbb{P}^2(k)$, counted with multiplicities.

Corollary 2.6. If F, G are as in Theorem 2.5, and for some k , $F = 0$ and $G = 0$ meet at more than ab points with multiplicity, then F and G have a common factor.

Definition 2.7. If $f \in k[x_1, \dots, x_n]$ is a polynomial, and let $P = (a_1, \dots, a_n) \in k^n$ with $f(P) = 0$. Then P is a **smooth** point of the graph of f , or a **nonsingular** point, if for some i , $(\partial f / \partial x_i)(P) \neq 0$. A singular point is one for which all $(\partial f / \partial x_i)(P) = 0$ for $1 \leq i \leq n$.

Example. $(\partial f / \partial x_2)(x_1 x_2^5) = 5x_1 x_2^4$.

Example. If $f = x^2 - y^5$, $\partial f / \partial x = 2x$ and $\partial f / \partial y = -5y^4$. $(0, 0)$ is singular and $(1, 1)$ is nonsingular.

Motivation is that tangent space to $f = 0$ at P should be $\sum_{i=1}^n (\partial f / \partial x_i)(P)(x_i - a_i) = 0$. Generalising to projective space, consider $F \in k[X_1, \dots, X_{n+1}]$ homogeneous. Consider P with $F(P) = 0$. Say that P is nonsingular if at least one $(\partial F / \partial X_i)(P) \neq 0$ for $1 \leq i \leq n+1$, singular if all $(\partial F / \partial X_i)(P) = 0$.

A fact is $Im(P) \in Im(\phi_i)$ for $\phi_i : k^n \rightarrow \mathbb{P}^n(k)$ insert 1 at i -th coordinate. Then P is nonsingular as a point of $F = 0$ iff the corresponding point $\phi_i^{-1}(P)$ is nonsingular for the i -th dehomogenisation of F . A key fact is if F is homogeneous of degree d , then $dF = \sum_{i=0}^{n+1} X_i (\partial F / \partial X_i)$.

Definition 2.8. Say the graph of f or F is nonsingular, or smooth, if it is nonsingular at every point of k^n or $\mathbb{P}^n(k)$ respectively.

Example. Any polynomial of degree one has a nonsingular graph. If $f = \sum_{i=1}^n a_i x_i$, then $\partial f / \partial x_i = a_i$.

Example. Let $k = \mathbb{Q}$.

1. $y^2 = x^2(x+1) = x^3 + x^2$. If $f = y^2 - x^3 - x^2$, then $\partial f / \partial x = -3x^2 - 2x$ and $\partial f / \partial y = 2y$. So if f is singular, $y = 0$, so $x = 0$ or $x = -1$, so $x = 0$. So $(0, 0)$ is the only singular point.
2. If $f = y^2$, then $\partial f / \partial x = 0$ and $\partial f / \partial y = 2y$. So any point $(x, 0)$ is singular.
3. $y^2 = (x^2 - 2)^2$ has singular points $(\pm\sqrt{2}, 0)$. Still say that this is singular, even though the singular points are not defined over \mathbb{Q} .

Example. If $y = x + 1$, then $F = Y - X - Z$. $\partial F / \partial X = -1$, $\partial F / \partial Y = 1$, $\partial F / \partial Z = -1$. This is non singular.

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Example. If $y = x^3$, then $F = YZ^2 - X^3$. $\partial F/\partial X = -3X^2$, $\partial F/\partial Y = Z^2$, $\partial F/\partial Z = 2YZ$. So $X = Z = 0$, that is $[0 : 1 : 0]$ is a singular point. Homogenise with respect to Y gives $z^2 - x^3 = 0$ singular at $(x, z) = (0, 0)$.

Theorem 2.9. If $f, g \in k[x, y]$ then a point P on $f = 0$ and $g = 0$ has multiplicity one iff

1. P is a nonsingular point for both $f = 0$ and $g = 0$, and
2. the tangent lines to $f = 0$ and $g = 0$ at P are distinct.

Tangent line at P is

$$(\partial f/\partial x)(P)(x - x(P)) + (\partial f/\partial y)(P)(y - y(P)) = 0.$$

3 Plane conics

Let k be a field. $k[X]$ is a PID, so a UFD. $k[X_1, \dots, X_n]$ is a UFD, but not a PID.

Example. (X_1, X_2) in $k[X_1, X_2]$ is not principal.

If k is algebraically closed, then the prime ideals of $k[X]$ are just (0) and $(X - \alpha)$ for $\alpha \in K$. If k is not algebraically closed, you have more.

Example. $(X^2 - 2)$ is a prime ideal of $\mathbb{Q}[X]$.

Even if k is algebraically closed, there are many prime ideals in $k[X, Y]$ than in $k[X]$.

Example. $X^2 - Y^2 - 1$ does not factor in $\mathbb{C}[X, Y]$.

A plane conic is just the graph of some $f \in k[x, y]$ where f has degree two. Same terminology also covers $F = 0$ for $F \in k[X, Y, Z]$ homogeneous of degree two. The next steps are

1. to understand when this is singular,
2. to understand how to find all solutions to $f = 0$ or $F = 0$ if it is singular, and
3. to understand how to find all solutions to $f = 0$ in the nonsingular case given one solution.

Then in the next section we will specialise to $k = \mathbb{Q}$ and see how to find solutions, or prove there are not any.

Example. $x^2 - y^2$ is singular, because $(0, 0)$ is a singular point.

Algorithm 3.1 (Checking if $f = 0$ is singular). Suppose $f \in k[x, y]$ degree two. Then any singular points are given by solving $\partial f/\partial x = 0$, $\partial f/\partial y = 0$. These are linear, so solve them and substitute back into $f = 0$.

Example. Let $f = x^2 - y^2 - 1$. $\partial f/\partial x = 2x$ and $\partial f/\partial y = -2y$ gives $x = y = 0$, so $f(0, 0) \neq 0$. So f is nonsingular.

Example. Let $f = x^2 - 2xy + y^2 = (x - y)^2$. $\partial f/\partial x = 2x - 2y$ and $\partial f/\partial y = 2y - 2x$. If characteristic of $k \neq 2$, solutions are $x = y$, and any such point is a solution. So every point is singular.

Theorem 3.2. Let k be algebraically closed, and let $f \in k[X, Y, Z]$ be homogeneous of degree two. Then f has a singular point in $\mathbb{P}^2(k)$ iff F is a product of two linear factors.

Proof. If F factors, then any point of intersection of the two lines is a singular point. (TODO Exercise) The lines do meet, by Bézout. Suppose conversely that P is a singular point of $F = 0$. Let Q be any other point of $F = 0$, and let L be the line joining P and Q , so L is given by some equation $G = 0$, where G has degree one. If $F = GH$ for some H , then we are done. Otherwise, F and G are coprime because G has degree one, so has no proper factors. So by Bézout's theorem, $F = 0$ and $G = 0$ meet in exactly two points, counted with multiplicity. Since P is a singular point of $F = 0$, it follows from Theorem 2.9 that the multiplicity of P is at least $2 + 1 = 3$, a contradiction. So G is a factor of F , as required. \square

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Algorithm 3.3 (Finding all rational points on a singular conic).

1. Find the singular point.
2. Write down another point.
3. Factor the conic as a product of linear factors.
4. Solve the two linear equations.

Example. Let $F = X^2 - 2Y^2$. Singular point is $2X = -4Y = 0$, so $X = Y = 0$. Any other point is $Y = 1$ and $X = \sqrt{2}$. $X = \sqrt{2}Y$ is the line through these two points. So $X - \sqrt{2}Y$ is a factor of $X^2 - 2Y^2$, and indeed $X^2 - 2Y^2 = (X + \sqrt{2}Y)(X - \sqrt{2}Y)$. So if $k = \mathbb{C}$, then the solutions are $(\pm\sqrt{2}t, t)$ for $t \in \mathbb{C}$. If $k = \mathbb{Q}$, the only solution is $(0, 0)$.

Example. Now for a nonsingular case, let $x^2 + 2y^2 = 6$. Start with one solution, such as $(2, 1)$. Find all solutions by drawing lines through this point.

Algorithm 3.4. Let F be a nonsingular conic over some field k , and suppose that we are given a point with coordinates in k where $F = 0$. Then we can find all points by drawing lines with rational slope through this point, and finding the second point of intersection with $F = 0$.

Note. If k is algebraically closed, then Bézout tells us that we have exactly two points of intersection, when counted with multiplicity. Since f is nonsingular, and any line is nonsingular, the only way you can have a multiplicity is if the line you draw is the tangent line at that point. If k is not algebraically closed, just need to check that the other point of intersection has coordinates in k iff the slope of the line joining the points is in k . (TODO Exercise)

Lecture 10 is a problem class.

4 The Hasse principle

We have seen that for any field k , and any plane conic, there are algorithms to

1. check if the conic is singular or not,
2. if it is singular, write it as a product of two linear factors, and so find all the points over k , and
3. if it is nonsingular, or smooth, find all points over k given one such point, by drawing lines through this point.

What is missing is given a nonsingular curve, determine if it has any points over k , and find such a point if there is one. For $k = \mathbb{Q}$, we will do this by using the cases $k = \mathbb{Q}_p$ for all primes p . Over \mathbb{Q}_p , it is not so hard to check if there are points. Use Hensel's lemma to reduce modulo p , use quadratic reciprocity. Very easy over \mathbb{R} . Hasse's principle says a nonsingular plane conic over \mathbb{Q} has points over \mathbb{Q} if and only if it does over \mathbb{Q}_p and \mathbb{R} for all p . Forward direction is trivial, converse direction is harder. Crucially, this holds for conics, that is polynomials of degree two, but not true for degree three or higher.

Example. $3X^3 + 4Y^3 + 5Z^3$.

Example. $X^2 + Y^2 = 6$ has no solution in \mathbb{Q}_3 gives none in \mathbb{Q} . $X^2 + Y^2 = -1$ has no solution in \mathbb{R} gives none in \mathbb{Q} .

Basic idea is if we have solutions to some equation in every \mathbb{Q}_p , we have solutions modulo N for all $N \in \mathbb{Z}$. Use Chinese remainder theorem to reduce to $N = p^k$, Hensel's Lemma says that equations have solutions in \mathbb{Z}_p iff modulo p^k for all k . We would like a technique for upgrading solutions to congruences to solutions to equations in \mathbb{Q} or in \mathbb{Z} . Aim is to prove that if $p \equiv 1 \pmod{4}$ then $p = x^2 + y^2$ for some $x, y \in \mathbb{Z}$. Input is n such that $n^2 \equiv -1 \pmod{p}$.

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Lemma 4.1. If $U \subset \mathbb{R}^n$ is measurable, or just open, and its measure, or volume, is bigger than m , then we can find distinct points c_0, \dots, c_m such that for all i , $c_0 - c_i \in \mathbb{Z}^n$.

Proof. Let C be the unit cube, $C = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \forall i, 0 \leq x_i < 1\}$. By assumption $\int_{\mathbb{R}^n} 1_U(x) dx > m$ where

$$1_U(x) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases}.$$

$\int_{\mathbb{R}^n} 1_U(x) dx = \int_C \sum_{t \in \mathbb{Z}^n} 1_U(x+t) dx > m$. The measure of C is 1, so there exists $x \in C$ such that $\sum_{t \in \mathbb{Z}^n} 1_U(x+t) > m$. So there exists $t_0, \dots, t_m \in \mathbb{Z}^n$, $x \in C$ such that $x+t_i \in U$ for all i . Set $c_i = x+t_i$. \square

Definition 4.2. Say U is **convex** if $x, y \in U$ and $\lambda \in [0, 1]$ gives $\lambda x + (1-\lambda)y \in U$. Say that U is **symmetric** if $x \in U$ gives $-x \in U$.

Corollary 4.3 (Minkowski). If $\Lambda \in \mathbb{Z}^n$ is a finite index subgroup, and if V is convex and symmetric, with measure greater than $2^n [\mathbb{Z}^n : \Lambda]$, then $V \cap \Lambda \neq \{0\}$.

Remark 4.4. $0 \in V$, because given $x \in V$, $-x \in V$, so $\frac{1}{2}(x + (-x)) \in V$.

A **lattice** is a finite index subgroup of \mathbb{Z}^n .

Proof. Set $U = \frac{1}{2}V = \{\frac{1}{2}x \mid x \in V\}$. So measure of U is greater than $[\mathbb{Z}^n : \Lambda] = m$. By Lemma 4.1, there exists $c_0, \dots, c_m \in U$ such that $c_i - c_0 \in \mathbb{Z}^n$ for all i . Consider $c_0 - c_0, \dots, c_m - c_0 \in \mathbb{Z}^n$. We have $m+1$ differences, so there exists $i \neq j$ such that $(c_i - c_0) - (c_j - c_0) \in \Lambda$, that is $c_i - c_j \in \Lambda$, $c_i - c_j \neq 0$. Since $c_i - c_j = \frac{1}{2}(2c_i - 2c_j)$, $2c_i, 2c_j \in V$ by definition, and $-2c_j \in V$ by symmetry, so $\frac{1}{2}(2c_i - 2c_j) \in V$ by convexity. \square

How do you use this to prove that $p = x^2 + y^2$?

Theorem 4.5. If $n \in \mathbb{Z}$ and there exists t such that $t^2 \equiv -1 \pmod{n}$, then there exists $x, y \in \mathbb{Z}$ such that $x^2 + y^2 = n$.

Proof. $\Lambda = \{(x, y) \mid y \equiv tx \pmod{n}\} \subseteq \mathbb{Z}^2$. $\Lambda = \text{Ker}(\mathbb{Z}^2 \rightarrow \mathbb{Z}/n\mathbb{Z})$ by $(x, y) \mapsto y - tx$. This is surjective, so by the first isomorphism theorem, $\mathbb{Z}^2/\Lambda \cong \mathbb{Z}/n\mathbb{Z}$. So $[\mathbb{Z}^2 : \Lambda] = \#(\mathbb{Z}/n\mathbb{Z}) = n$. Let $V = \{(x, y) \mid x^2 + y^2 < 2n\}$. Measure of V is $\pi(\sqrt{n})^2 = 2\pi n = 2^2 [\mathbb{Z}^2 : \Lambda] = 4n$ because $\pi > 2$. So by Corollary 4.3, there exists $(x, y) \in \Lambda \cap V$ such that $(x, y) \neq (0, 0)$. $(x, y) \in V$ gives $0 < x^2 + y^2 < 2n$. $(x, y) \in \Lambda$ gives $x^2 + y^2 \equiv x^2(t^2 + 1) \equiv 0 \pmod{n}$. So $x^2 + y^2 = n$, as required. \square

Want to consider plane conics over \mathbb{Q} , that is homogeneous equations of degree two in X, Y, Z . Completing the square, that is existence of orthogonal basis, can write this in the form $aX^2 + bY^2 + cZ^2$. If this is nonsingular, $abc \neq 0$. Conversely if $abc \neq 0$, this is nonsingular. Rescaling X, Y, Z , we can change a, b, c by squares of elements of \mathbb{Q}^* . So without loss of generality, a, b, c are all in \mathbb{Z} . We can then furthermore arrange by rescaling again that each of a, b, c is squarefree, that is not divisible by the square of any prime. If a, b, c share a common factor, we can divide by it, so we can assume that $(a, b, c) = 1$. If $p \mid a$, $p \mid b$, so $p \mid c$, we can replace (a, b, c) by $(a/p, b/p, pc)$ since $aX^2 + bY^2 + cZ^2 = p((a/p)X^2 + (b/p)Y^2 + pc(Z/p)^2)$. Repeating this, we can assume that $(a, b) = 1$, $(b, c) = 1$, $(c, a) = 1$. So without loss of generality $aX^2 + bY^2 + cZ^2$, a, b, c nonzero squarefree integers, pairwise coprime. Let $\Sigma = \{p \mid 2abc\}$ for p prime.

Lemma 4.6. Write $F = aX^2 + bY^2 + cZ^2$ as above. Then the following are equivalent.

1. $F = 0$ has infinitely many solutions in $\mathbb{P}^2(\mathbb{Q})$.
2. $F = 0$ has a solution in $\mathbb{P}^2(\mathbb{Q})$.
3. $F = 0$ has solutions in $\mathbb{P}^2(\mathbb{Q}_p)$ for all p and in $\mathbb{P}^2(\mathbb{R})$.
4. $F = 0$ has solutions in $\mathbb{P}^2(\mathbb{Q}_p)$ for all $p \in \Sigma$.

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Proof. $1 \implies 2 \implies 3 \implies 4$. $2 \implies 1$ by drawing lines through a point. So we need to do $4 \implies 2$. Use Corollary 4.3 in \mathbb{Z}^3 . Want to find V, Λ with measure of V greater than $2^3 [\mathbb{Z}^3 : \Lambda]$, so that $V \cap \Lambda = \{0\}$. Want to know that we will have $ax^2 + by^2 + cz^2 = 0$ if $(x, y, z) \in V \cap \Lambda$. Choose $V = \{|a|x^2 + |b|y^2 + |c|z^2 < 4|abc|\}$. This has measure $\frac{\pi}{3} \cdot 2^3 \cdot 4|abc|$. So to prove the theorem, it is enough to find Λ with $[\mathbb{Z}^3 : \Lambda] = 4|abc|$, and satisfying $(x, y, z) \in \Lambda$ gives $ax^2 + by^2 + cz^2 \equiv 0 \pmod{4|abc|}$. So if $(x, y, z) \in V \cap \Lambda$ then $0 \leq |ax^2 + by^2 + cz^2| < |a|x^2 + |b|y^2 + |c|z^2 < 4|abc|$, so $ax^2 + by^2 + cz^2 = 0$. \square

Lemma 4.7. If $p \mid a$ then there is a solution to $b + cz^2 \equiv 0 \pmod{p}$.

Proof. $p \mid a$ gives $p \in \Sigma$ so there exists $x, y, z \in \mathbb{Q}_p$ not all zero such that $ax^2 + by^2 + cz^2 = 0$. By multiplying by an appropriate power of p , we can assume that $\max(|x|_p, |y|_p, |z|_p) = 1$. $p \mid a$ and $(a, b) = (a, c) = 1$ gives $p \nmid b$, $p \nmid c$, that is $|a|_p < 1$ and $|b|_p = |c|_p = 1$. If $|y|_p < 1$ then $|by^2|_p < 1$, $|ax^2|_p \leq |a|_p < 1$, so $|cz^2|_p = |ax^2 + by^2| < 1$, so $|z|_p < 1$. Then $|x|_p = 1$, so $|ax^2|_p = |a|_p$ but $|ax^2|_p = |by^2 + cz^2|_p \leq \max(|y|_p^2, |z|_p^2) \leq 1/p^2$. So $p^2 \mid a$, a contradiction, as a is squarefree. So $|y|_p = 1$. Now divide by y , $a(x/y)^2 + b + c(z/y)^2 = 0$ and $x/y, z/y \in \mathbb{Z}_p$. Reduce modulo p gives $b + c(z/y)^2 \equiv 0 \pmod{p}$, as claimed. \square

If $p \in \Sigma$ and $p > 2$, then by symmetry without loss of generality $p \mid a$. Then the lemma applies, so there exists α with $b + c\alpha^2 \equiv 0 \pmod{p}$. So we impose the condition that $z \equiv \alpha y \pmod{p}$. Then if $x, y, z \in \mathbb{Z}$ satisfying $z \equiv \alpha y \pmod{p}$, then $ax^2 + by^2 + c \equiv by^2 + cz^2 \equiv by^2 + c\alpha^2 y^2 \equiv y^2(b + c\alpha^2) \equiv 0 \pmod{p}$. Now have to deal with $p = 2$. Suppose $2 \mid a$. Lemma gives a solution to $b + cz^2 \equiv 0 \pmod{2}$, and a solution to $ax^2 + by^2 + cz^2 = 0$ in \mathbb{Q}_2 with $|y|_2 = 1$, so $y^2 \equiv 1 \pmod{8}$. $(2m+1)^2 = 4m(m+1) + 1 \equiv 1 \pmod{8}$. In fact, squares modulo 8 are 0, 1, 4. Similarly, $|z|_2 = 1$ gives $z^2 \equiv 1 \pmod{8}$. Since $ax^2 + by^2 + cz^2 = 0$, $ax^2 + by^2 + cz^2 \equiv 0 \pmod{8}$, that is $ax^2 + b + c \equiv 0 \pmod{8}$. If $|x|_2 = 1$ then $x^2 \equiv 1 \pmod{8}$, so $a + b + c \equiv 0 \pmod{8}$. Impose

$$y \equiv z \pmod{4}, \quad x \equiv y \pmod{2}.$$

If $(x, y, z) \in \mathbb{Z}^3$ satisfies this condition, then either x, y, z all odd, and $ax^2 + by^2 + cz^2 \equiv a + b + c \equiv 0 \pmod{8}$, or x, y, z all even, and $ax^2 + by^2 + cz^2 \equiv by^2 + cz^2 \pmod{8}$. $y \equiv z \pmod{4}$ gives either $4 \mid y$, $4 \mid z$, which is fine, or $y \equiv z \equiv 2 \pmod{4}$, and $by^2 + cz^2 \equiv 4(b + c) \equiv -4a \equiv 0 \pmod{8}$. If $|x|_2 < 1$, then $ax^2 \equiv 0 \pmod{8}$, so $b + c \equiv 0 \pmod{8}$. Impose

$$y \equiv z \pmod{4}, \quad x \equiv 0 \pmod{2}.$$

Then if these conditions hold, $ax^2 + by^2 + cz^2 \equiv by^2 + cz^2 \equiv (b + c)y^2 \equiv 0 \pmod{8}$. The cases $2 \nmid b$, $2 \nmid c$ are dealt with by symmetry. Now suppose that $2 \nmid abc$. Again $2 \in \Sigma$ gives by assumption $x, y, z \in \mathbb{Q}_2$ not all zero with $ax^2 + by^2 + cz^2 = 0$. Again, without loss of generality $\max(|x|_2, |y|_2, |z|_2) = 1$ or without loss of generality $|z|_2 = 1$, so $|cz^2|_2 = 1$, so at least one of $|x|_2, |y|_2$ is 1. Without loss of generality $|y|_2 = 1$. If $|x|_2 = 1$ then $ax^2 + by^2 + cz^2 \equiv 1 + 1 + 1 \pmod{2}$, a contradiction. So $|x|_2 < 1$. So $ax^2 + by^2 + cz^2 \equiv b + c \pmod{4}$, that is $b + c \equiv 0 \pmod{4}$. Impose

$$x \equiv 0 \pmod{2}, \quad y \equiv z \pmod{2}.$$

If these hold, $ax^2 + by^2 + cz^2 \equiv by^2 + cz^2 \equiv (b + c)y^2 \equiv 0 \pmod{4}$. What have we done? If $p > 2$ and $p \in \Sigma$, we found a congruence modulo p which guaranteed that $ax^2 + by^2 + cz^2 \equiv 0 \pmod{p}$. If $2 \mid abc$, found conditions guaranteeing that $ax^2 + by^2 + cz^2 \equiv 0 \pmod{8}$. If $2 \nmid abc$, $ax^2 + by^2 + cz^2 \equiv 0 \pmod{4}$. Define Λ as the subgroup of \mathbb{Z}^3 satisfying all of these congruence conditions. Since abc is squarefree, if $(x, y, z) \in \Lambda$ then $ax^2 + by^2 + cz^2 \equiv 0 \pmod{4abc}$. It remains to check that $[\mathbb{Z}^3 : \Lambda] = 4|abc|$. To see this, write Λ as the kernel of a map from \mathbb{Z}^3 to a product of a group of order four or eight and $\prod_{p>2, p \mid abc} \mathbb{Z}/p\mathbb{Z}$ and use the Chinese remainder theorem.

Lecture 14 is a problem class.

Lecture 13
Thursday
01/11/18

Lecture 14
Friday
02/11/18