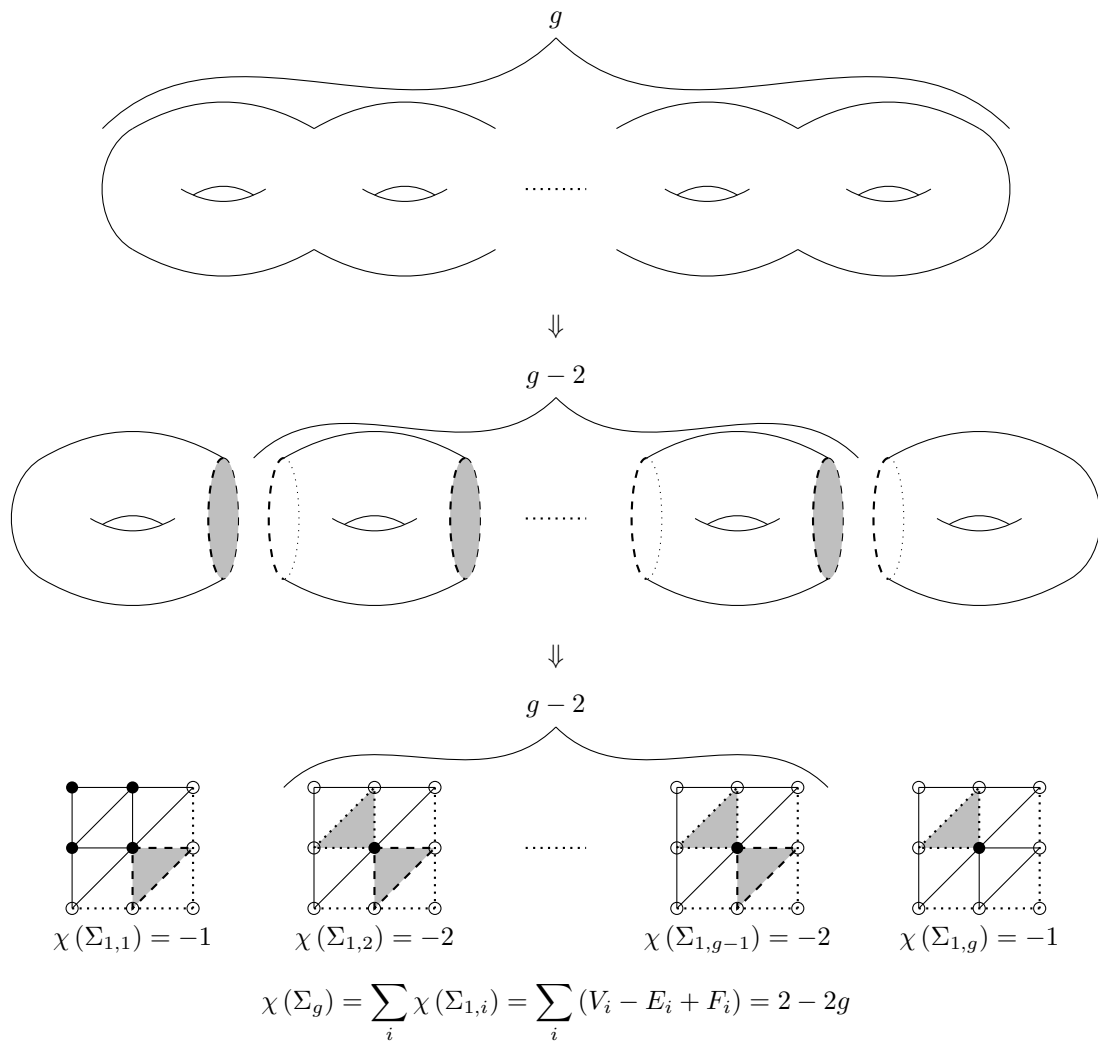


M3P5 Geometry of Curves and Surfaces

Lectured by Prof Tom Coates
 Typed by David Kurniadi Angdinata

Autumn 2018



Syllabus

Parametrisations of curves in three-dimensional space. Curvature. Torsion. Frenet–Serret formulae. Winding number. Charts of surfaces. Tangent vectors. Tangent planes. Smooth maps between surfaces. Normal vectors. The first fundamental form. The second fundamental form. Christoffel symbols. Normal curvature. Gaussian curvature. Mean curvature. Gauss’s Theorema Egregium. Area of surfaces. Length-minimising curves. Geodesic curvature. The Gauss–Bonnet theorem and applications.

Contents

0	Introduction	3
1	Curves in three-dimensional space	4
1.1	What is a curve?	4
1.2	Curvature	7
1.3	Space curves	8
1.4	Plane curves	11
2	Surfaces	13
2.1	Regular surfaces	13
2.2	Tangent vectors and tangent planes	16
2.3	Normal vectors	20
3	Curvature	21
3.1	The second fundamental form	21
3.2	Normal curvature, Gaussian curvature, and mean curvature	22
3.3	What does Gaussian curvature mean?	24
3.4	How to compute Gaussian and mean curvature?	25
3.5	The first fundamental form	27
3.6	Christoffel symbols	29
3.7	Theorema Egregium	30
4	Area of surfaces	32
4.1	Area	32
5	Geodesics	34
5.1	Geodesic curvature	34
5.2	Length-minimising curves	36
6	The Gauss-Bonnet theorem and applications	37
6.1	Local version of Gauss-Bonnet	37
6.2	Gauss-Bonnet for curvilinear triangles	39
6.3	Gauss-Bonnet theorem	40

0 Introduction

Lecture 1

A question is what does it mean for a surface to be curved, more curved, less curved, or differently curved? In fact, what does it mean for a curve to be curved? The goal is to answer these questions. Touches on

Friday

05/10/18

- manifolds, smooth shapes, differential geometry,
- topology, and
- Riemannian geometry, general relativity.

The following are references.

- C Bär, Elementary differential geometry, 2010

A tentative outline of the material that we will cover is as follows. This may change as the term progresses. If so then an updated outline will be given during lectures.

- Curves in three-dimensional space.
 - Parametrisations.
 - Curvature and torsion, Frenet–Serret formulae.
 - Curves are determined by curvature and torsion.
 - Winding number.
- Surfaces.
 - Charts.
 - Tangent vectors and tangent planes.
 - Smooth maps between surfaces.
 - Normal vectors.
- Curvature.
 - The first and second fundamental forms.
 - Christoffel symbols.
 - Normal curvature, Gaussian curvature, and mean curvature.
 - Gauss’s Theorema Egregium.
- Area of surfaces.
- Geodesics.
 - Length-minimising curves.
 - Existence, non-existence, and examples.
 - Geodesic curvature.
- The Gauss–Bonnet theorem and applications.
- The topological classification of surfaces.
- Vector fields and the Poincaré–Hopf theorem.

1 Curves in three-dimensional space

Ultimate goal is surfaces. First is curves. What does it mean for a curve to be curved? In fact, what is a curve?

1.1 What is a curve?

Definition 1.1.1. The n -dimensional Euclidean space \mathbb{R}^n consists of

$$(x_1, \dots, x_n) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in \mathbb{R}.$$

\mathbb{R}^n is a vector space and an inner product space. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ then

$$x \cdot y = \sum_{i=1}^n x_i y_i$$

is the **inner product** of x and y . The **length** of $x \in \mathbb{R}^n$ is

$$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^n x_i^2}.$$

Focus on \mathbb{R}^2 and \mathbb{R}^3 .

Definition 1.1.2. A **parametrised curve** in \mathbb{R}^n is a smooth map $\phi : [a, b] \rightarrow \mathbb{R}^n$. It is **regular** if $\phi'(t) \neq 0$ for all $t \in [a, b]$.

The curve is $\phi([a, b])$. The parametrised curve says where a particle moving along this curve is positioned at time t . This is $\phi(t)$.

Example.

$$\begin{array}{ll} \phi : [0, 2\pi] & \longrightarrow \mathbb{R}^2 \\ t & \longmapsto (\cos(t), \sin(t)) \end{array}, \quad \begin{array}{ll} \phi : [0, 1] & \longrightarrow \mathbb{R}^3 \\ t & \longmapsto (\cos(4\pi t), \sin(4\pi t), t^4) \end{array}$$

are curves.

Example.

$$\begin{array}{ll} \phi : [-1, 1] & \longrightarrow \mathbb{R}^2 \\ t & \longmapsto (t, |t|) \end{array}$$

is not a curve.

For us, all parametrised curves will be regular.

Example.

$$\begin{array}{ll} \phi : [-1, 1] & \longrightarrow \mathbb{R}^2 \\ t & \longmapsto (t^2, t^3) \end{array}$$

is not a curve, since $\phi'(t) = (2t, 3t^2)$ and $\phi'(0) = (0, 0)$.

We are not interested in the parametrisation, just the curve.

Example.

$$\begin{array}{ll} \phi : [0, 1] & \longrightarrow \mathbb{R}^2 \\ t & \longmapsto (\cos(2\pi t), \sin(2\pi t)) \end{array}.$$

Definition 1.1.3. Given a regular parametrised curve $\phi : [a, b] \rightarrow \mathbb{R}^n$ in \mathbb{R}^n and a smooth map $f : [c, d] \xrightarrow{\sim} [a, b]$, with $f'(t) \neq 0$ for all t , the curve $\varphi : [c, d] \rightarrow \mathbb{R}^n$, where $\varphi = \phi \circ f$, is called a **reparametrisation** of ϕ .

Proposition 1.1.4. If ϕ is regular and $f : [c, d] \xrightarrow{\sim} [a, b]$ is smooth with $f'(t) \neq 0$ for all t , then $\varphi = \phi \circ f$ is also a regular parametrised curve.

Proof. Chain rule implies that

$$\varphi'(t) = \phi'(f(t)) \cdot f'(t).$$

Then $\phi'(f(t))$ is never zero because ϕ is regular and $f'(t)$ is never zero by assumption. Thus $\varphi'(t) \neq 0$. \square

So will study parametrised curves up to reparametrisation and study reparametrisation-invariant properties of parametrised curves.

Remark. Reparametrisations $f : [c, d] \xrightarrow{\sim} [a, b]$ with $f'(t) \neq 0$ form a groupoid under composition of functions. The groupoid acts on parametrised curves.

Say that $\phi \sim \varphi$ if and only if ϕ is a reparametrisation of φ . Then \sim is an equivalence relation. A **curve** is an equivalence class of parametrised curves. The **trace** of a curve $\phi : [a, b] \rightarrow \mathbb{R}^n$ is $\phi([a, b])$, that is the image in \mathbb{R}^n . If $\phi \sim \varphi$ then the trace of ϕ is equal to the trace of φ , so this is well-defined.

Lecture 2
Monday
08/10/18

Example. The tangent line to a curve at a point.

Definition 1.1.5. The **tangent line** L to ϕ at $\phi(t_0)$ is

$$L = \{\phi(t_0) + s\phi'(t_0) \mid s \in \mathbb{R}\}.$$

Check that this is a reparametrisation-invariant. Suppose $f : [c, d] \xrightarrow{\sim} [a, b]$ is a reparametrisation and $\varphi = \phi \circ f$. Let $s_0 \in [c, d]$ satisfy $f(s_0) = t_0$. Then

$$\varphi(s_0) = \phi(f(s_0)) = \phi(t_0), \quad \varphi'(s_0) = \phi'(f(s_0)) \cdot f'(s_0) = \phi'(t_0) \cdot f'(s_0),$$

so

$$L = \{\varphi(s_0) + s'\varphi'(s_0) \mid s' \in \mathbb{R}\}.$$

Definition 1.1.6. The **length** of a parametrised curve $\phi : [a, b] \rightarrow \mathbb{R}^n$ is

$$L(\phi) = \int_a^b |\phi'(t)| \, dt.$$

Chopping up $[a, b]$ into N intervals of size $\Delta t = (b - a)/N$,

$$L(\phi) \approx \sum_{i=0}^N |\phi'(a + i\Delta t)| \Delta t.$$

Example. Let

$$\begin{aligned} \phi & : [0, 2\pi] \longrightarrow \mathbb{R}^2 \\ t & \longmapsto (\cos(t), \sin(t)) \end{aligned}.$$

Then $\phi'(t) = (-\sin(t), \cos(t))$, so

$$|\phi'(t)| = \sqrt{\sin^2(t) + \cos^2(t)} = 1.$$

Thus

$$L(\phi) = \int_0^{2\pi} 1 \, dt = 2\pi,$$

so the length is 2π .

Proposition 1.1.7. $L(\phi)$ is invariant under reparametrisation.

Proof. For simplicity, suppose $n = 3$, so \mathbb{R}^3 . Then

$$\begin{aligned} \phi &: [a, b] \longrightarrow \mathbb{R}^3 \\ t &\longmapsto (x(t), y(t), z(t)) \end{aligned}.$$

Thus

$$L(\phi) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt.$$

Given a reparametrisation $f : [c, b] \xrightarrow{\sim} [a, b]$, so f is smooth and $f'(t) \neq 0$ for all t , set $\varphi = \phi \circ f$ and write

$$\varphi(s) = (X(t), Y(t), Z(t)).$$

Then

$$L(\varphi) = \int_c^d \sqrt{\left(\frac{dX}{ds}\right)^2 + \left(\frac{dY}{ds}\right)^2 + \left(\frac{dZ}{ds}\right)^2} ds.$$

Thus use the change of variable formula and

$$\frac{dX}{ds}(s) = \frac{dx}{dt}(f(s)) \frac{df}{ds}(s),$$

by the chain rule. □

If $|\phi'(t)| = 1$ for all t then

$$L(\phi) = \int_a^b |\phi'(t)| dt = b - a,$$

and in fact $L(\phi|_{[a,t]}) = t - a$, so the curve is parametrised by arc-length. This is an **arc-length parametrisation** or a **unit speed parametrisation**.

Proposition 1.1.8. Let $\phi : [a, b] \rightarrow \mathbb{R}^n$ be a regular curve. Then there exists an arc-length parametrisation of ϕ .

Proof. Let

$$\begin{aligned} l &: [a, b] \longrightarrow [0, L] \\ t &\longmapsto \int_0^t |\phi'(s)| ds \end{aligned}.$$

Set $f = l^{-1}$ as the inverse function. Then if $\varphi = \phi \circ f$ we have

$$|\varphi'(t)| = |\phi'(f(t))| f'(t).$$

Also $l(f(t)) = t$, so $l'(f(t)) f'(t) = 1$. Thus

$$f'(t) = \frac{1}{l'(f(t))} = \frac{1}{|\phi'(f(t))|},$$

by the fundamental theorem of calculus, so we have $|\varphi'(t)| = 1$ as required. □

Is an arc-length parametrisation unique? No. Suppose that $\phi : [a, b] \rightarrow \mathbb{R}^n$ is an arc-length parametrisation of a curve, and that $f : [c, d] \xrightarrow{\sim} [a, b]$ is a reparametrisation such that $\varphi = \phi \circ f$ is also an arc-length parametrisation. Then $|\phi'(t)| = 1$ for all $t \in [a, b]$ and $|\varphi'(t)| = 1$ for all $t \in [c, d]$, so

$$|\phi'(f(t))| |f'(t)| = 1, \quad t \in [c, d],$$

so $|f'(t)| = 1$. Thus $f'(t) = \pm 1$, so

$$f(t) = \pm t + C,$$

for some constant C .

1.2 Curvature

Definition 1.2.1. Let $\phi : [a, b] \rightarrow \mathbb{R}^n$ be a curve parametrised by arc-length. The **curvature** of ϕ at $\phi(t)$ is $\kappa(t) = |\phi''(t)|$. The **curvature vector** is $\vec{\kappa}(t) = \phi''(t)$.

Claim that this depends only on the curve not on the parametrisation ϕ . If $\varphi = \phi \circ f$ is another arc-length parametrisation then $f(t) = \pm t + C$. Putting $s = f(t)$ we have

$$\frac{d\varphi}{ds} = \pm \frac{d\phi}{dt}, \quad \frac{d^2\varphi}{ds^2} = \frac{d^2\phi}{dt^2},$$

so $\vec{\kappa}(t)$ is independent of the choice of parametrisation.

Proposition 1.2.2. $\kappa(t) = 0$ if and only if the curve is a straight line.

Proof. Let $\phi : [a, b] \rightarrow \mathbb{R}^n$ be an arc-length parametrisation of the curve. Then $\kappa(t) = 0$, so $\phi''(t) = 0$. Thus

$$\phi(t) = \vec{a} + \vec{b}t, \quad \vec{a}, \vec{b} \in \mathbb{R}^n.$$

□

Proposition 1.2.3. For curves in \mathbb{R}^n , the vector curvature $\vec{\kappa}(t)$ is perpendicular to the tangent line at $\phi(t)$ for all t , where ϕ is an arc-length parametrisation.

Proof. The tangent line to the curve at $\phi(t)$ is

$$L = \{\phi(t) + s\phi'(t) \mid s \in \mathbb{R}\},$$

where $\phi'(t)$ is the direction vector of the tangent line. Need to show $\phi'(t) \cdot \phi''(t) = 0$ for all t . Know $|\phi'(t)| = 1$ for all t , that is $\phi'(t) \cdot \phi'(t) = 1$ for all t . Differentiating,

$$\phi''(t) \cdot \phi'(t) + \phi'(t) \cdot \phi''(t) = 0,$$

so $2\phi'(t) \cdot \phi''(t) = 0$. Thus $\phi'(t) \cdot \phi''(t) = 0$. □

Example. The curvature of a circle in \mathbb{R}^2 centred at the origin of radius R . Need an arc-length parametrisation. Try

$$\begin{aligned} \phi &: [0, 2\pi] \longrightarrow \mathbb{R} \\ t &\longmapsto (R \cos(t), R \sin(t)) \end{aligned}$$

Then

$$|\phi'(t)| = |(-R \sin(t), R \cos(t))| = R.$$

Oops. Try

$$\begin{aligned} \phi &: [0, 2\pi R] \longrightarrow \mathbb{R} \\ t &\longmapsto \left(R \cos\left(\frac{t}{R}\right), R \sin\left(\frac{t}{R}\right) \right) \end{aligned}$$

Checking,

$$|\phi'(t)| = \left| \left(-\sin\left(\frac{t}{R}\right), \cos\left(\frac{t}{R}\right) \right) \right| = 1.$$

The vector curvature is

$$\vec{\kappa}(t) = \left(-\frac{1}{R} \cos\left(\frac{t}{R}\right), -\frac{1}{R} \sin\left(\frac{t}{R}\right) \right),$$

so the curvature is

$$\kappa(t) = |\vec{\kappa}(t)| = \frac{1}{R}.$$

The curvature $\kappa(t)$ does not determine the curve.

1.3 Space curves

Let $\phi : [a, b] \rightarrow \mathbb{R}^3$ be an arc-length parametrisation of a curve. Set

$$T(t) = \phi'(t),$$

the **unit tangent vector** to the curve at $\phi(t)$ and assume $T'(t) \neq 0$ for all t . Set

$$N(t) = \frac{T'(t)}{|T'(t)|},$$

the **principal normal vector** to the curve at $\phi(t)$. Set

$$B(t) = T(t) \times N(t),$$

the **binormal vector** to the curve at $\phi(t)$. Check that $|B(t)| = 1$. Thus (T, N, B) is a positively oriented orthonormal basis for \mathbb{R}^3 , for each $t \in [a, b]$. This is the **Frenet frame** or **Serret-Frenet frame**.

Lecture 4
Friday
12/10/18

$$T'(t) = |\phi''(t)| N(t) = \kappa(t) N(t).$$

$B'(t) = T'(t) \times N(t) + T(t) \times N'(t) = T(t) \times N'(t)$, so $B'(t)$ is perpendicular to $T(t)$. $B(t) \cdot B(t) = 1$ for all t , so $B(t)$ is perpendicular to $B'(t)$. Thus

$$B'(t) = -\tau(t) N(t),$$

for some function $\tau(t)$. $\tau(t)$ is called the **torsion** of ϕ at the point $\phi(t)$. Then $N(t) = B(t) \times T(t)$ so

$$N'(t) = B'(t) \times T(t) + B(t) \times T'(t) = -\tau(t) N(t) \times T(t) + B(t) \times \kappa(t) N(t) = \tau(t) B(t) - \kappa(t) T(t).$$

Thus

- $T'(t) = \kappa(t) \cdot N(t)$,
- $B'(t) = -\tau(t) \cdot N(t)$, and
- $N'(t) = \tau(t) \cdot B(t) - \kappa(t) \cdot T(t)$.

So the time evolution of the frame depends only on the curvature, the torsion, and the frame itself.

Proposition 1.3.1. *Let $\phi : [a, b] \rightarrow \mathbb{R}^3$ be a curve parametrised by arc-length. Suppose $\phi''(t) \neq 0$ for all $t \in [a, b]$. Need this for the Frenet frame to exist. Then $\tau(t) = 0$ for all t if and only if ϕ is planar.*

Proof.

\implies Suppose $\tau(t) = 0$. Then $B'(t) = 0$, so $B(t) = v$ is constant. Claim that $\phi(t) \cdot v$ is constant for all t .

$$(\phi(t) \cdot v)' = \phi'(t) \cdot v + \phi(t) \cdot 0 = \phi'(t) \cdot v = T(t) \cdot v = 0,$$

since T is perpendicular to B . Thus ϕ is moving in a plane perpendicular to v .

\impliedby Suppose ϕ is planar. Then $\phi(t) \cdot v = a$ for some constant $v \in \mathbb{R}^3$ and constant $a \in \mathbb{R}$. $\phi'(t) \cdot v = 0$, so $T(t)$ is perpendicular to v . $\phi''(t) \cdot v = 0$, so $\kappa(t) \cdot N(t)$ is perpendicular to v . $\kappa(t) \neq 0$, so $N(t)$ is perpendicular to v . Thus $B(t) = \pm v/|v|$ is constant, so $\tau(t) = 0$.

□

Example.

- The unit circle in the (x, y) plane is

$$\phi(t) = (\cos(t), \sin(t), 0), \quad t \in [0, 2\pi].$$

Compute $T(t), N(t), B(t), \tau(t), \kappa(t)$. Verify the Frenet formulae.

- The helix is

$$\phi(t) = \left(\cos\left(\frac{t}{\sqrt{2}}\right), \sin\left(\frac{t}{\sqrt{2}}\right), \frac{t}{\sqrt{2}} \right), \quad t \in \mathbb{R}.$$

Check it is an arc-length parametrisation.

$$\phi'(t) = \left(-\frac{1}{\sqrt{2}} \sin\left(\frac{t}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \cos\left(\frac{t}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \right).$$

So $|\phi'(t)| = 1$.

$$\begin{aligned} T(t) &= \phi'(t) = \left(-\frac{1}{\sqrt{2}} \sin\left(\frac{t}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \cos\left(\frac{t}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \right), \\ N(t) &= \frac{\phi''(t)}{|\phi''(t)|} = 2 \left(-\frac{1}{2} \cos\left(\frac{t}{\sqrt{2}}\right), -\frac{1}{2} \sin\left(\frac{t}{\sqrt{2}}\right), 0 \right) = \left(-\cos\left(\frac{t}{\sqrt{2}}\right), -\sin\left(\frac{t}{\sqrt{2}}\right), 0 \right), \\ B(t) &= T(t) \times N(t) = \left(\frac{1}{\sqrt{2}} \sin\left(\frac{t}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}} \cos\left(\frac{t}{\sqrt{2}}\right), \frac{1}{\sqrt{2}} \right). \end{aligned}$$

Then $\kappa(t) = |\phi''(t)| = 1/2$ and $\tau(t) = 1/2$, since

$$B'(t) = \left(\frac{1}{2} \cos\left(\frac{t}{\sqrt{2}}\right), \frac{1}{2} \sin\left(\frac{t}{\sqrt{2}}\right), 0 \right) = -\tau(t) \cdot N(t).$$

In general κ and τ are not constants.

Theorem 1.3.2 (Fundamental theorem of the local theory of curves). *Given two differentiable functions $\kappa : [a, b] \rightarrow \mathbb{R}$ and $\tau : [a, b] \rightarrow \mathbb{R}$ such that $\kappa(t) > 0$ for all t , there exists a regular curve $\phi : [a, b] \rightarrow \mathbb{R}^3$ parametrised by arc-length with curvature $\kappa(t)$ and torsion $\tau(t)$. Any other such curve $\varphi : [a, b] \rightarrow \mathbb{R}^3$ differs from ϕ by a **rigid motion**, that is*

$$\varphi = g \cdot \phi + \vec{c}, \quad g \in SO(3), \quad \vec{c} \in \mathbb{R}^3.$$

Proof.

- Proof of uniqueness. Rigid motions preserve arc-length, curvature, and torsion. If $\tilde{\phi} = g \cdot \phi + \vec{c}$ then

$$\begin{aligned} -|\tilde{\phi}'| &= |g \cdot \phi'| = |\phi'| = 1, \\ -\kappa_{\tilde{\phi}} &= |\tilde{\phi}''| = |g \cdot \phi''| = |\phi''| = \kappa_{\phi}, \text{ and} \\ -T_{\tilde{\phi}} &= \tilde{\phi}' = g \cdot \phi' = g \cdot T_{\phi} \text{ and } N_{\tilde{\phi}} = T'_{\tilde{\phi}} / |T'_{\tilde{\phi}}| = g \cdot T'_{\phi} / |T'_{\phi}| = g \cdot N_{\phi}, \text{ so} \end{aligned}$$

$$B_{\tilde{\phi}} = T_{\tilde{\phi}} \times N_{\tilde{\phi}} = g \cdot T_{\phi} \times g \cdot N_{\phi} = g \cdot (T_{\phi} \times N_{\phi}) = g \cdot B_{\phi},$$

so

$$-\tau_{\tilde{\phi}} \cdot N_{\tilde{\phi}} = B'_{\tilde{\phi}} = (g \cdot B_{\phi})' = g \cdot B'_{\phi} = g \cdot (-\tau_{\phi} \cdot N_{\phi}) = -\tau_{\phi} \cdot N_{\tilde{\phi}}.$$

Thus $\tau_{\tilde{\phi}} = \tau_{\phi}$.

We can apply a rigid motion to φ which sends $\varphi(a)$ to $\phi(a)$ and Frenet frame $(T_{\varphi}(a), N_{\varphi}(a), B_{\varphi}(a))$ to $(T_{\phi}(a), N_{\phi}(a), B_{\phi}(a))$. We can assume that these are equal without loss of generality. We compute

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\langle T_{\phi} - T_{\varphi}, T_{\phi} - T_{\varphi} \rangle + \langle N_{\phi} - N_{\varphi}, N_{\phi} - N_{\varphi} \rangle + \langle B_{\phi} - B_{\varphi}, B_{\phi} - B_{\varphi} \rangle) \\ &= \langle T'_{\phi} - T'_{\varphi}, T_{\phi} - T_{\varphi} \rangle + \langle N'_{\phi} - N'_{\varphi}, N_{\phi} - N_{\varphi} \rangle + \langle B'_{\phi} - B'_{\varphi}, B_{\phi} - B_{\varphi} \rangle \\ &= \kappa \langle T_{\phi} - T_{\varphi}, N_{\phi} - N_{\varphi} \rangle + (\tau \langle N_{\phi} - N_{\varphi}, B_{\phi} - B_{\varphi} \rangle - \kappa \langle N_{\phi} - N_{\varphi}, T_{\phi} - T_{\varphi} \rangle) - \tau \langle B_{\phi} - B_{\varphi}, N_{\phi} - N_{\varphi} \rangle \\ &= 0, \end{aligned}$$

by the Frenet equations and ϕ and φ are solutions to the problem in the statement.

Lecture 5
Monday
15/10/18

Then $\langle T_\phi - T_\varphi, T_\phi - T_\varphi \rangle + \langle N_\phi - N_\varphi, N_\phi - N_\varphi \rangle + \langle B_\phi - B_\varphi, B_\phi - B_\varphi \rangle$ is constant in time and is zero at $t = a$, so it is zero for all $t \in [a, b]$, so

$$T_\phi = T_\varphi, \quad N_\phi = N_\varphi, \quad B_\phi = B_\varphi, \quad t \in [a, b].$$

In particular

$$\phi(t) = \phi(a) + \int_a^t T_\phi(s) \, ds = \varphi(a) + \int_a^t T_\varphi(s) \, ds = \varphi(t).$$

Conclusion is $\phi = \varphi$.

- Proof of existence. Given $\kappa(t) > 0$ and $\tau(t)$, we can pick any positive orthonormal frame (T_a, N_a, B_a) and use the existence theorem for solutions of linear differential equations to find $T, N, B : [a, b] \rightarrow \mathbb{R}^3$ such that

$$\begin{cases} (T(a), N(a), B(a)) = (T_a, N_a, B_a) \\ \begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \end{cases}.$$

We check that (T, N, B) is an orthonormal frame at all $t \in [a, b]$. This is true at $t = a$. We consider

$$M = \begin{pmatrix} T & N & B \end{pmatrix}, \quad M' = \begin{pmatrix} T' & N' & B' \end{pmatrix} = \begin{pmatrix} T & N & B \end{pmatrix} \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} = M \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

As M is orthonormal if and only if $M^T \cdot M = Id$, we want to prove that $M^T \cdot M = Id$ at $t = a$, and

$$\frac{d}{dt} (M^T \cdot M) = M'^T \cdot M + M^T \cdot M' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \cdot M^T \cdot M + M^T \cdot M \cdot \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

The linear system

$$\begin{cases} \frac{d}{dt} (A) = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} A + A \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \\ A(a) = Id \end{cases}$$

has solution $A(t) = Id$. Hence by uniqueness, $A(t) = Id = M^T \cdot M$.

□

(Exercise: why is $(T(t), N(t), B(t))$ orthonormal for all t equivalent to

$$\begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

is skew-symmetric?)

Example. Show that any curve $\phi : [a, b] \rightarrow \mathbb{R}^3$ with zero torsion and constant curvature $c > 0$ is an arc of a circle of radius $1/c$. First check that an arc of a circle has this property.

$$\begin{aligned} \phi(t) &= \left(R \cos\left(\frac{t}{R}\right), R \sin\left(\frac{t}{R}\right), 0 \right), & \phi'(t) &= \left(-\sin\left(\frac{t}{R}\right), \cos\left(\frac{t}{R}\right), 0 \right), \\ \phi''(t) &= \left(-\frac{1}{R} \cos\left(\frac{t}{R}\right), -\frac{1}{R} \sin\left(\frac{t}{R}\right), 0 \right), \end{aligned}$$

so $\kappa(t) = 1/R = c$ and $\tau(t) = 0$, since the circle is a planar curve. Then apply the fundamental theorem of the local theory of curves.

Lecture 6
Tuesday
16/10/18

1.4 Plane curves

Let $\phi : [a, b] \rightarrow \mathbb{R}^3$.

- $\phi'(t) = 0$ is a point.
- $\kappa(t) = 0$ is a line.
- $\tau(t) = 0$ is a plane. This is the topic of today.

Let

$$\begin{aligned} \phi & : [a, b] \longrightarrow \mathbb{R}^2 \\ t & \longmapsto (x(t), y(t)) \end{aligned}$$

be parametrised by arc-length, so $\phi'(t) = (x'(t), y'(t)) = 1$ and $N(t) = (-y'(t), x'(t))$. Then $\phi''(t) = \kappa(t) N(t)$, so

$$\begin{aligned} \kappa(t) &= \kappa(t) \langle N(t), N(t) \rangle = \langle \kappa(t) N(t), N(t) \rangle = \langle \phi''(t), N(t) \rangle \\ &= \langle (x''(t), y''(t)), (-y'(t), x'(t)) \rangle = x'(t) y''(t) - y'(t) x''(t). \end{aligned}$$

Proposition 1.4.1. For arbitrary ϕ ,

$$\kappa(t) = \frac{\phi''(t) \cdot N(t)}{|\phi'(t)|^2}.$$

Proof. Let

$$\varphi(s) = \phi(f(s)) = (x(f(s)), y(f(s)))$$

be a reparametrisation by arc-length. Then

$$\begin{aligned} \kappa_\varphi(s) &= (x(f(s)))' (y(f(s)))'' - y(f(s))' x(f(s))'' \\ &= (x'(f(s)) y''(f(s)) - y'(f(s)) x''(f(s))) \cdot (f'(s))^3 \\ &= \phi''(f(s)) \cdot (-(y(f(s)))', (x(f(s)))') \cdot (f'(s))^2 \\ &= (\phi''(f(s)) \cdot N_\varphi(s)) \cdot (f'(s))^2. \end{aligned}$$

We have

$$f = l^{-1}, \quad l(t) = \int_a^t |\phi'(s)| \, ds,$$

so in particular $f'(s) = 1/|\phi'(f(s))|$, and hence, replacing $f(s) = t$,

$$\kappa_\phi(t) = \frac{\phi''(t) \cdot N_\phi(t)}{|\phi'(t)|^2}.$$

□

Example. Let

$$\begin{aligned} \phi & : [a, b] \longrightarrow \mathbb{R}^2 \\ t & \longmapsto (t, f(t)) \end{aligned}$$

be a graph of a smooth function f . Then

$$\phi'(t) = (1, f'(t)), \quad \phi''(t) = (0, f''(t)), \quad |\phi'(t)| = 1 + |f'(t)|^2, \quad N(t) = \frac{(-f'(t), 1)}{\sqrt{1 + |f'(t)|^2}},$$

so

$$\kappa(t) = \frac{f''(t)}{(1 + (f'(t))^2)^{3/2}}.$$

Suppose that $\phi : [a, b] \rightarrow \mathbb{R}^2 \cong \mathbb{C}$ is a smooth closed curve, so $\phi(a) = \phi(b)$ and $\phi^{(k)}(a) = \phi^{(k)}(b)$ for all k . The **winding number** of ϕ is an invariant measuring the number of clockwise rotations of the curve. Let $z = x + iy$. We know from complex analysis

$$\begin{aligned}
 w(\phi) &= \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z} dz \\
 &= \frac{1}{2\pi i} \int_a^b \frac{1}{x(t) + iy(t)} d(x(t) + iy(t)) \\
 &= \frac{1}{2\pi i} \int_a^b \frac{x'(t) + iy'(t)}{x(t) + iy(t)} \cdot \frac{x(t) - iy(t)}{x(t) - iy(t)} dt \\
 &= \frac{1}{2\pi i} \int_a^b \frac{(xx' + yy') + i(y'x - x'y)}{x^2 + y^2} dt \\
 &= \frac{1}{4\pi i} [\ln(x^2 + y^2)]_a^b + \frac{1}{2\pi} \int_a^b \frac{y'x - x'y}{x^2 + y^2} dt \\
 &= \frac{1}{2\pi} \int_a^b \frac{y'x - x'y}{x^2 + y^2} dt,
 \end{aligned}$$

since ϕ is a closed curve. Then

$$w(\phi') = \frac{1}{2\pi} \int_a^b \frac{y''x' - x''y'}{|\phi'(t)|^2} dt.$$

If ϕ is parametrised by arc-length, so $|\phi'| = 1$,

$$w(\phi') = \frac{1}{2\pi} \int_a^b \kappa(t) dt.$$

Proposition 1.4.2. The winding number $w(T(t))$ of the unit tangent vector $T(t) = \phi'(t)/|\phi'(t)|$ is equal to the **total curvature** $\int_a^b \kappa(t) dt$, divided by 2π .

Definition 1.4.3. The winding number $w(T(t))$ is called the **index** or **turning number** of ϕ , and it is denoted $Ind(\phi)$. Thus

$$Ind(\phi) = \frac{1}{2\pi} \int_a^b \kappa(t) dt.$$

If $\phi'(t) = e^{i\theta t}$, then $\kappa(t) = \dot{\theta}(t)$, so

$$Ind(\phi) = \frac{1}{2\pi} \int_a^b \dot{\theta}(t) dt.$$

2 Surfaces

Lecture 7
Friday
19/10/18

2.1 Regular surfaces

Definition 2.1.1. A **regular surface** $S \subset \mathbb{R}^3$ is a subset of \mathbb{R}^3 such that for all $p \in S$ there exists

- an open neighbourhood $V \subset \mathbb{R}^3$ of p ,
- an open set $U \subset \mathbb{R}^2$, and
- a smooth map $\phi : U \rightarrow \mathbb{R}^3$ such that
 1. $\phi(U) = V \cap S$,
 2. ϕ is a homeomorphism onto its image, and
 3. for all $q \in U$, $d\phi_q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is injective.

(U, ϕ) is called a **chart** near p or at p .

Write

$$\phi(u, v) = (x(u, v), y(u, v), z(u, v)),$$

then $d\phi_q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ has a matrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \bigg|_q,$$

and is injective if and only if the matrix has rank two, if and only if the columns are linearly independent. $(1, 0)$ at $q \in U \subset \mathbb{R}^2$ maps to

$$d\phi_q(1, 0) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \bigg|_q$$

at $\phi(q) \in S \subset \mathbb{R}^3$. Each column matrix of the matrix $d\phi_q$ is a tangent vector to S at $\phi(q)$, so the regularity condition 3 for surfaces ensures that a regular surface has a tangent plane at every point, since the columns of $d\phi_q$ span a two-dimensional space, that is a plane in \mathbb{R}^3 .

Example.

- Let $\mathbb{R}^2 \subset \mathbb{R}^3$. Take the chart (U, ϕ) with $U = \mathbb{R}^2$ and $\phi(u, v) = (u, v, 0)$.
- Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth function. The **graph** of f is

$$\Gamma_f = \{(x, y, f(x, y)) \mid x, y \in \mathbb{R}^2\}.$$

Take a chart (U, ϕ) where $U = \mathbb{R}^2$ and

$$\phi(u, v) = (u, v, f(u, v)).$$

Check

1. this is smooth,
2. this is a homeomorphism onto its image, by projecting to (x, y) or (u, v) plane, and
3. regularity, since

$$d\phi_{(u,v)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix}$$

has rank two.

Example. Non-examples.

- $S = \{z^2 = x^2 + y^2 \mid z \geq 0\}$ has no smooth chart near $(0,0,0)$. Suppose there exists a smooth chart (U, ϕ) near $(0,0,0)$. Consider the curve $z = |y|$ and $x = 0$. It has a discontinuous velocity at $t = 0$. Any smooth curve through $q \in U$, when $\phi(q) = 0$, maps to a smooth curve in \mathbb{R}^2 , by assumption, since we assumed ϕ is smooth, a contradiction. Thus curves like this are not smooth.

- Define

$$\begin{aligned} \phi &: (0, 2\pi) \times (-1, 1) \longrightarrow \mathbb{R}^3 \\ (u, v) &\longmapsto (\sin(u), \sin(2u), v) \end{aligned}$$

ϕ is not a homeomorphism onto its image.

The following is a very common situation.

Definition 2.1.2. Let $F: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth function. We say that $S \subset \mathbb{R}^3$ is a **regular level set** of F if $S = F^{-1}(c)$ for some $c \in \mathbb{R}$ and $\nabla F(p) \neq 0$ for all $p \in S$.

Example. Let

$$S = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3.$$

Here $F = x^2 + y^2 + z^2$ for $c = 1$ and $\nabla F = (2x, 2y, 2z)$. Note $\nabla F(p) \neq 0$ for all $p \in S$.

Proposition 2.1.3. *Regular level sets are regular surfaces.*

For this we need the inverse function theorem.

Theorem 2.1.4 (Inverse function theorem). *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be smooth on an open neighbourhood V of $p \in \mathbb{R}^n$, and suppose that $df_p: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is invertible. Then there exists an open neighbourhood $U \subset V$ of p such that $f|_U: U \rightarrow f(U)$ is a **diffeomorphism**, which is a smooth bijection with a smooth inverse.*

Proof of Proposition 2.1.3. The goal is to construct a chart on S near each $p \in S$. Recall $S = F^{-1}(c)$. Now $\nabla F(p) \neq 0$, so at least one of

$$\frac{\partial F}{\partial x}(p), \quad \frac{\partial F}{\partial y}(p), \quad \frac{\partial F}{\partial z}(p)$$

is non-zero. Without loss of generality assume $\frac{\partial F}{\partial z}(p) \neq 0$. Consider

$$\begin{aligned} g &: \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ (x, y, z) &\longmapsto (x, y, F(x, y, z)) \end{aligned}$$

Then g is smooth, and

$$dg_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial F}{\partial x}(p) & \frac{\partial F}{\partial y}(p) & \frac{\partial F}{\partial z}(p) \end{pmatrix}$$

is invertible. On a neighbourhood U of p , g is a diffeomorphism. Let

$$W = \{(u, v) \in \mathbb{R}^2 \mid (u, v, c) \in g(U)\}.$$

Identify this with $\{z = c\} \cap g(U)$. Let $V = U \cap F^{-1}(c)$ and let

$$\begin{aligned} \phi &: W \longrightarrow V \\ (u, v) &\longmapsto g^{-1}(u, v, c) \end{aligned}$$

Then ϕ is a chart on S near p .

1. ϕ is smooth.
2. ϕ is a homeomorphism onto its image.
3. $d\phi_q$ is injective for all $q \in W$, guaranteed by the inverse function theorem.

□

Lecture 8
Monday
22/10/18

Example. The torus is a regular surface. Consider the surface obtained by rotating the circle of radius one in the xz plane, centred at $(2, 0)$, about z -axis. Writing this in cylindrical polars,

$$S = (r - 2)^2 + z^2 = 1.$$

Therefore $S = F^{-1}(1)$ where

$$F(x, y, z) = \left(\sqrt{x^2 + y^2} - 2 \right)^2 + z^2.$$

Now, is this a regular surface?

$$\nabla F = \left(2 \left(\sqrt{x^2 + y^2} - 2 \right) \frac{x}{\sqrt{x^2 + y^2}}, 2 \left(\sqrt{x^2 + y^2} - 2 \right) \frac{y}{\sqrt{x^2 + y^2}}, 2z \right).$$

If $\nabla F = 0$ then $2z = 0$ and either $x = y = 0$, so $r = 0$, or $\sqrt{x^2 + y^2} = 2$, so $r = 2$. So $\nabla F(p) \neq 0$ for all $p \in S$. Thus S is a regular surface.

Proposition 2.1.5. *If S is a regular surface then, for every $p \in S$, there exists a neighbourhood V of p in S such that V is the graph of a smooth function $z = f(x, y)$ or $y = g(x, z)$ or $x = h(y, z)$.*

Proof. Take a chart $\phi : U \subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$ on S at p . Write

$$\phi(u, v) = (x(u, v), y(u, v), z(u, v)).$$

Let $q = \phi^{-1}(p)$. Now

$$d\phi_q = \begin{pmatrix} \frac{\partial x}{\partial u}(q) & \frac{\partial x}{\partial v}(q) \\ \frac{\partial y}{\partial u}(q) & \frac{\partial y}{\partial v}(q) \\ \frac{\partial z}{\partial u}(q) & \frac{\partial z}{\partial v}(q) \end{pmatrix}$$

has rank two. So one of the 2×2 minors of $d\phi_q$ is non-zero. Without loss of generality assume

$$\left| \begin{pmatrix} \frac{\partial x}{\partial u}(q) & \frac{\partial x}{\partial v}(q) \\ \frac{\partial y}{\partial u}(q) & \frac{\partial y}{\partial v}(q) \end{pmatrix} \right| \neq 0.$$

Consider

$$g : \begin{array}{ccc} U & \longrightarrow & \mathbb{R}^2 \\ (u, v) & \longmapsto & (x(u, v), y(u, v)) \end{array}.$$

This is a diffeomorphism near q , by the inverse function theorem.

$$\begin{array}{ccccc} \mathbb{R}^2 & \xleftarrow{g} & U & \xrightarrow{\phi} & \mathbb{R}^3 \\ (x(u, v), y(u, v)) & \longleftarrow & (u, v) & \mapsto & (x, y, F(x, y)) \end{array},$$

so

$$(\phi \circ g^{-1})(x, y) = (x, y, F(x, y)), \quad F(x, y) = z(u(x, y), v(x, y)).$$

Thus, near p , S is the graph $z = F(x, y)$. □

2.2 Tangent vectors and tangent planes

Proposition 2.2.1. *Let S be a regular surface, let $p \in S$, and let $\alpha : (-\epsilon, \epsilon) \rightarrow \mathbb{R}^3$ be a regular curve in S with $\alpha(0) = p$. Let $\phi : U \rightarrow \mathbb{R}^3$ be a chart on S near p . There exist smooth functions*

Lecture 9
Tuesday
23/10/18

$$u : (-\epsilon', \epsilon') \rightarrow U, \quad v : (-\epsilon', \epsilon') \rightarrow U,$$

such that

$$\alpha(t) = \phi(u(t), v(t)), \quad t \in (-\epsilon', \epsilon').$$

$$\begin{array}{ccc} & (-\epsilon, \epsilon) & \\ (u(t), v(t)) \swarrow & & \searrow \alpha \\ q \in U & \xrightarrow{\phi} & S \ni p \end{array}.$$

Proof. Without loss of generality S has the form $z = f(x, y)$. Then

$$\phi(u, v) = (x(u, v), y(u, v), f(x(u, v), y(u, v))).$$

Let $q = \phi^{-1}(p)$, and set $q = (u_0, v_0)$. Define

$$\begin{aligned} F : U \times (-c, c) &\longrightarrow \mathbb{R}^3 \\ (u, v, t) &\longmapsto (x(u, v), y(u, v), f(x(u, v), y(u, v)) + t). \end{aligned}$$

Then $F(u_0, v_0, 0) = p$, and

$$dF_{(u_0, v_0, 0)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} & \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} & 1 \end{pmatrix}.$$

The first two columns of $d\phi_{(u_0, v_0)}$ has rank two, so

$$\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right), \quad \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right)$$

are linearly independent, so $dF_{(u_0, v_0, 0)}$ is invertible. Applying the inverse function theorem, F is a diffeomorphism in a neighbourhood of $(u_0, v_0, 0)$. Let $G = F^{-1}$ near p . Then

$$G(x(u, v), y(u, v), f(x(u, v), y(u, v))) = (u, v, 0).$$

Now $\alpha(t) = (x(t), y(t), f(x(t), y(t)))$ for some smooth functions x and y .

$$\begin{array}{ccc} & (-\epsilon, \epsilon) & \\ (u(t), v(t)) \swarrow & & \searrow \alpha \\ q = (u_0, v_0) \in U & \xrightarrow{\phi} & S \ni p \end{array}.$$

\curvearrowright
 G

Setting $(u(t), v(t)) = (G \circ \alpha)(t)$ we find

- u and v are smooth, because G and α are, and
- $\alpha(t) = \phi(u(t), v(t))$.

□

Definition 2.2.2. Let S be a regular surface and $p \in S$. A **tangent vector** to S at p is a vector in \mathbb{R}^3 of the form $\alpha'(0)$, where $\alpha : (-\epsilon, \epsilon) \rightarrow S$ is a regular curve with $\alpha(0) = p$. The **tangent plane** $T_p S$ to S at p is the set of all tangent vectors.

This is independent of charts. How to compute this?

Proposition 2.2.3. If $\phi : U \rightarrow \mathbb{R}^3$ is a chart for S at p , with $\phi(q) = p$, then $T_p S$ is spanned by the columns of $d\phi_q$.

Proof.

- Span of columns of $d\phi_q$ is in $T_p S$. Write

$$d\phi_q = \begin{pmatrix} w_1 = \frac{\partial \phi}{\partial u} & w_2 = \frac{\partial \phi}{\partial v} \end{pmatrix}.$$

For all $a, b \in \mathbb{R}$, need to show $aw_1 + bw_2 \in T_p S$. Consider $\alpha(t) = \phi(q + t(a, b))$. Chain rule implies that $\alpha'(0) = d\phi_q \cdot (a, b) = aw_1 + bw_2$. So $aw_1 + bw_2 \in T_p S$.

- Conversely, need to show any element of $T_p S$ is of the form $aw_1 + bw_2$ for some $a, b \in \mathbb{R}$. This is of the form $\alpha'(0)$ where $\alpha : (-\epsilon, \epsilon) \rightarrow S$ is a regular curve with $\alpha(0) = p$. Given the discussion earlier, we know that there exist smooth functions

$$u : (-\epsilon', \epsilon') \rightarrow U, \quad v : (-\epsilon', \epsilon') \rightarrow U,$$

such that

$$\alpha(t) = \phi(u(t), v(t)), \quad t \in (-\epsilon', \epsilon').$$

Then

$$\alpha'(0) = d\phi_{(u(0), v(0))} \cdot \left(\frac{du}{dt}(0), \frac{dv}{dt}(0) \right) = d\phi_q \cdot \begin{pmatrix} a \\ b \end{pmatrix}, \quad a = \frac{du}{dt}(0), \quad b = \frac{dv}{dt}(0).$$

Thus $\alpha'(0) = aw_1 + bw_2$ as claimed. □

Example. Consider

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}, \quad p = (0, 0, 1).$$

Use the chart

$$\begin{aligned} \phi : U &\longrightarrow \mathbb{R}^3 \\ (u, v) &\longmapsto (u, v, \sqrt{1 - u^2 - v^2}) \end{aligned}$$

where U is the open unit disc in \mathbb{R}^2 to compute $T_p S$ as the xy plane.

Proposition 2.2.4. Let $S = F^{-1}(c)$ for $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ then for any $p \in S$

$$T_p S = (\nabla F(p))^\perp.$$

Proof. Let $\alpha : (-\epsilon, \epsilon) \rightarrow S$ such that $\alpha(0) = p$, so $\alpha'(0) \in T_p S$ and $F(\alpha(t)) = c$. Let $\alpha = (x(t), y(t), z(t))$.

$$0 = \frac{d}{dt} F(\alpha(t)) \Big|_{t=0} = \left(\frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} \right) \Big|_{t=0} = \nabla F(\alpha(0)) \cdot \alpha'(0),$$

so $\alpha'(0) \in (\nabla F(p))^\perp$, so $T_p S \subseteq (\nabla F(p))^\perp$, which is two-dimensional. This implies that $T_p S = (\nabla F(p))^\perp$. □

Example. Let S be the paraboloid

$$\{z = x^2 + y^2\}.$$

Compute $T_p S$ for $p = (1, 3, 10)$. S is a regular level set $F^{-1}(0)$ for $F = x^2 + y^2 - z$ and $\nabla F = (2x, 2y, -1) \neq 0$. Then $\nabla F(p) = (2, 6, -1)$, so

$$T_p S = (\nabla F(p))^\perp = \{2x + 6y - z = 0\}.$$

Lecture 10
Friday
26/10/18

Definition 2.2.5. Let S_1 and S_2 be regular surfaces.

- A map $F : S_1 \rightarrow \mathbb{R}^3$ is **smooth** if for every chart $\phi : U \rightarrow S_1$ the composition $U \xrightarrow{\phi} S_1 \xrightarrow{F} \mathbb{R}^3$ is smooth.
- A map $F : S_1 \rightarrow S_2$ is **smooth** if it is smooth when viewed as a map $F : S_1 \rightarrow \mathbb{R}^3$.

The **differential** of a smooth map $F : S_1 \rightarrow S_2$ at $p \in S_1$, denoted $dF_p : T_p S_1 \rightarrow T_{F(p)} S_2$, is defined as follows. Let $v = \alpha'(0)$ for $\alpha : (-\epsilon, \epsilon) \rightarrow S_1$ such that $\alpha(0) = p$ and $\beta(t) = F(\alpha(t)) : (-\epsilon, \epsilon) \rightarrow S_2$. This is a regular curve in S_2 , so $\beta'(0) \in T_{F(p)} S_2$, since $\beta(0) = F(p)$. We define

$$dF_p(v) = \beta'(0).$$

Proposition 2.2.6. *This definition is independent of the choice of curve α .*

Proof. Let $\alpha_1, \alpha_2 : (-\epsilon, \epsilon) \rightarrow S_1$ such that $\alpha_1(0) = p$ and $\alpha_1'(0) = \alpha_2'(0)$. Let $\alpha_1(t) = \phi(u_1(t), v_1(t))$ and $\alpha_2(t) = \phi(u_2(t), v_2(t))$, so

$$\frac{\partial \phi}{\partial u_1} \frac{du_1}{dt}(0) + \frac{\partial \phi}{\partial v_1} \frac{dv_1}{dt}(0) = \alpha_1'(0) = \alpha_2'(0) = \frac{\partial \phi}{\partial u_2} \frac{du_2}{dt}(0) + \frac{\partial \phi}{\partial v_2} \frac{dv_2}{dt}(0).$$

Recall $\frac{d\phi}{du}$ and $\frac{d\phi}{dv}$ are linearly independent. This implies that

$$\frac{du_1}{dt}(0) = \frac{du_2}{dt}(0), \quad \frac{dv_1}{dt}(0) = \frac{dv_2}{dt}(0). \quad (1)$$

Want $dF_p(\alpha_1'(0)) = dF_p(\alpha_2'(0))$.

$$dF_p(\alpha_i(0)) = \left. \frac{d}{dt} F(\phi(u_i, v_i)) \right|_{t=0} = \left(\frac{\partial (F \circ \phi)}{\partial u} \frac{du_i}{dt} + \frac{\partial (F \circ \phi)}{\partial v} \frac{dv_i}{dt} \right) \Big|_{t=0}, \quad i = 1, 2,$$

so $dF_p(\alpha_1'(0)) = dF_p(\alpha_2'(0))$ by (1). It follows that $dF_p(v)$ is the same if we define it via α_1 or α_2 . \square

Proposition 2.2.7. $dF_p : T_p S_1 \rightarrow T_{F(p)} S_2$ is linear.

Proof. Let $v, w \in T_p S_1$ and $c, d \in \mathbb{R}$. Want $dF_p(cv + dw) = cdF_p(v) + ddF_p(w)$. Let $\alpha_1, \alpha_2 : (-\epsilon, \epsilon) \rightarrow S_1$ such that $v = \alpha_1'(0)$ and $w = \alpha_2'(0)$. Let $\phi : U \rightarrow S_1$ be a chart. Without loss of generality we assume $(0, 0) \in U$ and $\phi(0, 0) = p$. Let $\alpha_i(t) = \phi(u_i(t), v_i(t))$. Find $\alpha_3(t)$ such that $\alpha_3'(0) = cv + dw$. Let

$$\alpha_3(t) = \phi(cu_1(t) + du_2(t), cv_1(t) + dv_2(t)),$$

so $\alpha_3(0) = \phi(0, 0) = p$ and

$$\begin{aligned} \alpha_3'(0) &= \frac{\partial \phi}{\partial u} (cu_1'(0) + du_2'(0)) + \frac{\partial \phi}{\partial v} (cv_1'(0) + dv_2'(0)) \\ &= c \left(u_1'(0) \frac{\partial \phi}{\partial u} + v_1'(0) \frac{\partial \phi}{\partial v} \right) + d \left(u_2'(0) \frac{\partial \phi}{\partial u} + v_2'(0) \frac{\partial \phi}{\partial v} \right) \\ &= c\alpha_1'(0) + d\alpha_2'(0) = cv + dw. \end{aligned}$$

Then

$$\begin{aligned} dF_p(cv + dw) &= dF_p(\alpha_3'(0)) = ((F \circ \phi)(cu_1(t) + du_2(t), cv_1(t) + dv_2(t)))'(0) \\ &= \frac{\partial (F \circ \phi)}{\partial u} (cu_1'(0) + du_2'(0)) + \frac{\partial (F \circ \phi)}{\partial v} (cv_1'(0) + dv_2'(0)) \\ &= c \left(\frac{\partial (F \circ \phi)}{\partial u} u_1'(0) + \frac{\partial (F \circ \phi)}{\partial v} v_1'(0) \right) + d \left(\frac{\partial (F \circ \phi)}{\partial u} u_2'(0) + \frac{\partial (F \circ \phi)}{\partial v} v_2'(0) \right) \\ &= c(F \circ \alpha_1)'(0) + d(F \circ \alpha_2)'(0) = cdF_p(v) + ddF_p(w), \end{aligned}$$

since

$$dF_p(\alpha_i'(0)) = (F \circ \alpha_i)'(0) = \left(\frac{\partial (F \circ \phi)}{\partial u} \right) \frac{du_i}{dt} + \left(\frac{\partial (F \circ \phi)}{\partial v} \right) \frac{dv_i}{dt}.$$

\square

Remark. We will use the following identity. Given a chart $\phi : U \rightarrow S_1$ and $F : S_1 \rightarrow S_2$

$$dF_p \left(\frac{\partial \phi}{\partial u} (u_0, v_0) \right) = \frac{\partial (F \circ \phi)}{\partial u} (u_0, v_0), \quad dF_p \left(\frac{\partial \phi}{\partial v} (u_0, v_0) \right) = \frac{\partial (F \circ \phi)}{\partial v} (u_0, v_0).$$

Similarly, we define the **differential** of a smooth function $f : S \rightarrow \mathbb{R}$ if $v \in T_p S$ satisfies $v = \alpha'(0)$ for $\alpha = (-\epsilon, \epsilon) \rightarrow S$ as

$$df_p(v) = f(\alpha(t))'(0).$$

(Exercise: show that df_p is well-defined, so independent of choice of α , and that it is linear)

Example. Let

$$S = \{x^2 + y^2 + z^2 = 1\}, \quad p = (0, 1, 0), \quad f : \begin{array}{ccc} S & \longrightarrow & \mathbb{R} \\ (x, y, z) & \longmapsto & z \end{array}.$$

Compute $T_p S$ and $df_p : T_p S \rightarrow \mathbb{R}$. $S = F^{-1}(1)$ for

$$F : \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ (x, y, z) & \longmapsto & x^2 + y^2 + z^2 \end{array}.$$

$\nabla F(p) = (0, 2, 0)$, so

$$T_p S = (\nabla F(p))^\perp = \langle (1, 0, 0), (0, 0, 1) \rangle.$$

But to compute df_p use the chart near p

$$\phi_p : \begin{array}{ccc} U & \longrightarrow & S \\ (u, v) & \longmapsto & (u, \sqrt{1 - u^2 - v^2}, v) \end{array}, \quad U = \{u^2 + v^2 < 1\}.$$

Then $(f \circ \phi)(u, v) = v$, so

$$df_p \left(\frac{\partial \phi}{\partial u} (0, 0) \right) = 0, \quad df_p \left(\frac{\partial \phi}{\partial v} (0, 0) \right) = 1.$$

This determines $df_p : T_p S \rightarrow \mathbb{R}$, a linear map.

Proposition 2.2.8. Let $f : S_1 \rightarrow S_2$ be a smooth map. If $df_p : T_p S_1 \rightarrow T_{f(p)} S_2$ is an isomorphism, then there exists an open neighbourhood $V \subseteq S_1$ of p such that $f : V \rightarrow f(V)$ is a diffeomorphism.

Proof. Take charts $\phi_1 : U_1 \rightarrow S_1$ at p and $\phi_2 : U_2 \rightarrow S_2$ at $f(p)$.

$$\begin{array}{ccc} \phi_1(q_1) = p \in \phi_1(U_1) \subseteq S_1 & \xrightarrow{f} & \phi_2(q_2) = f(p) \in \phi_2(U_2) \subseteq S_2 \\ \uparrow \phi_1 & & \uparrow \phi_2 \\ q_1 \in U_1 \subseteq \mathbb{R}^2 & \xrightarrow{g} & q_2 \in U_2 \subseteq \mathbb{R}^2 \end{array}.$$

Check that the map

$$g : U_1 \rightarrow S_1 \rightarrow S_2 \rightarrow U_2$$

satisfies

- $g(q_1) = q_2$, and
- dg_{q_1} is invertible.

Use the inverse function theorem for g . Then $f = \phi_2 \circ g \circ \phi_1^{-1}$ will be a diffeomorphism on some neighbourhood of $\phi_1(q_1) = p$. \square

Lecture 11
Monday
29/10/18

2.3 Normal vectors

A regular surface $S \subseteq \mathbb{R}^2$ has two normal vectors at p in S . If $S = F^{-1}(c)$ for some smooth function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$, $T_p S = (\nabla F(p))^\perp$. We can define the **unit normal vector**

$$N(p) = \frac{\nabla F(p)}{|\nabla F(p)|}.$$

In general we can only do this locally, in a chart. Given a chart $\phi : U \rightarrow S$ such that $\phi(q) = p$, we know that $T_p S$ is spanned by $\frac{\partial \phi}{\partial u}(q)$ and $\frac{\partial \phi}{\partial v}(q)$. Define

$$N(p) = \frac{\frac{\partial \phi}{\partial u}(q) \times \frac{\partial \phi}{\partial v}(q)}{\left| \frac{\partial \phi}{\partial u}(q) \times \frac{\partial \phi}{\partial v}(q) \right|}.$$

This can always be done locally, but not globally. On the Möbius strip, you cannot pick $N(p)$ continuously.

Definition 2.3.1. S is **orientable** if it admits a continuous choice of unit normal vector $N(p) \in S$. If S is orientable, we get a map $N : S \rightarrow S^2$. N is called the **Gauss map**. If this exists, it is smooth.

Example. Let $S = S^2$ be the unit sphere. As $S = F^{-1}(1)$ for $F(x, y, z) = x^2 + y^2 + z^2$,

$$N(x, y, z) = \frac{\nabla F(x, y, z)}{|\nabla F(x, y, z)|} = \frac{(2x, 2y, 2z)}{\sqrt{4x^2 + 4y^2 + 4z^2}} = (x, y, z).$$

Or, near the north pole $(0, 0, 1)$ we have a chart $\phi(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$, so

$$\frac{\partial \phi}{\partial u} = \left(1, 0, \frac{-u}{\sqrt{1 - u^2 - v^2}}\right), \quad \frac{\partial \phi}{\partial v} = \left(0, 1, \frac{-v}{\sqrt{1 - u^2 - v^2}}\right),$$

so

$$\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} = \left(\frac{u}{\sqrt{1 - u^2 - v^2}}, \frac{v}{\sqrt{1 - u^2 - v^2}}, 1\right).$$

Thus

$$N(\phi(u, v)) = \frac{\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}}{\left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right|} = (u, v, \sqrt{1 - u^2 - v^2}) = \phi(u, v).$$

Example. Let

$$S = \{ax + by + cz = d\}$$

be a plane, so $F(x, y, z) = ax + by + cz$ and $S = F^{-1}(d)$. Then

$$N(x, y, z) = \frac{\nabla F(x, y, z)}{|\nabla F(x, y, z)|} = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}$$

is a constant map.

Let $N : S \rightarrow S^2$. What is $dN_p : T_p S \rightarrow T_{N(p)} S^2$? $T_{N(p)} S^2$ is all vectors orthogonal to $N(p)$, which is $T_p S$. We use this identification to write $dN_p : T_p S \rightarrow T_p S$.

Example. Let

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = r^2\}$$

be a sphere of radius r and $p = (x, y, z)$, so $S = F^{-1}(r^2)$ for $F = x^2 + y^2 + z^2$. Then

$$N(p) = \frac{\nabla F(p)}{|\nabla F(p)|} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{p}{r}.$$

Let $\alpha : (-\epsilon, \epsilon) \rightarrow S$ be a curve with $\alpha(0) = p$, so

$$dN_p(\alpha'(0)) = \frac{d((N \circ \alpha)(t))}{dt} \Big|_{t=0} = \frac{1}{r} \alpha'(0).$$

Thus $dN_p : T_p S \rightarrow T_p S$ is $dN_p = Id/r$.

3 Curvature

We will try to define the curvature using the Gauss map.

Lecture 12
Tuesday
30/10/18

3.1 The second fundamental form

Example. Let

$$S = \{ax + by + cz = d\}$$

be a plane and $p \in S$. Then

$$N(p) = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}$$

is constant with respect to p , so $dN_p : T_p S \rightarrow T_{N(p)} S^2$ is the zero map.

Example. Let

$$S = \{x^2 + 2y^2 + z^2 = r^2\}.$$

Compute dN_p in $(1, 0, 0)$ and in $(0, 1, 0)$.

Definition 3.1.1. Let S be a regular orientable surface, let $p \in S$. The **second fundamental form** at p is

$$\begin{aligned} A : T_p S \times T_p S &\longrightarrow \mathbb{R} \\ (X, Y) &\longmapsto -X \cdot dN_p(Y), \end{aligned}$$

which is how much $N_p(Y)$ is changing in the direction of X .

Example. If S is a sphere of radius r then

$$A(X, Y) = -\frac{1}{r} X \cdot Y.$$

Proposition 3.1.2. A is a symmetric bilinear form.

Proof.

- Bilinearity.

$$\begin{aligned} A(X, cY_1 + dY_2) &= -X \cdot dN_p(cY_1 + dY_2) = -X \cdot (cdN_p(Y_1) + ddN_p(Y_2)) \\ &= c(-X \cdot dN_p(Y_1)) + d(-X \cdot dN_p(Y_2)) = cA(X, Y_1) + dA(X, Y_2). \end{aligned}$$

Similarly $A(cX_1 + dX_2, Y) = cA(X_1, Y) + dA(X_2, Y)$.

- Symmetry. Take a chart $\phi : U \rightarrow S$ near p . Then $\frac{d\phi}{du}(q)$ and $\frac{d\phi}{dv}(q)$, where $\phi(q) = p$, form a basis for the tangent space $T_p S$. Sufficient to prove that

$$A\left(\frac{\partial\phi}{\partial u}(q), \frac{\partial\phi}{\partial v}(q)\right) = A\left(\frac{\partial\phi}{\partial v}(q), \frac{\partial\phi}{\partial u}(q)\right).$$

Observe that

$$N(\phi(p)) \cdot \frac{\partial\phi}{\partial u} = 0, \quad N(\phi(p)) \cdot \frac{\partial\phi}{\partial v} = 0,$$

since $\frac{d\phi}{du}$ and $\frac{d\phi}{dv}$ are in $T_p S$ and $N(\phi(p))$ is normal to $T_p S$. Applying $\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial u}$,

$$\frac{\partial}{\partial v}(N \circ \phi) \cdot \frac{\partial\phi}{\partial u} + (N \circ \phi) \cdot \frac{\partial^2\phi}{\partial v \partial u} = 0, \quad \frac{\partial}{\partial u}(N \circ \phi) \cdot \frac{\partial\phi}{\partial v} + (N \circ \phi) \cdot \frac{\partial^2\phi}{\partial u \partial v} = 0.$$

Then

$$A\left(\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\right) = (N \circ \phi) \cdot \frac{\partial^2\phi}{\partial v \partial u}, \quad A\left(\frac{\partial\phi}{\partial v}, \frac{\partial\phi}{\partial u}\right) = (N \circ \phi) \cdot \frac{\partial^2\phi}{\partial u \partial v}.$$

Thus both these are equal, so

$$A\left(\frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}\right) = A\left(\frac{\partial\phi}{\partial v}, \frac{\partial\phi}{\partial u}\right).$$

□

Lecture 13 is a problem class.

Lecture 13
Friday
02/11/18

3.2 Normal curvature, Gaussian curvature, and mean curvature

There exists an orthonormal basis x_1, x_2 for $T_p S$ such that

$$dN_p(x_i) = -\lambda_i x_i, \quad i \in \{1, 2\}.$$

Then

- $A(x_1, x_1) = -x_1 \cdot dN_p(x_1) = x_1 \cdot (\lambda_1 x_1) = \lambda_1$,
- $A(x_1, x_2) = -x_1 \cdot dN_p(x_2) = x_1 \cdot (\lambda_2 x_2) = 0$, and
- $A(x_2, x_2) = -x_2 \cdot dN_p(x_2) = x_2 \cdot (\lambda_2 x_2) = \lambda_2$.

We call x_1, x_2 the **principal directions** in $T_p S$ and λ_1, λ_2 the **principal curvatures**.

Lemma 3.2.1. *Let S be a regular surface and $\lambda_1(p) \leq \lambda_2(p)$ be the principal curvatures at $p \in S$. Then*

$$\lambda_1 = \min \{A(x, x) \mid x \in T_p S, |x| = 1\}, \quad \lambda_2 = \max \{A(x, x) \mid x \in T_p S, |x| = 1\}.$$

Proof. Consider $x \in T_p S$ with $|x| = 1$. Write $x = c_1 x_1 + c_2 x_2$ where $c \in \mathbb{R}$, then $c_1^2 + c_2^2 = 1$. Then

$$\begin{aligned} A(x, x) &= A(c_1 x_1 + c_2 x_2, c_1 x_1 + c_2 x_2) \\ &= c_1^2 A(x_1, x_1) + c_1 c_2 A(x_1, x_2) + c_2 c_1 A(x_2, x_1) + c_2^2 A(x_2, x_2) \\ &= \lambda_1 c_1^2 + \lambda_2 c_2^2 \leq \lambda_2 c_1^2 + \lambda_2 c_2^2 = \lambda_2, \end{aligned}$$

with equality if and only if $c_1 = 0$ and $c_2 = 1$. Similarly

$$A(x, x) = \lambda_1 c_1^2 + \lambda_2 c_2^2 \geq \lambda_1 c_1^2 + \lambda_1 c_2^2 = \lambda_1,$$

with equality if and only if $c_1 = 1$ and $c_2 = 0$. □

Example. Let S be a sphere of radius r . Then $N(p) = p/r$ and $dN(p) = Id/r$, so principal curvatures are $\lambda_1 = \lambda_2 = 1/r$, which are independent of p .

Example. Suppose S is a connected regular surface and that $\lambda_1(p) = \lambda_2(p) = 0$ for all $p \in S$. Then $dN_p = 0$, so $N(p)$ is constant for all $p \in S$. Suppose $N(p) = \vec{v}$ for all $p \in S$. Then S is in the plane perpendicular to \vec{v} through p . (Exercise)

Definition 3.2.2. Suppose $C \subset S$ is a curve through $p \in S$. Suppose N is the unit normal vector to S at p and n is the unit normal vector to C at p . The **normal curvature** of C at p is

$$k_n(p) = \kappa \cos(\theta),$$

where θ is the angle between N and n and κ is the curvature of C at p , so $k_n(p)$ is the length of the projection of $\vec{\kappa}$, the vector curvature of C at p , along N .

Proposition 3.2.3. *Let S be a regular surface and $C \subset S$ be a curve with unit tangent vector $v \in T_p S$. Then*

$$A(v, v) = k_n(p).$$

Proof. Let $\alpha : (-\epsilon, \epsilon) \rightarrow S$ be an arc-length parametrisation of C near p . Then $\alpha(0) = p$ and $\alpha'(0) = v$. We have $\langle \alpha'(t), N(\alpha(t)) \rangle = 0$ for all t . Differentiating,

$$\langle \alpha''(t), N(\alpha(t)) \rangle + \langle \alpha'(t), dN_{\alpha(t)}(\alpha'(t)) \rangle,$$

for all t . Setting $t = 0$,

$$\langle \kappa n_p, N(p) \rangle + \langle v, dN_p(v) \rangle = 0.$$

Thus

$$A(v, v) = -\langle v, dN_p(v) \rangle = \langle \kappa n_p, N(p) \rangle = k_n(p).$$

□

Definition 3.2.4. A point $p \in S$ is **umbilical** if and only if the principal curvatures $\lambda_1(p)$ and $\lambda_2(p)$ are equal.

Proposition 3.2.5. If S is a connected regular surface such that every point in S is umbilical then S is contained in a plane or a sphere.

Lecture 15
Tuesday
06/11/18

Proof. By assumption, there exists a smooth function $\lambda : S \rightarrow \mathbb{R}$ such that $\lambda_1(p) = \lambda_2(p) = \lambda(p)$ for all $p \in S$.

Step 1. Prove $\lambda(p)$ is constant. Take a chart $\phi : U \rightarrow S$ near p . Then

$$\frac{\partial}{\partial u}(N \circ \phi) = dN_{\phi(u,v)} \cdot \frac{\partial \phi}{\partial u} = -(\lambda \circ \phi) \frac{\partial \phi}{\partial u}, \quad \frac{\partial}{\partial v}(N \circ \phi) = dN_{\phi(u,v)} \cdot \frac{\partial \phi}{\partial v} = -(\lambda \circ \phi) \frac{\partial \phi}{\partial v},$$

since $dN_{\phi(u,v)}$ is some multiple of Id . Now

$$\frac{\partial^2}{\partial u \partial v}(N \circ \phi) = \frac{\partial^2}{\partial v \partial u}(N \circ \phi),$$

so

$$-\frac{\partial}{\partial u}(\lambda \circ \phi) \frac{\partial \phi}{\partial v} = -\frac{\partial}{\partial v}(\lambda \circ \phi) \frac{\partial \phi}{\partial u}.$$

The vectors $\frac{d\phi}{dv}$ and $\frac{d\phi}{du}$ are linearly independent, so

$$-\frac{\partial}{\partial u}(\lambda \circ \phi) = -\frac{\partial}{\partial v}(\lambda \circ \phi) = 0,$$

so $\lambda \circ \phi$ is locally constant, that is λ is constant on $\phi(U)$. S is connected, and covered by such charts, so λ is constant.

Step 2. If $\lambda(p) \equiv 0$ for all $p \in S$ then we already know that S is contained in a plane. Otherwise $\lambda(p) \equiv \lambda_0$ with $\lambda_0 \neq 0$. Without loss of generality $\lambda_0 > 0$. Now

$$\frac{\partial}{\partial u}(N \circ \phi) = -\lambda_0 \frac{\partial \phi}{\partial u}, \quad \frac{\partial}{\partial v}(N \circ \phi) = -\lambda_0 \frac{\partial \phi}{\partial v},$$

so

$$\frac{\partial}{\partial u} \left(\frac{1}{\lambda_0} (N \circ \phi) + \phi \right) = 0, \quad \frac{\partial}{\partial v} \left(\frac{1}{\lambda_0} (N \circ \phi) + \phi \right) = 0,$$

so there exists $c_0 \in \mathbb{R}^3$ such that $(N \circ \phi) / \lambda_0 + \phi = c_0$. Then $(N \circ \phi) / \lambda_0$ has length $1/\lambda_0$ and c_0 is constant, so $\phi(u, v)$ lies in the sphere centred at c_0 radius $1/\lambda_0$.

□

What if $\lambda_1(p) \neq \lambda_2(p)$?

Lecture 16
Friday
09/11/18

Definition 3.2.6. Let $p \in S$ be a point on a regular surface. Let $\lambda_1(p), \lambda_2(p)$ be the principal curvatures. Then

$$K(p) = \lambda_1(p) \lambda_2(p) = \det(dN_p)$$

is the **Gaussian curvature** at p and

$$H(p) = \frac{1}{2}(\lambda_1(p) + \lambda_2(p)) = -\frac{1}{2} \text{tr}(dN_p)$$

is the **mean curvature** at p .

Reversing the orientation of S , that is replacing N by $-N$, sends $H(p)$ to $-H(p)$ and $K(p)$ to $K(p)$.

Example. If S is a sphere of radius r then $K = 1/r^2$ and $H = 1/r$. A plane has $K = H = 0$.

3.3 What does Gaussian curvature mean?

Let S be a regular surface and $p \in S$. Choose a chart $\phi : U \rightarrow S$ on S near p . Choose a curve through p , $\gamma(t) = \phi(ct, dt)$. Now $\gamma'(t) \cdot N(\gamma(t)) = 0$ for all t . Differentiating,

$$\gamma''(t) \cdot N(\gamma(t)) + \gamma'(t) \cdot dN_{\gamma(t)}(\gamma'(t)) = 0,$$

or in other words

$$A(\gamma'(t), \gamma'(t)) = \gamma''(t) \cdot N(\gamma(t)).$$

At $t = 0$

$$\gamma'(t) = c \frac{\partial \phi}{\partial u} + d \frac{\partial \phi}{\partial v}, \quad \gamma''(t) = c^2 \frac{\partial^2 \phi}{\partial u^2} + 2cd \frac{\partial^2 \phi}{\partial u \partial v} + d^2 \frac{\partial^2 \phi}{\partial v^2}.$$

Thus

$$A\left(c \frac{\partial \phi}{\partial u} + d \frac{\partial \phi}{\partial v}, c \frac{\partial \phi}{\partial u} + d \frac{\partial \phi}{\partial v}\right) = \left(c^2 \frac{\partial^2 \phi}{\partial u^2} + 2cd \frac{\partial^2 \phi}{\partial u \partial v} + d^2 \frac{\partial^2 \phi}{\partial v^2}\right) \cdot N(p).$$

Proposition 3.3.1. *If $K(p) > 0$ then all points of S near p lie on the same side of $T_p S$. If $K(p) < 0$ then there are points of S near p on each side of $T_p S$.*

Proof. Consider the Taylor series

$$\phi(u, v) = p + \left(\frac{\partial \phi}{\partial u}(0, 0)u + \frac{\partial \phi}{\partial v}(0, 0)v\right) + \frac{1}{2} \left(\frac{\partial^2 \phi}{\partial u^2}(0, 0)u^2 + 2 \frac{\partial^2 \phi}{\partial u \partial v}(0, 0)uv + \frac{\partial^2 \phi}{\partial v^2}(0, 0)v^2\right) + R(u, v),$$

where

$$\lim_{(u,v) \rightarrow (0,0)} \frac{R(u, v)}{u^2 + v^2} = 0.$$

Consider

$$\begin{aligned} \langle \phi(u, v) - p, N(p) \rangle &= \frac{1}{2} \left\langle \frac{\partial^2 \phi}{\partial u^2}(0, 0)u^2 + 2 \frac{\partial^2 \phi}{\partial u \partial v}(0, 0)uv + \frac{\partial^2 \phi}{\partial v^2}(0, 0)v^2, N(p) \right\rangle + \langle R(u, v), N(p) \rangle \\ &= \frac{1}{2} A(w, w) + \langle R(u, v), N(p) \rangle, \quad w = \frac{\partial \phi}{\partial u}(0)u + \frac{\partial \phi}{\partial v}(0)v. \end{aligned}$$

$A(w, w)$ is quadratic in (u, v) and $R(u, v)$ is worse than quadratic in (u, v) . So for u and v small, when we are close to p , the function $\langle \phi(u, v) - p, N(p) \rangle$ has the same sign as $A(w, w)$. Write $w = ax_1 + bx_2$ where x_1, x_2 are the principal directions then $A(w, w) = \lambda_1 a^2 + \lambda_2 b^2$.

- $\lambda_1, \lambda_2 > 0$ gives $A(w, w) > 0$.
- $\lambda_1 > 0$ and $\lambda_2 < 0$ gives that $A(w, w) > 0$ can be positive or negative.
- $\lambda_1, \lambda_2 < 0$ gives $A(w, w) < 0$.

□

In fact we can do slightly better than this. Consider the chart map $\phi : U \rightarrow S$. After a translation of S , without loss of generality $\phi(0, 0) = (0, 0, 0)$, after a rigid motion to $S \subseteq \mathbb{R}^3$, without loss of generality

$$\frac{\partial \phi}{\partial u}(0, 0) = (1, 0, 0), \quad \frac{\partial \phi}{\partial v}(0, 0) = (0, 1, 0).$$

Taylor series implies that

$$\begin{aligned} \phi(u, v) &= (0, 0, 0) + \left(\frac{\partial \phi}{\partial u}(0, 0)u + \frac{\partial \phi}{\partial v}(0, 0)v\right) + \frac{1}{2} \left(\frac{\partial^2 \phi}{\partial u^2}(0, 0)u^2 + 2 \frac{\partial^2 \phi}{\partial u \partial v}(0, 0)uv + \frac{\partial^2 \phi}{\partial v^2}(0, 0)v^2\right) + \dots \\ &= (u, v, 0) + \frac{1}{2} \left(u^2 \frac{\partial^2 \phi}{\partial u^2}(0, 0) + 2uv \frac{\partial^2 \phi}{\partial u \partial v}(0, 0) + v^2 \frac{\partial^2 \phi}{\partial v^2}(0, 0)\right) + \dots \end{aligned}$$

Near $(u, v) = (0, 0)$ the quadratic term satisfies

$$\left(u^2 \frac{\partial^2 \phi}{\partial u^2}(0, 0) + 2uv \frac{\partial^2 \phi}{\partial u \partial v}(0, 0) + v^2 \frac{\partial^2 \phi}{\partial v^2}(0, 0) \right) \cdot N(0, 0, 0) = A((u, v, 0), (u, v, 0)) = \lambda_1 u^2 + \lambda_2 v^2,$$

because we chose $(1, 0, 0)$ and $(0, 1, 0)$ as our principal directions. So near $(u, v) = (0, 0)$, the surface is approximated by the graph

$$\phi(u, v) \approx \left(u, v, \frac{1}{2}(\lambda_1 u^2 + \lambda_2 v^2) \right).$$

Say $p \in S$ is

- **elliptic** if $K(p) > 0$, so either $\lambda_1, \lambda_2 > 0$ or $\lambda_1, \lambda_2 < 0$,
- **hyperbolic** if $K(p) < 0$, so either $\lambda_1 < 0, \lambda_2 > 0$ or vice versa,
- **parabolic** if $K(p) = 0$ and $H(p) = 0$, so $\lambda_1 = 0, \lambda_2 \neq 0$ or vice versa, and
- **planar** if $K(p) = H(p) = 0$, so $\lambda_1 = \lambda_2 = 0$.

Example. Monkey saddle. Consider the surface with equation $z = x^3 - 3x^2y$, so

$$z = \Re((x + iy)^3) = \Re(r^3 e^{i3\theta}) = r^3 \cos(3\theta),$$

using polar coordinates $x + iy = re^{i\theta}$. At $(0, 0, 0)$, the principal curvatures are $\lambda_1 = \lambda_2 = 0$. So the previous theorem says, to second order near $p = (0, 0, 0)$, the surface S looks like the plane $z = 0$. True, but not so informative. This has an umbilical point with $\lambda_1 = \lambda_2 = 0$ at $(0, 0, 0)$, and hyperbolic points elsewhere.

3.4 How to compute Gaussian and mean curvature?

We start with a simple observation.

Lemma 3.4.1. *Let V be a two-dimensional vector space with inner product $\langle \cdot, \cdot \rangle$. Let e_1, e_2 be an orthonormal basis for V . Let x, y be elements of V and write $x = x_1 e_1 + x_2 e_2$ and $y = y_1 e_1 + y_2 e_2$. Then*

$$\langle x, y \rangle = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

That is, we can compute inner products in V using the ordinary matrix multiplication between row vectors and column vectors, provided that we remember to use an orthonormal basis when writing elements of V as vectors.

Proof. Just write everything out.

$$\langle x, y \rangle = \langle x_1 e_1 + x_2 e_2, y_1 e_1 + y_2 e_2 \rangle = x_1 y_1 \langle e_1, e_1 \rangle + x_1 y_2 \langle e_1, e_2 \rangle + x_2 y_1 \langle e_2, e_1 \rangle + x_2 y_2 \langle e_2, e_2 \rangle = x_1 y_1 + x_2 y_2,$$

as required. The final equality here uses the fact that e_1 and e_2 are orthonormal. \square

Remark. There is nothing special about the fact that V is two-dimensional here. The obvious generalisation of Lemma 3.4.1 holds for any finite-dimensional inner product space.

Proposition 3.4.2. *Let $\phi : U \rightarrow S$ be a chart on a regular surface S . Define 2×2 matrices*

$$g = \begin{pmatrix} \phi_u \cdot \phi_u & \phi_u \cdot \phi_v \\ \phi_v \cdot \phi_u & \phi_v \cdot \phi_v \end{pmatrix}, \quad A = \begin{pmatrix} A(\phi_u, \phi_u) & A(\phi_u, \phi_v) \\ A(\phi_v, \phi_u) & A(\phi_v, \phi_v) \end{pmatrix}, \quad \phi_u = \frac{\partial \phi}{\partial u}, \quad \phi_v = \frac{\partial \phi}{\partial v}.$$

Then, writing $\sigma = g^{-1}A$, we have

$$K = \det(\sigma) = \frac{\det(A)}{\det(g)}, \quad H = \frac{1}{2} \text{tr}(\sigma).$$

Remark. If ϕ_u, ϕ_v are the principal directions at $p \in S$ then this is immediate, because in that case g is the identity matrix and Proposition 3.4.2 just restates the definitions of K and H .

Lecture 17
Monday
12/11/18

Proof. Let p be a point on S . Let us work in a basis for $T_p S$ given by the principal directions $x_1(p), x_2(p)$. Then, because this is an orthonormal basis, we can apply Lemma 3.4.1 to find,

$$A = \begin{pmatrix} \phi_u \\ \phi_v \end{pmatrix} \begin{pmatrix} -dN_p(\phi_u) & -dN_p(\phi_v) \end{pmatrix}.$$

Here (ϕ_u) is a row vector of size two and $(-dN_p(\phi_u))$ is a column vector of size two. Now write the matrix of $dN_p : T_p S \rightarrow T_p S$ with respect to the basis $x_1(p), x_2(p)$ too,

$$A = \begin{pmatrix} \phi_u \\ \phi_v \end{pmatrix} M \begin{pmatrix} \phi_u & \phi_v \end{pmatrix},$$

where M , the matrix of dN_p with respect to the basis $x_1(p), x_2(p)$, is a 2×2 matrix. Then

$$\det(A) = \det \begin{pmatrix} \phi_u \\ \phi_v \end{pmatrix} \det(M) \det \begin{pmatrix} \phi_u & \phi_v \end{pmatrix} = \det(M) \det \left(\begin{pmatrix} \phi_u \\ \phi_v \end{pmatrix} \begin{pmatrix} \phi_u & \phi_v \end{pmatrix} \right) = \det(M) \det(g),$$

where for the last equality we used Lemma 3.4.1 again. Since $\det(M) = K$, by definition, this proves that $K = \det(A) / \det(g)$. Furthermore,

$$A = \begin{pmatrix} \phi_u \\ \phi_v \end{pmatrix} \begin{pmatrix} \phi_u & \phi_v \end{pmatrix} \begin{pmatrix} \phi_u & \phi_v \end{pmatrix}^{-1} M \begin{pmatrix} \phi_u & \phi_v \end{pmatrix} = gB,$$

where $B = \begin{pmatrix} \phi_u & \phi_v \end{pmatrix}^{-1} M \begin{pmatrix} \phi_u & \phi_v \end{pmatrix}$ is conjugate to M . Thus $\sigma = g^{-1}A$ is conjugate to M , and so $H = \text{tr}(M) = \text{tr}(\sigma)$ because conjugate matrices have the same trace. \square

Example. Let S be the surface in \mathbb{R}^3 defined by the equation $z = x^2 - y^2$. Compute the Gaussian and mean curvature at each point of S . Choose the chart

$$\begin{aligned} \phi : U = \mathbb{R}^2 &\longrightarrow S \\ (u, v) &\longmapsto (u, v, u^2 - v^2) \end{aligned}.$$

Now $\phi_u = (1, 0, 2u)$ and $\phi_v = (0, 1, -2v)$, so

$$g = \begin{pmatrix} \phi_u \cdot \phi_u & \phi_u \cdot \phi_v \\ \phi_v \cdot \phi_u & \phi_v \cdot \phi_v \end{pmatrix} = \begin{pmatrix} 1 + 4u^2 & -4uv \\ -4uv & 1 + 4v^2 \end{pmatrix}.$$

Also,

$$\phi_u \times \phi_v = \left| \begin{pmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & -2v \end{pmatrix} \right| = (-2u, 2v, 1),$$

and so

$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \left(\frac{-2u}{\sqrt{1 + 4u^2 + 4v^2}}, \frac{2v}{\sqrt{1 + 4u^2 + 4v^2}}, \frac{1}{\sqrt{1 + 4u^2 + 4v^2}} \right).$$

Now $\phi_{uu} = (1, 0, 2)$, $\phi_{uv} = (0, 0, 0)$, $\phi_{vv} = (1, 0, -2)$, so

$$A = \begin{pmatrix} N \cdot \phi_{uu} & N \cdot \phi_{uv} \\ N \cdot \phi_{vu} & N \cdot \phi_{vv} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{1 + 4u^2 + 4v^2}} & 0 \\ 0 & \frac{-2}{\sqrt{1 + 4u^2 + 4v^2}} \end{pmatrix} = \frac{1}{\sqrt{1 + 4u^2 + 4v^2}} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then

$$\sigma = g^{-1}A = \frac{1}{1 + 4u^2 + 4v^2} \begin{pmatrix} 1 + 4v^2 & 4uv \\ 4uv & 1 + 4u^2 \end{pmatrix} \cdot A = \frac{2}{(1 + 4u^2 + 4v^2)^{3/2}} \begin{pmatrix} 1 + 4v^2 & -4uv \\ 4uv & -1 - 4u^2 \end{pmatrix}.$$

So

$$K = \det(\sigma) = 4 \left(\frac{-(1 + 4u^2)(1 + 4v^2) + 16u^2v^2}{(1 + 4u^2 + 4v^2)^3} \right) = \frac{-4}{(1 + 4u^2 + 4v^2)^2},$$

and

$$H = \frac{1}{2} \text{tr}(\sigma) = \frac{4v^2 - 4u^2}{(1 + 4u^2 + 4v^2)^{3/2}}.$$

Lecture 18
Tuesday
13/11/18

3.5 The first fundamental form

Definition 3.5.1. Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. The **first fundamental form** at p , also called the **metric** at p , is the bilinear map

$$g : T_p S \times T_p S \longrightarrow \mathbb{R} \\ (v, w) \longmapsto \langle v, w \rangle .$$

g is symmetric, bilinear, and non-degenerate. If $\phi : U \rightarrow S$ is a chart near $p \in S$, then ϕ_u, ϕ_v are a basis for $T_p S$ and in this basis, g has the form

$$\begin{aligned} g(a\phi_u + b\phi_v, c\phi_u + d\phi_v) &= acg(\phi_u, \phi_u) + adg(\phi_u, \phi_v) + bcg(\phi_v, \phi_u) + bdg(\phi_v, \phi_v) \\ &= \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} g(\phi_u, \phi_u) & g(\phi_u, \phi_v) \\ g(\phi_v, \phi_u) & g(\phi_v, \phi_v) \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} g \begin{pmatrix} c \\ d \end{pmatrix}, \end{aligned}$$

that is the metric is represented by the symmetric matrix that we called g before. Metric determines arc-length. If $\alpha : [a, b] \rightarrow S$ is a curve contained in $\phi(U)$, so that $\alpha(t) = \phi(u(t), v(t))$ for some smooth functions $u, v : [a, b] \rightarrow \mathbb{R}$, then

$$L(\alpha) = \int_a^b |\alpha'(t)| dt = \int_a^b \left| \frac{\partial \phi}{\partial u} \frac{du}{dt} + \frac{\partial \phi}{\partial v} \frac{dv}{dt} \right| dt = \int_a^b \sqrt{\left(\frac{du}{dt} \quad \frac{dv}{dt} \right) \cdot g \cdot \left(\frac{du}{dt} \quad \frac{dv}{dt} \right)^T} dt.$$

Definition 3.5.2. A smooth map $F : S_1 \rightarrow S_2$ is called a **local isometry** if it preserves the first fundamental form, so

$$\langle dF_p(x), dF_p(y) \rangle = \langle x, y \rangle, \quad x, y \in T_p S_1, \quad p \in S_1.$$

F is an **isometry** if it is also bijective.

Proposition 3.5.3. *Local isometries are local diffeomorphisms.*

Proof. If F is a local isometry then, for each $p \in S$, dF_p is injective. Suppose not. Then there exists $0 \neq v \in T_p S$ such that $dF_p(v) = 0$, so $\langle dF_p(v), dF_p(w) \rangle = 0$ for all $w \in T_p S$, so $\langle v, w \rangle = 0$ for all $w \in T_p S$, so $v = 0$, a contradiction. Now $dF_p : T_p S_1 \rightarrow T_{F(p)} S_2$ is linear and injective, and $\dim T_p S_1 = \dim T_p S_2 = 2$, so dF_p is an isomorphism. From before, since dF_p is an isomorphism, F is a local diffeomorphism, that is, there is an open neighbourhood U of p in S_1 such that $F|_U : U \rightarrow F(U)$ is a diffeomorphism. \square

Proposition 3.5.4. *A smooth map $F : S_1 \rightarrow S_2$ between regular surfaces is a local isometry if and only if it preserves lengths of curves, that is if and only if for all $\alpha : [a, b] \rightarrow S_1$ smooth we have*

$$L(\alpha) = L(F \circ \alpha).$$

Proof.

\implies Suppose that F is a local isometry. Then

$$\begin{aligned} L(F \circ \alpha) &= \int_a^b |(F \circ \alpha)'(t)| dt = \int_a^b \sqrt{\langle (F \circ \alpha)'(t), (F \circ \alpha)'(t) \rangle} dt \\ &= \int_a^b \sqrt{\langle dF_{\alpha(t)}(\alpha'(t)), dF_{\alpha(t)}(\alpha'(t)) \rangle} dt = \int_a^b \sqrt{\langle \alpha'(t), \alpha'(t) \rangle} dt \\ &= \int_a^b |\alpha'(t)| dt = L(\alpha), \end{aligned}$$

by definition of $dF_{\alpha(t)}$ and as F is a local isometry.

\impliedby Given $v \in T_p S_1$ and a curve $\alpha : (-\epsilon, \epsilon) \rightarrow S_1$ with $\alpha(0) = p$ and $\alpha'(0) = v$. Consider the curve $\alpha|_{[-\delta, t]} : [-\delta, t] \rightarrow S_1$, so

$$\begin{array}{ccc} & p \in S_1 & \xrightarrow{F} S_2 \\ \alpha \nearrow & \uparrow \alpha|_{[\delta, t]} & \nearrow \\ (-\epsilon, \epsilon) & \supset [-\delta, t] & \end{array} .$$

Lecture 19
Friday
16/11/18

Know $L\left(\alpha|_{[-\delta,t]}\right) = L\left((F \circ \alpha)|_{[-\delta,t]}\right)$, that is

$$\int_{-\delta}^t |\alpha'(s)| \, ds = \int_{-\delta}^t |dF_{\alpha(s)}(\alpha'(s))| \, ds.$$

Taking $\frac{d}{dt}$ and setting $t = 0$, $|\alpha'(0)| = |dF_p(\alpha'(0))|$, that is

$$\langle v, v \rangle = \langle dF_p(v), dF_p(v) \rangle, \quad v \in T_p S_1, \quad p \in S_1.$$

Now,

$$\langle X + Y, X + Y \rangle = \langle X, X \rangle + \langle X, Y \rangle + \langle Y, X \rangle + \langle Y, Y \rangle = \langle X, X \rangle + 2\langle X, Y \rangle + \langle Y, Y \rangle,$$

so

$$\langle X, Y \rangle = \frac{1}{2} (\langle X + Y, X + Y \rangle - \langle X, X \rangle - \langle Y, Y \rangle).$$

Set $X = dF_p(v)$ and $Y = dF_p(w)$, so

$$\begin{aligned} \langle dF_p(v), dF_p(w) \rangle &= \frac{1}{2} (\langle dF_p(v+w), dF_p(v+w) \rangle - \langle dF_p(v), dF_p(v) \rangle - \langle dF_p(w), dF_p(w) \rangle) \\ &= \frac{1}{2} (\langle v+w, v+w \rangle - \langle v, v \rangle - \langle w, w \rangle) = \langle v, w \rangle. \end{aligned}$$

Therefore F is a local isometry. □

Example. Let $S \subset \mathbb{R}^3$ be any regular surface. Let $B \in SO(3)$, 3×3 matrices B such that $B^T B = Id$ and $\det(B) = 1$. Set

$$S' = \{B\vec{x} \mid \vec{x} \in S\}.$$

Then

$$\begin{array}{ccc} B & : & S \longrightarrow S' \\ & & \vec{x} \longmapsto B\vec{x} \end{array}$$

is an isometry. Need to check the following.

- Bijective.
- Local isometry. Let $p \in S$ and $v, w \in T_p S$ be arbitrary. Then $dF_p(v) = Bv$ and

$$\langle dF_p(v), dF_p(w) \rangle = \langle Bv, Bw \rangle = (Bv)^T Bw = v^T B^T Bw = v^T w = \langle v, w \rangle,$$

so B is a local isometry.

Example. Let

$$S_1 = \{(x, y, 0) \in \mathbb{R}^3 \mid x \in \mathbb{R}, y \in \mathbb{R}\}, \quad S_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

Define

$$\begin{array}{ccc} F & : & S_1 \longrightarrow S_2 \\ & & (u, v, 0) \longmapsto (\cos(u), \sin(u), v) \end{array}$$

At $p = (u, v, 0) \in S_1$ a tangent vector $\vec{v} = (a, b, 0)$ can be written as $\alpha'(0)$ where $\alpha(t) = (u + at, v + bt, 0)$. Now

$$(F \circ \alpha)(t) = (\cos(u + at), \sin(u + at), v + bt),$$

so

$$(F \circ \alpha)'(0) = (-a \sin(u), a \cos(u), b).$$

Then

$$\langle dF_p(\vec{v}), dF_p(\vec{v}) \rangle = \langle (-a \sin(u), a \cos(u), b), (-a \sin(u), a \cos(u), b) \rangle = a^2 + b^2 = \langle \vec{v}, \vec{v} \rangle, \quad \vec{v} \in T_p S_1,$$

and so, as before,

$$\langle \vec{v}, \vec{w} \rangle = \langle dF_p(\vec{v}), dF_p(\vec{w}) \rangle, \quad \vec{v}, \vec{w} \in T_p S_1.$$

Thus F is a local isometry. It is not an isometry because it is not bijective.

3.6 Christoffel symbols

Now, $\frac{d\phi}{dx_1}$ and $\frac{d\phi}{dx_2}$ span $T_p S$ and $N(p)$ is orthogonal to $T_p S$ so $\frac{d\phi}{dx_1}, \frac{d\phi}{dx_2}, N$ form a basis for \mathbb{R}^3 . Write

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \Gamma_{ij}^1 \frac{\partial \phi}{\partial x_1} + \Gamma_{ij}^2 \frac{\partial \phi}{\partial x_2} + A_{ij} N, \quad (2)$$

for some coefficients Γ_{ij}^k and A_{ij} .

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i},$$

so $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all i, j, k .

Definition 3.6.1. The Γ_{ij}^k are called **Christoffel symbols**.

Taking (2) $\cdot N$, we get

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} \cdot N = A_{ij}.$$

Note. $\nabla_i e_j = \Gamma_{ij}^k e_k$, where ∇_i is the metric connection in the local frame e_j .

Proposition 3.6.2. The Christoffel symbols are determined only by the first fundamental form, or metric.

Proof. From (2) $\cdot \frac{d\phi}{dx_k}$,

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} \cdot \frac{\partial \phi}{\partial x_k} = \Gamma_{ij}^1 g_{1k} + \Gamma_{ij}^2 g_{2k}, \quad g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

Also,

$$\frac{\partial}{\partial x_i} \left(\frac{\partial \phi}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k} \right) = \frac{\partial^2 \phi}{\partial x_i \partial x_j} \cdot \frac{\partial \phi}{\partial x_k} + \frac{\partial \phi}{\partial x_j} \cdot \frac{\partial^2 \phi}{\partial x_i \partial x_k}.$$

Taking $j = k$,

$$\frac{\partial}{\partial x_i} (g_{jj}) = 2 (\Gamma_{ij}^1 g_{1j} + \Gamma_{ij}^2 g_{2j}).$$

If $j \neq k$, then either $i = j$ or $i = k$. Taking $i = j \neq k$,

$$\frac{\partial}{\partial x_i} (g_{ik}) = \frac{\partial^2 \phi}{\partial x_i \partial x_i} \cdot \frac{\partial \phi}{\partial x_k} + \frac{\partial \phi}{\partial x_i} \cdot \frac{\partial^2 \phi}{\partial x_i \partial x_k} = \frac{\partial^2 \phi}{\partial x_i \partial x_i} \cdot \frac{\partial \phi}{\partial x_k} + \frac{1}{2} \frac{\partial}{\partial x_k} (g_{ii}).$$

Rewriting,

$$\Gamma_{ij}^1 g_{1k} + \Gamma_{ij}^2 g_{2k} = \frac{\partial^2 \phi}{\partial x_i \partial x_i} \cdot \frac{\partial \phi}{\partial x_k} = \frac{\partial}{\partial x_i} (g_{ik}) - \frac{1}{2} \frac{\partial}{\partial x_k} (g_{ii}).$$

We have lots of equations involving Γ_{ij}^k , g_{ij} , and derivatives of g_{ij} . In fact we can solve these for Γ_{ij}^k . $i = j = 1$ and $k \in \{1, 2\}$ gives

$$\Gamma_{11}^1 g_{11} + \Gamma_{11}^2 g_{21} = \frac{1}{2} \frac{\partial g_{11}}{\partial x_1}, \quad \Gamma_{11}^1 g_{12} + \Gamma_{11}^2 g_{22} = \frac{\partial g_{12}}{\partial x_1} - \frac{1}{2} \frac{\partial g_{11}}{\partial x_2}.$$

That is,

$$g \begin{pmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \frac{\partial g_{11}}{\partial x_1} \\ \frac{\partial g_{12}}{\partial x_1} - \frac{1}{2} \frac{\partial g_{11}}{\partial x_2} \end{pmatrix}.$$

g is invertible, because $\frac{d\phi}{dx_1}$ and $\frac{d\phi}{dx_2}$ are linearly independent. Similarly,

$$g \begin{pmatrix} \Gamma_{21}^1 \\ \Gamma_{21}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \frac{\partial g_{11}}{\partial x_2} \\ \frac{1}{2} \frac{\partial g_{22}}{\partial x_1} \end{pmatrix}, \quad g \begin{pmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial g_{21}}{\partial x_2} - \frac{1}{2} \frac{\partial g_{22}}{\partial x_1} \\ \frac{1}{2} \frac{\partial g_{22}}{\partial x_2} \end{pmatrix}.$$

Thus can solve these equations to write all Γ_{ij}^k in terms of g_{ij} 's and $\frac{\partial g_{jk}}{\partial x_i}$'s. \square

3.7 Theorema Egregium

Lecture 21
Tuesday
20/11/18

Recall that $K = \det(A_{ij}) / \det(g_{ij})$. The goal is that K depends only on g .

Notation. Set $x_1 = u$ and $x_2 = v$. Write $\phi_u = \frac{d\phi}{dx_1} = \frac{d\phi}{du}$, without confusion with indices.

Mixed partial derivatives are equal, so

$$\frac{\partial}{\partial x_2} \left(\frac{\partial^2 \phi}{\partial x_1 \partial x_1} \right) = \frac{\partial}{\partial x_1} \left(\frac{\partial^2 \phi}{\partial x_1 \partial x_2} \right),$$

so

$$\frac{\partial}{\partial x_2} \left(\Gamma_{11}^1 \frac{\partial \phi}{\partial x_1} + \Gamma_{11}^2 \frac{\partial \phi}{\partial x_2} + A_{11} N \right) = \frac{\partial}{\partial x_1} \left(\Gamma_{12}^1 \frac{\partial \phi}{\partial x_1} + \Gamma_{12}^2 \frac{\partial \phi}{\partial x_2} + A_{12} N \right).$$

Thus

$$\Gamma_{11}^1 \phi_{uv} + \Gamma_{11v}^1 \phi_u + \Gamma_{11}^2 \phi_{vv} + \Gamma_{11v}^2 \phi_v + A_{11v} N + A_{11} N_v = \Gamma_{12}^1 \phi_{uu} + \Gamma_{12u}^1 \phi_u + \Gamma_{12}^2 \phi_{vu} + \Gamma_{12u}^2 \phi_v + A_{12u} N + A_{12} N_u. \quad (3)$$

Will take the ϕ_u -component of (3), and use

$$\begin{aligned} \phi_{uv} &= \Gamma_{12}^1 \phi_u + \Gamma_{12}^2 \phi_v + A_{12} N, & \phi_{vv} &= \Gamma_{22}^1 \phi_u + \Gamma_{22}^2 \phi_v + A_{22} N, & \Gamma_{uu} &= \Gamma_{11}^1 \phi_u + \Gamma_{11}^2 \phi_v + A_{11} N, \\ A_{11} &= -\frac{\partial \phi}{\partial u} \cdot N_u, & A_{21} &= -\frac{\partial \phi}{\partial v} \cdot N_u, & A_{12} &= -\frac{\partial \phi}{\partial u} \cdot N_v, & A_{22} &= -\frac{\partial \phi}{\partial v} \cdot N_v. \end{aligned}$$

Now

$$N_u = a\phi_u + b\phi_v, \quad N_v = c\phi_u + d\phi_v,$$

since $N \cdot N = 1$, so $N_u \cdot N + N \cdot N_u = 0$ and $N_v \cdot N + N \cdot N_v = 0$, so $N_u, N_v \in T_p S$. Then

$$-A_{11} = ag_{11} + bg_{12}, \quad -A_{21} = ag_{21} + bg_{22}, \quad -A_{12} = cg_{11} + dg_{12}, \quad -A_{22} = cg_{21} + dg_{22},$$

so

$$-A = g \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

so

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -g^{-1} A = \frac{-1}{g_{11}g_{12} - g_{12}^2} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

so

$$a = \frac{g_{12}A_{21} - g_{22}A_{11}}{g_{11}g_{22} - g_{12}^2}, \quad c = \frac{g_{12}A_{22} - g_{22}A_{12}}{g_{11}g_{22} - g_{12}^2}.$$

Now, take (3), dot with ϕ_u , and use these equations above. We find,

$$\Gamma_{11}^1 \Gamma_{12}^1 + \Gamma_{11v}^1 + \Gamma_{11}^2 \Gamma_{22}^1 + A_{11} \left(\frac{g_{12}A_{22} - g_{22}A_{12}}{\det(g)} \right) = \Gamma_{12}^1 \Gamma_{11}^1 + \Gamma_{12u}^1 + \Gamma_{12}^2 \Gamma_{12}^1 + A_{12} \left(\frac{g_{12}A_{21} - g_{22}A_{11}}{\det(g)} \right).$$

Rearranging,

$$\Gamma_{11v}^1 - \Gamma_{12u}^1 = \Gamma_{12}^2 \Gamma_{12}^1 - \Gamma_{11}^2 \Gamma_{22}^1 + \frac{(A_{12}A_{21} - A_{11}A_{22})g_{12}}{\det(g)}.$$

In other words,

$$Kg_{12} = \Gamma_{12}^2 \Gamma_{12}^1 - \Gamma_{11}^2 \Gamma_{22}^1 + \Gamma_{12u}^1 - \Gamma_{11v}^1.$$

Similarly, taking the ϕ_v component of (3)

$$Kg_{11} = \Gamma_{11}^1 \Gamma_{21}^2 + \Gamma_{11}^2 \Gamma_{22}^2 - \Gamma_{12}^1 \Gamma_{11}^2 - \Gamma_{12}^2 \Gamma_{12}^2 + \Gamma_{11v}^2 - \Gamma_{12u}^2.$$

These are the **Gauss equations**.

Note. Kg_{12} and Kg_{11} depend only on g . But g is invertible, so g_{11} and g_{12} cannot simultaneously vanish.

We have proved the following.

Theorem 3.7.1 (Theorema Egregium, Gauss). *The Gaussian curvature K depends only on the first fundamental form.*

In other words, it is an intrinsic invariant of the surface (S, g) , as a Riemannian manifold.

Corollary 3.7.2. *If $f : S_1 \rightarrow S_2$ is a local isometry then $K_{S_1}(p) = K_{S_2}(f(p))$ for all $p \in S_1$.*

Corollary 3.7.3. *There is no local isometry from a plane to a sphere.*

Proof. A plane has $K = 0$ everywhere and a sphere has $K = 1/r^2$ everywhere, where r is the radius. \square

In particular, there is no way to draw a map of the Earth on a flat sheet of paper without distorting lengths or angles. This is also why it is hard to gift-wrap a ball.

Theorem 3.7.4. *Let S be a compact surface with $K > 0$, such that K is constant, then S is umbilical.*

Proof. Let $x \in S$, and $\lambda_1(x) \leq \lambda_2(x)$ be principal curvatures at x . The aim is that $\lambda_1(x) = \lambda_2(x)$. The idea of the proof is to assume $\lambda_1 < \lambda_2$ for a contradiction. Let $p \in S$ be a point maximising $\lambda_2(x)$, so minimises $\lambda_1(x)$ and

$$\lambda_1(p) \leq \lambda_1(x) \leq \lambda_2(x) \leq \lambda_2(p).$$

The aim is if $\lambda_1(p) = \lambda_2(p)$, then $\lambda_1(x) = \lambda(x)$, for all $x \in S$. Assume $\lambda_1(p) < \lambda_2(p)$, and want to get a contradiction. After a rigid motion, around $p = (0, 0, 0)$,

$$\phi(u, v) = (u, v, F(u, v)) + \dots = \frac{1}{2}(\lambda_1 u^2 + \lambda_2 v^2) + \dots,$$

by Taylor expansion. At $(0, 0)$, $F(0, 0) = F_u(0, 0) = F_v(0, 0) = 0$, so

$$\frac{\partial \phi}{\partial u}(0, 0) = (1, 0, F_u(0, 0)) = (1, 0, 0), \quad \frac{\partial \phi}{\partial v}(0, 0) = (0, 1, F_v(0, 0)) = (0, 1, 0),$$

so

$$N = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{(-F_u, -F_v, 1)}{\sqrt{1 + |\nabla F|^2}} = (0, 0, 1),$$

at $(0, 0)$. Then

$$A = \begin{pmatrix} F_{uu} & F_{uv} \\ F_{vu} & F_{vv} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad g = \begin{pmatrix} \phi_u \cdot \phi_u & \phi_u \cdot \phi_v \\ \phi_v \cdot \phi_u & \phi_v \cdot \phi_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $K = \det(A) / \det(g) = \lambda_1 \lambda_2$. Let

$$E_1 = \frac{(1, 0, F_u)}{\sqrt{1 + F_u^2}}, \quad E_2 = \frac{(0, 1, F_v)}{\sqrt{1 + F_v^2}}.$$

Let

$$h_1(t) = A(E_1(0, t), E_1(0, t)), \quad h_2(t) = A(E_2(t, 0), E_2(t, 0)).$$

Check (Exercise)

$$h_1(t) = \frac{1}{1 + F_u^2} \cdot \left(\frac{F_{uu}}{\sqrt{1 + |\nabla F|^2}} \right) \Bigg|_{(0, t)}, \quad h_2(t) = \frac{1}{1 + F_v^2} \cdot \left(\frac{F_{vv}}{\sqrt{1 + |\nabla F|^2}} \right) \Bigg|_{(t, 0)}.$$

Recall that

$$\lambda_1(p) \leq \lambda_1(\phi(0, t)) = \min \{A(v, v) \mid v \in T_{\phi(0, t)}S, |v| = 1\} \leq A(E_1(0, t), E_1(0, t)) = h_1(t),$$

and

$$\lambda_2(p) \geq \lambda_2(\phi(t, 0)) = \max \{A(v, v) \mid v \in T_{\phi(t, 0)}S, |v| = 1\} \geq A(E_2(t, 0), E_2(t, 0)) = h_2(t),$$

so $h_1(t)$ is minimum at $t = 0$ and $h_2(t)$ is maximum at $t = 0$, so $h_1''(0) \geq 0$ and $h_2''(0) \leq 0$, so

$$h_1''(0) - h_2''(0) \geq 0.$$

Assume $\lambda_1 < \lambda_2$. Thus $\lambda_1 \lambda_2 (\lambda_1 - \lambda_2) < 0$, (Exercise) a contradiction. \square

Corollary 3.7.5. *Let $K > 0$ be constant. Then S is compact and connected implies that $S = S^2$.*

Lecture 22
Friday
23/11/18

4 Area of surfaces

4.1 Area

Let $S \subset \mathbb{R}^3$ be a regular surface and $\phi : U \rightarrow S$ be a chart. How to measure the area on S ? Consider a tiny rectangle in U with sides $(\delta u, 0)$ and $(0, \delta v)$. The area of this, in U , is $\delta u \delta v$. If δu and δv are small then the area of the image of this rectangle in $\phi(U)$ is approximately the area of the parallelogram in \mathbb{R}^3 with sides $\left(\frac{d\phi}{du}\right) \delta u$ and $\left(\frac{d\phi}{dv}\right) \delta v$, that is

$$\left| \left(\frac{\partial \phi}{\partial u} \delta u \right) \times \left(\frac{\partial \phi}{\partial v} \delta v \right) \right| = \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right| \delta u \delta v.$$

Definition 4.1.1. If $D \subset U$ is compact, then

$$\text{area}(\phi(D)) = \iint_D \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right| du dv.$$

Need to show this integral

- converges, and
- is independent of choice of chart.

Proposition 4.1.2. *This definition does not depend on the choice of coordinate chart $\phi : U \rightarrow S$.*

Proof. Let $\phi : U \rightarrow S$ and $\psi : U' \rightarrow S$ be charts whose images contain $\phi(D) = \psi(D)$. After shrinking U, U' if necessary, we have that $f = \phi^{-1} \circ \psi : U' \rightarrow U$ is well-defined and smooth. Write

$$f(u, v) = (x(u, v), y(u, v)),$$

so

$$\begin{array}{ccc} D \subset U & \xrightarrow{\phi} & S \\ f \uparrow & \nearrow \psi & \\ D' \subset U' & & \end{array}.$$

Then $\psi = \phi \circ f$ so

$$\frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v} = \left(\frac{\partial \phi}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial u} \right) \times \left(\frac{\partial \phi}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial v} \right) = \Delta \left(\frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y} \right), \quad \Delta(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}.$$

So

$$\begin{aligned} \text{area}(\psi(D')) &= \iint_{D'} \left| \frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v} \right| du dv \\ &= \iint_{D'} \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right| \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} \right| du dv \\ &= \iint_D \left| \frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y} \right| dx dy = \text{area}(\phi(D)), \end{aligned}$$

by the change of variable formula. □

Note. $S \subset \mathbb{R}^3$, and \mathbb{R}^3 has a distinguished 3-form $dx \wedge dy \wedge dz$. If S is oriented, so we have chosen a normal vector field N on S , then can consider the 2-form $\omega = 2_N(dx \wedge dy \wedge dz)$, where 2_N is the contraction, and measure areas on S using this 2-form. Up to sign, this gives the same notion of area.

Lemma 4.1.3.

$$\text{area}(\phi(D)) = \iint_D \sqrt{\det(g)} \, du \, dv.$$

Proof. $|a \times b|^2 = |a|^2 |b|^2 - (a \cdot b)^2$, so

$$\left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right|^2 = \left| \frac{\partial \phi}{\partial u} \right|^2 \left| \frac{\partial \phi}{\partial v} \right|^2 - \left(\frac{\partial \phi}{\partial u} \cdot \frac{\partial \phi}{\partial v} \right)^2.$$

But

$$g = \begin{pmatrix} \frac{\partial \phi}{\partial u} \cdot \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial u} \cdot \frac{\partial \phi}{\partial v} \\ \frac{\partial \phi}{\partial v} \cdot \frac{\partial \phi}{\partial u} & \frac{\partial \phi}{\partial v} \cdot \frac{\partial \phi}{\partial v} \end{pmatrix},$$

so

$$\left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right|^2 = g_{11}g_{22} - g_{21}^2 = \det(g).$$

Thus

$$\text{area}(\phi(D)) = \iint_D \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right| \, du \, dv = \iint_D \sqrt{\det(g)} \, du \, dv.$$

□

Definition 4.1.4. Assume S is compact. If $f : S \rightarrow \mathbb{R}$ is a smooth function on S , $\phi : U \rightarrow S$ is a chart, and $D \subset U$ is a compact subset, then we define

$$\int_{\phi(D)} f \, dA = \iint_D (f \circ \phi)(u, v) \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right| \, du \, dv = \iint_D f \sqrt{\det(g)} \, du \, dv.$$

This does not depend on chart $\phi : U \rightarrow S$. To integrate over all of S we divide S into pieces

$$S = S_1 \cup \cdots \cup S_k, \quad S_i \cap S_j \subset (\partial S_i) \cap (\partial S_j), \quad i \neq j,$$

such that for each i there exists a chart $\phi : U_i \rightarrow S$ such that $S_i \subset \phi(U_i)$ and $S_i = \phi(D_i)$ for some compact set $D_i \subset U_i$. By compactness of S , such a partition $S = S_1 \cup \cdots \cup S_k$ always exists.

Definition 4.1.5. Define

$$\int_S f \, dA = \sum_{i=1}^k \int_{S_i} f \, dA.$$

Claim that this is independent of the choice of decomposition.

Example. What is the surface area of the unit sphere in \mathbb{R}^3 ? Use spherical polar coordinates

$$\psi(\theta, \phi) = (\cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi)), \quad \epsilon \leq \theta \leq 2\pi - \epsilon, \quad \epsilon \leq \phi \leq \pi - \epsilon.$$

Is this a regular surface?

$$(\psi_\theta \quad \psi_\phi) = \begin{pmatrix} -\sin(\theta) \sin(\phi) & \cos(\theta) \cos(\phi) \\ \cos(\theta) \sin(\phi) & \sin(\theta) \cos(\phi) \\ 0 & -\sin(\phi) \end{pmatrix}.$$

Is this matrix rank two, that is are the columns linearly independent? If $\sin(\phi) \neq 0$ then the first two rows of the first column is not both zero and the last row of the second column is not zero. So this surface is regular, because $\sin(\phi) \neq 0$ in our chart, since we took $\phi \in (0, \pi)$.

$$\begin{aligned} \text{area}(\psi(D)) &= \iint_D |\psi_\theta \times \psi_\phi| \, d\theta \, d\phi = \int_\epsilon^{\pi-\epsilon} \int_\epsilon^{2\pi-\epsilon} \sin(\phi) \, d\theta \, d\phi \\ &= (2\pi - 2\epsilon) [\cos(\phi)]_{\pi-\epsilon}^\epsilon = (2\pi - 2\epsilon) (\cos(\epsilon) - \cos(\pi - \epsilon)). \end{aligned}$$

Then

$$\lim_{\epsilon \rightarrow 0} \text{area}(\psi(D)) = 4\pi.$$

Compute the area of the rest by using another chart, which tends to zero as $\epsilon \rightarrow 0$. Thus $\text{area}(S^2) = 4\pi$.

Lecture 24
Tuesday
27/11/18

5 Geodesics

5.1 Geodesic curvature

Geodesics are the shortest paths. Let $\gamma : [a, b] \rightarrow S$ be a regular curve in a regular surface S . Suppose that S is oriented, with normal vector field N , and that γ is parametrised by arc-length. Then

$$(\gamma'(t), N(\gamma(t)) \times \gamma'(t), N(\gamma(t)))$$

is an orthonormal basis for \mathbb{R}^3 . Recall that the curvature of γ is $\vec{\kappa}(t) = \gamma''(t)$ and that γ'' is orthogonal to γ' , because $\gamma'(t) \cdot \gamma'(t) = 1$, so $\frac{d}{dt}(\gamma''(t) \cdot \gamma'(t)) = 0$, so $2\gamma''(t) \cdot \gamma'(t) = 0$. So

$$\vec{\kappa}(t) = k_n(t)N + k_g(t)(N \times \gamma'),$$

where

$$k_n(t) = \vec{\kappa}(t) \cdot N$$

is the **normal curvature** of γ at t , and

$$k_g(t) = \vec{\kappa}(t) \cdot (N \times \gamma')$$

is the **geodesic curvature** of γ at t . $k_g(t)$ is the amount that γ is curving along S . γ is a **geodesic** if the geodesic curvature is zero.

Example. Let S be the xy plane and $N = (0, 0, 1)$. Consider a curve $\gamma(t) = (x(t), y(t), 0)$ parametrised by arc-length, so $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1$. Then the normal curvature is

$$k_n = \vec{\kappa} \cdot N = (x'', y'', 0) \cdot (0, 0, 1) = 0,$$

and

$$N \times \gamma' = (0, 0, 1) \times (x', y', 0) = (-y', x', 0),$$

so the geodesic curvature is

$$k_g = \vec{\kappa} \cdot (N \times \gamma') = (x'', y'', 0) \cdot (-y', x', 0) = x'y'' - x''y'.$$

$\vec{\kappa} = k_g(N \times \gamma')$, so $k_g = 0$ if and only if $\vec{\kappa} = 0$, if and only if $x'' = y'' = 0$. That is, geodesics in the plane take the form

$$(x(t), y(t)) = (a_0, b_0) + t(a_1, b_1),$$

which are straight lines.

Example. Let S be the unit sphere

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Consider a latitude of radius r

$$\gamma(t) = \left(r \cos\left(\frac{t}{r}\right), r \sin\left(\frac{t}{r}\right), \sqrt{1-r^2}\right), \quad 0 \leq t \leq 2\pi r.$$

Compute the geodesic curvature $k_g = \vec{\kappa} \cdot (N \times \gamma')$.

$$\gamma'(t) = \left(-\sin\left(\frac{t}{r}\right), \cos\left(\frac{t}{r}\right), 0\right), \quad N(\gamma(t)) = \gamma(t) = \left(r \cos\left(\frac{t}{r}\right), r \sin\left(\frac{t}{r}\right), \sqrt{1-r^2}\right),$$

so

$$N \times \gamma' = \left(-\sqrt{1-r^2} \cos\left(\frac{t}{r}\right), -\sqrt{1-r^2} \sin\left(\frac{t}{r}\right), r\right), \quad \vec{\kappa}(t) = \gamma''(t) = \frac{1}{r} \left(-\cos\left(\frac{t}{r}\right), -\sin\left(\frac{t}{r}\right), 0\right).$$

Lecture 25
Friday
30/11/18

Thus

$$\begin{aligned} k_g &= \frac{1}{r} \left(-\cos\left(\frac{t}{r}\right), -\sin\left(\frac{t}{r}\right), 0 \right) \cdot \left(-\sqrt{1-r^2} \cos\left(\frac{t}{r}\right), -\sqrt{1-r^2} \sin\left(\frac{t}{r}\right), r \right) \\ &= \frac{\sqrt{1-r^2}}{r} \left(\cos^2\left(\frac{t}{r}\right) + \sin^2\left(\frac{t}{r}\right) \right) = \frac{\sqrt{1-r^2}}{r}. \end{aligned}$$

So a latitude is a geodesic if and only if $r = 1$, if and only if it is the equator.

Proposition 5.1.1. *Local isometries send geodesics to geodesics.*

Proof. Let $\gamma : [a, b] \rightarrow S$ be a geodesic parametrised by arc-length, and $F : S \rightarrow S'$ a local isometry, so

$$\begin{array}{ccc} \gamma(t) \subset S & \xrightarrow{F} & (F \circ \gamma)(t) \subset S' \\ \uparrow \phi & \nearrow \psi = F \circ \phi & \\ U \subset \mathbb{R}^2 & & \end{array}.$$

From

$$\vec{\kappa}(t) = k_n N(\gamma(t)) + k_g (N(\gamma(t)) \times \gamma'(t)),$$

we see that γ is a geodesic if and only if $\gamma''(t)$ is a multiple of $N(\gamma(t))$, if and only if

$$\gamma''(t) \cdot \phi_u = \gamma''(t) \cdot \phi_v = 0,$$

for $\phi : U \rightarrow \mathbb{R}^2$ a chart at $\gamma(t)$. Write $\gamma(t) = \phi(u(t), v(t))$, then $\gamma'(t) = \phi_u u' + \phi_v v'$, so

$$\gamma''(t) = (\phi_{uu} u' + \phi_{uv} v') u' + (\phi_{uv} u' + \phi_{vv} v') v',$$

so

$$\begin{aligned} 0 &= \gamma'' \cdot \phi_u = (\phi_u \cdot \phi_{uu}) (u')^2 + 2(\phi_u \cdot \phi_{uv}) (u'v') + (\phi_u \cdot \phi_{vv}) (v')^2 \\ &= \frac{1}{2} \frac{\partial}{\partial u} (\phi_u \cdot \phi_u) (u')^2 + \frac{\partial}{\partial v} (\phi_u \cdot \phi_u) (u'v') + \left(\frac{\partial}{\partial v} (\phi_u \cdot \phi_v) - \frac{1}{2} \frac{\partial}{\partial u} (\phi_v \cdot \phi_v) \right) (v')^2 \\ &= \frac{1}{2} g_{11u} (u')^2 + g_{11v} (u'v') + \left(g_{12v} - \frac{1}{2} g_{22u} \right) (v')^2, \end{aligned}$$

and similarly for $\gamma'' \cdot \phi_v$. These are determined by u , v , and the first fundamental form g . Since F is an isometry, it preserves g with respect to the chart $\psi = F \circ \phi$, and $F(\gamma(t)) = \psi(u(t), v(t))$ implies that

$$(F \circ \gamma)'' \cdot \psi_u = \gamma'' \cdot \phi_u = 0, \quad (F \circ \gamma)'' \cdot \psi_v = \gamma'' \cdot \phi_v = 0.$$

□

Let $\phi : U \rightarrow S$ be a chart on a regular surface, and let $\gamma(t) = \phi(u(t), v(t))$ be a curve on S . Then γ is a geodesic if and only if

$$(g_{11} u' + g_{12} v')' = \frac{1}{2} \left(g_{11u} (u')^2 + 2g_{12u} (u'v') + g_{22u} (v')^2 \right),$$

and

$$(g_{21} u' + g_{22} v')' = \frac{1}{2} \left(g_{11v} (u')^2 + 2g_{12v} (u'v') + g_{22v} (v')^2 \right).$$

(Exercise) These are the **geodesic equations**.

5.2 Length-minimising curves

Claim that geodesics minimise arc-length, locally. That is, for any small perturbation β of γ , $L(\beta) \geq L(\gamma)$.

Definition 5.2.1. A **variation** of $\gamma : [0, L] \rightarrow S$ is a smooth map

$$\begin{aligned} [0, L] \times [-\epsilon, \epsilon] &\longrightarrow S \\ (t, s) &\longmapsto \gamma_s(t) \end{aligned} ,$$

such that

$$\gamma_0(t) = \gamma(t), \quad \gamma_s(0) = \gamma(0), \quad \gamma_s(L) = \gamma(L), \quad t \in [0, L], \quad s \in [-\epsilon, \epsilon].$$

Proposition 5.2.2. *Geodesics minimise arc-length locally. That is, if $\gamma : [0, L] \rightarrow S$ is a local minimum for arc-length between $\gamma(0)$ and $\gamma(L)$ and is parametrised by arc-length, then γ is a geodesic.*

Proof. Let $\gamma_s(t)$ be a variation of γ . Then $s \mapsto L(\gamma_s)$ is minimised at $s = 0$. This is a smooth function of s . So

$$\begin{aligned} 0 &= \frac{d}{ds} (L(\gamma_s)) \Big|_{s=0} = \frac{d}{ds} \int_0^L \sqrt{\gamma'_s(t) \cdot \gamma'_s(t)} dt \Big|_{s=0} \\ &= \int_0^L \frac{d}{ds} \sqrt{\gamma'_s(t) \cdot \gamma'_s(t)} \Big|_{s=0} dt && \sqrt{\gamma'_s(t) \cdot \gamma'_s(t)} \text{ is smooth} \\ &= \int_0^L \frac{\frac{d}{ds} (\gamma'_s(t) \cdot \gamma'_s(t))}{2\sqrt{\gamma'_s(t) \cdot \gamma'_s(t)}} \Big|_{s=0} dt = \int_0^L \frac{d}{ds} (\gamma'_s(t) \cdot \gamma'_s(t)) \Big|_{s=0} dt && |\gamma'_0(t)| = |\gamma'(t)| = 1 \\ &= \int_0^L \frac{d}{ds} \left(\frac{d\gamma_s}{dt} \right) \Big|_{s=0} \cdot \frac{d\gamma_0}{dt} dt = \int_0^L \frac{d}{dt} \left(\frac{d\gamma_s}{ds} \right) \Big|_{s=0} \cdot \frac{d\gamma_0}{dt} dt && (s, t) \mapsto \gamma_s(t) \text{ is smooth} \\ &= \left[\frac{d\gamma_s}{ds} \Big|_{s=0} \cdot \frac{d\gamma_0}{dt} \right]_0^L - \int_0^L \frac{d\gamma_s}{ds} \Big|_{s=0} \cdot \frac{d^2\gamma_0}{dt^2} dt && \text{integration by parts} \\ &= - \int_0^L \frac{d\gamma_s}{ds} \Big|_{s=0} \cdot (k_n(t) N(\gamma(t)) + k_g(t) (N(\gamma(t)) \times \gamma'(t))) dt && \frac{d\gamma_s}{ds}(0) = \frac{d\gamma_s}{ds}(L) = 0 \\ &= \int_0^L \frac{d\gamma_s}{ds} \Big|_{s=0} \cdot (k_g(t) (N(\gamma(t)) \times \gamma'(t))) dt && N(\gamma(t)) \perp T_{\gamma(t)}S. \end{aligned}$$

Let $g : [0, L] \rightarrow \mathbb{R}$ be any smooth function such that $g(0) = g(L) = 0$. Can find a variation $\gamma_s(t)$ with

$$\frac{d\gamma_s}{ds} \Big|_{s=0} = g(t) (N(\gamma(t)) \times \gamma'(t)).$$

Thus we have

$$\int_0^L k_g(t) g(t) dt = 0,$$

and, since k_g is continuous as a function of t , and g is arbitrary, it follows that $k_g(t) = 0$ for all t , that is γ was a geodesic. \square

Remark.

- Geodesics need not be global minima for arc-length between two points. In S^2 with usual metric, a major arc is a geodesic but not a global minimum for arc-length.
- Geodesics between two points need not be unique. All great circles are geodesics from N to S .
- Geodesics connecting two points need not exist. In $\mathbb{R}^2 \setminus \{0\}$, 1 and -1 has no geodesic. $\mathbb{R}^2 \setminus \{0\}$ is locally isometric to \mathbb{R}^2 . Local isometries preserve geodesics, so geodesics in $\mathbb{R}^2 \setminus \{0\}$ are straight lines.

In general, it is hard to find geodesics explicitly. Need to solve the geodesic equation.

6 The Gauss-Bonnet theorem and applications

Equates geometry and topology. Will prove this from its local version.

Lecture 27
Tuesday
04/12/18

6.1 Local version of Gauss-Bonnet

Theorem 6.1.1 (Local Gauss-Bonnet). *Let $\phi : U \rightarrow S$ be a chart which is smooth in \bar{U} , the closure of U in \mathbb{R}^2 , such that $S = \phi(\bar{U})$ is an oriented surface with boundary $\partial S = \phi(\partial \bar{U})$. Suppose that \bar{U} is diffeomorphic to a disc. Then,*

$$\int_S K \, dA + \int_{\partial S} k_g \, ds = 2\pi,$$

where A is the area, s is the arc-length, and ∂S is positively oriented with respect to S .

- A **surface with boundary** is the same as a regular surface except that some points $p \in S$ have charts $\phi : U \subset \mathbb{R}^2 \rightarrow S$, where U is the intersection of an open neighbourhood of $(0,0)$ in \mathbb{R}^2 and $\{y \geq 0\}$. ϕ here is smooth, which means there exists an open set $V \subset \mathbb{R}^3$ and $\Phi : V \rightarrow \mathbb{R}^3$ smooth such that $\phi = \Phi|_U$ and $U \subseteq V$.
- ∂S is the collection of all points with charts like this. This is a collection of regular curves. ∂S is **positively oriented** if in some parametrisation $\gamma : [a, b] \rightarrow \partial S$, $N \times \gamma'$ points into S .

Proof. The idea is to recall

$$\int_S K \, dA = \iint_U (K \circ \phi) \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right| du \, dv.$$

We will find functions $M, L : U \rightarrow \mathbb{R}$ such that

$$\iint_U (K \circ \phi) \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right| du \, dv = \iint_U \left(\frac{\partial M}{\partial u} \times \frac{\partial L}{\partial v} \right) du \, dv = \int_{\partial U} L \, du + M \, dv = 2\pi - \int_{\partial U} k_g \, ds,$$

by applying Green's theorem. For a precise version, make an orthonormal basis for $T_p S$. Take $E_1 = \phi_u / |\phi_u|$ and $E_2 = N \times E_1$, so $N = E_1 \times E_2$. Have orthonormal basis E_1, E_2 for $T_p S$ along ∂S . So there exists a continuous function $\theta : [0, L] \rightarrow \mathbb{R}$ such that

$$\gamma'(t) = \cos(\theta(t)) E_1(\gamma(t)) + \sin(\theta(t)) E_2(\gamma(t)).$$

Here L is the length of the boundary, and $\gamma : [0, L] \rightarrow \partial S$ is an arc-length parametrisation of ∂S .

- Claim that

$$k_g(\gamma(t)) = \theta'(t) - E_1(t) \cdot E_2'(t).$$

$k_g = \gamma'' \cdot (N \times \gamma')$. Then

$$\gamma''(t) = (-\sin(\theta) E_1 + \cos(\theta) E_2) \theta'(t) + \cos(\theta) E_1' + \sin(\theta) E_2', \quad N \times \gamma'(t) = -\sin(\theta) E_1 + \cos(\theta) E_2,$$

so

$$k_g(\gamma(t)) = \gamma''(t) \cdot (N \times \gamma'(t)) = \theta'(t) + (\cos(\theta) E_1' + \sin(\theta) E_2') \cdot (-\sin(\theta) E_1 + \cos(\theta) E_2).$$

Now,

$$E_1(t) \cdot E_1(t) = 1, \quad E_1(t) \cdot E_2(t) = 0, \quad E_2(t) \cdot E_2(t) = 1.$$

Differentiating,

$$E_1 \cdot E_1' = 0, \quad E_1 \cdot E_2' + E_1' \cdot E_2 = 0, \quad E_2 \cdot E_2' = 0,$$

so $k_g = \theta' - E_1 \cdot E_2'$, which was the claim.

- Claim that

$$\int_0^L E_1(t) \cdot E_2'(t) \, dt = \iint_U (K \circ \phi) |\phi_u \times \phi_v| \, du \, dv.$$

This will help because it implies

$$\int_S K \, dA + \int_{\partial S} k_g \, ds = \int_0^L \theta'(t) \, dt.$$

Lecture 28
Friday
07/12/18

Let us show that

$$\frac{\partial E_1}{\partial u} \cdot \frac{\partial E_2}{\partial v} - \frac{\partial E_1}{\partial v} \cdot \frac{\partial E_2}{\partial u} = K |\phi_u \times \phi_v| \quad (4)$$

Assuming (4) then, setting $\gamma(t) = \phi(u(t), v(t))$, we have

$$\begin{aligned} \int_0^L E_1(t) \cdot E_2'(t) dt &= \int_0^L E_1(t) \cdot \left(\frac{\partial E_2}{\partial u} u' + \frac{\partial E_2}{\partial v} v' \right) dt = \int_{\partial U} E_1(t) \cdot \frac{\partial E_2}{\partial u} du + E_1(t) \cdot \frac{\partial E_2}{\partial v} dv \\ &= \iint_U \left(\frac{\partial}{\partial u} \left(E_1(t) \cdot \frac{\partial E_2}{\partial v} \right) - \frac{\partial}{\partial v} \left(E_1(t) \cdot \frac{\partial E_2}{\partial u} \right) \right) du dv \\ &= \iint_U \left(\frac{\partial E_1}{\partial u} \cdot \frac{\partial E_2}{\partial v} - \frac{\partial E_1}{\partial v} \cdot \frac{\partial E_2}{\partial u} \right) du dv = \iint_U K |\phi_u \times \phi_v| du dv, \end{aligned}$$

by Green's theorem and (4), which proves the claim. It remains to prove (4). Note

$$E_1 \cdot \frac{\partial E_1}{\partial u} = 0, \quad E_1 \cdot \frac{\partial E_2}{\partial u} + \frac{\partial E_1}{\partial u} \cdot E_2 = 0, \quad \dots$$

So

$$\frac{\partial E_1}{\partial u} = aE_2 + \left(\frac{\partial E_1}{\partial u} \cdot N \right) N = aE_2 - \left(E_1 \cdot \frac{\partial N}{\partial u} \right) N = aE_2 + A \left(E_1, \frac{\partial \phi}{\partial u} \right) N,$$

for some constant a . Similarly,

$$\frac{\partial E_1}{\partial v} = bE_2 + A \left(E_1, \frac{\partial \phi}{\partial v} \right) N, \quad \frac{\partial E_2}{\partial u} = cE_1 + A \left(E_2, \frac{\partial \phi}{\partial u} \right) N, \quad \frac{\partial E_2}{\partial v} = dE_1 + A \left(E_2, \frac{\partial \phi}{\partial v} \right) N,$$

for some constants b, c, d . Thus

$$\frac{\partial E_1}{\partial u} \cdot \frac{\partial E_2}{\partial v} - \frac{\partial E_1}{\partial v} \cdot \frac{\partial E_2}{\partial u} = A \left(E_1, \frac{\partial \phi}{\partial u} \right) A \left(E_2, \frac{\partial \phi}{\partial v} \right) - A \left(E_2, \frac{\partial \phi}{\partial u} \right) A \left(E_1, \frac{\partial \phi}{\partial v} \right).$$

Writing

$$\frac{\partial \phi}{\partial u} = c_{11}E_1 + c_{12}E_2, \quad \frac{\partial \phi}{\partial v} = c_{21}E_1 + c_{22}E_2,$$

this is

$$\begin{aligned} \frac{\partial E_1}{\partial u} \cdot \frac{\partial E_2}{\partial v} - \frac{\partial E_1}{\partial v} \cdot \frac{\partial E_2}{\partial u} &= (c_{11}A(E_1, E_1) + c_{12}A(E_1, E_2)) \cdot (c_{21}A(E_2, E_1) + c_{22}A(E_2, E_2)) \\ &\quad - (c_{11}A(E_2, E_1) + c_{12}A(E_2, E_2)) \cdot (c_{21}A(E_1, E_1) + c_{22}A(E_1, E_2)) \\ &= (c_{11}c_{22} - c_{12}c_{21}) \cdot (A(E_1, E_1)A(E_2, E_2) - A(E_1, E_2)A(E_2, E_1)) \\ &= \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right| \det(A) = \left| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right| K, \end{aligned}$$

as claimed.

So at this point we know

$$\int_S K dA + \int_{\partial S} k_g ds = \int_0^L \frac{d\theta}{dt} dt.$$

It remains to show that the right hand side is 2π .

$$\int_0^L \frac{d\theta}{dt} dt = \theta(L) - \theta(0)$$

is the total rotation of $\gamma'(t)$ along the path ∂S with respect to the frame $(E_1(t), E_2(t))$. The fact that this is 2π follows from the fact that \bar{U} is only diffeomorphic to a disc, so replace it by a disc. There is a **regular homotopy**

$$\begin{aligned} H &: [0, 1] \times [0, 1] \longrightarrow X \\ (s, t) &\longmapsto \gamma_s(t), \end{aligned}$$

smooth in s, t , from the boundary curve to an approximate tiny circle in $T_p S$. Since $Ind(\gamma) = \theta(L) - \theta(0)$ is invariant under regular homotopy, $Ind(\gamma)$ is the index of a small circle in the plane, which is 2π . \square

6.2 Gauss-Bonnet for curvilinear triangles

Definition 6.2.1. A **curvilinear triangle** in \mathbb{R}^2 is a continuous map $\beta : \mathbb{R} \rightarrow \mathbb{R}^2$ such that $\beta(t) = \beta(t+3)$ for all t , and that there exist

$$t_0, t_1, t_2 \in [0, 3), \quad 0 \leq t_0 < t_1 < t_2 < 3,$$

such that β is smooth on $(t_0, t_1), (t_1, t_2), (t_2, t_3)$ where $t_3 = t_0 + 3$, and $\beta|_{[0,3)}$ is injective, and the one-sided derivatives $\beta'_-(t_i)$ and $\beta'_+(t_i)$ exist and meet at an angle θ_i that is not a multiple of π .

Let $T \subset U \subset \mathbb{R}^2$ be a curvilinear triangle and let $\phi : U \rightarrow S$ be a chart on a regular surface S . At each vertex of $\phi(T)$, the edges incident to that vertex are parametrised by $\gamma_{in} : (-\epsilon, 0] \rightarrow S$ and $\gamma_{out} : [0, \epsilon) \rightarrow S$. The tangent vectors meet at some angle θ , for $-\pi < \theta < \pi$, so

$$\cos(\theta) = \frac{\gamma'_{in}(0) \cdot \gamma'_{out}(0)}{|\gamma'_{in}(0)| |\gamma'_{out}(0)|}.$$

Choose the sign of θ such that $\theta > 0$ if $(\gamma'_{in}(0), \gamma'_{out}(0), N)$ is a positive basis of \mathbb{R}^3 and $\theta < 0$ if $(\gamma'_{in}(0), \gamma'_{out}(0), N)$ is a negative basis of \mathbb{R}^3 . θ is the **exterior angle** at the vertex. The **interior angle** is $\pi - \theta$.

Theorem 6.2.2 (Gauss-Bonnet for curvilinear triangles). *Let $T \subset U \subset \mathbb{R}^2$ be a curvilinear triangle and let $\phi : U \rightarrow S$ be a chart on the regular surface S . Suppose that $\phi(\partial T) \subset S$ is oriented positively with respect to the orientation of S . Let $\theta_1, \theta_2, \theta_3$ be the exterior angles and let $\gamma_1, \gamma_2, \gamma_3$ be the edges of $\phi(T)$ parametrised by arc-length. Then,*

$$\sum_{i=1}^3 \int_{\gamma_i} k_g \, ds + \sum_{i=1}^3 \theta_i = 2\pi - \int_{\phi(T)} K \, dA.$$

Think of the left hand side as $\int_{\phi(\partial T)} k_g \, ds$ where $\phi(\partial T)$ is thought as having a δ -function of curvature of size θ_i at the vertex i .

Proof. Essentially the same as in local Gauss-Bonnet. We still integrate $k_g = \theta' - E_1 \cdot E'_2$, apply Green's theorem to the triangle T , and get

$$\int_{\phi(T)} K \, dA + \sum_{i=1}^3 \int_{\gamma_i} k_g \, ds = \int_{\partial S} \theta'(t) \, dt.$$

But now, by topology, we have

$$\int_{\partial S} \theta'(t) \, dt = 2\pi - \sum_{i=1}^3 \theta_i,$$

because the tangent vectors to the γ_i along $\phi(T) = \partial \bar{S}$ still rotate a total of 2π , but this includes jumps of $\theta_1, \theta_2, \theta_3$ at the three corners, so that $\int_{\partial S} \theta'(t) \, dt$ does not include these jumps. \square

The following are consequences. Consider a **geodesic triangle** on S of geodesic edges, and interior angles $\alpha_1, \alpha_2, \alpha_3$. Gauss-Bonnet says,

$$\int_{\phi(T)} K \, dA = 2\pi - \sum_{i=1}^3 \int_{\gamma_i} k_g \, ds - \sum_{i=1}^3 \theta_i = 2\pi - \sum_{i=1}^3 (\pi - \alpha_i) = \sum_{i=1}^3 \alpha_i - \pi.$$

For plane triangles, $K = 0$, so

$$\sum_{i=1}^3 \alpha_i = \pi.$$

For geodesic triangles T on a sphere of radius one, $K = 1$, so

$$\sum_{i=1}^3 \alpha_i = \pi + \text{area}(T) \geq \pi.$$

6.3 Gauss-Bonnet theorem

Proposition 6.3.1. *Every compact regular surface admits a **triangulation***

$$S = \bigcup_{i=1}^N T_i,$$

that is a partition of S into curvilinear triangles T_i such that whenever $T_i \cap T_j$ is non-empty then $T_i \cap T_j$ is an edge or vertex of both T_i and T_j , and each edge belongs to exactly two of the T_i if that edge lies in the interior of S , and in exactly one of the T_i if that edge lies in ∂S .

S is sewn together from curvilinear triangles.

Proof. By compactness, we can cover S by finitely many charts $\phi_i : U_i \rightarrow S$ such that U_i is homeomorphic to a disc. Without loss of generality we may assume

$$\phi_i(U_i) \not\subseteq \bigcup_{j \neq i} \phi_j(U_j),$$

because otherwise just remove (U_i, ϕ_i) from our cover. Then

$$V_i = \phi_i(U_i) \setminus \bigcup_{j \neq i} \phi_j(U_j)$$

is non-empty, and closed in S . Draw a curve $C_i \subset \phi_i(U_i)$ which bounds a neighbourhood of V_i . The union of the C_i divides S into regions, each of which is contained in a single chart, and that chart is homeomorphic to a disc in \mathbb{R}^2 . Now work in U_i . Now have a closed curve in $U_i \subset \mathbb{R}^2$. Triangulate the interior of this. \square

Lemma 6.3.2. *Three curves with the same vertex of interior angles α_1, α_2 imply that the sum of the interior angles is $\alpha_1 + \alpha_2$.*

Proof. Without loss of generality we can take the velocity vectors $\gamma'_{in}(0)$ and $\gamma'_{out}(0)$ as unit vectors. Let $(\cos(\alpha_1), \sin(\alpha_1))$ be a velocity vector of interior angle α_1 , $(\cos(\alpha_1 + \alpha_2), \sin(\alpha_1 + \alpha_2))$ be a velocity vector of interior angle α_2 , and ϕ be the difference of the interior angles. Then

$$\begin{aligned} \cos(\phi) &= (\cos(\alpha_1 + \alpha_2), \sin(\alpha_1 + \alpha_2)) \cdot (\cos(\alpha_1), \sin(\alpha_1)) \\ &= \cos(\alpha_1 + \alpha_2) \cos(\alpha_1) + \sin(\alpha_1 + \alpha_2) \sin(\alpha_1) = \cos((\alpha_1 + \alpha_2) - \alpha_1) = \cos(\alpha_2), \end{aligned}$$

so $\phi = \alpha_2$. \square

What is the **Euler characteristic**? Take a triangulation of S . Then

$$\chi(S) = V - E + F,$$

where V is the number of vertices in the triangulation, E is the number of edges in the triangulation, and F is the number of triangles in the triangulation. This is independent of the choice of triangulation of S . Follows from the Gauss-Bonnet theorem and can prove this topologically.

Example.

- Triangulate S^2 by an inflated tetrahedron. $V = 4, E = 6, F = 4$, so $\chi(S^2) = 4 - 6 + 4 = 2$.
- Triangulate T^2 by two triangles in a square. $V = 1, E = 3, F = 2$, so $\chi(T^2) = 1 - 3 + 2 = 0$.
- Exercise: triangulate T^2 by eight triangles in a square, and compute $\chi(T^2)$ using this triangulation.
- Exercise: triangulate Σ_g , a surface of genus g , by two tori without a disc and $g - 2$ tori without two discs, and show $\chi(\Sigma_g) = 2 - 2g$.

Lecture 30
Tuesday
11/12/18

Theorem 6.3.3 (Gauss-Bonnet). *Let $S \subset \mathbb{R}^3$ be a compact oriented surface, with, a possibly empty, positively oriented boundary. Then*

$$\int_S K \, dA + \int_{\partial S} k_g \, ds = 2\pi\chi(S),$$

where χ is the Euler characteristic. If $\partial S = \emptyset$, then

$$\int_S K \, dA = 2\pi\chi(S).$$

Proof. Triangulate S , so

$$S = \bigcup_{i=1}^N T_i,$$

where T_i is a curvilinear triangle with exterior angles θ_{ij} for $1 \leq j \leq 3$. Set the interior angles $\alpha_{ij} = \pi - \theta_{ij}$. Then

$$\int_{T_i} K \, dA + \int_{\partial T_i} k_g \, ds = 2\pi - \sum_{j=1}^3 \theta_{ij} = \sum_{j=1}^3 \alpha_{ij} - \pi,$$

by Gauss-Bonnet for T_i . The integral of k_g along the edge $T_i \cap T_j$ cancels because it occurs for T_i and T_j with opposite signs. All of the interior edges cancel out. At each interior vertex, the interior angles add up to 2π . Boundary edges do not cancel. At boundary vertices, the interior angles sum to π . Adding everything up,

$$\sum_{i=1}^N \int_{T_i} K \, dA + \sum_{i=1}^N \int_{\partial T_i} k_g \, ds = \sum_{i=1}^N \sum_{j=1}^3 \alpha_{ij} - \sum_{i=1}^N \pi,$$

so

$$\int_S K \, dA + \int_{\partial S} k_g \, ds = 2\pi V - \pi \# \{\text{boundary vertices}\} - \pi F.$$

But counting edges,

$$3F = \# \{\text{interior edges}\} + \# \{\text{boundary edges}\} = 2E - \# \{\text{boundary edges}\} = 2E - \# \{\text{boundary vertices}\}.$$

So

$$\int_S K \, dA + \int_{\partial S} k_g \, ds = 2\pi(V + F) - \pi(3F + \# \{\text{boundary vertices}\}) = 2\pi(V - E + F).$$

□

Corollary 6.3.4. *Let $S \subset \mathbb{R}^3$ be a compact oriented surface with $\partial S = \emptyset$, such that $K \geq 0$ on S . Then $S \cong S^2$.*

Proof. There exists some point with $K > 0$ because there exists an elliptic point on S , so

$$\chi(S) = \frac{1}{2\pi} \int_S K \, dA > 0.$$

But $\chi(S) = 2 - 2g$ with g the genus and so $g = 0$ because $\chi(S) > 0$.

□