

M4P63 Algebra IV

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Syllabus

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1 Modules over a ring

Lecture 1
Friday
10/01/20

Let R be an **associative ring with unity**, that is an abelian group written additively with a multiplication which is associative but not necessarily commutative, with an identity 1 and distributive laws $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$. Then

$$R^* = \{r \in R \mid \exists s \in R, rs = 1 = sr\}$$

is the unit group of R . If $R^* = R \setminus \{0\}$ then R is a **division ring**, or a **skew field**. In the case that R is commutative, R is a **field**.

Example.

- Fields \mathbb{C} , \mathbb{R} , \mathbb{Q} , and \mathbb{F}_q , the field with $q = p^a$ elements with p a prime and $a \geq 1$.
- Skew fields $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ where $i^2 = j^2 = k^2 = ijk = -1$.
- Other rings are polynomial rings $k[x]$ for k a field, more generally $k[x_1, \dots, x_p]$, and $\text{Mat}_n k$, the $n \times n$ matrices with entries from k , a field.

1.1 Modules over rings

Definition 1.1. Let R be a ring. A **left R -module** is an abelian group M , written additively, together with a function $*$: $R \times M \rightarrow M$ satisfying

$$r*(m_1 + m_2) = r*m_1 + r*m_2, \quad (r_1 + r_2)*m = r_1*m + r_2*m, \quad (r_1 r_2)*m = r_1*(r_2*m), \quad 1*m = m.$$

We write rm for $r*m$.

Example.

- R is itself a left R -module, with $*$ as ring multiplication. More generally, let I be a left ideal of R , so I is an additive subgroup, and $rI \subseteq I$ for all $r \in R$. Then I is an R -module with $*$ as ring multiplication.
- Let k be a field. Then any vector space over k is a k -module, and vice versa.
- Any abelian group is a \mathbb{Z} -module, with $*$ defined by $na = a + \dots + a$ for $n \in \mathbb{Z}^+$ and $a \in A$, and $(-n)a = -(na)$.
- Let k be a field. Let k^n be column vectors. Then k^n is a left $\text{Mat}_n k$ -module, with $*$ as the usual matrix-vector multiplication.
- Let $M \in \text{Mat}_n k$. Then we can define a left $k[x]$ -module structure on k^n by letting x act as M on k^n . So $(x^2 + 3x - 2)*v = M^2v + 3Mv - 2v$.
- Let G be a group. Any representation of G over the field k is a left module for $k[G]$, the **group algebra**, a vector space over k with elements of G as a basis, with multiplication derived from that of G .

Definition 1.2. A **right R -module** is defined similarly, with the R -multiplication on the right, so M an abelian group under $+$, and a map $M \times R \rightarrow M$ satisfying

$$(m_1 + m_2)*r = m_1*r + m_2*r, \quad m*(r_1 + r_2) = m*r_1 + m*r_2, \quad m*(r_1 r_2) = (m*r_1)*r_2, \quad m*1 = m.$$

Left and right modules are not quite the same. If we amend this definition by putting the ring multiplication on the left, the third axiom becomes $(r_1 r_2)m = r_2(r_1 m)$. But in a left module, we have $(r_1 r_2)m = r_1(r_2 m)$.

Definition 1.3. Let R be a ring. The **opposite ring** R^{op} is R with a redefined multiplication $r*s_{R^{\text{op}}}s = s*Rr$.

It is easy to see that a left R -module is the same as a right R^{op} -module and vice versa. If R is commutative then $R = R^{\text{op}}$.

Exercise. Show that $\text{Mat}_n k \cong \text{Mat}_n k^{\text{op}}$.

Except where otherwise stated, R -modules are assumed to be left R -modules.

1.2 Homomorphisms and submodules

Definition 1.4. Let M_1 and M_2 be R -modules. A map $f : M_1 \rightarrow M_2$ is an R -module **homomorphism** if

- f is a group homomorphism, with respect to the $+$ operation, and
- $f(rm) = rf(m)$, for $r \in R$ and $m \in M$.

If f is bijective, then it is an R -module **isomorphism**.

Definition 1.5. An additive subgroup $L \leq M$ is a **submodule** if $rL \leq L$ for $r \in R$. In this case we automatically get an R -module structure on the quotient M/L with multiplication given by $r(m + L) = rm + L$.

Theorem 1.6 (First isomorphism theorem). *Let $f : M_1 \rightarrow M_2$ be an R -module homomorphism. Then*

$$\text{Im } f \leq M_2, \quad \text{Ker } f \leq M_1, \quad \text{Im } f \cong M / \text{Ker } f.$$

The other isomorphism theorems have R -module versions too.

1.3 Direct products and direct sums

Let S be a set. We have a collection of R -modules $(M_s)_S$ indexed by S .

Definition 1.7. The **direct product** is

$$\prod_{s \in S} M_s = \{(m_s)_S \mid m_s \in M_s\},$$

with coordinate-wise addition and R -multiplication, so

$$(m_s)_S + (n_s)_S = (m_s + n_s)_S, \quad r(m_s)_S = (rm_s)_S.$$

If $M_s = M$ for all $s \in S$, then we write M^S for $\prod_{s \in S} M_s$.

Definition 1.8. The **direct sum** is

$$\bigoplus_{s \in S} M_s = \{(m_s)_S \mid \text{all but finitely many coordinates } m_s \text{ are zero}\} \leq \prod_{s \in S} M_s.$$

If S is finite then the direct product and the direct sum are equal.

Example. Let $M = \mathbb{Z}_2$, as a \mathbb{Z} -module, and let $S = \mathbb{N}$. Then $\bigoplus_{s \in \mathbb{N}} \mathbb{Z}_2$ is a countable \mathbb{Z} -module but $\prod_{s \in \mathbb{N}} \mathbb{Z}_2 = \mathbb{Z}_2^{\mathbb{N}}$ is uncountable.

When $|S| = 2$, generally we write $M_1 \oplus M_2$ for the direct sum or product. There are natural injective maps

$$\begin{aligned} \iota_A : A &\longrightarrow A \oplus B & \iota_B : B &\longrightarrow A \oplus B \\ a &\longmapsto (a, 0) & b &\longmapsto (0, b) \end{aligned}$$

and surjective maps

$$\begin{aligned} \pi_A : A \oplus B &\longrightarrow A & \pi_B : A \oplus B &\longrightarrow B \\ (a, b) &\longmapsto a & (a, b) &\longmapsto b \end{aligned}$$

1.4 Exact sequences

Definition 1.9. Suppose we have a sequence of R -modules

$$\dots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} \dots,$$

with maps $f_n : M_n \rightarrow M_{n+1}$. Say the sequence is **exact at M_n** if

$$\text{Im } f_{n-1} = \text{Ker } f_n.$$

The sequence is **exact** if it is exact everywhere. A **short exact sequence** is an exact sequence

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0.$$

Note. α is injective and β is surjective.

Lecture 2
Monday
13/01/20

The first isomorphism theorem implies that $B/\text{Im } \alpha \cong C$, where $\text{Im } \alpha \cong A$. An easy case is

$$B \cong A \oplus C,$$

with $\text{Im } \alpha = \text{Im } \iota_A = A \oplus 0$ and $\text{Im } \beta = \text{Im } \pi_B = C$. We say that the short exact sequence **splits** in this case.

Example. A non-split short exact sequence of \mathbb{Z} -modules, or abelian groups, is

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$

Proposition 1.10. *A short exact sequence*

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

is split if and only if there exists an R -module homomorphism $\sigma : C \rightarrow B$ such that $\beta \circ \sigma = \text{id}_C$.

Such a σ is called a **section** of β .

Proof.

\Rightarrow Suppose that the short exact sequence is split. So assume $B = A \oplus C$, with $\alpha = \iota_A$ and $\beta = \pi_C$. Now ι_C is a section for β .

\Leftarrow For the converse, suppose that σ is a section for β . We want $f : A \oplus C \xrightarrow{\sim} B$ such that $f \circ \iota_A = \alpha$ and $\beta \circ f = \pi_C$, so

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \begin{array}{c} \nearrow \iota_A \\ \searrow \alpha \end{array} & A \oplus C & \begin{array}{c} \nwarrow \pi_C \\ \searrow \beta \end{array} & C \longrightarrow 0 \\ & & & & \downarrow f & & \\ & & & & B & & \end{array}$$

Define

$$\begin{aligned} f : A \times C &\longrightarrow B \\ (a, c) &\longmapsto \alpha(a) + \sigma(c) \end{aligned}$$

Need to check the following.

- f is an R -module homomorphism. ¹
- f is injective. Suppose $f(a, c) = 0$. Then $\alpha(a) + \sigma(c) = 0$. Now $\alpha(a) \in \text{Im } \alpha = \text{Ker } \beta$, so $\beta(\alpha(a) + \sigma(c)) = \beta(\sigma(c)) = c$. Since $\alpha(a) + \sigma(c) = 0$, we have $c = 0$. Hence $\alpha(a) = 0$, and so $a = 0$ since α is injective. We have shown that f is injective.
- f is surjective. Let $b \in B$. Let $c = \beta(b)$. We have $(\beta \circ \sigma)(c) = c = \beta(b)$, so $b - \sigma(c) \in \text{Ker } \beta = \text{Im } \alpha$. So there exists $a \in A$ with $\alpha(a) = b - \sigma(c)$. Then $b = \alpha(a) + \sigma(c) = f(a, c)$.
- $f \circ \iota_A = \alpha$ and $\beta \circ f = \pi_C$. Immediate from the construction of f .

□

Proposition 1.11. *The short exact sequence*

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

is split if and only if there exists $\rho : B \rightarrow A$ such that $\rho \circ \alpha = \text{id}_A$.

Such a ρ is a **retraction** of α .

Proof.

\Rightarrow Once again, if the short exact sequence is split then the existence of ρ is clear.

\Leftarrow Suppose that ρ is a retraction for α . We define $f : B \xrightarrow{\sim} A \oplus C$ such that $f \circ \alpha = \iota_A$ and $\pi_C \circ f = \beta$. Do this by

$$\begin{aligned} g : B &\longrightarrow A \oplus C \\ b &\longmapsto (\rho(b), \beta(b)) \end{aligned}$$

□

¹Exercise

2 Projective and injective modules

2.1 Projective modules

Definition 2.1. An R -module M is **projective** if any surjective map $\beta : B \rightarrow M$ has a section. In other words, any short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow M \rightarrow 0$$

splits.

Example. The R -module R is projective. Let

$$0 \rightarrow A \rightarrow B \xrightarrow{\beta} R \rightarrow 0$$

be a short exact sequence. Since β is surjective, there exists $b \in B$ such that $\beta(b) = 1$. Now for all $r \in R$, $\beta(rb) = r$. Now define

$$\sigma : R \longrightarrow B \\ r \longmapsto rb.$$

Then σ is a section for β .

Proposition 2.2. An R -module M is projective if and only if whenever $\beta : B \rightarrow C$ is surjective, and $f : M \rightarrow C$, there exists $g : M \rightarrow B$ such that $f = \beta \circ g$, so

$$\begin{array}{ccccccc} & & & & M & & \\ & & & & \downarrow f & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow[\beta]{} & C \longrightarrow 0 \\ & & & & \nwarrow g & & \end{array}.$$

Such a g is called a **lift** of f .

Proof.

\Leftarrow Suppose that whenever $\beta : B \rightarrow C$ is surjective and $f : M \rightarrow C$ then there exists $g : M \rightarrow B$ with $f = \beta \circ g$. Suppose $\beta : B \rightarrow M$ is a surjective map. Define $f : M \rightarrow M$ to be id_M . Then there exists $g : M \rightarrow B$ such that $f = \beta \circ g$, so $\text{id}_M = \beta \circ g$. So g is a section for β , and so M is projective.

\Rightarrow For the converse, suppose $\beta : B \rightarrow C$ is surjective, and $f : M \rightarrow C$. We construct a module X to complete a commuting square

$$\begin{array}{ccc} X & \xrightarrow{\epsilon} & M \\ \delta \downarrow & & \downarrow f \\ B & \xrightarrow[\beta]{} & C \end{array}.$$

Let X be the submodule of $B \oplus M$ defined by

$$X = \{(b, m) \mid \beta(b) = f(m)\}.$$

The maps δ and ϵ are just π_B and π_M respectively, in their restrictions to X . It is clear that $X \leq B \oplus M$, and that the square above commutes. Now suppose that M is projective. Since β is surjective, we see that for all $m \in M$ there exists $b \in B$ with $\beta(b) = f(m)$. It follows that $\epsilon : X \rightarrow M$ is surjective. So ϵ has a section $\sigma : M \rightarrow X$. Define $g = \delta \circ \sigma : M \rightarrow B$, so

$$\begin{array}{ccc} X & \xrightarrow{\epsilon} & M \\ \delta \downarrow & \nearrow \sigma & \downarrow f \\ B & \xrightarrow[\beta]{} & C \end{array}.$$

Since $\beta \circ \delta = f \circ \epsilon$, we have

$$(\beta \circ g)(m) = (\beta \circ \delta \circ \sigma)(m) = (f \circ \epsilon \circ \sigma)(m) = (f \circ \text{id}_M)(m) = f(m), \quad m \in M.$$

So $\beta \circ g = f$ as required.

□

Such an X is the **pullback** of β and f , and there is a short exact sequence

$$0 \rightarrow A \rightarrow X \rightarrow M \rightarrow 0.$$

2.2 Free modules

Definition 2.3. An R -module M is **free** if M is a direct sum of copies of R , so

$$M = \bigoplus_{s \in S} R.$$

A **basis** for a module M is a set T of elements such that every element $m \in M$ has a unique expression as

$$m = \sum_{i=1}^m r_i t_i, \quad r_i \in R, \quad t_i \in T.$$

If $M = \bigoplus_{s \in S} R$, then M has a basis consisting of elements with exactly one coordinate one, and the rest zero. On the other hand, if M has a basis T then it is straightforward to show that $M \cong \bigoplus_{t \in T} R$.

Proposition 2.4. Let F be a free R -module with basis T . Let M be some R -module, and let $\psi : T \rightarrow M$ be a set map. Then ψ extends uniquely to an R -module homomorphism $\psi : F \rightarrow M$.

Proof. Each element of F has a unique expression as $\sum_i r_i t_i$ for $r_i \in R$ and $t_i \in T$. Now define

$$\begin{array}{ccc} \psi & : & F \longrightarrow M \\ & & \sum_i r_i t_i \longmapsto \sum_i r_i \psi(t_i) \end{array}$$

It is easy to check that this respects $+$ and R -multiplication. □

Proposition 2.5. A module M is projective if and only if there exists N such that $M \oplus N$ is free, so projective modules are direct summands of free modules.

Proof.

\Rightarrow Suppose M is projective. Let F be the free module with basis $\{b_m \mid m \in M\}$. Now the map $b_m \mapsto m$ extends to an R -module homomorphism $F \rightarrow M$, which is clearly surjective. Then if $K = \text{Ker } \psi$, we have a short exact sequence

$$0 \rightarrow K \rightarrow F \xrightarrow{\psi} M \rightarrow 0.$$

Since M is projective, there is a section σ for ψ , and so the short exact sequence splits, and $F \cong K \oplus M$. Lecture 4

\Leftarrow Suppose that $M \oplus N = F$, a free module with basis T . Suppose $\beta : B \rightarrow C$ is surjective, and that $f : M \rightarrow C$. Note that $f \circ \pi_M : F \rightarrow C$. For each $t \in T$, let $b_t \in B$ be such that $\beta(b_t) = (f \circ \pi_M)(t)$. The set map Friday
17/01/20

$$\begin{array}{ccc} T & \longrightarrow & B \\ t & \longmapsto & b_t \end{array}$$

extends to a homomorphism $\hat{g} : F \rightarrow B$. Now define $g : M \rightarrow B$ by $g = \hat{g} \circ \iota_M$. We need to show $f = \beta \circ g$. Take $m \in M$. Then $\iota_M(m) = (m, 0) \in F$ can be written as $\sum_i r_i t_i$, where $t_i \in T$ and $r_i \in R$. Applying π_M , $m = \sum_i r_i m_{t_i}$. Then

$$g(m) = (\hat{g} \circ \iota_M)(m) = \hat{g}\left(\sum_i r_i t_i\right) = \sum_i r_i b_{t_i}.$$

So

$$(\beta \circ g)(m) = \beta\left(\sum_i r_i b_{t_i}\right) = \sum_i r_i \beta(b_{t_i}) = \sum_i r_i f(m_{t_i}) = f\left(\sum_i r_i m_{t_i}\right) = f(m).$$

Hence $\beta \circ g = f$. So M is projective. □

2.3 Injective modules

Definition 2.6. Let M be an R -module. Then M is **injective** if whenever $\alpha : M \rightarrow B$ is an injective map, it has a retraction $\rho : B \rightarrow M$, so $\rho \circ \alpha = \text{id}_M$. Equivalently, every short exact sequence

$$0 \rightarrow M \rightarrow B \rightarrow C \rightarrow 0$$

splits.

Example. Let k be a field. Then k -modules are vector spaces. Every k -module is injective. Suppose M and N are k -vector spaces and $\alpha : M \rightarrow N$ is a injective map. Then $\text{Im } \alpha$ is a submodule, or subspace, of N . Take a basis for $\text{Im } \alpha$, and extend to a basis for N . The basis vectors not in $\text{Im } \alpha$ form a basis for a complementary subspace U , so $N = \text{Im } \alpha \oplus U$. Now $\pi_{\text{Im } \alpha}$ is surjective, and $\alpha : M \rightarrow \text{Im } \alpha$ is an isomorphism. This gives a retraction $N \rightarrow M$.

If R is a general ring, the module R need not be injective.

Example. Let $R = \mathbb{Z}$. Then R -modules are abelian groups. There exists an injective $\alpha : \mathbb{Z} \rightarrow \mathbb{Q}$. But \mathbb{Z} is not a quotient of \mathbb{Q} ,² so no retraction exists for α .

Proposition 2.7. An R -module M is injective if and only if whenever $\alpha : A \rightarrow B$ is injective, and $f : A \rightarrow M$, there exists $g : B \rightarrow M$ such that $f = g \circ \alpha$.

Proof.

\Leftarrow Suppose that whenever $\alpha : A \rightarrow B$ is injective, and $f : A \rightarrow M$, there exists $g : B \rightarrow M$ such that $f = g \circ \alpha$. Suppose that $\alpha : M \rightarrow B$ is injective. We have a map $M \rightarrow M$, namely id_M . There exists $g : B \rightarrow M$ such that $\text{id}_M = g \circ \alpha$. So g is a retraction for α , and so M is injective.

\Rightarrow For the converse, suppose $\alpha : A \rightarrow B$ is injective, and M is an injective module, with $f : A \rightarrow M$. We define a module Y completing a square

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ f \downarrow & & \downarrow \delta \\ M & \xrightarrow{\epsilon} & Y \end{array}$$

with $\epsilon \circ f = \delta \circ \alpha$. Let Y be a quotient of $B \oplus M$, by the kernel

$$K = \{(\alpha(a), -f(a)) \mid a \in A\}.$$

Let $\gamma : B \oplus M \rightarrow (B \oplus M)/K$ be the canonical quotient map. Then we define $\delta = \gamma \circ \iota_B$ and $\epsilon = \gamma \circ \iota_M$. By construction, we have

$$\begin{aligned} (\epsilon \circ f)(a) &= (\gamma \circ \iota_M \circ f)(a) = \gamma(0, f(a)) = (0, f(a)) + K \\ &= (\alpha(a), 0) + K = \gamma(\alpha(a), 0) = (\gamma \circ \iota_B \circ \alpha)(a) = (\delta \circ \alpha)(a). \end{aligned}$$

Hence $\epsilon \circ f = \delta \circ \alpha$. Claim that ϵ is injective. Suppose $\epsilon(m) = 0$. Then $\iota_M(m) \in K$, so $(0, m) = (\alpha(a), -f(a))$ for some $a \in A$. But $\alpha(a) = 0$ implies that $a = 0$, and so $m = -f(0) = 0$. Since M is injective, ϵ has a retraction $\rho : Y \rightarrow M$. Define $g : B \rightarrow M$ by $g = \rho \circ \delta$, so

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ f \downarrow & \begin{array}{c} \nearrow g \\ \nwarrow \rho \end{array} & \downarrow \delta \\ M & \xrightarrow[\epsilon]{\rho} & Y \end{array}$$

We know that $(\epsilon \circ f)(a) = (\delta \circ \alpha)(a)$ for all $a \in A$. So

$$f(a) = (\text{id}_M \circ f)(a) = (\rho \circ \epsilon \circ f)(a) = (\rho \circ \delta \circ \alpha)(a) = (g \circ \alpha)(a),$$

so $f = g \circ \alpha$ as required. □

²Exercise

We know that projectives are direct summands of free modules. We might hope for a dual version of this for injective modules. But there is no straightforward way of doing this.

Proposition 2.8 (Baer's criterion for injectivity). *Let M be an R -module. Then M is injective if and only if every R -module map $f : I \rightarrow M$, where I is a left ideal of R , has the form $f(x) = xm$ for some $m \in M$. Equivalently, every map $I \rightarrow M$ extends to a map $R \rightarrow M$.*

Why are these two conditions equivalent? If $f(x) = xm$ for $x \in I$, then we can extend f to R by $f(r) = rm$. Conversely, suppose that $f : I \rightarrow M$ extends to $f^+ : R \rightarrow M$. Let $m = f^+(1)$. Then for all $r \in R$, $f^+(r) = rm$, and so $f(x) = xm$ for $x \in I$. The proof requires Zorn's lemma.

Lemma 2.9 (Zorn's lemma). *Let X be a non-empty set, partially ordered by \leq . If every chain, or totally ordered subset, in X has an upper bound in X , then X has a maximal element.*

Proof.

\Leftarrow Suppose $\alpha : A \rightarrow B$, where α is injective. Suppose $f : A \rightarrow M$. We want to show there exists $g : B \rightarrow M$ such that $f = g \circ \alpha$. We have $\text{Im } \alpha \leq B$. Define

$$X = \{(L, h) \mid \text{Im } \alpha \leq L \leq B, h : L \rightarrow M, f = h \circ \alpha\}.$$

Note that $X \neq \emptyset$ since $(\text{Im } \alpha, f \circ \alpha^{-1})$ is in it. Define \leq on X by $(L_1, h_1) \leq (L_2, h_2)$ if $L_1 \leq L_2$ and h_2 extends h_1 , so $h_2|_{L_1} = h_1$. Suppose $\{(L_s, h_s) \mid s \in S\}$ is a chain in X . Set $L = \bigcup_{s \in S} L_s$. Then $\text{Im } \alpha \leq L \leq B$. Define

$$\begin{aligned} h &: L \longrightarrow M \\ l &\longmapsto h_s(l), \quad l \in L_s. \end{aligned}$$

This does not depend on the choice of s . Then (L, h) is an upper bound for the chain $\{(L_s, h_s) \mid s \in S\}$. Hence X has a maximal element, (L_0, h_0) . We want to show that $L_0 = B$. Then we may set $g = h_0$. Suppose that $L_0 \neq B$. Let $b \in B \setminus L_0$. Note that $Rb \leq B$. Consider

$$L_0 + Rb = \{l + rb \mid l \in L_0, r \in R\} \leq B.$$

We would like to extend h_0 to h_0^+ by specifying an image for $h_0^+(b)$. The problem is that $Rb \cap L_0$ may not be $\{0\}$, and if $rb \in L_0$ then we require $rh_0^+(b) = h_0(rb)$, otherwise h_0^+ will not be well-defined. Note that $I = \{r \in R \mid rb \in L_0\}$ is a left ideal for R . Suppose that M has the condition from Baer's criterion, so every map $I \rightarrow M$ has the form $x \mapsto xm$ for some $m \in M$. Note that $\{xb \mid x \in I\}$ is a submodule of L_0 . Define

$$\begin{aligned} \delta &: I \longrightarrow M \\ x &\longmapsto h_0(xb). \end{aligned}$$

This is an R -module homomorphism. So $\delta(x) = xm$ for some $m \in M$. Hence $h_0(xb) = xm$ for all $x \in I$. So we can safely define $h_0^+(b) = m$. Now $(L_0 + Rb, h_0^+) \in X$, and $(L_0, h_0) < (L_0 + Rb, h_0^+)$, which contradicts the maximality of (L_0, h_0) . Hence $L_0 = B$, and we are done.

\Rightarrow The converse is left as an exercise. ³

□

Example.

- Suppose R is a field. Then the only ideals of R are zero and R . Any map $0 \rightarrow M$, for M an R -module, can be extended to the zero map $R \rightarrow M$. Hence any R -module is injective.
- Let \mathbb{Z} be a module for itself. The ideals of \mathbb{Z} are $k\mathbb{Z}$ for $k \in \mathbb{Z}$. Define

$$\begin{aligned} f &: k\mathbb{Z} \longrightarrow \mathbb{Z} \\ km &\longmapsto m. \end{aligned}$$

If $k \neq 0, \pm 1$, then $f(k) = 1$, and so $f(x) \neq xm$ for $m \in \mathbb{Z}$, since one is not divisible by k in \mathbb{Z} . So Baer's criterion fails, and \mathbb{Z} is not injective. We already knew that $\mathbb{Z} \rightarrow \mathbb{Q}$ has no retraction.

- \mathbb{Q} is injective as a \mathbb{Z} -module. Suppose we have a map $f : k\mathbb{Z} \rightarrow \mathbb{Q}$. Let $q = f(k)$. Then $f(kt) = qt = (q/k)kt$. So $f(x) = x(q/k)$ for all x , so \mathbb{Q} satisfies Baer's criterion.

³Exercise

3 Hom and tensor products

3.1 Hom

Let A and B be two R -modules.

Definition 3.1. Define

$$\text{Hom}_R(A, B) = \{R\text{-module homomorphisms } A \rightarrow B\}.$$

We can define a natural addition on $\text{Hom}_R(A, B)$ by defining $f_1 + f_2$ by

$$(f_1 + f_2)(a) = f_1(a) + f_2(a), \quad f_1, f_2 \in \text{Hom}_R(A, B).$$

This gives $\text{Hom}_R(A, B)$ the structure of an abelian group. Why does $\text{Hom}_R(A, B)$ not carry an R -module structure in general? The only obvious candidate for rf is

$$(rf)(a) = rf(a) = f(ra), \quad r \in R, \quad f \in \text{Hom}_R(A, B).$$

Now suppose $s \in R$. We have $(rf)(sa) = rf(sa) = rsf(a)$. But for rf to be a homomorphism, we would need $(rf)(sa) = s(rf)(a) = sfr(a)$. If R is non-commutative, then rs may not be sr , and so rf is not an R -module homomorphism in general. Clearly, however, if R is commutative then rf is an R -module homomorphism, and $\text{Hom}_R(A, B)$ has an R -module structure. The following are observations.

Proposition 3.2. Suppose $A, A_1, A_2, B, B_1, B_2, M$ are R -modules, and $\alpha : A \rightarrow B$.

- $\text{Hom}_R(A_1 \oplus A_2, B) \cong \text{Hom}_R(A_1, B) \oplus \text{Hom}_R(A_2, B)$.
- $\text{Hom}_R(A, B_1 \oplus B_2) \cong \text{Hom}_R(A, B_1) \oplus \text{Hom}_R(A, B_2)$.
- Then we can define

$$\begin{array}{ccc} \alpha_* : \text{Hom}_R(M, A) & \longrightarrow & \text{Hom}_R(M, B) \\ f & \longmapsto & \alpha \circ f \end{array}, \quad f : M \rightarrow A.$$

- We can also define

$$\begin{array}{ccc} \alpha^* : \text{Hom}_R(B, M) & \longrightarrow & \text{Hom}_R(A, M) \\ g & \longmapsto & g \circ \alpha \end{array}, \quad g : B \rightarrow M.$$

Thus Hom is a bifunctor between the category of R -modules and the category of abelian groups, additive in both arguments, covariant in the second argument and contravariant in the first argument.

- Bi means Hom takes two arguments.
- Functor means that homomorphisms between R -modules turn into abelian group homomorphisms.
- Covariant means the homomorphism goes in the same direction.
- Contravariant means the direction gets reversed.
- Additive in both arguments means Hom respects direct sums.

Proposition 3.3. Suppose $\alpha : A \rightarrow B$ is surjective. Then $\alpha^* : \text{Hom}_R(B, M) \rightarrow \text{Hom}_R(A, M)$ is injective.

Proof. Suppose $f_1, f_2 : B \rightarrow M$ are such that $\alpha^*(f_1) = \alpha^*(f_2)$. Then $f_1 \circ \alpha = f_2 \circ \alpha$, so $(f_1 \circ \alpha)(a) = (f_2 \circ \alpha)(a)$ for all $a \in A$. Let $b \in B$. Then $b = \alpha(a)$ for some a , since α is surjective, so $f_1(b) = (f_1 \circ \alpha)(a) = (f_2 \circ \alpha)(a) = f_2(b)$, so $f_1 = f_2$. \square

Proposition 3.4. Suppose $\alpha : A \rightarrow B$ is injective. Then $\alpha_* : \text{Hom}_R(M, A) \rightarrow \text{Hom}_R(M, B)$ is injective.

Proof. Suppose $f_1, f_2 : M \rightarrow A$, and $\alpha_*(f_1) = \alpha_*(f_2)$. Then $\alpha \circ f_1 = \alpha \circ f_2$, so $(\alpha \circ f_1)(m) = (\alpha \circ f_2)(m)$ for all $m \in M$. But α is injective, so this implies $f_1(m) = f_2(m)$ for all $m \in M$. \square

Proposition 3.5. *Suppose*

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

is a short exact sequence of R -modules. Then we have an exact sequence

$$0 \rightarrow \operatorname{Hom}_R(C, M) \xrightarrow{\beta^*} \operatorname{Hom}_R(B, M) \xrightarrow{\alpha^*} \operatorname{Hom}_R(A, M).$$

Proof. This is exact at $\operatorname{Hom}_R(C, M)$, since β^* is injective. Claim that the sequence is also exact at $\operatorname{Hom}_R(B, M)$, so it is an exact sequence. It is not necessarily a short exact sequence since α^* is not generally surjective. Let $g : B \rightarrow M$. We have

$$g \in \operatorname{Ker} \alpha^* \iff \alpha^*(g) = 0 \iff g \circ \alpha = 0 \iff g(\alpha(A)) = 0 \iff \operatorname{Im} \alpha \leq \operatorname{Ker} g \iff \operatorname{Ker} \beta \leq \operatorname{Ker} g,$$

Then $g \in \operatorname{Ker} \alpha^*$ if and only if for all $b_1, b_2 \in B$, $\beta(b_1) = \beta(b_2)$ implies that $g(b_1) = g(b_2)$, which is if and only if the map defined by

$$\begin{array}{ccc} f : C & \longrightarrow & M \\ c & \longmapsto & g(b) \end{array}, \quad \beta(b) = c$$

is well-defined, since β is surjective, and f is an R -module homomorphism. Thus

$$g \in \operatorname{Ker} \alpha^* \iff \exists f \in \operatorname{Hom}_R(C, M), \beta^*(f) = g \iff g \in \operatorname{Im} \beta^*.$$

Hence $\operatorname{Ker} \alpha^* = \operatorname{Im} \beta^*$. So the sequence is exact at $\operatorname{Hom}_R(B, M)$. □

Example. These examples show that $\alpha : A \rightarrow B$ is injective does not imply $\alpha^* : \operatorname{Hom}_R(B, M) \rightarrow \operatorname{Hom}_R(A, M)$ is surjective.

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- The inclusion $\alpha : \mathbb{Z} \rightarrow \mathbb{Q}$ is a \mathbb{Z} -module homomorphism. Let $M = \mathbb{Z}$. Then we get $\alpha^* : \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \rightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$. Then α is injective, but α^* is not surjective. Why is this? In fact $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$. Suppose

$$\begin{array}{ccc} f : \mathbb{Q} & \longrightarrow & \mathbb{Z} \\ 1 & \longmapsto & k \neq 0 \end{array}.$$

Suppose $p \nmid k$. Then there is no possible image for $1/p \in \mathbb{Q}$, since we would require $pf(1/p) = f(1) = k$. But $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$, so α^* is not surjective.

- Let $\alpha : k\mathbb{Z} \rightarrow \mathbb{Z}$ be the inclusion, so α is injective and not surjective. Let $M = \mathbb{Z}$. So we get $\alpha^* : \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \rightarrow \operatorname{Hom}_{\mathbb{Z}}(k\mathbb{Z}, \mathbb{Z})$. Suppose that $g \in \operatorname{Im} \alpha^*$. Then $g = f \circ \alpha$, where $f : \mathbb{Z} \rightarrow \mathbb{Z}$. Then $g(k) = f(k) = kf(1)$, so $\operatorname{Im} g \leq k\mathbb{Z}$. But there exists $g \in \operatorname{Hom}_{\mathbb{Z}}(k\mathbb{Z}, \mathbb{Z})$ such that $g(k) = 1$. So this $g \notin \operatorname{Im} \alpha^*$, so α^* is not surjective.

Proposition 3.6. *Let*

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

be exact. Then

$$0 \rightarrow \operatorname{Hom}_R(M, A) \xrightarrow{\alpha_*} \operatorname{Hom}_R(M, B) \xrightarrow{\beta_*} \operatorname{Hom}_R(M, C)$$

is exact.

Proof. We already know that α injective implies that α_* is injective, so the sequence is exact at $\operatorname{Hom}_R(M, A)$. We show that $\operatorname{Ker} \beta_* = \operatorname{Im} \alpha_*$. Suppose $g \in \operatorname{Hom}_R(M, B)$. Then

$$g \in \operatorname{Ker} \beta_* \iff (\beta \circ g)(M) = 0 \iff \operatorname{Im} g \leq \operatorname{Ker} \beta \iff \operatorname{Im} g \leq \operatorname{Im} \alpha.$$

Note there exists $\alpha^{-1} : \operatorname{Im} \alpha \rightarrow A$. If $\operatorname{Im} g \leq \operatorname{Im} \alpha$, then $\alpha^{-1} \circ g : M \rightarrow A$. If $f = \alpha^{-1} \circ g$, then $\alpha \circ f = g$, so $g \in \operatorname{Im} \alpha_*$. Conversely, if $g \in \operatorname{Im} \alpha_*$, then $g = \alpha \circ f$ for some $f \in \operatorname{Hom}_R(M, A)$ and so $\operatorname{Im} g \leq \operatorname{Im} \alpha$. So

$$g \in \operatorname{Ker} \beta_* \iff \operatorname{Im} g \leq \operatorname{Im} \alpha \iff g \in \operatorname{Im} \alpha_*.$$

Hence $\operatorname{Ker} \beta_* = \operatorname{Im} \alpha_*$. So the sequence is exact at $\operatorname{Hom}_R(M, B)$. □

Example. These examples show that $\beta : B \rightarrow C$ is surjective does not imply $\beta_* : \text{Hom}_R(M, B) \rightarrow \text{Hom}_R(M, C)$ is surjective.

- Let

$$\begin{array}{ccc} \beta & : & \sum_{q \in \mathbb{Q}} \mathbb{Z} \longrightarrow \mathbb{Q} \\ & & e_q \longmapsto q \end{array}.$$

In general $\beta : \sum_{m \in M} R \rightarrow M$ defined by mapping the basis vector e_m to m , is a surjective homomorphism, so β is surjective. Let $M = \mathbb{Q}$. So we get $\beta_* : \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \sum_{q \in \mathbb{Q}} \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$. Claim that $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \sum_{q \in \mathbb{Q}} \mathbb{Z})$ is trivial. Suppose $f : \mathbb{Q} \rightarrow \sum_{q \in \mathbb{Q}} \mathbb{Z}$ is not zero. Suppose $f(q_0) \neq 0$. Then there exist $q_1, \dots, q_t \in \mathbb{Q}$ and $a_1, \dots, a_t \in \mathbb{Z}$ such that $f(q_0) = \sum_{i=1}^t a_i e_{q_i}$. Now the projection of $\sum_{q \in \mathbb{Q}} \mathbb{Z}$ onto $\mathbb{Z}e_{q_1}$ is a non-trivial \mathbb{Z} -module homomorphism. But $\mathbb{Z}e_{q_1} \cong \mathbb{Z}$, and so no non-trivial map $\mathbb{Q} \rightarrow \mathbb{Z}e_{q_1}$ exists. But $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$ is not trivial, so β_* is not surjective.

- Let

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0$$

be a short exact sequence of \mathbb{Z} -modules. Then we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) & \xrightarrow{\alpha_*} & \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_4) & \xrightarrow{\beta_*} & \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2) \\ & & \downarrow \text{IR} & & \downarrow \text{IR} & & \downarrow \text{IR} \\ & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 \end{array}.$$

But there is no short exact sequence of abelian groups

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow 0,$$

and so β_* cannot be surjective.

Proposition 3.7. *Let M be an R -module. Then M is injective if and only if for every injective map $\alpha : A \rightarrow B$, we get $\alpha^* : \text{Hom}_R(B, M) \rightarrow \text{Hom}_R(A, M)$ is surjective.*

Proof. M is injective if and only if for all injective $\alpha : A \rightarrow B$, for all $f \in \text{Hom}_R(A, M)$, there exists $g \in \text{Hom}_R(B, M)$ such that $f = g \circ \alpha$, so $f = \alpha^*(g)$. This is if and only if for all injective $\alpha : A \rightarrow B$, $f \in \text{Im } \alpha^*$ for all $f \in \text{Hom}_R(A, M)$, which is if and only if α^* is surjective. \square

Proposition 3.8. *Let M be an R -module. Then M is projective if and only if whenever $\beta : B \rightarrow C$ is surjective, the map $\beta_* : \text{Hom}_R(M, B) \rightarrow \text{Hom}_R(M, C)$ is surjective.*

Proof. M is projective if and only if whenever $\beta : B \rightarrow C$ is surjective, and $f \in \text{Hom}_R(M, C)$, there exists $g \in \text{Hom}_R(M, B)$ such that $f = \beta \circ g$. This is if and only if whenever $\beta : B \rightarrow C$ is surjective, and $f \in \text{Hom}_R(M, C)$, then $f \in \text{Im } \beta_*$, which is if and only if β_* is surjective. \square

3.2 The snake lemma

Let $\alpha : A \rightarrow B$ be an R -module homomorphism. The **cokernel** of α is $B/\text{Im } \alpha$, written $\text{Coker } \alpha$. The sequence

$$0 \rightarrow \text{Ker } \alpha \rightarrow A \xrightarrow{\alpha} B \rightarrow \text{Coker } \alpha \rightarrow 0$$

is exact.

Lemma 3.9 (The snake lemma). *Suppose we have a commutative diagram*

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & X & \xrightarrow{\phi} & Y & \xrightarrow{\psi} & Z \end{array},$$

where the rows are exact. Then we obtain an exact sequence

$$\text{Ker } f \xrightarrow{\alpha} \text{Ker } g \xrightarrow{\beta} \text{Ker } h \xrightarrow{\delta} \text{Coker } f \xrightarrow{\bar{\phi}} \text{Coker } g \xrightarrow{\bar{\psi}} \text{Coker } h.$$

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Proof.

- The maps $\alpha : \text{Ker } f \rightarrow \text{Ker } g$ and $\beta : \text{Ker } g \rightarrow \text{Ker } h$ are obtained simply by restricting α and β respectively. Observe that if $a \in \text{Ker } f$ then $f(a) = 0$, so $(\phi \circ f)(a) = 0$. But $\phi \circ f = g \circ \alpha$, and so $(g \circ \alpha)(a) = 0$, so $\alpha(a) \in \text{Ker } g$, which is what we wanted.
- The maps $\bar{\phi} : \text{Coker } f \rightarrow \text{Coker } g$ and $\bar{\psi} : \text{Coker } g \rightarrow \text{Coker } h$ are induced from ϕ and ψ by

$$\bar{\phi}(x + \text{Im } f) = \phi(x) + \text{Im } g, \quad \bar{\psi}(y + \text{Im } g) = \psi(y) + \text{Im } h.$$

Check that these maps make sense. Suppose $x_1 + \text{Im } f = x_2 + \text{Im } f$. Then $x_1 - x_2 \in \text{Im } f$, so there exists $a \in A$ such that $f(a) = x_1 - x_2$. Now

$$\phi(x_1) - \phi(x_2) = \phi(x_1 - x_2) = (\phi \circ f)(a) = (g \circ \alpha)(a) \in \text{Im } g.$$

So $\phi(x_1) + \text{Im } g = \phi(x_2) + \text{Im } g$. So $\bar{\phi}$ is well-defined, and $\bar{\psi}$ is shown to be well-defined by a similar argument.

- How is the **connecting homomorphism** δ defined? Since β is surjective, for all $c \in C$, there exists $b \in B$ with $\beta(b) = c$. Suppose $c \in \text{Ker } h$. Then $(h \circ \beta)(b) = 0$, so $(\psi \circ g)(b) = 0$. Hence $g(b) \in \text{Ker } \psi = \text{Im } \phi$. Define

$$\delta(c) = x + \text{Im } f, \quad \phi(x) = g(b), \quad \beta(b) = c.$$

Check this is well-defined. Suppose b_1, b_2, x_1, x_2 are such that $\phi(x_1) = g(b_1)$ and $\phi(x_2) = g(b_2)$, and $\beta(b_1) = \beta(b_2) = c$. We have $b_1 - b_2 \in \text{Ker } \beta = \text{Im } \alpha$. So $b_1 - b_2 = \alpha(a)$ for some $a \in A$. Then

$$(\phi \circ f)(a) = (g \circ \alpha)(a) = g(b_1 - b_2) = g(b_1) - g(b_2) = \phi(x_1) - \phi(x_2) = \phi(x_1 - x_2).$$

But ϕ is injective, and so $f(a) = x_1 - x_2$, and so $x_1 + \text{Im } f = x_2 + \text{Im } f$. So δ is well-defined.

Exactness of the sequence is an exercise, on problem sheet. \square

3.3 Tensor products

Definition 3.10. Let M be a left R -module, and let L be a right R -module. The **tensor product** $L \otimes_R M$ is an abelian group generated as an abelian group by a set of **pure tensors**

$$\{l \otimes m \mid l \in L, m \in M\},$$

subject to the relations

$$\begin{aligned} l_1 \otimes m + l_2 \otimes m &= (l_1 + l_2) \otimes m, & l_1, l_2 \in L, & m \in M, \\ l \otimes m_1 + l \otimes m_2 &= l \otimes (m_1 + m_2), & l \in L, & m_1, m_2 \in M, \\ (lr) \otimes m &= l \otimes (rm), & l \in L, & m \in M, r \in R. \end{aligned}$$

The following are observations.

- In general, not every element of $L \otimes_R M$ is a pure tensor. A general element of $L \otimes_R M$ is a \mathbb{Z} -linear combination of pure tensors.
- If R is commutative, L can be a left module, since left and right modules are the same. Also, in this case, $L \otimes_R M$ has an R -module structure, by $r(l \otimes m) = rl \otimes m$.
- Suppose that S is a set of generators for L , as an abelian group, and T is a set of generators for M , as an abelian group. Then a smaller generating set for $L \otimes_R M$ is $\{s \otimes t \mid s \in S, t \in T\}$. This is because if

$$l = \sum_{i=1}^p a_i s_i, \quad m = \sum_{j=1}^q b_j t_j, \quad s_i \in S, \quad t_j \in T, \quad a_i, b_j \in \mathbb{Z},$$

then, from the relations,

$$l \otimes m = \sum_{i=1}^p \sum_{j=1}^q a_i b_j (s_i \otimes t_j).$$

Example. Tensor products can be counter intuitive, such as $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 = 0$. Why? Observe that for $x \in \mathbb{Z}_2$, $x3 = 3x = x$. So for all $x \in \mathbb{Z}_2$ and $y \in \mathbb{Z}_3$,

$$x \otimes y = x3 \otimes y = x \otimes 3y = x \otimes 0 = x \otimes y - x \otimes y = 0.$$

Theorem 3.11 (Universal property of tensor products). *Let A be a right R -module and B a left R -module. Let C be an abelian group. Let $f : A \times B \rightarrow C$ be a map, not necessarily a homomorphism, which is \mathbb{Z} -linear in both arguments, so*

$$f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b), \quad a_1, a_2 \in A, \quad b \in B,$$

$$f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2), \quad a \in A, \quad b_1, b_2 \in B,$$

and such that

$$f(ar, b) = f(a, rb), \quad a \in A, \quad b \in B, \quad r \in R.$$

Then there is a unique homomorphism

$$\begin{array}{ccc} g & : & A \otimes_R B \longrightarrow C \\ & & a \otimes b \longmapsto f(a, b) \end{array}.$$

Proof. In formal group theoretic terms, the tensor product $A \otimes_R B$ is a quotient F/K , where F is the free abelian group on the set of pure tensors $a \otimes b$, and K is the subgroup of F generated by elements of the form

$$(a_1 + a_2) \otimes b - a_1 \otimes b - a_2 \otimes b, \quad a \otimes (b_1 + b_2) - a \otimes b_1 - a \otimes b_2, \quad ar \otimes b - a \otimes rb.$$

The universal property of free abelian groups states that if F is free abelian on a set S , then any set map $S \rightarrow C$, for C an abelian group, extends uniquely to a homomorphism $F \rightarrow C$. In the situation under discussion, we have a map

$$g' : \{a \otimes b \mid a \in A, b \in B\} \rightarrow C.$$

So g' extends uniquely to a homomorphism $F \rightarrow C$. The conditions stipulated on f guarantee that $g'(K) = 0$. So g' induces a map $g : F/K \rightarrow C$, which is what we want, since $F/K = A \otimes_R B$. This establishes the existence of g . Since the images of the pure tensors under g are specified, it is clear that g is unique. \square

Corollary 3.12.

1. Let M be a left R -module. Then $R \otimes_R M \cong M$, via the map

$$\begin{array}{ccc} f & : & M \longrightarrow R \otimes_R M \\ & & m \longmapsto 1 \otimes m \end{array}.$$

2. Let M be a right R -module. Then $M \otimes_R R \cong M$.

Proof.

1. It is clear that f is a homomorphism of abelian groups. Now $r \otimes m = 1 \otimes rm$, so $R \otimes_R M$ is generated by $\{1 \otimes m \mid m \in M\}$, so f is surjective. For injectivity of f , we need the universal property. Define a bilinear map

$$\begin{array}{ccc} R \times M & \longrightarrow & M \\ (r, m) & \longmapsto & rm \end{array}.$$

This induces a homomorphism

$$\begin{array}{ccc} g & : & R \otimes_R M \longrightarrow M \\ & & r \otimes m \longmapsto rm \end{array}.$$

It is easy to check that g is an inverse for f , so f is bijective.

2. By the same argument as 1.

\square

Corollary 3.13. *Let A and B be right R -modules, and let C be a left R -module.*

1. $(A \oplus B) \otimes_R C \cong (A \otimes_R C) \oplus (B \otimes_R C)$, via the map

$$\begin{aligned} f : (A \oplus B) \otimes_R C &\longrightarrow (A \otimes_R C) \oplus (B \otimes_R C) \\ (a, b) \otimes c &\longmapsto (a \otimes c, b \otimes c) \end{aligned}.$$

2. $A \otimes_R (B \oplus C) \cong (A \otimes_R B) \oplus (A \otimes_R C)$.

Proof.

1. Take a bilinear map, that is \mathbb{Z} -bilinear in both arguments, and respecting R -multiplication,

$$\begin{aligned} A \oplus B \times C &\longrightarrow (A \otimes_R C) \oplus (B \otimes_R C) \\ ((a, b), c) &\longmapsto (a \otimes c, b \otimes c) \end{aligned}.$$

This induces a homomorphism $f : (A \oplus B) \otimes_R C \rightarrow (A \otimes_R C) \oplus (B \otimes_R C)$ with the description as given above. Now take the bilinear map given by

$$\begin{aligned} A \times C &\longrightarrow (A \oplus B) \otimes_R C \\ (a, c) &\longmapsto (a, 0) \otimes c \end{aligned}.$$

This induces a homomorphism $g_1 : A \otimes_R C \rightarrow (A \oplus B) \otimes_R C$. Similarly, we get a homomorphism $g_2 : B \otimes_R C \rightarrow (A \oplus B) \otimes_R C$. Now define

$$\begin{aligned} g = g_1 \oplus g_2 : (A \otimes_R C) \oplus (B \otimes_R C) &\longrightarrow (A \oplus B) \otimes_R C \\ (x, y) &\longmapsto g_1(x) + g_2(y) \end{aligned}.$$

It is easy to check that f and g are mutually inverse, so both isomorphisms.

2. Similarly. □

Corollary 3.14. *Let A be an abelian group. Then*

1. $\mathbb{Z}_n \otimes_{\mathbb{Z}} A \cong A/nA$, and
2. $A \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong A/nA$.

Proof.

1. Define a map by

$$\begin{aligned} f : A &\longrightarrow \mathbb{Z}_n \otimes_{\mathbb{Z}} A \\ a &\longmapsto 1 \otimes a \end{aligned}.$$

Suppose $a_0 \in A$ such that $a_0 = na$ for some a . Then $f(a_0) = 1 \otimes a_0 = 1 \otimes na = n \otimes a = 0$ so $nA \leq \text{Ker } f$. So f induces a map

$$\bar{f} : A/nA \rightarrow \mathbb{Z}_n \otimes_{\mathbb{Z}} A.$$

Notice that the pure tensor $k \otimes a$ is equal to $1 \otimes ka$, so $\mathbb{Z}_n \otimes_{\mathbb{Z}} A$ is generated by $\{1 \otimes a \mid a \in A\}$. So \bar{f} is surjective. For injectivity, use the universal property. We have a bilinear map

$$\begin{aligned} g : \mathbb{Z}_n \times A &\longrightarrow A/nA \\ (k, a) &\longmapsto ka + nA \end{aligned}.$$

This is well-defined and bilinear. So extends to a homomorphism

$$\bar{g} : \mathbb{Z}_n \otimes_{\mathbb{Z}} A \rightarrow A/nA.$$

It is easy to check that $\bar{g} \circ \bar{f} = \text{id}_{A/nA}$, so \bar{f} is injective.

2. Similarly. □

Proposition 3.15. *Let $\alpha : A \rightarrow B$ be a homomorphism of right R -modules. Let M be a left R -module. There is a unique abelian group homomorphism*

$$\begin{aligned} \alpha' : A \otimes_R M &\longrightarrow B \otimes_R M \\ a \otimes m &\longmapsto \alpha(a) \otimes m, \quad a \in A, \quad m \in M. \end{aligned}$$

Proof. The set map defined by

$$\begin{aligned} f : A \times M &\longrightarrow B \otimes_R M \\ (a, m) &\longmapsto \alpha(a) \otimes m \end{aligned}$$

is linear in both arguments, and we have

$$f(ar, m) = \alpha(ar) \otimes m = \alpha(a)r \otimes m = \alpha(a) \otimes rm = f(a, rm).$$

Now by the universal property of tensor products, f gives rise to a unique homomorphism $\alpha' : A \otimes_R M \rightarrow B \otimes_R M$ with the properties claimed. \square

Proposition 3.16. *Suppose $\alpha : A \rightarrow B$ is surjective. Then $\alpha' : A \otimes_R M \rightarrow B \otimes_R M$ is surjective.*

Proof. Since α is surjective, every pure tensor $b \otimes m \in B \otimes_R M$ is equal to $\alpha(a) \otimes m$ for some $a \in A$. So $b \otimes m = \alpha'(a \otimes m) \in \text{Im } \alpha'$. Since $B \otimes_R M$ is generated by its pure tensors, α' is surjective. \square

An observation is that it is not true that $A \rightarrow B$ is injective implies $A \otimes_R M \rightarrow B \otimes_R M$ is injective.

Example. Let

$$\begin{aligned} \alpha : \mathbb{Z}_2 &\longrightarrow \mathbb{Z}_4 \\ 1 &\longmapsto 2, \end{aligned}$$

which is injective. Consider

$$\begin{aligned} \alpha' : \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2 \cong \mathbb{Z}_2 &\longrightarrow \mathbb{Z}_4 \otimes_{\mathbb{Z}} \mathbb{Z}_2 \\ 1 \otimes 1 &\longmapsto 2 \otimes 1 = 1 \otimes 2 = 0. \end{aligned}$$

So α' is the zero map, which is not injective.

Proposition 3.17. *Let*

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$$

be a short exact sequence of right R -modules. Then the sequence

$$A \otimes_R M \xrightarrow{\alpha'} B \otimes_R M \xrightarrow{\beta'} C \otimes_R M \rightarrow 0$$

is exact.

Proof. Since β' is surjective, the sequence is exact at $C \otimes_R M$. We show it is exact at $B \otimes_R M$. Since β is surjective, for every $c \in C$, there exists $f(c) \in B$ such that $\beta(f(c)) = c$. Here f is a set map $C \rightarrow B$, which is not uniquely defined in general. Suppose that $\beta(b) = c$. Then $b - f(c) \in \text{Ker } \beta = \text{Im } \alpha$, so $f(c) + \text{Im } \alpha = b + \text{Im } \alpha$. Define a set map by

$$\begin{aligned} g : C \times M &\longrightarrow (B \otimes_R M) / \text{Im } \alpha' \\ (c, m) &\longmapsto f(c) \otimes m + \text{Im } \alpha'. \end{aligned}$$

Note that if $\beta(b) = c$, then $b \otimes m - f(c) \otimes m = \alpha(a) \otimes m \in \text{Im } \alpha'$ for some $a \in A$. We can check that g is linear in both arguments. For example, for the first argument, we have $g(c_1 + c_2, m) = f(c_1 + c_2) \otimes m + \text{Im } \alpha'$. Now $\beta(f(c_1 + c_2)) = c_1 + c_2 = \beta(f(c_1)) + \beta(f(c_2)) = \beta(f(c_1) + f(c_2))$ so

$$g(c_1 + c_2, m) = (f(c_1) + f(c_2)) \otimes m + \text{Im } \alpha' = f(c_1) \otimes m + f(c_2) \otimes m + \text{Im } \alpha' = g(c_1, m) + g(c_2, m).$$

Also, we have $g(cr, m) = f(cr) \otimes m + \text{Im } \alpha'$. But $\beta(f(cr)) = cr = \beta(f(c)r)$, so $f(cr) \otimes m + \text{Im } \alpha' = f(c)r \otimes m + \text{Im } \alpha'$. So

$$g(cr, m) = f(c)r \otimes m + \text{Im } \alpha' = f(c) \otimes rm + \text{Im } \alpha' = g(c, rm).$$

By the universal property, there is a unique homomorphism

$$\begin{aligned} \psi &: C \otimes_R M \longrightarrow (B \otimes_R M) / \text{Im } \alpha' \\ c \otimes m &\longmapsto f(c) \otimes m + \text{Im } \alpha' \end{aligned}$$

Next observe that $(\beta' \circ \alpha')(a \otimes m) = (\beta \circ \alpha)(a) \otimes m = 0$, since $\text{Im } \alpha = \text{Ker } \beta$. Since $A \otimes_R M$ is generated by pure tensors, we have $\beta' \circ \alpha' = 0$. So $\text{Im } \alpha' \leq \text{Ker } \beta'$. Hence β' induces a map

$$\phi : (B \otimes_R M) / \text{Im } \alpha' \rightarrow C \otimes_R M.$$

It is easy to check that ϕ and ψ are mutually inverse, and so both are isomorphisms. In particular ϕ is injective, and so $\text{Im } \alpha' = \text{Ker } \beta'$ as required. \square

3.4 Flat modules

Definition 3.18. A left R -module M is **flat** if $A \rightarrow B$ is injective implies that $A \otimes_R M \rightarrow B \otimes_R M$ is injective.

If M is flat then any short exact sequence of right R -modules

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

corresponds to a short exact sequence of abelian groups

$$0 \rightarrow A \otimes_R M \rightarrow B \otimes_R M \rightarrow C \otimes_R M \rightarrow 0.$$

Proposition 3.19. *Every projective module is flat.*

This follows from two lemmas.

Lemma 3.20. *$P \oplus Q$ is flat if and only if P and Q are both flat.*

Proof. Recall there is a canonical isomorphism

$$A \otimes_R (P \oplus Q) \cong (A \otimes_R P) \oplus (A \otimes_R Q).$$

Suppose $\alpha : A \rightarrow B$ is injective. Then $\alpha' : A \otimes_R (P \oplus Q) \rightarrow B \otimes_R (P \oplus Q)$ corresponds to

$$\begin{aligned} \overline{\alpha'} &: (A \otimes_R P) \oplus (A \otimes_R Q) \longrightarrow (B \otimes_R P) \oplus (B \otimes_R Q) \\ (a \otimes p, 0) &\longmapsto (\alpha(a) \otimes p, 0) \\ (0, a \otimes q) &\longmapsto (0, \alpha(a) \otimes q) \end{aligned}$$

It is clear from this that $\overline{\alpha'}$ is injective if and only if $A \otimes_R P \rightarrow B \otimes_R P$ and $A \otimes_R Q \rightarrow B \otimes_R Q$ are injective, and Lemma 3.20 follows immediately. \square

Lemma 3.21. *Every free R -module is flat.*

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Proof. We know $(A \oplus B) \otimes_R C \cong (A \otimes_R C) \oplus (B \otimes_R C)$. Similarly,

$$\left(\bigoplus_{s \in S} A_s \right) \otimes_R C \cong \bigoplus_{s \in S} (A_s \otimes_R C).$$

So Lemma 3.20 generalises, so $\bigoplus_{s \in S} A_s$ is flat if and only if all of the A_s is flat for $s \in S$. Let F be free. Then $F = \bigoplus_{s \in S} R$, and so F is flat if and only if R is flat. But for any R -module A , we have $A \otimes_R R \cong A$, so

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \text{Id} \downarrow & & \downarrow \text{Id} \\ A \otimes_R R & \xrightarrow[\alpha']{} & B \otimes_R R \end{array},$$

and it is easy to check that R is flat. \square

Proof of Proposition 3.19. Lemma 3.20 and Lemma 3.21 imply Proposition 3.19, since a projective module is a direct summand of a free module. \square

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4 Modules over a PID

There exist flat modules which are not projective. We will show that \mathbb{Q} as a module for \mathbb{Z} is flat, and it is easy to see it is not projective. To do this we will study the case of modules over a PID. Recall that R is an **integral domain** if R is commutative and $rs = 0$ implies that $r = 0$ or $s = 0$ for $r, s \in R$. An integral domain is a **PID** if every ideal is $\langle a \rangle = \{ra \mid r \in R\}$ for some $a \in R$.

Example. The ring \mathbb{Z} is an example of a PID.

4.1 Free and projective modules

Proposition 4.1. *Let R be a PID. Then every projective R -module is free. Equivalently, every summand of a free module is free.*

In fact we will show that any submodule of a free module is free. Moreover, if $F_1 \leq F_2$, where F_1 and F_2 are free, and if B_1 and B_2 are bases for F_1 and F_2 respectively, then $|B_1| \leq |B_2|$. In particular, if $M \leq R^n$, then $M \cong R^m$ for some $m \leq n$. For this, we will need the well-ordering theorem.

Theorem 4.2 (Well-ordering theorem). *Let X be a set. There exists a well-order \leq on X , that is a total order such that every non-empty subset of X has a least element.*

Corollary 4.3 (Transfinite induction). *Let X be a non-empty set well-ordered by \leq . Let x_0 be the least element of X . Let $S \subseteq X$. If $x_0 \in S$, and $s < t$ implies $s \in S$ implies that $t \in S$, then $S = X$.*

Proof. Let $F = \bigoplus_{s \in S} R$. Let \leq be a well-order on S . For $s \in S$, let π_s be the projection map $F \rightarrow R$ onto the s -coordinate. Let e_s be the element of F with one in coordinate s , and zero elsewhere. Suppose $U \leq F$ is an R -submodule of F . Define R_t to be the submodule of F generated by $\{e_s \mid s \leq t\}$, so

$$R_t = \text{sp}\{e_s \mid s \leq t\}.$$

So if $t_1 \leq t_2$ then $R_{t_1} \leq R_{t_2}$. Let

$$U_t = U \cap R_t.$$

So $t_1 < t_2$ implies that $U_{t_1} \leq U_{t_2}$. Consider $\pi_s(U_s)$. This is an ideal of R . Hence there exists $a_s \in R$ such that $\pi_s(U_s) = \langle a_s \rangle$, since R is a PID. For each s , let $u_s \in U_s$ be such that $\pi_s(u_s) = a_s$. In cases where $a_s = 0$, assume $u_s = 0$. Let

$$B = \{u_s \mid s \in S, u_s \neq 0\}.$$

- Claim that B generates U . We will actually prove that $B_t = \{u_s \mid s \leq t\}$ generates U_t , using transfinite induction. If s_0 is the least element of S , it is easy to see that $B_{s_0} = \{u_{s_0}\}$ generates U_{s_0} . Suppose B_t generates U_t for all $t < t_0$. Let $u \in U_{t_0}$. Then $\pi_{t_0}(u) = ra_{t_0}$. Hence $\pi_{t_0}(u - ru_{t_0}) = 0$. So $u - ru_{t_0}$ has zero in the t_0 -coordinate, so $u - ru_{t_0} \in \text{sp}\{e_s \mid s < t_0\}$. Clearly $u - ru_{t_0} \in U$. We have $u - ru_{t_0} = \sum_{i=1}^q r_i e_{s_i}$, where $s_i < t_0$, and $s_1 < \dots < s_q$. Then

$$u - ru_{t_0} \in U \cap R_{s_q} = U_{s_q} = \text{sp } B_{s_q},$$

by the inductive hypothesis. Hence $u \in \text{sp}(B_{s_q} \cup \{u_{t_0}\}) \subseteq \text{sp } B_{t_0}$. Hence B_{t_0} generates U_{t_0} , as required.

- Next we show the linear independence of B . Suppose we have a linear combination of elements of B equal to zero. Say $\sum_{i=1}^k r_i u_{s_i} = 0$. Assume $s_1 < \dots < s_k$. We have

$$\pi_{s_k} \left(\sum_{i=1}^k r_i u_{s_i} \right) = \sum_{i=1}^k r_i \pi_{s_k}(u_{s_i}).$$

Now $u_{s_i} \in U_{s_i} \subseteq R_{s_i}$, and so $\pi_{s_k}(u_{s_i}) = 0$ if $s_i < s_k$. Hence $r_k \pi_{s_k}(u_{s_k}) = 0$, so $r_k a_{s_k} = 0$. But $a_{s_k} \neq 0$, and R is an integral domain. So $r_k = 0$. It follows easily that $r_i = 0$ for all i , so B is linearly independent.

We have shown that B is a basis for U . Hence U is free. Since the elements of B are indexed by a subset of S , we have $|B| \leq |S|$. □

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4.2 Injective and divisible modules

Definition 4.4. Let R be an integral domain, and M an R -module. Let $m \in M$. Say that m is **infinitely divisible** if for all $r \in R \setminus \{0\}$ there exists $l \in M$ such that $rl = m$.

Proposition 4.5. *The divisible elements of M form a submodule $D(M)$.*

Proof. Easy. □

Definition 4.6. If $D(M) = M$, then M is **divisible**.

Proposition 4.7. *Let R be an integral domain. Then if an R -module M is injective then it is divisible.*

Proof. Recall that for an integral domain R , and $a \in R \setminus \{0\}$, the map

$$\begin{aligned} f : R &\longrightarrow \langle a \rangle \\ r &\longmapsto ra \end{aligned}$$

is an isomorphism. Suppose M is an injective R -module. Let

$$\begin{aligned} g : R &\longrightarrow M \\ 1 &\longmapsto m \end{aligned}.$$

Then $g \circ f^{-1}$ is a homomorphism $\langle a \rangle \rightarrow M$, and $(g \circ f^{-1})(a) = g(1) = m$. Now by Baer's criterion, there is a map $h : R \rightarrow M$ extending $g \circ f^{-1}$. Now $ah(1) = h(a) = (g \circ f^{-1})(a) = m$. Hence there exists $l \in M$ such that $al = m$. So m is a divisible element, and so M is divisible. □

Proposition 4.8. *Let R be a PID. If M is a divisible R -module then M is injective.*

So divisible equals injective when R is a PID.

Proof. We use Baer's criterion. Let I be an ideal of R , and $f : I \rightarrow M$ an R -module homomorphism. Since R is a PID, $I = \langle a \rangle$ for some $a \in R$. Suppose $f(a) = m$. If $a = 0$ there is nothing to prove, since the zero map $R \rightarrow M$ extends f . So assume $a \neq 0$. Since m is divisible, there exists $l \in M$ with $al = m$. Now the map given by

$$\begin{aligned} R &\longrightarrow M \\ 1 &\longmapsto l \end{aligned}$$

extends f . So Baer's criterion is satisfied, and so M is injective. □

4.3 Flat and torsion-free modules

Definition 4.9. Let R be an integral domain. Let M be an R -module. Say that $m \in M$ is a **torsion element** if there exists $r \in R \setminus \{0\}$ such that $rm = 0$.

Proposition 4.10. *The torsion elements of M form a submodule $T(M)$.*

Proof. Easy, using the fact that integral domains are commutative. □

Definition 4.11. If $T(M) = 0$, then M is **torsion-free**. If $T(M) = M$, then M is a **torsion module**.

Proposition 4.12. *Let R be an integral domain. Let M be a flat R -module. Then M is torsion-free.*

Proof. Let $a \in R \setminus \{0\}$. Then

$$\begin{aligned} f : R &\longrightarrow R \\ 1 &\longmapsto a \end{aligned}$$

is an injective R -module homomorphism. Suppose that M is flat. Then the map

$$\begin{aligned} g : R \otimes_R M &\longrightarrow R \otimes_R M \\ r \otimes m &\longmapsto ra \otimes m = r \otimes am \end{aligned}$$

is injective. But $R \otimes_R M$ is canonically isomorphic to M , under which the map g corresponds to $m \mapsto am$. Since g is injective, we have $am \neq 0$ for $m \neq 0$. Hence m is not a torsion element, if $m \neq 0$, and so M is torsion-free. □

We now build up to the following.

Proposition 4.13. *Let R be a PID. If M is a torsion-free R -module then M is flat.*

The following is the strategy. We want to prove that whenever $\alpha : A \rightarrow B$ is injective, so is $\alpha' : A \otimes_R M \rightarrow B \otimes_R M$, where M is torsion-free.

1. Prove this in the case that B is free, and A is a submodule of B , and α is the inclusion map, by
 - first reducing the problem to the case that A and B are finitely generated, so $B \cong \mathbb{R}^n$, and
 - then using induction on the rank n of B .
2. Show the general case follows from 1.

Lemma 4.14. *Let R be a PID, let $I = \langle a \rangle$ be an ideal of R , and let M be a torsion-free R -module. Then $g : I \otimes_R M \rightarrow R \otimes_R M$ is injective.*

Proof. The homomorphism given by

$$\begin{array}{ccc} R & \longrightarrow & I \\ r & \longmapsto & ra \end{array}$$

gives a map $f : R \otimes_R M \rightarrow I \otimes_R M$. Now $g \circ f$ is a map

$$\begin{array}{ccc} R \otimes_R M & \longrightarrow & R \otimes_R M \\ r & \longmapsto & ra \end{array}.$$

Now f is surjective, and $g \circ f$ is injective, since R is an integral domain. But this implies that g is injective, as required. \square

Lemma 4.15. *Let A be a right R -module. Let M be a left R -module. Suppose $\sum_{i=1}^t (a_i \otimes m_i) = 0$ in $A \otimes_R M$. There exists a finitely generated submodule $A_0 \leq A$ such that $a_i \in A_0$ for all i , and $\sum_{i=1}^t (a_i \otimes m_i) = 0$ in $A_0 \otimes_R M$.*

Proof. Recall that

$$A \otimes_R M = F_{\text{ab}}(A \times M) / K,$$

where K is generated by certain relators. If $\sum_{i=1}^t (a_i \otimes m_i) = 0$ in $A \otimes_R M$, then in $F_{\text{ab}}(A \times M)$, we have $\sum_{i=1}^t (a_i \otimes m_i) \in K$. So there exist relators s_1, \dots, s_q , or their negations, such that

$$\sum_{i=1}^t (a_i \otimes m_i) = \sum_{i=1}^q s_i.$$

Only finitely many elements of A are involved in the relators s_1, \dots, s_q . Let A_0 be generated by these together with a_1, \dots, a_t . Then certainly $a_i \in A_0$ for all i . And $\sum_{i=1}^t (a_i \otimes m_i) = \sum_{i=1}^q s_i$ in $F_{\text{ab}}(A_0 \times M)$ so $\sum_{i=1}^t (a_i \otimes m_i) = 0$ in $A_0 \otimes_R M$. Clearly A_0 is finitely generated. \square

Lemma 4.16. *Let $F = F(S) = \bigoplus_{s \in S} R$. Let U be a finitely generated submodule of F . Then there exists a finite $T \subseteq S$ such that $U \leq F(T)$, and for any M , the map $F(T) \otimes_R M \rightarrow F(S) \otimes_R M$ is injective.*

Proof. Let u_1, \dots, u_q be generators for U . Every u_i is an R -linear combination of elements of S . Since each of these linear combinations mentions only finitely many elements of S , there is a finite subset $T \subseteq S$ such that every u_i is an R -linear combination of elements of T . So $U \leq F(T)$. We have

$$F(S) = F(T) \oplus F(S \setminus T),$$

and so

$$F(S) \otimes_R M \cong (F(T) \otimes_R M) \oplus (F(S \setminus T) \otimes_R M).$$

It follows that the natural map $F(T) \otimes_R M \rightarrow F(S) \otimes_R M$ is injective. \square

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Lemma 4.15 and Lemma 4.16 tell us that if F is free and $U \leq F$, and if M is an R -module, if $U \otimes_R M \rightarrow F \otimes_R M$ is not injective, then there exists a finitely generated $U_0 < U$ and a finite rank free submodule $F_0 < F$ such that $U_0 \otimes_R M \rightarrow F_0 \otimes_R M$ is not injective.

Lemma 4.17. *Let R be a PID. Let F be free, and $U \leq F$. Let M be torsion-free. Then $U \otimes_R M \rightarrow F \otimes_R M$ is injective.*

Proof. We assume that $F = R^n$. We do this by induction on n .

Base case. Let $n = 1$. So F is R , and U is an ideal of R . By Lemma 4.14, $U \otimes_R M \rightarrow F \otimes_R M$ is injective in this case.

Inductive hypothesis. $U \leq F = R^{n-1}$ implies that $U \otimes_R M \rightarrow F \otimes_R M$ is injective.

Inductive step. Assume $U \leq F = R^n$. Write $R^n = R \oplus R^{n-1}$. So we have a short exact sequence

$$0 \rightarrow R \rightarrow R^n \rightarrow R^{n-1} \rightarrow 0.$$

We also have a short exact sequence

$$0 \rightarrow U_1 \rightarrow U \rightarrow \pi_{R^{n-1}}(U) \rightarrow 0,$$

where $U_1 = U \cap (R \oplus 0^{n-1})$. Identifying R with $R \oplus 0^{n-1}$, we get a commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & U_1 & \longrightarrow & U & \longrightarrow & \pi_{R^{n-1}}(U) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & R & \longrightarrow & R^n & \longrightarrow & R^{n-1} \longrightarrow 0 \end{array},$$

where the vertical maps are inclusions, and the rows are exact. Tensoring everything with M , we get a new commuting diagram

$$\begin{array}{ccccccc} U_1 \otimes_R M & \longrightarrow & U \otimes_R M & \longrightarrow & \pi_{R^{n-1}}(U) \otimes_R M & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & R \otimes_R M & \longrightarrow & R^n \otimes_R M & \longrightarrow & R^{n-1} \otimes_R M \longrightarrow 0 \end{array}.$$

The initial zero in the bottom row comes from the fact that

$$0 \rightarrow R \rightarrow R^n \rightarrow R^{n-1} \rightarrow 0$$

is split, since $R^n = R \oplus R^{n-1}$, and so

$$R^n \otimes_R M \cong (R \otimes_R M) \oplus (R^{n-1} \otimes_R M).$$

Now f is injective by Lemma 4.14, and h is injective by the inductive hypothesis. The snake lemma tells us that the sequence

$$\text{Ker } f \rightarrow \text{Ker } g \rightarrow \text{Ker } h$$

is exact at $\text{Ker } g$. So

$$0 \rightarrow \text{Ker } g \rightarrow 0$$

is exact, and so $\text{Ker } g = 0$. So g is injective, and this completes the induction. □

Proof of Proposition 4.13. Prove that if $\alpha : A \rightarrow B$ is injective, and M is torsion-free, over a PID R , then $\alpha' : A \otimes_R M \rightarrow B \otimes_R M$ is injective. There exists a free module F such that B is quotient of F . So there is a short exact sequence

$$0 \rightarrow K \rightarrow F \xrightarrow{\delta} B \rightarrow 0.$$

Now $A \cong \alpha A = \text{Im } \alpha$. Let F_A be the δ -preimage of αA . Then $K < F_A$, and we have another short exact sequence

$$0 \rightarrow K \rightarrow F_A \rightarrow \alpha A \rightarrow 0.$$

We have a commuting diagram

$$\begin{array}{ccccccc} & & & F_A & \longrightarrow & \alpha A & \longrightarrow 0 \\ & & \nearrow & \downarrow & & \downarrow & \\ 0 & \longrightarrow & K & & & & \\ & & \searrow & F & \longrightarrow & B & \longrightarrow 0 \end{array}.$$

Tensoring with M ,

$$\begin{array}{ccccccc} & & & F_A \otimes_R M & \xrightarrow{\gamma} & \alpha A \otimes_R M & \longrightarrow 0 \\ & & \nearrow \beta & \downarrow f & & \downarrow g & \\ K \otimes_R M & & & F \otimes_R M & \xrightarrow{\epsilon} & B \otimes_R M & \longrightarrow 0 \\ & & \searrow \delta & & & & \end{array}$$

is commuting, and exact along rows. Let $u \in \text{Ker } g \leq \alpha A \otimes_R M \cong A \otimes_R M$. Since γ is surjective, there is $w \in F_A \otimes_R M$ with $\gamma(w) = u$. So $(g \circ \gamma)(w) = 0$. So $(\epsilon \circ f)(w) = 0$. So $f(w) \in \text{Ker } \epsilon = \text{Im } \delta$, so $f(w) = \delta(k)$ for $k \in K \otimes_R M$. Since f is injective, by Lemma 4.17, we get $w = \beta(k) \in \text{Im } \beta$. So $w \in \text{Ker } \gamma$, so $u = 0$. Hence g is injective, as required. \square

We have shown that if R is a PID, and if M is torsion-free, then M is flat.

4.4 Modules over PIDs

For an R -module M

$$\text{free} \implies \text{projective} \implies \text{flat} \implies \text{torsion-free}, \quad \text{injective} \implies \text{divisible}.$$

Over a PID

$$\text{free} \iff \text{projective} \implies \text{flat} \iff \text{torsion-free}, \quad \text{injective} \iff \text{divisible}.$$

Do we have projective if and only if flat, over a general ring, or over a PID? The answer is no.

Example. The \mathbb{Z} -module \mathbb{Q} is torsion-free, so flat. Is \mathbb{Q} projective? Is \mathbb{Q} free, since \mathbb{Z} is a PID? Consider a free \mathbb{Z} -module $F = \bigoplus_{s \in S} \mathbb{Z}$. Let $s_0 \in S$. Then let

$$x = (x_s)_{s \in S} = \begin{cases} 1 & s = s_0 \\ 0 & \text{otherwise} \end{cases} \in F.$$

It is clear there are no $y \in F$ such that $2y = x$. So x is not a divisible element of F . Indeed, $D(F) = \{0\}$. But $D(\mathbb{Q}) = \mathbb{Q}$. Hence $\mathbb{Q} \not\cong F$. So \mathbb{Q} is an example of a flat module which is not projective.

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5 Projective and injective resolutions

Definition 5.1. Let M be an R -module. A **resolution**, or **left resolution**, for M is a sequence of R -modules A_0, A_1, A_2, \dots , with homomorphisms $d : A_{i+1} \rightarrow A_i$, and also a homomorphism $A_0 \rightarrow M$, such that

$$\dots \xrightarrow{d} A_2 \xrightarrow{d} A_1 \xrightarrow{d} A_0 \rightarrow M \rightarrow 0$$

is an exact sequence, where d is the **differential**. If all of the modules A_i have a property \mathcal{P} , we call this a **\mathcal{P} -resolution**.

So we can talk about free resolutions, projective resolutions, flat resolutions. We do not use the term injective resolution in this context.

Definition 5.2. A **right resolution**, or **coresolution**, for M is a sequence of R -modules A^0, A^1, A^2, \dots , with homomorphisms $d : A^i \rightarrow A^{i+1}$, and $M \rightarrow A^0$, such that

$$0 \rightarrow M \rightarrow A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \dots$$

is exact. If the modules A^i have a property \mathcal{P} , we can refer to a **right \mathcal{P} -resolution**.

An injective resolution always means a right injective resolution.

5.1 Existence of projective resolutions

Proposition 5.3. Let M be an R -module. Then M has free, projective, and flat resolutions.

Proof. Since free implies projective implies flat, it is enough to show that free resolutions exist. Use the fact that for any module L , there exists a free module F and $K \leq F$ such that $L \cong F/K$. So we get a short exact sequence

$$0 \rightarrow K \rightarrow F \rightarrow L \rightarrow 0.$$

It follows that we can find F_0, F_1, F_2, \dots , and $K_0 \leq F_0, K_1 \leq F_1, K_2 \leq F_2, \dots$ such that

$$0 \rightarrow K_0 \rightarrow F_0 \rightarrow M \rightarrow 0, \quad 0 \rightarrow K_1 \rightarrow F_1 \rightarrow K_0 \rightarrow 0, \quad 0 \rightarrow K_2 \rightarrow F_2 \rightarrow K_1 \rightarrow 0, \quad \dots$$

are all exact. Since $K_i \leq F_i$, we may consider the maps $F_{i+1} \rightarrow K_i$ as maps $F_{i+1} \rightarrow F_i$ with image K_i . But K_i is the kernel of the map $F_i \rightarrow K_{i-1}$, so the sequence

$$\dots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

is exact, and a free resolution for M . □

5.2 Existence of injective resolutions

Injective coresolutions exist too, but the proof is more intricate. It involves making use of properties of the abelian group \mathbb{Q}/\mathbb{Z} .

Proposition 5.4. Let A be an abelian group, and let $a \in A \setminus \{0\}$. There is a homomorphism $f : A \rightarrow \mathbb{Q}/\mathbb{Z}$ such that $f(a) \neq 0$.

Proof. Start by defining $f_0 : \langle a \rangle \rightarrow \mathbb{Q}/\mathbb{Z}$. If a has finite order t , then $f_0 : a \mapsto 1/t + \mathbb{Z}$. If a has infinite order, then $f_0 : a \mapsto \frac{1}{2} + \mathbb{Z}$. We will use Zorn's lemma. Let X be the set

$$\{(B, f) \mid B \leq A, a \in B, f : B \rightarrow \mathbb{Q}/\mathbb{Z}, f \text{ extends } f_0\}.$$

Then X is non-empty, since $(\langle a \rangle, f_0) \in X$. Define a partial order \leq on X by $(B_1, f_1) \leq (B_2, f_2)$ if $B_1 \leq B_2$ and f_2 extends f_1 . Let $\{(B_s, f_s) \mid s \in S\}$ be a chain in X , where S is a suitable indexing set. Then $\{B_s \mid s \in S\}$ is a chain of subgroups of A . So the union $B = \bigcup_{s \in S} B_s$ is a subgroup of A , containing a . Define

$$f : B \longrightarrow \mathbb{Q}/\mathbb{Z} \\ b \longmapsto f_s(b), \quad b \in B_s.$$

This is well-defined since if $b \in B_t$ then $f_s(b) = f_t(b)$. Now (B, f) is an upper bound for $\{B_s \mid s \in S\}$ in X . So by Zorn's lemma, X has a maximal element, which we will call (B, f) . We show that $B = A$. Since $f(a) = f_0(a)$, this will complete the proof. Suppose $x \in A \setminus B$. Then let $I < \mathbb{Z}$ be defined by

$$I = \{k \mid kx \in B\}.$$

Since \mathbb{Z} is a PID, we have $I = n\mathbb{Z}$ for some n . We have $\langle B, x \rangle \leq A$, and $\langle B, x \rangle \cong B \oplus \langle x \rangle / \langle nx - b_0 \rangle$, where $b_0 = nx$ in A . Define

$$\begin{aligned} \phi &: B \oplus \langle x \rangle \longrightarrow \mathbb{Q}/\mathbb{Z} \\ (b, kx) &\longmapsto f(b) + \frac{kf(b_0)}{n}, \end{aligned}$$

so sending x to $f(b_0)/n$. We see that $\phi(nx - b_0) = 0$, so ϕ induces a map $B \oplus \langle x \rangle / \langle nx - b_0 \rangle \rightarrow \mathbb{Q}/\mathbb{Z}$, and hence a map $f' : \langle B, x \rangle \rightarrow \mathbb{Q}/\mathbb{Z}$. But $f'(a) = f_0(a)$, so $(\langle B, x \rangle, f')$ is an element of X greater than (B, f) , contradicting maximality of (B, f) . Hence $B = A$ as required. \square

Proposition 5.5. *For every abelian group A , there is an injective abelian group I such that A is isomorphic to a subgroup of I .*

Proof. We know that \mathbb{Q}/\mathbb{Z} is injective, as a \mathbb{Z} -module, since it is divisible, and \mathbb{Z} is a PID. So $\prod_{s \in S} \mathbb{Q}/\mathbb{Z}$ is also injective. Take $S = A \setminus \{0\}$. Then define, for each $s \in S$, $f_s : A \rightarrow \mathbb{Q}/\mathbb{Z}$ such that $f_s(s) \neq 0$. Define

$$\begin{aligned} f &: A \longrightarrow \prod_{s \in S} \mathbb{Q}/\mathbb{Z} \\ a &\longmapsto (f_s(a))_{s \in S}. \end{aligned}$$

Now if $s \in A \setminus \{0\}$, then $f_s(s) \neq 0$, so $f(s) \neq 0$. So f is injective. It is easy to check that f is a homomorphism. \square

Proposition 5.6. *Let M be a right R -module, and let A be an abelian group. Then $\text{Hom}_{\mathbb{Z}}(M, A)$ is a left R -module, with the R -action defined by $(rf)(m) = f(mr)$.*

Proof. This is clearer if we write the map f on the right instead of the left. Then the proposition becomes $(m)(rf) = (mr)f$, and it is easy to see this works. \square

Proposition 5.7. *Let M be a left R -module, and A an abelian group. Then $\text{Hom}_{\mathbb{Z}}(R, A)$ is a left R -module, and there is a natural isomorphism*

$$\text{Hom}_R(M, \text{Hom}_{\mathbb{Z}}(R, A)) \cong \text{Hom}_{\mathbb{Z}}(M, A).$$

Proof. Write $H = \text{Hom}_{\mathbb{Z}}(R, A)$. Define

$$\begin{aligned} \Phi &: \text{Hom}_R(M, H) \longrightarrow \text{Hom}_{\mathbb{Z}}(M, A) \\ f &\longmapsto (m \mapsto f(m)(1)), \quad m \in M, \quad 1 \in R. \end{aligned}$$

Check the following.

- $\Phi(f)$ is a homomorphism, since

$$\begin{aligned} \Phi(f)(m_1 + m_2) &= f(m_1 + m_2)(1) \\ &= (f(m_1) + f(m_2))(1) \\ &= f(m_1)(1) + f(m_2)(1) && \text{definition of } + \text{ in } \text{Hom}_{\mathbb{Z}}(R, A) \\ &= \Phi(f)(m_1) + \Phi(f)(m_2). \end{aligned}$$

- Φ is a homomorphism, since

$$\begin{aligned} \Phi(f_1 + f_2)(m) &= (f_1 + f_2)(m)(1) \\ &= (f_1(m) + f_2(m))(1) && \text{definition of } + \text{ in } \text{Hom}_{\mathbb{Z}}(M, A) \\ &= f_1(m)(1) + f_2(m)(1) \\ &= \Phi(f_1)(m) + \Phi(f_2)(m) \\ &= (\Phi(f_1) + \Phi(f_2))(m) && \text{definition of } + \text{ in } \text{Hom}_{\mathbb{Z}}(M, A), \end{aligned}$$

so since m was arbitrary, $\Phi(f_1 + f_2) = \Phi(f_1) + \Phi(f_2)$.

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Now define

$$\begin{aligned} \Psi &: \operatorname{Hom}_{\mathbb{Z}}(M, A) \longrightarrow \operatorname{Hom}_R(M, H) \\ p &\longmapsto (m \mapsto (r \mapsto p(rm))) \end{aligned}, \quad m \in M, \quad r \in R.$$

Check the following.

- $\Psi(p)(m)$ is a homomorphism, since

$$\begin{aligned} \Psi(p)(m)(r_1 + r_2) &= p((r_1 + r_2)m) = p(r_1m + r_2m) \\ &= p(r_1m) + p(r_2m) = \Psi(p)(m)(r_1) + \Psi(p)(m)(r_2). \end{aligned}$$

- $\Psi(p)$ is an R -module homomorphism, since

$$\begin{aligned} \Psi(p)(m_1 + m_2)(r) &= p(r(m_1 + m_2)) = p(rm_1 + rm_2) = p(rm_1) + p(rm_2) \\ &= \Psi(p)(m_1)(r) + \Psi(p)(m_2)(r) = (\Psi(p)(m_1) + \Psi(p)(m_2))(r), \end{aligned}$$

so $\Psi(p)(m_1 + m_2) = \Psi(p)(m_1) + \Psi(p)(m_2)$, and for $h \in H$, we have $(sh)(r) = h(rs)$, by definition of the R -module structure on H , so

$$s\Psi(p)(m)(r) = \Psi(p)(m)(rs) = p(rsm) = \Psi(p)(sm)(r),$$

so $s\Psi(p)(m) = \Psi(p)(sm)$.

- Ψ is a homomorphism, since

$$\begin{aligned} \Psi(p_1 + p_2)(m)(r) &= (p_1 + p_2)(rm) = p_1(rm) + p_2(rm) \\ &= \Psi(p_1)(m)(r) + \Psi(p_2)(m)(r) = (\Psi(p_1) + \Psi(p_2))(m)(r), \end{aligned}$$

so $\Psi(p_1 + p_2) = \Psi(p_1) + \Psi(p_2)$.

Then $\Psi \circ \Phi = \operatorname{id}_{\operatorname{Hom}_R(M, H)}$ and $\Phi \circ \Psi = \operatorname{id}_{\operatorname{Hom}_{\mathbb{Z}}(M, A)}$.⁴ Hence Φ and Ψ are isomorphisms. \square

We are interested in the case $A = \mathbb{Q}/\mathbb{Z}$. Write $S = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

Proposition 5.8. *S is injective as a left R -module.*

Proof. Let M and N be R -modules, and $\alpha : M \rightarrow N$ an injective homomorphism. By identifying M with $\operatorname{Im} \alpha$, we may assume that $M \leq N$, and α is the inclusion map. Since \mathbb{Q}/\mathbb{Z} is injective as an abelian group, any \mathbb{Z} -module homomorphism $M \rightarrow S$ extends to a homomorphism $N \rightarrow S$. Define

$$\begin{aligned} \Theta &: \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \\ f &\longmapsto f|_M \end{aligned},$$

the restriction to M . We see that Θ is surjective. Similarly, we can define

$$\begin{aligned} \Theta' &: \operatorname{Hom}_R(N, S) \longrightarrow \operatorname{Hom}_R(M, S) \\ f &\longmapsto f|_M \end{aligned}.$$

Then Θ' is an abelian group homomorphism. But we know there is a naturally defined isomorphism between $\operatorname{Hom}_R(M, S)$ and $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. So we get

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\Theta} & \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \\ \Psi \downarrow \sim & & \sim \downarrow \Psi \\ \operatorname{Hom}_R(N, S) & \xrightarrow{\Theta'} & \operatorname{Hom}_R(M, S) \end{array}.$$

It is easy to see that this diagram commutes. It follows that Θ' is surjective. So any R -module homomorphism $M \rightarrow S$ extends to a homomorphism $N \rightarrow S$. Hence S is injective. \square

⁴Exercise

Proposition 5.9. *Let M be a left R -module, and $m \in M \setminus \{0\}$. Then there exists $f : M \rightarrow S$ such that $f(m) \neq 0$.*

Proof. We know there is an abelian group homomorphism $g : M \rightarrow \mathbb{Q}/\mathbb{Z}$ such that $g(m) \neq 0$. Now $\Psi(g) \in \text{Hom}_R(M, S)$, and $\Psi(g)(m)(1) = g(m) \neq 0$ for $1 \in R$, so $\Psi(g)(m)$ is not the zero map. \square

Proposition 5.10. *Let M be a left R -module. There exists an injective R -module I such that M is isomorphic to a submodule of I . Equivalently, there exists an injection $M \rightarrow I$.*

Proof. Same as abelian groups. Let $T = M \setminus \{0\}$. Then $I = \prod_{t \in T} S$ is injective. Let f_t be a homomorphism $M \rightarrow S$ such that $f_t(t) \neq 0$. Then

$$\begin{array}{ccc} f & : & M \longrightarrow I \\ & & m \longmapsto (f_t(m))_{t \in T} \end{array}$$

is injective, and a homomorphism. \square

Proposition 5.11. *Every R -module admits an injective resolution.*

Thus there exists injective I_0, I_1, I_2, \dots such that

$$0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \dots$$

is exact.

Proof. Let M be an R -module. Then M injects into some injective module I_0 . Let $C_0 = I_0 / \text{Im}(M \rightarrow I_0)$. Then C_0 injects into some injective I_1 . This induces a map $I_0 \rightarrow I_1$ whose kernel is $\text{Im}(M \rightarrow I_0)$. Further terms in the sequence are constructed in an identical manner. \square

5.3 Uniqueness of projective resolutions

Proposition 5.12. *Let M and N be R -modules, and $\phi : M \rightarrow N$. Let (P_i) be a projective resolution for M , and (Q_i) a projective resolution for N .*

1. *There exist R -module homomorphisms $f_i : P_i \rightarrow Q_i$ such that the diagram*

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{d_2} & P_2 & \xrightarrow{d_1} & P_1 & \xrightarrow{d_0} & P_0 & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow \phi & & \\ \dots & \xrightarrow{d'_2} & Q_2 & \xrightarrow{d'_1} & Q_1 & \xrightarrow{d'_0} & Q_0 & \xrightarrow{q} & N & \longrightarrow & 0 \end{array}$$

commutes.

2. *Let $g_i : P_i \rightarrow Q_i$ be such that the diagram*

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{d_2} & P_2 & \xrightarrow{d_1} & P_1 & \xrightarrow{d_0} & P_0 & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & g_2 \downarrow & & g_1 \downarrow & & g_0 \downarrow & & \downarrow \phi & & \\ \dots & \xrightarrow{d'_2} & Q_2 & \xrightarrow{d'_1} & Q_1 & \xrightarrow{d'_0} & Q_0 & \xrightarrow{q} & N & \longrightarrow & 0 \end{array}$$

commutes. Then there exist homomorphisms $s_i : P_i \rightarrow Q_{i+1}$ such that

$$g_i - f_i = \begin{cases} s_{i-1} \circ d_{i-1} + d'_i \circ s_i & i > 0 \\ d'_0 \circ s_0 & i = 0 \end{cases},$$

so

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{d_2} & P_2 & \xrightarrow{d_1} & P_1 & \xrightarrow{d_0} & P_0 & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \swarrow s_2 & & \swarrow s_1 & & \swarrow s_0 & & \downarrow \phi & & \\ \dots & \xrightarrow{d'_2} & Q_2 & \xrightarrow{d'_1} & Q_1 & \xrightarrow{d'_0} & Q_0 & \xrightarrow{q} & N & \longrightarrow & 0 \end{array}.$$

Proof.

1. The map $q : Q_0 \rightarrow N$ is surjective. There is a map $p : P_0 \rightarrow N$, given by composing $P_0 \rightarrow M$ with ϕ . Since P_0 is projective there exists $f_0 : P_0 \rightarrow Q_0$ such that $p = q \circ f_0$. Suppose the maps f_0, \dots, f_{t-1} have been constructed, so

$$\begin{array}{ccccccc} \dots & \xrightarrow{d_t} & P_t & \xrightarrow{d_{t-1}} & P_{t-1} & \xrightarrow{d_{t-2}} & P_{t-2} \longrightarrow \dots \\ & & \downarrow f_t & & \downarrow f_{t-1} & & \downarrow f_{t-2} \\ \dots & \xrightarrow{d'_t} & Q_t & \xrightarrow{d'_{t-1}} & Q_{t-1} & \xrightarrow{d'_{t-2}} & Q_{t-2} \longrightarrow \dots \end{array}$$

Observe that $d'_{t-2} \circ f_{t-1} \circ d_{t-1} = f_{t-2} \circ d_{t-2} \circ d_{t-1}$, since the existing squares of the diagram commute. But $d_{t-2} \circ d_{t-1} = 0$. So $d'_{t-2} \circ f_{t-1} \circ d_{t-1} = 0$, so $\text{Im}(f_{t-1} \circ d_{t-1}) \leq \text{Ker } d'_{t-2} = \text{Im } d'_{t-1}$. Now the map $d'_{t-1} : Q_t \rightarrow \text{Im } d'_{t-1}$ is obviously surjective, and P_t is projective. So there is a map $f_t : P_t \rightarrow Q_t$ such that $f_{t-1} \circ d_{t-1} = d'_{t-1} \circ f_t$. Now inductively, maps f_i exist for all i .

2. We want s_i such that $g_i - f_i = d'_i \circ s_i + s_{i-1} \circ d_{i-1}$. Let $h_i = g_i - f_i$. We see that the diagram

$$\begin{array}{ccccccccc} \dots & \xrightarrow{d_2} & P_2 & \xrightarrow{d_1} & P_1 & \xrightarrow{d_0} & P_0 & \xrightarrow{p} & M \longrightarrow 0 \\ & & \downarrow h_2 & & \downarrow h_1 & & \downarrow h_0 & & \downarrow 0 \\ \dots & \xrightarrow{d'_2} & Q_2 & \xrightarrow{d'_1} & Q_1 & \xrightarrow{d'_0} & Q_0 & \xrightarrow{q} & N \longrightarrow 0 \end{array}$$

commutes, since we want $h_i \circ d_i = d'_i \circ h_{i+1}$, but we have

$$h_i \circ d_i = g_i \circ d_i - f_i \circ d_i = d'_i \circ g_{i+1} - d'_i \circ f_{i+1} = d'_i \circ h'_{i+1},$$

so we are fine.

Base case. Let $x \in P_0$. Then $(q \circ h_0)(x) = (0 \circ p)(x) = 0$ so $\text{Im } h_0 \leq \text{Ker } q = \text{Im } d'_0$. We have a surjective map $d'_0 : Q_1 \rightarrow \text{Im } d'_0$, and a map $h_0 : P_0 \rightarrow \text{Im } d'_0$. Since P_0 is projective, there exists $s_0 : P_0 \rightarrow Q_1$ such that $h_0 = d'_0 \circ s_0$.

Inductive step. Suppose we have maps s_0, \dots, s_{t-1} , with $s_i : P_i \rightarrow Q_{i+1}$, and $h_i = d'_i \circ s_i + s_{i-1} \circ d_{i-1}$ for $i = 1, \dots, t-1$, so

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{d_{t+1}} & P_{t+1} & \xrightarrow{d_t} & P_t & \xrightarrow{d_{t-1}} & P_{t-1} & \xrightarrow{d_{t-2}} & P_{t-2} & \xrightarrow{d_{t-3}} & \dots \\ & & \downarrow h_{t+1} & \swarrow s_t & \downarrow h_t & \swarrow s_{t-1} & \downarrow h_{t-1} & \swarrow s_{t-2} & \downarrow h_{t-2} & \swarrow s_{t-3} & \\ \dots & \xrightarrow{d'_{t+1}} & Q_{t+1} & \xrightarrow{d'_t} & Q_t & \xrightarrow{d'_{t-1}} & Q_{t-1} & \xrightarrow{d'_{t-2}} & Q_{t-2} & \xrightarrow{d'_{t-3}} & \dots \end{array}$$

Look at $h_t - s_{t-1} \circ d_{t-1}$. We want to show that the image of this map is contained in $\text{Im } d'_t = \text{Ker } d'_{t-1}$. So check

$$\begin{aligned} d'_{t-1} \circ (h_t - s_{t-1} \circ d_{t-1}) &= d'_{t-1} \circ h_t - d'_{t-1} \circ s_{t-1} \circ d_{t-1} \\ &= h_{t-1} \circ d_{t-1} - (h_{t-1} - s_{t-2} \circ d_{t-2}) \circ d_{t-1} \\ &= h_{t-1} \circ d_{t-1} - h_{t-1} \circ d_{t-1} + s_{t-2} \circ d_{t-2} \circ d_{t-1}. \end{aligned}$$

Now $d_{t-2} \circ d_{t-1} = 0$, so we have $d'_{t-1} \circ (h_t - s_{t-1} \circ d_{t-1}) = 0$. So $h_t - s_{t-1} \circ d_{t-1} \in \text{Ker } d'_{t-1}$. Now we have the situation

$$\begin{array}{ccc} & P_t & \\ \swarrow s_t & \downarrow h_t - s_{t-1} \circ d_{t-1} & \\ Q_{t+1} & \xrightarrow{d'_t} & \text{Im } d'_t \end{array}$$

and since P_t is projective, there exists s_t such that $d'_t \circ s_t = h_t - s_{t-1} \circ d_{t-1}$, so $h_t = d'_t \circ s_t + s_{t-1} \circ d_{t-1}$ as required. □

5.4 Uniqueness of injective resolutions

The following is the equivalent result for injectives.

Proposition 5.13. *Let M and N be R -modules, and $\phi : M \rightarrow N$ a homomorphism. Let (I_t) be an injective resolution for M , and (J_t) another injective resolution for N . Then*

- *there exist maps $f_i : I_i \rightarrow J_i$ such that the diagram*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & I_0 & \xrightarrow{d_0} & I_1 & \xrightarrow{d_1} & I_2 & \xrightarrow{d_2} & \dots \\ & & \downarrow \phi & & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \\ 0 & \longrightarrow & N & \xrightarrow{j} & J_0 & \xrightarrow{d'_0} & J_1 & \xrightarrow{d'_1} & J_2 & \xrightarrow{d'_2} & \dots \end{array}$$

commutes, and

- *if (g_i) is another set of maps $g_i : I_i \rightarrow J_i$ such that the diagram*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & I_0 & \xrightarrow{d_0} & I_1 & \xrightarrow{d_1} & I_2 & \xrightarrow{d_2} & \dots \\ & & \downarrow \psi & & f_0 \downarrow & & f_1 \downarrow & & f_2 \downarrow & & \\ & & & & g_0 & & g_1 & & g_2 & & \\ 0 & \longrightarrow & N & \xrightarrow{j} & J_0 & \xrightarrow{d'_0} & J_1 & \xrightarrow{d'_1} & J_2 & \xrightarrow{d'_2} & \dots \end{array}$$

commutes, then there exist maps $s_i : I_{i+1} \rightarrow J_i$ such that

$$g_i - f_i = \begin{cases} s_i \circ d_i + d'_{i-1} \circ s_{i-1} & i > 0 \\ s_0 \circ d_0 & i = 0 \end{cases},$$

so

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & I_0 & \xrightarrow{d_0} & I_1 & \xrightarrow{d_1} & I_2 & \xrightarrow{d_2} & \dots \\ & & \downarrow \psi & & & & \swarrow s_0 & & \swarrow s_1 & & \swarrow s_2 \\ 0 & \longrightarrow & N & \xrightarrow{j} & J_0 & \xrightarrow{d'_0} & J_1 & \xrightarrow{d'_1} & J_2 & \xrightarrow{d'_2} & \dots \end{array}$$

Proof. Very similar to Proposition 5.12. □

Lecture 19 is a problems class.

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6 Chain and cochain complexes

6.1 Chain complexes

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Definition 6.1. A **chain complex** is a series $A_* = (A_i)$, with maps $d_i^A = d_i = d : A_{i+1} \rightarrow A_i$ such that $d^2 = 0$, that is $d_{i+1} \circ d_i = 0$, or $\text{Im } d_{i+1} \leq \text{Ker } d_i$.

Definition 6.2. A **cochain complex** is a series $A^* = (A^i)$ with maps $d_i^A = d_i = d : A^i \rightarrow A^{i+1}$ such that $d^2 = 0$, or $\text{Im } d_i \leq \text{Ker } d_{i+1}$.

Let A_* and B_* be chain complexes. Let $f = (f_i)$ be a family of R -module homomorphisms $f_i : A_i \rightarrow B_i$. Say that f is a **map of chain complexes** if $f \circ d = d \circ f$, that is $f_i \circ d_i^A = d_i^B \circ f_{i+1}$. So

$$\begin{array}{ccccccc} \dots & \xrightarrow{d_{n+1}^A} & A_{n+1} & \xrightarrow{d_n^A} & A_n & \xrightarrow{d_{n-1}^A} & A_{n-1} \xrightarrow{d_{n-2}^A} \dots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \dots & \xrightarrow{d_{n+1}^B} & B_{n+1} & \xrightarrow{d_n^B} & B_n & \xrightarrow{d_{n-1}^B} & B_{n-1} \xrightarrow{d_{n-2}^B} \dots \end{array}$$

commutes. Say that f **has property** \mathcal{P} if all f_i have property \mathcal{P} , where \mathcal{P} is injective, surjective, etc. A sequence

$$A_* \xrightarrow{f} B_* \xrightarrow{g} C_*$$

is **exact** at B_* if

$$A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$$

is exact at B_n for all n . A sequence of chain complexes is **exact** if it is exact everywhere. An **exact sequence**

$$0 \rightarrow A_* \rightarrow B_* \rightarrow C_* \rightarrow 0$$

is a short exact sequence of chain complexes.

6.2 Homology groups

Definition 6.3. Let A_* be a chain complex. The **n -th homology group** of A_* is $\text{Ker } d_{n-1} / \text{Im } d_n$. We write $H_n(A_*)$. Also write $H_*(A_*) = (H_n(A_*))$.

Definition 6.4. Let A^* be a cochain complex. The **n -th cohomology group** of A^* is $\text{Ker } d_n / \text{Im } d_{n-1}$. We write $H^n(A^*)$, and $H^*(A^*) = (H^n(A^*))$.

Example. Let $A_i = \mathbb{Z}^3$ for all i , and let $d(a, b, c) = (0, 0, a)$. Certainly $d^2 = 0$, so this is a chain complex. Then

$$\text{Ker } d = \{(0, b, c)\} = 0 \oplus \mathbb{Z}^2, \quad \text{Im } d = \{(0, 0, a)\} = 0^2 \oplus \mathbb{Z}.$$

Now

$$\text{Ker } d_{n-1} / \text{Im } d_n = \{(0, b, 0) + 0^2 \oplus \mathbb{Z}\}.$$

Proposition 6.5. A map of chain complexes $f : A_* \rightarrow B_*$ induces a map on the homology,

$$f_* : H_*(A_*) \rightarrow H_*(B_*),$$

given by

$$\begin{array}{ccccc} f_{*i} & : & H_i(A_*) & \longrightarrow & H_i(B_*) \\ & & x + \text{Im } d_i^A & \longmapsto & f_i(x) + \text{Im } d_i^B \end{array}$$

Proof. Let $x \in \text{Ker } d_{i-1}^A$. Then $(f_{i-1} \circ d_{i-1}^A)(x) = 0$, so $(d_{i-1}^B \circ f_i)(x) = 0$. Hence $f_i(x) \in \text{Ker } d_{i-1}^B$. So f_i certainly induces a map $\overline{f}_i : \text{Ker } d_{i-1}^A \rightarrow \text{Ker } d_{i-1}^B / \text{Im } d_i^B$. So there exists $y \in A_{i+1}$ with $d_i^A(y) = x$. Now $f_i(x) = (f_i \circ d_i^A)(y) = (d_i^B \circ f_{i+1})(y) \in \text{Im } d_i^B$, so $\overline{f}_i(x) = 0$. Hence $\text{Im } d_i^A \leq \text{Ker } \overline{f}_i$ so \overline{f}_i induces a map

$$\text{Ker } d_{i-1}^A / \text{Im } d_i^A = H_i(A_*) \rightarrow \text{Ker } d_{i-1}^B / \text{Im } d_i^B = H_i(B_*).$$

□

Let A_* and B_* be chain complexes, and let f and g be maps between them. We say that f and g are **equal up to homotopy** if there exist maps $s_i : A_i \rightarrow B_{i+1}$ such that

$$g_i - f_i = s_{i-1} \circ d_{i-1}^A + d_i^B \circ s_i.$$

Proposition 6.6. *If $f, g : A_* \rightarrow B_*$ are equal up to homotopy, then $f_* = g_*$, so f and g induce the same map on homology.*

Proof. Exercise. ⁵ □

6.3 The long exact sequence in homology

Proposition 6.7. *Let*

$$0 \rightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \rightarrow 0$$

be a short exact sequence. This induces a long exact sequence

$$\cdots \rightarrow H_{n+1}(A_*) \rightarrow H_{n+1}(B_*) \rightarrow H_{n+1}(C_*) \rightarrow H_n(A_*) \rightarrow H_n(B_*) \rightarrow H_n(C_*) \rightarrow \cdots$$

Proof. We have a commuting diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \longrightarrow 0 \\ & & \downarrow d_n^A & & \downarrow d_n^B & & \downarrow d_n^C \\ 0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n \longrightarrow 0 \end{array}.$$

Notice $\text{Im } d_n \leq \text{Ker } d_{n-1}$, so we can change this to

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \longrightarrow 0 \\ & & \downarrow d_n^A & & \downarrow d_n^B & & \downarrow d_n^C \\ 0 & \longrightarrow & \text{Ker } d_{n-1}^A & \xrightarrow{f_n} & \text{Ker } d_{n-1}^B & \xrightarrow{g_n} & \text{Ker } d_{n-1}^C \end{array}.$$

Now $\text{Im } d_{n+1} \leq \text{Ker } d_n$, so the maps $A_{n+1} \rightarrow \text{Ker } d_{n+1}$ induce maps $A_{n+1}/\text{Im } d_{n+1} \rightarrow \text{Ker } d_{n-1}$. So we get a diagram

$$\begin{array}{ccccccc} A_{n+1}/\text{Im } d_{n+1}^A & \xrightarrow{f_{n+1}} & B_{n+1}/\text{Im } d_{n+1}^B & \xrightarrow{g_{n+1}} & C_{n+1}/\text{Im } d_{n+1}^C & \longrightarrow & 0 \\ & & \downarrow \overline{d_n^A} & & \downarrow \overline{d_n^B} & & \downarrow \overline{d_n^C} \\ 0 & \longrightarrow & \text{Ker } d_{n-1}^A & \xrightarrow{f_n} & \text{Ker } d_{n-1}^B & \xrightarrow{g_n} & \text{Ker } d_{n-1}^C \end{array}.$$

We are now in the position to apply the snake lemma, so

$$\text{Ker } \overline{d_n^A} \rightarrow \text{Ker } \overline{d_n^B} \rightarrow \text{Ker } \overline{d_n^C} \rightarrow \text{Coker } \overline{d_n^A} \rightarrow \text{Coker } \overline{d_n^B} \rightarrow \text{Coker } \overline{d_n^C}$$

is an exact sequence. Then

$$\text{Ker } \overline{d_n^A} = \text{Ker } d_n^A / \text{Im } d_{n+1}^A = H_{n+1}(A_*), \quad \text{Coker } \overline{d_n^A} = \text{Ker } d_{n-1}^A / \text{Im } d_n^A = H_n(A_*).$$

Similarly for B_* and C_* . So we have an exact sequence

$$H_{n+1}(A_*) \rightarrow H_{n+1}(B_*) \rightarrow H_{n+1}(C_*) \rightarrow H_n(A_*) \rightarrow H_n(B_*) \rightarrow H_n(C_*).$$

Since consecutive values of i give a sequence overlapping in three terms we can glue them together, to give the long exact sequence in the proposition. □

⁵Exercise

7 Derived functors

7.1 Covariant and contravariant functors

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The following are two variations.

Definition 7.1. A **covariant functor** F from the category of left or right R -modules to the category of abelian groups is a map from R -modules to abelian groups such that if $\phi : M \rightarrow N$ is an R -module homomorphism then there exists an abelian group homomorphism

$$F(\phi) : F(M) \rightarrow F(N),$$

which respects identity maps, so $F(\text{id}_M) = \text{id}_{F(M)}$, and respects composition, so

$$F(\phi_1 \circ \phi_2) = F(\phi_1) \circ F(\phi_2).$$

The map F on homomorphisms is **additive** if $F(\phi_1 + \phi_2) = F(\phi_1) + F(\phi_2)$. If

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

be a short exact sequence, then F is **right exact** if

$$F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$$

is exact, **left exact** if

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C)$$

is exact. Then F is **exact** if both left and right exact.

Definition 7.2. A **contravariant functor** F from the category of left or right R -modules to the category of abelian groups is a map from R -modules to abelian groups such that if $\phi : M \rightarrow N$ is an R -module homomorphism then there exists an abelian group homomorphism

$$F(\phi) : F(N) \rightarrow F(M),$$

which respects identity maps, so $F(\text{id}_M) = \text{id}_{F(M)}$, and respects composition, so

$$F(\phi_1 \circ \phi_2) = F(\phi_2) \circ F(\phi_1).$$

Similarly, if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

be a short exact sequence, then F is **right exact** if

$$F(C) \rightarrow F(B) \rightarrow F(A) \rightarrow 0$$

is exact, and **left exact** if

$$0 \rightarrow F(C) \rightarrow F(B) \rightarrow F(A)$$

is exact.

Example. Some functors we have seen. Fix a left R -module M .

- $F(A) = \text{Hom}_R(M, A)$, where

$$\begin{array}{ccc} F(\phi) : F(A) = \text{Hom}_R(M, A) & \longrightarrow & F(B) = \text{Hom}_R(M, B) \\ f & \longmapsto & \phi \circ f \end{array}, \quad \phi : A \rightarrow B,$$

is covariant, left exact, and exact if and only if M is projective.

- $F(A) = \text{Hom}_R(A, M)$, where

$$\begin{array}{ccc} F(\phi) : F(B) = \text{Hom}_R(B, M) & \longrightarrow & F(A) = \text{Hom}_R(A, M) \\ f & \longmapsto & f \circ \phi \end{array}, \quad \phi : A \rightarrow B,$$

is contravariant, left exact, and exact if and only if M is injective.

- For a right R -module A , $F(A) = A \otimes_R M$ is covariant, right exact, and exact if and only if M is flat.

7.2 Left derived functors

Let F be the functor $F(A) = A \otimes_R M$, where M is a fixed R -module. Let $P_* \rightarrow A$ be a projective resolution for A . So $P_* = (P_i)_{i \geq 0}$ for projective P_i , and

$$\dots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{\phi} A \rightarrow 0$$

is exact. Consider the sequence

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0.$$

This is no longer exact, but it is a chain complex. And if we apply F , we get a chain complex $F(P_*)$,

$$\dots \rightarrow F(P_2) \rightarrow F(P_1) \rightarrow F(P_0) \rightarrow 0.$$

Define **left derived functors**

$$L_n F(A) = H_n(F(P_*)).$$

Theorem 7.3.

1. $L_n F(A)$ does not depend on the choice of resolution P_* .
2. $L_n F$ is an additive functor from right R -modules to abelian groups.
3. $L_0 F(A) = F(A)$.

Proof.

1. Let $P_* \rightarrow A$ and $Q_* \rightarrow A$ be projective resolutions. Then there exist maps of chain complexes $f : P_* \rightarrow Q_*$ and $g : Q_* \rightarrow P_*$. So $g \circ f : P_* \rightarrow P_*$, so

$$\begin{array}{ccccccccc} \dots & \xrightarrow{d_2} & P_2 & \xrightarrow{d_1} & P_1 & \xrightarrow{d_0} & P_0 & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow g_2 \circ f_2 & & \downarrow g_1 \circ f_1 & & \downarrow g_0 \circ f_0 & & \downarrow \text{id} \\ \dots & \xrightarrow{d_2} & P_2 & \xrightarrow{d_1} & P_1 & \xrightarrow{d_0} & P_0 & \longrightarrow & A \longrightarrow 0 \end{array},$$

and $g \circ f$ is equal to id up to homotopy. Apply F to everything. Since F is right exact,

$$\begin{array}{ccccccccc} \dots & \xrightarrow{F(d_2)} & F(P_2) & \xrightarrow{F(d_1)} & F(P_1) & \xrightarrow{F(d_0)} & F(P_0) & \longrightarrow & F(A) \longrightarrow 0 \\ & & \downarrow F(g_2) \circ F(f_2) & & \downarrow F(g_1) \circ F(f_1) & & \downarrow F(g_0) \circ F(f_0) & & \downarrow \text{id} \\ \dots & \xrightarrow{F(d_2)} & F(P_2) & \xrightarrow{F(d_1)} & F(P_1) & \xrightarrow{F(d_0)} & F(P_0) & \longrightarrow & F(A) \longrightarrow 0 \end{array}.$$

The diagram remains commutative, since F preserves composition. Now

$$g_i \circ f_i - \text{id} = s_{i-1} \circ d_{i-1} + d_i \circ s_i,$$

for suitable maps s_i . Then

$$F(g_i) \circ F(f_i) - \text{id} = F(s_{i-1}) \circ F(d_{i-1}) + F(d_i) \circ F(s_i).$$

So $F(g_i) \circ F(f_i)$ is id up to homotopy. Hence $F(g) \circ F(f)$ induces the identity on homology $H_*(F(P_*))$. Also $F(f) \circ F(g)$ induces the identity on $H_*(F(Q_*))$. Now we have

$$\overline{F(f_i)} : H_i(F(P_*)) \rightarrow H_i(F(Q_*)), \quad \overline{F(g_i)} : H_i(F(Q_*)) \rightarrow H_i(F(P_*)),$$

and $\overline{F(f_i)} \circ \overline{F(g_i)} = \text{id}$ and $\overline{F(g_i)} \circ \overline{F(f_i)} = \text{id}$, so $\overline{F(f_i)}$ and $\overline{F(g_i)}$ are isomorphisms. So

$$H_n(F(P_*)) \cong H_n(F(Q_*)),$$

as required. This argument tells us nothing about $H_0(F(P_*))$.

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2. Let $\phi : A \rightarrow B$. Let $P_* \rightarrow A$ and $Q_* \rightarrow B$ be projective resolutions. Then there exists $f : P_* \rightarrow Q_*$ such that

$$\begin{array}{ccccc} P_* & \longrightarrow & A & \longrightarrow & 0 \\ f \downarrow & & \downarrow \phi & & \\ Q_* & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

commutes. Then F is covariant and right exact. So

$$\begin{array}{ccccc} F(P_*) & \longrightarrow & F(A) & \longrightarrow & 0 \\ F(f) \downarrow & & \downarrow F(\phi) & & \\ F(Q_*) & \longrightarrow & F(B) & \longrightarrow & 0 \end{array}$$

is commutative, where $F(f) = (F(f_i))$. If $g : P_* \rightarrow Q_*$ is such that

$$\begin{array}{ccccc} P_* & \longrightarrow & A & \longrightarrow & 0 \\ g \downarrow & & \downarrow \phi & & \\ Q_* & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

commutes, then \bar{f} and \bar{g} , the induced maps on homology, are equal. So there exists a map $\overline{F(f_i)} : L_i F(A) \rightarrow L_i F(B)$, and is independent of the choice of f . So we can write $L_n F(\phi) = \overline{F(f_i)}$. Then $L_n F$ preserves identity and compositions and is additive, since F is an additive functor. ⁶

3. We have a short exact sequence

$$0 \rightarrow \text{Im } d_0 \xrightarrow{\subseteq} P_0 \xrightarrow{\phi} A \rightarrow 0.$$

Since F is right exact, we get an exact sequence

$$F(\text{Im } d_0) \rightarrow F(P_0) \rightarrow F(A) \rightarrow 0.$$

Now $d_0 : P_1 \rightarrow \text{Im } d_0$ is surjective, and F preserves surjectivity. So $F(d_0) : F(P_1) \rightarrow F(\text{Im } d_0)$ is surjective. So

$$F(P_1) \rightarrow F(P_0) \rightarrow F(A) \rightarrow 0$$

is exact. So, setting $P_{-1} = 0$, we get $L_0 F(P_*) = F(P_0) / \text{Im } F(d_0) = F(A)$.

□

⁶Exercise

7.3 The long exact sequence of left derived functors

Proposition 7.4 (Horseshoe lemma). *Suppose*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is an exact sequence of R -modules. Suppose $P_ \rightarrow A$ and $R_* \rightarrow C$ are projective resolutions. Define $Q_i = P_i \oplus R_i$. Then there exist maps $Q_{i+1} \rightarrow Q_i$ and $Q_0 \rightarrow B$ such that $Q_* \rightarrow B$ is a projective resolution, and such that*

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \iota & & \downarrow \iota & & \downarrow \iota & & \downarrow & & \\ \dots & \dashrightarrow & Q_2 & \dashrightarrow & Q_1 & \dashrightarrow & Q_0 & \dashrightarrow & B & \dashrightarrow & 0 \\ & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \downarrow & & \\ \dots & \longrightarrow & R_2 & \longrightarrow & R_1 & \longrightarrow & R_0 & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & 0 & & \end{array}$$

commutes, where if $x \in P_i$ and $y \in R_i$ then $\iota(x) = (x, 0)$ and $\pi(x, y) = y$.

Note. Q_i is a direct sum of projectives, so is itself projective.

Proof. We have the setup

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ P_0 & \xrightarrow{\phi} & A \longrightarrow 0 \\ \downarrow \iota & & \downarrow f \\ Q_0 & & B \\ \downarrow \pi & & \downarrow g \\ R_0 & \xrightarrow{\psi} & C \longrightarrow 0 \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}.$$

Since $B \rightarrow C$ is surjective, and R_0 is projective, there exists $h : R_0 \rightarrow B$ such that $g \circ h = \psi$. Now define

$$\chi : \begin{array}{ccc} Q_0 & \longrightarrow & B \\ (x, y) & \longmapsto & (f \circ \phi)(x) + h(y) \end{array}, \quad x \in P_0, \quad y \in R_0.$$

This construction guarantees that the squares are commutative. It is easy to see that χ is surjective, so

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ P_0 & \xrightarrow{\phi} & A \longrightarrow 0 \\ \downarrow \iota & & \downarrow f \\ Q_0 & \xrightarrow{\chi} B \dashrightarrow 0 \\ \downarrow \pi & \nearrow h & \downarrow g \\ R_0 & \xrightarrow{\psi} C \longrightarrow 0 \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}.$$

Now we have a short exact sequence

$$0 \rightarrow \text{Ker } \phi \xrightarrow{\iota} \text{Ker } \chi \xrightarrow{\pi} \text{Ker } \psi \rightarrow 0,$$

by the snake lemma. So now we can iterate, replacing A, B, C with these kernels, to construct a map $Q_1 \rightarrow Q_0$, and so on. \square

Proposition 7.5. *Let F be an additive functor, and let A and B be R -modules. There is a canonical isomorphism $F(A) \oplus F(B) \rightarrow F(A \oplus B)$.*

Proof. Let $M = A \oplus B$. Consider functions

$$\begin{array}{ccc} p_1 : M & \longrightarrow & M \\ (a, b) & \longmapsto & (a, 0) \end{array}, \quad \begin{array}{ccc} p_2 : M & \longrightarrow & M \\ (a, b) & \longmapsto & (0, b) \end{array}.$$

Then $p_i^2 = p_i$, $p_1 \circ p_2 = p_2 \circ p_1 = 0$, and $p_1 + p_2 = \text{id}_M$. If q_1 and q_2 are maps on a module M satisfying these relations, then $M = q_1(M) \oplus q_2(M)$. \square

Proposition 7.6. *Let*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

be a short exact sequence of right R -modules. This gives rise to a long exact sequence

$$\cdots \rightarrow L_n F(A) \rightarrow L_n F(B) \rightarrow L_n F(C) \rightarrow \cdots \rightarrow L_0 F(A) \rightarrow L_0 F(B) \rightarrow L_0 F(C) \rightarrow 0.$$

Proof. Let $P_* \rightarrow A$ be a projective resolution and $R_* \rightarrow C$ be a projective resolution. By the Horseshoe lemma, there exists a projective resolution $Q_* \rightarrow B$ such that

$$0 \rightarrow P_* \rightarrow Q_* \rightarrow R_* \rightarrow 0$$

is a split short exact sequence of chain complexes, that is $Q_i = P_i \oplus R_i$. Since $Q_i = P_i \oplus R_i$, and since F is an additive functor, we have $F(Q_i) = F(P_i) \oplus F(R_i)$. So

$$0 \rightarrow F(P_*) \rightarrow F(Q_*) \rightarrow F(R_*) \rightarrow 0$$

is a short exact sequence. Therefore we get a long exact sequence on homology,

$$\cdots \rightarrow H_n(F(P_*)) \rightarrow H_n(F(Q_*)) \rightarrow H_n(F(R_*)) \rightarrow \cdots$$

Since $H_n(F(P_*)) = L_n F(A)$ this gives the long sequence that we need. Since $L_0 F(A) = F(A)$, and F is right exact, the sequence terminates

$$L_0 F(A) \rightarrow L_0 F(B) \rightarrow L_0 F(C) \rightarrow 0,$$

as required. \square

7.4 General derived functors

Proposition 7.7.

- *Let F be any covariant, right exact, additive functor from left or right R -modules to abelian groups. Then the left derived functors $L_n F$ can be defined in just the same way as we did for the case $F(A) = A \otimes_R M$. All of the results we have proved follow in the more general case, by the same arguments.*
- *If F is a covariant, left exact, additive functor from R -modules to abelian groups, then we can define **right derived functors** $R^i F$ in a similar manner. Instead of working with a projective resolution, we use an injective resolution,*

$$0 \rightarrow A \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots$$

By similar arguments, we show that $R^i F(A)$ is independent of the choice of injective resolution. All of the results we proved for left derived functors have natural analogies for right derived functors. The argument requires a version of the Horseshoe lemma for injective resolutions, which is exactly what one might expect.

- *We can even construct derived functors for contravariant functors. If F is contravariant and right exact, so*

$$0 \rightarrow A \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots$$

is exact implies that

$$\cdots \rightarrow F(I_2) \rightarrow F(I_1) \rightarrow F(I_0) \rightarrow F(A) \rightarrow 0$$

is exact, we get a left derived functor, which is defined using an injective resolution. If F is contravariant and left exact, we get a right derived functor, which is defined using a projective resolution.

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8 Tor and Ext

8.1 Balancing theorems

Definition 8.1. Let F be the functor $F(A) = A \otimes_R B$. Then F is covariant, left exact, and additive. So $L_n F$ exists. Define

$$\mathrm{Tor}_i^R(A, B) = L_i F(A).$$

Fact. Let F' be the functor $F'(B) = A \otimes_R B$. Then F' is covariant, right exact, and additive. So $L_n F'$ exists. We have

$$L_i F'(B) \cong L_i F(A) = \mathrm{Tor}_i^R(A, B).$$

Definition 8.2. Let F be the functor $F(B) = \mathrm{Hom}_R(A, B)$. Then F is covariant, left exact, and additive, so $R^n F$ exists. Define

$$\mathrm{Ext}_R^i(A, B) = R^i F(B).$$

Fact. Let F' be the functor $F'(A) = \mathrm{Hom}_R(A, B)$. Then F' is contravariant, left exact, and additive, so $R^n F'$ exists. We have

$$R^i F'(A) \cong R^i F(B) = \mathrm{Ext}_R^i(A, B).$$

The two facts above are the **balancing theorems** for Tor and Ext. Their proof is beyond the scope of the course.

8.2 Tor

The following is an observation. Suppose A is projective. Then a projective resolution for A is

$$0 \rightarrow \cdots \rightarrow 0 \rightarrow A \xrightarrow{\mathrm{id}} A \rightarrow 0.$$

So $L_i F(A) = 0$ for $i \geq 1$, and $L_0 F(A) = F(A)$, for F possessing left derived functors. Similarly, if A is injective, then an injective resolution is

$$0 \rightarrow A \xrightarrow{\mathrm{id}} A \rightarrow 0 \rightarrow \cdots \rightarrow 0,$$

so $R^i F(A) = 0$ for $i \geq 1$, and $R^0 F(A) = F(A)$, for F possessing right derived functors. In fact the property $\mathrm{Tor}_i^R(A, B) = 0$ for all $i \geq 0$ characterises flat modules, so either A or B is flat.

Proposition 8.3. Let $F(A) = A \otimes_R B$. Then $L_i F(A) = \mathrm{Tor}_i^R(A, B) = 0$ for all $i \geq 1$ and for all A if and only if B is flat.

Similarly if $F'(B) = A \otimes_R B$, then $L_i F'(B) = 0$ for all $i \geq 1$ and for all B if and only if A is flat.

Proof.

\Leftarrow If B is flat then $F(A) = A \otimes_R B$ is exact, so

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

is exact implies that

$$0 \rightarrow F(L) \rightarrow F(M) \rightarrow F(N) \rightarrow 0$$

is exact, or F maps kernels to kernels and cokernels to cokernels. Let $P_* \rightarrow A$ be a projective resolution. Then

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0$$

is exact everywhere except P_0 . So

$$\cdots \rightarrow F(P_2) \rightarrow F(P_1) \rightarrow F(P_0) \rightarrow 0$$

is exact everywhere except $F(P_0)$. So $L_n F(P_*) = 0$ for $n \geq 1$. But $L_n F(P_*) = \mathrm{Tor}_n^R(A, B)$.

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\Rightarrow Conversely, suppose $\text{Tor}_i^R(A, B) = 0$ for all A . Let

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

be exact. This gives a long exact sequence of homology groups

$$\begin{array}{ccccccc} \dots & \rightarrow & L_1 F(L) & \rightarrow & L_1 F(M) & \rightarrow & L_1 F(N) \rightarrow L \otimes_R B \rightarrow M \otimes_R B \rightarrow N \otimes_R B \rightarrow 0 \\ & & & & & & \parallel \\ & & & & & & \text{Tor}_1^R(N, B) \end{array}$$

Since $\text{Tor}_1^R(N, B) = 0$, we get a short exact sequence

$$0 \rightarrow L \otimes_R B \rightarrow M \otimes_R B \rightarrow N \otimes_R B \rightarrow 0.$$

So $F(A) = A \otimes_R B$ is left exact, and so B is flat. □

Proposition 8.4. *Let A and B be abelian groups. Then $\text{Tor}_n^{\mathbb{Z}}(A, B) = 0$ for $n > 1$.*

Proof. A is a quotient of some free module K , say

$$K \xrightarrow{f} A \rightarrow 0.$$

Now $\text{Ker } f \leq K$, and since \mathbb{Z} is a PID, $\text{Ker } f$ is free, since it is a submodule of a free module. So

$$\dots \rightarrow 0 \rightarrow \text{Ker } f \rightarrow K \rightarrow A \rightarrow 0$$

is a projective resolution for A . Since all of the modules above P_1 in the resolution are zero, clearly $H_n(P_*) = 0$ for $n > 1$. □

Fact. $\text{Tor}_1^{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) = T(A) = \{a \in A \mid a \text{ has finite order}\}$. Proof omitted.

8.3 Ext

Proposition 8.5. *Let A and B be abelian groups. Then $\text{Ext}_R^n(A, B) = 0$ for $n > 1$.*

Proof. Problem sheet question. □

More generally, $\text{Ext}_R^1(A, C)$ tells us about **extensions** of C by A , that is B such that

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0.$$

Let B_1 and B_2 be two extensions of C by A . Write $B_1 \sim B_2$ if there exists $f : B_1 \rightarrow B_2$ such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \begin{array}{c} \nearrow \alpha_1 \\ \searrow \alpha_2 \end{array} & \begin{array}{c} B_1 \\ \downarrow f \\ B_2 \end{array} & \begin{array}{c} \nwarrow \beta_1 \\ \nearrow \beta_2 \end{array} & C \longrightarrow 0 \end{array}$$

commutes.

Proposition 8.6. *Any such f is an isomorphism.*

Proof.

- f is surjective. Suppose $y \in B_2$. Then $\beta_2(y) \in C$, and β_1 is surjective, so $\beta_2(y) = \beta_1(x)$ for some $x \in B_1$. Now $f(x) - y \in \text{Ker } \beta_2 = \text{Im } \alpha_2$, so $f(x) - y = \alpha_2(a)$ for some $a \in A$. So $f(x) - y = (f \circ \alpha_1)(a)$, and so $y = f(x) - (f \circ \alpha_1)(a) = f(x - \alpha_1(a))$.
- f is injective. Suppose $f(x) = f(y)$ for $x, y \in B_1$. Then $f(x - y) = 0$, so $(\beta_2 \circ f)(x - y) = 0$, so $\beta_1(x - y) = 0$. So $x - y \in \text{Ker } \beta_1 = \text{Im } \alpha_1$, so $x - y = \alpha_1(a)$ for some $a \in A$. Now $\alpha_2(a) = (f \circ \alpha_1)(a) = f(x - y) = 0$. But α_2 is injective, so $a = 0$, so $x - y = 0$. □

Hence the relation \sim is an equivalence relation. Write $E_c(A)$ for the set of \sim -equivalence classes. We will put an abelian group structure on $E_c(A)$. Let B_1 and B_2 be extensions, so

$$0 \rightarrow A \xrightarrow{\alpha_1} B_1 \xrightarrow{\beta_1} C \rightarrow 0, \quad 0 \rightarrow A \xrightarrow{\alpha_2} B_2 \xrightarrow{\beta_2} C \rightarrow 0.$$

Define maps α^* and β^* by

$$\begin{aligned} \alpha^* : A &\longrightarrow B_1 \oplus B_2 \\ a &\longmapsto (\alpha_1(a), -\alpha_2(a)) \end{aligned} \quad , \quad \begin{aligned} \beta^* : B_1 \oplus B_2 &\longrightarrow C \\ (b_1, b_2) &\longmapsto \beta_1(b_1) - \beta_2(b_2) \end{aligned} .$$

Now $\beta^* \circ \alpha^* = 0$. So

$$0 \rightarrow A \xrightarrow{\alpha^*} B_1 \oplus B_2 \xrightarrow{\beta^*} C \rightarrow 0$$

is a chain complex. Define H , the **Baer sum** of $[B_1]$ and $[B_2]$, to be the homology group at $B_1 \oplus B_2$, that is $H = \text{Ker } \beta^* / \text{Im } \alpha^*$. Now define maps

$$\begin{aligned} \alpha : A &\longrightarrow H \\ a &\longmapsto (\alpha_1(a), 0) + \text{Im } \alpha^* \end{aligned} \quad , \quad \begin{aligned} \beta : H &\longrightarrow C \\ (x, y) + \text{Im } \alpha^* &\longmapsto \beta_1(x) - \beta_2(y) \end{aligned} .$$

Claim that

$$0 \rightarrow A \xrightarrow{\alpha} H \xrightarrow{\beta} C \rightarrow 0$$

is a short exact sequence.