

M4P57 Complex Manifolds

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Spring 2020

Syllabus

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1 Introduction

Lecture 1
Thursday
09/01/20

The following are references.

- O Biquard and A Höring, Kähler geometry and Hodge theory, 2008.
- J P Demailly, Complex analytic and differential geometry, 2012.
- C Voisin, Hodge theory and complex algebraic geometry, 2002.
- R O Wells, Differential analysis on complex manifolds, 1973.
- A Gathmann, Algebraic geometry, 2002
- P Griffiths and J Harris, Principles of algebraic geometry, 1978.

Complex manifolds are manifolds over \mathbb{C}^n .

Example 1.1. \mathbb{C}^1 is a complex manifold. Any open $U \subset \mathbb{C}^n$ is a complex manifold.

Example 1.2. The sphere $S^2 \subset \mathbb{R}^3$ is a complex manifold by

$$S^2 \cong \mathbb{C} \cup \{\infty\} = \mathbb{P}_{\mathbb{C}}^1 = \mathbb{CP}^1.$$

More in general $\mathbb{P}_{\mathbb{C}}^n$ is a complex manifold for all n .

Example 1.3. The torus

$$S^1 \times S^1 = \mathbb{R}^2 / \mathbb{Z}^2 = \mathbb{C} / \mathbb{Z}^2$$

is a complex manifold. More in general a $2n$ -dimensional torus \mathbb{C}^n / Λ for a lattice $\Lambda \cong \mathbb{Z}^{2n}$ is a complex manifold.

Example 1.4. Compact Riemannian surfaces of genus $g > 1$, called **hyperbolics**, are all complex manifolds.

Example 1.5. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic. The graph of f ,

$$\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{C}\} \subset \mathbb{C} \times \mathbb{C},$$

is a complex manifold. From Γ_f we can recover f , by

$$f(x) = q(p^{-1}(x) \cap \Gamma_f),$$

where $p, q : \mathbb{C}^2 \rightarrow \mathbb{C}$ are the projections to the first and second factors. This allows us to define f^{-1} . Assume f is bijective. Define

$$\tau : \begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ (x, y) & \longmapsto & (y, x) \end{array}.$$

Define

$$\Gamma_{f^{-1}} = \tau(\Gamma_f).$$

Then f^{-1} is the function induced by $\Gamma_{f^{-1}}$. This makes sense even if f is not bijective. Then we get a multivalued function, such as $\log z$ as the inverse of $\exp z$.

Example 1.6. Generalising Example 1.5, we can consider two complex manifolds M and N and we can consider holomorphisms $f : M \rightarrow N$. Given M ,

$$\text{Aut } M = \{f : M \rightarrow M \text{ holomorphic bijective and } f^{-1} \text{ holomorphic}\}.$$

If $M = \mathbb{C}$, there are lots of C^∞ -functions $\mathbb{C} \rightarrow \mathbb{C}$ but the automorphisms of \mathbb{C} are just affine linear maps. If $M = \mathbb{C}/\mathbb{Z}^2$, then $\text{Aut } M$ is interesting.

Example 1.7. Algebraic geometry is the zeroes of polynomials. That is, fix m , and take polynomials f_1, \dots, f_k in m variables. Define

$$M = \{(x_1, \dots, x_m) \in \mathbb{C}^m \mid f_1(x_1, \dots, x_m) = \dots = f_k(x_1, \dots, x_m) = 0\}.$$

Then M is called an **algebraic variety**. If M is smooth then M is a complex manifold. Fix m , take homogeneous polynomials F_1, \dots, F_k in $m + 1$ variables, where F is **homogeneous** if it is the sum of monomials of the same degree. Consider

$$N = \{(x_0, \dots, x_m) \in \mathbb{P}_{\mathbb{C}}^m \mid F_1(x_0, \dots, x_m) = \dots = F_k(x_0, \dots, x_m) = 0\}.$$

Then N is called a **projective variety**. If N is smooth then N is a complex manifold.

The idea is if M is a differentiable manifold, then M contains lots of submanifolds N . This is not true for complex manifolds. There exist complex manifolds without any proper complex submanifolds, which is not true for projective varieties. The following are questions.

- What can we say about the topology of complex manifolds? For example, what is $\pi_1(M)$? What is the cohomology of M ?
- Assume that M and N are complex manifolds which are diffeomorphic. Are they also isomorphic, so there exists a biholomorphism $M \rightarrow N$?

What is next?

- Hodge decomposition theorem. Understand the cohomology of M by using the complex structure.
- Kodaira embedding theorem. Understand when a compact complex manifold is projective.

Note. If $M \subset \mathbb{P}_{\mathbb{C}}^m$ is a compact complex manifold then M is projective.

Example. Let $M = \Gamma_{\exp}$ for $\exp : \mathbb{C} \rightarrow \mathbb{C}$. Then $M \subset \mathbb{C}^2$ but it is not algebraic.

2 Local theory

2.1 Holomorphic functions in several variables

Notation 2.1. Given $z_0 \in \mathbb{C}$ and $r > 0$, the **disc** is

$$D(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$

and $\partial D(z_0, r)$ is the boundary of $D(z_0, r)$.

Definition 2.2. Let $U \subset \mathbb{C}$, and let $f : U \rightarrow \mathbb{C}$ be a function. Then f is **holomorphic at** $z_0 \in U$ if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Theorem 2.3 (Cauchy). *Let $U \subset \mathbb{C}$ be open, let f be holomorphic on U , and let $z_0 \in U$. Assume that if $D = D(z_0, r) \subset U$ then $\overline{D} \subset U$. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz.$$

Notation 2.4. Fix $z_0 = (z_{01}, \dots, z_{0n}) \in \mathbb{C}^n$ and $R = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$. Then the **polydisc** is

$$D(z_0, R) = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i - z_{0i}| < r_i \text{ for each } i\},$$

where R is the **polyradius**.

Definition 2.5. Let $U \subset \mathbb{C}^n$ be open, let $f : U \rightarrow \mathbb{C}$ be a continuous function, and let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Then f is **holomorphic at** z , if assuming that $D = D(z, R) \subset U$ for some $R = (r_1, \dots, r_n)$ then

$$f(z_1, \dots, z_{i-1}, \cdot, z_{i+1}, \dots, z_n) : D(z_i, r_i) \rightarrow \mathbb{C}$$

is holomorphic for all i .

Example 2.6. Any convergent power series in n -variables is holomorphic.

The opposite is also true.

Theorem 2.7 (Cauchy). *Let $U \subset \mathbb{C}^n$ be an open set, let $f : U \rightarrow \mathbb{C}$ be holomorphic, and let $z = (z_1, \dots, z_n) \in U$. Assume that if $D = D(z, R)$ for some $R = (r_1, \dots, r_n)$ then $\overline{D} \subset U$. If $z' = (z'_1, \dots, z'_n) \in D$ then*

$$f(z') = \frac{1}{(2\pi i)^n} \int_{\partial D(z_1, r_1)} \cdots \int_{\partial D(z_n, r_n)} \frac{f(z)}{(z - z'_1) \cdots (z - z'_n)} dz_n \cdots dz_1.$$

Proof. Use induction on n and Cauchy theorem at each step. □

Corollary 2.8. *Let $U \subset \mathbb{C}^n$ be open, let $f : U \rightarrow \mathbb{C}$ be holomorphic, and let $z = (z_1, \dots, z_n) \in U$. Then there exists $D = D(z, R) \subset U$ for some $R = (r_1, \dots, r_n)$ and there exists*

$$p(w) = \sum_{m_1, \dots, m_n \geq 0} a_{m_1, \dots, m_n} (w_1 - z_1)^{m_1} \cdots (w_n - z_n)^{m_n},$$

such that p is convergent on D and $f(w) = p(w)$ inside D .

Proof. The idea is to use Theorem 2.7 and $1/(1-w) = \sum_{k \geq 0} w^k$. □

Definition 2.9. Let $U \subset \mathbb{C}^n$ be open. Then $f : U \rightarrow \mathbb{C}^m$ is **holomorphic** if $f_i = p_i \circ f$ is holomorphic for any $i = 1, \dots, m$ where $p_i : \mathbb{C}^m \rightarrow \mathbb{C}$ is the i -th projection, so $f = (f_1, \dots, f_m)$.

Fact. If $f : U \rightarrow \mathbb{C}^m$ is holomorphic and $g : V \rightarrow \mathbb{C}^p$ is holomorphic where $V \supset f(U)$ then $g \circ f$ is holomorphic.

Definition 2.10. Let $U \subset \mathbb{C}^n$ be open. A holomorphic function $f : U \rightarrow \mathbb{C}^m$ is **biholomorphic at** $z_0 \in U$ if there exists an open neighbourhood $V \subset U$ of z_0 such that $f : V \rightarrow f(V)$ is bijective and $f^{-1} : f(V) \rightarrow V$ is holomorphic. Then f is **biholomorphic** if f is bijective and f is biholomorphic at any point.

Note. $f(V)$ is automatically open in \mathbb{C}^m if $m = n$.

Example 2.11. Let $\Phi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be linear such that $\det \Phi \neq 0$. Then Φ is a biholomorphism.

Example 2.12. Let $U = \mathbb{C} \setminus \{0\}$ and

$$\begin{array}{ccc} f & : & U \longrightarrow U \\ z & \longmapsto & z^2 \end{array}.$$

Check that f is biholomorphic at any point of U but f is not biholomorphic.

Remark. $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Then a holomorphic $f : U \subset \mathbb{C}^n \rightarrow \mathbb{C}^m$ is also a diffeomorphism $U \subset \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2m}$.

Theorem 2.13 (Hartogs). *Let $n \geq 2$, let $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ and $\delta = (\delta_1, \dots, \delta_n)$ such that $\epsilon_i > \delta_i > 0$, let $U = D(0, \epsilon) \setminus \overline{D(0, \delta)}$, and let $f : U \rightarrow \mathbb{C}^m$ be holomorphic. Then there exists a holomorphic $\bar{f} : D(0, \epsilon) \rightarrow \mathbb{C}^m$ such that $\bar{f}|_U = f$.*

Example. Hartogs theorem is false for $n = 1$. If $f(z) = 1/z$, for all $\epsilon > \delta > 0$, then f cannot be extended.

2.2 Cauchy formula in one variable

Let $\omega = x + iy \in \mathbb{C}$ for $x, y \in \mathbb{R}$, and let $f : U \rightarrow \mathbb{C}$ be C^∞ for some $U \subset \mathbb{C}$. Recall that

$$\frac{\partial f}{\partial \omega} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f, \quad \frac{\partial f}{\partial \bar{\omega}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f.$$

Recall that f is holomorphic if and only if $\frac{\partial f}{\partial \bar{\omega}} = 0$ on U . More in general, let $U \subset \mathbb{C}^n$ be open, let $z_i = x_i + iy_i$, and let $f : U \rightarrow \mathbb{C}$ be a C^∞ -function. Then f is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}_i} = 0$ for all $i = 1, \dots, n$. Let $\omega \in \mathbb{C}$. Since $dx \wedge dy = -dy \wedge dx$, let

$$dA = \frac{i}{2} d\omega \wedge d\bar{\omega} = \frac{i}{2} (dx + idy) \wedge (dx - idy) = dx \wedge dy,$$

which is the Lebesgue measure on $\mathbb{R}^2 \cong \mathbb{C}$.

Proposition 2.14. *Let $f : U \rightarrow \mathbb{C}$ for $U \subset \mathbb{C}$ be a C^∞ -function, and let $D = D(z, r)$ such that $\bar{D} \subset U$. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f}{\omega - z} d\omega - \frac{1}{\pi} \int_D \frac{1}{\omega - z} \frac{\partial f}{\partial \bar{\omega}} dA.$$

Proof. Assume $z = 0$. Recall that $f(\omega) = 1/\omega$ is locally integrable around zero, so

$$\int_D \frac{1}{\omega} \frac{\partial f}{\partial \bar{\omega}} dA = \lim_{\epsilon \rightarrow 0} \int_{D \setminus D(0, \epsilon)} \frac{1}{\omega} \frac{\partial f}{\partial \bar{\omega}} dA.$$

Away from zero

$$\begin{aligned} d \left(\frac{f}{\omega} d\omega \right) &= \frac{1}{\omega} df \wedge d\omega + f d \left(\frac{1}{\omega} \right) \wedge d\omega = \frac{1}{\omega} \left(\frac{\partial f}{\partial \omega} d\omega + \frac{\partial f}{\partial \bar{\omega}} d\bar{\omega} \right) \wedge d\omega + f \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \right) d\omega \wedge d\omega \\ &= \frac{1}{\omega} \frac{\partial f}{\partial \bar{\omega}} d\bar{\omega} \wedge d\omega = \frac{2i}{\omega} \frac{\partial f}{\partial \bar{\omega}} dA. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{\pi} \int_D \frac{1}{\omega} \frac{\partial f}{\partial \bar{\omega}} dA &= \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{D \setminus D(0, \epsilon)} \frac{1}{\omega} \frac{\partial f}{\partial \bar{\omega}} dA \\ &= \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \int_{D \setminus D(0, \epsilon)} d \left(\frac{f}{\omega} d\omega \right) & \frac{1}{\omega} \frac{\partial f}{\partial \bar{\omega}} dA &= \frac{1}{2i} d \left(\frac{f}{\omega} d\omega \right) \\ &= \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \left(\int_{\partial D} \frac{f}{\omega} d\omega - \int_{\partial D(0, \epsilon)} \frac{f}{\omega} d\omega \right) & \text{Stokes' theorem} \\ &= \frac{1}{2\pi i} \left(\int_{\partial D} \frac{f}{\omega} d\omega - 2\pi i f(0) \right) & \lim_{\epsilon \rightarrow 0} \int_{\partial D(0, \epsilon)} \frac{1}{\omega} d\omega &= 2\pi i. \end{aligned}$$

□

If f is holomorphic, then $\frac{\partial f}{\partial \bar{\omega}} = 0$, which implies Theorem 2.3.

2.3 Rank theorem

Let $U \subset \mathbb{C}^n$ be open, and let $f : U \rightarrow \mathbb{C}^m$ be holomorphic. Then the **Jacobian** is

$$J_f = \left(\frac{\partial f_j}{\partial z_i}(z) \right),$$

where $f_j = p_j \circ f$ and $p_j : \mathbb{C}^m \rightarrow \mathbb{C}$ is the j -th projection.

Exercise. Show that the real Jacobian, which is $2n \times 2n$, has non-negative determinants.

Theorem 2.15 (Rank theorem). *Let $z \in U$ such that $r = \text{rk } J_f(z')$ is constant around z . Then there exist open $z \in V \subset U \subset \mathbb{C}^n$ and $f(z) \in W \subset f(U) \subset \mathbb{C}^m$ such that $\phi : D(0, 1)^n \rightarrow V$ and $\psi : D(0, 1)^m \rightarrow W$ are biholomorphisms such that*

$$\eta = \psi^{-1} \circ f \circ \phi : \begin{array}{ccc} D(0, 1)^n & \longrightarrow & D(0, 1)^m \\ (z_1, \dots, z_n) & \longmapsto & (z_1, \dots, z_r, 0, \dots, 0) \end{array},$$

so

$$\begin{array}{ccccc} \mathbb{C}^n \supset U & \supset & V \ni z & \xrightarrow{f} & f(z) \in W \subset f(U) \subset \mathbb{C}^m \\ & & \uparrow \phi & & \uparrow \psi \\ & & D(0, 1)^n & \xrightarrow{\eta} & D(0, 1)^m \end{array}.$$

Corollary 2.16 (Inverse function theorem). *Let $f : U \rightarrow \mathbb{C}^n$ be holomorphic for $U \subset \mathbb{C}^n$, and let $z \in U$ such that $\det J_f(z) \neq 0$. Then f is a biholomorphism at z .*

Proof. $\det J_f(z) \neq 0$ if and only if $\text{rk } J_f(z) = n$, so $\text{rk } J_f(z') = n$ around z , since $\det J_f(z)$ is a continuous function. Let ϕ and ψ as in the theorem. Then $\eta = \psi^{-1} \circ f \circ \phi = \text{id}$, so on V , $f = \psi \circ \phi^{-1}$ is a composition of biholomorphisms, which is a biholomorphism. \square

Remark 2.17. Let $f : U \rightarrow \mathbb{C}^n$ for $U \subset \mathbb{C}^n$. Then $\det J_f(z)$ is a holomorphism, so

$$Z = \{z \in U \mid \det J_f(z) = 0\}$$

is closed.

2.4 Holomorphic differential forms

Let $U \subset \mathbb{C}^n$ be open.

Definition 2.18. A **holomorphic vector field** on U is the expression

$$X = \sum_i a_i \frac{\partial}{\partial z_i},$$

where a_i are holomorphic functions on U .

For all $x \in U$, the **tangent space** is

$$T_x U = \left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle \cong \mathbb{C}^n.$$

If $x \in U$, then $X(x) \in T_x U$.

Notation 2.19.

$$H^0(U, \mathcal{O}_U) = \{\text{holomorphic functions } f : U \rightarrow \mathbb{C}\}, \quad H^0(U, T_U) = \{\text{holomorphic vector fields on } U\}.$$

Remark. $R = H^0(U, \mathcal{O}_U)$ is a ring and $M = H^0(U, T_U)$ is a module over R . That is, if $X \in H^0(U, T_U)$ and $f \in H^0(U, \mathcal{O}_U)$, then $fX \in H^0(U, T_U)$.

Definition 2.20. Let R be a ring and M be an R -module for $p \geq 1$. The p -th exterior power $\Lambda^p M$ of M is the R -module $M^{\otimes p}$ with the relations

$$m_1 \otimes \cdots \otimes m_p - \epsilon(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(p)}, \quad m_1, \dots, m_p \in M, \quad \sigma \in \mathcal{S}_p,$$

where $\epsilon(\sigma) = (-1)^m$ is the signature of σ and m is the number of transpositions defining σ . Then $M^* = \text{Hom}_R(M, R)$ is the **dual** of M as an R -module.

Let $R = H^0(U, \mathcal{O}_U)$ and $M = H^0(U, T_U)$.

Definition 2.21. Let $p > 0$. We define a **holomorphic p -form**, as an element of

$$H^0(U, \Omega_U^p) = \Lambda^p M^*.$$

If $p = 0$, by convention a **holomorphic 0-form** is just an element in R .

Let (z_1, \dots, z_n) be coordinates for U . Recall $\eta \in M$ is given by $\eta = \sum_i a_i \frac{\partial}{\partial z_i}$ for holomorphic functions $a_i \in R$. Then $\omega \in M^*$ is given by the expression

$$\sum_i b_i dz_i, \quad b_i \in R, \quad dz_i \left(\frac{\partial}{\partial z_j} \right) = \delta_{ij}.$$

More in general $\omega \in H^0(U, \Omega_U^p)$ is given by

$$\omega = \sum_{|I|=p} f_I dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad f_I \in R, \quad I = (i_1, \dots, i_p), \quad i_1 < \cdots < i_p,$$

where $dz_{i_1}, \dots, dz_{i_p}$ is an R -basis of $H^0(U, \Omega_U^p)$.

Example.

$$H^0(U, \Omega_U^p) \cong \Lambda^p H^0(U, \Omega_U^1)$$

is an isomorphism as R -modules. This is not true for complex manifolds in general.

The **exterior product** is

$$\begin{aligned} H^0(U, \Omega_U^p) \otimes H^0(U, \Omega_U^q) &\longrightarrow H^0(U, \Omega_U^{p+q}) \\ \omega_1 \otimes \omega_2 &\longmapsto \omega_1 \wedge \omega_2 \end{aligned},$$

where we just need to define

$$\omega_1 \wedge \omega_2 = f dz_{i_1} \wedge dz_{i_p} \otimes g dz_{j_1} \wedge dz_{j_q} = f g dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_q},$$

by linearity. Then $\omega_1 \wedge \omega_2 = 0$ if $\{i_1, \dots, i_p\} \cap \{j_1, \dots, j_q\} \neq \emptyset$, since $dz_i \wedge dz_i = 0$.

Exercise. Check that this definition coincides with the definition in M4P54.

The **exterior derivative** is

$$\begin{aligned} d : H^0(U, \Omega_U^p) &\longrightarrow H^0(U, \Omega_U^{p+1}) \\ f dz_{i_1} \wedge \cdots \wedge dz_{i_p} &\longmapsto \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p}. \end{aligned}$$

By definition d is \mathbb{C} -linear, but not R -linear. That is,

$$d(a\omega_1 + b\omega_2) = ad\omega_1 + bd\omega_2, \quad \omega_1, \omega_2 \in H^0(U, \Omega_U^p), \quad a, b \in \mathbb{C}.$$

Proposition 2.22. Let $U \subset \mathbb{C}^n$ be open. Then

- the Leibnitz rule

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2, \quad \omega_1 \in H^0(U, \Omega_U^p), \quad \omega_2 \in H^0(U, \Omega_U^q),$$

- $d^2 = 0$, that is

$$d(d\omega) = 0, \quad \omega \in H^0(U, \Omega_U^p).$$

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Definition 2.23. Let $f : U \subset \mathbb{C}^n \rightarrow \mathbb{C}^m$ be holomorphic, let $f_i = p_i \circ f : U \rightarrow \mathbb{C}$ where $p_i : \mathbb{C}^m \rightarrow \mathbb{C}$ is the i -th projection, and let $f(U) \subset V \subset \mathbb{C}^m$ be open. Then if

$$\omega = h dz_{i_1} \wedge \cdots \wedge dz_{i_p} \in H^0(V, \Omega_V^p), \quad h \in H^0(U, \mathcal{O}_U),$$

then we can define the **pull-back** of ω ,

$$f^*\omega = h \circ f df_{i_1} \wedge \cdots \wedge df_{i_p} \in H^0(U, \Omega_U^p),$$

since $f_i \in H^0(V, \mathcal{O}_V) = H^0(V, \Omega_V^0)$ implies that $df_i \in H^0(V, \Omega_V^1)$, so

$$\begin{array}{ccc} U & \xrightarrow{f} & f(U) \subset V \\ & \searrow h \circ f \in H^0(U, \mathcal{O}_U) & \downarrow h \\ & & \mathbb{C} \end{array} .$$

This is linear over \mathbb{C} and over $H^0(U, \mathcal{O}_U)$.

Proposition 2.24. Let $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$, and $W \subset \mathbb{C}^{m'}$ be open, let $f : U \rightarrow \mathbb{C}^m$ and $g : V \rightarrow \mathbb{C}^{m'}$ be holomorphic such that $V \supset f(U)$ and $W \supset g(V)$, and let $\omega \in H^0(V, \Omega_V^p)$ and $\eta \in H^0(W, \Omega_W^q)$. Then

- $f^*(\omega + \eta) = f^*\omega + f^*\eta$ if $p = q$,
- $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$,
- $df^*\omega = f^*d\omega$, and
- $f^*g^*\omega = (g \circ f)^*\omega$.

Let $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, and let $z_i = x_i + iy_i$ for $i = 1, \dots, n$ and $x_i, y_i \in \mathbb{R}$. Then

$$dz_i = dx_i + idy_i,$$

so any holomorphic form is a differentiable form on \mathbb{R}^{2n} . A (p, q) -**form** is a differentiable $(p + q)$ -form of the expression

$$\omega = \sum_{|I|=p, |J|=q} f_{I,J} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}, \quad f_{I,J} : U \rightarrow \mathbb{C} \cong \mathbb{R}^2 \in C^\infty,$$

where $d\bar{z}_j = dx_j - idy_j$. We denote

$$C^\infty(U, \Omega_U^{p,q}) = \{\text{differentiable } (p + q)\text{-forms on } U\}.$$

If ω is a (p, q) -form, then the **conjugate** $\bar{\omega}$ of ω is the (q, p) -form defined by

$$\bar{\omega} = \sum_{|I|=p, |J|=q} \overline{f_{I,J}} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_q}.$$

3 Complex manifolds

3.1 Complex manifolds

Definition 3.1. A **complex manifold** of dimension n is a connected Hausdorff topological space X , with a countable open cover $\{U_\alpha\}$ of X such that for all α , there exists $\phi_\alpha : U_\alpha \rightarrow \mathbb{C}^n$ such that $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$ is a homeomorphism and

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta)$$

is a biholomorphism for each α and β , so

$$\begin{array}{ccc} & U_\alpha \cap U_\beta & \\ \phi_\alpha \swarrow & & \searrow \phi_\beta \\ \mathbb{C}^n \supset \phi_\alpha(U_\alpha \cap U_\beta) & \xrightarrow{\phi_\alpha \circ \phi_\beta^{-1}} & \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n \end{array} .$$

The pair (U_α, ϕ_α) is called a **holomorphic chart**. The set $\{(U_\alpha, \phi_\alpha)\}$ is called a **holomorphic atlas** or a **complex structure**.

Recall X is Hausdorff if for all $x, y \in X$ there exist U and V open in X such that $U \cap V = \emptyset$ and $x \in U$ and $y \in V$.

Example 3.2.

- If $U \subset \mathbb{C}^n$ is an open set then U is a complex manifold. More in general if X is a complex manifold and $U \subset X$ is open then U is a complex manifold. Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on M . Then

$$\{(\overline{U_\alpha}, \overline{\phi_\alpha})\} = \{(U_\alpha \cap U, \phi_\alpha|_{\overline{U_\alpha}})\}$$

is a complex structure of M .

- If X and Y are complex manifolds, then $X \times Y$ is a complex manifold.

Example 3.3. The projective space $\mathbb{P}_{\mathbb{C}}^n$ or \mathbb{CP}^n . Let $V^* = \mathbb{C}^{n+1} \setminus \{0\}$, with coordinates (z_0, \dots, z_n) . Define an equivalence on V^* as

$$v_1 \sim v_2 \iff \exists \lambda \in \mathbb{C}, v_1 = \lambda v_2.$$

Check that \sim is an equivalence. Consider the Euclidean topology on V^* . Then there exists an induced topology on $X = V^*/\sim = \{[v] \mid v \in V^*\}$, with quotient map

$$\begin{array}{ccc} q & : & V^* \longrightarrow X \\ & & v \longmapsto [v] \end{array} .$$

Given $v = (z_0, \dots, z_n) \in V^*$ we denote $[v] = [z_0, \dots, z_n]$ such that $z_i \neq 0$ for some i . Two elements $[x_0, \dots, x_n]$ and $[y_0, \dots, y_n]$ of X define the same point if and only if there exists λ such that $x_i = \lambda y_i$ for all i . Let

$$V_i = \{(z_0, \dots, z_n) \in V^* \mid z_i \neq 0\},$$

which is open in V^* , and let $U_i = q(V_i)$, which is open in X , such that $\{U_i\}$ is a cover of X , that is $\bigcup_i U_i = X$. Let

$$H_i = \{(z_0, \dots, z_n) \in V^* \mid z_i = 1\}.$$

Then there exists a homeomorphism

$$\begin{array}{ccc} r_i & : & H_i \longrightarrow \mathbb{C}^n \\ & & (z_0, \dots, z_n) \longmapsto [z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_n] \end{array} ,$$

and let

$$\begin{array}{ccc} q_i = q|_{H_i} & : & H_i \subset V^* \longrightarrow U_i \subset X \\ & & (z_0, \dots, z_n) \longmapsto [z_0, \dots, z_n] \end{array}$$

be also a homeomorphism.

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- q_i is surjective. Take $[x_0, \dots, x_n] \in U_i$. Then $x_i \neq 0$, so choose $\lambda = 1/x_i$. Then

$$[x_0, \dots, x_n] = \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right] = q(z_0, \dots, z_n), \quad z_j = \frac{x_j}{x_i},$$

and in particular $z_i = 1$, so there exists $(z_0, \dots, z_n) \in H_i$ such that $q_i(z_0, \dots, z_n) = [x_0, \dots, x_n]$.

- q_i is injective.¹

For all i , $q_i^{-1} : U_i \rightarrow H_i$ and $r_i : H_i \rightarrow \mathbb{C}^n$ are homeomorphisms, so $\phi_i = r_i \circ q_i^{-1} : U_i \rightarrow \mathbb{C}^n$ is also a homeomorphism. We want to show that (U_i, ϕ_i) define a holomorphic atlas, so

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)$$

is a biholomorphism. Consider the case $j = 0$ and $i = 1$. Then $\phi_0(U_0 \cap U_1) = \{(x_1, \dots, x_n) \mid x_1 \neq 0\}$, so

$$\begin{aligned} \phi_1 \circ \phi_0^{-1} : \phi_0(U_0 \cap U_1) &\longrightarrow \phi_1(U_0 \cap U_1) \\ (x_1, \dots, x_n) &\longmapsto \left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right) \end{aligned}$$

is a biholomorphism. Thus X is a compact complex manifold. If $n = 1$, then $\mathbb{P}_{\mathbb{C}}^1 \cong \mathbb{S}^2$.

Example 3.4. The complex torus. Let

$$\begin{aligned} \Lambda = \mathbb{Z}^{2n} &\longrightarrow \mathbb{C}^n \\ (a_1, \dots, a_n, b_1, \dots, b_n) &\longmapsto (a_1 + ib_1, \dots, a_n + ib_n) \end{aligned}$$

Define an equivalence on \mathbb{C}^n by

$$v_1 \sim v_2 \iff v_1 - v_2 \in \Lambda.$$

Then $X = \mathbb{C}^n / \sim$ with quotient map $q : \mathbb{C}^n \rightarrow X$ is Hausdorff and compact. Topologically $X \cong [0, 1]^{2n} / \sim$. For each $x \in \mathbb{C}^n$, we can find an open set $x \in U \subset \mathbb{C}^n$ such that $q|_U : U \rightarrow X$ is a homeomorphism. The idea is if $x \in (0, 1)^{2n}$ then we can take $U = (0, 1)^{2n}$. If not, there exists a translation of $\mathbb{C}^n \rightarrow \mathbb{C}^n$ such that the property holds. We define

$$\phi_V = q|_U^{-1} : V \subset \mathbb{C}^n / \Lambda \rightarrow U \subset \mathbb{C}^n, \quad V = q(U).$$

Show that (V, ϕ_V) define a complex structure on X .² This is also a compact complex manifold. More in general \mathbb{C}^n / Λ where $\Lambda \cong \mathbb{Z}^{2n}$ is a lattice is a compact complex manifold.

3.2 Holomorphic functions on complex manifolds

Definition 3.5. Let $f : X \rightarrow Y$ be a continuous morphism between complex manifolds. Then f is **holomorphic** if there exists a complex structure $\{(U_\alpha, \phi_\alpha)\}$ on Y and for all $y \in Y$ there exists a holomorphic chart (V_α, ψ_α) such that $x \in V_\alpha$ and $f(V_\alpha) \subset U_\alpha$ around any point x of $f^{-1}(y)$ and $\phi_\alpha \circ f \circ \psi_\alpha^{-1}$ is holomorphic, so

$$\begin{array}{ccc} X \supset V_\alpha & \xrightarrow{f} & U_\alpha \subset Y \\ \psi_\alpha \downarrow & & \downarrow \phi_\alpha \\ \psi_\alpha(V_\alpha) & \xrightarrow{\tilde{f}} & \phi_\alpha(U_\alpha) \end{array}$$

Then $J_f = J_{\tilde{f}}$, and a **holomorphic function on X** is a holomorphic function $f : X \rightarrow \mathbb{C}$.

Exercise 3.6. If X is a compact complex manifold then any holomorphic function $f : X \rightarrow \mathbb{C}$ is constant.

¹Exercise

²Exercise

Definition 3.7. Let $f : X \rightarrow Y$ be a holomorphic function between complex manifolds. Then f is

- a **submersion** if $\dim Y \geq \dim X = r$ and $\text{rk } J_f = r$ at any point,
- an **immersion** if $r = \dim X \leq \dim Y$ and $\text{rk } J_f = r$ at any point, and
- an **embedding** if it is an immersion and $f : X \rightarrow f(X)$ is a homeomorphism.

Example 3.8. Let $f_2, \dots, f_n : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic, and let

$$\begin{aligned} f &: \mathbb{C} \longrightarrow \mathbb{C}^n \\ z &\longmapsto (z, f_2(z), \dots, f_n(z)) \end{aligned}$$

Then f is an embedding.

Example 3.9. Let $X = \mathbb{C}^2 / \Lambda$ for $\Lambda = \mathbb{Z}^4 \subset \mathbb{C}^2$, and let $q : \mathbb{C}^2 \rightarrow X$. Fix $\lambda \in \mathbb{C}$. Let

$$\begin{aligned} f &: \mathbb{C} \longrightarrow \mathbb{C}^2 \\ z &\longmapsto (z, \lambda z) \end{aligned}$$

Then $\tilde{f} = q \circ f : \mathbb{C} \rightarrow X$ is an immersion.

- If $\lambda = 0$ or $\lambda = \frac{1}{2}$, then $\tilde{f}(\mathbb{C})$ is a closed submanifold.
- If λ is general then $\tilde{f}(\mathbb{C})$ is dense inside X , so it is not closed. Thus it is not a complex submanifold of X .

3.3 Complex submanifolds

Definition 3.10. Let $i : X \rightarrow Y$ be an embedding of complex manifolds. If $i(X) \subset Y$ is closed then $i(X)$ is called a **complex submanifold** of Y . The **codimension** of X in Y is $\dim Y - \dim X$.

Theorem 3.11.

1. Let $i : X \rightarrow Y$ be a submanifold of codimension k , and let $n = \dim X$. Then for all $x \in X$, there exists an open neighbourhood $x \in U \subset Y$ and a submersion $f : U \rightarrow \mathbb{D}(0, 1)^k \subset \mathbb{C}^k$ such that $X \cap U = f^{-1}(0)$.
2. If $X \subset Y$ is a closed subset such that for all $x \in X$ there exists $U \ni x$ open in Y and a submersion $f : U \rightarrow \mathbb{D}(0, 1)^k$ such that $X \cap U = f^{-1}(0)$, then X is a complex submanifold.

Proof.

1. We can assume that if there exists a holomorphic chart (U, ψ) on Y such that $x \in U$ and if $V = i^{-1}(U)$ then there exists $\phi : V \rightarrow \mathbb{C}^n$ such that (V, ϕ) is a holomorphic chart on X . After possibly shrinking U smaller, by the rank theorem, there exist biholomorphic $a : \psi(U) \rightarrow \mathbb{D}(0, 1)^{n+k}$ and $b : \phi(U) \rightarrow \mathbb{D}(0, 1)^n$ such that the induced morphism is given by

$$\begin{aligned} \mathbb{D}(0, 1)^n &\longrightarrow \mathbb{D}(0, 1)^{n+k} \\ (z_1, \dots, z_n) &\longmapsto (z_1, \dots, z_n, 0, \dots, 0) \end{aligned}$$

Let

$$\begin{aligned} c &: \mathbb{D}(0, 1)^{n+k} \longrightarrow \mathbb{D}(0, 1)^k \\ (z_1, \dots, z_{n+k}) &\longmapsto (z_{n+1}, \dots, z_{n+k}) \end{aligned}$$

so

$$\begin{array}{ccccc} Y & \supset & U & \xrightarrow{\phi} & \phi(U) & \xrightarrow{b} & \mathbb{D}(0, 1)^n \subset \mathbb{C}^n & \xleftarrow{c} \\ \uparrow i & & \uparrow i & & & & \downarrow & \\ X & \supset & V & \xrightarrow{\psi} & \psi(U) & \xrightarrow{a} & \mathbb{D}(0, 1)^{n+k} \subset \mathbb{C}^{n+k} & \end{array}$$

Then f is the composition $c \circ a \circ \psi : U \rightarrow \mathbb{D}(0, 1)^n$.

2. Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on Y , and let $V_\alpha = X \cap U_\alpha$ and $\psi_\alpha = \phi_\alpha|_{V_\alpha}$. The goal is to show that $\{(V_\alpha, \psi_\alpha)\}$ defines a complex structure on X . By assumption,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k} \rightarrow \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k}$$

is biholomorphic. Let $U' = \phi_\beta(U)$, let $X' = \phi_\beta(X \cap U)$, and let $f' = f \circ \phi_\beta^{-1}$, so

$$\begin{array}{ccccccc} & & & \phi_\alpha(U) & \subset & \phi_\alpha(U_\alpha \cap U_\beta) & \subset \mathbb{C}^{n+k} \\ & & & \nearrow \phi_\alpha & & \uparrow \phi_\alpha \circ \phi_\beta^{-1} & \\ Y & \supset & U_\alpha \cap U_\beta & \supset & U & \xrightarrow{\phi_\beta} & U' \subset \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k} \\ \uparrow i & & \cup & & \cup & \searrow f & \\ X & \supset & X \cap U_\alpha \cap U_\beta & \supset & X \cap U & \xrightarrow{f} & X' \subset D(0,1)^k \subset \mathbb{C}^k \end{array}$$

Then $f'^{-1}(0) = \phi_\beta(X \cap U_\alpha \cap U_\beta)$ and f' is also a submersion. By the rank theorem, we may assume that $U' = D(0,1)^{n+k}$ and $f'(z_1, \dots, z_{n+k}) = (z_1, \dots, z_k)$, so $\phi_\beta(X' \cap U_\alpha \cap U_\beta) = f'^{-1}(0)$. Thus

$$(\psi_\alpha \circ \psi_\beta^{-1})(z_1, \dots, z_n) = (\phi_\alpha \circ \phi_\beta^{-1})(z_1, \dots, z_n, 0, \dots, 0)$$

is also a biholomorphism. □

3.4 Examples of complex manifolds

Example 3.12. Let $U \subset \mathbb{C}^n$ be open, let $k \leq n$, let $f_1, \dots, f_k : U \rightarrow \mathbb{C}$, and let

$$V = \{x \in \mathbb{C}^n \mid f_1(x) = \dots = f_k(x) = 0\}.$$

Assume that $\left(\frac{\partial f_i}{\partial z_j}\right)$ has maximal rank k at any point of U . Then V is a complex submanifold of U . The idea is if $f = (f_1, \dots, f_k) : U \rightarrow \mathbb{C}^k$, then f is a submersion around any point of V , and use the previous Theorem 3.11.

Example 3.13. Let $f : X \rightarrow Y$ be a holomorphism between complex manifolds, and let $W \subset X$ be a submanifold. Then $f|_W : W \rightarrow Y$ is holomorphic.

Exercise 3.14. Let $X = \mathbb{C}^n$. Show that all the compact submanifolds of X are zero-dimensional, that is points.

Exercise 3.15. Let X and Y be compact manifolds. Recall that $X \times Y$ is also a complex manifold. Assume $f : X \rightarrow Y$, so

$$\Gamma_f = \{(x, f(x)) \mid x \in X\} \subset X \times Y.$$

Show that Γ_f is a complex submanifold.

Example 3.16. Let $n, m > 0$, and let

$$\text{Mat}_{n,m} \mathbb{C} = \{(n \times m)\text{-matrices}\} \cong \mathbb{C}^{n \cdot m}.$$

Then $\text{Mat}_{n,m} \mathbb{C}$ is a complex manifold. Let

$$\text{GL}_n \mathbb{C} = \{(n \times n)\text{-matrices } A \mid A \text{ invertible}\}.$$

Then $\text{GL}_n \mathbb{C}$ is a complex manifold, open in $\text{Mat}_{n,n} \mathbb{C}$.

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Example 3.17. Projective manifolds. Let $R = \mathbb{C}[x_0, \dots, x_n]$ be the ring of polynomials, and let $X = \mathbb{P}_{\mathbb{C}}^n$ be the complex projective space. Then $f \in R$ is homogeneous of degree d if $f(\lambda x) = \lambda^d f(x)$. Let $q : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}_{\mathbb{C}}^n$, let F_1, \dots, F_k be homogeneous polynomials in R , and let

$$V = \{F_1 = \dots = F_k = 0\} \subset \mathbb{C}^{n+1} \setminus \{0\}, \quad W = q(V) \subset \mathbb{P}_{\mathbb{C}}^n,$$

so $q^{-1}(W) = V$, because F_i are homogeneous. Since V is closed in $\mathbb{C}^{n+1} \setminus \{0\}$, W is closed in $\mathbb{P}_{\mathbb{C}}^n$. Claim that if V is a submanifold of $\mathbb{C}^{n+1} \setminus \{0\}$ then W is a compact submanifold of $\mathbb{P}_{\mathbb{C}}^n$. If $\{U_i\}$ is the open covering given by

$$U_i = \{[x_0, \dots, x_n] \mid x_i \neq 0\},$$

then it is enough to show that $W \cap U_i$ is a complex submanifold of U_i for all i . Assume $i = n$. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then $q(x) = \mathbb{C}^*$ for all $x \in X$ but $\mathbb{P}_{\mathbb{C}}^n \times \mathbb{C}^* \neq \mathbb{C}^{n+1} \setminus \{0\}$. We want to show there exists a biholomorphism

$$\begin{aligned} \phi_n : \quad U_n \times \mathbb{C}^* &\longrightarrow q^{-1}(U_n) = \{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} \mid x_n \neq 0\} \\ ([x_0, \dots, x_n], t) &\longmapsto \left(\frac{tx_0}{x_n}, \dots, \frac{tx_{n-1}}{x_n}, t \right), \end{aligned}$$

such that

$$\begin{aligned} \phi_n^{-1} : \quad q^{-1}(U_n) &\longrightarrow U_n \times \mathbb{C}^* \\ (y_0, \dots, y_n) &\longmapsto (q(y_0, \dots, y_n), y_n) = ([y_0, \dots, y_n], y_n). \end{aligned}$$

From this, it follows that $V \cap q^{-1}(U_n) \cong (W \cap U_n) \times \mathbb{C}^*$, so the claim follows.

Example 3.18. Plane curves. Let $X = \mathbb{P}_{\mathbb{C}}^2$, let $F \in R[x_0, x_1, x_2]$ be homogeneous of degree d , and let $W = \{F = 0\} \subset \mathbb{P}_{\mathbb{C}}^2$. Then W is a compact complex submanifold if and only if for all $x \in \mathbb{P}_{\mathbb{C}}^2$, $\partial_{x_i} F(x) \neq 0$ for some i .

$d = 1$. W is the projective line, so $F = ax_0 + bx_1 + cx_2$ for a, b, c not all zero. Then W is a complex submanifold. There exists a biholomorphism $\mathbb{P}_{\mathbb{C}}^1 \rightarrow W$.

$d = 2$. W is a conic, so F is a degree two polynomial. Then $F = x_0x_1$ does not define a manifold. If $F = x_0x_1 - x_2^2$, then W is a complex submanifold of X . There exists

$$\begin{aligned} \mathbb{P}_{\mathbb{C}}^1 &\longrightarrow W \subset X \\ [t_0, t_1] &\longmapsto [t_0^2, t_1^2, t_0t_1]. \end{aligned}$$

Check that it is a biholomorphism. ³ This is true for any f of degree two such that W is a complex submanifold.

$d \geq 3$. If W is a complex submanifold then we will show that W is not biholomorphic to $\mathbb{P}_{\mathbb{C}}^1$.

3.5 Tangent spaces of complex manifolds

Definition 3.19. Let X be a complex manifold of dimension n , and let $x \in X$. Then there exists a chart (U, ϕ) around x such that $\phi(U) \subset \mathbb{C}^n$. The **holomorphic tangent space** $T_x X$ of X at x , is the vector space over \mathbb{C} generated by

$$\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right).$$

Let X be a real manifold. The **real tangent space** $T_x^{\mathbb{R}} X$ is the vector space over \mathbb{R} defined by

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right),$$

where $(x_1, \dots, x_n, y_1, \dots, y_n)$ are coordinates of \mathbb{R}^{2n} . The **complex tangent space** $T_x^{\mathbb{C}} X$ is the vector space over \mathbb{C} generated by

$$\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right),$$

a $2n$ -dimensional vector space over \mathbb{C} . Then $T_x^{\mathbb{C}} X = T_x^{\mathbb{R}} X \otimes_{\mathbb{R}} \mathbb{C}$.

³Exercise

3.6 Holomorphic differential forms on complex manifolds

Definition 3.20. Let X be a complex manifold of dimension n . Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on X . A **holomorphic p -form** on X is the data ω_α , the p -forms on $\phi_\alpha(U_\alpha) \subset \mathbb{C}^n$ such that if

$$h_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta),$$

then $h_{\alpha\beta}^* \omega_\beta = \omega_\alpha$ for all α and β .

Notation 3.21.

$$\Omega_x^p(X) = H^0(X, \Omega_x^p) = \{\text{holomorphic } p\text{-forms on } X\},$$

$$\mathcal{O}_x(X) = H^0(X, \mathcal{O}_x) = \{\text{holomorphic functions on } X\}.$$

$R = \mathcal{O}_x(X)$ is a ring and $M = \Omega_x^p(X)$ is an R -module.

Lemma 3.22. Let $f : X \rightarrow Y$ be holomorphic. Then $f^* : \Omega^p(Y) \rightarrow \Omega^p(X)$.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on Y . We can write $f^{-1}(U_\alpha) = \bigcup_{\alpha,\beta} V_{\alpha,\beta}$ where $\{(V_{\alpha,\beta}, \psi_{\alpha,\beta})\}$ is a complex structure on X , so

$$\mathbb{C}^n \xleftarrow{\psi_{\alpha,\beta}} V_{\alpha,\beta} \xrightarrow{f|_{V_{\alpha,\beta}}} U_\alpha \xrightarrow{\phi_\alpha} \mathbb{C}^n.$$

Assume ω is defined by ω_α on $\phi_\alpha(U_\alpha)$. Let

$$\omega_{\alpha,\beta} = \left(\left(\psi_{\alpha,\beta}^{-1} \right)^* \circ f^* \circ \phi_\alpha^* \right) (\omega_\alpha)$$

be a p -form on $\psi_{\alpha,\beta}(V_{\alpha,\beta})$. Check that $\omega_{\alpha,\beta}$ are compatible with respect to the atlas on X .⁴ □

As in the local case, we can define

$$\begin{array}{ccc} \Omega_x^p(X) \otimes \Omega_x^q(X) & \longrightarrow & \Omega_x^{p+q}(X) \\ \omega_1 \otimes \omega_2 & \longmapsto & \omega_1 \wedge \omega_2 \end{array}.$$

Similarly there exists $d : \Omega_x^p(X) \rightarrow \Omega_x^{p+1}(X)$.

⁴Exercise

4 Vector bundles

4.1 Holomorphic vector bundles

Definition 4.1. Let X be a complex manifold. A **holomorphic vector bundle** E of rank r on X is a complex manifold E , a holomorphism $\pi : E \rightarrow X$, and an open covering U_α of X such that there exists a biholomorphism

$$\psi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{C}^r,$$

such that if $p_\alpha : U_\alpha \times \mathbb{C}^r \rightarrow U_\alpha$ is the projection then $\pi|_{\pi^{-1}(U_\alpha)} = p_\alpha \circ \psi_\alpha$, so

$$\begin{array}{ccc} E & \supset & \pi^{-1}(U_\alpha) \xrightarrow{\psi_\alpha} U_\alpha \times \mathbb{C}^r \\ \pi \downarrow & & \downarrow \pi \swarrow p_\alpha \\ X & \supset & U_\alpha \end{array} .$$

A vector bundle of rank one is called a **line bundle**.

For any $x \in X$, there exists α such that $x \in U_\alpha$, so

$$\begin{array}{ccc} \pi^{-1}(x) & \xrightarrow{\psi_\alpha} & \{x\} \times \mathbb{C}^r \\ \pi \downarrow & & \swarrow p_\alpha \\ x & & \end{array} .$$

Then $E(x) = \pi^{-1}(x)$ is a vector space of rank r over \mathbb{C} . Let $U_\alpha \ni x \in U_\beta$. There exists a biholomorphism

$$\mathbb{C}^r \cong p_\alpha^{-1}(x) \rightarrow p_\beta^{-1}(x) \cong \mathbb{C}^r,$$

because they are both biholomorphic to $\pi^{-1}(x)$, so $g_{\alpha\beta}(x) \in \mathrm{GL}_r \mathbb{C}$ because all the biholomorphisms from $\mathbb{C}^r \rightarrow \mathbb{C}^r$ are linear. The holomorphisms

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \mathrm{GL}_r \mathbb{C}$$

are called **transition functions**. Then

$$\begin{array}{ccc} p_\alpha^{-1}(x) & \xrightarrow{\mathrm{id}} & p_\alpha^{-1}(x) \\ & \searrow & \swarrow \\ & p_\beta^{-1}(x) & \end{array} ,$$

so

$$(g_{\alpha\beta} \circ g_{\beta\alpha})(x) = x, \quad x \in U_\alpha \cap U_\beta,$$

and

$$\begin{array}{ccc} p_\alpha^{-1}(x) & \xrightarrow{g_{\alpha\gamma}} & p_\gamma^{-1}(x) \\ & \searrow & \swarrow \\ & p_\beta^{-1}(x) & \end{array} ,$$

so

$$(g_{\alpha\beta} \circ g_{\beta\gamma})(x) = g_{\alpha\gamma}(x), \quad x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Definition 4.2. Let X be a complex manifold, and let E and F be vector bundles on X of rank r and s respectively, with $\pi : E \rightarrow X$ and $\pi' : F \rightarrow X$. A **holomorphic map** $f : E \rightarrow F$ is a holomorphic function $E \rightarrow F$ such that $\pi = \pi' \circ f$ and such that the rank of the induced linear map $E(x) \rightarrow F(x)$ is independent of $x \in X$, so

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \pi \searrow & & \swarrow \pi' \\ & X & \end{array} , \quad \begin{array}{ccc} E(x) = \pi^{-1}(x) & \xrightarrow{f} & \pi'^{-1}(x) = F(x) \\ \pi \searrow & & \swarrow \pi' \\ & x & \end{array} .$$

4.2 Examples of holomorphic vector bundles

Example 4.3. $\pi : E = X \times \mathbb{C}^r \rightarrow X$ is a vector bundle of rank r , called **trivial**.

Example 4.4. Algebra of vector bundles. Let $\pi : E \rightarrow X$ and $\pi'^{-1} : F \rightarrow X$ be vector bundles on X of rank r and s respectively.

- The **direct sum** $E \oplus F$ is the $(r + s)$ -vector bundle such that

$$(E \oplus F)(x) = E(x) \oplus F(x), \quad x \in X.$$

The idea is to take an open cover which trivialises both E and F . Find the transition function of $E \oplus F$.⁵

- The **tensor product** $E \otimes F$ is the $(r \cdot s)$ -vector bundle such that

$$(E \otimes F)(x) = E(x) \otimes F(x), \quad x \in X.$$

- The **p -th exterior power** of E is the vector bundle $\Lambda^p E$ such that

$$(\Lambda^p E)(x) = \Lambda^p(E(x)), \quad x \in X.$$

If $p = r = \text{rk } E$ then $\det E = \Lambda^r E$ is a line bundle on X .

- The **dual** of E is the rank r vector bundle E^* such that

$$E^*(x) = (E(x))^*, \quad x \in X,$$

the dual $\text{Hom}(E(x), \mathbb{C})$ of $E(x)$.

- Let $f : E \rightarrow F$ be a holomorphic map. Then the **kernel** $\text{Ker } f$ is a vector bundle such that

$$(\text{Ker } f)(x) = \text{Ker } f(x) \subset E(x), \quad x \in X.$$

The **cokernel** $\text{Coker } f$ is a vector bundle such that

$$(\text{Coker } f)(x) = \text{Coker } f(x) \subset F(x), \quad x \in X.$$

Example 4.5. Let $X = \mathbb{P}_{\mathbb{C}}^1$, and let

$$\mathcal{O}(-1) = \{(x, v) \mid x = [x_0, \dots, x_n] \in \mathbb{P}_{\mathbb{C}}^n, v = \mu(x_0, \dots, x_n), \mu \in \mathbb{C}\} \subset \mathbb{P}_{\mathbb{C}}^n \times \mathbb{C}^{n+1}.$$

Then $\pi = p_1 : \mathcal{O}(-1) \rightarrow \mathbb{P}_{\mathbb{C}}^n$, so

$$\pi^{-1}([x_0, \dots, x_n]) = \{v = \mu(x_0, \dots, x_n) \mid \mu \in \mathbb{C}\} \cong \mathbb{C}^1.$$

Let $\{U_i\}$ be an open covering of X given by $U_i = \{[x_0, \dots, x_n] \mid x_i \neq 0\}$. We define

$$\begin{aligned} \psi_i : \quad \pi^{-1}(U_i) &\longrightarrow U_i \times \mathbb{C} \\ ([x_0, \dots, x_n], (v_0, \dots, v_n)) &\longmapsto ([x_0, \dots, x_n], v_i) \end{aligned}$$

which is a biholomorphism. Thus $\mathcal{O}(-1)$ is a complex manifold and $\mathcal{O}(-1)$ is a line bundle. The **tautological line bundle** $\mathcal{O}(1)$ is the dual of $\mathcal{O}(-1)$. Let

$$\mathcal{O}(k) = \begin{cases} X \times \mathbb{C} & k = 0 \\ \mathcal{O}(1)^{\otimes k} & k > 0 \\ \mathcal{O}(-1)^{\otimes k} & k < 0 \end{cases}.$$

Then $\mathcal{O}(k) = \mathcal{O}(-k)^*$.⁶ On $\mathbb{P}_{\mathbb{C}}^n$ these are the only line bundles. That is, if \mathcal{L} is a line bundle on $\mathbb{P}_{\mathbb{C}}^1$, there exists $k \in \mathbb{Z}$ such that $\mathcal{L} \cong \mathcal{O}(k)$. Let $X = \mathbb{P}_{\mathbb{C}}^1$, and let E be a line bundle of rank r on X . Then

$$E \cong \bigoplus_{i=1}^r \mathcal{O}(a_i), \quad a_1, \dots, a_r \in \mathbb{Z}.$$

This is false for $X = \mathbb{P}_{\mathbb{C}}^n$, with $n \geq 2$.

⁵Exercise

⁶Exercise

Definition 4.6. Let $f : Y \rightarrow X$ be a holomorphism between complex manifolds, and let E be a vector bundle of rank r on X . Then there exists a vector bundle f^*E of rank r on Y defined by

$$f^*E = \{(y, v) \in Y \times E \mid f(y) = \pi(v)\},$$

the **fibre product** of E and Y over X , such that

$$\begin{array}{ccc} f^*E & \xrightarrow{f'} & E \\ \pi' \downarrow & & \downarrow \pi \\ Y & \xrightarrow{f} & X \end{array}$$

Let $U = \{U_i\}$ be an open cover of X which trivialises E , so

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\psi_i} & U_i \times \mathbb{C}^r \\ & \searrow \pi \quad \swarrow p_1 & \\ & U_i & \end{array}$$

Then $U' = \{f^{-1}(U_i)\}$ is an open covering of Y , so

$$\begin{array}{ccccccc} \pi'^{-1}(f^{-1}(U_i)) & \xrightarrow{f'} & \pi^{-1}(U_i) & \xrightarrow{\psi_i} & U_i \times \mathbb{C}^r & \xrightarrow{p_2} & \mathbb{C}^r \\ \pi' \downarrow & & \downarrow \pi & & & & \\ f^{-1}(U_i) & \xrightarrow{f} & U_i & & & & \end{array},$$

and

$$\begin{aligned} \pi'^{-1}(f^{-1}(U_i)) &= \{(y, v) \in f^{-1}(U_i) \times \pi^{-1}(U_i) \mid f(y) = \pi(v)\} \longrightarrow f^{-1}(U_i) \times \mathbb{C}^r \\ (y, v) &\longmapsto (y, p_2(\psi_i(v))) \end{aligned}$$

is a biholomorphism. Thus f^*E is a vector bundle, where

$$f^*E(y) = \pi'^{-1}(y) = E(f(y)), \quad y \in Y.$$

Notation 4.7. Let $f : Y \rightarrow X$ be a morphism, and let E be a vector bundle on X . Then $f^*E = E|_Y$, mostly used if $f : Y \hookrightarrow X$.

Definition 4.8. Let E be a holomorphic vector bundle on a complex manifold X , and let $\pi : E \rightarrow X$. A **section** of E is a holomorphic function $s : X \rightarrow E$ such that $\pi \circ s = \text{id}_X$.

Example 4.9. Let $E = X \times \mathbb{C}^r$ be the trivial vector bundle of rank r . Fix $v \in \mathbb{C}^r$. Then

$$\begin{aligned} s_v &: X \longrightarrow E \\ x &\longmapsto (x, v) \end{aligned}$$

is a section of E . If v_1, \dots, v_r is a basis of \mathbb{C}^r then s_{v_1}, \dots, s_{v_r} have the property that $s_{v_1}(x), \dots, s_{v_r}(x)$ forms a basis of $E(x)$. Vice versa, assume E is a vector bundle on X of rank r such that there exist sections s_1, \dots, s_r of E such that for all $x \in X$, $s_1(x), \dots, s_r(x)$ is a basis of $E(x)$. Then $E \cong X \times \mathbb{C}^r$, since

$$\begin{aligned} X \times \mathbb{C}^r &\longrightarrow E \\ (x, (v_1, \dots, v_r)) &\longmapsto \sum_i v_i s_i(x) \end{aligned}$$

is a biholomorphism. Then s_1, \dots, s_r is called a **holomorphic frame** for E . Recall that for all $E \rightarrow X$ and for all $x \in X$ there exists open $U \ni x$ such that $E|_U$ is trivial, so there exists a frame on U for $E|_U$. This is called a **local frame** around x .

Example 4.10. Let X be a complex manifold of dimension n , and let (z_1, \dots, z_n) be coordinates on \mathbb{C}^n . There exists an atlas $\{(U_\alpha, \phi_\alpha)\}$ for $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{C}^n$. For all $x \in U_\alpha$, $T_x U_\alpha \rightarrow T_{\phi_\alpha(x)} V_\alpha$, and $T_{\phi_\alpha(x)} V_\alpha = \left\langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right\rangle$ is a frame of T_{V_α} . Let

$$T_X = \bigcup_{x \in X} T_x X,$$

and let $\pi^{-1} : T_X \rightarrow X$ such that $\pi^{-1}(x) = T_x X$. Then T_X is a holomorphic vector bundle of rank n called the **tangent bundle**, where $U = \{U_\alpha\}$ and

$$\psi_\alpha : \pi^{-1}(U_\alpha) = T_X|_{U_\alpha} \rightarrow T_{\mathbb{C}^n}|_{V_\alpha} \cong V_\alpha \times \mathbb{C}^r \rightarrow U_\alpha \times \mathbb{C}^r$$

defines the trivialisation. The **cotangent bundle** of X is

$$\Omega_X^1 = T_X^*,$$

and let

$$\Omega_X^p = \Lambda^p \Omega_X^1, \quad p \geq 1.$$

A holomorphic p -form on X is a section of Ω_X^p .⁷

4.3 Complexified tangent bundles

Let X be a complex manifold. How to view X as a differentiable manifold? Let V be a vector space of dimension m over \mathbb{R} . An **almost complex structure** on V is a linear map $J : V \rightarrow V$ such that $J^2 = -\text{id}_V$. If V admits an almost complex structure, then V can be seen as a vector space over \mathbb{C} . Let $\lambda = a + ib$ for $a, b \in \mathbb{R}$, and let $v \in V$. Define

$$\lambda v = av + bJ(v).$$

If $\lambda_1, \lambda_2 \in \mathbb{C}$, then $\lambda_1(\lambda_2 v) = (\lambda_1 \lambda_2)v$.⁸ Let $v_1, \dots, v_n \in V$ be a basis over \mathbb{C} . Then

$$v_1, \dots, v_n, J(v_1), \dots, J(v_n)$$

is a basis of V over \mathbb{R} . The idea is to assume that $a_i, b_i \in \mathbb{R}$ such that $\sum_i a_i v_i + \sum_i b_i J(v_i) = 0$, then

$$0 = \sum_i a_i v_i + \sum_i b_i J(v_i) = \sum_i (a_i v_i + b_i J(v_i)) = \sum_i (a_i + ib_i) v_i,$$

so $a_i + ib_i = 0$ for all i . Thus $a_i = b_i = 0$, so $m = 2n$. On a vector space an almost complex structure is a complex structure. Let V be a vector space of dimension $2n$ over \mathbb{R} . Then the **complexification** $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of V is a \mathbb{C} -vector space of dimension $2n$ over \mathbb{C} , where

$$\begin{aligned} \lambda & : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} \\ v \otimes \mu & \longmapsto v \otimes \mu \lambda, \quad \lambda \in \mathbb{C}. \end{aligned}$$

Let J be an almost complex structure on V . Then we can extend J to a linear map

$$\begin{aligned} J & : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} \\ v \otimes \mu & \longmapsto J(v) \otimes \mu, \end{aligned}$$

such that $J^2 = -\text{id}_{V_{\mathbb{C}}}$,⁹ so $J^2 + \text{id}_{V_{\mathbb{C}}} = 0$. Thus the eigenvalues of J on $V_{\mathbb{C}}$ are $\pm i$. Let $V^{1,0}$ be the eigenspace for i and $V^{0,1}$ be the eigenspace for $-i$, so

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}.$$

The **conjugation**

$$\begin{aligned} \bar{\cdot} & : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} \\ v \otimes \mu & \longmapsto v \otimes \bar{\mu} \end{aligned}$$

on $V_{\mathbb{C}}$ is linear over \mathbb{R} , such that $\overline{V^{1,0}} = V^{0,1}$ and $\overline{V^{0,1}} = V^{1,0}$,¹⁰ so $V^{1,0}$ and $V^{0,1}$ are \mathbb{C} -vector spaces of dimension n .

⁷Exercise

⁸Exercise

⁹Exercise

¹⁰Exercise

Example 4.11. Let $W = \mathbb{C}^n$ with coordinates (z_1, \dots, z_n) , and let $z_j = x_j + iy_j$ with coordinates $(x_1, y_1, \dots, x_n, y_n)$ for \mathbb{R}^{2n} . Define

$$J : \begin{array}{ccc} \mathbb{R}^{2n} & \longrightarrow & \mathbb{R}^{2n} \\ (x_1, y_1, \dots, x_n, y_n) & \longmapsto & (-y_1, x_1, \dots, -y_n, x_n) \end{array}.$$

Then $J^2 = \text{id}_{\mathbb{R}^{2n}}$, and J is the **standard almost complex structure** on \mathbb{R}^{2n} . Let $V = \mathbb{R}^{2n}$, so $V_{\mathbb{C}} \cong \mathbb{C}^{2n}$ with complex coordinates $(x_1, y_1, \dots, x_n, y_n)$. Then $V^{0,1}$ is spanned by $x_j - iy_j$ and $V^{1,0}$ is spanned by $x_j + iy_j$, where $\overline{x_j + iy_j} = x_j - iy_j$ for $j = 1, \dots, n$.

Definition 4.12. Let X be a differentiable manifold. A **real, or complex, vector bundle** of rank r is a differentiable manifold E with a smooth morphism $\pi : E \rightarrow X$ such that if $K = \mathbb{R}$, or $K = \mathbb{C}$, then there exists an open covering $U = \{U_i\}$ of X such that

- for all $x \in X$, the fibre of π , $E(x) = \pi^{-1}(x)$, is a vector space of rank r over K ,
- for all i there exists a diffeomorphism h_i such that

$$\begin{array}{ccccc} \pi^{-1}(U_i) & \xrightarrow{h_i} & U_i \times K^r & \xrightarrow{p_2} & K^r \\ & \searrow \pi & \swarrow p_1 & & \\ & & U_i & & \end{array},$$

and for all x , $p_2 \circ h_i : E(x) \rightarrow K^r$ is an isomorphism of vector spaces.

Pull-backs, sections, exterior powers, tensors, direct sums, frames, etc are the same as holomorphic vector bundles, where holomorphic becomes smooth and biholomorphic becomes diffeomorphic, and for all X there exists a tangent bundle T_X . Assume X is a complex manifold of dimension n . Let T_X be the holomorphic tangent bundle of X . Then X is also a differentiable manifold of dimension $2n$, so let $T_X^{\mathbb{R}}$ be the real tangent bundle of X , which is a rank $2n$ vector bundle, and let $T_X^{\mathbb{C}} = T_X^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be the complex tangent bundle of X , which is a non-holomorphic complex vector bundle of rank $2n$. Smooth morphisms of real or complex vector bundles are defined similarly as holomorphisms between holomorphic vector bundles such that the rank of the image is constant, so

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \pi \searrow & & \swarrow \pi' \\ & X & \end{array}.$$

Let X be a differentiable manifold of dimension $m = 2n$. Then an **almost complex structure** on X is a smooth morphism between the real tangent bundle $J : T_X^{\mathbb{R}} \rightarrow T_X^{\mathbb{R}}$ such that $J^2 = -\text{id}$. In particular, $J(x) : T_x^{\mathbb{R}} X \rightarrow T_x^{\mathbb{R}} X$ is an almost complex structure for all $x \in X$.

Proposition 4.13. *Let X be a complex manifold. Then the underlying differentiable manifold admits an almost complex structure $J : T_X^{\mathbb{R}} \rightarrow T_X^{\mathbb{R}}$ such that $J^2 = -\text{id}$.*

Proof. Let $x \in X$, and let (U, ϕ) be a complex chart around x such that

$$\begin{array}{ccc} \phi : U & \longrightarrow & V \\ x & \longmapsto & 0 \end{array}.$$

Fix holomorphic coordinates (z_1, \dots, z_n) on U . The tangent bundle of X on U is trivial, with a local frame $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$, so

$$T_X|_U \xrightarrow{\sim} T_V = V \times \mathbb{C}^n.$$

Define $x_i = \text{Re } z_i$ and $y_i = \text{Im } z_i$. Then $(x_1, y_1, \dots, x_n, y_n)$ are smooth coordinates $U \rightarrow \mathbb{R}$ around x , and $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$ define a local smooth frame of $T_X^{\mathbb{R}}$ on U , so

$$T_X^{\mathbb{R}}|_U \xrightarrow{\sim} T_V = V \times \mathbb{R}^{2n}.$$

In particular, there exists an almost complex structure J_U for $T_V \cong T_X^{\mathbb{R}}|_U$, so

$$J_U : T_X^{\mathbb{R}}|_U \rightarrow T_X^{\mathbb{R}}|_U, \quad J_U^2 = -\text{id}.$$

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Let $f : V \rightarrow V$ be a biholomorphism, so

$$\begin{array}{ccc} & U \cap U' & \\ \phi \swarrow & & \searrow \phi \\ V & \xrightarrow{f} & V \end{array},$$

and let z'_1, \dots, z'_n be local holomorphic coordinates given by

$$z'_i = f_i(z_1, \dots, z_n), \quad f_i = p_i \circ f,$$

where $p_i : \mathbb{C}^n \rightarrow \mathbb{C}$ is the i -th projection. Define

$$x'_i = \operatorname{Re} z'_i = \operatorname{Re} f_i(z_1, \dots, z_n) = u_i(z_1, \dots, z_n), \quad y'_i = \operatorname{Im} z'_i = \operatorname{Im} f_i(z_1, \dots, z_n) = v_i(z_1, \dots, z_n),$$

so $f_j = u_j + iv_j$. The real Jacobian J_f of f is given by the derivatives of u_j and v_j with respect to $x_1, y_1, \dots, x_n, y_n$, a $2n \times 2n$ matrix of $n \times n$ blocks of 2×2 blocks of

$$\begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix}.$$

These define the transition function of $T_X^{\mathbb{R}}$. To show that J extends to X , it is enough to show that J commutes with J_f at each point, so

$$\begin{array}{ccc} T_X^{\mathbb{R}}|_{U \cap U'} & \xrightarrow{J_f} & T_X^{\mathbb{R}}|_{U \cap U'} \\ J \downarrow & & \downarrow J \\ T_X^{\mathbb{R}}|_{U \cap U'} & \xrightarrow{J_f} & T_X^{\mathbb{R}}|_{U \cap U'} \end{array}.$$

Since f_j is holomorphic $\frac{\partial f_j}{\partial z_k} = 0$ for all j and k , so the Cauchy-Riemann equations

$$\frac{\partial u_j}{\partial x_k} - \frac{\partial v_j}{\partial y_k} = 0, \quad \frac{\partial v_j}{\partial x_k} + \frac{\partial u_j}{\partial y_k} = 0,$$

or

$$\begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_j}{\partial y_k} & \frac{\partial u_j}{\partial y_k} \\ -\frac{\partial u_j}{\partial y_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix}.$$

Since J is the standard almost complex structure on \mathbb{R}^{2n} , where $x_j \mapsto y_j$ and $y_j \mapsto -x_j$,

$$J = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & \ddots & \\ & 0 & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Check that J_f commutes with J .¹¹

□

Corollary 4.14. *Every complex manifold is orientable.*

Proof. We prove that if $T_X^{\mathbb{R}}$ admits an almost complex structure then X is an orientable manifold. For all $x \in X$ choose the orientation on $T_x^{\mathbb{R}}X$, a vector space of dimension $2n$ over \mathbb{R} , given by any ordered basis of the form

$$v_1, \dots, v_n, J(v_1), \dots, J(v_n).$$

Assume that $v_1, \dots, v_n, J(v_1), \dots, J(v_n)$ and $w_1, \dots, w_n, J(w_1), \dots, J(w_n)$ are ordered bases. Show that the determinant of the matrix given by the change of basis is positive.¹²

□

¹¹Exercise

¹²Exercise

Let X be a complex manifold. Then there exists an almost complex structure $J : T_X^{\mathbb{R}} \rightarrow T_X^{\mathbb{R}}$ on X . Then J extends to

$$J : \begin{array}{ccc} T_X^{\mathbb{C}} & \longrightarrow & T_X^{\mathbb{C}} \\ v \otimes \mu & \longmapsto & J(v) \otimes \mu \end{array}.$$

For all x , $J(x)$ has two eigenvalues $\pm i$, so

$$T_X^{\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1},$$

which are complex vector bundles and **eigenbundles**. Locally $T_X^{1,0}$ and $T_X^{0,1}$ are spanned by the frames $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$ and $\frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}$ respectively. Moreover there exists a conjugation

$$\begin{array}{ccc} T_X^{\mathbb{C}} & \longrightarrow & T_X^{\mathbb{C}} \\ v \otimes \mu & \longmapsto & v \otimes \bar{\mu} \end{array}$$

over \mathbb{R} , such that $\overline{T_X^{1,0}} = T_X^{0,1}$ and $\overline{T_X^{0,1}} = T_X^{1,0}$. Let

$$\Omega_{X,\mathbb{C}}^1 = (T_X^{\mathbb{C}})^*$$

be the dual of the complex vector bundle $T_X^{\mathbb{C}}$. Then

$$\Omega_{X,\mathbb{C}}^1 = \Omega_{X,\mathbb{R}}^1 \otimes_{\mathbb{R}} \mathbb{C} = \Omega_{X,\mathbb{C}}^{1,0} \oplus \Omega_{X,\mathbb{C}}^{0,1} = (T_X^{1,0})^* \oplus (T_X^{0,1})^*.$$

Exercise. Let V and W be vector spaces. Show that

$$\Lambda^k(V \oplus W) = \bigoplus_{p+q=k} \Lambda^p V \otimes \Lambda^q W$$

is a canonical isomorphism.

Thus, for all $k \geq 0$

$$\Omega_{X,\mathbb{C}}^k = \Lambda^k \Omega_{X,\mathbb{C}}^1 = \bigoplus_{p+q=k} \Omega_X^{p,q}, \quad \Omega_X^{p,q} = \Lambda^p \Omega_X^{1,0} \otimes \Lambda^q \Omega_X^{0,1},$$

where $\Omega_X^{p,q}$ is a complex vector bundle for any p and q .

Definition 4.15. The sections of $\Omega_X^{p,q}$ are called (p, q) -forms on X , or **forms of type (p, q)** .

Locally, let $x \in X$, and let $(U \ni x, \phi)$ be a holomorphic chart for $\phi : U \xrightarrow{\sim} V \subset \mathbb{C}^n$. A (p, q) -form on U can be locally written as

$$\omega = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J = \sum_{|I|=p, |J|=q} \alpha_{I,J} dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q},$$

where $\alpha_{I,J}$ are smooth functions on U . Let X be a manifold. If E is a complex vector bundle then

$$C^\infty(X, E) = \{\text{smooth sections of } E\}.$$

The **differential**

$$d : C^\infty(X, \Omega_X^k) \rightarrow C^\infty(X, \Omega_X^{k+1})$$

satisfies the Leibnitz rule and $d^2 = 0$, so $d(d\omega) = 0$. If $\omega \in C^\infty(X, \Omega_X^{p,q})$, then $d\omega \in C^\infty(X, \Omega_{X,\mathbb{C}}^{p+q+1})$.

Assume that locally $\omega = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J$. Then

$$d\omega = \sum_{I,J} d\alpha_{I,J} dz_I \wedge d\bar{z}_J, \quad d\alpha_{I,J} = \sum_{i=1}^n \frac{\partial}{\partial z_i} \alpha_{I,J} dz_i + \sum_{i=1}^n \frac{\partial}{\partial \bar{z}_i} \alpha_{I,J} d\bar{z}_i.$$

Let

$$\partial\alpha_{I,J} = \sum_{i=1}^n \frac{\partial}{\partial z_i} \alpha_{I,J} dz_i \in C^\infty(X, \Omega_X^{1,0}), \quad \bar{\partial}\alpha_{I,J} = \sum_{i=1}^n \frac{\partial}{\partial \bar{z}_i} \alpha_{I,J} d\bar{z}_i \in C^\infty(X, \Omega_X^{0,1}).$$

Then $d = \partial + \bar{\partial}$ for smooth functions. Back to $d\omega$. Then

$$d\omega = \sum_{I,J} d\alpha_{I,J} dz_I \wedge d\bar{z}_J = \sum_{I,J} \partial\alpha_{I,J} dz_I \wedge d\bar{z}_J + \sum_{I,J} \bar{\partial}\alpha_{I,J} dz_I \wedge d\bar{z}_J.$$

Let

$$\partial\omega = \sum_{I,J} \partial\alpha_{I,J} dz_I \wedge d\bar{z}_J, \quad \bar{\partial}\omega = \sum_{I,J} \bar{\partial}\alpha_{I,J} dz_I \wedge d\bar{z}_J.$$

Then $d = \partial + \bar{\partial}$ for ω .

Lemma 4.16. *The maps*

$$\partial : C^\infty(X, \Omega_X^{p,q}) \rightarrow C^\infty(X, \Omega_X^{p+1,q}), \quad \bar{\partial} : C^\infty(X, \Omega_X^{p,q}) \rightarrow C^\infty(X, \Omega_X^{p,q+1})$$

are linear. They satisfy the Leibnitz rule. That is, if $\omega \in C^\infty(X, \Omega_X^{p,q})$ and $\eta \in C^\infty(X, \Omega_X^{p',q'})$, then

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{p+q} \omega \wedge \partial\eta, \quad \bar{\partial}(\omega \wedge \eta) = \bar{\partial}\omega \wedge \eta + (-1)^{p+q} \omega \wedge \bar{\partial}\eta.$$

Proof. d satisfies the Leibnitz rule

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{p+q} \omega \wedge d\eta,$$

since $\omega \in C^\infty(X, \Omega_{X,\mathbb{C}}^{p+q})$, so

$$\begin{aligned} \partial(\omega \wedge \eta) + \bar{\partial}(\omega \wedge \eta) &= (\partial\omega + \bar{\partial}\omega) \wedge \eta + (-1)^{p+q} \omega \wedge (\partial\eta + \bar{\partial}\eta) \\ &= (\partial\omega \wedge \eta + (-1)^{p+q} \omega \wedge \partial\eta) + (\bar{\partial}\omega \wedge \eta + (-1)^{p+q} \omega \wedge \bar{\partial}\eta). \end{aligned}$$

Then $\partial(\omega \wedge \eta)$ and $\partial\omega \wedge \eta + (-1)^{p+q} \omega \wedge \partial\eta$ are $(p+1, q)$ -forms, and $\bar{\partial}(\omega \wedge \eta)$ and $\bar{\partial}\omega \wedge \eta + (-1)^{p+q} \omega \wedge \bar{\partial}\eta$ are $(p, q+1)$ -forms. Forms of the same type in the decomposition of $d(\omega \wedge \eta)$ must coincide. \square

Lemma 4.17. $\partial^2 = \bar{\partial}^2 = \bar{\partial}\partial + \partial\bar{\partial} = 0$.

Proof. Let $\omega \in C^\infty(X, \Omega_X^{p,q})$. Because $d^2 = 0$,

$$0 = d^2\omega = (\partial + \bar{\partial})((\partial + \bar{\partial})\omega) = \partial^2\omega + \partial\bar{\partial}\omega + \bar{\partial}\partial\omega + \bar{\partial}^2\omega.$$

Then $d^2\omega$ is a $(p+q+2)$ -form, $\partial^2\omega$ is a $(p+2, q)$ -form, $\partial\bar{\partial}\omega + \bar{\partial}\partial\omega$ is a $(p+1, q+1)$ -form, and $\bar{\partial}^2\omega$ is a $(p, q+2)$ -form. Forms of the same type in the decomposition of $d^2\omega$ must coincide. \square

Let X be a complex manifold. Fix $p, q \geq 0$. Let

$$\begin{aligned} \mathcal{Z}^{p,q}(X) &= \text{Ker} \left(\bar{\partial} : C^\infty(X, \Omega_X^{p,q}) \rightarrow C^\infty(X, \Omega_X^{p,q+1}) \right) \\ &= \{ \omega \in C^\infty(X, \Omega_X^{p,q}) \mid \bar{\partial}\omega = 0 \} \end{aligned}$$

and let

$$\begin{aligned} \mathcal{B}^{p,q}(X) &= \text{Im} \left(\bar{\partial} : C^\infty(X, \Omega_X^{p,q-1}) \rightarrow C^\infty(X, \Omega_X^{p,q}) \right) \\ &= \{ \omega \in C^\infty(X, \Omega_X^{p,q}) \mid \exists \eta \in C^\infty(X, \Omega_X^{p,q-1}), \omega = \bar{\partial}\eta \}. \end{aligned}$$

Since $\bar{\partial}^2 = 0$ we have $\mathcal{B}^{p,q}(X) \subset \mathcal{Z}^{p,q}(X)$ for all p and q . The **Dolbeault cohomology** is

$$H^{p,q}(X) = \mathcal{Z}^{p,q}(X) / \mathcal{B}^{p,q}(X).$$

Exercise. Assume X and Y are biholomorphic complex manifolds. Then $H^{p,q}(X) = H^{p,q}(Y)$.

If $H^{p,q}(X)$ is finite dimensional then we define the **Hodge numbers** of X as

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X).$$

Lemma 4.18. *Let X be a complex manifold. Then*

$$H^{p,0}(X) \cong H^0(X, \Omega_X^p) = \{\text{holomorphic sections of } \Omega_X^p\} = \{\text{holomorphic } p\text{-forms on } X\}.$$