M4P54 Differential Topology

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Syllabus

Contents

0 Introduction

Lecture 1
Thursday
09/01/20

Differential topology is the study of the topology of a manifold using analysis. The topics are

- a review of differential forms,
- de Rham cohomology,
- Morse theory, and
- singular homology.

The following are references.

- J M Lee, Introduction to smooth manifolds, 2000
- L W Tu, Introduction to smooth manifolds, 2008
- J Milnor, Morse theory, 1960
- A Banyaga and D Hurtubise, Lectures on Morse homology, 2004

1 Differential forms on manifolds

1.1 Alternating p-forms on a vector space

Let V be a vector space over $\mathbb{R}$, and let $p \geq 0$. Then $V^p = V \times \cdots \times V$.

Definition 1.1. A multilinear map $\omega : V^p \to \mathbb{R}$ is called an **alternating p-form** if we have

$$\omega\left(v_{\sigma(1)}, \dots, v_{\sigma(p)}\right) = \epsilon(\sigma)\,\omega(v_1, \dots, v_p), \qquad v_1, \dots, v_p \in V \qquad \sigma \in \mathcal{S}_p,$$

where $\mathcal{S}_p$ is the group of permutations of p elements and $\epsilon(\sigma)$ is the signature of σ.

Recall that if m is the number of transpositions in a decomposition of σ, then $\epsilon(\sigma) = (-1)^m$, where a **transposition** is $(a_i a_j)$ for $a_i \neq a_j$.

Notation 1.2. $\Lambda^p V^* = \{\text{alternating } p\text{-forms } \omega \text{ on } V\}$ is called the **p-th exterior power** of V.

Check that it is a vector space. [1]

Example 1.3.

- $\Lambda^0 V^* = \mathbb{R}$.
- $\Lambda^1 V^* = V^* = \mathrm{Hom}(V, \mathbb{R})$, the **dual** of V.

Definition 1.4. Let $\omega_1 \in \Lambda^p V^*$ and $\omega_2 \in \Lambda^q V^*$. We define the **exterior product** $\omega_1 \wedge \omega_2 \in \Lambda^{p+q} V^*$ of ω_1 and ω_2 by

$$\omega_1 \wedge \omega_2 (v_1, \dots, v_{p+q}) = \sum_{\sigma \in \mathcal{S}_{p,q}} \epsilon(\sigma)\,\omega_1\left(v_{\sigma(1)}, \dots, v_{\sigma(p)}\right) \omega_2\left(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}\right), \qquad v_1, \dots, v_{p+q} \in V,$$

where

$$\mathcal{S}_{p,q} = \{\sigma \in \mathcal{S}_{p+q} \mid \sigma(1) < \cdots < \sigma(p),\ \sigma(p+1) < \cdots < \sigma(p+q)\}.$$

Example 1.5.

- Assume $\omega_1, \omega_2 \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \omega_2 (v_1, v_2) = \omega_1(v_1)\,\omega_2(v_2) - \omega_1(v_2)\,\omega_2(v_1), \qquad v_1, v_2 \in V.$$

- Assume $\omega_1, \dots, \omega_p \in \Lambda^1 V^*$. Then

$$\omega_1 \wedge \cdots \wedge \omega_p (v_1, \dots, v_p) = \det\left(\omega_i(v_j)\right)_{i,j=1,\dots,p}, \qquad v_1, \dots, v_p \in V.$$

Proposition 1.6. *Let $\omega_i \in \Lambda^{p_i} V^*$ for $i = 1, 2, 3$.*

- *Associativity* $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$.
- *Distributivity* $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$, *assuming* $p_2 = p_3$.
- *Supercommutativity* $\omega_1 \wedge \omega_2 = (-1)^{p_1 \cdot p_2}\,\omega_2 \wedge \omega_1$.

Definition 1.7. Let $\Phi : V \to W$ be a linear map between vector spaces over $\mathbb{R}$. Let $\omega \in \Lambda^p W^*$. Then the **pull-back** $\Phi^*(\omega) \in \Lambda^p V^*$ of ω is an alternating p-form on V defined by

$$\Phi^*(\omega)(v_1, \dots, v_p) = \omega(\Phi(v_1), \dots, \Phi(v_p)), \qquad v_1, \dots, v_p \in V.$$

[1] Exercise

Proposition 1.8. *Given $\Phi : V \to W$ a linear map,*

- *the pull-back*
$$\begin{array}{cccc} \Phi^* & : & \Lambda^p W^* & \longrightarrow & \Lambda^p V^* \\ & & \omega & \longmapsto & \Phi^*(\omega) \end{array}$$
is a linear map that preserves exterior products, that is
$$\Phi^*(\omega_1 \wedge \omega_2) = \Phi^*(\omega_1) \wedge \Phi^*(\omega_2), \qquad \omega_1 \in \Lambda^p W^*, \qquad \omega_2 \in \Lambda^q W^*,$$
- *if $\Psi : W \to Z$ is linear then*
$$(\Psi \circ \Phi)^*(\omega) = \Phi^*(\Psi^*(\omega)), \qquad \omega \in \Lambda^p Z^*,$$
- *assuming $V = W$ and $p = \dim V$, then*
$$\Phi^*(\omega) = (\det \Phi)\,\omega, \qquad \omega \in \Lambda^p V^*.$$

1.2 Differential forms on manifolds

Let M be a smooth manifold of dimension n, and let $x \in M$. Then the tangent space $\mathrm{T}_x M$ of M at x is a vector space of dimension n.

Notation 1.9. Let
$$\Lambda^p \mathrm{T}_x^* M = \Lambda^p (\mathrm{T}_x M)^*.$$
Consider the set
$$\Lambda^p \mathrm{T}^* M = \bigsqcup_{x \in M} \Lambda^p \mathrm{T}_x^* M,$$
the **p-th exterior bundle** on M. There exists a morphism $\pi : \Lambda^p \mathrm{T}^* M \to M$ such that for all $x \in M$, $\pi^{-1}(x) = \Lambda^p \mathrm{T}_x^* M$, so $\Lambda^p \mathrm{T}^* M$ is a vector bundle and it is a smooth manifold, and π is a smooth morphism.

Example 1.10.

- $\Lambda^0 \mathrm{T}^* M = M \times \mathbb{R}$.
- $\Lambda^1 \mathrm{T}^* M$ is the **cotangent bundle**, the dual of the tangent bundle.

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Definition 1.11. A **differential p-form** ω on M is a smooth section of π. That is, it is a smooth morphism $\omega : M \to \Lambda^p \mathrm{T}^* M$ such that $\pi \circ \omega = \mathrm{id}_M$.

Thus, $\omega(x) \in \Lambda^p \mathrm{T}_x^* M$.

Notation 1.12.
$$\Omega^p(M) = \{\text{differential } p\text{-forms } \omega \text{ on } M\}, \qquad \Omega^\bullet(M) = \bigoplus_p \Omega^p(M).$$

Example 1.13.
$$\Omega^0(M) \cong \{f : M \to \mathbb{R}\ \mathrm{C}^\infty\text{-function}\}.$$

Exercise. If $n = \dim M$, then $\Omega^{n+1}(M) = 0$.

The algebra is the same as last week.

Definition 1.14. Let $\omega_1 \in \Omega^p(M)$ and $\omega_2 \in \Omega^q(M)$. Then $\omega_1 \wedge \omega_2 \in \Omega^{p+q}(M)$ is defined by
$$\omega_1 \wedge \omega_2(x) = \omega_1(x) \wedge \omega_2(x) \in \Lambda^{p+q} \mathrm{T}_x^* M, \qquad x \in M.$$

By Proposition 1.6, associativity, distributivity, and supercommutativity hold for $\Omega^p(M)$. Let $F : M \to N$ be a smooth morphism between manifolds. Then for all $x \in M$, the differential of F at x is the linear map
$$\mathrm{D}F_x : \mathrm{T}_x M \to \mathrm{T}_{F(x)} N.$$

Thus, for all $p \geq 0$, we have a natural map, called the **pull-back**,

$$F_x^* \quad : \quad \begin{array}{rcl} \Lambda^p \mathrm{T}_{F(x)}^* N & \longrightarrow & \Lambda^p \mathrm{T}_x^* M \\ \omega\left(v_1, \dots, v_p\right) & \longmapsto & \omega\left(\mathrm{D}F_x\left(v_1\right), \dots, \mathrm{D}F_x\left(v_p\right)\right) \end{array}, \qquad \omega \in \Lambda^p \mathrm{T}_{F(x)}^* N, \qquad v_1, \dots, v_p \in \mathrm{T}_x^* M.$$

Thus, we can define

$$F^* \quad : \quad \begin{array}{rcl} \Omega^p\left(N\right) & \longrightarrow & \Omega^p\left(M\right) \\ \omega\left(x\right) & \longmapsto & F^*\left(\omega\left(F\left(x\right)\right)\right) \end{array}, \qquad \omega \in \Omega^p\left(N\right).$$

By Proposition 1.8, the pull-back preserves the exterior product, so

$$F^*\left(\omega_1 \wedge \omega_2\right) = F^*\left(\omega_1\right) \wedge F^*\left(\omega_2\right).$$

If $G : N \to P$,

$$\left(G \circ F\right)^*\left(\omega\right) = F^*\left(G^*\left(\omega\right)\right).$$

1.3 Local description of p-forms

Let M be a manifold of dimension n, let $x_0 \in M$, let (U, ϕ) be a local chart around x_0, and let $(x_1, \dots, x_n)$ be local coordinates around x_0. A basis of $\mathrm{T}_{x_0} M$ is given by

$$\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}.$$

A basis of $\mathrm{T}_{x_0}^* M$ is given by

$$\left\{\mathrm{d}x_1, \dots, \mathrm{d}x_n\right\}, \qquad \mathrm{d}x_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}.$$

A basis of $\Lambda^p \mathrm{T}_{x_0}^* M$ is

$$\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad i_1 < \dots < i_p.$$

Thus, $\omega \in \Omega^p\left(M\right)$ is locally given by

$$\omega\left(x\right) = \sum_{|I|=p} f_I\left(x\right) \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad I = \left(i_1, \dots, i_p\right), \qquad i_1 < \dots < i_p,$$

where f_I is a C^∞-function on U for all I.

Example 1.15. Let $F : M \to N$ be a smooth morphism between manifolds of dimension n, and let $\omega \in \Omega^n\left(N\right)$. Locally,

$$\omega\left(y\right) = f\left(y\right) \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n, \qquad y \in N$$

for some $f \in \mathrm{C}^\infty$. Proposition 1.8 implies that

$$F^*\left(\omega\right)\left(x\right) = \left(f \circ F\right)\left(x\right) \det \mathrm{D}F_x \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n, \qquad x \in M,$$

where $y_i = p_i \circ F$ and $p_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th projection.

Let $f : M \to \mathbb{R}$ be a smooth function, so $f \in \Omega^0\left(M\right)$. Locally, the **differential** is

$$\mathrm{d} \quad : \quad \begin{array}{rcl} \Omega^0\left(M\right) & \longrightarrow & \Omega^1\left(M\right) \\ f & \longmapsto & \sum_{i=1}^n \frac{\partial f}{\partial x_i} \mathrm{d}x_i \end{array}.$$

Check that $\mathrm{d}f \in \Omega^1\left(M\right)$, so $\mathrm{d}f$ is a 1-form on M. Alternatively, $\mathrm{d}f = f^*\left(\mathrm{d}x\right)$ for $\mathrm{d}x$ a 1-form on $\mathbb{R}$, or $\mathrm{d}f\left(X\right) = X\left(f\right)$ for any vector field X on M. More in general, let $\omega \in \Omega^p\left(M\right)$. Locally,

$$\omega = \sum_{|I|=p} f_I \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p}, \qquad f_I \in \mathrm{C}^\infty,$$

so $\mathrm{d}\omega \in \Omega^{p+1}\left(M\right)$. Then the **de Rham differential** is

$$\mathrm{d} \quad : \quad \begin{array}{rcl} \Omega^p\left(M\right) & \longrightarrow & \Omega^{p+1}\left(M\right) \\ \omega & \longmapsto & \sum_{|I|=p} \mathrm{d}f_I \wedge \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_p} \end{array}.$$

Proposition 1.16.

- *The Leibniz rule*
$$\mathrm{d}\left(\omega_1 \wedge \omega_2\right) = \mathrm{d}\omega_1 \wedge \omega_2 + \left(-1\right)^p \omega_1 \wedge \mathrm{d}\omega_2, \qquad w_1 \in \Omega^p\left(M\right), \qquad \omega_2 \in \Omega^q\left(M\right).$$
- $\mathrm{d}^2 = 0$*, that is*
$$\mathrm{d}\left(\mathrm{d}\omega\right) = 0, \qquad \omega \in \Omega^p\left(M\right).$$
- *Let $F : M \to N$ be a smooth morphism between manifolds. Then*
$$F^*\left(\mathrm{d}\omega\right) = \mathrm{d}\left(F^*\left(\omega\right)\right), \qquad \omega \in \Omega^p\left(M\right)$$
so
$$\begin{array}{ccc} \Omega^p\left(M\right) & \xrightarrow{\mathrm{d}} & \Omega^{p+1}\left(M\right) \\ {\scriptstyle F^*}\uparrow & & \uparrow{\scriptstyle F^*} \\ \Omega^p\left(N\right) & \xrightarrow[\mathrm{d}]{} & \Omega^{p+1}\left(N\right) \end{array}.$$

Definition 1.17. Lecture 3 Tuesday 14/01/20

- $\omega \in \Omega^p\left(M\right)$ is **closed** if $\mathrm{d}\omega = 0$.
- $\omega \in \Omega^p\left(M\right)$ is **exact** if there exists $\omega' \in \Omega^{p-1}\left(M\right)$ such that $\mathrm{d}\omega' = \omega$.

ω is exact implies that ω is closed, since if $\omega = \mathrm{d}\omega'$ then $\mathrm{d}\omega = \mathrm{d}^2\omega' = 0$.

1.4 Integrations on manifolds

Let M be a manifold of dimension n, let $F : M \to M$ be a smooth morphism, and let $\omega \in \Omega^n\left(M\right)$. Then
$$F^*\left(\omega\right)\left(x\right) = \det \mathrm{D}F_x \omega\left(F\left(x\right)\right).$$
Locally, assume $\omega = f \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n$ for some coordinates $y_1, \dots, y_n$ and $f \in \mathrm{C}^\infty$. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas of M, where $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$. Then
$$h_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha\left(U_\alpha \cap U_\beta\right) \subset \mathbb{R}^n \to \phi_\beta\left(U_\alpha \cap U_\beta\right) \subset \mathbb{R}^n,$$
such that
$$h_{\alpha\beta}^*\left(\omega\right)\left(x\right) = \left(f \circ h_{\alpha\beta}\right)\left(x\right) \det \left(\mathrm{D}h_{\alpha\beta}\right)_x \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$
Let $D \subset \mathbb{R}^n$ be compact such that ∂D has zero measure, so D is a domain of integration, let $f : U \to \mathbb{R}$ be a C^∞-function where $U \subset \mathbb{R}^n$ is open such that $D \subset U$, and let $h : U \to h\left(U\right)$ be a diffeomorphism. Then
$$\int_{h^{-1}(D)} f\left(y\right)\, \mathrm{d}y_1 \dots \mathrm{d}y_n = \int_{h^{-1}(D)} f\left(y\right)\, \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n = \int_D \left(f \circ h\right)\left(x\right) \lvert \det \mathrm{D}h_x \rvert\, \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$
Let us assume that $\omega = f\left(y\right) \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n$ on U. We define
$$\int_D \omega = \int_D f\left(y\right)\, \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n, \qquad D \subset U.$$
Definition 1.18. Let $U \subset \mathbb{R}^n$ be an open set. We define the **support** of ω as
$$\operatorname{supp} \omega = \overline{\{x \in U \mid \omega\left(x\right) \ne 0\}}, \qquad \omega\left(x\right) \in \Lambda^p \mathrm{T}_x^* U.$$
Then ω has **compact support**, if $\operatorname{supp} \omega$ is compact.

Fact. Under this assumption, we can define
$$\int_U \omega = \int_D \omega \in \mathbb{R},$$
which is well-defined. Under the same assumption, if $\phi : V \to U$ is a diffeomorphism, provided that $\det \mathrm{D}\phi_x > 0$, since $\det \mathrm{D}\phi_x \ne 0$ for all x, then
$$\int_U \omega = \int_V \phi^*\left(\omega\right).$$

1.5 Orientation

Let V be a vector space over $\mathbb{R}$ of dimension n, and let $B = (b_1, \dots, b_n) \subset V$ and $B' = (b_1', \dots, b_n') \subset V$ be ordered bases of V. Then B and B' have the **same orientation** if $\det T > 0$ where

$$\begin{array}{rccc} T & : & V & \longrightarrow & V \\ & & b_i & \longmapsto & b_i' \end{array}$$

is a linear map. Let $\omega \in \Lambda^n V^*$ for $\omega \neq 0$. Then B and B' have the same orientation if and only if $\omega(b_1, \dots, b_n)$ has the same sign as $\omega(b_1', \dots, b_n')$, by Proposition 1.8. An **orientation** Λ of V is a set of all the ordered basis of V with the same orientation. Let $\phi : V \to W$ be an isomorphism of vector spaces with fixed orientations Λ_v and Λ_w respectively. We say that ϕ is **orientation preserving** if an ordered basis of V induces an ordered basis of W, so Λ_v induces Λ_w. Let $V = \mathbb{R}^n$, and let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. Then $e_1, \dots, e_n$ defines an orientation of V called **positive**. Let M be a manifold. The idea is to find an orientation Λ_x of $\mathrm{T}_x M$ for all $x \in M$.

Special case. Let $M = U \subset \mathbb{R}^n$ be open. There exists a natural isomorphism $\phi_x : \mathrm{T}_x U \to \mathbb{R}^n$. Let Λ_x^+ be an orientation on $\mathrm{T}_x U$ such that ϕ_x is orientation preserving with respect to the positive orientation on $\mathbb{R}^n$. Let $\Lambda^+ = \{\Lambda_x^+\}$.

General case. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M. On U_α, we define the orientation so that $(\mathrm{D}\phi_\alpha)_x : \mathrm{T}_x U_\alpha \to \mathrm{T}_{\phi_\alpha(x)} \phi_\alpha(U) \subset \mathbb{R}^n$ is orientation preserving. This is called the positive orientation on the chart (U_α, ϕ_α). We define Λ on M, which is a collection of Λ^+ on $\mathrm{T}_x M$ for all $x \in M$. Then M is **orientable** if there exists an atlas with positive orientation charts. This coincides in assuming that $\det \mathrm{D}\left(\phi_\beta^{-1} \circ \phi_\alpha\right) > 0$ for all α and β.

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Notation 1.19. For all $p \geq 0$,

$$\Omega_{\mathrm{c}}^p(M) = \{\omega \in \Omega^p(M) \mid \operatorname{supp} M \text{ is compact}\}.$$

If M is compact $\Omega_{\mathrm{c}}^p(M) = \Omega^p(M)$. Let $\omega \in \Omega_{\mathrm{c}}^r(M)$. Assume $\operatorname{supp} \omega \subset U$ where (U, ϕ) is a chart of M, and $\phi : U \to \phi(U) \subset \mathbb{R}^n$. Assume also that (U, ϕ) is positively oriented. Let $\phi^{-1} : \phi(U) \to U$ such that $\left(\phi^{-1}\right)^*(\omega) \in \Omega_{\mathrm{c}}^n(\phi(U))$, that is $\operatorname{supp}\left(\phi^{-1}\right)^*(\omega) \subset \phi(U)$. We define

$$\int_M \omega = \int_{\phi(U)} \left(\phi^{-1}\right)^*(\omega). \tag{1}$$

We need to show that, under the assumptions above, $\int_M \omega$ does not depend on (U, ϕ). Let $\left(\overline{U}, \overline{\phi}\right)$ be also a positively oriented chart such that $\operatorname{supp} \omega \subset \overline{U}$. We want to show that

$$\int_{\phi(U)} \left(\phi^{-1}\right)^*(\omega) = \int_{\overline{\phi}(\overline{U})} \left(\overline{\phi}^{-1}\right)^*(\omega).$$

Let $\overline{\phi} \circ \phi^{-1} : \phi\left(U \cap \overline{U}\right) \to \overline{\phi}\left(U \cap \overline{U}\right)$, so

$$\begin{tikzcd} & U \cap \overline{U} \arrow[dl, "\phi"'] \arrow[dr, "\overline{\phi}"] & \\ \mathbb{R}^n \supset \phi\left(U \cap \overline{U}\right) \arrow[rr, "\overline{\phi} \circ \phi^{-1}"'] & & \overline{\phi}\left(U \cap \overline{U}\right) \subset \mathbb{R}^n \end{tikzcd}.$$

Since both charts are positively oriented the determinant of the differential $\mathrm{D}\left(\overline{\phi} \circ \phi^{-1}\right)$ is positive, so

$$\begin{aligned} \int_{\overline{\phi}(U)} \left(\overline{\phi}^{-1}\right)^*(\omega) &= \int_{\overline{\phi}(U \cap \overline{U})} \left(\overline{\phi}^{-1}\right)^*(\omega) = \int_{\overline{\phi}(U \cap \overline{U})} \left(\overline{\phi} \circ \phi^{-1}\right)^* \left(\overline{\phi}^{-1}\right)^*(\omega) = \int_{\overline{\phi}(U \cap \overline{U})} \left(\phi^{-1}\right)^* \overline{\phi}^* \left(\overline{\phi}^{-1}\right)^*(\omega) \\ &= \int_{\overline{\phi}(U \cap \overline{U})} \left(\phi^{-1}\right)^* \left(\overline{\phi}^{-1} \circ \overline{\phi}\right)^*(\omega) = \int_{\overline{\phi}(U \cap \overline{U})} \left(\phi^{-1}\right)^*(\omega) = \int_{\overline{\phi}(U)} \left(\phi^{-1}\right)^*(\omega), \end{aligned}$$

by a property of the pull-back and since $\left(\overline{\phi}^{-1}\right)^*(\omega) = 0$ outside $\overline{\phi}\left(U \cap \overline{U}\right)$.

1.6 Partitions of unity

Definition 1.20. Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering. A **partition of unity** with respect to U is a collection of smooth functions $f_\alpha : M \to [0,1]$ such that

1. $\operatorname{supp} f_\alpha = \overline{\{x \in M \mid f_\alpha(x) = 0\}} \subset U_\alpha$ for all α,
2. $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$, and
3. for all $x \in M$, there exists $U \ni x$ open such that $\operatorname{supp} f_\alpha \cap U \neq \emptyset$ for only finitely many α.

Remark. 3 implies that 2 is a finite sum.

Example 1.21. Let

$$M = \mathrm{S}^1 = \left\{x \in \mathbb{R}^2 \mid |x| = 1\right\}, \qquad U_1 = \mathrm{S}^1 \setminus \{(1,0)\}, \qquad U_2 = \mathrm{S}^1 \setminus \{(-1,0)\},$$

so $\{U_i\}$ is a cover. Let

$$f_1(\cos\theta, \sin\theta) = \frac{1}{2} - \frac{1}{2}\cos\theta, \qquad f_2(\cos\theta, \sin\theta) = \frac{1}{2} + \frac{1}{2}\cos\theta.$$

Then f_i is a partition of unity.

Theorem 1.22. *Let M be a manifold, and let $U = \{U_\alpha\}$ be an open covering of M. Then there exists a partition of unity f_α with respect to U.*

Proof. We omit the proof. ☐

Theorem 1.23. *Let M be a manifold, and let $n = \dim M$. Then M is orientable if and only if there exists $\omega \in \Omega^n(M)$ which is never vanishing on M, so $\omega(x) \neq 0$ for all $x \in M$.*

ω is called a **volume form** on M.

Proof.

$\Longleftarrow$ Assume $\omega \in \Omega^n(M)$ is a volume form. We want to construct an orientation Λ on M, that is Λ_x on $\mathrm{T}_x M$ for all $x \in M$. Given an oriented basis $v_1, \dots, v_n$ of $\mathrm{T}_x M$ we say that it is **positively oriented** if $\omega(x)(v_1, \dots, v_n) > 0$. For all $x \in M$, we define the orientation Λ_x on $\mathrm{T}_x M$ by considering the class of positively oriented ordered basis of $\mathrm{T}_x M$ which is compatible with the choice of an atlas on M. Take any atlas $\{(U_\alpha, \phi_\alpha)\}$, where $\phi_\alpha : U_\alpha \to \mathbb{R}^n$. On U_α,

$$\omega = g_\alpha \phi_\alpha^* (\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n).$$

Since $\omega \neq 0$, $g_\alpha > 0$ or $g_\alpha < 0$. If $g_\alpha < 0$ then switch x_1 with x_2, so $g_\alpha > 0$. After this change of coordinates, (U_α, ϕ_α) is positively oriented, so M is orientable.

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$\Longrightarrow$ Assume that M is orientable, that is there exists an atlas $\{(U_\alpha, \phi_\alpha)\}$ of positively oriented charts. On U_α, we consider

$$\omega_\alpha = \phi_\alpha^* (\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n).$$

Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Let $\widetilde{\omega_\alpha} = f_\alpha \omega_\alpha \in \Omega^n(U_\alpha)$. We may assume that $\widetilde{\omega_\alpha} \in \Omega^n(M)$ by extending equal to zero outside U_α. We define $\omega = \sum_\alpha \widetilde{\omega_\alpha} \in \Omega^n(M)$. For all α, since $\sum_\alpha f_\alpha = 1$ there exists α such that $\widetilde{\omega_\alpha} \neq 0$, so $\omega \neq 0$.

☐

Let M be an orientable manifold of dimension n, and let $\omega \in \Omega_\mathrm{c}^n(M)$. We want to define $\int_M \omega$. So far we defined for ω such that $\operatorname{supp}\omega \subset U_\alpha$ where (U_α, ϕ_α) is a chart.

Definition 1.24. Let $\{(U_\alpha, \phi_\alpha)\}$ be a positively oriented atlas on M, and let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $\operatorname{supp} f_\alpha \omega \subset U_\alpha$, so let

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega.$$

Remark 1.25. Note that for each α, we have that the support of $f_\alpha \omega$ is contained in U_α and therefore each term of the sum is well-defined as in (1). Indeed, we have

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} \left(\phi^{-1}\right)^* (f_\alpha).$$

Lemma 1.26. *$\int_M \omega$ does not depend on $\{(U_\alpha, \phi_\alpha)\}$ and f_α.*

Proof. Under the assumption that $\operatorname{supp} \omega \subset U_\alpha$ then we showed $\int_{U_\alpha} \omega$ does not depend on (U_α, ϕ_α). Let $\{(U_\alpha, \phi_\alpha)\}$ and $\left\{\left(\overline{U_\alpha}, \overline{\phi_\alpha}\right)\right\}$ be two atlases with positively oriented charts, and let f_α and $\overline{f_\alpha}$ be two partitions of unity with respect to $\{U_\alpha\}$ and $\left\{\overline{U_\alpha}\right\}$ respectively. Then $\sum_\alpha f_\alpha = \sum_\alpha \overline{f_\alpha} = 1$, so $\int_M f_\alpha \omega = \sum_\beta \int_M \overline{f_\beta} f_\alpha \omega$. Thus

$$\int_M \omega = \sum_\alpha \int_M f_\alpha \omega = \sum_{\alpha,\beta} \int_M \overline{f_\beta} f_\alpha \omega = \sum_\beta \int_M \sum_\alpha f_\alpha \overline{f_\beta} \omega = \sum_\beta \int_M \overline{f_\beta} \omega.$$

□

Proposition 1.27. *Let M and N be orientable manifolds of dimension n, and let $\omega, \eta \in \Omega_c^n(M)$.*

1. *Linearity*

$$\int_M (a\omega + b\eta) = a \int_M \omega + b \int_M \eta.$$

2. *Orientation reversal. Let $\overline{M}$ be the manifold M with opposite orientation $\Lambda^- = \{\Lambda_x^- \mid x \in M\}$, which is the orientation opposite than the one induced by M with orientation Λ. Then*

$$\int_M \omega = - \int_{\overline{M}} \omega.$$

3. *Positivity. Let ω be the volume form on M. Then*

$$\int_M \omega > 0.$$

4. *Diffeomorphism invariance. Let $F : N \to M$ be an orientation preserving diffeomorphism. Then*

$$\int_M \omega = \int_N F^*(\omega).$$

Proof.

1. Exercise. [2]

2. Exercise. [3]

3. Choose a positively oriented chart (U_α, ϕ_α) on U_α, so

$$\omega = g_\alpha \phi_\alpha^* (\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n), \qquad g_\alpha > 0.$$

Then $\int_M \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega$ where f_α is a partition of unity. For all $x \in M$ there exists α such that $x \in U_\alpha$ and $\int_{U_\alpha} f_\alpha \omega > 0$, so $\int_M \omega > 0$.

4. Let (U_α, ϕ_α) be a positively oriented atlas on M. Then $\left(F^{-1}(U_\alpha), \phi_\alpha \circ F\right)$ is an atlas on N which is positively oriented. Let f_α be a partition of unity with respect to $\{U_\alpha\}$. Then $f_\alpha \circ F$ is a partition of the unity with respect to $\left\{F^{-1}(U_\alpha)\right\}$, so

$$\int_N F^*(\omega) = \sum_\alpha \int_N (f_\alpha \circ F) F^*(\omega) = \sum_\alpha \int_N F^*(f_\alpha \omega) = \sum_\alpha \int_M f_\alpha \omega = \int_M \omega.$$

□

[2] Exercise
[3] Exercise

1.7 Manifolds with boundary

Denote
$$\mathbb{R}^n_{\geq 0} = (\mathbb{R}_{\geq 0})^n, \qquad \mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Let $U \subset \mathbb{R}^n_+$ be open, and let $F : U \to \mathbb{R}^m$ be a function. Then F is C^∞ if it can be extended to a C^∞-function $\widetilde{F} : \widetilde{U} \to \mathbb{R}^m$ where $\widetilde{U} \supset U$ and $\widetilde{U}$ is open.

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Definition 1.28. A **manifold with boundary** of dimension n is a Hausdorff topological space M such that there exists an open covering $\{U_\alpha\}$, and for all α, there exists a homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ such that for all α and β,
$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta (U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+ \to \phi_\alpha (U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+$$
is a diffeomorphism, so
$$\begin{tikzcd} & U_\alpha \cap U_\beta \arrow[dl, "\phi_\alpha"'] \arrow[dr, "\phi_\beta"] & \\ \mathbb{R}^n_+ \supset \phi_\alpha (U_\alpha \cap U_\beta) \arrow[rr, "\phi_\alpha \circ \phi_\beta^{-1}"'] & & \phi_\beta (U_\alpha \cap U_\beta) \subset \mathbb{R}^n_+ \end{tikzcd}.$$

The **boundary** of M is
$$\partial M = \left\{x \in M \;\middle|\; \exists \alpha,\ \phi_\alpha (x) \in \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}\right\}.$$

Then (U_α, ϕ_α) is called a **chart** and $\{(U_\alpha, \phi_\alpha)\}$ is called an **atlas**.

Remark 1.29.

- ∂M is closed in M.
- $\mathring{M} = M \setminus \partial M$ is a manifold of dimension n.

Example 1.30.

- $M = [0, 1]$ is a manifold with boundary $\partial M = \{0, 1\}$.
- The closed disc $\mathrm{D} = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ is a manifold with boundary $\partial \mathrm{D} = \mathrm{S}^{n-1}$.
- $M = [0, 1] \times \mathrm{S}^1$ is a manifold with boundary $\partial M = \mathrm{S}^1 \sqcup \mathrm{S}^1$.

Remark 1.31.

- We can define tangent spaces and differential forms exactly in the same way as usual manifolds.
- The definition of orientability is the same. If M is orientable, then ∂M is also orientable. As a convention, the positive orientation on the boundary of $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$ is given by $(-1)^n \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_{n-1}$. This induces a positive orientation on ∂M.
- Also partitions of unity for any open cover U_α of M is defined the same way. If M is orientable, for any manifold with boundary, for all open covering $U = \{U_\alpha\}$, there exists a partition of unity f_α. This implies that if $\omega \in \Omega^n_\mathrm{c} (M)$, then $\int_M \omega$ is defined the same way for manifolds.

1.8 Stokes' theorem

Theorem 1.32 (Stokes). *For any manifold with boundary M of dimension n, and for any $\omega \in \Omega^{n-1}_\mathrm{c} (M)$ we have*
$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \in \Omega^n_\mathrm{c} (M).$$

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas, and let $f_\alpha : M \to \mathbb{R}$ be a partition of unity with respect to this cover. Then $\sum_\alpha f_\alpha = 1$ on M, so

$$\int_M \mathrm{d}\omega = \int_M \mathrm{d}\left(\sum_\alpha f_\alpha \omega\right) = \sum_\alpha \int_M \mathrm{d}(f_\alpha \omega) = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} \left(\phi_\alpha^{-1}\right)^* \left(\mathrm{d}\left(f_\alpha \omega\right)\right).$$

Proposition 1.16 implies that

$$\left(\phi_\alpha^{-1}\right)^* \left(\mathrm{d}\left(f_\alpha \omega\right)\right) = \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right).$$

Then $\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)$ is an $(n-1)$-form on $\phi_\alpha\left(U_\alpha\right)$. In coordinates,

$$\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right) = \sum_{j=1}^n \widetilde{f_\alpha} \omega_j \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n,$$

where ω_j is a smooth function on $\phi_\alpha\left(U_\alpha\right)$ and

$$\begin{tikzcd} U_\alpha \arrow[r, "\widetilde{\phi_\alpha}"] \arrow[d, "f_\alpha"'] & \phi_\alpha\left(U_\alpha\right) \arrow[dl, "\widetilde{f_\alpha}"] \\ {[0,1]} & \end{tikzcd}.$$

Then

$$\begin{aligned} \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right) &= \mathrm{d}\left(\sum_{j=1}^n \widetilde{f_\alpha} \omega_j \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n\right) \\ &= \sum_{j=1}^n \sum_{k=1}^n \frac{\partial}{\partial x_k} \left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_k \wedge \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n \\ &= \sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_j \wedge \mathrm{d}x_1 \wedge \dots \wedge \widehat{\mathrm{d}x_j} \wedge \dots \wedge \mathrm{d}x_n \\ &= \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n, \end{aligned}$$

so

$$\sum_\alpha \int_{\phi_\alpha(U_\alpha)} \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right) = \sum_\alpha \int_{\mathbb{R}_+^n} \mathrm{d}\left(\left(\phi_\alpha^{-1}\right)^* \left(f_\alpha \omega\right)\right),$$

because $\widetilde{f_\alpha} = 0$ outside $\phi_\alpha\left(U_\alpha\right)$. Thus

$$\begin{aligned} \int_M \mathrm{d}\omega &= \sum_\alpha \int_{\mathbb{R}_+^n} \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n \\ &= \sum_\alpha \int_{-\infty}^\infty \dots \int_{-\infty}^\infty \int_0^\infty \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha} \omega_j\right) \mathrm{d}x_n \, \mathrm{d}x_{n-1} \dots \mathrm{d}x_1 \\ &= \sum_\alpha \sum_{j=1}^n \int_{-\infty}^\infty \dots \widehat{\int_{-\infty}^\infty} \dots \int_{-\infty}^\infty \int_0^\infty (-1)^{j-1} \frac{\partial}{\partial x_j} \left(\widetilde{f_\alpha} \omega_j\right)\Big|_{x_n=0} \mathrm{d}x_n \, \mathrm{d}x_{n-1} \dots \widehat{\mathrm{d}x_j} \dots \mathrm{d}x_1 \\ &= \sum_\alpha \int_{-\infty}^\infty \dots \int_{-\infty}^\infty (-1)^{n-1} \left(\widetilde{f_\alpha} \omega_j\right)\Big|_{x_n=0} \mathrm{d}x_{n-1} \dots \mathrm{d}x_1, \end{aligned}$$

since $\left(f_\alpha \omega_j\right)|_{x_n=0} = 0$ for $j = 1, \dots, n-1$, so

$$\int_M \mathrm{d}\omega = \sum_\alpha \int_{-\infty}^\infty \dots \int_{-\infty}^\infty (-1)^{n-1} \left(\widetilde{f_\alpha} \omega_j\right)\Big|_{x_n=0} \mathrm{d}x_{n-1} \dots \mathrm{d}x_1 = \sum_\alpha \int_{\partial U_\alpha} f_\alpha|_{\partial U_\alpha} \omega = \int_{\partial M} \omega,$$

where $\partial U_\alpha = U_\alpha \cap \partial M$. ∎

1.9 Applications of Stokes' theorem

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Theorem 1.33 (Integration by parts). *Let M be an orientable n-dimensional manifold with boundary, let $\omega \in \Omega_{\mathrm{c}}^{p}(M)$, let $\eta \in \Omega_{\mathrm{c}}^{n-p-1}(M)$, and let $p \in \{0, \dots, n-1\}$. Then*

$$\int_{\partial M} \omega \wedge \eta = \int_M \mathrm{d}\omega \wedge \eta + (-1)^p \int_M \omega \wedge \mathrm{d}\eta.$$

Proof.

$$\int_{\partial M} \omega \wedge \eta = \int_M \mathrm{d}(\omega \wedge \eta) = \int_M \left(\mathrm{d}\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}\eta\right),$$

by Stokes and the Leibniz rule. ☐

Theorem 1.34 (Brouwer's fixed point theorem). *Let $D = \{x \in \mathbb{R}^n \mid |x| \le 1\}$ such that $\partial D = \mathrm{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$, and let $f : D \to D$ be a smooth morphism. Then f admits a fixed point, that is there exists $x \in D$ such that $f(x) = x$.*

Proof. Assume that $f(x) \ne x$ for all $x \in D$. For any $x \in D$, consider the ray starting from $f(x)$ and passing through x. Let $g(x)$ be the point where this ray intersects ∂D away from $f(x)$. Note that if $x \in \partial D$ then $g(x) = x$. Then $g : D \to \partial D$. It is easy to check that g is smooth. Since $\partial D = \mathrm{S}^{n-1}$ is orientable by Theorem 1.23 there exists a volume form $\omega \in \Omega^{n-1}(\partial D)$, so $\omega(x) \ne 0$. Since $\omega \in \Omega^{n-1}(\partial D)$, $\mathrm{d}\omega \in \Omega^n(\partial D)$, which is an n-dimensional manifold, so $\mathrm{d}\omega = 0$. Thus

$$0 < \int_{\partial D} \omega = \int_{\partial D} g^*(\omega) = \int_D \mathrm{d}g^*(\omega) = \int_D g^*(\mathrm{d}\omega) = 0,$$

by Stokes, a contradiction. ☐

Example 1.35. Recall any exact form is closed, since $\mathrm{d}^2 = 0$. But the opposite is not always true. Let $M = \mathbb{R}^2 \setminus \{0\}$, and let

$$\omega = \frac{x}{x^2+y^2}\mathrm{d}y - \frac{y}{x^2+y^2}\mathrm{d}x \in \Omega^1(M).$$

Then ω is closed, since

$$\mathrm{d}\omega = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)\mathrm{d}x \wedge \mathrm{d}y - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\mathrm{d}y \wedge \mathrm{d}x = 0.$$

We want to show that ω is not exact. Assume that

$$\omega = \mathrm{d}f, \qquad f \in \Omega^0(M) = \{\mathrm{C}^\infty\text{-function}\}.$$

In particular $\omega = \mathrm{d}f$ on $\mathrm{S}^1 \subset M$. Let

$$\begin{array}{cccc} \gamma & : & [0, 2\pi] & \longrightarrow & \mathrm{S}^1 \\ & & \theta & \longmapsto & (\cos\theta, \sin\theta) \end{array}.$$

Then

$$\int_{\mathrm{S}^1} \omega = \int_0^{2\pi} \gamma^*(\omega) = \int_0^{2\pi} \left(\left(\frac{\cos\theta}{\cos^2\theta + \sin^2\theta}\right)\cos\theta \mathrm{d}\theta - \left(\frac{\sin\theta}{\cos^2\theta + \sin^2\theta}\right)(-\sin\theta)\,\mathrm{d}\theta\right) = \int_0^{2\pi} \mathrm{d}\theta = 2\pi,$$

and

$$\int_{\mathrm{S}^1} \omega = \int_{\mathrm{S}^1} \mathrm{d}f = \int_{\partial \mathrm{S}^1} f = \int_{\emptyset} f = 0,$$

so ω is not exact.

Theorem 1.36. *Let M be an orientable manifold of dimension n without boundary, and let $\omega \in \Omega_{\mathrm{c}}^n(M)$. Assume ω is exact. Then*

$$\int_M \omega = 0.$$

Proof. Easy from Stokes. ☐

Theorem 1.37. *Let M be an orientable manifold of dimension n with boundary, and let $\omega \in \Omega_{\mathrm{c}}^{n-1}(M)$ be a closed form. Then*

$$\int_{\partial M} \omega = 0.$$

Proof. Easy from Stokes. □

Definition 1.38. Let M be an orientable manifold of dimension n, let $\omega \in \Omega_{\mathrm{c}}^{k}(M)$, and let $N \subset M$ be a submanifold of dimension k. We can define

$$\int_{M} \omega = \int_{N} i^* (\omega),$$

where $i : N \hookrightarrow M$ is the inclusion. We will denote

$$\omega|_N = i^* (\omega) \in \Omega_{\mathrm{c}}^{k}(N).$$

Theorem 1.39. *Let M be an oriented manifold of dimension n, let $\omega \in \Omega_{\mathrm{c}}^{k}(M)$, and let $S \subset M$ be a compact orientable submanifold of dimension k such that $\partial S = \emptyset$ and $\int_S \omega \neq 0$. Then*

- *ω is not exact,*
- *$\omega|_S$ is not exact, and*
- *S is not the boundary of an orientable manifold $N \subset M$ of dimension $k+1$.*

Proof. Exercise. [4] □

[4] Exercise