M4P55 Commutative Algebra

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Syllabus

Contents

0 Introduction

Lecture 1 Thursday 03/10/19

The prerequisites are

- groups,
- rings,
- fields, and
- a solid linear algebra.

This course is good for

- algebraic geometry, and
- algebraic number theory.

The following are books.

- M Reid, Undergraduate commutative algebra, 1995
- M F Atiyah and I G Macdonald, Introduction to commutative algebra, 1969

The following is the structure of the course.

- Generalities on rings, such as ideals, and examples.
- Localisation of rings between a ring R and the fraction field K of R, such as $\mathbb{Z}$ and $\mathbb{Q}$.
- Finiteness conditions of Noetherian rings and Artinian rings.
- Integral closure and normal rings, such as $\mathbb{Z}[i] \subset \mathbb{Q}(i)$ and $\mathbb{Z}\left[\sqrt{-3}\right] \subset \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right] \subset \mathbb{Q}\left(\sqrt{-3}\right)$.
- Discrete valuation rings.
- Completion of rings with topology.

1 Rings and ideals

Definition 1.1. A **commutative ring** is a set $(A, +, \cdot, 0, 1)$ such that

1. $(A, +, 0)$ is an abelian group,
2. for all $x, y, z \in A$,
   - $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
   - $x \cdot y = y \cdot x$,
   - $x \cdot (y + z) = x \cdot y + x \cdot z$, and
3. for all $x \in A$, $x \cdot 1 = 1 \cdot x = x$.

Remark 1.2.

- One is uniquely determined by 3, since $1' = 1' \cdot 1 = 1$.
- If $1 = 0$, then $0 = x \cdot 0 = x \cdot 1 = x$, since

$$x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0,$$

so $x \cdot 0 = 0$. So every element is zero. Hence $R = \{0\}$.

Definition 1.3. A **homomorphism of rings** $f : A \to B$ is a map such that for all $x, y \in A$,

$$f(x + y) = f(x) + f(y), \qquad f(xy) = f(x) f(y), \qquad f(1) = 1.$$

Example. If $A \subset B$ is closed under $+$ and $\cdot$, and $1 \in A$, then

$$\begin{array}{ccc} A & \longrightarrow & B \\ x & \longmapsto & x \end{array}$$

is a homomorphism.

Remark 1.4.

- A composition of homomorphisms is a homomorphism.
- An **isomorphism** is a bijective homomorphism.

Definition 1.5. A subset I of a ring A is an **ideal** if I is a subgroup of the additive group $(A, +)$ which is closed under multiplication by elements of A, so $xI \subset I$ for any $x \in A$. Sometimes this is written as $I \triangleleft A$. In this case the **quotient group** A/I is naturally a ring, where $(x + I)(y + I)$ is defined as $xy + I$.

Proposition 1.6. *Let I be an ideal of a commutative ring A. Then there is a natural bijection between the ideals $J \subset A$ such that $I \subset J$ and the ideals of A/I.*

Proof. Let

$$\begin{array}{ccc} A & \longrightarrow & A/I \\ x & \longmapsto & x + I \end{array}$$

be the natural surjective map. Send J to its image under this map. $\square$

Definition 1.7. If $f : A \to B$ is a homomorphism, then

$$\operatorname{Ker} f = \{x \in A \mid f(x) = 0\}$$

is an ideal in A, and

$$\operatorname{Im} f = f(A) \cong A / \operatorname{Ker} f \subset B.$$

2 Polynomials and formal power series

Lecture 2
Tuesday
08/10/19

Definition 2.1. Let R be a ring. The **polynomial ring** with coefficients in R is

$$R[x] = \{a_0 + \cdots + a_n x^n \mid a_i \in R,\ n \in \mathbb{Z}_{\geq 0}\}.$$

The addition is coefficient-wise, and the multiplication is given by the formula

$$\left(\sum_{i\geq 0} a_i x^i\right)\left(\sum_{j\geq 0} b_j x^j\right) = \sum_{i\geq 0}\left(\sum_{j+k=i,\ j\geq 0,\ k\geq 0} a_j b_k\right) x^i,$$

where all but finitely many coefficients are zero. Define

$$R[x_1,\dots,x_n] = R[x_1]\dots[x_n] = \left\{ \sum_{i_1,\dots,i_n\geq 0} a_{i_1,\dots,i_n} x_1^{i_1}\dots x_n^{i_n} \,\middle|\, a_{i_1,\dots,i_n} \in R \right\},$$

where all but finitely many coefficients $a_{i_1,\dots,i_n}$ are equal to zero.

Definition 2.2. The **ring of formal power series** with coefficients in R is

$$R[[t]] = \{a_0 + a_1 t + \dots \mid a_i \in R\}.$$

The addition is coefficient-wise, and the multiplication is given by the formula

$$\left(\sum_{i\geq 0} a_i t^i\right)\left(\sum_{j\geq 0} b_j t^j\right) = \sum_{i\geq 0}\left(\sum_{j+k=i,\ j\geq 0,\ k\geq 0} a_j b_k\right) x^i.$$

Define

$$R[[t_1,\dots,t_n]] = R[[t_1]]\dots[[t_n]].$$

In $R[[t]]$ many products equal one unlike in $R[t]$, for example $(1-t)(1+t+\dots) = 1$.

3 Zero-divisors, nilpotents, units

Definition 3.1. Let A be a ring. An element $x \in A$ is a **zero-divisor** if $x \neq 0$ but $xy = 0$ for some $y \neq 0$ in A. A ring without zero-divisors is called an **integral domain**. An element $x \in A$ is **nilpotent** if $x^n = 0$ for some $n \in \mathbb{Z}_{>0}$. A **unit** $x \in A$ is an element such that $xy = 1$ for some $y \in A$. The units of A form a group under multiplication, denoted by A^*, or $A^\times$.

Definition 3.2. Let $x \in A$. Then the set

$$\langle x\rangle = \{xy \mid y \in A\}$$

is an ideal. Such ideals are called **principal ideals**.

Remark. $x \in A^*$ if and only if $\langle x\rangle = A$, and R is a field if and only if $R^* = R \setminus \{0\}$.

Proposition 3.3. *Let A be a non-zero ring. Then the following are equivalent.*

1. *A is a field.*
2. *There are no ideals in A other than $\langle 0\rangle$ and A.*
3. *Every non-zero homomorphism $f : A \to B$ is injective.*

Proof.

$1 \implies 2$ Clear.

$2 \implies 3$ $\operatorname{Ker} f \subset A$ is an ideal. Since $f \neq 0$, $\operatorname{Ker} f \neq A$. Hence $\operatorname{Ker} f = 0$.

$3 \implies 1$ Take any $x \neq 0$ in A. Look at $\langle x\rangle$. Define $B = A/\langle x\rangle$. Then take $f : A \to B$ to be the natural surjective map. If f is not identically zero, we get a contradiction with 3.

□

4 Prime ideals and maximal ideals

Definition 4.1. An ideal $I \subset A$ is called **prime** if $I \neq A$ and if whenever $xy \in I$, then $x \in I$ or $y \in I$. An ideal $J \subset A$ is called **maximal** if there is no ideal J' such that $J \subsetneq J' \subsetneq A$.

Notation. The set of prime ideals of A is called the **spectrum** of A and is denoted by $\operatorname{Spec} A$.

Lemma 4.2. *An ideal $I \subset A$ is prime if and only if A/I is an integral domain.*

Proof. Obvious. ☐

Lemma 4.3. *An ideal $J \subset A$ is maximal if and only if A/J is a field.*

Proof. Obvious. ☐

Proposition 4.4. *If $f : A \to B$ is a ring homomorphism and $I \subset B$ is a prime ideal, then $f^{-1}(I)$ is a prime ideal of A.*

Proof. It is easy to see that $f^{-1}(I)$ is an ideal in A. Suppose $xy \in f^{-1}(I)$ for some $x, y \in A$. Then $f(x) f(y) = f(xy) \in I$. Since I is prime, $f(x) \in I$ or $f(y) \in I$, so $x \in f^{-1}(I)$ or $y \in f^{-1}(I)$. ☐

So we get a canonical map

$$\begin{array}{cccc} f^* : & \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\ & I \subset B & \longmapsto & f^{-1}(I) \subset A \end{array}.$$

Lecture 3 Wednesday 09/10/19

Remark 4.5. If $f : A \to B$ is a ring homomorphism, then $f^{-1}(\mathfrak{p})$, where $\mathfrak{p} \subset B$ is a prime ideal, is a prime ideal. But this is false for maximal ideals. Let $A = \mathbb{Z}$, let $B = \mathbb{Q}$, and let $f(x) = x$. Then $\langle 0 \rangle \subset \mathbb{Q}$ is a maximal ideal and $f^{-1}(\langle 0 \rangle) = \langle 0 \rangle \subset \mathbb{Z}$ is not a maximal ideal. For example, $\langle 0 \rangle \subsetneq \langle 2 \rangle \subsetneq \mathbb{Z}$.

Theorem 4.6. *Let A be a non-zero ring. Then A has at least one maximal ideal. In particular,* $\operatorname{Spec} A$ *is not empty.*

The proof is based on Zorn's lemma. Let S be a set. Then a **partial order** is a binary relation $\leq$ such that

- $x \leq x$ for all $x \in S$,
- $x \leq y \leq z$ implies that $x \leq z$, and
- $x \leq y$ and $y \leq x$ imply that $x = y$,

where not all pairs are comparable. A **chain** $T \subset S$ is a subset in which every two elements are comparable.

Lemma 4.7 (Zorn)**.** *Suppose that S is a partially ordered set such that every chain $T \subset S$ has an upper bound, that is an element $t \in S$ such that $x \leq t$ for all $x \in T$. Then S has a maximal element, that is there exists $s \in S$ such that if $x \in S$ and $x \geq s$, then $x = s$.*

Zorn's lemma is equivalent to the axiom of choice.

Proof of Theorem 4.6. Let Σ be the set of all ideals of A which are not equal to A. Then $\langle 0 \rangle \in \Sigma$, so $\Sigma \neq \emptyset$. Equip Σ with partial order given by inclusion. Enough to check the assumption of Zorn's lemma. Suppose T is a chain of ideals, so it is a collection of ideals J_i for $i \in T$. Consider instead

$$I = \bigcup_{i \in T} J_i.$$

Claim that T is a chain implies that I is an ideal. Then $x \in I$ implies that $x \in J_i$ for some i. Take any $x, y \in I$. Then $x \in J_i$ and $y \in J_k$ for some $i, k \in T$, so T is a chain, hence $i \leq k$ or $k \leq i$, so $J_i \subset J_k$ or $J_k \subset J_i$. Without loss of generality assume $J_i \subset J_k$. Then $x, y \in J_k$, so $x + y \in J_k \subset I$. Clearly, I is an upper bound. ☐

Corollary 4.8. *Any ideal of A is contained in a maximal ideal of A.*

Proof. If $I \subset A$ is an ideal, apply Theorem 4.6 to A/I. □

Corollary 4.9. *Any non-unit of A is contained in a maximal ideal.*

Proof. Apply Corollary 4.8 to $\langle a \rangle$. □

Example. The maximal ideals of $\mathbb{Z}$ are $\langle p \rangle$, where p is prime.

Definition 4.10. A ring A is **local** if A has exactly one maximal ideal.

Example. Any field is a local ring. If k is a field, then $k[[t]]$ is a local ring.

Lemma 4.11 (Prime avoidance)**.** *Let A be a ring and let $\mathfrak{p} \subset A$ be a prime ideal. Suppose that $I_1, \dots, I_n$ are ideals in A such that $\bigcap_{j=1}^n I_j \subset \mathfrak{p}$. Then $I_j \subset \mathfrak{p}$ for some j. If, moreover, $\bigcap_{j=1}^k I_j = \mathfrak{p}$, then $I_j = \mathfrak{p}$ for some j.*

Proof. Suppose that I_j is not a subset of $\mathfrak{p}$ for any j. Then there exists $x_j \in I_j$ such that $x_j \notin \mathfrak{p}$. Hence

$$x_1, \dots, x_n \in I_1 \dots I_n \subset \bigcap_{j=1}^n I_j \subset \mathfrak{p},$$

so $x_1 (x_2 \dots x_n) \in \mathfrak{p}$. Then $x_1 \notin \mathfrak{p}$ implies that $x_2 \dots x_n \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime we get a contradiction. For the second claim, we know that some $I_j \subset \mathfrak{p}$. But $\mathfrak{p} = \bigcap_{j=1}^k I_j \subset I_k$ for all k. Hence $\mathfrak{p} = I_j$. □

5 Nilradical and the Jacobson radical

Lecture 4 Thursday 10/10/19

Proposition 5.1. *The set $\mathcal{N}(A)$ consisting of all nilpotents of the ring A and zero is an ideal. Then $\mathcal{N}(A)$ is called the **nilradical** of A. The quotient $A/\mathcal{N}(A)$ has no nilpotents.*

Proof. Suppose $x \in A$ is nilpotent, so $x^n = 0$. For any $a \in A$, $(ax)^n = a^n x^n = 0$. Let x and y be nilpotents. Say $x^n = y^m = 0$. Then

$$(x+y)^{n+m} = \sum_{i,j \geq 0,\ i+j=n+m} a_{ij} x^i y^j, \qquad a_{ij} \in A.$$

Clearly, either $i \geq n$ or $j \geq m$. Then $a_{ij} x^i y^j = 0$. Therefore, $(x+y)^{n+m} = 0$, hence $x + y \in \mathcal{N}(A)$. If $x + \mathcal{N}(A)$ is nilpotent in $A/\mathcal{N}(A)$, then $x^n + \mathcal{N}(A) = \mathcal{N}(A)$ is the trivial coset. Hence $x^n \in \mathcal{N}(A)$. Thus $(x^n)^m = 0$ for some m. □

Definition 5.2. A ring A such that $\mathcal{N}(A) = 0$ is called a **reduced ring**.

Proposition 5.3. *$\mathcal{N}(A)$ is the intersection of all prime ideals of A.*

Proof.

- $\subset$ Let I be the intersection of all prime ideals of A. Let $f \in A$ be such that $f^n = 0$. Take any prime ideal $\mathfrak{p} \subset A$. We know that $f^n = 0 \in \mathfrak{p}$. Then $f(f \dots f) \in \mathfrak{p}$ and $\mathfrak{p}$ prime implies that $f \in \mathfrak{p}$, so $f \in I$.

- $\supset$ Let us prove the converse. Suppose f is not nilpotent, so $f^n \neq 0$ for all $n \geq 1$. We will show that there exists a prime ideal $\mathfrak{p} \subset A$ that does not contain f. Let us consider all ideals of A that do not contain f^m, where $m \in \mathbb{Z}_{>0}$. Let Σ be the set of ideals $J \subset A$ such that

$$J \cap \{f^m \mid m \geq 1\} = \emptyset.$$

The zero ideal $\langle 0 \rangle$ is in Σ. So $\Sigma \neq \emptyset$. Equip Σ with a partial order given by inclusion. Applying Zorn's lemma we obtain that Σ contains a maximal element. Call it $\mathfrak{p}$. By construction, $\mathfrak{p} \cap \{f^m \mid m \geq 1\} = \emptyset$, so $f \notin \mathfrak{p}$. It remains to prove that $\mathfrak{p}$ is prime. Enough to prove that if $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$, then $xy \notin \mathfrak{p}$. Consider the ideal $\mathfrak{p} + \langle x \rangle \supsetneq \mathfrak{p}$. Since $\mathfrak{p}$ is maximal in Σ, thus $\mathfrak{p} + \langle x \rangle$ is not in Σ. By definition of Σ there exists $n \geq 1$ such that $f^n \in \mathfrak{p} + \langle x \rangle$. Similarly, there exists $m \geq 1$ such that $f^m \in \mathfrak{p} + \langle y \rangle$. Then $(\mathfrak{p} + \langle x \rangle)(\mathfrak{p} + \langle y \rangle) \subset \mathfrak{p} + \langle xy \rangle$. In particular, $f^{n+m} = f^n \cdot f^m \in \mathfrak{p} + \langle xy \rangle$. If $xy \in \mathfrak{p}$, then $f^{n+m} \in \mathfrak{p}$, which is not possible. Therefore, $xy \notin \mathfrak{p}$. So $\mathfrak{p}$ is a prime ideal that does not contain f.

□

Definition 5.4. The **Jacobson radical** $\mathcal{J}(A)$ is the intersection of all maximal ideals of A.

Proposition 5.5. *$x \in \mathcal{J}(A)$ if and only if $1 - xy \in A^*$ for all $y \in A$.*

Proof.

$\implies$ Let $x \in \mathcal{J}(A)$. Suppose there exists $y \in A$ such that $1 - xy$ is not a unit. By Corollary 4.9 every non-unit is contained in a maximal ideal. Say $M \subset A$ is a maximal ideal and $1 - xy \in M$. But $x \in \mathcal{J}(A) \subset M$. Then $1 = (1 - xy) + xy \in M$, but then $M \neq A$. A contradiction.

$\impliedby$ Given $x \in A$ such that $1 - xy \in A^*$ for all $y \in A$, we must have $x \in \mathcal{J}(A)$. If $x \notin \mathcal{J}(A)$, then there exists a maximal ideal $M \subset A$ such that $x \notin M$. Then $M + \langle x \rangle = A \ni 1$. Thus $1 = m + xy$, where $y \in A$. But by assumption $1 - xy \in A^*$, so $m \in A^*$. But then $M = A$. A contradiction.

$\square$

Definition 5.6. Let I be an ideal of A. The **radical** of I is the set

$$\operatorname{rad} I = \{x \in A \mid \exists n \geq 1,\ x^n \in I\}.$$

Proposition 5.7. *The radical of I is the intersection of all prime ideals of A that contain I.*

Proof. Apply Proposition 5.3 to A/I. $\square$

Lecture 5 Tuesday 15/10/19

Definition 5.8. Let I be an indexing set. For each $i \in I$ we are given a ring R_i. Consider the product set $\prod_{i \in I} R_i$. This is $(x_i)_{i \in I}$ for $x_i \in R_i$. Define

$$0 = (0)_{i \in I} \in \prod_{i \in I} R_i, \qquad 1 = (1)_{i \in I} \in \prod_{i \in I} R_i.$$

Define addition and multiplication coordinate-wise, so

$$(a_i)_{i \in I} + (b_i)_{i \in I} = (a_i + b_i)_{i \in I}, \qquad (a_i)_{i \in I} \cdot (b_i)_{i \in I} = (a_i \cdot b_i)_{i \in I}, \qquad (a_i)_{i \in I}, (b_i)_{i \in I} \in \prod_{i \in I} R_i.$$

Then $\prod_{i \in I} R_i$ is a ring, the **product of rings**.

A warning is if I has at least two elements, then $\prod_{i \in I} R_i$ has zero-divisors.

Example. $R_1 \times R_2$ has $(1, 0) \cdot (0, 1) = (0, 0) = 0$.

If $h_i : R \to R_i$ is a ring homomorphism for $i \in I$, then $(h_i)_{i \in I}$ is a ring homomorphism $R \to \prod_{i \in I} R_i$.

Remark 5.9. Let $\mathfrak{p}_i$ for $i \in I$ be all prime ideals of R. Let $h_i : R \to R/\mathfrak{p}_i$. Then

$$h = (h_i)_{i \in I} : R \to \prod_{i \in I} R/\mathfrak{p}_i$$

is a homomorphism, and

$$\operatorname{Ker} h = \bigcap_{i \in I} \operatorname{Ker} h_i = \bigcap_{i \in I} \mathfrak{p}_i = \mathcal{N}(R).$$

So there is an injective map

$$R/\mathcal{N}(R) \hookrightarrow \prod_{i \in I} R/\mathfrak{p}_i,$$

a product of integral domains. Now take $f_j : R \to R/M_j$, so if we take the indexing set J to be the set of all maximal ideals of R, then we obtain an injective map

$$R/\mathcal{J}(R) \hookrightarrow \prod_{j \in J} R/M_j,$$

a product of fields.

6 Localisation of rings

Example. Fix a prime p. Then

$$\mathbb{Z} \subset \left\{ \frac{m}{p^k} \;\middle|\; m \in \mathbb{Z},\ k \in \mathbb{Z}_{\geq 0} \right\} \subset \mathbb{Q}.$$

Definition 6.1. A subset S of a ring A is called a **multiplicative set** if $1 \in S$ and $0 \notin S$, and S is closed under multiplication.

Example 6.2.

- Let $a \in A$ be a non-nilpotent. Then $\{1, a, \dots\}$ is a multiplicative set.
- Let $\mathfrak{p} \subsetneq A$ be a prime ideal. Then $A \setminus \mathfrak{p}$ is a multiplicative set. Indeed, if $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$ then $xy \notin \mathfrak{p}$ by the definition of a prime ideal.
- If we have a family $\mathfrak{p}_i$ for $i \in I$ of prime ideals, then $A \setminus \bigcup_{i \in I} \mathfrak{p}_i$ is a multiplicative set.
- A^* is a multiplicative set.
- All non-zero-divisors in A form a multiplicative set.
- Let $I \subsetneq A$ be an ideal. Then $1 + I = \{1 + x \mid x \in I\}$ is a multiplicative set.

Definition 6.3. Consider $A \times S$ and the equivalence relation on $A \times S$ defined as

$$(a, s) \sim (b, t) \qquad \iff \qquad \exists u \in S,\ u\,(at - bs) = 0.$$

Check that this is indeed an equivalence relation. [1] The following is some notation.

- The equivalence class of (a, s) is written as a/s. For example, if $t \in S$, then $a/s = at/st$.
- The set of equivalence classes is denoted by $S^{-1}A$.

Define

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \qquad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

Need to check that these operations are well-defined. [2] Define $\frac{0}{1}$ as the zero of $S^{-1}A$, and $\frac{1}{1}$ as the one of $S^{-1}A$. Then $S^{-1}A$ is a ring, the **localisation of A with respect to a multiplicative set S**.

Lemma 6.4. *There is a ring homomorphism*

$$\begin{array}{cccc} f & : & A & \longrightarrow & S^{-1}A \\ & & x & \longmapsto & \dfrac{x}{1} \end{array}.$$

This f is injective if and only if S has no zero-divisors.

Proof. If S contains a zero-divisor, say u, then there exists $a \in A$ for $a \neq 0$ such that $ua = 0$. Then

$$f(a) = \frac{a}{1} = \frac{au}{u} = \frac{0}{u} = 0.$$

So $\operatorname{Ker} f$ contains a, hence f is not injective. If f has no zero-divisors, then $u \cdot a = u\,(a - 0) \neq 0$ if $a \neq 0$ and any $u \in S$. Hence $f(a) \neq 0$. ☐

Lecture 6 Thursday 16/10/19

If A is an integral domain, then $\operatorname{Ker} f = 0$. So $A \hookrightarrow S^{-1}A$.

[1] Exercise

[2] Exercise

Example. Let $R = \mathbb{Z}$.

- If $S = \{1, a, \dots\}$, then
$$S^{-1}\mathbb{Z} = \left\{ \frac{n}{a^m} \;\middle|\; n \in \mathbb{Z},\ m \in \mathbb{Z}_{\ge 0} \right\}.$$
- If $S = \mathbb{Z} \setminus p\mathbb{Z}$, then
$$S^{-1}\mathbb{Z} = \left\{ \frac{n}{m} \;\middle|\; p \nmid m \right\}.$$
- If $S = \mathbb{Z} \setminus \bigcup_{p_i \text{ prime}} p_i\mathbb{Z}$, then
$$S^{-1}\mathbb{Z} = \left\{ \frac{n}{m} \;\middle|\; p_i \nmid m \right\}.$$
- If $S = \mathbb{Z}^* = \{\pm 1\}$, then $S^{-1}\mathbb{Z} = \mathbb{Z}$.
- If $S = \{\text{all non-zero elements}\}$, then $S^{-1}\mathbb{Z} = \mathbb{Q}$.
- If $S = \{1 + I \mid I \subset \mathbb{Z} \text{ ideal}\} = \{1 + nk \mid k \in \mathbb{Z}\}$, then
$$S^{-1}\mathbb{Z} = \left\{ \frac{m}{1 + nk} \;\middle|\; m, k \in \mathbb{Z} \right\},$$
where n is fixed.

Example. Let $R = k[x]$, where k is a field.

- If $S = k[x]^* = k^*$, then $S^{-1}k[x] = k[x]$.
- If $S = \{\text{all non-zero elements}\}$, then
$$S^{-1}k[x] = k(x) = \left\{ \frac{f(x)}{g(x)} \;\middle|\; g(x) \text{ arbitrary non-zero polynomial} \right\}.$$

Example 6.5. Let k be a field, and let $A = k[x, y] / \langle xy \rangle$. Note that A has zero-divisors, since $xy = 0$ in A, but $x \ne 0$ in A and $y \ne 0$ in A. Then $S = \{1, x, \dots\}$ is a multiplicative set, since $x^n \ne 0$ in A for $n = 1, 2, \dots$, because no power of the polynomial x is in $\langle xy \rangle$. What is $S^{-1}A$? Let $f : A \to S^{-1}A$. Then $a \in \operatorname{Ker} f$ if and only if $a/1 = 0/1$, if and only if $u \cdot (a \cdot 1 - 0 \cdot 1) = 0$ for some $u \in S$, if and only if $ua = 0$. Let $a \ne 0$. Then $u = 1$ is not interesting. Take $u = x$ and $a = y$, then $xy = 0$, hence $y \in \operatorname{Ker} f$. Then f is a homomorphism, hence $\operatorname{Ker} f$ is an ideal. So $\langle y \rangle = yA \subset \operatorname{Ker} f$. In general,
$$a = \sum_{i,j \ge 0} a_{ij} x^i y^j \equiv a_{00} + \sum_{i \ge 1} a_{i0} x^i + \sum_{j \ge 1} a_{0j} y^j \mod \langle xy \rangle.$$
Then $\operatorname{Ker} f = yA = \langle y \rangle$, since $\sum_{j \ge 1} a_{0j} y^j$ goes to zero, since it is annihilated by x, and $x^n \cdot \sum_{i \ge 0} a_i x^i$ is never zero in A. Thus $f(A) = k[x]$, and
$$S^{-1}A = \left\{ \frac{f(x)}{x^n} \;\middle|\; f(x) \in k[x],\ n \ge 0 \right\} = k\left[x, x^{-1}\right] = \left\{ \sum_{i \in \mathbb{Z},\ a_i = 0 \text{ for almost all } i} a_i x^i \;\middle|\; a_i \in k \right\}.$$

Lecture 7 Thursday 17/10/19

Lemma 6.6 (Universal property of localisation)**.** *Let A be a ring, and $S \subset A$ a multiplicative set. Let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in B for all $s \in S$. Then there exists a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$ where $f : A \to S^{-1}A$ is the canonical map, so*

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "g"] & \\ S^{-1}A \arrow[r, "\exists! h"'] & B \end{tikzcd}.$$

Proof. Define

$$\begin{array}{rcccl} h & : & S^{-1}A & \longrightarrow & B \\ & & \dfrac{a}{s} & \longmapsto & \dfrac{g(a)}{g(s)}, \qquad a \in A, \qquad s \in S. \end{array}$$

This is well-defined, that is if $a/s = b/t$ then $g(a)\,g(s)^{-1} = g(b)\,g(t)^{-1}$. [3] This is a ring homomorphism. [4] Now easy to check that

$$(h \circ f)(a) = h\left(\frac{a}{1}\right) = \frac{g(a)}{g(1)} = \frac{g(a)}{1} = g(a), \qquad a \in A.$$

Moreover, if $h' : S^{-1}A \to B$ and $g = h' \circ f$ then for all $a \in A$ we have $(h' \circ f)(a) = g(a)$. Since h' is a ring homomorphism, for all $s \in S$, $h'(1/s) = 1/h'(s/1) = 1/g(s)$. Hence

$$h'\left(\frac{a}{s}\right) = h'\left(\frac{a}{1}\right) h'\left(\frac{1}{s}\right) = \frac{h'(f(a))}{h'(f(s))} = \frac{g(a)}{g(s)} = h\left(\frac{a}{s}\right).$$

∎

For all ideal $I \subseteq A$, set

$$S^{-1}I = \left\{ \frac{i}{s} \in S^{-1}A \;\middle|\; i \in I,\ s \in S \right\},$$

the ideal of $S^{-1}A$ generated by $f(I)$.

Proposition 6.7. *Let $S \subset A$ be a multiplicative subset, and let $I_1, \dots, I_n$ be ideals of A. Then*

1. $S^{-1}(I_1 + \dots + I_n) = S^{-1}I_1 + \dots + S^{-1}I_n$,
2. $S^{-1}(I_1 \cdot \dots \cdot I_n) = S^{-1}I_1 \cdot \dots \cdot S^{-1}I_n$,
3. $S^{-1}\left(\bigcap_{i=1}^n I_i\right) = \bigcap_{j=1}^n S^{-1}I_j$, *and*
4. $S^{-1}(\operatorname{rad} I) = \operatorname{rad} S^{-1}I$ *for every ideal I.*

Proof. Exercise. [5] ∎

There is a map

$$\{\text{ideals } I \text{ of } A\} \to \left\{\text{ideals } S^{-1}I \text{ of } S^{-1}A\right\}.$$

Proposition 6.8. *Every ideal of $S^{-1}A$ is of the form $S^{-1}I$ for some ideal $I \subseteq A$.*

Proof. Let J be any ideal of $S^{-1}A$. Define $I = f^{-1}A$. Know I is an ideal of A. Claim that $J = S^{-1}I$. Say $a/s \in J$. Since J is an ideal, $s(a/s) \in J$, so $a/1 \in J$, so $a \in I$. Hence $a/s \in S^{-1}I$. So $J \subseteq S^{-1}I$. Conversely, $f(I) = f\left(f^{-1}(J)\right) \subseteq J$. Thus $S^{-1}I \subseteq J$. ∎

Theorem 6.9. *The only prime ideals of $S^{-1}A$ are of the form $S^{-1}\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal of A such that $\mathfrak{p} \cap S = \emptyset$. Hence there is a bijection*

$$\left\{ \text{ prime ideals of } S^{-1}A \right\} \quad \leftrightsquigarrow \quad \left\{ \text{ prime ideals of } A \text{ that do not intersect } S \right\}.$$

Proof. Prove $S^{-1}\mathfrak{p}$ is prime if $\mathfrak{p}$ is prime and $\mathfrak{p} \cap S = \emptyset$. Say $a/s \cdot b/t \in S^{-1}\mathfrak{p}$ for $a/s, b/t \in S^{-1}A$. This implies $v(abu - cst) = 0$ for some $u, v \in S$ and $c \in \mathfrak{p}$. Hence $abuv = cstv \in \mathfrak{p}$, so $ab \in \mathfrak{p}$, as u and v are units, so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Hence $S^{-1}\mathfrak{p}$ is prime. Next note that $f^{-1}\left(S^{-1}\mathfrak{p}\right) = \mathfrak{p}$, assuming $\mathfrak{p} \cap S = \emptyset$. For if $a \in A$ lies in $S^{-1}\mathfrak{p}$ then by definition there exists $s \in S$ such that $sa \in \mathfrak{p}$. Then s is a unit and so $a \in \mathfrak{p}$. Hence $\mathfrak{p}$ is uniquely determined by $S^{-1}\mathfrak{p}$. Now let $\mathfrak{q}$ be an arbitrary prime ideal of $S^{-1}A$. Then certainly $\mathfrak{q} = S^{-1}I$ for $I = f^{-1}(\mathfrak{q})$. But the preimage of a prime ideal is prime. So I is prime. Moreover, $I \cap S = \emptyset$ as no $s \in S$ is in $\mathfrak{q}$, since $\mathfrak{q}$ is prime, so $\mathfrak{q}$ contains no units. ∎

[3] Exercise
[4] Exercise
[5] Exercise