M4P57 Complex Manifolds

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Syllabus

Contents

1 Introduction

Lecture 1 Thursday 09/01/20

The following are references.

- O Biquard and A Höring, Kähler geometry and Hodge theory, 2008.
- J P Demailly, Complex analytic and differential geometry, 2012.
- C Voisin, Hodge theory and complex algebraic geometry, 2002.
- R O Wells, Differential analysis on complex manifolds, 1973.
- A Gathmann, Algebraic geometry, 2002
- P Griffiths and J Harris, Principles of algebraic geometry, 1978.

Complex manifolds are manifolds over $\mathbb{C}^n$.

Example 1.1. $\mathbb{C}^1$ is a complex manifold. Any open $U \subset \mathbb{C}^n$ is a complex manifold.

Example 1.2. The sphere $\mathrm{S}^2 \subset \mathbb{R}^3$ is a complex manifold by

$$\mathrm{S}^2 \cong \mathbb{C} \cup \{\infty\} = \mathbb{P}^1_{\mathbb{C}} = \mathbb{CP}^1.$$

More in general $\mathbb{P}^n_{\mathbb{C}}$ is a complex manifold for all n.

Example 1.3. The torus

$$\mathrm{S}^1 \times \mathrm{S}^1 = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{C}/\mathbb{Z}^2$$

is a complex manifold. More in general a $2n$-dimensional torus $\mathbb{C}^n/\Lambda$ for a lattice $\Lambda \cong \mathbb{Z}^{2n}$ is a complex manifold.

Example 1.4. Compact Riemannian surfaces of genus $g > 1$, called **hyperbolics**, are all complex manifolds.

Example 1.5. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. The graph of f,

$$\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{C}\} \subset \mathbb{C} \times \mathbb{C},$$

is a complex manifold. From Γ_f we can recover f, by

$$f(x) = q\left(p^{-1}(x) \cap \Gamma_f\right),$$

where $p, q : \mathbb{C}^2 \to \mathbb{C}$ are the projections to the first and second factors. This allows us to define f^{-1}. Assume f is bijective. Define

$$\begin{array}{cccc} \tau & : & \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ & & (x, y) & \longmapsto & (y, x) \end{array}.$$

Define

$$\Gamma_{f^{-1}} = \tau(\Gamma_f).$$

Then f^{-1} is the function induced by $\Gamma_{f^{-1}}$. This makes sense even if f is not bijective. Then we get a multivalued function, such as $\log z$ as the inverse of $\exp z$.

Example 1.6. Generalising Example 1.5, we can consider two complex manifolds M and N and we can consider holomorphisms $f : M \to N$. Given M,

$$\mathrm{Aut}\, M = \left\{f : M \to M \text{ holomorphic bijective and } f^{-1} \text{ holomorphic}\right\}.$$

If $M = \mathbb{C}$, there are lots of C^∞-functions $\mathbb{C} \to \mathbb{C}$ but the automorphisms of $\mathbb{C}$ are just affine linear maps. If $M = \mathbb{C}/\mathbb{Z}^2$, then $\mathrm{Aut}\, M$ is interesting.

Example 1.7. Algebraic geometry is the zeroes of polynomials. That is, fix m, and take polynomials $f_1, \dots, f_k$ in m variables. Define

$$M = \{(x_1, \dots, x_m) \in \mathbb{C}^m \mid f_1(x_1, \dots, x_m) = \dots = f_k(x_1, \dots, x_m) = 0\}.$$

Then M is called an **algebraic variety**. If M is smooth then M is a complex manifold. Fix m, take homogeneous polynomials $F_1, \dots, F_k$ in $m+1$ variables, where F is **homogeneous** if it is the sum of monomials of the same degree. Consider

$$N = \{(x_0, \dots, x_m) \in \mathbb{P}_{\mathbb{C}}^m \mid F_1(x_0, \dots, x_m) = \dots = F_k(x_0, \dots, x_m) = 0\}.$$

Then N is called a **projective variety**. If N is smooth then N is a complex manifold.

The idea is if M is a differentiable manifold, then M contains lots of submanifolds N. This is not true for complex manifolds. There exist complex manifolds without any proper complex submanifolds, which is not true for projective varieties. The following are questions.

- What can we say about the topology of complex manifolds? For example, what is $\pi_1(M)$? What is the cohomology of M?
- Assume that M and N are complex manifolds which are diffeomorphic. Are they also isomorphic, so there exists a biholomorphism $M \to N$?

What is next?

- Hodge decomposition theorem. Understand the cohomology of M by using the complex structure.
- Kodaira embedding theorem. Understand when a compact complex manifold is projective.

Note. If $M \subset \mathbb{P}_{\mathbb{C}}^m$ is a compact complex manifold then M is projective.

Example. Let $M = \Gamma_{\exp}$ for $\exp : \mathbb{C} \to \mathbb{C}$. Then $M \subset \mathbb{C}^2$ but it is not algebraic.

2 Local theory

2.1 Holomorphic functions in several variables

Lecture 2 Thursday 09/01/20

Notation 2.1. Given $z_0 \in \mathbb{C}$ and $r > 0$, the **disc** is

$$\mathrm{D}(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$

and $\partial \mathrm{D}(z_0, r)$ is the boundary of $\mathrm{D}(z_0, r)$.

Definition 2.2. Let $U \subset \mathbb{C}$, and let $f : U \to \mathbb{C}$ be a function. Then f is **holomorphic at** $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Theorem 2.3 (Cauchy)**.** *Let $U \subset \mathbb{C}$ be open, let f be holomorphic on U, and let $z_0 \in U$. Assume that if $D = \mathrm{D}(z_0, r) \subset U$ then $\overline{D} \subset U$. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} \, \mathrm{d}z.$$

Notation 2.4. Fix $z_0 = (z_{01}, \dots, z_{0n}) \in \mathbb{C}^n$ and $R = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$. Then the **polydisc** is

$$\mathrm{D}(z_0, R) = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i - z_{0i}| < r_i \text{ for each } i\},$$

where R is the **polyradius**.

Definition 2.5. Let $U \subset \mathbb{C}^n$ be open, let $f : U \to \mathbb{C}$ be a continuous function, and let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Then f is **holomorphic** at z, if assuming that $D = \mathrm{D}(z, R) \subset U$ for some $R = (r_1, \dots, r_n)$ then

$$f(z_1, \dots, z_{i-1}, \cdot, z_{i+1}, \dots, z_n) : \mathrm{D}(z_i, r_i) \to \mathbb{C}$$

is holomorphic for all i.

Example 2.6. Any convergent power series in n-variables is holomorphic.

The opposite is also true.

Theorem 2.7 (Cauchy)**.** *Let $U \subset \mathbb{C}^n$ be an open set, let $f : U \to \mathbb{C}$ be holomorphic, and let $z = (z_1, \dots, z_n) \in U$. Assume that if $D = \mathrm{D}(z_0, R)$ for some $R = (r_1, \dots, r_n)$ then $\overline{D} \subset U$. If $z' = (z'_1, \dots, z'_n) \in D$ then*

$$f(z') = \frac{1}{(2\pi i)^n} \int_{\partial \mathrm{D}(z_1, r_1)} \dots \int_{\partial \mathrm{D}(z_n, r_n)} \frac{f(z)}{(z - z'_1) \dots (z - z'_n)} \, \mathrm{d}z_n \dots \mathrm{d}z_1.$$

Proof. Use induction on n and Cauchy theorem at each step. □

Corollary 2.8. *Let $U \subset \mathbb{C}^n$ be open, let $f : U \to \mathbb{C}$ be holomorphic, and let $z = (z_1, \dots, z_n) \in U$. Then there exists $D = \mathrm{D}(z, R) \subset U$ for some $R = (r_1, \dots, r_n)$ and there exists*

$$p(w) = \sum_{m_1, \dots, m_n \geq 0} a_{m_1, \dots, m_n} (w_1 - z_1)^{m_1} \dots (w_n - z_n)^{m_n},$$

such that p is convergent on D and $f(w) = p(w)$ inside D.

Proof. The idea is to use Theorem 2.7 and $1/(1 - w) = \sum_{k \geq 0} w^k$. □

Definition 2.9. Let $U \subset \mathbb{C}^n$ be open. Then $f : U \to \mathbb{C}^m$ is **holomorphic** if $f_i = p_i \circ f$ is holomorphic for any $i = 1, \dots, m$ where $p_i : \mathbb{C}^m \to \mathbb{C}$ is the i-th projection, so $f = (f_1, \dots, f_m)$.

Fact. If $f : U \to \mathbb{C}^m$ is holomorphic and $g : V \to \mathbb{C}^p$ is holomorphic where $V \supset f(U)$ then $g \circ f$ is holomorphic.

Definition 2.10. Let $U \subset \mathbb{C}^n$ be open. A holomorphic function $f : U \to \mathbb{C}^m$ is **biholomorphic at** $z_0 \in U$ if there exists an open neighbourhood $V \subset U$ of z_0 such that $f : V \to f(V)$ is bijective and $f^{-1} : f(V) \to V$ is holomorphic. Then f is **biholomorphic** if f is bijective and f is biholomorphic at any point.

Note. $f(V)$ is automatically open in $\mathbb{C}^m$ if $m = n$.

Example 2.11. Let $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ be linear such that $\det \Phi \neq 0$. Then Φ is a biholomorphism.

Example 2.12. Let $U = \mathbb{C} \setminus \{0\}$ and

$$\begin{array}{rcccc} f & : & U & \longrightarrow & U \\ & & z & \longmapsto & z^2 \end{array}.$$

Check that f is biholomorphic at any point of U but f is not biholomorphic.

Remark. $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Then a holomorphic $f : U \subset \mathbb{C}^n \to \mathbb{C}^m$ is also a diffeomorphism $U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2m}$.

Theorem 2.13 (Hartogs)**.** *Let $n \geq 2$, let $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ and $\delta = (\delta_1, \dots, \delta_n)$ such that $\epsilon_i > \delta_i > 0$, let $U = \mathrm{D}(0, \epsilon) \setminus \overline{\mathrm{D}(0, \delta)}$, and let $f : U \to \mathbb{C}^m$ be holomorphic. Then there exists a holomorphic $\overline{f} : \mathrm{D}(0, \epsilon) \to \mathbb{C}^m$ such that $\overline{f}\big|_U = f$.*

Example. Hartogs theorem is false for $n = 1$. If $f(z) = 1/z$, for all $\epsilon > \delta > 0$, then f cannot be extended.

2.2 Cauchy formula in one variable

Lecture 3 Tuesday 14/01/20

Let $\omega = x + iy \in \mathbb{C}$ for $x, y \in \mathbb{R}$, and let $f : U \to \mathbb{C}$ be C^∞ for some $U \subset \mathbb{C}$. Recall that

$$\frac{\partial f}{\partial \omega} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f, \qquad \frac{\partial f}{\partial \overline{\omega}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f.$$

Recall that f is holomorphic if and only if $\frac{\partial f}{\partial \overline{\omega}} = 0$ on U. More in general, let $U \subset \mathbb{C}^n$ be open, let $z_i = x_i + iy_i$, and let $f : U \to \mathbb{C}$ be a C^∞-function. Then f is holomorphic if and only if $\frac{\partial f}{\partial \overline{z_i}} = 0$ for all $i = 1, \dots, n$. Let $\omega \in \mathbb{C}$. Since $\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$, let

$$\mathrm{d}A = \frac{i}{2}\mathrm{d}\omega \wedge \mathrm{d}\overline{\omega} = \frac{i}{2}(\mathrm{d}x + i\mathrm{d}y) \wedge (\mathrm{d}x - i\mathrm{d}y) = \mathrm{d}x \wedge \mathrm{d}y,$$

which is the Lebesgue measure on $\mathbb{R}^2 \cong \mathbb{C}$.

Proposition 2.14. *Let $f : U \to \mathbb{C}$ for $U \subset \mathbb{C}$ be a C^∞-function, and let $D = \mathrm{D}(z, r)$ such that $\overline{D} \subset U$. Then*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f}{\omega - z}\,\mathrm{d}\omega - \frac{1}{\pi}\int_D \frac{1}{\omega - z}\frac{\partial f}{\partial \overline{\omega}}\,\mathrm{d}A.$$

Proof. Assume $z = 0$. Recall that $f(\omega) = 1/\omega$ is locally integrable around zero, so

$$\int_D \frac{1}{\omega}\frac{\partial f}{\partial \overline{\omega}}\,\mathrm{d}A = \lim_{\epsilon \to 0}\int_{D \setminus \mathrm{D}(0,\epsilon)} \frac{1}{\omega}\frac{\partial f}{\partial \overline{\omega}}\,\mathrm{d}A.$$

Away from zero

$$\begin{aligned} \mathrm{d}\left(\frac{f}{\omega}\mathrm{d}\omega\right) &= \frac{1}{\omega}\mathrm{d}f \wedge \mathrm{d}\omega + f\mathrm{d}\left(\frac{1}{\omega}\right) \wedge \mathrm{d}\omega = \frac{1}{\omega}\left(\frac{\partial f}{\partial \omega}\mathrm{d}\omega + \frac{\partial f}{\partial \overline{\omega}}\mathrm{d}\overline{\omega}\right) \wedge \mathrm{d}\omega + f\frac{\partial}{\partial \omega}\left(\frac{1}{\omega}\right)\mathrm{d}\omega \wedge \mathrm{d}\omega \\ &= \frac{1}{\omega}\frac{\partial f}{\partial \overline{\omega}}\mathrm{d}\overline{\omega} \wedge \mathrm{d}\omega = \frac{2i}{\omega}\frac{\partial f}{\partial \overline{\omega}}\mathrm{d}A. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{\pi}\int_D \frac{1}{\omega}\frac{\partial f}{\partial \overline{\omega}}\,\mathrm{d}A &= \frac{1}{\pi}\lim_{\epsilon \to 0}\int_{D \setminus \mathrm{D}(0,\epsilon)} \frac{1}{\omega}\frac{\partial f}{\partial \overline{\omega}}\,\mathrm{d}A \\ &= \frac{1}{2\pi i}\lim_{\epsilon \to 0}\int_{D \setminus \mathrm{D}(0,\epsilon)} \mathrm{d}\left(\frac{f}{\omega}\mathrm{d}\omega\right) && \frac{1}{\omega}\frac{\partial f}{\partial \overline{\omega}}\mathrm{d}A = \frac{1}{2i}\mathrm{d}\left(\frac{f}{\omega}\mathrm{d}\omega\right) \\ &= \frac{1}{2\pi i}\lim_{\epsilon \to 0}\left(\int_{\partial D} \frac{f}{\omega}\,\mathrm{d}\omega - \int_{\partial \mathrm{D}(0,\epsilon)} \frac{f}{\omega}\,\mathrm{d}\omega\right) && \text{Stokes' theorem} \\ &= \frac{1}{2\pi i}\left(\int_{\partial D} \frac{f}{\omega}\,\mathrm{d}\omega - 2\pi i f(0)\right) && \lim_{\epsilon \to 0}\int_{\partial \mathrm{D}(0,\epsilon)} \frac{1}{\omega}\,\mathrm{d}\omega = 2\pi i. \end{aligned}$$

$\square$

If f is holomorphic, then $\frac{\partial f}{\partial \overline{\omega}} = 0$, which implies Theorem 2.3.

2.3 Rank theorem

Let $U \subset \mathbb{C}^n$ be open, and let $f : U \to \mathbb{C}^m$ be holomorphic. Then the **Jacobian** is

$$\mathrm{J}_f = \left(\frac{\partial f_j}{\partial z_i}(z)\right),$$

where $f_j = p_j \circ f$ and $p_j : \mathbb{C}^m \to \mathbb{C}$ is the j-th projection.

Exercise. Show that the real Jacobian, which is $2n \times 2n$, has non-negative determinants.

Theorem 2.15 (Rank theorem)**.** *Let $z \in U$ such that $r = \operatorname{rk} \mathrm{J}_f(z')$ is constant around z. Then there exist open $z \in V \subset U \subset \mathbb{C}^n$ and $f(z) \in W \subset f(U) \subset \mathbb{C}^m$ such that $\phi : \mathrm{D}(0,1)^n \to V$ and $\psi : \mathrm{D}(0,1)^m \to W$ are biholomorphisms such that*

$$\eta = \psi^{-1} \circ f \circ \phi \quad : \quad \begin{array}{ccc} \mathrm{D}(0,1)^n & \longrightarrow & \mathrm{D}(0,1)^m \\ (z_1, \dots, z_n) & \longmapsto & (z_1, \dots, z_r, 0, \dots, 0) \end{array},$$

so

$$\begin{array}{ccccccc} \mathbb{C}^n \supset U & \supset & V \ni z & \xrightarrow{f} & f(z) \in W & \subset & f(U) \subset \mathbb{C}^m \\ & & \uparrow{\scriptstyle\phi} & & \uparrow{\scriptstyle\psi} & & \\ & & \mathrm{D}(0,1)^n & \xrightarrow[\eta]{} & \mathrm{D}(0,1)^m & & \end{array}.$$

Theorem 2.16 (Inverse function theorem)**.** *Let $f : U \to \mathbb{C}^n$ be holomorphic for $U \subset \mathbb{C}^n$, and let $z \in U$ such that $\det \mathrm{J}_f(z) \neq 0$. Then f is a biholomorphism at z.*

Proof. $\det \mathrm{J}_f(z) \neq 0$ if and only if $\operatorname{rk} \mathrm{J}_f(z) = n$, so $\operatorname{rk} \mathrm{J}_f(z') = n$ around z, since $\det \mathrm{J}_f(z)$ is a continuous function. Let ϕ and ψ as in the theorem. Then $\eta = \psi^{-1} \circ f \circ \phi = \mathrm{id}$, so on V, $f = \psi \circ \phi^{-1}$ is a composition of biholomorphisms, which is a biholomorphism. ☐

Remark 2.17. Let $f : U \to \mathbb{C}^n$ for $U \subset \mathbb{C}^n$. Then $\det \mathrm{J}_f(z)$ is a holomorphism, so

$$Z = \{z \in U \mid \det \mathrm{J}_f(z) = 0\}$$

is closed.

2.4 Holomorphic differential forms

Let $U \subset \mathbb{C}^n$ be open.

Definition 2.18. A **holomorphic vector field** on U is the expression

$$X = \sum_i a_i \frac{\partial}{\partial z_i},$$

where a_i are holomorphic functions on U.

For all $x \in U$, the **tangent space** is

$$\mathrm{T}_x U = \left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle \cong \mathbb{C}^n.$$

If $x \in U$, then $X(x) \in \mathrm{T}_x U$.

Notation 2.19.

$$\mathrm{H}^0(U, \mathcal{O}_U) = \{\text{holomorphic functions } f : U \to \mathbb{C}\}, \qquad \mathrm{H}^0(U, \mathrm{T}_U) = \{\text{holomorphic vector fields on } U\}.$$

Remark. $R = \mathrm{H}^0(U, \mathcal{O}_U)$ is a ring and $M = \mathrm{H}^0(U, \mathrm{T}_U)$ is a module over R. That is, if $X \in \mathrm{H}^0(U, \mathrm{T}_U)$ and $f \in \mathrm{H}^0(U, \mathcal{O}_U)$, then $fX \in \mathrm{H}^0(U, \mathrm{T}_U)$.

Definition 2.20. Let R be a ring and M be an R-module for $p \geq 1$. The **p-th exterior power** $\Lambda^p M$ of M is the R-module $M^{\otimes p}$ with the relations

$$m_1 \otimes \cdots \otimes m_p - \epsilon(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(p)}, \qquad m_1, \ldots, m_p \in M, \qquad \sigma \in \mathcal{S}_p,$$

where $\epsilon(\sigma) = (-1)^m$ is the signature of σ and m is the number of transpositions defining σ. Then $M^* = \mathrm{Hom}_R(M, R)$ is the **dual** of M as an R-module.

Let $R = \mathrm{H}^0(U, \mathcal{O}_U)$ and $M = \mathrm{H}^0(U, \mathrm{T}_U)$. Lecture 4 Thursday 16/01/20

Definition 2.21. Let $p > 0$. We define a **holomorphic p-form**, as an element of

$$\mathrm{H}^0(U, \Omega_U^p) = \Lambda^p M^*.$$

If $p = 0$, by convention a **holomorphic 0-form** is just an element in R.

Let $z_1, \ldots, z_n$ be coordinates for U. Recall $\eta \in M$ is given by $\eta = \sum_i a_i \frac{\partial}{\partial z_i}$ for holomorphic functions $a_i \in R$. Then $\omega \in M^*$ is given by the expression

$$\sum_i b_i \mathrm{d}z_i, \qquad b_i \in R, \qquad \mathrm{d}z_i \left(\frac{\partial}{\partial z_j}\right) = \delta_{ij}.$$

More in general $\omega \in \mathrm{H}^0(U, \Omega_U^p)$ is given by

$$\omega = \sum_{|I|=p} f_I \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p}, \qquad f_I \in R, \qquad I = (i_1, \ldots, i_p), \qquad i_1 < \cdots < i_p,$$

where $\mathrm{d}z_{i_1}, \ldots, \mathrm{d}z_{i_p}$ is an R-basis of $\mathrm{H}^0(U, \Omega_U^p)$.

Example.

$$\mathrm{H}^0(U, \Omega_U^p) \cong \Lambda^p \mathrm{H}^0\left(U, \Omega_U^1\right)$$

is an isomorphism as R-modules. This is not true for complex manifolds in general.

The **exterior product** is

$$\begin{array}{ccc} \mathrm{H}^0(U, \Omega_U^p) \otimes \mathrm{H}^0(U, \Omega_U^q) & \longrightarrow & \mathrm{H}^0\left(U, \Omega_U^{p+q}\right) \\ \omega_1 \otimes \omega_2 & \longmapsto & \omega_1 \wedge \omega_2 \end{array},$$

where we just need to define

$$\omega_1 \wedge \omega_2 = f \mathrm{d}z_{i_1} \wedge \mathrm{d}z_{i_p} \otimes g \mathrm{d}z_{j_1} \wedge \mathrm{d}z_{j_q} = fg \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}z_{j_1} \wedge \cdots \wedge \mathrm{d}z_{j_q},$$

by linearity. Then $\omega_1 \wedge \omega_2 = 0$ if $\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\} \neq \emptyset$, since $\mathrm{d}z_i \wedge \mathrm{d}z_i = 0$.

Exercise. Check that this definition coincides with the definition in M4P54.

The **exterior derivative** is

$$\begin{array}{cccc} \mathrm{d} : & \mathrm{H}^0(U, \Omega_U^p) & \longrightarrow & \mathrm{H}^0\left(U, \Omega_U^{p+1}\right) \\ & f \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} & \longmapsto & \sum_{j=1}^n \frac{\partial f}{\partial z_j} \mathrm{d}z_j \wedge \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \end{array}.$$

By definition d is $\mathbb{C}$-linear, but not R-linear. That is,

$$\mathrm{d}(a\omega_1 + b\omega_2) = a\mathrm{d}\omega_1 + b\mathrm{d}\omega_2, \qquad \omega_1, \omega_2 \in \mathrm{H}^0(U, \Omega_U^p), \qquad a, b \in \mathbb{C}.$$

Theorem 2.22. *Let $U \subset \mathbb{C}^n$ be open. Then*

- *the Leibniz rule*

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \mathrm{d}\omega_2, \qquad \omega_1 \in \mathrm{H}^0(U, \Omega_U^p), \qquad \omega_2 \in \mathrm{H}^0(U, \Omega_U^q),$$

- $\mathrm{d}^2 = 0$, *that is*

$$\mathrm{d}(\mathrm{d}\omega) = 0, \qquad \omega \in \mathrm{H}^0(U, \Omega_U^p).$$

Definition 2.23. Let $f : U \subset \mathbb{C}^n \to \mathbb{C}^m$ be holomorphic, let $f_i = p_i \circ f : V \to \mathbb{C}$ where $p_i : \mathbb{C}^m \to \mathbb{C}$ is the i-th projection, and let $f(U) \subset V \subset \mathbb{C}^m$ be open. Then if

$$\omega = h \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \in \mathrm{H}^0 \left(V, \Omega_V^p\right), \qquad h \in \mathrm{H}^0 \left(U, \mathcal{O}_U\right),$$

then we can define the **pull-back** of ω,

$$f^* (\omega) = h \circ f \mathrm{d}f_{i_1} \wedge \cdots \wedge \mathrm{d}f_{i_p} \in \mathrm{H}^0 \left(U, \Omega_U^p\right),$$

since $f_i \in \mathrm{H}^0 \left(V, \mathcal{O}_V\right) = \mathrm{H}^0 \left(V, \Omega_V^0\right)$ implies that $\mathrm{d}f_i \in \mathrm{H}^0 \left(V, \Omega_V^1\right)$, so

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "h \circ f \in \mathrm{H}^0(U, \mathcal{O}_U)"'] & f(U) \subset V \arrow[d, "h"] \\ & \mathbb{C} \end{tikzcd}.$$

This is linear over $\mathbb{C}$ and over $\mathrm{H}^0 \left(U, \mathcal{O}_U\right)$.

Proposition 2.24. *Let $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$, and $W \subset \mathbb{C}^{m'}$ be open, let $f : U \to \mathbb{C}^m$ and $g : V \to \mathbb{C}^{m'}$ be holomorphic such that $V \supset f(U)$ and $W \supset g(V)$, and let $\omega \in \mathrm{H}^0 \left(V, \Omega_V^p\right)$ and $\eta \in \mathrm{H}^0 \left(V, \Omega_V^q\right)$. Then*

- $f^* (\omega + \eta) = f^* (\omega) + f^* (\eta)$ *if* $p = q$,
- $f^* (\omega \wedge \eta) = f^* (\omega) \wedge f^* (\eta)$,
- $\mathrm{d}f^* (\omega) = f^* (\mathrm{d}\omega)$, *and*
- $f^* (g^* (\omega)) = (g \circ f)^* (\omega)$.

Let $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, and let $z_i = x_i + iy_i$ for $i = 1, \dots, n$ and $x_i, y_i \in \mathbb{R}$. Then

$$\mathrm{d}z_i = \mathrm{d}x_i + i\mathrm{d}y_i,$$

so any holomorphic form is a differentiable form on $\mathbb{R}^{2n}$. A (p, q)**-form** is a differentiable $(p + q)$-form of the expression

$$\omega = \sum_{|I|=p,\ |J|=q} f_{I,J} \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z_{j_1}} \wedge \cdots \wedge \mathrm{d}\overline{z_{j_q}}, \qquad \mathrm{d}\overline{z_j} = \mathrm{d}x_j - i\mathrm{d}y_j, \qquad f_{I,J} : U \to \mathbb{C} \cong \mathbb{R}^2 \in \mathrm{C}^\infty.$$

We denote

$$\mathrm{C}^\infty \left(U, \Omega_U^{p,q}\right) = \{\text{differentiable } (p + q)\text{-forms on } U\}.$$

If ω is a (p, q)-form, then the **conjugate** $\overline{\omega}$ of ω is the (q, p)-form defined by

$$\overline{\omega} = \sum_{|I|=p,\ |J|=q} \overline{f_{I,J}} \mathrm{d}\overline{z_{i_1}} \wedge \cdots \wedge \mathrm{d}\overline{z_{i_p}} \wedge \mathrm{d}z_{j_1} \wedge \cdots \wedge \mathrm{d}z_{j_q}.$$

3 Complex manifolds

3.1 Objects

Definition 3.1. A **complex manifold** of dimension n is a connected Hausdorff topological space X, with a countable open cover $\{U_\alpha\}$ of X such that for all α, there exists $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism and

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is a biholomorphism for each α and β, so

$$\begin{tikzcd} & U_\alpha \cap U_\beta \arrow[dl, "\phi_\alpha"'] \arrow[dr, "\phi_\beta"] & \\ \mathbb{C}^n \supset \phi_\alpha(U_\alpha \cap U_\beta) \arrow[rr, "\phi_\alpha \circ \phi_\beta^{-1}"'] & & \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n \end{tikzcd}.$$

The pair (U_α, ϕ_α) is called a **holomorphic chart**. The set $\{(U_\alpha, \phi_\alpha)\}$ is called a **holomorphic atlas** or a **complex structure**.

Recall X is Hausdorff if for all $x, y \in X$ there exist U and V open in X such that $U \cap V = \emptyset$ and $x \in U$ and $y \in V$.

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Example 3.2. If $U \subset \mathbb{C}^n$ is an open set then U is a complex manifold. More in general if X is a complex manifold and $U \subset X$ is open then U is a complex manifold. Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on M. Then

$$\left\{\left(\overline{U_\alpha}, \overline{\phi_\alpha}\right)\right\} = \left\{\left(U_\alpha \cap U, \phi_\alpha|_{\overline{U_\alpha}}\right)\right\}$$

is a complex structure of M.

Example 3.3. If X and Y are complex manifolds, then $X \times Y$ is a complex manifold.

Example 3.4. The projective space $\mathbb{P}^n_{\mathbb{C}}$ or $\mathbb{CP}^n$. Let

$$V^* = \mathbb{C}^{n+1} \setminus \{0\},$$

with coordinates $(z_0, \dots, z_n)$. Define an equivalence on V^* as $v_1 \sim v_2$ for $v_1, v_2 \in V^*$ if there exists $\lambda \in \mathbb{C}$ such that $v_1 = \lambda v_2$. Check that $\sim$ is an equivalence. Consider the Euclidean topology on V^*. Then there exists an induced topology on

$$X = V^* / \sim = \{[v] \mid v \in V^*\}.$$

with quotient map

$$\begin{array}{cccc} q : & V^* & \longrightarrow & X \\ & v & \longmapsto & [v] \end{array}.$$

Given $v = (z_0, \dots, z_n) \in V^*$ we denote $[v] = [z_0, \dots, z_n]$ such that $z_i \neq 0$ for some i. Two elements $[x_0, \dots, x_n]$ and $[y_0, \dots, y_n]$ of X define the same point if and only if there exists λ such that $x_i = \lambda y_i$ for all i. Let

$$V_i = \{(z_0, \dots, z_n) \in V^* \mid z_i \neq 0\},$$

which is open in V^*, and let

$$U_i = q(V_i),$$

which is open in X, such that $\{U_i\}$ is a cover of X, that is $\bigcup_i U_i = X$. Let

$$H_i = \{(z_0, \dots, z_n) \in V^* \mid z_i = 1\}.$$

Then there exists a homeomorphism

$$\begin{array}{cccc} r_i : & H_i & \longrightarrow & \mathbb{C}^n \\ & (z_0, \dots, z_n) & \longmapsto & [z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_n] \end{array},$$

and let

$$\begin{array}{cccc} q_i = q|_{H_i} : & H_i \subset V^* & \longrightarrow & U_i \subset X \\ & (z_0, \dots, z_n) & \longmapsto & [z_0, \dots, z_n] \end{array}$$

be also a homeomorphism.

- q_i is surjective. Take $[x_0, \dots, x_n] \in U_i$. Then $x_i \neq 0$, so choose $\lambda = 1/x_i$. Then
$$[x_0, \dots, x_n] = \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] = q(z_0, \dots, z_n), \qquad z_j = \frac{x_j}{x_i},$$
and in particular $z_i = 1$, so there exists $(z_0, \dots, z_n) \in H_i$ such that $q_i(z_0, \dots, z_n) = [x_0, \dots, x_n]$.
- q_i is injective. [1]

For all i, $q_i^{-1} : U_i \to H_i$ and $r_i : H_i \to \mathbb{C}^n$ are homeomorphisms, so $\phi_i = r_i \circ q_i^{-1} : U_i \to \mathbb{C}^n$ is also a homeomorphism. We want to show that (U_i, ϕ_i) define a holomorphic atlas, so
$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$
is a biholomorphism. Consider the case $j = 0$ and $i = 1$. Then $\phi_0(U_0 \cap U_1) = \{(x_1, \dots, x_n) \mid x_1 \neq 0\}$, so
$$\begin{array}{cccc} \phi_1 \circ \phi_0^{-1} & : & \phi_0(U_0 \cap U_1) & \longrightarrow & \phi_1(U_0 \cap U_1) \\ & & (x_1, \dots, x_n) & \longmapsto & \left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right) \end{array}$$
is a biholomorphism. Thus X is a compact complex manifold. If $n = 1$, then $\mathbb{P}^1_{\mathbb{C}} \cong \mathrm{S}^2$.

Example 3.5. The complex torus. Let
$$\begin{array}{ccc} \Lambda = \mathbb{Z}^{2n} & \longrightarrow & \mathbb{C}^n \\ (a_1, \dots, a_n, b_1, \dots, b_n) & \longmapsto & (a_1 + ib_1, \dots, a_n + ib_n) \end{array}.$$
Define an equivalence on $\mathbb{C}^n$ by $v_1 \sim v_2$ for $v_1, v_2 \in \mathbb{C}^n$ if $v_1 - v_2 \in \Lambda$. Then
$$X = \mathbb{C}^n / \sim,$$
with quotient map $q : \mathbb{C}^n \to X$ is Hausdorff and compact. Topologically $X \cong [0,1]^{2n} / \sim$. For each $x \in \mathbb{C}^n$, we can find an open set $x \in U \subset \mathbb{C}^n$ such that $q|_U : U \to X$ is a homeomorphism. The idea is if $x \in (0,1)^{2n}$ then we can take $U = (0,1)^{2n}$. If not, there exists a translation of $\mathbb{C}^n \to \mathbb{C}^n$ such that the property holds. We define
$$\phi_V = q|_U^{-1} : V \subset \mathbb{C}^n/\Lambda \to U \subset \mathbb{C}^n, \qquad V = q(U).$$
Show that (V, ϕ_V) define a complex structure on X. [2] This is also a compact complex manifold. More in general $\mathbb{C}^n/\Lambda$ where $\Lambda \cong \mathbb{Z}^{2n}$ is a lattice is a compact complex manifold.

3.2 Morphisms

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Definition 3.6. Let $f : X \to Y$ be a continuous morphism between complex manifolds. Then f is **holomorphic** if there exists a complex structure $\{(U_\alpha, \phi_\alpha)\}$ on Y and for all $y \in Y$ there exists a holomorphic chart (V_α, ψ_α) such that $x \in V_\alpha$ and $f(V_\alpha) \subset U_\alpha$ around any point x of $f^{-1}(y)$ and $\phi_\alpha \circ f \circ \psi_\alpha^{-1}$ is holomorphic, so
$$\begin{tikzcd} X \supset V_\alpha \arrow[r, "f"] \arrow[d, "\psi_\alpha"'] & U_\alpha \subset Y \arrow[d, "\phi_\alpha"] \\ \psi_\alpha(V_\alpha) \arrow[r, "\widetilde{f}"'] & \phi_\alpha(U_\alpha) \end{tikzcd}.$$
Then $\mathrm{J}_f = \mathrm{J}_{\widetilde{f}}$, and a **holomorphic function on** X is a holomorphic function $f : X \to \mathbb{C}$.

Exercise. If X is a compact complex manifold then any holomorphic function $f : X \to \mathbb{C}$ is constant.

[1] Exercise
[2] Exercise

Definition 3.7. Let $f : X \to Y$ be a holomorphic function between complex manifolds. Then f is

- a **submersion** if $\dim Y \geq \dim Y = r$ and $\operatorname{rk} \mathrm{J}_f = r$ at any point,
- an **immersion** if $r = \dim X \leq \dim Y$ and $\operatorname{rk} \mathrm{J}_f = r$ at any point, and
- an **embedding** if it is an immersion and $f : X \to f(X)$ is a homeomorphism.

Example 3.8. Let $f_2, \dots, f_n : \mathbb{C} \to \mathbb{C}$ be holomorphic, and let

$$\begin{array}{cccc} f & : & \mathbb{C} & \longrightarrow & \mathbb{C}^n \\ & & z & \longmapsto & (z, f_2(z), \dots, f_n(z)) \end{array}.$$

Then f is an immersion.

Definition 3.9. Let $i : X \to Y$ be an embedding of complex manifolds. If $i(X) \subset Y$ is closed then $i(X)$ is called a **complex submanifold** of Y. The **codimension** of X in Y is $\dim Y - \dim X$.

Example 3.10. Let $X = \mathbb{C}^2 / \Lambda$ for $\Lambda = \mathbb{Z}^4 \subset \mathbb{C}^2$, and let $q : \mathbb{C}^2 \to X$. Fix $\lambda \in \mathbb{C}$. Let

$$\begin{array}{ccccc} f & : & \mathbb{C} & \longrightarrow & \mathbb{C}^2 \\ & & z & \longmapsto & (z, \lambda z) \end{array}.$$

Then $\widetilde{f} = q \circ f : \mathbb{C} \to X$ is an embedding.

- If $\lambda = 0$ or $\lambda = \frac{1}{2}$, then $\widetilde{f}(\mathbb{C})$ is a closed submanifold.
- If λ is general then $\widetilde{f}(\mathbb{C})$ is dense inside X, so it is not closed. Thus it is not a complex submanifold of X.

Theorem 3.11.

1. *Let $i : X \to Y$ be a submanifold of codimension k, and let $n = \dim X$. Then for all $x \in X$, there exists an open neighbourhood $x \in U \subset Y$ and a submersion $f : U \to \mathrm{D}(0,1)^k \subset \mathbb{C}^k$ such that $X \cap U = f^{-1}(0)$.*
2. *If $X \subset Y$ is a closed subset such that for all $x \in X$ there exists $U \ni x$ open in Y and a submersion $f : U \to \mathrm{D}(0,1)^k$ such that $X \cap U = f^{-1}(0)$, then X is a complex submanifold.*

Proof.

1. We can assume that if there exists a holomorphic chart (U, ψ) on Y such that $x \in U$ and if $V = i^{-1}(U)$ then there exists $\phi : V \to \mathbb{C}^n$ such that (V, ϕ) is a holomorphic chart on X. After possibly shrinking U smaller, by the rank theorem, there exist biholomorphic $a : \psi(U) \to \mathrm{D}(0,1)^{n+k}$ and $b : \phi(U) \to \mathrm{D}(0,1)^n$ such that the induced morphism is given by

$$\begin{array}{ccc} \mathrm{D}(0,1)^n & \longrightarrow & \mathrm{D}(0,1)^{n+k} \\ (z_1, \dots, z_n) & \longmapsto & (z_1, \dots, z_n, 0, \dots, 0) \end{array}.$$

   Let

$$\begin{array}{ccccc} c & : & \mathrm{D}(0,1)^{n+k} & \longrightarrow & \mathrm{D}(0,1)^k \\ & & (z_1, \dots, z_{n+k}) & \longmapsto & (z_{n+1}, \dots, z_{n+k}) \end{array},$$

   so

$$\begin{tikzcd} Y & \supset & U \arrow[r, "\phi"] & \phi(U) \arrow[r, "b"] & \mathrm{D}(0,1)^n \subset \mathbb{C}^n \arrow[d] \\ X \arrow[u, hook, "i"] & \supset & V \arrow[u, hook, "i"] \arrow[r, "\psi"'] & \psi(U) \arrow[r, "a"'] & \mathrm{D}(0,1)^{n+k} \subset \mathbb{C}^{n+k} \arrow[u, dashed, bend right, "c"'] \end{tikzcd}.$$

   Then f is the composition $c \circ a \circ \psi : U \to \mathrm{D}(0,1)^n$.

2. Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on Y, and let $V_\alpha = X \cap U_\alpha$ and $\psi_\alpha = \phi_\alpha|_{V_\alpha}$. The goal is to show that $\{(V_\alpha, \psi_\alpha)\}$ defines a complex structure on X. By assumption,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta (U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k} \to \phi_\alpha (U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k}$$

is biholomorphic. Let $U' = \phi_\beta (U)$, let $X' = \phi_\beta (X \cap U)$, and let $f' = f \circ \phi_\beta^{-1}$, so

$$\begin{tikzcd}
& & & & & \phi_\alpha(U) & \subset & \phi_\alpha(U_\alpha\cap U_\beta)\subset\mathbb{C}^{n+k} \\
Y & \supset & U_\alpha\cap U_\beta & \supset & U \arrow[ur, "\phi_\alpha"] \arrow[r, "\phi_\beta"] \arrow[ddrrr, "f"] & U' \arrow[ddrr, "f'"] & \subset & \phi_\beta(U_\alpha\cap U_\beta)\subset\mathbb{C}^{n+k} \arrow[u, "\phi_\alpha\circ\phi_\beta^{-1}"'] \\
& & \cup & & \cup & \cup & & \\
X \arrow[uu, hook, "i"] & \supset & X\cap U_\alpha\cap U_\beta & \supset & X\cap U & X' & & \mathrm{D}(0,1)^k\subset\mathbb{C}^k
\end{tikzcd}.$$

Then $f'^{-1}(0) = \phi_\beta (X \cap U_\alpha \cap U_\beta)$ and f' is also a submersion. By the rank theorem, we may assume that $U' = \mathrm{D}(0,1)^{n+k}$ and $f'(z_1, \dots, z_{n+k}) = (z_1, \dots, z_k)$, so $\phi_\beta (X' \cap U_\alpha \cap U_\beta) = f'^{-1}(0)$. Thus

$$\left(\psi_\alpha \circ \psi_\beta^{-1}\right)(z_1, \dots, z_n) = \left(\phi_\alpha \circ \phi_\beta^{-1}\right)(z_1, \dots, z_n, 0, \dots, 0)$$

is also a biholomorphism.

$\square$