

M3P21 Geometry II: Algebraic Topology

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0 Some underlying geometric notions

0.1 Introduction

Lecture 1
Friday
11/01/19

Combines topological spaces with algebraic objects, groups.

- How to show that a torus is not homeomorphic to a sphere?
- How to show that $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n \neq m$?

Content is fundamental groups and homology. We will follow chapter one and two from

- A Hatcher, Algebraic topology, 2002

Prerequisites are the following.

- Point set topology. Topological spaces, continuous maps, product and quotient topologies, Hausdorff spaces, etc.
- Basic group theory. Normal subgroups and quotients, isomorphism theorems, free groups, presentation of groups, etc.

0.2 Homotopy

Let X, Y be topological spaces and $I = [0, 1]$.

Definition. A **homotopy** is a continuous map $F : X \times I \rightarrow Y$. For every $t \in I$ we obtain a continuous map

$$\begin{aligned} f_t : X &\rightarrow Y \\ x &\mapsto f_t(x) = F(x, t) \end{aligned} .$$

Definition. Two continuous maps $f_0, f_1 : X \rightarrow Y$ are **homotopic** if there exists a homotopy $F : X \times I \rightarrow Y$ such that

$$f_0(x) = F(x, 0), \quad f_1(x) = F(x, 1),$$

for all $x \in X$. We write $f_0 \cong f_1$. (Exercise: this is an equivalence relation)

Definition. Let $A \subseteq X$ be a subspace. A **retraction** of X onto A is a continuous map $r : X \rightarrow A$ such that

- $r(X) = A$, and
- $r|_A = id_A$.

Example. If $X \neq \emptyset$, $p \in X$, then X retracts to p by the constant map $X \rightarrow \{p\}$.

Definition. A **deformation retraction** of X onto $A \subseteq X$ is a retraction that is homotopic to the identity. That is, there is a continuous map

$$\begin{aligned} F : X \times I &\rightarrow A \\ (x, t) &\mapsto f_t(x) \end{aligned} ,$$

such that $f_0 = id_X$ and $f_1 : X \rightarrow A$ is the deformation retraction.

Example. The closed n -dimensional **n -disc**

$$D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$

deformation retracts to $(0, \dots, 0) \in \mathbb{R}^n$. Let $f_t(x) = t \cdot x$. $t = 1$ gives $f_1 = id_{D^n}$ and $t = 0$ gives $f_0 : D^n \rightarrow (0, \dots, 0)$.

Example. Let S^n be the **n -sphere**,

$$\partial D^{n+1} = S^n = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

The cylinder $S^n \times I$ deformation retracts to $S^n \times \{0\}$, by defining $f_t(x, r) = (x, t \cdot r)$.

An observation is if X is a topological space, and $f : X \rightarrow \{p\}$ for $p \in X$ is a deformation retraction of X to p , then X is path connected. Indeed, if $F : X \times I \rightarrow X$ is a homotopy from id_X to f and $x \in X$ is a point, then this gives a path

$$\begin{aligned} I &\rightarrow X \\ t &\mapsto F(x, t) \end{aligned}$$

that connects x to p . This implies that not all retractions are deformation retractions.

Example. A retraction that is not a deformation retraction. Take a space that is not path connected and retract it to a point. Let $X = \{0, 1\}$ with discrete topology. $x \mapsto 0$ is a retraction, but not a deformation retraction because X is not path connected.

Definition. A continuous map $f : X \rightarrow Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \rightarrow X$ such that $fg \cong id_Y$ and $gf \cong id_X$. If there exists a homotopy equivalence between X and Y , X and Y are **homotopy equivalent** or they have the same **homotopy type**.

Lemma 0.1. A deformation retraction $f : X \rightarrow A$ is a homotopy equivalence.

Proof. Let $i : A \hookrightarrow X$ be the inclusion map. Then $fi = id_A$ and $if = f \cong id_X$ by definition. \square

Example. The disc with two holes is equivalent to ∞ .

Example. \mathbb{R}^n deformation retracts to a point, by $f_t(x) = t \cdot x$.

Definition.

- X is **contractible** if it is homotopy equivalent to a point.
- A continuous map is **nullhomotopic** if it is homotopy equivalent to a constant map.

0.3 Cell complexes

Example. The torus $S^1 \times S^1$ is the union of a point, two open intervals, and the open disc $Int(D^2)$.

These are called **cells**. Can think of discs D^n glued together.

Definition. A **CW-complex**, or **cell complex**, is a topological space X such that there exists a decomposition

$$X = \bigcup_{n \in \mathbb{N}} X^n,$$

where the X^n are constructed inductively in the following way.

- X^n is a discrete set.
- For each $n \geq 0$ there is an collection of closed n -discs $\{D_\alpha^n\}$ together with continuous maps $\phi_\alpha : \partial D_\alpha^n \rightarrow X^{n-1}$, such that

$$X^n = \frac{X^{n-1} \sqcup \bigsqcup_\alpha D_\alpha^n}{\sim},$$

where $x \sim \phi_\alpha(x)$ for all $x \in \partial D_\alpha^n$ for all α .

- A subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n .

Remark.

- As a set,

$$X^n = X^{n-1} \sqcup \bigsqcup_\alpha e_\alpha^n,$$

where each e_α^n is homeomorphic to an open n -disc. These e_α^n are called the **n -cells** of X .

- If $X = X^m$ for some m , then X is called **finite dimensional**. The minimal m such that $X = X^m$ is the **dimension** of X .

Lecture 2
Tuesday
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Example.

- $[0, 1]$ is a CW-complex.
- \mathbb{R} is a CW-complex.
- S^1 is a CW-complex.
- A graph is a CW-complex.
- $S^n = D^n / \partial D^n$ is a CW-complex. See worksheet 1.

Can also decompose CW-complexes.

- The sphere S^2 is one 0-cell, one 1-cell, and two 2-cells.
- The torus $S^1 \times S^1$ is one 0-cell, two 1-cells, and one 2-cell.
- The Möbius strip is two 0-cells, three 1-cells, and one 2-cell.
- The Klein bottle is one 0-cell, two 1-cells, and one 2-cell.

Definition. If X is a CW-complex with finitely many cells the **Euler characteristic** $\chi(X)$ of X is the number of even cells minus the number of odd cells.

Fact. $\chi(X)$ does not depend of the choice of cells decomposition.

Example.

- $\chi(S^n) = 0$ if n is odd and $\chi(S^n) = 2$ if n is even.
- $\chi(S^1 \times S^1) = 0$.

This is the generalisation of the following observation by Leonhard Euler. Let P be a convex polyhedron, where

- V is the number of vertices of P ,
- E is the number of edges of P , and
- F is the number of faces of P .

Then $V - E + F = 2$.

Example. A topological space that is not a CW-complex. $X = \{0, 1\}$ with trivial topology does not contain any closed points.

Fact. CW-complexes are always Hausdorff.

1 The fundamental group

1.1 Paths and homotopy

Let X be a topological space. A **path** is a continuous map $f : I \rightarrow X$, where $I = [0, 1]$.

Definition. Two paths f_0, f_1 are **homotopic** if there exists a homotopy between f_0 and f_1 preserving the endpoints, that is a continuous map

$$F : I \times I \rightarrow X \\ (s, t) \mapsto f_t(s) ,$$

such that

$$f_t(0) = f_0(0), \quad f_t(1) = f_0(1),$$

for all $t \in I$, and

$$F(s, 0) = f_0(s), \quad F(s, 1) = f_1(s),$$

for all $s \in I$.

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set. Then all the paths in X are homotopic if they have the same endpoints.

Proof. Let $f_0, f_1 : I \rightarrow X$ be two paths such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$. Define

$$f_t(s) = (1-t)f_0(s) + tf_1(s).$$

□

Lemma 1.1. *Being homotopic is an equivalence relation on the set of paths with fixed endpoints. We will write $f_0 \cong f_1$ for two homotopic paths f_0 and f_1 .*

Proof.

- f is homotopic to f .
- If f_0 is homotopic to f_1 by a homotopy f_t , then f_1 is homotopic to f_0 by the homotopy f_{1-t} .
- If f_0 is homotopic to f_1 by a homotopy f_t and $f_1 = g_0$ is homotopic to g_1 by a homotopy g_t , then f_0 is homotopic to g_1 by the homotopy

$$h_t = \begin{cases} f_{2t} & 0 \leq t \leq \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Then

$$H : I \times I \rightarrow X \\ (s, t) \mapsto h_t(s)$$

is continuous because its restriction to the closed subsets $I \times [0, 1/2]$ and $I \times [1/2, 1]$ is continuous, since if the restriction to two closed subsets is continuous then the restriction to the union of these subsets is continuous.

□

Let X be a topological space and $I = [0, 1]$. If $f : I \rightarrow X$ is a path, $[f]$ is the class of all paths on X homotopic to f .

Definition. Let $f, g : I \rightarrow X$ be two paths such that $f(1) = g(0)$. The **product path** $f \cdot g$ is the path

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}.$$

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A convention is that whenever we write $f \cdot g$ we implicitly assume $f(1) = g(0)$.

Lemma 1.2. *Let f_0, f_1, g_0, g_1 be paths on X such that $f_1 \cong f_0$ and $g_0 \cong g_1$. Then $f_0 \cdot g_0 \cong f_1 \cdot g_1$.*

Proof.

$$\begin{aligned} I \times I &\rightarrow X \\ (s, t) &\mapsto (f_t \cdot g_t)(s) \end{aligned}$$

is a homotopy between $f_0 \cdot g_0$ and $f_1 \cdot g_1$. □

Remark. Let $\phi : [0, 1] \rightarrow [0, 1]$ be continuous such that $\phi(0) = 0$ and $\phi(1) = 1$. If $f : I \rightarrow X$ is a path, then $f \circ \phi \cong f$. This is a **reparametrisation**.

Proof. Define

$$\phi_t(s) = (1 - t)\phi(s) + ts,$$

then $f \circ \phi_t$ is a homotopy between $f \circ \phi$ and f . □

For $x \in X$, let the **constant path** at x be

$$\begin{aligned} c_x : I &\rightarrow X \\ s &\mapsto x \end{aligned}.$$

For a path $f : I \rightarrow X$, define

$$\begin{aligned} f^{-1} : I &\rightarrow X \\ s &\mapsto f(1 - s) \end{aligned}.$$

Lemma 1.3. *Let $f, g, h : I \rightarrow X$ be paths. Then*

1. $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$,
2. $f \cdot c_{f(1)} \cong f$ and $c_{f(0)} \cdot f \cong f$, and
3. $f \cdot f^{-1} \cong c_{f(0)}$ and $f^{-1} \cdot f \cong c_{f(1)}$.

Proof.

1. $((f \cdot g) \cdot h) \phi = f \cdot (g \cdot h)$, where

$$\phi(s) = \begin{cases} \frac{s}{2} & s \in [0, \frac{1}{2}] \\ s - \frac{1}{4} & s \in [\frac{1}{2}, \frac{3}{4}] \\ 2s - 1 & s \in [\frac{3}{4}, 1] \end{cases},$$

so $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$ by reparametrisation.

2. Again reparametrisation, by

$$\psi(s) = \begin{cases} 2s & s \in [0, \frac{1}{2}] \\ 1 & s \in [\frac{1}{2}, 1] \end{cases}, \quad \chi(s) = \begin{cases} 0 & s \in [0, \frac{1}{2}] \\ 2s - 1 & s \in [\frac{1}{2}, 1] \end{cases}.$$

3. Define

$$H(s, t) = \begin{cases} f(\max\{1 - 2s, t\}) & s \in [0, \frac{1}{2}] \\ f(\max\{2s - 1, t\}) & s \in [\frac{1}{2}, 1] \end{cases}.$$

H is continuous, and

$$H(s, 0) = f^{-1} \cdot f, \quad H(s, 1) = c_{f(1)}.$$

The inverse is similar. □

Definition. A **loop** with **basepoint** $x_0 \in X$ is a path $f : I \rightarrow X$ such that $f(0) = f(1) = x_0$.

Definition. Denote by $\pi_1(X, x_0)$ the set of homotopy classes $[f]$ of loops $f : I \rightarrow X$ with basepoint x_0 .

Proposition 1.4. $\pi_1(X, x_0)$ is a group with product $[f][g] = [f \cdot g]$ and neutral element $c_{x_0} : I \rightarrow X$, the constant path at x_0 .

Proof. Follows directly from Lemma 1.2 and Lemma 1.3. \square

Definition. $\pi_1(X, x_0)$ is the **fundamental group** of X at x_0 .

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set and $x_0 \in X$. Then $\pi_1(X, x_0) = 0$.

Proof. X is convex gives that all loops are homotopic to each other. \square

Example.

- The fundamental group of a space X with the trivial topology is trivial, since X is simply connected, because all maps $f : I \rightarrow X$ are continuous, so path connected and all paths are homotopic.
- The fundamental group of a space X with the discrete topology is trivial, since $f : I \rightarrow X$ continuous gives f constant.

Assume $x_0, x_1 \in X$ such that x_0 and x_1 are in the same path component of X . Let $h : I \rightarrow X$ be a path such that $h(0) = x_0$ and $h(1) = x_1$. Define

$$\begin{aligned} \beta_h : \pi_1(X, x_1) &\rightarrow \pi_1(X, x_0) \\ [f] &\mapsto [h \cdot f \cdot h^{-1}] \end{aligned} .$$

This is well-defined by Lemma 1.2.

Proposition 1.5. $\beta_h : \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$ is an isomorphism.

Proof. It is a homomorphism.

$$\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot h^{-1}] = [h \cdot f \cdot h^{-1}] [h \cdot g \cdot h^{-1}] = \beta_h[f] \cdot \beta_h[g],$$

and $\beta_h[c_{x_1}] = [c_{x_1}]$. It is bijective with $(\beta_h)^{-1} = \beta_{h^{-1}}$. \square

If X is path connected, we often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Definition. X is **simply connected** if it is path connected and $\pi_1(X) = 0$.

Proposition 1.6. X is simply connected if and only if there exists a unique homotopy class of paths between any two points of X .

Proof.

\implies There exists a path between any two points. Let f, g be two paths from x_0 to x_1 for $x_0, x_1 \in X$. $f \cdot g^{-1} \cong g \cdot g^{-1}$ gives $f \cong f \cdot g^{-1} \cdot g \cong g \cdot g^{-1} \cdot g \cong g$.

\impliedby X is path connected. $x_1 = x_0$ gives that all loops at x_0 are homotopic to each other, so $\pi_1(X) = 0$. \square

1.2 The fundamental group of the circle

Goal is to show that $\pi_1(S^1) \cong \mathbb{Z}$.

Lecture 4
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Definition. A **covering space** of a space X is a space \tilde{X} and a continuous map $p : \tilde{X} \rightarrow X$ such that for each $x \in X$ there is an open $U \subseteq X$ such that

- $p^{-1}(U) = \bigcup_{j \in J} \tilde{U}_j$, where $\tilde{U}_j \subseteq \tilde{X}$ is open,
- $\tilde{U}_i \cap \tilde{U}_j = \emptyset$ if $i \neq j$, and
- $p|_{\tilde{U}_j} : \tilde{U}_j \rightarrow U$ is a homeomorphism for all $j \in J$.

Such a U is called **evenly covered**. The \tilde{U}_j are called **sheets**.

Example.

$$\begin{aligned} p : \mathbb{R} &\rightarrow S^1 \\ s &\mapsto (\cos(2\pi s), \sin(2\pi s)) \end{aligned}$$

Definition. Let $p : \tilde{X} \rightarrow X$ be a covering space. A **lift** of a continuous map $f : Y \rightarrow X$ is a continuous map $\tilde{f} : Y \rightarrow \tilde{X}$ such that $p\tilde{f} = f$, so

$$\begin{array}{ccc} & \tilde{X} & \\ \tilde{f} \nearrow & \downarrow p & \\ Y & \xrightarrow{f} & X \end{array}$$

Proposition 1.7 (Unique lifting property). *Let $p : \tilde{X} \rightarrow X$ be a covering space and $f : Y \rightarrow X$ be a continuous map. If there are two lifts $\tilde{f}_1, \tilde{f}_2 : Y \rightarrow \tilde{X}$ of f such that $\tilde{f}_1(y) = \tilde{f}_2(y)$ for some $y \in Y$ and if Y is connected, then $\tilde{f}_1 = \tilde{f}_2$.*

Proof. Let $y \in Y$ and $U \subseteq X$ be an evenly covered neighbourhood of $f(y)$. Then

$$p^{-1}(U) = \bigcup_j \tilde{U}_j.$$

Let \tilde{U}_1 be the sheet such that $\tilde{f}_1(y) \in \tilde{U}_1$, and let \tilde{U}_2 be the sheet such that $\tilde{f}_2(y) \in \tilde{U}_2$. Let $N \subseteq Y$ be open and $y \in N$ such that $\tilde{f}_1(N) \subseteq \tilde{U}_1$ and $\tilde{f}_2(N) \subseteq \tilde{U}_2$. We have $p\tilde{f}_1 = p\tilde{f}_2$.

$$\tilde{f}_1(y) = \tilde{f}_2(y) \iff \tilde{U}_1 = \tilde{U}_2 \iff \tilde{f}_1|_N = \tilde{f}_2|_N.$$

Let

$$A = \{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\},$$

so A is open and $Y \setminus A$ is open. Thus $A \neq \emptyset$ gives $A = Y$. \square

Proposition 1.8 (Homotopy lifting property). *Let $p : \tilde{X} \rightarrow X$ be a covering space and $F : Y \times I \rightarrow X$ be a continuous map such that there exists a lift $\tilde{f}_0 : Y \times \{0\} \rightarrow \tilde{X}$ of $F|_{Y \times \{0\}}$. Then there is a unique lift $\tilde{F} : Y \times I \rightarrow \tilde{X}$ of F such that $\tilde{F}|_{Y \times \{0\}} = \tilde{f}_0$.*

Proof. Let $y_0 \in Y$ and $t \in I$. There are open $y_0 \in N_t \subseteq Y$ and $t \in (a_t, b_t) \subseteq I$ such that $F(N_t \times (a_t, b_t)) \subseteq U \subseteq X$, where $U \subseteq X$ is open and evenly covered. Compactness of I gives that there exist

$$0 = t_0 < \dots < t_m = 1,$$

and there exists $y_0 \in N \subseteq Y$ open such that $F(N \times [t_i, t_{i+1}]) \subseteq U_i \subseteq X$, where $U_i \subseteq X$ is open and evenly covered. We inductively construct a lift $\tilde{F}|_{N \times I}$ of $F|_{N \times I}$.

- $\tilde{F}|_{N \times [0,0]} = \tilde{f}_0|_{N \times [0,0]}$ exists.
- Assume the lift has been constructed on $N \times [0, t_i]$. Let $\tilde{U}_i \subseteq \tilde{X}$ be such that $p|_{\tilde{U}_i}: \tilde{U}_i \rightarrow U_i$ such that $\tilde{F}(y_0, t_i) \subseteq \tilde{U}_i$. After shrinking N , may assume $\tilde{F}(N \times \{t_i\}) \subseteq \tilde{U}_i$. Define \tilde{F} on $N \times [t_i, t_{i+1}]$ to be composition of F with the homeomorphism $p^{-1}: U_i \rightarrow \tilde{U}_i$.

After finitely many steps we obtain a lift $\tilde{F}: N \times I \rightarrow \tilde{X}$, where $y_0 \in N \subseteq Y$ is open, so for each $y \in Y$ there is a neighbourhood $N_y \subseteq Y$ such that $F|_{N_y \times I}: N_y \times I \rightarrow X$ lifts. For all $y \in Y$, $\{y\} \times I$ is connected and can be lifted, so Proposition 1.7 gives that the lift of $N \times I$ is unique. Thus there is a unique lift $\tilde{F}: Y \times I \rightarrow \tilde{X}$. \square

Example. Let X be a topological space and A be discrete. Then $p: X \times A \rightarrow X$ is a covering space. This is the **trivial covering**. (Exercise: show the unique lifting property and the homotopy lifting property for the trivial covering)

Corollary 1.9. Let $f: I \rightarrow X$ be a path, $f(0) = x_0$, and $p: \tilde{X} \rightarrow X$ be a covering space. For each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f}: I \rightarrow \tilde{X}$ such that $\tilde{f}(0) = \tilde{x}_0$.

Proof. Proposition 1.8 for Y a point. \square

Theorem 1.10. Let $x_0 = (1, 0) \in S^1$. $\pi_1(S^1, x_0)$ is the infinite cyclic group generated by the homotopy class of the loop

$$\begin{aligned} \omega: I &\rightarrow S^1 \\ s &\mapsto (\cos(2\pi s), \sin(2\pi s)) \end{aligned}$$

Remark.

- $[\omega]^n = [\omega_n]$, where

$$\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns)).$$

-

$$\begin{aligned} p: \mathbb{R} &\rightarrow S^1 \\ s &\mapsto (\cos(2\pi s), \sin(2\pi s)) \end{aligned}$$

is a covering space.

- ω_n lifts to

$$\begin{aligned} \tilde{\omega}_n: I &\rightarrow \mathbb{R} \\ s &\mapsto ns \end{aligned}$$

such that $\tilde{\omega}_n(0) = 0$ and $\tilde{\omega}_n(1) = n$.

Proof of Theorem 1.10.

- If $f: I \rightarrow S^1$ be a loop at x_0 , then the homotopy lifting property gives that there exists a lift $\tilde{f}: I \rightarrow \mathbb{R}$ such that $\tilde{f}(0) = 0$. Since $p(\tilde{f}(1)) = f(1) = x_0$, then $\tilde{f}(1) = n$ for some $n \in \mathbb{Z}$. $\tilde{\omega}_n: I \rightarrow \mathbb{R}$ is another path such that $\tilde{\omega}_n(0) = 0$ and $\tilde{\omega}_n(1) = n$, so $\tilde{f} \cong \tilde{\omega}_n$. Let $F: I \times I \rightarrow \mathbb{R}$ be a homotopy equivalence between \tilde{f} and $\tilde{\omega}_n$. Then $pF: I \times I \rightarrow S^1$ gives a homotopy between $p\tilde{f} = f$ and $p\tilde{\omega}_n = \omega_n$.
- Let $m, n \in \mathbb{Z}$ and assume $\omega_m \cong \omega_n$. Let $F: I \times I \rightarrow S^1$ be a homotopy.

$$F(0, t) = \omega_m(t), \quad F(1, t) = \omega_n(t), \quad F(s, 0) = F(s, 1) = x_0,$$

for all $s, t \in I$. The unique lifting property gives that $\tilde{\omega}_n, \tilde{\omega}_m: I \rightarrow \mathbb{R}$ are unique lifts such that $\tilde{\omega}_n(0) = 0 = \tilde{\omega}_m(0)$. The homotopy lifting property gives that F lifts uniquely to a homotopy $\tilde{F}: I \times I \rightarrow \mathbb{R}$ between $\tilde{\omega}_n$ and $\tilde{\omega}_m$, and $\tilde{F}(s, 1) \in \mathbb{Z}$ for all $s \in I$. Thus $\tilde{F}(s, 1) = n = m$, so $\omega_m \cong \omega_n$ if and only if $n = m$. \square

Lecture 5 is a problem class.

Theorem 1.11. *Every non-constant polynomial $p \in \mathbb{C}[z]$ has a root in \mathbb{C} .*

Proof. May assume

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n.$$

Assume p has no roots in \mathbb{C} . For each $r \in \mathbb{R}_{\geq 0}$ we obtain a loop

$$\begin{aligned} f_r : I &\rightarrow \mathbb{C} \\ s &\mapsto \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}, \end{aligned}$$

so $|f_r(s)| = 1$. $f_r(0) = 1$ and $f_r(1) = 1$, so f_r is a loop based at 1. f_0 is the constant loop at 1. $f_r(s)$ depends continuously on r , so $f_r \cong f_0$ for all $r \in \mathbb{R}_{\geq 0}$ and $[f_r] = [f_0] = 0 \in \pi_1(S^1)$. Fix $r \in \mathbb{R}_{\geq 0}$ such that $r > 1$ and $r > |a_1| + \cdots + |a_n|$. For $|z| = r$ we have

$$|z^n| > (|a_1| + \cdots + |a_n|) |z^{n-1}| \geq |a_1 z^{n-1}| + \cdots + |a_n| \geq |a_1 z^{n-1} + \cdots + a_n|.$$

Hence, for $0 \leq t \leq 1$ the polynomial $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$ has no root z with $|z| = r$. Define

$$F_r(t, s) = \frac{p_t(re^{2\pi i s})/p_t(r)}{|p_t(re^{2\pi i s})/p_t(r)|}.$$

$F_r(0, s) = \omega_n(s)$ and $F_r(1, s) = f_r(s)$, so $[\omega_n] = [f_r] = 0 \in \pi_1(S^1)$. Theorem 1.10 gives that $n = 0$, so p is constant. \square

See Hatcher Theorem 1.9 and Theorem 1.10 for more applications.

Proposition 1.12. *Let X, Y be topological spaces, $x_0 \in X$, and $y_0 \in Y$. Then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. A map

$$\begin{aligned} f : Z &\rightarrow X \times Y \\ z &\mapsto (g(z), h(z)) \end{aligned}$$

is continuous if and only if $g : Z \rightarrow X$ and $h : Z \rightarrow Y$ are continuous. For $Z = I$,

$$\{ \text{loops in } X \times Y \text{ based at } (x_0, y_0) \} \quad \longleftrightarrow \quad \{ \text{loops in } X \text{ based at } x_0 \} \times \{ \text{loops in } Y \text{ based at } y_0 \}.$$

Two loops

$$\begin{aligned} f_1 : I &\rightarrow X \times Y \\ s &\mapsto (g_1(s), h_1(s)) \end{aligned} \quad , \quad \begin{aligned} f_2 : I &\rightarrow X \times Y \\ s &\mapsto (g_2(s), h_2(s)) \end{aligned}$$

are homotopic if and only if $g_1 \cong g_2$ and $h_1 \cong h_2$, so there is a bijection

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

$f_1 \cdot f_2 = (g_1 \cdot g_2, h_1 \cdot h_2)$ and the constant loop is mapped to the constant loop, so this is also a group isomorphism. \square

Example. The torus $S^1 \times S^1$ has

$$\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z}^2.$$

1.3 Induced homomorphisms

Let X, Y be topological spaces, $x_0 \in X$, and $\phi : X \rightarrow Y$. An observation is that ϕ induces a homomorphism

$$\begin{array}{ccc} \phi_* : \pi_1(X, x_0) & \rightarrow & \pi_1(Y, \phi(x_0)) \\ [f] & \mapsto & [\phi f] \end{array} .$$

ϕ_* is well-defined, since if f_t is a homotopy between the loops f_0 and f_1 based at x_0 , then ϕf_t is a homotopy of loops between ϕf_0 and ϕf_1 . Moreover,

$$\phi(f \cdot g) = (\phi f) \cdot (\phi g),$$

and ϕ maps the constant path at x_0 to the constant path at $\phi(x_0)$, so ϕ is a homomorphism.

Proposition 1.13.

1. Let $\psi : X \rightarrow Y$ and $\phi : Y \rightarrow Z$ be continuous maps between topological spaces, $x_0 \in X$, and

$$\begin{aligned} \psi_* : \pi_1(X, x_0) &\rightarrow \pi_1(Y, \psi(x_0)), & \phi_* : \pi_1(Y, \psi(x_0)) &\rightarrow \pi_1(Z, \phi\psi(x_0)), \\ (\phi\psi)_* : \pi_1(X, x_0) &\rightarrow \pi_1(Z, \phi\psi(x_0)). \end{aligned}$$

$$\text{Then } (\phi\psi)_* = \phi_*\psi_*.$$

2. Let $id_X : X \rightarrow X$ be the identity then

$$(id_X)_* : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$$

is the identity.

Proof.

1. Let $f : I \rightarrow X$ be a loop at x_0 , then

$$(\phi\psi)_*([f]) = [(\phi\psi)f] = [\phi(\psi f)] = \phi_*([\psi f]) = \phi_*\psi_*([f]).$$

2. $(id_X)_*([f]) = [id_X f] = [f]$.

□

These two observations yield in particular that if $\phi : X \rightarrow Y$ is a homeomorphism with inverse $\psi : Y \rightarrow X$, then

$$\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \phi(x_0))$$

is an isomorphism with inverse ψ_* .

Proposition 1.14. If $n \geq 2$, then $\pi_1(S^n) = 0$.

Proposition 1.15. Let $\phi : X \rightarrow Y$ be a homotopy equivalence. Then

$$\phi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \phi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Recall that if $x_0, x_1 \in X$ and $h : I \rightarrow X$ is a path such that $h(0) = x_0$ and $h(1) = x_1$, then we obtain an isomorphism

$$\begin{array}{ccc} \beta_h : \pi_1(X, x_1) & \rightarrow & \pi_1(X, x_0) \\ [f] & \mapsto & [h \cdot f \cdot h^{-1}] \end{array} .$$

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Lemma 1.16. Let $\phi_t : X \rightarrow Y$ be a homotopy and $x_0 \in X$. Define the path

$$\begin{aligned} h : I &\rightarrow Y \\ s &\mapsto \phi_s(x_0) \end{aligned} ,$$

where $h(0) = \phi_0(x_0)$ and $h(1) = \phi_1(x_0)$. Then $(\phi_0)_* = \beta_h(\phi_1)_*$, that is the following diagram commutes.

$$\begin{array}{ccc} & \pi_1(Y, \phi_1(x_0)) & \\ (\phi_1)_* \nearrow & \downarrow \sim \beta_h & \\ \pi_1(X, x_0) & & \\ (\phi_0)_* \searrow & \downarrow & \\ & \pi_1(Y, \phi_0(x_0)) & \end{array} .$$

Proof. For $t \in I$, define the path

$$\begin{aligned} h_t : I &\rightarrow X \\ s &\mapsto h(ts) \end{aligned} ,$$

where $h_t(0) = \phi_0(x_0)$ and $h_t(1) = h(t) = \phi_t(x_0)$. Let f be a loop at x_0 . Define

$$F_t = h_t \cdot (\phi_t f) \cdot h_t^{-1}.$$

Then F_t is a loop at $\phi_0(x_0)$, which is continuous in t . So F_t is a homotopy of loops between

$$F_0 = h_0 \cdot (\phi_0 f) \cdot h_0^{-1} \cong \phi_0 f, \quad F_1 = h_1 \cdot (\phi_1 f) \cdot h_1^{-1} = h \cdot (\phi_1 f) \cdot h^{-1}.$$

Hence

$$(\phi_0)_*([f]) = [\phi_0 f] = [h \cdot (\phi_1 f) \cdot h^{-1}] = \beta_h([\phi_1 f]) = \beta_h(\phi_1)_*([f]).$$

□

Lemma 1.16 implies in particular the following.

Corollary 1.17. If $\psi : X \rightarrow X$ is continuous and $\psi \cong id_X$, then

$$\psi_* : \pi_1(X, x_0) \rightarrow \pi_1(X, \psi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Proof. By Lemma 1.16 there is a path h from $\psi(x_0)$ to x_0 such that

$$\begin{array}{ccc} & \pi_1(X, x_0) & \\ (id_X)_* \nearrow & \downarrow \sim \beta_h & \\ \pi_1(X, x_0) & & \\ \psi_* \searrow & \downarrow & \\ & \pi_1(X, \psi(x_0)) & \end{array} ,$$

so $\psi_* = \beta_h$ hence an isomorphism. □

Proof of Proposition 1.15. Let $\phi : X \rightarrow Y$ be a homotopy equivalence. Let $\psi : Y \rightarrow X$ be a homotopy inverse of ϕ , that is $\phi\psi \cong id_Y$ and $\psi\phi \cong id_X$.

$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\phi(x_0)) \xrightarrow{\phi_*} \pi_1(Y, \psi\phi\psi(x_0)).$$

Have to show that ϕ_* is bijective. The observation above gives that $(\psi\phi)_* = \psi_*\phi_*$ is an isomorphism, so ϕ_* is injective and ψ_* is surjective. Similarly $(\phi\psi)_* = \phi_*\psi_*$ is an isomorphism, so ψ_* is injective and ϕ_* is surjective. □

Lemma 1.18. *Let X be a topological space and $x_0 \in X$. Assume*

$$X = \bigcup_{\alpha \in \Lambda} A_\alpha,$$

such that

- *the A_α are all open and path-connected,*
- *$x_0 \in A_\alpha$ for all $\alpha \in \Lambda$, and*
- *all the intersections $A_\alpha \cap A_\beta$ are path-connected for all $\alpha, \beta \in \Lambda$.*

If f is a loop in X at x_0 , then we can write $[f] = [h_1] \dots [h_m]$, such that the h_i are loops at x_0 , and each contained in a single A_{α_i} .

Proof. f is continuous, so for all $s \in I$ there is an open neighbourhood V_s such that $f(V_s)$ such that $f(V_s) \subseteq A_\alpha$ for some α . We can choose V_s to be an interval (a_s, b_s) such that $f([a_s, b_s]) \subseteq A_\alpha$. I is compact gives that a finite number of such intervals cover I , so there is a partition

$$0 = s_0 < \dots < s_m = 1,$$

such that $f([s_{i-1}, s_i]) \subseteq A_{\alpha_i}$ for some α_i . Let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$, and rescaling.

$$f \cong f_1 \cdot \dots \cdot f_m,$$

for $f_i \subseteq A_{\alpha_i}$ and $A_{\alpha_i} \cap A_{\alpha_j}$ is path-connected. Let g_i be a path from x_0 to $f(s_i)$ in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$. Let g_0, g_m be the constant loops at x_0 . $h_i = g_{i-1} \cdot f_i \cdot g_i^{-1}$ is a loop based at x_0 and $h_i \subseteq A_{\alpha_i}$. Thus

$$f \cong (g_0 \cdot f_1 \cdot g_1^{-1}) \cdot \dots \cdot (g_{m-1} \cdot f_m \cdot g_m^{-1}),$$

so $[f] = [h_1] \dots [h_m]$. □

A Quotient topology

Recall that if X is a set with equivalence relation \sim , there is a quotient set X/\sim . The quotient map

$$\begin{array}{ccc} \pi : X & \rightarrow & \frac{X}{\sim} \\ x & \mapsto & [x] \end{array}$$

is characterised by the following universal property. For every map $g : X \rightarrow Y$ such that

$$a \sim b \implies g(a) = g(b),$$

there exists a unique $f : X/\sim \rightarrow Y$ such that $g = f \cdot \pi$, so

$$\begin{array}{ccc} X & & \\ \pi \downarrow & \searrow g & \\ \frac{X}{\sim} & \xrightarrow{\exists! f} & Y \end{array}.$$

Let X be a topological space and \sim be an equivalence relation on X . We define a topology on X/\sim by

$$U \subseteq \frac{X}{\sim} \text{ open} \iff \pi^{-1}(U) \text{ open.}$$

Remark.

- This is the largest topology on X/\sim such that π is continuous. Exercise 1 states that if Z is a topological space and $f : X/\sim \rightarrow Z$ is a map, then f is continuous if and only if $f\pi : X \rightarrow Z$ is continuous. This implies that the topological quotient $\pi : X \rightarrow X/\sim$ is characterised by the following universal property. For any topological space Z and a continuous $g : X \rightarrow Z$ such that

$$a \sim b \implies g(a) = g(b),$$

there exists a unique continuous map $f : X/\sim \rightarrow Z$ such that $gf \cdot \pi$, so

$$\begin{array}{ccc} X & & \\ \pi \downarrow & \searrow g & \\ \frac{X}{\sim} & \xrightarrow{\exists! f} & Z \end{array}.$$

- The quotient map is in general not open. For example, if $\pi : [0, 1] \rightarrow S^1$, then $[0, 1] \subset [0, 1]$ is open but $\pi([0, 1)) \subseteq S^1$ is not open.
- If X is Hausdorff, in general X/\sim is not Hausdorff.
- If \sim is the trivial relation, then $\pi : X \rightarrow X/\sim$ is a homeomorphism. Exercise 3 states that if X, Y are topological spaces, X is compact, Y is Hausdorff, and $\pi : X \rightarrow Y$ is surjective and continuous, then π is a quotient, that is there exists \sim on X and $\pi : X \rightarrow Y \cong X/\sim$ is a quotient map.
- In particular, if $\pi : X \rightarrow Y$ is bijective, then π is a homeomorphism. Exercise 4, 5, 6 state that if f is continuous and surjective, $f(\partial D^n)$ is a point, and f is a bijection on $D^n \setminus \partial D^n$, then

$$\begin{array}{ccc} D^n & & \\ \pi \downarrow & \searrow f & \\ \frac{D^n}{\partial D^n} & \xrightarrow{\sim} & S^1 \end{array}.$$