M4P58 Modular Forms

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Syllabus

Contents

0 Introduction

Lecture 1 Friday 04/10/19

The following are textbooks.

- Serre, A course in arithmetic, 1973
- J Shurman and F Diamond, A first course in modular forms, 2005

Let

$$f = q \prod_{n=1}^{\infty} (1 - q^n)^2 \left(1 - q^{11n}\right)^2 = \sum_{n=1} b_n q^n = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \dots,$$

and let a_n be the number of solutions modulo n to the elliptic curve

$$E = \left\{(x, y) \in \mathbb{Z} \mid y^2 + y = x^3 - x^2 - 10x - 20\right\}.$$

- Modulo 2, there are $a_2 = 4$ solutions $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$.
- Modulo 3, there are $a_3 = 4$ solutions $(1,0)$, $(1,-1)$, $(-1,0)$, $(-1,-1)$.
- Modulo 5, there are $a_5 = 4$ solutions $(0,0)$, $(0,-1)$, $(1,0)$, $(-1,-1)$.
- Modulo 7, there are $a_7 = 9$ solutions $(1,3)$, $(2,2)$, $(2,-3)$, $(-1,1)$, $(-1,-2)$, $(-2,1)$, $(-2,-2)$, $(-3,1)$, $(-3,-2)$.

If $p \neq 11$, then

$$a_p - p = -b_p.$$

The following are some questions.

- What is the relationship between E and f?
- Can we find similar relationships for other E?
- How does one prove something like this?

Let

$$\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R},\ y > 0\} \subseteq \mathbb{C}.$$

Then $\mathbb{H}$ has an action of

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}.$$

Modular forms are complex functions on $\mathbb{H}$ with a high degree of symmetry. These functions are symmetric under the action of large discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$, in particular

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\} \subseteq \mathrm{SL}_2(\mathbb{R}).$$

Why are these interesting to number theorists? Power series expansions often involve expressions of interest to number theorists. For example,

- Bernoulli numbers,
- divisor functions $\sigma_k(n) = \sum_{d \mid n} d^k$,
- number of points on elliptic curves, and
- traces of Galois representations.

1 Modular forms of level one

1.1 Modular functions and forms

1.1.1 Modular actions

Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}.$$

Then $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{C} \cup \{\infty\}$ by

$$\gamma \cdot z = \begin{cases} \frac{az+b}{cz+d} & z \ne -\frac{d}{c} \\ \infty & z = -\frac{d}{c} \end{cases} \qquad \gamma \cdot \infty = \frac{a}{c}.$$

One checks that this gives a bijection from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$, where inverse is given by the inverse matrix

$$\gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and $\gamma \cdot (\gamma' \cdot z) = \gamma\gamma' \cdot z$. One obtains a left action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{C} \cup \{\infty\}$. An observation is

$$\operatorname{Im} \gamma z = \operatorname{Im} \frac{az+b}{cz+d} = \operatorname{Im} \frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = \frac{\operatorname{Im}(az+b)(c\overline{z}+d)}{|cz+d|^2} = \frac{(ad-bc)\operatorname{Im} z}{|cz+d|^2}.$$

In particular, if $\gamma \in \mathrm{SL}_2(\mathbb{R})$, then Lecture 2 Friday 04/10/19

$$\operatorname{Im} \gamma z = \frac{\operatorname{Im} z}{|cz+d|^2}.$$

So $\mathrm{SL}_2(\mathbb{R})$ preserves $\mathbb{H} \cup \{\infty\}$. More generally, if $\gamma \in \mathrm{GL}_2(\mathbb{R})$, then

$$\operatorname{Im} \gamma z = \frac{\det \gamma \operatorname{Im} z}{|cz+d|^2}.$$

So

$$\mathrm{GL}_2(\mathbb{R})_+ = \{\gamma \in \mathrm{GL}_2(\mathbb{R}) \mid \det \gamma > 0\}$$

preserves $\mathbb{H} \cup \{\infty\}$. Define

$$\begin{array}{cccc} f|_{k,\gamma} & : & \mathbb{H} & \longrightarrow & \mathbb{C} \\ & & z & \longmapsto & \det \gamma^{k-1} f(\gamma z)(cz+d)^{-k} \end{array}, \qquad f : \mathbb{H} \to \mathbb{C}, \qquad \gamma \in \mathrm{GL}_2(\mathbb{R})_+, \qquad k \in \mathbb{Z},$$

where $\det \gamma^{k-1}$ is the fudge factor, which is one for $\gamma \in \mathrm{SL}_2(\mathbb{R})$, and $(cz+d)^{-k}$ is the twisted action on functions. Check that

$$f|_{k,\mathrm{id}} = f, \qquad \left(f|_{k,\gamma}\right)\Big|_{k,\gamma'} = f|_{k,\gamma'\gamma}.$$

This gives, for each k, a left action of $\mathrm{GL}_2(\mathbb{R})_+$ on functions $\mathbb{H} \to \mathbb{C}$, a **modular action of weight** k. A modular form of weight k will be a sufficiently nice function $f : \mathbb{H} \to \mathbb{C}$ such that $f|_{k,\gamma} = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. That is, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and all $z \in \mathbb{H}$,

$$f(\gamma z)(cz+d)^{-k} = f(z), \qquad \Longrightarrow \qquad f(\gamma z) = f(z)(cz+d)^k,$$

the **modular transformation law of weight** k. The following are some observations.

- Let $k = 0$. Then constant functions satisfy $f(\gamma z) = f(z)$. It will turn out that all functions of weight zero are constant.
- Let k be odd, and $\gamma = -\mathrm{id}$. Then $\gamma z = z$ for all z and $cz + d = -1$, so $f(\gamma z) = f(z)(cz+d)^k$ gives $f(z) = f(z)(-1)^k$, so $f(z) = -f(z)$, so $f(z) = 0$ for all z. So no functions $f : \mathbb{H} \to \mathbb{C}$ satisfy the modular transformation law of weight k, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, when k is odd.

1.1.2 Review of complex analysis

Let $f : U \to \mathbb{C}$, for $U \subseteq \mathbb{C}$ open, and let $p \in U$.

Definition 1.1.1. f is **holomorphic** at p if

$$f'(p') = \lim_{\epsilon \to 0,\ \epsilon \in \mathbb{C}} \frac{f(p' + \epsilon) - f(p')}{\epsilon}$$

exists for all p' in a neighbourhood of p.

Proposition 1.1.2. *f is holomorphic at p implies that f is continuous.*

Proposition 1.1.3. *f is holomorphic at p implies that f is infinitely differentiable at p, that is $f^{(n)}(p)$ exists for all $n \geq 0$. Moreover, we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (z-p)^n = f(p) + f'(p)(z-p) + \frac{f'(p)}{2}(z-p)^2 + \dots,$$

for all z in a neighbourhood of p.

Corollary 1.1.4. *If f is holomorphic and not identically zero on an open set U, then the zeroes of f are isolated on U.*

More generally is the following.

Definition 1.1.5. f is **meromorphic** at p if there exists a neighbourhood U of p and $g, h : U \to \mathbb{C}$ holomorphic on U such that $f = g/h$ on $U \setminus \{p\}$. Such an f has a **Laurent series expansion** at p,

$$f(z) = \sum_{i=-N}^{\infty} c_i (z-p)^i.$$

The smallest i such that $c_i \neq 0$ is denoted by $\operatorname{ord}_p f$, the **order of vanishing** of f at p.

- If $\operatorname{ord}_p f = -n$ for $n > 0$, we say f has a **pole of order** n.
- If $\operatorname{ord}_p f = n$ for $n > 0$, we say f has a **zero of order** n.

Proposition 1.1.6.

- $\operatorname{ord}_p fg = \operatorname{ord}_p f + \operatorname{ord}_p g$.
- $\operatorname{ord}_p (f+g) \geq \min\{\operatorname{ord}_p f, \operatorname{ord}_p g\}$, *with equality if* $\operatorname{ord}_p f \neq \operatorname{ord}_p g$.

If f is holomorphic on $U \setminus \{p\}$ for U a neighbourhood of p, then f may or may not be meromorphic at p.

Example. $f(z) = e^{-1/z^2}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, but not meromorphic at zero.

Theorem 1.1.7. *Let f be holomorphic on $U \setminus \{p\}$, and there exists $n > 0$ such that*

$$\lim_{x \to p} (x-p)^n f(x)$$

exists. Then f is meromorphic on U, and $\operatorname{ord}_p f \geq -n$.

1.1.3 Modular functions

Definition 1.1.8. $f : \mathbb{H} \to \mathbb{C}$ is a **weakly modular function of weight** k if

- f is meromorphic on $\mathbb{H}$, and
- f satisfies the modular transformation law of weight k.

Consider

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $\gamma z = z + 1$ and $cz + d = 1$. The modular transformation law gives $f(z+1) = f(z)$. Let

$$\mathrm{D} = \{q \mid |q| < 1\}.$$

Can define a function

$$\begin{array}{rccc} g & : & \mathrm{D} \setminus \{0\} & \longrightarrow & \mathbb{C} \\ & & q & \longmapsto & f\left(\dfrac{\log q}{2\pi i}\right) \end{array},$$

that is $f(z) = g\left(e^{2\pi i z}\right)$ for $z \in \mathbb{H}$, where g is holomorphic or meromorphic on $\{z \mid 0 < |z| < 1\}$ if and only if f is holomorphic or meromorphic on $\mathbb{H}$.

Definition 1.1.9. $f : \mathbb{H} \to \mathbb{C}$ is a **modular form of weight** k if

1. f satisfies the modular transformation law of weight k,
2. f is holomorphic on $\mathbb{H}$, and
3. f is holomorphic at ∞, so the function $g : \mathrm{D} \setminus \{0\} \to \mathbb{C}$, which is holomorphic on $\mathrm{D} \setminus \{0\}$ by 2, extends to a holomorphic function on D.

Then $q \to 0$ in D if and only if $\operatorname{Im} z \to +\infty$. Then 3 means $g(q)$ is bounded as $q \to 0$ so $f(z)$ is bounded as $\operatorname{Im} z \to +\infty$. For f satisfying 3, $g : \mathrm{D} \setminus \{0\} \to \mathbb{C}$ has a series expansion

$$g(q) = \sum_n a_n q^n = a_0 + a_1 q + \dots$$

in $q = e^{2\pi i z}$. We call this the **q-expansion** for f.

Lecture 3 Monday 07/10/19

Definition 1.1.10. $f : \mathbb{H} \to \mathbb{C}$ is a **meromorphic modular form of weight** k if the same conditions 1 to 3 hold, but with holomorphic weakened to meromorphic.

Note. If f is only meromorphic at ∞ then a finite number of negative powers of q can appear.

Example.

- The **modular discriminant**

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

  is a modular form of weight 12.

- The j**-invariant**

$$\mathrm{j}(z) = \frac{1}{q} + 744 + 196844q + 21493760q^2 + \dots$$

  is a meromorphic modular form of weight 0.

1.1.4 Lattice functions

How can we construct modular forms?

Definition 1.1.11. A **lattice** in $\mathbb{C}$ is an abelian subgroup of $\mathbb{C}$ of the form $\mathbb{Z}w_1 + \mathbb{Z}w_2$, where $w_1, w_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent. More generally if V is an $\mathbb{R}$-vector space, a **lattice** L in V is a discrete abelian subgroup of V that spans V over $\mathbb{R}$. For $L \subseteq \mathbb{C}$ a lattice and $\lambda \in \mathbb{C}^\times$, let

$$\lambda L = \{\lambda x \mid x \in L\} \subseteq \mathbb{C}.$$

We say that L and λL are **homothetic**. For $z \in \mathbb{H}$, let

$$\mathrm{L}_{z,1} = \mathbb{Z} + \mathbb{Z}z = \{az + b \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}.$$

A question is when is $\mathrm{L}_{z,1}$ homothetic to $\mathrm{L}_{z',1}$, and what is a homothety factor?

- Suppose $\mathrm{L}_{z,1} = \lambda \mathrm{L}_{z',1}$. Then there exist a, b, c, d such that $\lambda z' = az + b$ and $\lambda = cz + d$, so

$$\begin{pmatrix} \lambda z' \\ \lambda \end{pmatrix} = \gamma \begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{1}$$

  On the other hand there exist a', b', c', d' such that $z = a'\lambda z' + b'\lambda$ and $1 = c'\lambda z' + d'\lambda$, so

$$\gamma' \begin{pmatrix} \lambda z' \\ \lambda \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{2}$$

  (1) and (2) imply that

$$\gamma'\gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix},$$

  so $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Moreover (1) implies that $z' = (az + b) / (cz + d)$.

- Conversely, if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $\gamma z = (az + b) / (cz + d)$, so

$$\mathrm{L}_{\gamma z,1} = (cz + d)^{-1} \mathrm{L}_{az+b,cz+d}.$$

  But certainly $\mathrm{L}_{az+b,cz+d} \subseteq \mathrm{L}_{z,1}$. On the other hand if γ' is inverse to γ,

$$\begin{pmatrix} z \\ 1 \end{pmatrix} = \gamma'\gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \gamma \begin{pmatrix} az + b \\ cz + d \end{pmatrix} = \begin{pmatrix} a'(az + b) + b'(cz + d) \\ c'(az + b) + d'(cz + d) \end{pmatrix},$$

  so $z \in \mathrm{L}_{az+b,cz+d}$ and $1 \in \mathrm{L}_{az+b,cz+d}$. So $\mathrm{L}_{az+b,cz+d} = \mathrm{L}_{z,1}$, so $\mathrm{L}_{\gamma z,1} = (cz + d)^{-1} \mathrm{L}_{z,1}$.

Definition 1.1.12. A **lattice function of weight** k is a function $F : \{\text{lattices in } \mathbb{C}\} \to \mathbb{C}$ such that

$$F(\lambda L) = \lambda^{-k} F(L),$$

for all lattices L. Given such an F, can define

$$\begin{array}{cccc} f & : & \mathbb{H} & \longrightarrow & \mathbb{C} \\ & & z & \longmapsto & F(\mathrm{L}_{z,1}) \end{array}.$$

If F has weight k, then

$$f(\gamma z) = F(\mathrm{L}_{\gamma z,1}) = F\left((cz + d)^{-1} \mathrm{L}_{z,1}\right) = (cz + d)^k F(\mathrm{L}_{z,1}) = (cz + d)^k f(z).$$

1.2 Eisenstein series

Lecture 4
Friday
11/10/19

Definition 1.2.1. For $L \in \mathbb{C}$, define the **Eisenstein series**

$$\mathrm{G}_k(L) = \sum_{w \in L,\ w \neq 0} \frac{1}{w^k}, \qquad \mathrm{g}_k(z) = \mathrm{G}_k(\mathrm{L}_{z,1}) = \sum_{\substack{m=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k}.$$

Then

$$\mathrm{G}_k(\lambda L) = \sum_{w' \in \lambda L,\ w' \neq 0} \frac{1}{w'^k} = \sum_{w \in L,\ w \neq 0} \frac{1}{(\lambda w)^k} = \lambda^{-k} \mathrm{G}_k(L).$$

Corollary 1.2.2. *g_k satisfies the modular transformation law of weight k.*

The following are some questions.

- Does G_k, or g_k, converge?
- Is g_k holomorphic or meromorphic on $\mathbb{H}$?
- Is g_k holomorphic at ∞?
- What is the q-expansion of g_k?

1.2.1 Convergence and holomorphy on $\mathbb{H}$

Definition 1.2.3. Let $U \subseteq \mathbb{C}$ be open. A sequence of functions $f_n : U \to \mathbb{C}$ **converges uniformly on compact sets** to f if for all $C \subseteq U$ compact and all $\epsilon > 0$, there exists $N \in \mathbb{Z}$ such that for all $n > N$,

$$|f(z) - f_n(z)| < \epsilon, \qquad z \in C.$$

Theorem 1.2.4. *A uniform limit of holomorphic functions is holomorphic. If f_n converges to f uniformly on compact sets and f_n is holomorphic on U, then f is holomorphic on U.*

Theorem 1.2.5. *Let $k \geq 4$. The series $\mathrm{g}_k(z)$ converges absolutely and uniformly on compact subsets of $\mathbb{H}$.*

Proof. Let

$$P_{z,r} = \{az + b \mid a, b \in \mathbb{R},\ \max(|a|, |b|) = r\} \subseteq \mathbb{C},$$

so $P_{z,r} = rP_{z,1}$, and there are $8r$ points on $P_{z,r} \cap \mathrm{L}_{z,1}$. Then

$$\mathrm{g}_k(z) = \sum_{r=1}^{\infty} \sum_{w \in \mathrm{L}_{z,1} \cap P_{z,r}} \frac{1}{w^k}.$$

The function $z \mapsto |z|$ attains a non-zero minimum $\delta(z)$ on $P_{z,1}$, so on $P_{z,1}$, have $|z| > \delta(z)$, so $1/|z|^k < 1/\delta(z)^k$. On $P_{z,r}$, have $|z| > r\delta(z)$, so $1/|z|^k < 1/r^k \delta(z)^k$. Let $C \subseteq \mathbb{H}$ be compact. Then $z \mapsto \delta(z)$ is a continuous function on C and attains a minimum δ_C. For all $z \in C$ and all $w \in P_{z,r}$, get $|w| > r\delta_C$, so

$$\frac{1}{|w|^k} < \frac{1}{r^k \delta_C^k}.$$

Thus for $z \in C$, $\mathrm{g}_k(z)$ is dominated by

$$\sum_{r=1}^{\infty} \frac{8r}{r^k \delta_C^k} = \frac{8}{\delta_C^k} \sum_{r=1}^{\infty} \frac{1}{r^{k-1}},$$

which converges absolutely for $k \geq 4$. ∎

Corollary 1.2.6. *$\mathrm{g}_k(z)$ is holomorphic on $\mathbb{H}$.*

1.2.2 q-expansion and holomorphy at ∞

The idea is to understand series of the form

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k}.$$

Theorem 1.2.7. *A bounded holomorphic function on all of $\mathbb{C}$ is constant.*

Lemma 1.2.8.

1.

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

2.

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Proof.

1. The right hand side converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$, so the right hand side is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$. Locally around $z = n$, the series looks like

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \dots + \frac{1}{(z-n+1)^2} + \frac{1}{(z-n)^2} + \frac{1}{(z-n-1)^2} + \dots = \frac{1}{(z-n)^2} + h_1(z),$$

where $h_1(z)$ is holomorphic in a neighbourhood of $z = n$. Similarly, the left hand side is meromorphic on $\mathbb{C}$, and the Laurent series near $z = n$ is

$$\frac{\pi^2}{\sin^2 \pi z} = \pi \left(\frac{1}{\pi^2 (z-n)^2} + \frac{1}{3} + \frac{1}{15} \pi^2 (z-n)^2 + \dots \right) = \frac{1}{(z-n)^2} + h_2(z),$$

where $h_2(z)$ is a holomorphic function. So the difference

$$g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} - \frac{\pi^2}{\sin^2 \pi z}$$

is meromorphic on $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \mathbb{Z}$, and the Laurent expression around $z = n$ is

$$g(z) = \frac{1}{(z-n)^2} + h_1(z) - \left(\frac{1}{(z-n)^2} + h_2(z) \right) = h_1(z) - h_2(z),$$

so $g(z)$ is holomorphic at $z = n$ for all n. Consider $t \to \pm\infty$ for $z = a + it$. The right hand side is

$$R = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=a-N}^{a+N} \frac{1}{(z-n)^2} + \sum_{n=-\infty}^{a-N-1} \frac{1}{(z-n)^2} + \sum_{n=a+N+1}^{\infty} \frac{1}{(z-n)^2} = R_0 + R_- + R_+,$$

where R_0 has finitely many terms that converge to less than $\epsilon/2$ as $t \to \pm\infty$ and $R_- + R_+ < \epsilon/2$ for $N \gg 0$ independent of t, so $R < \epsilon$ converges to zero. Similarly, the left hand side is

$$\left| \frac{\pi^2}{\sin^2 \pi z} \right| = \left| \frac{2\pi^2}{e^{\pi i z} - e^{-\pi i z}} \right| \to 0,$$

so $\lim_{t \to \infty} g(a + it) = 0$. Moreover, $g(z+1) = g(z)$ for all z. Then

$$S = \{ z \in \mathbb{C} \mid n - 1 \le \operatorname{Re} z \le n, \ -N \le \operatorname{Im} z \le N \}, \qquad n \in \mathbb{Z}$$

is compact, so $|g(z)|$ attains a maximum in S, so $g(z)$ is bounded in S. Since $g(z)$ is also bounded in $R_- + R_+$, $g(z)$ is bounded in $\mathbb{C}$, so g is constant. Since $\lim_{t \to \infty} g(a + it) = 0$, $g = 0$.

2. Check that the right hand side converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$, so the right hand side is meromorphic on $\mathbb{C} \setminus \mathbb{Z}$. Similarly, the left hand side is also meromorphic on $\mathbb{C} \setminus \mathbb{Z}$. Comparing derivatives, Lecture 5 Friday 11/10/19

$$-\frac{\pi^2}{\sin^2 \pi z} = -\frac{1}{z^2} - \sum_{n=1}^{\infty} \left(\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} \right),$$

so the difference is constant. Let $z = \frac{1}{2}$. The left hand side is $\pi \cot \pi/2 = 0$ and the right hand side is

$$\frac{2}{1} + \left(-\frac{2}{1} + \frac{2}{3} \right) + \left(-\frac{2}{3} + \frac{2}{5} \right) + \dots \to 0, \qquad n \to \infty,$$

so the difference is zero.

$\square$

Thus

$$\frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \pi \cot \pi z = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = \pi i \frac{q+1}{q-1} = \pi i - \frac{2\pi i}{1-q} = \pi i - 2\pi i \sum_{n=0}^{\infty} q^n.$$

Take $\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}$. For $k \ge 2$ even, get

$$-(k-1)! \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = -(2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} q^n,$$

so

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n.$$

Collecting powers of q,

$$\begin{aligned}
\mathrm{g}_k(z) &= \sum_{\substack{m=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\
&= 2 \sum_{n=1}^{\infty} \frac{1}{n^k} + 2 \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\
&= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{k-1} q^{nm} && \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \\
&= 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n && \sigma_{k-1}(n) = \sum_{d \mid n,\ d>0} d^{k-1}.
\end{aligned}$$

Corollary 1.2.9. *$\mathrm{g}_k(z)$ is holomorphic at ∞. In particular, g_k is a modular form of weight k.*

1.2.3 Bernoulli numbers

Definition 1.2.10. The **Bernoulli numbers** b_k are defined by

$$\sum_{k=0}^{\infty} b_k \frac{x^k}{k!} = \frac{x}{e^x - 1},$$

a formal power series with rational coefficients.

Then

$$b_0 = 1, \qquad b_1 = -\tfrac{1}{2}, \qquad b_2 = \tfrac{1}{6}, \qquad b_3 = 0, \qquad b_4 = -\tfrac{1}{20}, \qquad \dots, \qquad b_{2k} \in \mathbb{Q}, \qquad b_{2k+1} = 0, \qquad \dots.$$

Proposition 1.2.11. *For all even k,*

$$\zeta(k) = -b_k \frac{(2\pi i)^k}{2k!}.$$

Proof. On one hand,

$$\pi z \cot \pi z = \pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1} = \pi i z + \sum_{k=0}^{\infty} b_k \frac{(2\pi i z)^k}{k!}.$$

On the other hand,

$$\begin{aligned}
\pi \cot \pi z &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \frac{1}{z} - \frac{2z}{n^2} \sum_{n=1}^{\infty} \frac{1}{1 - z^2/n^2} \\
&= \frac{1}{z} - \sum_{n=1}^{\infty} \frac{2}{z} \sum_{k=1}^{\infty} \left(\frac{z^2}{n^2}\right)^k = \frac{1}{z} - \frac{2}{z} \sum_{k=1}^{\infty} z^{2k} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{1}{z} - \frac{2}{z} \sum_{k=1}^{\infty} \zeta(2k) z^{2k},
\end{aligned}$$

so

$$\pi i z + \sum_{k=0}^{\infty} b_k \frac{(2\pi i z)^k}{k!} = \pi z \cot \pi z = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k}.$$

Comparing,

$$b_{2k} \frac{(2\pi i)^{2k}}{(2k)!} = -2\zeta(2k),$$

get the desired formula. □

So

$$\mathrm{g}_k(z) = \frac{-b_k (2\pi i)^k}{k!} + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Set the **normalised Eisenstein series**

$$\mathrm{E}_k = \frac{\mathrm{g}_k}{2\zeta(k)} = 1 - \frac{2k}{b_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Example.

$$\mathrm{E}_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \qquad \mathrm{E}_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

$$\mathrm{E}_8 = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, \qquad \mathrm{E}_{12} = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n.$$