M4P57 Complex Manifolds

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Syllabus

Contents

1 Introduction

Lecture 1
Thursday
09/01/20

The following are references.

- O Biquard and A Höring, Kähler geometry and Hodge theory, 2008.
- J P Demailly, Complex analytic and differential geometry, 2012.
- C Voisin, Hodge theory and complex algebraic geometry, 2002.
- R O Wells, Differential analysis on complex manifolds, 1973.
- A Gathmann, Algebraic geometry, 2002
- P Griffiths and J Harris, Principles of algebraic geometry, 1978.

Complex manifolds are manifolds over $\mathbb{C}^n$.

Example 1.1. $\mathbb{C}^1$ is a complex manifold. Any open $U \subset \mathbb{C}^n$ is a complex manifold.

Example 1.2. The sphere $\mathrm{S}^2 \subset \mathbb{R}^3$ is a complex manifold by

$$\mathrm{S}^2 \cong \mathbb{C} \cup \{\infty\} = \mathbb{P}^1_{\mathbb{C}} = \mathbb{CP}^1.$$

More in general $\mathbb{P}^n_{\mathbb{C}}$ is a complex manifold for all n.

Example 1.3. The torus

$$\mathrm{S}^1 \times \mathrm{S}^1 = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{C}/\mathbb{Z}^2$$

is a complex manifold. More in general a $2n$-dimensional torus $\mathbb{C}^n/\Lambda$ for a lattice $\Lambda \cong \mathbb{Z}^{2n}$ is a complex manifold.

Example 1.4. Compact Riemannian surfaces of genus $g > 1$, called **hyperbolics**, are all complex manifolds.

Example 1.5. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. The graph of f,

$$\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{C}\} \subset \mathbb{C} \times \mathbb{C},$$

is a complex manifold. From Γ_f we can recover f, by

$$f(x) = q\left(p^{-1}(x) \cap \Gamma_f\right),$$

where $p, q : \mathbb{C}^2 \to \mathbb{C}$ are the projections to the first and second factors. This allows us to define f^{-1}. Assume f is bijective. Define

$$\begin{array}{cccc} \tau & : & \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ & & (x, y) & \longmapsto & (y, x) \end{array}.$$

Define

$$\Gamma_{f^{-1}} = \tau(\Gamma_f).$$

Then f^{-1} is the function induced by $\Gamma_{f^{-1}}$. This makes sense even if f is not bijective. Then we get a multivalued function, such as $\log z$ as the inverse of $\exp z$.

Example 1.6. Generalising Example 1.5, we can consider two complex manifolds M and N and we can consider holomorphisms $f : M \to N$. Given M,

$$\operatorname{Aut} M = \left\{f : M \to M \text{ holomorphic bijective and } f^{-1} \text{ holomorphic}\right\}.$$

If $M = \mathbb{C}$, there are lots of C^∞-functions $\mathbb{C} \to \mathbb{C}$ but the automorphisms of $\mathbb{C}$ are just affine linear maps. If $M = \mathbb{C}/\mathbb{Z}^2$, then $\operatorname{Aut} M$ is interesting.

Example 1.7. Algebraic geometry is the zeroes of polynomials. That is, fix m, and take polynomials $f_1, \dots, f_k$ in m variables. Define

$$M = \{(x_1, \dots, x_m) \in \mathbb{C}^m \mid f_1(x_1, \dots, x_m) = \dots = f_k(x_1, \dots, x_m) = 0\}.$$

Then M is called an **algebraic variety**. If M is smooth then M is a complex manifold. Fix m, take homogeneous polynomials $F_1, \dots, F_k$ in $m+1$ variables, where F is **homogeneous** if it is the sum of monomials of the same degree. Consider

$$N = \{(x_0, \dots, x_m) \in \mathbb{P}_{\mathbb{C}}^m \mid F_1(x_0, \dots, x_m) = \dots = F_k(x_0, \dots, x_m) = 0\}.$$

Then N is called a **projective variety**. If N is smooth then N is a complex manifold.

The idea is if M is a differentiable manifold, then M contains lots of submanifolds N. This is not true for complex manifolds. There exist complex manifolds without any proper complex submanifolds, which is not true for projective varieties. The following are questions.

- What can we say about the topology of complex manifolds? For example, what is $\pi_1(M)$? What is the cohomology of M?
- Assume that M and N are complex manifolds which are diffeomorphic. Are they also isomorphic, so there exists a biholomorphism $M \to N$?

What is next?

- Hodge decomposition theorem. Understand the cohomology of M by using the complex structure.
- Kodaira embedding theorem. Understand when a compact complex manifold is projective.

Note. If $M \subset \mathbb{P}_{\mathbb{C}}^m$ is a compact complex manifold then M is projective.

Example. Let $M = \Gamma_{\exp}$ for $\exp : \mathbb{C} \to \mathbb{C}$. Then $M \subset \mathbb{C}^2$ but it is not algebraic.

2 Local theory

2.1 Holomorphic functions in several variables

Lecture 2 Thursday 09/01/20

Notation 2.1. Given $z_0 \in \mathbb{C}$ and $r > 0$, the **disc** is

$$\mathrm{D}(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\},$$

and $\partial\mathrm{D}(z_0, r)$ is the boundary of $\mathrm{D}(z_0, r)$.

Definition 2.2. Let $U \subset \mathbb{C}$, and let $f : U \to \mathbb{C}$ be a function. Then f is **holomorphic at** $z_0 \in U$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Theorem 2.3 (Cauchy)**.** *Let $U \subset \mathbb{C}$ be open, let f be holomorphic on U, and let $z_0 \in U$. Assume that if $D = \mathrm{D}(z_0, r) \subset U$ then $\overline{D} \subset U$. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} \, \mathrm{d}z.$$

Notation 2.4. Fix $z_0 = (z_{01}, \dots, z_{0n}) \in \mathbb{C}^n$ and $R = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$. Then the **polydisc** is

$$\mathrm{D}(z_0, R) = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i - z_{0i}| < r_i \text{ for each } i\},$$

where R is the **polyradius**.

Definition 2.5. Let $U \subset \mathbb{C}^n$ be open, let $f : U \to \mathbb{C}$ be a continuous function, and let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Then f is **holomorphic** at z, if assuming that $D = \mathrm{D}(z, R) \subset U$ for some $R = (r_1, \dots, r_n)$ then

$$f(z_1, \dots, z_{i-1}, \cdot, z_{i+1}, \dots, z_n) : \mathrm{D}(z_i, r_i) \to \mathbb{C}$$

is holomorphic for all i.

Example 2.6. Any convergent power series in n-variables is holomorphic.

The opposite is also true.

Theorem 2.7 (Cauchy)**.** *Let $U \subset \mathbb{C}^n$ be an open set, let $f : U \to \mathbb{C}$ be holomorphic, and let $z = (z_1, \dots, z_n) \in U$. Assume that if $D = \mathrm{D}(z_0, R)$ for some $R = (r_1, \dots, r_n)$ then $\overline{D} \subset U$. If $z' = (z'_1, \dots, z'_n) \in D$ then*

$$f(z') = \frac{1}{(2\pi i)^n} \int_{\partial\mathrm{D}(z_1, r_1)} \dots \int_{\partial\mathrm{D}(z_n, r_n)} \frac{f(z)}{(z - z'_1) \dots (z - z'_n)} \, \mathrm{d}z_n \dots \mathrm{d}z_1.$$

Proof. Use induction on n and Cauchy theorem at each step. □

Corollary 2.8. *Let $U \subset \mathbb{C}^n$ be open, let $f : U \to \mathbb{C}$ be holomorphic, and let $z = (z_1, \dots, z_n) \in U$. Then there exists $D = \mathrm{D}(z, R) \subset U$ for some $R = (r_1, \dots, r_n)$ and there exists*

$$p(w) = \sum_{m_1, \dots, m_n \ge 0} a_{m_1, \dots, m_n} (w_1 - z_1)^{m_1} \dots (w_n - z_n)^{m_n},$$

such that p is convergent on D and $f(w) = p(w)$ inside D.

Proof. The idea is to use Theorem 2.7 and $1/(1 - w) = \sum_{k \ge 0} w^k$. □

Definition 2.9. Let $U \subset \mathbb{C}^n$ be open. Then $f : U \to \mathbb{C}^m$ is **holomorphic** if $f_i = \mathrm{p}_i \circ f$ is holomorphic for any $i = 1, \dots, m$ where $\mathrm{p}_i : \mathbb{C}^m \to \mathbb{C}$ is the i-th projection, so $f = (f_1, \dots, f_m)$.

Fact. If $f : U \to \mathbb{C}^m$ is holomorphic and $g : V \to \mathbb{C}^p$ is holomorphic where $V \supset f(U)$ then $g \circ f$ is holomorphic.

Definition 2.10. Let $U \subset \mathbb{C}^n$ be open. A holomorphic function $f : U \to \mathbb{C}^m$ is **biholomorphic at** $z_0 \in U$ if there exists an open neighbourhood $V \subset U$ of z_0 such that $f : V \to f(V)$ is bijective and $f^{-1} : f(V) \to V$ is holomorphic. Then f is **biholomorphic** if f is bijective and f is biholomorphic at any point.

Note. $f(V)$ is automatically open in $\mathbb{C}^m$ if $m = n$.

Example 2.11. Let $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ be linear such that $\det \Phi \neq 0$. Then Φ is a biholomorphism.

Example 2.12. Let $U = \mathbb{C} \setminus \{0\}$ and

$$\begin{array}{cccc} f & : & U & \longrightarrow & U \\ & & z & \longmapsto & z^2 \end{array}.$$

Check that f is biholomorphic at any point of U but f is not biholomorphic.

Remark. $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Then a holomorphic $f : U \subset \mathbb{C}^n \to \mathbb{C}^m$ is also a diffeomorphism $U \subset \mathbb{R}^{2n} \to \mathbb{R}^{2m}$.

Theorem 2.13 (Hartogs)**.** *Let $n \geq 2$, let $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ and $\delta = (\delta_1, \dots, \delta_n)$ such that $\epsilon_i > \delta_i > 0$, let $U = \mathrm{D}(0, \epsilon) \setminus \overline{\mathrm{D}(0, \delta)}$, and let $f : U \to \mathbb{C}^m$ be holomorphic. Then there exists a holomorphic $\overline{f} : \mathrm{D}(0, \epsilon) \to \mathbb{C}^m$ such that $\left.\overline{f}\right|_U = f$.*

Example. Hartogs theorem is false for $n = 1$. If $f(z) = 1/z$, for all $\epsilon > \delta > 0$, then f cannot be extended.

2.2 Cauchy formula in one variable

Lecture 3 Tuesday 14/01/20

Let $\omega = x + iy \in \mathbb{C}$ for $x, y \in \mathbb{R}$, and let $f : U \to \mathbb{C}$ be C^∞ for some $U \subset \mathbb{C}$. Recall that

$$\frac{\partial}{\partial \omega} f = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f, \qquad \frac{\partial}{\partial \overline{\omega}} f = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f.$$

Recall that f is holomorphic if and only if $\frac{\partial}{\partial \overline{\omega}} f = 0$ on U. More in general, let $U \subset \mathbb{C}^n$ be open, let $z_i = x_i + iy_i$, and let $f : U \to \mathbb{C}$ be a C^∞-function. Then f is holomorphic if and only if $\frac{\partial}{\partial \overline{z_i}} f = 0$ for all $i = 1, \dots, n$. Let $\omega \in \mathbb{C}$. Since $\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$, let

$$\mathrm{d}A = \frac{i}{2} \mathrm{d}\omega \wedge \mathrm{d}\overline{\omega} = \frac{i}{2} (\mathrm{d}x + i\mathrm{d}y) \wedge (\mathrm{d}x - i\mathrm{d}y) = \mathrm{d}x \wedge \mathrm{d}y,$$

which is the Lebesgue measure on $\mathbb{R}^2 \cong \mathbb{C}$.

Proposition 2.14. *Let $f : U \to \mathbb{C}$ for $U \subset \mathbb{C}$ be a C^∞-function, and let $D = \mathrm{D}(z, r)$ such that $\overline{D} \subset U$. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f}{\omega - z} \, \mathrm{d}\omega - \frac{1}{\pi} \int_D \frac{1}{\omega - z} \frac{\partial}{\partial \overline{\omega}} f \, \mathrm{d}A.$$

Proof. Assume $z = 0$. Recall that $f(\omega) = 1/\omega$ is locally integrable around zero, so

$$\int_D \frac{1}{\omega} \frac{\partial}{\partial \overline{\omega}} f \, \mathrm{d}A = \lim_{\epsilon \to 0} \int_{D \setminus \mathrm{D}(0, \epsilon)} \frac{1}{\omega} \frac{\partial}{\partial \overline{\omega}} f \, \mathrm{d}A.$$

Away from zero

$$\begin{aligned} \mathrm{d}\left(\frac{f}{\omega} \mathrm{d}\omega \right) &= \frac{1}{\omega} \mathrm{d}f \wedge \mathrm{d}\omega + f \mathrm{d}\left(\frac{1}{\omega} \right) \wedge \mathrm{d}\omega = \frac{1}{\omega} \left(\frac{\partial}{\partial \omega} f \mathrm{d}\omega + \frac{\partial}{\partial \overline{\omega}} f \mathrm{d}\overline{\omega} \right) \wedge \mathrm{d}\omega + f \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \right) \mathrm{d}\omega \wedge \mathrm{d}\omega \\ &= \frac{1}{\omega} \frac{\partial}{\partial \overline{\omega}} f \mathrm{d}\overline{\omega} \wedge \mathrm{d}\omega = \frac{2i}{\omega} \frac{\partial}{\partial \overline{\omega}} f \mathrm{d}A. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{\pi} \int_D \frac{1}{\omega} \frac{\partial}{\partial \overline{\omega}} f \, \mathrm{d}A &= \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{D \setminus \mathrm{D}(0, \epsilon)} \frac{1}{\omega} \frac{\partial}{\partial \overline{\omega}} f \, \mathrm{d}A \\ &= \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{D \setminus \mathrm{D}(0, \epsilon)} \mathrm{d}\left(\frac{f}{\omega} \mathrm{d}\omega \right) && \frac{1}{\omega} \frac{\partial}{\partial \overline{\omega}} f \mathrm{d}A = \frac{1}{2i} \mathrm{d}\left(\frac{f}{\omega} \mathrm{d}\omega \right) \\ &= \frac{1}{2\pi i} \lim_{\epsilon \to 0} \left(\int_{\partial D} \frac{f}{\omega} \, \mathrm{d}\omega - \int_{\partial \mathrm{D}(0, \epsilon)} \frac{f}{\omega} \, \mathrm{d}\omega \right) && \text{Stokes' theorem} \\ &= \frac{1}{2\pi i} \left(\int_{\partial D} \frac{f}{\omega} \, \mathrm{d}\omega - 2\pi i f(0) \right) && \lim_{\epsilon \to 0} \int_{\partial \mathrm{D}(0, \epsilon)} \frac{1}{\omega} \, \mathrm{d}\omega = 2\pi i. \end{aligned}$$

$\square$

If f is holomorphic, then $\frac{\partial}{\partial \overline{\omega}} f = 0$, which implies Theorem 2.3.

2.3 Rank theorem

Let $U \subset \mathbb{C}^n$ be open, and let $f : U \to \mathbb{C}^m$ be holomorphic. Then the **Jacobian** is

$$\mathrm{J}_f = \left(\frac{\partial}{\partial z_i} f_j(z) \right),$$

where $f_j = \mathrm{p}_j \circ f$ and $\mathrm{p}_j : \mathbb{C}^m \to \mathbb{C}$ is the j-th projection.

Exercise. Show that the real Jacobian, which is $2n \times 2n$, has non-negative determinants.

Theorem 2.15 (Rank theorem)**.** *Let $z \in U$ such that $r = \operatorname{rk} \mathrm{J}_f(z')$ is constant around z. Then there exist open $z \in V \subset U \subset \mathbb{C}^n$ and $f(z) \in W \subset f(U) \subset \mathbb{C}^m$ such that $\phi : \mathrm{D}(0,1)^n \to V$ and $\psi : \mathrm{D}(0,1)^m \to W$ are biholomorphisms such that*

$$\eta = \psi^{-1} \circ f \circ \phi \quad : \quad \begin{array}{ccc} \mathrm{D}(0,1)^n & \longrightarrow & \mathrm{D}(0,1)^m \\ (z_1, \dots, z_n) & \longmapsto & (z_1, \dots, z_r, 0, \dots, 0) \end{array},$$

so

$$\begin{tikzcd} \mathbb{C}^n \supset U & \supset & V \ni z \arrow[r, "f"] & f(z) \in W & \subset & f(U) \subset \mathbb{C}^m \\ & & \mathrm{D}(0,1)^n \arrow[u, "\phi"] \arrow[r, "\eta"'] & \mathrm{D}(0,1)^m \arrow[u, "\psi"'] & & \end{tikzcd}.$$

Corollary 2.16 (Inverse function theorem)**.** *Let $f : U \to \mathbb{C}^n$ be holomorphic for $U \subset \mathbb{C}^n$, and let $z \in U$ such that $\det \mathrm{J}_f(z) \neq 0$. Then f is a biholomorphism at z.*

Proof. $\det \mathrm{J}_f(z) \neq 0$ if and only if $\operatorname{rk} \mathrm{J}_f(z) = n$, so $\operatorname{rk} \mathrm{J}_f(z') = n$ around z, since $\det \mathrm{J}_f(z)$ is a continuous function. Let ϕ and ψ as in the theorem. Then $\eta = \psi^{-1} \circ f \circ \phi = \mathrm{id}$, so on V, $f = \psi \circ \phi^{-1}$ is a composition of biholomorphisms, which is a biholomorphism. $\square$

Remark 2.17. Let $f : U \to \mathbb{C}^n$ for $U \subset \mathbb{C}^n$. Then $\det \mathrm{J}_f(z)$ is a holomorphism, so

$$Z = \{ z \in U \mid \det \mathrm{J}_f(z) = 0 \}$$

is closed.

2.4 Holomorphic differential forms

Let $U \subset \mathbb{C}^n$ be open.

Definition 2.18. A **holomorphic vector field** on U is the expression

$$X = \sum_i a_i \frac{\partial}{\partial z_i},$$

where a_i are holomorphic functions on U.

For all $x \in U$, the **tangent space** is

$$\mathrm{T}_x U = \left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle \cong \mathbb{C}^n.$$

If $x \in U$, then $X(x) \in \mathrm{T}_x U$.

Notation 2.19.

$$\mathrm{H}^0(U, \mathcal{O}_U) = \{\text{holomorphic functions } f : U \to \mathbb{C}\}, \qquad \mathrm{H}^0(U, \mathrm{T}_U) = \{\text{holomorphic vector fields on } U\}.$$

Remark. $R = \mathrm{H}^0(U, \mathcal{O}_U)$ is a ring and $M = \mathrm{H}^0(U, \mathrm{T}_U)$ is a module over R. That is, if $X \in \mathrm{H}^0(U, \mathrm{T}_U)$ and $f \in \mathrm{H}^0(U, \mathcal{O}_U)$, then $fX \in \mathrm{H}^0(U, \mathrm{T}_U)$.

Definition 2.20. Let R be a ring and M be an R-module for $p \geq 1$. The **p-th exterior power** $\Lambda^p M$ of M is the R-module $M^{\otimes p}$ with the relations

$$m_1 \otimes \cdots \otimes m_p - \epsilon(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(p)}, \qquad m_1, \ldots, m_p \in M, \qquad \sigma \in \mathcal{S}_p,$$

where $\epsilon(\sigma) = (-1)^m$ is the signature of σ and m is the number of transpositions defining σ. Then $M^* = \mathrm{Hom}_R(M, R)$ is the **dual** of M as an R-module.

Let $R = \mathrm{H}^0(U, \mathcal{O}_U)$ and $M = \mathrm{H}^0(U, \mathrm{T}_U)$. Lecture 4 Thursday 16/01/20

Definition 2.21. Let $p > 0$. We define a **holomorphic p-form**, as an element of

$$\mathrm{H}^0(U, \Omega_U^p) = \Lambda^p M^*.$$

If $p = 0$, by convention a **holomorphic 0-form** is just an element in R.

Let $(z_1, \ldots, z_n)$ be coordinates for U. Recall $\eta \in M$ is given by $\eta = \sum_i a_i \frac{\partial}{\partial z_i}$ for holomorphic functions $a_i \in R$. Then $\omega \in M^*$ is given by the expression

$$\sum_i b_i \mathrm{d}z_i, \qquad b_i \in R, \qquad \mathrm{d}z_i \left(\frac{\partial}{\partial z_j} \right) = \delta_{ij}.$$

More in general $\omega \in \mathrm{H}^0(U, \Omega_U^p)$ is given by

$$\omega = \sum_{|I|=p} f_I \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p}, \qquad f_I \in R, \qquad I = (i_1, \ldots, i_p), \qquad i_1 < \cdots < i_p,$$

where $\mathrm{d}z_{i_1}, \ldots, \mathrm{d}z_{i_p}$ is an R-basis of $\mathrm{H}^0(U, \Omega_U^p)$.

Example.

$$\mathrm{H}^0(U, \Omega_U^p) \cong \Lambda^p \mathrm{H}^0\left(U, \Omega_U^1\right)$$

is an isomorphism as R-modules. This is not true for complex manifolds in general.

The **exterior product** is

$$\begin{array}{ccc} \mathrm{H}^0(U, \Omega_U^p) \otimes \mathrm{H}^0(U, \Omega_U^q) & \longrightarrow & \mathrm{H}^0\left(U, \Omega_U^{p+q}\right) \\ \omega_1 \otimes \omega_2 & \longmapsto & \omega_1 \wedge \omega_2 \end{array},$$

where we just need to define

$$\omega_1 \wedge \omega_2 = f \mathrm{d}z_{i_1} \wedge \mathrm{d}z_{i_p} \otimes g \mathrm{d}z_{j_1} \wedge \mathrm{d}z_{j_q} = fg \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}z_{j_1} \wedge \cdots \wedge \mathrm{d}z_{j_q},$$

by linearity. Then $\omega_1 \wedge \omega_2 = 0$ if $\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\} \neq \emptyset$, since $\mathrm{d}z_i \wedge \mathrm{d}z_i = 0$.

Exercise. Check that this definition coincides with the definition in M4P54.

The **exterior derivative** is

$$\begin{array}{cccc} \mathrm{d} : & \mathrm{H}^0(U, \Omega_U^p) & \longrightarrow & \mathrm{H}^0\left(U, \Omega_U^{p+1}\right) \\ & f \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} & \longmapsto & \sum_{j=1}^n \dfrac{\partial}{\partial z_j} f \mathrm{d}z_j \wedge \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \end{array}.$$

By definition d is $\mathbb{C}$-linear, but not R-linear. That is,

$$\mathrm{d}(a\omega_1 + b\omega_2) = a\mathrm{d}\omega_1 + b\mathrm{d}\omega_2, \qquad \omega_1, \omega_2 \in \mathrm{H}^0(U, \Omega_U^p), \qquad a, b \in \mathbb{C}.$$

Proposition 2.22. *Let $U \subset \mathbb{C}^n$ be open. Then*

- *the Leibniz rule*

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \mathrm{d}\omega_2, \qquad \omega_1 \in \mathrm{H}^0(U, \Omega_U^p), \qquad \omega_2 \in \mathrm{H}^0(U, \Omega_U^q),$$

- $\mathrm{d}^2 = 0$, *that is*

$$\mathrm{d}(\mathrm{d}\omega) = 0, \qquad \omega \in \mathrm{H}^0(U, \Omega_U^p).$$

Definition 2.23. Let $f : U \subset \mathbb{C}^n \to \mathbb{C}^m$ be holomorphic, let $f_i = \mathrm{p}_i \circ f : V \to \mathbb{C}$ where $\mathrm{p}_i : \mathbb{C}^m \to \mathbb{C}$ is the i-th projection, and let $f(U) \subset V \subset \mathbb{C}^m$ be open. Then if

$$\omega = h \mathrm{d}z_{i_1} \wedge \dots \wedge \mathrm{d}z_{i_p} \in \mathrm{H}^0\left(V, \Omega_V^p\right), \qquad h \in \mathrm{H}^0\left(U, \mathcal{O}_U\right),$$

then we can define the **pull-back** of ω,

$$f^* \omega = h \circ f \mathrm{d}f_{i_1} \wedge \dots \wedge \mathrm{d}f_{i_p} \in \mathrm{H}^0\left(U, \Omega_U^p\right),$$

since $f_i \in \mathrm{H}^0\left(V, \mathcal{O}_V\right) = \mathrm{H}^0\left(V, \Omega_V^0\right)$ implies that $\mathrm{d}f_i \in \mathrm{H}^0\left(V, \Omega_V^1\right)$, so

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "h \circ f \in \mathrm{H}^0(U, \mathcal{O}_U)"'] & f(U) \subset V \arrow[d, "h"] \\ & \mathbb{C} \end{tikzcd}.$$

This is linear over $\mathbb{C}$ and over $\mathrm{H}^0\left(U, \mathcal{O}_U\right)$.

Proposition 2.24. *Let $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$, and $W \subset \mathbb{C}^{m'}$ be open, let $f : U \to \mathbb{C}^m$ and $g : V \to \mathbb{C}^{m'}$ be holomorphic such that $V \supset f(U)$ and $W \supset g(V)$, and let $\omega \in \mathrm{H}^0\left(V, \Omega_V^p\right)$ and $\eta \in \mathrm{H}^0\left(V, \Omega_V^q\right)$. Then*

- *$f^*(\omega + \eta) = f^*\omega + f^*\eta$ if $p = q$,*
- *$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$,*
- *$\mathrm{d}f^*\omega = f^*\mathrm{d}\omega$, and*
- *$f^* g^* \omega = (g \circ f)^* \omega$.*

Let $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, and let $z_i = x_i + iy_i$ for $i = 1, \dots, n$ and $x_i, y_i \in \mathbb{R}$. Then

$$\mathrm{d}z_i = \mathrm{d}x_i + i\mathrm{d}y_i,$$

so any holomorphic form is a differentiable form on $\mathbb{R}^{2n}$. A (p, q)**-form** is a differentiable $(p+q)$-form of the expression

$$\omega = \sum_{|I|=p,\ |J|=q} f_{I,J} \mathrm{d}z_{i_1} \wedge \dots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z_{j_1}} \wedge \dots \wedge \mathrm{d}\overline{z_{j_q}}, \qquad f_{I,J} : U \to \mathbb{C} \cong \mathbb{R}^2 \in \mathrm{C}^\infty,$$

where $\mathrm{d}\overline{z_j} = \mathrm{d}x_j - i\mathrm{d}y_j$. We denote

$$\mathrm{C}^\infty\left(U, \Omega_U^{p,q}\right) = \left\{\text{differentiable } (p+q)\text{-forms on } U\right\}.$$

If ω is a (p, q)-form, then the **conjugate** $\overline{\omega}$ of ω is the (q, p)-form defined by

$$\overline{\omega} = \sum_{|I|=p,\ |J|=q} \overline{f_{I,J}} \mathrm{d}\overline{z_{i_1}} \wedge \dots \wedge \mathrm{d}\overline{z_{i_p}} \wedge \mathrm{d}z_{j_1} \wedge \dots \wedge \mathrm{d}z_{j_q}.$$

3 Complex manifolds

3.1 Complex manifolds

Definition 3.1. A **complex manifold** of dimension n is a connected Hausdorff topological space X, with a countable open cover $\{U_\alpha\}$ of X such that for all α, there exists $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism and

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is a biholomorphism for each α and β, so

$$\begin{tikzcd}
 & U_\alpha \cap U_\beta \arrow[dl, "\phi_\alpha"'] \arrow[dr, "\phi_\beta"] & \\
\mathbb{C}^n \supset \phi_\alpha(U_\alpha \cap U_\beta) \arrow[rr, "\phi_\alpha \circ \phi_\beta^{-1}"'] & & \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n
\end{tikzcd} .$$

The pair (U_α, ϕ_α) is called a **holomorphic chart**. The set $\{(U_\alpha, \phi_\alpha)\}$ is called a **holomorphic atlas** or a **complex structure**.

Recall X is Hausdorff if for all $x, y \in X$ there exist U and V open in X such that $U \cap V = \emptyset$ and $x \in U$ and $y \in V$.

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Example 3.2.

- If $U \subset \mathbb{C}^n$ is an open set then U is a complex manifold. More in general if X is a complex manifold and $U \subset X$ is open then U is a complex manifold. Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on X. Then

  $$\left\{\left(\overline{U_\alpha}, \overline{\phi_\alpha}\right)\right\} = \left\{\left(U_\alpha \cap U, \phi_\alpha|_{\overline{U_\alpha}}\right)\right\}$$

  is a complex structure of X.

- If X and Y are complex manifolds, then $X \times Y$ is a complex manifold.

Example 3.3. The projective space $\mathbb{P}^n_{\mathbb{C}}$ or $\mathbb{CP}^n$. Let $V^* = \mathbb{C}^{n+1} \setminus \{0\}$, with coordinates $(z_0, \dots, z_n)$. Define an equivalence on V^* as

$$v_1 \sim v_2 \qquad \iff \qquad \exists \lambda \in \mathbb{C}, \ v_1 = \lambda v_2.$$

Check that $\sim$ is an equivalence. Consider the Euclidean topology on V^*. Then there exists an induced topology on $X = V^* / \sim = \{[v] \mid v \in V^*\}$, with quotient map

$$\begin{array}{cccccc} q & : & V^* & \longrightarrow & X \\ & & v & \longmapsto & [v] \end{array} .$$

Given $v = (z_0, \dots, z_n) \in V^*$ we denote $[v] = [z_0, \dots, z_n]$ such that $z_i \neq 0$ for some i. Two elements $[x_0, \dots, x_n]$ and $[y_0, \dots, y_n]$ of X define the same point if and only if there exists λ such that $x_i = \lambda y_i$ for all i. Let

$$V_i = \{(z_0, \dots, z_n) \in V^* \mid z_i \neq 0\},$$

which is open in V^*, and let $U_i = q(V_i)$, which is open in X, such that $\{U_i\}$ is a cover of X, that is $\bigcup_i U_i = X$. Let

$$H_i = \{(z_0, \dots, z_n) \in V^* \mid z_i = 1\}.$$

Then there exists a homeomorphism

$$\begin{array}{ccccc} r_i & : & H_i & \longrightarrow & \mathbb{C}^n \\ & & (z_0, \dots, z_n) & \longmapsto & [z_0, \dots, z_{i-1}, z_{i+1}, \dots, z_n] \end{array} ,$$

and let

$$\begin{array}{ccccc} q_i = q|_{H_i} & : & H_i \subset V^* & \longrightarrow & U_i \subset X \\ & & (z_0, \dots, z_n) & \longmapsto & [z_0, \dots, z_n] \end{array}$$

be also a homeomorphism.

- q_i is surjective. Take $[x_0, \dots, x_n] \in U_i$. Then $x_i \neq 0$, so choose $\lambda = 1/x_i$. Then

$$[x_0, \dots, x_n] = \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] = q(z_0, \dots, z_n), \qquad z_j = \frac{x_j}{x_i},$$

  and in particular $z_i = 1$, so there exists $(z_0, \dots, z_n) \in H_i$ such that $q_i(z_0, \dots, z_n) = [x_0, \dots, x_n]$.

- q_i is injective. [1]

For all i, $q_i^{-1} : U_i \to H_i$ and $r_i : H_i \to \mathbb{C}^n$ are homeomorphisms, so $\phi_i = r_i \circ q_i^{-1} : U_i \to \mathbb{C}^n$ is also a homeomorphism. We want to show that (U_i, ϕ_i) define a holomorphic atlas, so

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is a biholomorphism. Consider the case $j = 0$ and $i = 1$. Then $\phi_0(U_0 \cap U_1) = \{(x_1, \dots, x_n) \mid x_1 \neq 0\}$, so

$$\begin{array}{cccc} \phi_1 \circ \phi_0^{-1} & : & \phi_0(U_0 \cap U_1) & \longrightarrow & \phi_1(U_0 \cap U_1) \\ & & (x_1, \dots, x_n) & \longmapsto & \left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right) \end{array}$$

is a biholomorphism. Thus X is a compact complex manifold. If $n = 1$, then $\mathbb{P}^1_{\mathbb{C}} \cong \mathrm{S}^2$.

Example 3.4. The complex torus. Let

$$\begin{array}{ccc} \Lambda = \mathbb{Z}^{2n} & \longrightarrow & \mathbb{C}^n \\ (a_1, \dots, a_n, b_1, \dots, b_n) & \longmapsto & (a_1 + ib_1, \dots, a_n + ib_n) \end{array}.$$

Define an equivalence on $\mathbb{C}^n$ by

$$v_1 \sim v_2 \qquad \iff \qquad v_1 - v_2 \in \Lambda.$$

Then $X = \mathbb{C}^n / \sim$ with quotient map $q : \mathbb{C}^n \to X$ is Hausdorff and compact. Topologically $X \cong [0,1]^{2n} / \sim$. For each $x \in \mathbb{C}^n$, we can find an open set $x \in U \subset \mathbb{C}^n$ such that $q|_U : U \to X$ is a homeomorphism. The idea is if $x \in (0,1)^{2n}$ then we can take $U = (0,1)^{2n}$. If not, there exists a translation of $\mathbb{C}^n \to \mathbb{C}^n$ such that the property holds. We define

$$\phi_V = q|_U^{-1} : V \subset \mathbb{C}^n/\Lambda \to U \subset \mathbb{C}^n, \qquad V = q(U).$$

Show that (V, ϕ_V) define a complex structure on X. [2] This is also a compact complex manifold. More in general $\mathbb{C}^n/\Lambda$ where $\Lambda \cong \mathbb{Z}^{2n}$ is a lattice is a compact complex manifold.

3.2 Holomorphic functions on complex manifolds

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Definition 3.5. Let $f : X \to Y$ be a continuous morphism between complex manifolds. Then f is **holomorphic** if there exists a complex structure $\{(U_\alpha, \phi_\alpha)\}$ on Y and for all $y \in Y$ there exists a holomorphic chart (V_α, ψ_α) such that $x \in V_\alpha$ and $f(V_\alpha) \subset U_\alpha$ around any point x of $f^{-1}(y)$ and $\phi_\alpha \circ f \circ \psi_\alpha^{-1}$ is holomorphic, so

$$\begin{tikzcd} X \supset V_\alpha \arrow[r, "f"] \arrow[d, "\psi_\alpha"'] & U_\alpha \subset Y \arrow[d, "\phi_\alpha"] \\ \psi_\alpha(V_\alpha) \arrow[r, "\widetilde{f}"'] & \phi_\alpha(U_\alpha) \end{tikzcd}.$$

Then $\mathrm{J}_f = \mathrm{J}_{\widetilde{f}}$, and a **holomorphic function on** X is a holomorphic function $f : X \to \mathbb{C}$.

Exercise 3.6. If X is a compact complex manifold then any holomorphic function $f : X \to \mathbb{C}$ is constant.

[1] Exercise
[2] Exercise

Definition 3.7. Let $f : X \to Y$ be a holomorphic function between complex manifolds. Then f is

- a **submersion** if $\dim Y \geq \dim Y = r$ and $\operatorname{rk} \mathrm{J}_f = r$ at any point,
- an **immersion** if $r = \dim X \leq \dim Y$ and $\operatorname{rk} \mathrm{J}_f = r$ at any point, and
- an **embedding** if it is an immersion and $f : X \to f(X)$ is a homeomorphism.

Example 3.8. Let $f_2, \dots, f_n : \mathbb{C} \to \mathbb{C}$ be holomorphic, and let

$$\begin{array}{cccccc} f & : & \mathbb{C} & \longrightarrow & \mathbb{C}^n \\ & & z & \longmapsto & (z, f_2(z), \dots, f_n(z)) \end{array}.$$

Then f is an embedding.

Example 3.9. Let $X = \mathbb{C}^2/\Lambda$ for $\Lambda = \mathbb{Z}^4 \subset \mathbb{C}^2$, and let $q : \mathbb{C}^2 \to X$. Fix $\lambda \in \mathbb{C}$. Let

$$\begin{array}{cccccc} f & : & \mathbb{C} & \longrightarrow & \mathbb{C}^2 \\ & & z & \longmapsto & (z, \lambda z) \end{array}.$$

Then $\widetilde{f} = q \circ f : \mathbb{C} \to X$ is an immersion.

- If $\lambda = 0$ or $\lambda = \frac{1}{2}$, then $\widetilde{f}(\mathbb{C})$ is a closed submanifold.
- If λ is general then $\widetilde{f}(\mathbb{C})$ is dense inside X, so it is not closed. Thus it is not a complex submanifold of X.

3.3 Complex submanifolds

Definition 3.10. Let $i : X \to Y$ be an embedding of complex manifolds. If $i(X) \subset Y$ is closed then $i(X)$ is called a **complex submanifold** of Y. The **codimension** of X in Y is $\dim Y - \dim X$.

Theorem 3.11.

1. *Let $i : X \to Y$ be a submanifold of codimension k, and let $n = \dim X$. Then for all $x \in X$, there exists an open neighbourhood $x \in U \subset Y$ and a submersion $f : U \to \mathrm{D}(0,1)^k \subset \mathbb{C}^k$ such that $X \cap U = f^{-1}(0)$.*
2. *If $X \subset Y$ is a closed subset such that for all $x \in X$ there exists $U \ni x$ open in Y and a submersion $f : U \to \mathrm{D}(0,1)^k$ such that $X \cap U = f^{-1}(0)$, then X is a complex submanifold.*

Proof.

1. We can assume that if there exists a holomorphic chart (U, ψ) on Y such that $x \in U$ and if $V = i^{-1}(U)$ then there exists $\phi : V \to \mathbb{C}^n$ such that (V, ϕ) is a holomorphic chart on X. After possibly shrinking U smaller, by the rank theorem, there exist biholomorphic $a : \psi(U) \to \mathrm{D}(0,1)^{n+k}$ and $b : \phi(U) \to \mathrm{D}(0,1)^n$ such that the induced morphism is given by

   $$\begin{array}{ccc} \mathrm{D}(0,1)^n & \longrightarrow & \mathrm{D}(0,1)^{n+k} \\ (z_1, \dots, z_n) & \longmapsto & (z_1, \dots, z_n, 0, \dots, 0) \end{array}.$$

   Let

   $$\begin{array}{cccccc} c & : & \mathrm{D}(0,1)^{n+k} & \longrightarrow & \mathrm{D}(0,1)^k \\ & & (z_1, \dots, z_{n+k}) & \longmapsto & (z_{n+1}, \dots, z_{n+k}) \end{array},$$

   so

   $$\begin{tikzcd} Y & \supset & U \arrow[r, "\phi"] & \phi(U) \arrow[r, "b"] & \mathrm{D}(0,1)^n \subset \mathbb{C}^n \arrow[d] \\ X \arrow[u, hook, "i"] & \supset & V \arrow[u, hook, "i"] \arrow[r, "\psi"'] & \psi(U) \arrow[r, "a"'] & \mathrm{D}(0,1)^{n+k} \subset \mathbb{C}^{n+k} \arrow[u, dashed, bend right=60, "c"'] \end{tikzcd}.$$

   Then f is the composition $c \circ a \circ \psi : U \to \mathrm{D}(0,1)^n$.

2. Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on Y, and let $V_\alpha = X \cap U_\alpha$ and $\psi_\alpha = \phi_\alpha|_{V_\alpha}$. The goal is to show that $\{(V_\alpha, \psi_\alpha)\}$ defines a complex structure on X. By assumption,

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta (U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k} \to \phi_\alpha (U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k}$$

is biholomorphic. Let $U' = \phi_\beta (U)$, let $X' = \phi_\beta (X \cap U)$, and let $f' = f \circ \phi_\beta^{-1}$, so

$$\begin{tikzcd}
 & & & & & \phi_\alpha (U) & \subset & \phi_\alpha (U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k} \\
Y & \supset & U_\alpha \cap U_\beta & \supset & U \arrow[ur, "\phi_\alpha"] \arrow[r, "\phi_\beta"] \arrow[drrr, "f"] & U' \arrow[drr, "f'"] & \subset & \phi_\beta (U_\alpha \cap U_\beta) \subset \mathbb{C}^{n+k} \arrow[u, "\phi_\alpha \circ \phi_\beta^{-1}"'] \\
 & & \cup & & \cup & \cup & & \\
X \arrow[uu, hook, "i"] & \supset & X \cap U_\alpha \cap U_\beta & \supset & X \cap U & X' & & \mathrm{D}(0,1)^k \subset \mathbb{C}^k
\end{tikzcd}.$$

Then $f'^{-1} (0) = \phi_\beta (X \cap U_\alpha \cap U_\beta)$ and f' is also a submersion. By the rank theorem, we may assume that $U' = \mathrm{D}(0,1)^{n+k}$ and $f' (z_1, \dots, z_{n+k}) = (z_1, \dots, z_k)$, so $\phi_\beta (X' \cap U_\alpha \cap U_\beta) = f'^{-1} (0)$. Thus

$$\left(\psi_\alpha \circ \psi_\beta^{-1}\right) (z_1, \dots, z_n) = \left(\phi_\alpha \circ \phi_\beta^{-1}\right) (z_1, \dots, z_n, 0, \dots, 0)$$

is also a biholomorphism.

□

3.4 Examples of complex manifolds

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Example 3.12. Let $U \subset \mathbb{C}^n$ be open, let $k \leq n$, let $f_1, \dots, f_k : U \to \mathbb{C}$, and let

$$V = \{x \in \mathbb{C}^n \mid f_1 (x) = \dots = f_k (x) = 0\}.$$

Assume that $\left(\frac{\partial}{\partial z_j} f_i\right)$ has maximal rank k at any point of U. Then V is a complex submanifold of U. The idea is if $f = (f_1, \dots, f_k) : U \to \mathbb{C}^k$, then f is a submersion around any point of V, and use the previous Theorem 3.11.

Example 3.13. Let $f : X \to Y$ be a holomorphism between complex manifolds, and let $W \subset X$ be a submanifold. Then $f|_W : W \to Y$ is holomorphic.

Exercise 3.14. Let $X = \mathbb{C}^n$. Show that all the compact submanifolds of X are zero-dimensional, that is points.

Exercise 3.15. Let X and Y be compact manifolds. Recall that $X \times Y$ is also a complex manifold. Assume $f : X \to Y$, so

$$\Gamma_f = \{(x, f (x)) \mid x \in X\} \subset X \times Y.$$

Show that Γ_f is a complex submanifold.

Example 3.16. Let $n, m > 0$, and let

$$\mathrm{Mat}_{n,m} \mathbb{C} = \{(n \times m)\text{-matrices}\} \cong \mathbb{C}^{n \cdot m}.$$

Then $\mathrm{Mat}_{n,m} \mathbb{C}$ is a complex manifold. Let

$$\mathrm{GL}_n \mathbb{C} = \{(n \times n)\text{-matrices } A \mid A \text{ invertible}\}.$$

Then $\mathrm{GL}_n \mathbb{C}$ is a complex manifold, open in $\mathrm{Mat}_{n,n} \mathbb{C}$.

Example 3.17. Projective manifolds. Let $R = \mathbb{C}[x_0, \dots, x_n]$ be the ring of polynomials, and let $X = \mathbb{P}^n_{\mathbb{C}}$ be the complex projective space. Then $f \in R$ is homogeneous of degree d if $f(\lambda x) = \lambda^d f(x)$. Let $q : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n_{\mathbb{C}}$, let $F_1, \dots, F_k$ be homogeneous polynomials in R, and let

$$V = \{F_1 = \dots = F_k = 0\} \subset \mathbb{C}^{n+1} \setminus \{0\}, \qquad W = q(V) \subset \mathbb{P}^n_{\mathbb{C}},$$

so $q^{-1}(W) = V$, because F_i are homogeneous. Since V is closed in $\mathbb{C}^{n+1} \setminus \{0\}$, W is closed in $\mathbb{P}^n_{\mathbb{C}}$. Claim that if V is a submanifold of $\mathbb{C}^{n+1} \setminus \{0\}$ then W is a compact submanifold of $\mathbb{P}^n_{\mathbb{C}}$. If $\{U_i\}$ is the open covering given by

$$U_i = \{[x_0, \dots, x_n] \mid x_i \neq 0\},$$

then it is enough to show that $W \cap U_i$ is a complex submanifold of U_i for all i. Assume $i = n$. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then $q(x) = \mathbb{C}^*$ for all $x \in X$ but $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^* \neq \mathbb{C}^{n+1} \setminus \{0\}$. We want to show there exists a biholomorphism

$$\begin{array}{cccc} \phi_n & : & U_n \times \mathbb{C}^* & \longrightarrow & q^{-1}(U_n) = \left\{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} \mid x_n \neq 0\right\} \\ & & ([x_0, \dots, x_n], t) & \longmapsto & \left(\dfrac{tx_0}{x_n}, \dots, \dfrac{tx_{n-1}}{x_n}, t\right) \end{array},$$

such that

$$\begin{array}{cccc} \phi_n^{-1} & : & q^{-1}(U_n) & \longrightarrow & U_n \times \mathbb{C}^* \\ & & (y_0, \dots, y_n) & \longmapsto & (q(y_0, \dots, y_n), y_n) = ([y_0, \dots, y_n], y_n) \end{array}.$$

From this, it follows that $V \cap q^{-1}(U_n) \cong (W \cap U_n) \times \mathbb{C}^*$, so the claim follows.

Example 3.18. Plane curves. Let $X = \mathbb{P}^2_{\mathbb{C}}$, let $F \in R[x_0, x_1, x_2]$ be homogeneous of degree d, and let $W = \{F = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$. Then W is a compact complex submanifold if and only if for all $x \in \mathbb{P}^2_{\mathbb{C}}$, $\partial_{x_i} F(x) \neq 0$ for some i.

$d = 1$. W is the projective line, so $F = ax_0 + bx_1 + cx_2$ for a, b, c not all zero. Then W is a complex submanifold. There exists a biholomorphism $\mathbb{P}^1_{\mathbb{C}} \to W$.

$d = 2$. W is a conic, so F is a degree two polynomial. Then $F = x_0 x_1$ does not define a manifold. If $F = x_0 x_1 - x_2^2$, then W is a complex submanifold of X. There exists

$$\begin{array}{ccc} \mathbb{P}^1_{\mathbb{C}} & \longrightarrow & W \subset X \\ [t_0, t_1] & \longmapsto & \left[t_0^2, t_1^2, t_0 t_1\right] \end{array}.$$

Check that it is a biholomorphism. [3] This is true for any f of degree two such that W is a complex submanifold.

$d \geq 3$. If W is a complex submanifold then we will show that W is not biholomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

3.5 Tangent spaces of complex manifolds

Definition 3.19. Let X be a complex manifold of dimension n, and let $x \in X$. Then there exists a chart (U, ϕ) around x such that $\phi(U) \subset \mathbb{C}^n$. The **holomorphic tangent space** $\mathrm{T}_x X$ of X at x, is the vector space over $\mathbb{C}$ generated by

$$\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right).$$

Let X be a real manifold. The **real tangent space** $\mathrm{T}^{\mathbb{R}}_x X$ is the vector space over $\mathbb{R}$ defined by

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right),$$

where $(x_1, \dots, x_n, y_1, \dots, y_n)$ are coordinates of $\mathbb{R}^{2n}$. The **complex tangent space** $\mathrm{T}^{\mathbb{C}}_x X$ is the vector space over $\mathbb{C}$ generated by

$$\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z_1}}, \dots, \frac{\partial}{\partial \overline{z_n}}\right),$$

a $2n$-dimensional vector space over $\mathbb{C}$. Then $\mathrm{T}^{\mathbb{C}}_x X = \mathrm{T}^{\mathbb{R}}_x X \otimes_{\mathbb{R}} \mathbb{C}$.

[3] Exercise

3.6 Holomorphic differential forms on complex manifolds

Definition 3.20. Let X be a complex manifold of dimension n. Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on X. A **holomorphic p-form** on X is the data ω_α, the p-forms on $\phi_\alpha(U_\alpha) \subset \mathbb{C}^n$ such that if

$$h_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta),$$

then $h_{\alpha\beta}^* \omega_\beta = \omega_\alpha$ for all α and β.

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Notation 3.21.

$$\Omega_x^p(X) = \mathrm{H}^0(X, \Omega_x^p) = \{\text{holomorphic } p\text{-forms on } X\},$$
$$\mathcal{O}_x(X) = \mathrm{H}^0(X, \mathcal{O}_x) = \{\text{holomorphic functions on } X\}.$$

$R = \mathcal{O}_x(X)$ is a ring and $M = \Omega_x^p(X)$ is an R-module.

Lemma 3.22. *Let $f : X \to Y$ be holomorphic. Then $f^* : \Omega^p(Y) \to \Omega^p(X)$.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be a complex structure on Y. We can write $f^{-1}(U_\alpha) = \bigcup_{\alpha,\beta} V_{\alpha,\beta}$ where $\{(V_{\alpha,\beta}, \psi_{\alpha,\beta})\}$ is a complex structure on X, so

$$\mathbb{C}^n \xleftarrow{\psi_{\alpha,\beta}} V_{\alpha,\beta} \xrightarrow{f|_{V_{\alpha,\beta}}} U_\alpha \xrightarrow{\phi_\alpha} \mathbb{C}^n.$$

Assume ω is defined by ω_α on $\phi_\alpha(U_\alpha)$. Let

$$\omega_{\alpha,\beta} = \left(\left(\psi_{\alpha,\beta}^{-1}\right)^* \circ f^* \circ \phi_\alpha^*\right)(\omega_\alpha)$$

be a p-form on $\psi_{\alpha,\beta}(V_{\alpha,\beta})$. Check that $\omega_{\alpha,\beta}$ are compatible with respect to the atlas on X. [4] $\square$

As in the local case, we can define

$$\begin{array}{ccc} \Omega_x^p(X) \otimes \Omega_x^q(X) & \longrightarrow & \Omega_x^{p+q}(X) \\ \omega_1 \otimes \omega_2 & \longmapsto & \omega_1 \wedge \omega_2 \end{array}.$$

Similarly there exists $\mathrm{d} : \Omega_x^p(X) \to \Omega_x^{p+1}(X)$.

[4] Exercise

4 Vector bundles

4.1 Holomorphic vector bundles

Definition 4.1. Let X be a complex manifold. A **holomorphic vector bundle** E of rank r on X is a complex manifold E, a holomorphism $\pi : E \to X$, and an open covering U_α of X such that there exists a biholomorphism

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r,$$

such that if $\mathrm{p}_\alpha : U_\alpha \times \mathbb{C}^r \to U_\alpha$ is the projection then $\pi|_{\pi^{-1}(U_\alpha)} = \mathrm{p}_\alpha \circ \psi_\alpha$, so

$$\begin{tikzcd} E \arrow[d, "\pi"'] & \supset & \pi^{-1}(U_\alpha) \arrow[r, "\psi_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times \mathbb{C}^r \arrow[dl, "\mathrm{p}_\alpha"] \\ X & \supset & U_\alpha & \end{tikzcd}.$$

A vector bundle of rank one is called a **line bundle**.

For any $x \in X$, there exists α such that $x \in U_\alpha$, so

$$\begin{tikzcd} \pi^{-1}(x) \arrow[r, "\psi_\alpha"] \arrow[d, "\pi"'] & \{x\} \times \mathbb{C}^r \arrow[dl, "\mathrm{p}_\alpha"] \\ x & \end{tikzcd}.$$

Then $E(x) = \pi^{-1}(x)$ is a vector space of rank r over $\mathbb{C}$. Let $U_\alpha \ni x \in U_\beta$. There exists a biholomorphism

$$\mathbb{C}^r \cong \mathrm{p}_\alpha^{-1}(x) \to \mathrm{p}_\beta^{-1}(x) \cong \mathbb{C}^r,$$

because they are both biholomorphic to $\pi^{-1}(x)$, so $g_{\alpha\beta}(x) \in \mathrm{GL}_r\,\mathbb{C}$ because all the biholomorphisms from $\mathbb{C}^r \to \mathbb{C}^r$ are linear. The holomorphisms

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_r\,\mathbb{C}$$

are called **transition functions**. Then

$$\begin{tikzcd} \mathrm{p}_\alpha^{-1}(x) \arrow[rr, "\mathrm{id}"] \arrow[dr] & & \mathrm{p}_\alpha^{-1}(x) \\ & \mathrm{p}_\beta^{-1}(x) \arrow[ur] & \end{tikzcd},$$

so

$$(g_{\alpha\beta} \circ g_{\beta\alpha})(x) = x, \qquad x \in U_\alpha \cap U_\beta,$$

and

$$\begin{tikzcd} \mathrm{p}_\alpha^{-1}(x) \arrow[rr, "g_{\alpha\gamma}"] \arrow[dr] & & \mathrm{p}_\gamma^{-1}(x) \\ & \mathrm{p}_\beta^{-1}(x) \arrow[ur] & \end{tikzcd},$$

so

$$(g_{\alpha\beta} \circ g_{\beta\gamma})(x) = g_{\alpha\gamma}(x), \qquad x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Definition 4.2. Let X be a complex manifold, and let E and F be vector bundles on X of rank r and s respectively, with $\pi : E \to X$ and $\pi' : F \to X$. A **holomorphic map** $f : E \to F$ is a holomorphic function $E \to F$ such that $\pi = \pi' \circ f$ and such that the rank of the induced linear map $E(x) \to F(x)$ is independent of $x \in X$, so

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & F \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}, \qquad \begin{tikzcd} E(x) = \pi^{-1}(x) \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \pi'^{-1}(x) = F(x) \arrow[dl, "\pi'"] \\ & x & \end{tikzcd}.$$

4.2 Examples of holomorphic vector bundles

Example 4.3. $\pi : E = X \times \mathbb{C}^r \to X$ is a vector bundle of rank r, called **trivial**.

Example 4.4. Algebra of vector bundles. Let $\pi : E \to X$ and $\pi'^{-1} : F \to X$ be vector bundles on X of rank r and s respectively.

- The **direct sum** $E \oplus F$ is the $(r+s)$-vector bundle such that
$$(E \oplus F)(x) = E(x) \oplus F(x), \qquad x \in X.$$
The idea is to take an open cover which trivialises both E and F. Find the transition function of $E \oplus F$. [5]
- The **tensor product** $E \otimes F$ is the $(r \cdot s)$-vector bundle such that
$$(E \otimes F)(x) = E(x) \otimes F(x), \qquad x \in X.$$
- The p**-th exterior power** of E is the vector bundle $\Lambda^p E$ such that
$$(\Lambda^p E)(x) = \Lambda^p(E(x)), \qquad x \in X.$$
If $p = r = \operatorname{rk} E$ then $\det E = \Lambda^r E$ is a line bundle on X.
- The **dual** of E is the rank r vector bundle E^* such that
$$E^*(x) = (E(x))^*, \qquad x \in X,$$
the dual $\operatorname{Hom}(E(x), \mathbb{C})$ of $E(x)$.
- Let $f : E \to F$ be a holomorphic map. Then the **kernel** $\operatorname{Ker} f$ is a vector bundle such that
$$(\operatorname{Ker} f)(x) = \operatorname{Ker} f(x) \subset E(x), \qquad x \in X.$$
The **cokernel** $\operatorname{Coker} f$ is a vector bundle such that
$$(\operatorname{Coker} f)(x) = \operatorname{Coker} f(x) \subset F(x), \qquad x \in X.$$

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Example 4.5. Let $X = \mathbb{P}^1_{\mathbb{C}}$, and let
$$\mathcal{O}(-1) = \{(x, v) \mid x = [x_0, \dots, x_n] \in \mathbb{P}^n_{\mathbb{C}},\ v = \mu(x_0, \dots, x_n),\ \mu \in \mathbb{C}\} \subset \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}.$$
Then $\pi = \mathrm{p}_1 : \mathcal{O}(-1) \to \mathbb{P}^n_{\mathbb{C}}$, so
$$\pi^{-1}([x_0, \dots, x_n]) = \{v = \mu(x_0, \dots, x_n) \mid \mu \in \mathbb{C}\} \cong \mathbb{C}^1.$$
Let $\{U_i\}$ be an open covering of X given by $U_i = \{[x_0, \dots, x_n] \mid x_i \neq 0\}$. We define
$$\begin{array}{cccc} \psi_i & : & \pi^{-1}(U_i) & \longrightarrow & U_i \times \mathbb{C} \\ & & ([x_0, \dots, x_n], (v_0, \dots, v_n)) & \longmapsto & ([x_0, \dots, x_n], v_i) \end{array},$$
which is a biholomorphism. Thus $\mathcal{O}(-1)$ is a complex manifold and $\mathcal{O}(-1)$ is a line bundle. The **tautological line bundle** $\mathcal{O}(1)$ is the dual of $\mathcal{O}(-1)$. Let
$$\mathcal{O}(k) = \begin{cases} X \times \mathbb{C} & k = 0 \\ \mathcal{O}(1)^{\otimes k} & k > 0 \\ \mathcal{O}(-1)^{\otimes k} & k < 0 \end{cases}.$$
Then $\mathcal{O}(k) = \mathcal{O}(-k)^*$. [6] On $\mathbb{P}^n_{\mathbb{C}}$ these are the only line bundles. That is, if $\mathcal{L}$ is a line bundle on $\mathbb{P}^1_{\mathbb{C}}$, there exists $k \in \mathbb{Z}$ such that $\mathcal{L} \cong \mathcal{O}(k)$. Let $X = \mathbb{P}^1_{\mathbb{C}}$, and let E be a line bundle of rank r on X. Then
$$E \cong \bigoplus_{i=1}^r \mathcal{O}(a_i), \qquad a_1, \dots, a_r \in \mathbb{Z}.$$
This is false for $X = \mathbb{P}^n_{\mathbb{C}}$, with $n \geq 2$.

[5] Exercise
[6] Exercise

Definition 4.6. Let $f : Y \to X$ be a holomorphism between complex manifolds, and let E be a vector bundle of rank r on X. Then there exists a vector bundle f^*E of rank r on Y defined by

$$f^*E = \{(y, v) \in Y \times E \mid f(y) = \pi(v)\},$$

the **fibre product** of E and Y over X, such that

$$\begin{tikzcd} f^*E \arrow[r, "f'"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}.$$

Let $U = \{U_i\}$ be an open cover of X which trivialises E, so

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\psi_i"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{C}^r \arrow[dl, "\mathrm{p}_1"] \\ & U_i & \end{tikzcd}.$$

Then $U' = \left\{f^{-1}(U_i)\right\}$ is an open covering of Y, so

$$\begin{tikzcd} \pi'^{-1}\left(f^{-1}(U_i)\right) \arrow[r, "f'"] \arrow[d, "\pi'"'] & \pi^{-1}(U_i) \arrow[r, "\psi_i"] \arrow[d, "\pi"] & U_i \times \mathbb{C}^r \arrow[r, "\mathrm{p}_2"] & \mathbb{C}^r \\ f^{-1}(U_i) \arrow[r, "f"'] & U_i & & \end{tikzcd},$$

and

$$\begin{array}{rcl} \pi'^{-1}\left(f^{-1}(U_i)\right) = \left\{(y, v) \in f^{-1}(U_i) \times \pi^{-1}(U_i) \;\middle|\; f(y) = \pi(v)\right\} & \longrightarrow & f^{-1}(U_i) \times \mathbb{C}^r \\ (y, v) & \longmapsto & (y, \mathrm{p}_2(\psi_i(v))) \end{array}$$

is a biholomorphism. Thus f^*E is a vector bundle, where

$$f^*E(y) = \pi'^{-1}(y) = E(f(y)), \qquad y \in Y.$$

Notation 4.7. Let $f : Y \to X$ be a morphism, and let E be a vector bundle on X. Then $f^*E = E|_Y$, mostly used if $f : Y \hookrightarrow X$.

Definition 4.8. Let E be a holomorphic vector bundle on a complex manifold X, and let $\pi : E \to X$. A **section** of E is a holomorphic function $s : X \to E$ such that $\pi \circ s = \mathrm{id}_X$.

Example 4.9. Let $E = X \times \mathbb{C}^r$ be the trivial vector bundle of rank r. Fix $v \in \mathbb{C}^r$. Then

$$\begin{array}{cccccc} s_v & : & X & \longrightarrow & E \\ & & x & \longmapsto & (x, v) \end{array}$$

is a section of E. If $v_1, \ldots, v_r$ is a basis of $\mathbb{C}^r$ then $s_{v_1}, \ldots, s_{v_r}$ have the property that $s_{v_1}(x), \ldots, s_{v_r}(x)$ forms a basis of $E(x)$. Vice versa, assume E is a vector bundle on X of rank r such that there exist sections $s_1, \ldots, s_r$ of E such that for all $x \in X$, $s_1(x), \ldots, s_r(x)$ is a basis of $E(x)$. Then $E \cong X \times \mathbb{C}^r$, since

$$\begin{array}{ccc} X \times \mathbb{C}^r & \longrightarrow & E \\ (x, (v_1, \ldots, v_r)) & \longmapsto & \sum_i v_i s_i(x) \end{array}$$

is a biholomorphism. Then $s_1, \ldots, s_r$ is called a **holomorphic frame** for E. Recall that for all $E \to X$ and for all $x \in X$ there exists an open $U \ni x$ such that $E|_U$ is trivial, so there exists a frame on U for $E|_U$. This is called a **local frame** around x.

Example 4.10. Let X be a complex manifold of dimension n, and let $(z_1, \dots, z_n)$ be coordinates on $\mathbb{C}^n$. There exists an atlas $\{(U_\alpha, \phi_\alpha)\}$ for $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n$. For all $x \in U_\alpha$, $\mathrm{T}_x U_\alpha \to \mathrm{T}_{\phi_\alpha(x)} V_\alpha$, and $\mathrm{T}_{\phi_\alpha(x)} V_\alpha = \left\langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right\rangle$ is a frame of T_{V_α}. Let

$$\mathrm{T}_X = \bigcup_{x \in X} \mathrm{T}_x X,$$

and let $\pi^{-1} : \mathrm{T}_X \to X$ such that $\pi^{-1}(x) = \mathrm{T}_x X$. Then T_X is a holomorphic vector bundle of rank n called the **tangent bundle**, where $U = \{U_\alpha\}$ and

$$\psi_\alpha : \pi^{-1}(U_\alpha) = \mathrm{T}_X|_{U_\alpha} \to \mathrm{T}_{\mathbb{C}^n}|_{V_\alpha} \cong V_\alpha \times \mathbb{C}^r \to U_\alpha \times \mathbb{C}^r$$

defines the trivialisation. The **cotangent bundle** of X is

$$\Omega_X^1 = \mathrm{T}_X^*,$$

and let

$$\Omega_X^p = \Lambda^p \Omega_X^1, \qquad p \geq 1.$$

A holomorphic p-form on X is a section of Ω_X^p. [7]

4.3 Complexification of tangent bundles

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Let X be a complex manifold. How to view X as a differentiable manifold? Let V be a vector space of dimension m over $\mathbb{R}$. An **almost complex structure** on V is a linear map $J : V \to V$ such that $J^2 = -\mathrm{id}_V$. If V admits an almost complex structure, then V can be seen as a vector space over $\mathbb{C}$. Let $\lambda = a + ib$ for $a, b \in \mathbb{R}$, and let $v \in V$. Define

$$\lambda v = av + bJ(v).$$

If $\lambda_1, \lambda_2 \in \mathbb{C}$, then $\lambda_1(\lambda_2 v) = (\lambda_1 \lambda_2) v$. [8] Let $v_1, \dots, v_n \in V$ be a basis over $\mathbb{C}$. Then

$$v_1, \dots, v_n, J(v_1), \dots, J(v_n)$$

is a basis of V over $\mathbb{R}$. The idea is to assume that $a_i, b_i \in \mathbb{R}$ such that $\sum_i a_i v_i + \sum_i b_i J(v_i) = 0$, then

$$0 = \sum_i a_i v_i + \sum_i b_i J(v_i) = \sum_i (a_i v_i + b_i J(v_i)) = \sum_i (a_i + ib_i) v_i,$$

so $a_i + ib_i = 0$ for all i. Thus $a_i = b_i = 0$, so $m = 2n$. On a vector space an almost complex structure is a complex structure. Let V be a vector space of dimension $2n$ over $\mathbb{R}$. Then the **complexification** $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$ of V is a $\mathbb{C}$-vector space of dimension $2n$ over $\mathbb{C}$, where

$$\begin{array}{cccc} \lambda & : & V_\mathbb{C} & \longrightarrow & V_\mathbb{C} \\ & & v \otimes \mu & \longmapsto & v \otimes \mu\lambda \end{array}, \qquad \lambda \in \mathbb{C}.$$

Let J be an almost complex structure on V. Then we can extend J to a linear map

$$\begin{array}{ccccc} J & : & V_\mathbb{C} & \longrightarrow & V_\mathbb{C} \\ & & v \otimes \mu & \longmapsto & J(v) \otimes \mu \end{array},$$

such that $J^2 = -\mathrm{id}_{V_\mathbb{C}}$, [9] so $J^2 + \mathrm{id}_{V_\mathbb{C}} = 0$. Thus the eigenvalues of J on $V_\mathbb{C}$ are $\pm i$. Let $V^{1,0}$ be the eigenspace for i and $V^{0,1}$ be the eigenspace for $-i$, so

$$V_\mathbb{C} = V^{1,0} \oplus V^{0,1}.$$

The **conjugation**

$$\begin{array}{ccccc} \bar{\cdot} & : & V_\mathbb{C} & \longrightarrow & V_\mathbb{C} \\ & & v \otimes \mu & \longmapsto & v \otimes \overline{\mu} \end{array}$$

on $V_\mathbb{C}$ in linear over $\mathbb{R}$, such that $\overline{V^{1,0}} = V^{0,1}$ and $\overline{V^{0,1}} = V^{1,0}$, [10] so $V^{1,0}$ and $V^{0,1}$ are $\mathbb{C}$-vector spaces of dimension n.

[7] Exercise
[8] Exercise
[9] Exercise
[10] Exercise

Example 4.11. Let $W = \mathbb{C}^n$ with coordinates $(z_1, \dots, z_n)$, and let $z_j = x_j + iy_j$ with coordinates $(x_1, y_1, \dots, x_n, y_n)$ for $\mathbb{R}^{2n}$. Define

$$\begin{array}{cccc} J & : & \mathbb{R}^{2n} & \longrightarrow & \mathbb{R}^{2n} \\ & & (x_1, y_1, \dots, x_n, y_n) & \longmapsto & (-y_1, x_1, \dots, -y_n, x_n) \end{array}.$$

Then $J^2 = \mathrm{id}_{\mathbb{R}^{2n}}$, and J is the **standard almost complex structure** on $\mathbb{R}^{2n}$. Let $V = \mathbb{R}^{2n}$, so $V_{\mathbb{C}} \cong \mathbb{C}^{2n}$ with complex coordinates $(x_1, y_1, \dots, x_n, y_n)$. Then $V^{0,1}$ is spanned by $x_j - iy_j$ and $V^{1,0}$ is spanned by $x_j + iy_j$, where $\overline{x_j + iy_j} = x_j - iy_j$ for $j = 1, \dots, n$.

Definition 4.12. Let X be a differentiable manifold. A **real, or complex, vector bundle** of rank r is a differentiable manifold E with a smooth morphism $\pi : E \to X$ such that if $K = \mathbb{R}$, or $K = \mathbb{C}$, then there exists an open covering $U = \{U_i\}$ of X such that

- for all $x \in X$, the fibre of π, $E(x) = \pi^{-1}(x)$, is a vector space of rank r over K,
- for all i there exists a diffeomorphism h_i such that

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "h_i"] \arrow[dr, "\pi"'] & & U_i \times K^r \arrow[r, "\mathrm{p}_2"] \arrow[dl, "\mathrm{p}_1"] & K^r \\ & U_i & & \end{tikzcd},$$

  and for all x, $\mathrm{p}_2 \circ h_i : E(x) \to K^r$ is an isomorphism of vector spaces.

Pull-backs, sections, exterior powers, tensors, direct sums, frames, etc are the same as holomorphic vector bundles, where holomorphic becomes smooth and biholomorphic becomes diffeomorphic, and for all X there exists a tangent bundle T_X. Assume X is a complex manifold of dimension n. Let T_X be the holomorphic tangent bundle of X. Then X is also a differentiable manifold of dimension $2n$, so let $\mathrm{T}_{X,\mathbb{R}}$ be the **real tangent bundle** of X, which is a rank $2n$ vector bundle, and let $\mathrm{T}_{X,\mathbb{C}} = \mathrm{T}_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be the **complex tangent bundle** of X, which is a non-holomorphic complex vector bundle of rank $2n$. Smooth morphisms of real or complex vector bundles are defined similarly as holomorphisms between holomorphic vector bundles such that the rank of the image is constant, so

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & F \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}.$$

Let X be a differentiable manifold of dimension $m = 2n$. Then an **almost complex structure** on X is a smooth morphism between the real tangent bundle $J : \mathrm{T}_{X,\mathbb{R}} \to \mathrm{T}_{X,\mathbb{R}}$ such that $J^2 = -\mathrm{id}$. In particular, $J(x) : \mathrm{T}_x^{\mathbb{R}} X \to \mathrm{T}_x^{\mathbb{R}} X$ is an almost complex structure for all $x \in X$. Lecture 11 Thursday 30/01/20

Proposition 4.13. *Let X be a complex manifold. Then the underlying differentiable manifold admits an almost complex structure $J : \mathrm{T}_{X,\mathbb{R}} \to \mathrm{T}_{X,\mathbb{R}}$ such that $J^2 = -\mathrm{id}$.*

Proof. Let $x \in X$, and let (U, ϕ) be a complex chart around x such that

$$\begin{array}{ccccc} \phi & : & U & \longrightarrow & V \\ & & x & \longmapsto & 0 \end{array}.$$

Fix holomorphic coordinates $(z_1, \dots, z_n)$ on U. The tangent bundle of X on U is trivial, with a local frame $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$, so

$$\mathrm{T}_X|_U \xrightarrow{\sim} \mathrm{T}_V = V \times \mathbb{C}^n.$$

Define $x_i = \operatorname{Re} z_i$ and $y_i = \operatorname{Im} e_i$. Then $(x_1, y_1, \dots, x_n, y_n)$ are smooth coordinates $U \to \mathbb{R}$ around x, and $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$ define a local smooth frame of $\mathrm{T}_{X,\mathbb{R}}$ on U, so

$$\mathrm{T}_{X,\mathbb{R}}|_U \xrightarrow{\sim} \mathrm{T}_V = V \times \mathbb{R}^{2n}.$$

In particular, there exists an almost complex structure J_U for $\mathrm{T}_V \cong \mathrm{T}_{X,\mathbb{R}}|_U$, so

$$J_U : \mathrm{T}_{X,\mathbb{R}}|_U \to \mathrm{T}_{X,\mathbb{R}}|_U, \qquad J_U^2 = -\mathrm{id}.$$

Let $f : V \to V$ be a biholomorphism, so

$$
\begin{tikzcd}
 & U \cap U' \arrow[dl, "\phi"'] \arrow[dr, "\phi"] & \\
V \arrow[rr, "f"'] & & V
\end{tikzcd},
$$

and let $z_1', \dots, z_n'$ be local holomorphic coordinates given by

$$z_i' = f_i(z_1, \dots, z_n), \qquad f_i = \mathrm{p}_i \circ f,$$

where $\mathrm{p}_i : \mathbb{C}^n \to \mathbb{C}$ is the i-th projection. Define

$$x_i' = \operatorname{Re} z_i' = \operatorname{Re} f_i(z_1, \dots, z_n) = u_i(z_1, \dots, z_n), \qquad y_i' = \operatorname{Im} z_i' = \operatorname{Im} f_i(z_1, \dots, z_n) = v_i(z_1, \dots, z_n),$$

so $f_j = u_j + iv_j$. The real Jacobian J_f of f is given by the derivatives of u_j and v_j with respect to $x_1, y_1, \dots, x_n, y_n$, a $(2n \times 2n)$-matrix of $n \times n$ blocks of 2×2 blocks of

$$\begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix}.$$

These define the transition function of $\mathrm{T}_{X,\mathbb{R}}$. To show that J extends to X, it is enough to show that J commutes with J_f at each point, so

$$
\begin{tikzcd}
\mathrm{T}_{X,\mathbb{R}}|_{U \cap U'} \arrow[r, "\mathrm{J}_f"] \arrow[d, "J"'] & \mathrm{T}_{X,\mathbb{R}}|_{U \cap U'} \arrow[d, "J"] \\
\mathrm{T}_{X,\mathbb{R}}|_{U \cap U'} \arrow[r, "\mathrm{J}_f"'] & \mathrm{T}_{X,\mathbb{R}}|_{U \cap U'}
\end{tikzcd}.
$$

Since f_j is holomorphic $\frac{\partial}{\partial \overline{z_k}} f_j = 0$ for all j and k, so the Cauchy-Riemann equations

$$\frac{\partial u_j}{\partial x_k} - \frac{\partial v_j}{\partial y_k} = 0, \qquad \frac{\partial v_j}{\partial x_k} + \frac{\partial u_j}{\partial y_k} = 0,$$

or

$$\begin{pmatrix} \frac{\partial u_j}{\partial x_k} & \frac{\partial u_j}{\partial y_k} \\ \frac{\partial v_j}{\partial x_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_j}{\partial y_k} & \frac{\partial u_j}{\partial y_k} \\ -\frac{\partial u_j}{\partial y_k} & \frac{\partial v_j}{\partial y_k} \end{pmatrix},$$

hold. Since J is the standard almost complex structure on $\mathbb{R}^{2n}$, where $x_j \mapsto y_j$ and $y_j \mapsto -x_j$,

$$J = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Check that J_f commutes with J. [11] $\square$

Corollary 4.14. *Every complex manifold is orientable.*

Proof. We prove that if $\mathrm{T}_{X,\mathbb{R}}$ admits an almost complex structure then X is an orientable manifold. For all $x \in X$ choose the orientation on $\mathrm{T}_x^{\mathbb{R}} X$, a vector space of dimension $2n$ over $\mathbb{R}$, given by any ordered basis of the form

$$v_1, \dots, v_n, J(v_1), \dots, J(v_n).$$

Assume that $v_1, \dots, v_n, J(v_1), \dots, J(v_n)$ and $w_1, \dots, w_n, J(w_1), \dots, J(w_n)$ are ordered bases. Show that the determinant of the matrix given by the change of basis is positive. [12] $\square$

[11] Exercise

[12] Exercise

4.4 Differential forms on complex tangent bundles

Let X be a complex manifold. Then there exists an almost complex structure $J : \mathrm{T}_{X,\mathbb{R}} \to \mathrm{T}_{X,\mathbb{R}}$ on X. Then J extends to

$$\begin{array}{cccc} J & : & \mathrm{T}_{X,\mathbb{C}} & \longrightarrow & \mathrm{T}_{X,\mathbb{C}} \\ & & v \otimes \mu & \longmapsto & J(v) \otimes \mu \end{array}.$$

For all x, $J(x)$ has two eigenvalues $\pm i$, so

$$\mathrm{T}_{X,\mathbb{C}} = \mathrm{T}_X^{1,0} \oplus \mathrm{T}_X^{0,1},$$

which are complex vector bundles and **eigenbundles**. Locally $\mathrm{T}_X^{1,0}$ and $\mathrm{T}_X^{0,1}$ are spanned by the frames $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$ and $\frac{\partial}{\partial \overline{z_1}}, \dots, \frac{\partial}{\partial \overline{z_n}}$ respectively. Moreover there exists a conjugation

$$\begin{array}{ccc} \mathrm{T}_{X,\mathbb{C}} & \longrightarrow & \mathrm{T}_{X,\mathbb{C}} \\ v \otimes \mu & \longmapsto & v \otimes \overline{\mu} \end{array}$$

over $\mathbb{R}$, such that $\overline{\mathrm{T}_X^{1,0}} = \mathrm{T}_X^{0,1}$ and $\overline{\mathrm{T}_X^{0,1}} = \mathrm{T}_X^{1,0}$. Let

$$\Omega_{X,\mathbb{C}}^1 = \mathrm{T}_{X,\mathbb{C}}^*$$

be the dual of the complex vector bundle $\mathrm{T}_{X,\mathbb{C}}$. Then

$$\Omega_{X,\mathbb{C}}^1 = \Omega_{X,\mathbb{R}}^1 \otimes_{\mathbb{R}} \mathbb{C} = \Omega_X^{1,0} \oplus \Omega_X^{0,1} = \left(\mathrm{T}_X^{1,0}\right)^* \oplus \left(\mathrm{T}_X^{0,1}\right)^*.$$

Exercise. Let V and W be vector spaces. Show that

$$\Lambda^k (V \oplus W) = \bigoplus_{p+q=k} \Lambda^p V \otimes \Lambda^q W$$

is a canonical isomorphism.

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Thus,

$$\Omega_{X,\mathbb{C}}^k = \Lambda^k \Omega_{X,\mathbb{C}}^1 = \bigoplus_{p+q=k} \Omega_X^{p,q}, \qquad \Omega_X^{p,q} = \Lambda^p \Omega_X^{1,0} \otimes \Lambda^q \Omega_X^{0,1}, \qquad k \geq 0,$$

where $\Omega_X^{p,q}$ is a complex vector bundle for any p and q.

Definition 4.15. The sections of $\Omega_X^{p,q}$ are called (p,q)**-forms** on X, or **forms of type** (p,q).

Locally, let $x \in X$, and let $(U \ni x, \phi)$ be a holomorphic chart for $\phi : U \xrightarrow{\sim} V \subset \mathbb{C}^n$. A (p,q)-form on U can be locally written as

$$\omega = \sum_{I,J} \alpha_{I,J} \mathrm{d}z_I \wedge \mathrm{d}\overline{z_J} = \sum_{|I|=p,\ |J|=q} \alpha_{I,J} \mathrm{d}z_{i_1} \wedge \dots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z_{j_1}} \wedge \dots \wedge \mathrm{d}\overline{z_{j_q}},$$

where $\alpha_{I,J}$ are smooth functions on U. Let X be a manifold. If E is a complex vector bundle then

$$\mathrm{C}^\infty (X, E) = \{\text{smooth sections of } E\}.$$

The **differential**

$$\mathrm{d} : \mathrm{C}^\infty \left(X, \Omega_{X,\mathbb{C}}^k\right) \to \mathrm{C}^\infty \left(X, \Omega_{X,\mathbb{C}}^{k+1}\right)$$

satisfies the Leibnitz rule and $\mathrm{d}^2 = 0$, so $\mathrm{d}(\mathrm{d}\omega) = 0$. If $\omega \in \mathrm{C}^\infty \left(X, \Omega_X^{p,q}\right)$, then $\mathrm{d}\omega \in \mathrm{C}^\infty \left(X, \Omega_{X,\mathbb{C}}^{p+q+1}\right)$. Assume that locally $\omega = \sum_{I,J} \alpha_{I,J} \mathrm{d}z_I \wedge \mathrm{d}\overline{z_J}$. Then

$$\mathrm{d}\omega = \sum_{I,J} \mathrm{d}\alpha_{I,J} \mathrm{d}z_I \wedge \mathrm{d}\overline{z_J}, \qquad \mathrm{d}\alpha_{I,J} = \sum_{i=1}^n \frac{\partial}{\partial z_i} \alpha_{I,J} \mathrm{d}z_i + \sum_{i=1}^n \frac{\partial}{\partial \overline{z_i}} \alpha_{I,J} \mathrm{d}\overline{z_i}.$$

Let
$$\partial\alpha_{I,J} = \sum_{i=1}^n \frac{\partial}{\partial z_i} \alpha_{I,J} \mathrm{d}z_i \in \mathrm{C}^\infty \left(X, \Omega_X^{1,0}\right), \qquad \overline{\partial}\alpha_{I,J} = \sum_{i=1}^n \frac{\partial}{\partial z_i} \alpha_{I,J} \mathrm{d}\overline{z_i} \in \mathrm{C}^\infty \left(X, \Omega_X^{0,1}\right).$$
Then $\mathrm{d} = \partial + \overline{\partial}$ for smooth functions. Back to $\mathrm{d}\omega$. Then
$$\mathrm{d}\omega = \sum_{I,J} \mathrm{d}\alpha_{I,J}\mathrm{d}z_I \wedge \mathrm{d}\overline{z_J} = \sum_{I,J} \partial\alpha_{I,J}\mathrm{d}z_I \wedge \mathrm{d}\overline{z_J} + \sum_{I,J} \overline{\partial}\alpha_{I,J}\mathrm{d}z_I \wedge \mathrm{d}\overline{z_J}.$$
Let
$$\partial\omega = \sum_{I,J} \partial\alpha_{I,J}\mathrm{d}z_I \wedge \mathrm{d}\overline{z_J}, \qquad \overline{\partial}\omega = \sum_{I,J} \overline{\partial}\alpha_{I,J}\mathrm{d}z_I \wedge \mathrm{d}\overline{z_J}.$$
Then $\mathrm{d} = \partial + \overline{\partial}$ for ω.

Lemma 4.16. *The linear maps*
$$\partial : \mathrm{C}^\infty \left(X, \Omega_X^{p,q}\right) \to \mathrm{C}^\infty \left(X, \Omega_X^{p+1,q}\right), \qquad \overline{\partial} : \mathrm{C}^\infty \left(X, \Omega_X^{p,q}\right) \to \mathrm{C}^\infty \left(X, \Omega_X^{p,q+1}\right)$$
satisfy the Leibnitz rule. That is, if $\omega \in \mathrm{C}^\infty \left(X, \Omega_X^{p,q}\right)$ *and* $\eta \in \mathrm{C}^\infty \left(X, \Omega_X^{p',q'}\right)$*, then*
$$\partial (\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^{p+q} \omega + \partial\eta, \qquad \overline{\partial} (\omega \wedge \eta) = \overline{\partial}\omega \wedge \eta + (-1)^{p+q} \omega + \overline{\partial}\eta.$$

Proof. d satisfies the Leibnitz rule
$$\mathrm{d} (\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^{p+q} \omega \wedge \mathrm{d}\eta,$$
since $\omega \in \mathrm{C}^\infty \left(X, \Omega_{X,\mathbb{C}}^{p+q}\right)$, so
$$\begin{aligned}\partial (\omega \wedge \eta) + \overline{\partial} (\omega \wedge \eta) &= \left(\partial\omega + \overline{\partial}\omega\right) \wedge \eta + (-1)^{p+q} \omega \wedge \left(\partial\eta + \overline{\partial}\eta\right) \\ &= \left(\partial\omega \wedge \eta + (-1)^{p+q} \omega \wedge \partial\eta\right) + \left(\overline{\partial}\omega \wedge \eta + (-1)^{p+q} \omega \wedge \overline{\partial}\eta\right).\end{aligned}$$
Then $\partial (\omega \wedge \eta)$ and $\partial\omega \wedge \eta + (-1)^{p+q} \omega \wedge \partial\eta$ are $(p+1,q)$-forms, and $\overline{\partial} (\omega \wedge \eta)$ and $\overline{\partial}\omega \wedge \eta + (-1)^{p+q} \omega \wedge \overline{\partial}\eta$ are $(p,q+1)$-forms. Forms of the same type in the decomposition of $\mathrm{d} (\omega \wedge \eta)$ must coincide. ☐

Lemma 4.17. $\partial^2 = \overline{\partial}^2 = \overline{\partial}\partial + \partial\overline{\partial} = 0$.

Proof. Let $\omega \in \mathrm{C}^\infty \left(X, \Omega_X^{p,q}\right)$. Because $\mathrm{d}^2 = 0$,
$$0 = \mathrm{d}^2\omega = \left(\partial + \overline{\partial}\right) \left(\left(\partial + \overline{\partial}\right) \omega\right) = \partial^2\omega + \partial\overline{\partial}\omega + \overline{\partial}\partial\omega + \overline{\partial}^2\omega.$$
Then $\mathrm{d}^2\omega$ is a $(p+q+2)$-form, $\partial^2\omega$ is a $(p+2,q)$-form, $\partial\overline{\partial}\omega + \overline{\partial}\partial\omega$ is a $(p+1,q+1)$-form, and $\overline{\partial}^2\omega$ is a $(p,q+2)$-form. Forms of the same type in the decomposition of $\mathrm{d}^2\omega$ must coincide. ☐

4.5 Dolbeault cohomology

Let X be a complex manifold. Fix $p, q \ge 0$. Let
$$\begin{aligned}\mathcal{Z}^{p,q}(X) &= \mathrm{Ker}\left(\overline{\partial} : \mathrm{C}^\infty \left(X, \Omega_X^{p,q}\right) \to \mathrm{C}^\infty \left(X, \Omega_X^{p,q+1}\right)\right) \\ &= \left\{\omega \in \mathrm{C}^\infty \left(X, \Omega_X^{p,q}\right) \;\middle|\; \overline{\partial}\omega = 0\right\}\end{aligned}$$
and let
$$\begin{aligned}\mathcal{B}^{p,q}(X) &= \mathrm{Im}\left(\overline{\partial} : \mathrm{C}^\infty \left(X, \Omega_X^{p,q-1}\right) \to \mathrm{C}^\infty \left(X, \Omega_X^{p,q}\right)\right) \\ &= \left\{\omega \in \mathrm{C}^\infty \left(X, \Omega_X^{p,q}\right) \;\middle|\; \exists \eta \in \mathrm{C}^\infty \left(X, \Omega_X^{p,q-1}\right),\ \omega = \overline{\partial}\eta\right\}.\end{aligned}$$
Since $\overline{\partial}^2 = 0$ we have $\mathcal{B}^{p,q}(X) \subset \mathcal{Z}^{p,q}(X)$ for all p and q. The **Dolbeault cohomology group** of X is
$$\mathrm{H}^{p,q}(X) = \mathcal{Z}^{p,q}(X) / \mathcal{B}^{p,q}(X).$$

Exercise. Assume X and Y are biholomorphic complex manifolds. Then

$$\mathrm{H}^{p,q}(X) = \mathrm{H}^{p,q}(Y).$$

If $\mathrm{H}^{p,q}(X)$ is finite dimensional then we define the **Hodge numbers** of X as

$$\mathrm{h}^{p,q}(X) = \dim_{\mathbb{C}} \mathrm{H}^{p,q}(X).$$

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Our goal is if X is Kähler and compact

$$\bigoplus_{p+q=k} \mathrm{H}^{p,q}(X) = \mathrm{H}^{p+q}(X),$$

as the de Rham cohomology. In particular this is true if X is projective. How to compute $\mathrm{H}^{p,q}(X)$? We need to use analysis.

Proposition 4.18. *Let X be a complex manifold. Then there exists an isomorphism*

$$\mathrm{H}^{p,0}(X) \cong \mathrm{H}^0\left(X, \Omega_X^p\right) = \{\textit{holomorphic sections of } \Omega_X^p\} = \{\textit{holomorphic } p\textit{-forms on } X\}, \qquad p \geq 0.$$

Remark. If X is compact then $\mathrm{H}^{0,0}(X) = \mathbb{C}$ because $\mathrm{H}^{0,0}(X) = \mathrm{H}^0(X, \mathcal{O}_X)$ are constants.

Proof.

$$\mathrm{H}^{p,0}(X) = \mathcal{Z}^{p,0}(X) / \mathcal{B}^{p,0}(X) = \mathcal{Z}^{p,0}(X) = \left\{\omega \in \mathrm{C}^\infty\left(X, \Omega_X^{p,0}\right) \;\middle|\; \overline{\partial}\omega = 0\right\}.$$

Locally $\omega = \sum_{|I|=p} \alpha_I \mathrm{d}z_I$. Then

$$\overline{\partial}\omega = \sum_{|I|=p} \overline{\partial}\alpha_I \mathrm{d}z_I = \sum_{|I|=p} \sum_{i=1}^n \frac{\partial}{\partial \overline{z_j}} \alpha_I \mathrm{d}\overline{z_j} \wedge \mathrm{d}z_I,$$

where $\mathrm{d}\overline{z_j} \wedge \mathrm{d}z_I$ are linearly independent. For all I and for all j, the Cauchy-Riemann equations $\frac{\partial}{\partial \overline{z_j}} \alpha_I = 0$ hold, so for all I, α_I is holomorphic. Then $\omega = \sum_{|I|=p} \alpha_I \mathrm{d}z_I$ is a holomorphic p-form, so $\omega \in \mathrm{H}^0\left(X, \Omega_X^p\right)$. $\square$

5 Connection, curvature, and metric

5.1 Connections

Let X be a differentiable manifold, and let E be a complex vector bundle. Then

$$\mathrm{C}^\infty(X, E) = \{\mathrm{C}^\infty\text{-sections of } E\}.$$

Is there a way to compute the derivatives of these sections?

Definition 5.1. Let X and E be as above. A **connection** of E is a $\mathbb{C}$-linear map

$$\nabla : \mathrm{C}^\infty(X, E) \to \mathrm{C}^\infty\left(X, \Omega^1_{X,\mathbb{C}} \otimes E\right)$$

such that the Leibniz rule holds, so

$$\nabla(f\omega) = f \cdot \nabla\omega + \mathrm{d}f \otimes \omega, \qquad f \in \mathrm{C}^\infty(X), \qquad \omega \in \mathrm{C}^\infty(X, E).$$

The following is the idea. Let $\omega \in \mathrm{C}^\infty(X, E)$. Then

$$\nabla\omega = \sum_i \eta_i \otimes \omega_i,$$

where η_i are 1-forms on X and ω_i are sections of E. Let $x \in X$, and let $v \in \mathrm{T}_x X$. Then

$$\nabla_v \omega_x = \sum_i \eta_i(v)\,\omega_i$$

is a section of E at x. The goal is to study connections locally. Let $x \in X$, and let (U, ϕ) be a chart around x that trivialises E, so $\pi^{-1}(U) = U \times \mathbb{C}^r$ for $\pi : E \to X$ and $r = \operatorname{rk} E$. Then there exists a frame $s_1, \dots, s_r \in \mathrm{C}^\infty(U, E)$ of E on U. Let $\sigma \in \mathrm{C}^\infty(X, E)$ be any section. Locally on U we write

$$\sigma \overset{U}{=} f = (f_1, \dots, f_r), \qquad \sigma = \sum_{i=1}^r f_i s_i, \qquad f_1, \dots, f_r \in \mathrm{C}^\infty(U).$$

By the Leibniz rule, on U,

$$\nabla\sigma = \sum_{i=1}^r \nabla(f_i s_i) = \sum_{i=1}^r (f_i \cdot \nabla s_i + \mathrm{d}f_i \otimes s_i) \in \mathrm{C}^\infty\left(U, \Omega^1_{X,\mathbb{C}} \otimes E\right).$$

Notation. $\mathrm{d}f = (\mathrm{d}f_1, \dots, \mathrm{d}f_r)$.

Then

$$\nabla s_j = \sum_{i=1}^r a_{ij} \otimes s_i, \qquad a_{ij} \in \mathrm{C}^\infty\left(U, \Omega^1_{X,\mathbb{C}}\right).$$

Notation. $A = (a_{ij})$ is an $(r \times r)$-matrix with coefficients 1-forms.

With this notation, this becomes

$$\nabla\sigma \overset{U}{=} A \cdot f + \mathrm{d}f.$$

- A depends very much on the choice of the frame.
- Locally on U, ∇ is determined by A.

Consider another chart (U', ϕ') which also gives a trivialisation of E. So we can choose a corresponding frame $s'_1, \dots, s'_r$. Assume $\sigma \in \mathrm{C}^\infty(U \cap U', E)$. Then

$$\sigma \overset{U'}{=} f' = (f'_1, \dots, f'_r), \qquad \sigma = \sum_{j=1}^r f'_j s'_j, \qquad f'_1, \dots, f'_r \in \mathrm{C}^\infty(U).$$

Let A' be the matrix with respect to this frame. Then

$$\nabla\sigma \overset{U'}{=} A' \cdot f' + \mathrm{d}f'.$$

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$$g : (U \cap U') \times \mathbb{C}^r \to (U \cap U') \times \mathbb{C}^r$$

be the transition function from the trivialisation of U' to the trivialisation of U. Then $g(x) \in \mathrm{GL}_r \mathbb{C}$ for all $x \in U \cap U'$, and $f = g \cdot f'$. Denote by $\mathrm{D}g$ the differential of g. Then

$$\mathrm{d}f = \mathrm{d}(g \cdot f') = \mathrm{D}g \cdot f' + g \cdot \mathrm{d}f' = g \cdot \left(g^{-1} \cdot \mathrm{D}g \cdot f' + \mathrm{d}f'\right),$$

by the Leibniz rule. Thus,

$$\begin{aligned} A' \cdot f' + \mathrm{d}f' &\overset{U'}{=} A \cdot f + \mathrm{d}f \overset{U}{=} A \cdot g \cdot f' + g \cdot \left(g^{-1} \cdot \mathrm{D}g \cdot f' + \mathrm{d}f'\right) \overset{U}{=} g \cdot \left(\left(g^{-1} \cdot \mathrm{D}g + g^{-1} \cdot A \cdot g\right) f' + \mathrm{d}f'\right) \\ &\overset{U'}{=} \left(g^{-1} \cdot \mathrm{D}g + g^{-1} \cdot A \cdot g\right) \cdot f' + \mathrm{d}f', \end{aligned}$$

so

$$A' = g^{-1} \cdot \mathrm{D}g + g^{-1} \cdot A \cdot g.$$

5.2 Curvature operators

What is ∇^2? The idea is

$$\mathrm{C}^\infty(X, E) \xrightarrow{\nabla} \mathrm{C}^\infty\left(X, \Omega^1_{X,\mathbb{C}} \otimes E\right) \xrightarrow{\nabla} \mathrm{C}^\infty\left(X, \Omega^1_{X,\mathbb{C}} \otimes \Omega^1_{X,\mathbb{C}} \otimes E\right) \xrightarrow{\wedge} \mathrm{C}^\infty\left(X, \Omega^2_{X,\mathbb{C}} \otimes E\right).$$

The **curvature tensor** is

$$\nabla^2 : \mathrm{C}^\infty(X, E) \to \mathrm{C}^\infty\left(X, \Omega^2_{X,\mathbb{C}} \otimes E\right).$$

Remark. If X has dimension one, then $\Omega^2_{X,\mathbb{C}} = 0$, so $\nabla^2 = 0$.

Again for all $x \in X$, take U as above. Let $s_1, \dots, s_r$ be a frame, let $A = (a_{ij})$ be the $(r \times r)$-matrix of 1-forms, and let $\mathrm{D}A$ be the differential of A.

Notation. $A \wedge A = \left(\sum_{k=1}^r (a_{ik} \wedge a_{kj})\right)$ is an $(r \times r)$-matrix of 2-forms.

Let $\sigma \overset{U}{=} (f_1, \dots, f_r) = \sum_i f_i s_i$ on U. Then

$$\begin{aligned} \nabla^2 \sigma &= \nabla(A \cdot f + \mathrm{d}f) = A \wedge (A \cdot f + \mathrm{d}f) + \mathrm{d}(A \cdot f + \mathrm{d}f) \\ &= A \wedge A \cdot f + A \wedge \mathrm{d}f + \mathrm{D}A \cdot f - A \wedge \mathrm{d}f + \mathrm{d}^2 f = (A \wedge A + \mathrm{D}A) \cdot f \end{aligned}$$

is C^∞-linear, so $\nabla^2(h\sigma) = h\nabla^2\sigma$. The **curvature operator** is

$$\Theta_\nabla \overset{U}{=} A \wedge A + DA,$$

so $\Theta_\nabla(\sigma) = \nabla^2\sigma$.

5.3 Hermitian metrics

Definition 5.2. Let V be a vector space over $\mathbb{C}$. A **Hermitian inner product** on V is a map

$$\begin{array}{ccc} V \times V & \longrightarrow & \mathbb{C} \\ (v, w) & \longmapsto & \langle v, w\rangle \end{array},$$

such that

- $\langle v, w\rangle = \overline{\langle w, v\rangle}$,
- it is linear on the first factor, and
- $\langle v, v\rangle \ge 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$.

Example. $V = \mathbb{C}$ and $\langle z_1, z_2\rangle = z_1 \cdot \overline{z_2}$.

Definition 5.3. Let X be a manifold, and let E be a complex vector bundle on X. A **Hermitian metric** h, or $\langle \cdot, \cdot \rangle$, on E is a choice of a Hermitian inner product

$$h_x = \langle \cdot, \cdot \rangle_x : E(x) \times E(x) \to \mathbb{C}, \qquad x \in X,$$

such that for any open set $U \subset X$ and for $s, t \in \mathrm{C}^\infty(U, E)$, $\langle s(x), t(x) \rangle_x$ is a C^∞-function with respect to x on U. The pair $(E, \langle \cdot, \cdot \rangle) = (E, h)$ is called a **Hermitian vector bundle**.

Let (E, h) be a Hermitian vector bundle, and let $x \in X$. Locally, let $s_1, \dots, s_r$ be a frame on $U \ni x$. For any $x \in U$, $\langle s_i(x), s_j(x) \rangle_x = h_{ij}(x)$ is a smooth function for all i and j, so

$$H = (h_{ij})_{i,j=1}^r$$

is an $(r \times r)$-matrix of smooth functions. Let $\sigma, \sigma' \in \mathrm{C}^\infty(U, E)$, and let $\sigma \overset{U}{=} f = (f_1, \dots, f_r)$ and $\sigma' \overset{U}{=} f' = (f_1', \dots, f_r')$. Then

$$\langle \sigma(x), \sigma'(x) \rangle_x = f^\intercal \cdot H \cdot \overline{f'}.$$

Now assume that U' is a different open set with frame $(s_1', \dots, s_r')$. Assume

$$g : (U \cap U') \times \mathbb{C}^r \to (U \cap U') \times \mathbb{C}^r$$

is the transition function from the trivialisation on U' to the trivialisation on U. Let H' be the matrix of h with respect to $s_1', \dots, s_r'$. Then

$$H' = g^\intercal \cdot H \cdot \overline{g}.$$

Proposition 5.4. *Let $\pi : E \to X$ be a complex vector bundle on X. Then E always admits a Hermitian metric.*

Before proving the proposition, we recall the definition of a partition of the unity.

Definition 5.5. Let M be a manifold and let $U = \{U_\alpha\}$ be an open covering. A **partition of unity** with respect to U is a collection of smooth functions $f_\alpha : M \to [0, 1]$ such that

- $\operatorname{supp} f_\alpha \subset U_\alpha$ for all α, in particular, $f_\alpha = 0$ outside U_α,
- $\sum_\alpha f_\alpha(x) = 1$ for all $x \in M$, and
- for all $x \in M$, there exists an open neighbourhood V of x such that $\operatorname{supp} f_\alpha \cap V \neq 0$ for only finitely many α.

It can be shown that if M is a manifold and $U = \{U_\alpha\}$ is an open cover of M, then there exists a partition of the unity $\{f_\alpha\}$ with respect to such a cover.

Proof. Let $U = \{U_i\}$ be an open cover of open sets of X, trivialising E, so $\phi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}^r$, and let $f_i : X \to [0, 1]$ be a partition of unity with respect to U. For each i, consider a Hermitian metric on $\mathbb{C}^r$. Then there is a Hermitian metric $\widetilde{h_i}$ on $U_i \times \mathbb{C}^r$. Let h_i be the Hermitian metric on $E|_{U_i}$ induced by ϕ_i. Take $h = \sum_i f_i h_i$. Check that h defines a Hermitian metric on X. [13] ☐

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Let $E \to X$ be a complex Hermitian vector bundle of rank r. Fix $p, q \geq 0$. There exists a bilinear **cup product**

$$\begin{array}{cccc} \{\cdot, \cdot\} & : & \mathrm{C}^\infty\left(X, \Omega^p_{X,\mathbb{C}} \otimes E\right) \times \mathrm{C}^\infty\left(X, \Omega^q_{X,\mathbb{C}} \otimes E\right) & \longrightarrow & \mathrm{C}^\infty\left(X, \Omega^{p+q}_{X,\mathbb{C}}\right) \\ & & (\sigma, \tau) & \longmapsto & \{\sigma, \tau\} \end{array},$$

where $\{\sigma, \tau\}$ is defined as follows. Let $x \in X$, let $s_1, \dots, s_r$ be a frame of E around x, let H be the matrix associated to the Hermitian metric with respect to the frame, and let

$$\sigma = \sum_i \sigma_i \otimes s_i, \qquad \tau = \sum_i \tau_i \otimes s_i, \qquad \sigma_i \in \mathrm{C}^\infty\left(X, \Omega^p_{X,\mathbb{C}}\right), \qquad \tau_i \in \mathrm{C}^\infty\left(X, \Omega^q_{X,\mathbb{C}}\right).$$

[13] Exercise

Then we define, around x,

$$\{\sigma, \tau\} = \sigma^\intercal \cdot H \cdot \overline{\tau} = \sum_{i,j=1}^{r} h_{ij} \sigma_i \wedge \overline{\tau_j}.$$

This is uniquely defined, and does not depend on the frame, so it extends to X. In particular $\{\sigma, \tau\}$ is a smooth $(p+q)$-form.

Definition 5.6. Let E be a complex Hermitian vector bundle on X, and let ∇ be a connection on E. We say that ∇ is **Hermitian**, or **compatible with the metric**, if the Leibnitz rule holds, so we have

$$\mathrm{d}\{\sigma, \tau\} = \{\nabla\sigma, \tau\} + (-1)^p \{\sigma, \nabla\tau\}, \qquad \sigma \in \mathrm{C}^\infty\left(X, E \otimes \Omega_{X,\mathbb{C}}^p\right), \qquad \tau \in \mathrm{C}^\infty\left(X, E \otimes \Omega_{X,\mathbb{C}}^q\right).$$

Let $x \in X$, and let $s_1, \dots, s_r$ be a local frame of E. Assume $s_1, \dots, s_r$ is an orthonormal frame around $x \in X$. Let ∇ be a connection compatible with the metric, and let A be the associated matrix with respect to $s_1, \dots, s_r$. Gram-Schmidt is an algorithm that gives an orthonormal basis of $E(x)$ for all x, which is C^∞, say $s_1', \dots, s_r'$. Then with respect to this frame $H = \mathrm{id}_r$ because $\langle s_i', s_j' \rangle_x = \delta_{ij}$.

Proposition 5.7. *A is anti-autodual, that is*

$$\overline{A}^\intercal = -A.$$

Proof. Let σ and τ be as before, and let $\sigma_1, \dots, \sigma_r$ and $\tau_1, \dots, \tau_r$ be the components of σ and τ with respect to the frame $s_1, \dots, s_r$. Then $\{\sigma, \tau\} = \sigma^\intercal \wedge \overline{\tau}$. Since ∇ is Hermitian, the Leibnitz rule holds, so

$$\mathrm{d}\{\sigma, \tau\} = \mathrm{d}(\sigma^\intercal \wedge \overline{\tau}) = \mathrm{d}\sigma^\intercal \wedge \overline{\tau} + (-1)^p \sigma^\intercal \wedge \mathrm{d}\overline{\tau},$$

by the usual Leibnitz rule for d. Then

$$\{\nabla\sigma, \tau\} = \{A \wedge \sigma + \mathrm{d}\sigma, \tau\} = \{A \wedge \sigma, \tau\} + \{\mathrm{d}\sigma, \tau\} = (A \wedge \sigma)^\intercal \wedge \overline{\tau} + \mathrm{d}\sigma^\intercal \wedge \overline{\tau} = (-1)^p \sigma^\intercal \wedge A^\intercal \wedge \overline{\tau} + \mathrm{d}\sigma^\intercal \wedge \overline{\tau},$$

and

$$\{\sigma, \nabla\tau\} = \sigma^\intercal \wedge \overline{\nabla\tau} = \sigma^\intercal \wedge \left(\overline{A \wedge \tau + \mathrm{d}\tau}\right) = \sigma^\intercal \wedge \overline{A} \wedge \overline{\tau} + \sigma^\intercal \wedge \mathrm{d}\overline{\tau}.$$

By the Leibnitz rule,

$$\sigma^\intercal \wedge \left(A^\intercal + \overline{A}\right) \wedge \overline{\tau} = 0.$$

This is true for all σ and τ, so $A^\intercal + \overline{A} = 0$. $\square$

Exercise. Let $s_1, \dots, s_r$ be any frame, let H be the matrix given by the metric with respect to $s_1, \dots, s_r$, and let A be the matrix given by the connection with respect to $s_1, \dots, s_r$ where the connection is Hermitian. Then

$$\mathrm{D}H = A^\intercal \cdot H + H \cdot \overline{A},$$

where if $H = (h_{ij})$ then $\mathrm{D}H = (\mathrm{d}h_{ij})$. A hint is to do the same calculation.

Theorem 5.8. *If $E \to X$ is a complex Hermitian vector bundle, then there exists a connection ∇ compatible with h.*

5.4 Holomorphic vector bundles

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Proposition 5.9. *Let X be a complex manifold, and let $\pi : E \to X$ be a holomorphic vector bundle of rank r. Then for all $q \geq 0$ there exists a $\mathbb{C}$-linear map*

$$\overline{\partial_E} : \mathrm{C}^\infty\left(X, \Omega_X^{0,q} \otimes E\right) \to \mathrm{C}^\infty\left(X, \Omega_X^{0,q+1} \otimes E\right),$$

which satisfies the Leibnitz rule and $\overline{\partial_E} = 0$. Moreover if σ is a holomorphic section of $\Omega_X^{0,q} \otimes E$ then $\overline{\partial_E}\sigma = 0$.

The idea is to do it locally in a canonical way, so does not depend on the choice of the trivialisation.

Proof. Let $x \in X$. There exists a holomorphic frame $s_1, \dots, s_r$ of E locally around x in U. Let $\sigma \in \mathrm{C}^\infty\left(X, \Omega_X^{0,q} \otimes E\right)$. Then locally, $\sigma \overset{U}{=} \sum_{i=1}^r f_i \otimes s_i$ where $f_i \in \mathrm{C}^\infty(U)$ are $(0,q)$-forms locally around x. We define

$$\overline{\partial_E} x \overset{U}{=} \sum_{i=1}^r \overline{\partial} f_i \otimes s_i \in \mathrm{C}^\infty\left(U, \Omega_X^{0,q+1} \otimes E\right).$$

We want to show that it can be extended to X. Let $U' \subset X$ be open, let $s_1', \dots, s_r'$ be a holomorphic frame on U' of E, and let

$$g : (U \cap U') \times \mathbb{C}^r \to (U \cap U') \times \mathbb{C}^r$$

be the transition map from the trivialisation of U' to the trivialisation of U. Then $\sigma \overset{U}{=} \sum_{i=1}^r f_i' \otimes s_i'$, and

$$\overline{\partial_E} x \overset{U'}{=} \sum_{i=1}^r \overline{\partial} f_i' \otimes s_i'.$$

Since g is holomorphic, that is $\overline{\partial} g = 0$, this implies that $\overline{\partial_E}$ on U coincides with $\overline{\partial_E}$ on U'. Recall for ∇ the change of frame was

$$A' = g^{-1} \cdot \mathrm{D} g + g^{-1} \cdot A \cdot g,$$

so $\overline{\partial_E}$ extends to X. Let σ be a holomorphic section of $\Omega_X^{0,q} \otimes E$. Then, on U if s_i and f_i are as before, then f_i are holomorphic $(0,q)$-forms. Thus $\overline{\partial} f_i = 0$, so $\overline{\partial_E} \sigma = 0$. ☐

Vice versa if $\nabla : \mathrm{C}^\infty(X, E) \to \mathrm{C}^\infty\left(X, \Omega_{X,\mathbb{C}}^1 \otimes E\right)$ is a connection and X is a complex manifold, then

$$\Omega_{X,\mathbb{C}}^1 \xrightarrow{\sim} \Omega_X^{1,0} \oplus \Omega_X^{0,1}, \qquad \Omega_{X,\mathbb{C}}^1 \otimes E = \left(\Omega_X^{1,0} \otimes E\right) \oplus \left(\Omega_X^{0,1} \otimes E\right).$$

Then for all σ,

$$\nabla \sigma = \nabla^{1,0} \sigma + \nabla^{0,1} \sigma,$$

where

$$\nabla^{1,0} : \mathrm{C}^\infty(X, E) \to \mathrm{C}^\infty\left(X, \Omega_X^{1,0} \otimes E\right), \qquad \nabla^{0,1} : \mathrm{C}^\infty(X, E) \to \mathrm{C}^\infty\left(X, \Omega_X^{0,1} \otimes E\right).$$

Theorem 5.10. *Assume X is a complex manifold and E is a holomorphic Hermitian vector bundle of rank r. Then there exists a unique connection*

$$\nabla_E : \mathrm{C}^\infty(X, E) \to \mathrm{C}^\infty\left(X, \Omega_{X,\mathbb{C}}^1 \otimes E\right),$$

such that $\nabla_E^{0,1} = \overline{\partial_E}$, defined in Proposition 5.9, and ∇_E is compatible with h.

∇_E is called the **Chern connection** and ∇_E^2 is called the **Chern curvature**.

Proof. Fix $x \in X$, on $U \ni x$. There exists a local holomorphic frame $s_1, \dots, s_r$. Let H be the matrix defining the metric h on U, so $H = (h_{ij})$ is an $(r \times r)$-matrix for $h_{ij} \in \mathrm{C}^\infty(U)$. Define the $(r \times r)$-matrix $\partial H = (\partial h_{ij})$ for $\partial h_{ij} \in \mathrm{C}^\infty\left(U, \Omega_X^{1,0}\right)$. We define

$$A = \overline{H}^{-1} \cdot \partial \overline{H},$$

an $(r \times r)$-matrix of 1-forms on U. This A will be the matrix defining ∇_E.

- Let $\sigma \overset{U}{=} \sum_i f_i s_i \in \mathrm{C}^\infty(U, E)$ where $f_i \in \mathrm{C}^\infty(U)$. Then

  $$\nabla_E \sigma \overset{U}{=} A \cdot f + \mathrm{d} f.$$

  Let $A = (a_{ij})$ where by definition of A, a_{ij} are $(1,0)$-forms. Thus

  $$\nabla_E^{0,1} \sigma = A^{0,1} \cdot f + \overline{\partial} f \overset{U}{=} \overline{\partial_E} \sigma.$$

- Recall that ∇ associated to A is compatible with h if and only if $\mathrm{D}H = A^\intercal \cdot H + H \cdot \overline{A}$. Since H is Hermitian, it follows that $H^\intercal = \overline{H}$, so
$$A^\intercal \cdot H = \left(\overline{H}^{-1} \cdot \partial \overline{H}\right)^\intercal \cdot H = \left(\partial \overline{H}\right)^\intercal \cdot \left(\overline{H}^{-1}\right)^\intercal \cdot H = \partial H \cdot H^{-1} \cdot H = \partial H,$$
and
$$H \cdot \overline{A} = H \cdot \overline{\overline{H}^{-1} \cdot \partial \overline{H}} = H \cdot H^{-1} \cdot \overline{\partial} H = \overline{\partial} H.$$
Thus
$$\mathrm{D}H = (\mathrm{d}h_{ij}) = \left(\partial h_{ij} + \overline{\partial} h_{ij}\right) = \partial H + \overline{\partial H} = A^\intercal \cdot H + H \cdot \overline{A},$$
so on U, ∇_E is compatible with h. Lecture 17 Thursday 13/02/20

- Let ∇ be another connection satisfying $\nabla^{0,1} = \overline{\partial_E}$ and ∇ is compatible with h. As before $s_1, \dots, s_r$ is the local holomorphic frame on U. Let $B = (b_{ij})$ be the $(r \times r)$-matrix of 1-forms associated to ∇, and let $B = B^{1,0} + B^{0,1}$, so $b_{ij} = b_{ij}^{1,0} + b_{ij}^{0,1}$. For all $f = (f_1, \dots, f_r)$ if $\sigma = \sum_i f_i s_i$ then
$$\nabla \sigma \overset{U}{=} B \cdot f + \mathrm{d}f,$$
so
$$B^{0,1} \cdot f + \overline{\partial} f \overset{U}{=} \nabla^{0,1} \sigma = \overline{\partial_E} \sigma \overset{U}{=} \overline{\partial} f.$$
Then for all f, $B^{0,1} \cdot f = 0$, so $B^{0,1} = 0$ and $B = B^{1,0}$. Since ∇ is compatible with h, $\mathrm{D}H = B^\intercal \cdot H + H \cdot \overline{B}$, so $\overline{\mathrm{D}H} = \overline{B^\intercal} \cdot \overline{H} + \overline{H} \cdot B$. Then
$$B = B^{1,0} = \left(\overline{H}^{-1} \cdot \left(\overline{\mathrm{D}H} - \overline{B^\intercal} \cdot \overline{H}\right)\right)^{1,0} = \overline{H}^{-1} \cdot \partial \overline{H} + \overline{H}^{-1} \cdot 0 \cdot \overline{H} = \overline{H}^{-1} \cdot \partial \overline{H} = A,$$
since $\overline{\mathrm{D}H}^{1,0} = \overline{(\mathrm{d}h_{ij})}^{1,0} = \left(\mathrm{d}\overline{h_{ij}}\right)^{1,0} = \partial \overline{h_{ij}}$ and $\left(\overline{B}\right)^{1,0} = \overline{B^{1,0}} = 0$, so $\nabla = \nabla_E$. We define in the same way ∇_E^U on any open U of X. On $U \cap U'$, by unicity $\nabla_E^U = \nabla_E^{U'}$. Thus ∇_E can be extended to X.

$\square$

Corollary 5.11. *Let X be a complex manifold, let (E, h) be a Hermitian vector bundle on X, let ∇_E be the Chern connection, and let $\Theta_E = \nabla_E^2$ be the Chern curvature. Locally at $x \in U$, let $s_1, \dots, s_r$ be a holomorphic frame, and let A be the matrix associated to ∇_E. Then*

1. *A is of type $(1,0)$ and $\partial A = -A \wedge A$,*
2. *$\Theta_E = \overline{\partial} A$ is of type $(1,1)$, and*
3. *$\overline{\partial} \Theta_E = 0$.*

Proof.

1. Let H be as above. Recall $A = \overline{H}^{-1} \cdot \partial \overline{H}$ is a $(1,0)$-form matrix. Then
$$0 = \partial I = \partial \left(\overline{H} \cdot \overline{H}^{-1}\right) = \overline{H} \cdot \partial \overline{H}^{-1} + \partial \overline{H} \cdot \overline{H}^{-1},$$
so
$$\begin{aligned} \partial A &= \partial \left(\overline{H}^{-1} \cdot \partial \overline{H}\right) = \partial \overline{H}^{-1} \wedge \partial \overline{H} + \overline{H}^{-1} \cdot \partial^2 \overline{H} \\ &= -\left(\overline{H}^{-1} \cdot \partial \overline{H} \cdot \overline{H}^{-1}\right) \wedge \partial \overline{H} = -\left(\overline{H}^{-1} \cdot \partial \overline{H}\right) \wedge \left(\overline{H}^{-1} \cdot \partial \overline{H}\right) = -A \wedge A. \end{aligned}$$
2. Recall $\Theta_E = A \wedge A + \mathrm{D}A = A \wedge A + \partial A + \overline{\partial} A = \overline{\partial} A$, by 1.
3. By 2, $\overline{\partial} \Theta_E = \overline{\partial}\left(\overline{\partial} A\right) = 0$.

$\square$

Lemma 5.12. *Let X be a complex manifold of dimension n, let (E, h) be a Hermitian vector bundle on X of rank r, let ∇_E be the Chern connection compatible with h such that $\nabla_E^{0,1} = \overline{\partial_E}$, and let $\Theta_E = \nabla_E^2$ be the Chern curvature. Locally around $x \in X$, there exists an open neighbourhood $U \ni x$ with local coordinates $z_1, \dots, z_n$ such that $x = (0, \dots, 0)$ and there exists a holomorphic frame $s_1, \dots, s_r$ for E on U such that if H is the matrix associated to the metric with respect to $s_1, \dots, s_r$ then*

1. $H(z) = \mathrm{id} + \mathcal{O}\left(|z|^2\right)$, *and*
2. $\Theta_E(0) \overset{U}{=} -\partial\overline{\partial H}(0)$.

1 means $h_{ij} = \delta_{ij} + \mathcal{O}\left(|z|^2\right)$, where $(h_{ij} - \delta_{ij}) / |z|^2 < C$ for some C.

Proof.

1. Let $U \ni x$ be an open set, and let $t_1, \dots, t_r$ be a holomorphic frame for E on U. Let H_1 be the matrix associated to h with respect to $t_1, \dots, t_r$, so $H_1(0)$ is a Hermitian matrix which gives a metric on $E(x)$. There exists an orthonormal basis of $E(x)$, that is there exists an $(r \times r)$-matrix $B \in \mathrm{GL}_r \mathbb{C}$ such that
$$B^\intercal \cdot H_1(0) \cdot \overline{B} = \mathrm{id}.$$
Let $t_i' = B \cdot t_i$, so $t_1', \dots, t_r'$ is a holomorphic frame. If H_2 is the matrix of h associated to the frame $t_1', \dots, t_r'$ then
$$H_2(0) = \mathrm{id}, \qquad H_2(z) = \mathrm{id} + \mathcal{O}(|z|).$$
The goal is to find a new local frame. We want to apply a change of basis given by the matrix $C(z) = \mathrm{id} + C_0(z)$ where $C_0(z)$ has coefficients linear in z. Recall that with respect to the new frame $s_1, \dots, s_r$,
$$H(z) = \left(\mathrm{id} + C_0^\intercal\right) \cdot H_2(z) \cdot \left(\mathrm{id} + \overline{C_0}\right).$$
In order to prove 1, we want $\mathrm{D}H(0) = 0$. Recall $H_2(0) = \mathrm{id}$. Then
$$\mathrm{D}H = \mathrm{D}H_2 + \mathrm{D}\left(\mathrm{id} + C_0^\intercal\right) \cdot H_2 + H_2 \cdot \mathrm{D}\left(\mathrm{id} + \overline{C_0}\right) + \mathcal{O}(|z|),$$
so
$$\mathrm{D}H(0) = \mathrm{D}H_2(0) + \mathrm{D}C_0^\intercal(0) + \mathrm{D}\overline{C_0}(0) = \left(\partial H_2(0) + \mathrm{D}C_0^\intercal(0)\right) + \left(\overline{\partial} H_2(0) + \mathrm{D}\overline{C_0}(0)\right).$$
Write $C_0 = (c_{ij})$, where
$$c_{ij} = -\sum_{l=1}^n \frac{\partial}{\partial z_l} (H_2)_{ji}(0)\, z_l.$$
Then
$$\mathrm{d}c_{ij} = -\sum_{l=1}^n \sum_{k=1}^n \frac{\partial}{\partial z_l} (H_2)_{ji}(0) \frac{\partial}{\partial z_k} z_l \mathrm{d}z_k = -\sum_{l=1}^n \frac{\partial}{\partial z_l} (H_2)_{ji}(0)\, \mathrm{d}z_l,$$
so
$$\mathrm{D}C_0^\intercal(0) = \partial C_0^\intercal(0) = -\partial H_2(0).$$
Similarly
$$\mathrm{D}\overline{C_0}(0) = \overline{\partial C_0}(0) = -\overline{\partial} H_2(0).$$
With this choice, we get $\mathrm{D}H(0) = 0$, so $H(z) = \mathrm{id} + \mathcal{O}\left(|z|^2\right)$.

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2. When we constructed ∇_E, we set $A = \overline{H}^{-1} \cdot \partial \overline{H}$ and we proved $\Theta_E(z) = \overline{\partial} A(z)$ in Corollary 5.11. Since $H(0) = \mathrm{id}$, $\mathrm{D}H(0) = 0$, so $\partial H(0) = 0$ and $\overline{\partial} H(0) = 0$. Then
$$\Theta_E(0) = \overline{\partial} A(0) = \overline{\partial}\left(\overline{H}^{-1} \cdot \partial \overline{H}\right)(0) = \overline{\partial H}^{-1}(0) \cdot \partial \overline{H}(0) + \overline{H}^{-1}(0) \cdot \overline{\partial} \partial \overline{H}(0) = \overline{\partial} \partial \overline{H}(0) = -\partial \overline{\partial H}(0),$$
since $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

∎

5.5 De Rham cohomology

Given a complex manifold X, we define

$$\begin{aligned} \mathcal{Z}^k(X) &= \mathrm{Ker}\left(\mathrm{d} : \mathrm{C}^\infty\left(X, \Omega^k_{X,\mathbb{C}}\right) \to \mathrm{C}^\infty\left(X, \Omega^{k+1}_{X,\mathbb{C}}\right)\right), \qquad k \ge 0 \\ &= \left\{\omega \in \mathrm{C}^\infty\left(X, \Omega^k_{X,\mathbb{C}}\right) \mid \mathrm{d}\omega = 0\right\}, \end{aligned}$$

and we define

$$\begin{aligned} \mathcal{B}^k(X) &= \mathrm{Im}\left(\mathrm{d} : \mathrm{C}^\infty\left(X, \Omega^{k-1}_{X,\mathbb{C}}\right) \to \mathrm{C}^\infty\left(X, \Omega^k_{X,\mathbb{C}}\right)\right), \qquad k \ge 1 \\ &= \left\{\omega \in \mathrm{C}^\infty\left(X, \Omega^k_{X,\mathbb{C}}\right) \;\middle|\; \exists \eta \in \mathrm{C}^\infty\left(X, \Omega^{k-1}_{X,\mathbb{C}}\right),\ \omega = \mathrm{d}\eta\right\}. \end{aligned}$$

For convenience, we define $\mathcal{B}^0 = 0$. Since $\mathrm{d} \circ \mathrm{d} = 0$, it follows that $\mathcal{B}^k(X) \subset \mathcal{Z}^k(X)$ for each $k \ge 0$. Thus, we may define

$$\mathrm{H}^k(X, \mathbb{C}) = \mathcal{Z}^k(X) / \mathcal{B}^k(X).$$

The group $\mathrm{H}^k(X, \mathbb{C})$ is called the **de Rham cohomology group** of X. If it is finite dimensional, then their dimension

$$\mathrm{h}^k(X) = \dim \mathrm{H}^k(X, \mathbb{C})$$

is called the **Betti number** of X.

Remark 5.13. If X and X' are diffeomorphic complex manifolds then $\mathrm{H}^k(X, \mathbb{C}) \cong \mathrm{H}^k(X', \mathbb{C})$ for any $k \ge 0$. The same result is not true for the Dolbeault cohomology groups.

5.6 Holomorphic line bundles

Let X be a complex manifold, let L be a complex line bundle, a vector bundle of rank one, and let ∇ be a connection on L. Then Θ_∇ is a C^∞-linear operator. The idea is $L^* \otimes L = \mathrm{Hom}(L, L)$, so $\Theta_\nabla \in \mathrm{C}^\infty\left(X, \Omega^2_{X,\mathbb{C}}\right)$.

Proposition 5.14.

1. *The curvature of ∇ defines a global 2-form $\Theta_\nabla \in \mathrm{C}^\infty\left(X, \Omega^2_{X,\mathbb{C}}\right)$ such that $\mathrm{d}\Theta_\nabla = 0$.*
2. *If ∇' is also a connection then there exists a 1-form η such that $\Theta_{\nabla'} - \Theta_\nabla = \mathrm{d}\eta$.*

Proof.

1. Let $x \in X$, and let $x \in U$ be open with a non-zero local section $s \in \mathrm{C}^\infty(U, L)$. There exists $A = (a)$ with the 1-form a on U representing ∇. That is, if $\sigma = fs \in \mathrm{C}^\infty(U, L)$ where $f \in \mathrm{C}^\infty(U)$, then $\nabla\sigma \overset{U}{=} f \cdot a + \mathrm{d}f$ and $\nabla^2\sigma = f \cdot \Theta_\nabla$. Recall that $\Theta_\nabla = a \wedge a + \mathrm{d}a = \mathrm{d}a$ is a 2-form on U, since a is a 1-form. Note that $\mathrm{d}\Theta_\nabla = \mathrm{d}^2 a = 0$, so 1 holds. Let $U' \subset X$ be another open set trivialising L, and let

   $$g : (U \cap U') \times \mathbb{C} \to (U \cap U') \times \mathbb{C}$$

   be the transition. Recall that if $(a') = A'$ is the matrix representing ∇ with respect to the trivialisation on U, then $a' = g^{-1} \cdot \mathrm{d}g + a$, so

   $$\mathrm{d}a' = \mathrm{d}\left(g^{-1} \cdot \mathrm{d}g\right) + \mathrm{d}a = g^{-2} \cdot \mathrm{d}g \wedge \mathrm{d}g + g^{-1} \cdot \mathrm{d}^2 g + \mathrm{d}a = \mathrm{d}a,$$

   since $\mathrm{d}g^{-1} = g^{-2} \cdot \mathrm{d}g$. [14] Thus $\Theta_{\nabla'} = \mathrm{d}a' = \mathrm{d}a$ does not depend on U, so $\Theta_{\nabla'}$ is a global 2-form on X. Lecture 19 Thursday 20/02/20
2. Let ∇' be also a connection on L. On U, let b be the 1-form representing ∇' so that $\Theta_{\nabla'} \overset{U}{=} \mathrm{d}b$, so $\Theta_{\nabla'} - \Theta_\nabla \overset{U}{=} \mathrm{d}(b - a)$. Let U', g, and a' be as above, and let b' be the 1-form representing ∇' on U'. Then

   $$b' - a' = \left(g^{-1} \cdot \mathrm{d}g + b\right) - \left(g^{-1} \cdot \mathrm{d}g + a\right) = b - a.$$

   Thus $\eta = b - a$ is a global 1-form.

∎

[14] Exercise

Remark 5.15. Thus, if L is a line bundle on a complex manifold X, there exists a 2-form Θ_∇ on X such that $[\Theta_\nabla]$ does not depend on ∇, and depends only on L, as an element in $\mathrm{H}^2(X, \mathbb{C})$, the de Rham cohomology over $\mathbb{C}$. We can define

$$\mathrm{c}_1(L) = \left[\frac{i}{2\pi}\Theta_\nabla\right] \in \mathrm{H}^2(X, \mathbb{C}),$$

the **first Chern class** of L. For vector bundles E of rank r on X, then we can define

$$\mathrm{c}_1(E) = \mathrm{c}_1(\Lambda^r E).$$

Let X be a complex manifold, and let (L, h) be a Hermitian holomorphic line bundle. Then there exists a unique Chern connection ∇_L compatible with h and such that $\nabla_L^{0,1} = \overline{\partial_L}$. Fix a non-vanishing section $s \in \mathrm{C}^\infty(U, L)$. Then $h(x) = \langle s, s \rangle_x : U \to \mathbb{R}$, because $\langle v, w \rangle = \overline{\langle w, v \rangle}$ and h is positive definite, so

$$\phi = -\log h(x),$$

the **weight** of (L, h) on U with respect to s, is well-defined, and $h = e^{-\phi}$. Let a be the 1-form defining ∇_L. Recall that

$$a = h^{-1} \cdot \partial h = e^\phi \cdot \partial e^{-\phi} = e^\phi \cdot \left(-e^\phi\right) \cdot \partial\phi = -\partial\phi,$$

so

$$\Theta_L = \Theta_{\nabla_L} = \mathrm{d}a = \left(\partial + \overline{\partial}\right)(-\partial\phi) = -\overline{\partial}\partial\phi = \partial\overline{\partial}\phi.$$

In particular Θ_L is a $(1,1)$-form on X.

Remark. Linear algebra. Let V be a vector space over $\mathbb{C}$ of dimension n. Then $V_\mathbb{R}$ is a vector space over $\mathbb{R}$ of dimension $2n$, so $V_\mathbb{C} = V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ is a vector space over $\mathbb{C}$ of dimension $2n$. There exists a conjugation

$$\begin{array}{ccc} V_\mathbb{C} & \longrightarrow & V_\mathbb{C} \\ v & \longmapsto & \overline{v} \end{array}.$$

Let $W_\mathbb{C} = V_\mathbb{C}^* = W^{1,0} \oplus W^{0,1}$. Then

$$\Lambda^k W_\mathbb{C} = \bigoplus_{p+q=k} W^{p,q}, \qquad W^{p,q} = \Lambda^p W^{1,0} \otimes \Lambda^q W^{0,1}.$$

There exists a conjugation on $\Lambda^k W_\mathbb{C}$. Then the eigenspace with respect to the eigenvalue one via the conjugation is the real forms on V.

Example. Let $V = \mathbb{C}^n$. Then $\mathrm{d}z_j + \mathrm{d}\overline{z_j}$ is real and $i(\mathrm{d}z_j - \mathrm{d}\overline{z_j})$ is real.

Back to L. Then

$$\overline{i\Theta_L} = -i\overline{\partial}\partial\phi = i\partial\overline{\partial}\phi = i\Theta_L,$$

so $\frac{i}{2\pi}\Theta_L$ is a real $(1,1)$-form. Thus the first Chern class of a holomorphic line bundle is defined by a real $(1,1)$-form.

Remark 5.16. Assume (L, h) is a holomorphic line bundle with $h = e^{-\phi}$ locally at $x \in X$. Then if $\left(L^{-1}, h'\right)$ is with respect to the induced frame, we can write $h' = e^\phi$.

Definition 5.17. Let (L, h) be a Hermitian holomorphic line bundle on X. Then L is **positive** if for all $x \in X$, $h = e^{-\phi}$ locally at x, such that

$$\left(\frac{\partial^2}{\partial z_i \partial \overline{z_j}}\phi\right)$$

is positive definite, where $z_1, \dots, z_n$ are local coordinates around x.

Example 5.18. Let $L = X \times \mathbb{C}$, let s be the constant section, and let $\phi = 1$ on X. Then $\Theta_L = 0$, so $\mathrm{c}_1(L) = 0$.

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Example 5.19. The Fubini-Study metric. Let $X = \mathbb{P}^n_{\mathbb{C}}$, let

$$\mathcal{O}(-1) = \{([x], v) \mid [x] \in \mathbb{P}^n_{\mathbb{C}},\ v = \lambda x,\ \lambda \in \mathbb{C}\},$$

and let $U_i = \{[x] \in \mathbb{P}^n_{\mathbb{C}} \mid x_i \neq 0\} \subset \mathbb{P}^n_{\mathbb{C}}$ be a trivialising open set. Then $\pi^{-1}(U_i) \cong U_i \times \mathbb{C}$. Define

$$\phi_i([x_0, \dots, x_n]) = -\log \frac{\sum_j |x_j|^2}{|x_i|^2} \in (0, \infty).$$

Then ϕ_i is well-defined. Claim that it defines h on $\mathcal{O}(-1)$. Let

$$g_{ij} : (U_j \cap U_i) \times \mathbb{C} \to (U_j \cap U_i) \times \mathbb{C}$$

be the transition $g_{ij} = x_i / x_j$ from U_j to U_i, and let $h_i = e^{-\phi_i}$ on U_i. Then

$$h_j = g_{ij} \cdot h_i \cdot \overline{g_{ij}},$$ [15]

which extends globally to X, is a metric on $\mathcal{O}(-1)$. Let $\mathcal{O}(1)$ be the dual of $\mathcal{O}(-1)$. Define

$$\psi_i = -\phi_i = \log \frac{\sum_j |x_j|^2}{|x_i|^2}.$$

Then ψ_i defines a metric on $\mathcal{O}(1)$. Claim that $\mathcal{O}(1)$ is positive, so

$$\left(\frac{\partial^2}{\partial z_i \partial \overline{z_j}} \phi \right)$$

is positive definite. Let us take $i = 0$, so $z_j = x_j / x_0$ are coordinates on $U_0 \cong \mathbb{C}^n$, and

$$\psi_0 = \log \left(1 + \sum_{j=1}^n |z_j|^2 \right).$$

Then

$$\frac{\partial}{\partial z_k} \left(\frac{\partial}{\partial \overline{z_l}} \psi_0 \right) = \frac{\partial}{\partial z_k} \left(\frac{z_l}{1 + \sum_j |z_j|^2} \right) = \frac{\delta_{kl} \left(1 + \sum_j |z_j|^2 \right) - z_l \overline{z_k}}{\left(1 + \sum_j |z_j|^2 \right)^2}.$$

Fix $z \in \mathbb{C}^n$, and let

$$T = \left(\frac{\partial^2}{\partial z_k \partial \overline{z_l}} \psi_0(z) \right).$$

We want to show that T is positive definite. If $n = 1$, then $T = \left(1 + |z|^2 \right)^{-2} > 0$, so ok. If $n > 1$, let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{C}^n$ and let $\| \cdot \|$ be the norm induced by it. For each $w \in \mathbb{C}^n$, we have

$$\langle Tw, w \rangle = \frac{\left(1 + \|z\|^2 \right) \|w\|^2 - |\langle z, w \rangle|^2}{\left(1 + \|z\|^2 \right)^2}$$

The Cauchy-Schwarz inequality implies $|\langle z, w \rangle|^2 \le \|z\|^2 \|w\|^2$. Thus,

$$\langle Tw, w \rangle \ge \frac{\|w\|^2}{\left(1 + \|z\|^2 \right)} \ge 0.$$

and the equality holds if and only if $w = 0$. Thus T is positive definite and $\mathcal{O}(1)$ is a positive line bundle.

[15] Exercise

6 Kähler manifolds

The idea is if (X, ω) is compact Kähler and $k \geq 0$, then

$$\mathrm{H}^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} \mathrm{H}^{p,q}(X).$$

6.1 Kähler manifolds

Definition 6.1. Let $V \subset \mathbb{C}^n$ be open. A **positive** real $(1,1)$-form on V is a real $(1,1)$-form

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk} \mathrm{d}z_j \wedge \mathrm{d}\overline{z_k},$$

such that (h_{jk}) is positive definite.

If $z_1 = x_1 + iy_1$, then $\frac{i}{2} \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} = \mathrm{d}x_1 \wedge \mathrm{d}y_1$. Then ω defines a Hermitian metric on $\mathrm{T}_{V,\mathbb{C}} = \left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_j}} \right\rangle$. Let ω be a positive real $(1,1)$-form as in the above. Then ω^n is a real (n,n)-form such that if $z_j = x_j + iy_j$ then

$$\omega^n = \det h_{ij} \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n$$

is a volume form.

Definition 6.2. Globally, let X be a complex manifold, and let $\omega \in \mathrm{C}^\infty\left(X, \Omega_X^{1,1}\right)$ be a real $(1,1)$-form. Then ω is said to be **positive** if for all $x \in X$, there exists an open $U \ni x$ and there exists a biholomorphism $\phi : U \to V \subset \mathbb{C}^n$ such that $\left(\phi^{-1}\right)^* \omega$ is a positive $(1,1)$-form on V.

In particular ω^n is a volume form on X, so X is oriented.

Definition 6.3. A complex manifold X is called **Kähler** if there exists a positive real $(1,1)$-form ω on X such that $\mathrm{d}\omega = 0$. Such ω is called a **Kähler form** on X.

Notation. (X, ω), where X is a Kähler manifold and ω is a Kähler form.

Example 6.4. Let $X = \mathbb{C}^n$, and let $\omega = \frac{i}{2} \sum_j \mathrm{d}z_j \wedge \mathrm{d}\overline{z_j}$. Then ω is Kähler, so $\mathbb{C}^n$ is Kähler.

Example 6.5. Let $X = \mathbb{C}^n / \Lambda$ be the complex torus for a lattice $\Lambda \subset \mathbb{C}^n$. Claim that X is Kähler. Let ω be as in the previous example. Consider

$$\begin{array}{cccc} \psi & : & \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ & & z & \longmapsto & z + \lambda \end{array},$$

for some fixed $\lambda \in \Lambda$. Then $\psi^* \omega = \omega$, so ω descends to a positive closed real $(1,1)$-form on X, that is there exists ω' on X such that $q^* \omega' = \omega$ for $q : \mathbb{C}^n \to X$. Thus X is Kähler.

Example 6.6. Let $X = \mathbb{P}^n_{\mathbb{C}}$. Recall that if h is the Fubini-Study metric on $\mathcal{O}(1)$. Then $i\Theta_h$ is a real positive $(1,1)$-form and $\mathrm{d}\Theta_h = 0$, so X is Kähler.

Lemma 6.7. *Let X be a complex manifold, let ω be a Kähler form on X, and let $i : Y \hookrightarrow X$ be an immersion for a complex submanifold Y. Then $i^* \omega$ is a Kähler form on Y. In particular Y is Kähler.*

Proof. Exercise. [16] □

Corollary 6.8. *Let X be a projective manifold. Then X is Kähler.*

Proof. X, by definition, is a complex submanifold of $\mathbb{P}^n_{\mathbb{C}}$. By the previous example $\mathbb{P}^n_{\mathbb{C}}$ is Kähler, so X is Kähler by Lemma 6.7. □

Fact. Every compact complex submanifold of $\mathbb{P}^n_{\mathbb{C}}$ is a projective manifold.

Example. Let X be a complex manifold of dimension one. Then X is Kähler. [17]

Lecture 21 is a problems class.

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[16] Exercise
[17] Exercise

Let (X, ω) be compact Kähler. For all $x \geq 1$, $\omega^k = \omega \wedge \dots \wedge \omega$ is closed by Leibnitz rule. Claim that $\left[\omega^k\right] \neq 0$ in $\mathrm{H}^k(X, \mathbb{C})$. Assume $\left[\omega^k\right] = 0$, so there exists a $(2k-1)$-form η such that $\omega^k = \mathrm{d}\eta$. Since ω is closed and ω^n is a volume form, Lecture 22 Thursday 27/02/20

$$0 < \int_X \omega^n = \int_X \omega^{n-k} \wedge \mathrm{d}\eta = \int_X \mathrm{d}\left(\omega^{n-k} \wedge \eta\right) = \int_{\partial X} \omega^{n-k} \wedge \eta = 0,$$

by the Leibnitz rule. Thus

$$\mathrm{H}^k(X, \mathbb{C}) \neq 0, \qquad k \in 2\mathbb{Z}.$$

Example 6.9. Pick $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < 1$. Then $\mathbb{Z}$ acts on $\mathbb{C}^n \setminus \{0\}$ by

$$\begin{array}{ccc} (\mathbb{Z} \times (\mathbb{C}^n \setminus \{0\})) & \longrightarrow & \mathbb{C}^n \setminus \{0\} \\ (n, z) & \longmapsto & \lambda^n z \end{array}.$$

Let $X = \mathbb{C}^n \setminus \{0\} / \mathbb{Z}$. Similarly to the case of complex tori, X can be shown to have a complex structure. Then $\mathrm{S}^{2n-1} \subset \mathbb{R}^{2n} \setminus \{0\} = \mathbb{C}^n \setminus \{0\} \cong \mathrm{S}^{2n-1} \times \mathbb{R}_{>0}$, so $X \sim \mathrm{S}^{2n-1} \times \mathrm{S}^1$. Thus if $n \geq 2$, then $\mathrm{H}^k(X, \mathbb{C}) = 0$, so X is not Kähler.

6.2 Hodge $\star$ operator

Let V be a vector space over $\mathbb{C}$ of dimension n with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$. There is a canonical inner product on $\Lambda^p V$ for all $p \geq 1$,

$$\langle v_1 \wedge \dots \wedge v_p, w_1 \wedge \dots \wedge w_p \rangle = \det\left(\langle v_i, w_j \rangle\right).$$

In particular there exists a unique up to orientation $\omega \in \Lambda^n V$ such that $\|\omega\| = 1$. The **Hodge $\star$ operator** $\star : \Lambda^p V \to \Lambda^{n-p} V$ for $p \geq 0$ is such that

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega \in \Lambda^n V, \qquad \alpha, \beta \in \Lambda^p V.$$

Let $e_1, \dots, e_n$ be an orthonormal basis of V. Then

- $\star 1 = \omega$,
- $\star \omega = 1$,
- $\star e_1 = e_1 \wedge \dots \wedge e_n$,
- $\star e_i = (-1)^{i-1} e_1 \wedge \dots \wedge \widehat{e_i} \wedge \dots \wedge e_n$, and
- more in general if $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ is ordered such that $e_I = e_{i_1} \wedge \dots \wedge e_{i_k}$, and $\mathcal{C}I = \{1, \dots, n\} \setminus I$ such that $\sigma : \{1, \dots, n\} \to \{I, \mathcal{C}I\}$ is a permutation, then

  $$\star e_I = \epsilon(\sigma) e_{\mathcal{C}I},$$

  where ϵ is the signature of σ, so

  $$\star \star \eta = (-1)^{k(n-k)} \eta, \qquad \eta \in \Lambda^k V.$$

Let X be a complex manifold, and let E be a Hermitian holomorphic vector bundle. Recall that we defined

$$\{\cdot, \cdot\} : \mathrm{C}^\infty\left(X, \Omega^p_{X,\mathbb{C}} \otimes E\right) \times \mathrm{C}^\infty\left(X, \Omega^q_{X,\mathbb{C}} \otimes E\right) \to \mathrm{C}^\infty\left(X, \Omega^{p+q}_{X,\mathbb{C}}\right).$$

Take $p = q = 0$ and $E = \Omega^k_{X,\mathbb{C}}$. If ω is a positive real $(1,1)$-form then ω induces a Hermitian metric on $\mathrm{T}_{X,\mathbb{C}}$. Locally, let $e_1, \dots, e_n$ be an orthonormal frame of $\mathrm{T}_{X,\mathbb{C}}$. Then $e_1^*, \dots, e_n^*$ define a metric on $\Omega^1_{X,\mathbb{C}}$ locally, where $e_i^*(e_j) = \delta_{ij}$. It is easy to check that such a choice is canonical, so the metric on $\Omega^1_{X,\mathbb{C}}$ extends to X. This induces a metric on $\Omega^k_{X,\mathbb{C}}$ for all $k \geq 0$, so there exists a cup product

$$\{\cdot, \cdot\} : \mathrm{C}^\infty\left(X, \Omega^k_{X,\mathbb{C}}\right) \times \mathrm{C}^\infty\left(X, \Omega^k_{X,\mathbb{C}}\right) \to \mathrm{C}^\infty(X).$$

Lemma 6.10. *Let (X, ω) be Kähler of dimension n. Then there exists a $\mathbb{C}$-linear*

$$\star : \mathrm{C}^\infty\left(X, \Omega^k_{X,\mathbb{C}}\right) \to \mathrm{C}^\infty\left(X, \Omega^{n-k}_{X,\mathbb{C}}\right), \qquad k \geq 0,$$

such that

$$\alpha \wedge \star\beta = \{\alpha, \beta\}\,\omega^n, \qquad \alpha, \beta \in \mathrm{C}^\infty\left(X, \Omega^k_{X,\mathbb{C}}\right).$$

Check that it is defined globally on X.

Definition 6.11. Let E be a vector bundle. Then let

$$\mathrm{C}^\infty_\mathrm{c}(X, E) = \{s \in \mathrm{C}^\infty(X, E) \mid s \text{ has compact support}\}.$$

Let E be Hermitian, and let ω be a positive real $(1,1)$-form. Then let

$$(\alpha, \beta)_E = \int_X \{\alpha, \beta\}\,\omega^n, \qquad \alpha, \beta \in \mathrm{C}^\infty_\mathrm{c}(X, E).$$

Let (X, ω) be Kähler, let E and F be Hermitian vector bundles on X, and let $P : \mathrm{C}^\infty_\mathrm{c}(X, E) \to \mathrm{C}^\infty_\mathrm{c}(X, F)$ be $\mathbb{C}$-linear. Then the **adjoint** of P is a $\mathbb{C}$-linear map $P^* : \mathrm{C}^\infty_\mathrm{c}(X, F) \to \mathrm{C}^\infty_\mathrm{c}(X, E)$ such that

$$(P\alpha, \beta)_F = (\alpha, P^*\beta)_E, \qquad \alpha \in \mathrm{C}^\infty_\mathrm{c}(X, E), \qquad \beta \in \mathrm{C}^\infty_\mathrm{c}(X, E).$$

In particular $\mathrm{d} : \mathrm{C}^\infty_\mathrm{c}\left(X, \Omega^k_{X,\mathbb{C}}\right) \to \mathrm{C}^\infty_\mathrm{c}\left(X, \Omega^{k+1}_{X,\mathbb{C}}\right)$ gives $\mathrm{d}^* : \mathrm{C}^\infty_\mathrm{c}\left(X, \Omega^{k+1}_{X,\mathbb{C}}\right) \to \mathrm{C}^\infty_\mathrm{c}\left(X, \Omega^k_{X,\mathbb{C}}\right)$.

Lemma 6.12. *Let (X, ω) be Kähler of dimension n. Then*

$$\mathrm{d}^*\beta = (-1)^{nk+1} \star \mathrm{d}(\star\beta), \qquad \beta \in \mathrm{C}^\infty_\mathrm{c}\left(X, \Omega^{k+1}_{X,\mathbb{C}}\right).$$