M4P61 Infinite Groups

Lectured by Dr Isabel Müller
Typed by David Kurniadi Angdinata

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Syllabus

Contents

1 Introduction

Lecture 1 Thursday 03/10/19

Groups are ubiquitous throughout almost all areas in mathematics and many areas in physics. They arise naturally as the symmetries of classical mathematical objects, that is bijective maps which preserve the structure of the object studied. Well known groups include $\mathcal{S}_n$, the group of symmetries of a set of size n, or $\mathcal{D}_n$, the group of symmetries of a regular n-gon. From linear algebra we also know $\mathrm{GL}_n(\mathbb{R})$, the group of all invertible linear transformations of the vector space $\mathbb{R}^n$, and $\mathrm{O}(n)$, its subgroup of isometries. Historically, groups appeared for the first time in the work of Galois, when he tried to understand solutions of polynomial equations by studying the group of symmetries of their roots. He was the first to use the word group in the modern sense and that work dates back to 1829, when he was 18 years old. Another main contribution to the study of groups in mathematics came from Felix Klein's Erlangen program in 1872, in which he aimed to understand and classify euclidean, affine, projective, etc geometries by studying their group of symmetries. A huge milestone in the study of groups has been the classification of finite simple groups, which is a result based on the accumulated work of more than 100 authors on tens of thousands of pages published between 1954 and 2004.

This course will focus on infinite groups. More specifically, we will aim to study and understand groups by their actions on geometric objects. In that sense, we can consider the course program as an inverse of Klein's Erlangen program. This area of mathematics goes back to the 1980s, hence is comparably new, and is nowadays wider known as geometric group theory. The two leading questions will roughly be the following.

- If we know that a given group G admits an action with properties P on a space of type T, what does this tell me about the group G itself?
- Assume we are given a group G. Does it act on a given space T with properties P?

Our main goal in the first part will be the fundamental theorem of Bass-Serre theory, which states that a group acting on a tree is the fundamental group of a graph of groups. We first will introduce the notion of graphs in the sense of Serre and study group actions on these graphs. Afterwards, we will introduce free groups as the universal object in the class of groups and study how groups can arise as fundamental groups of graphs. We will see that groups can be presented by giving a set of generators accompanied with a set of relators and point out advantages and disadvantages of this viewpoint on groups.

In the second chapter, we will learn how to construct new groups out of given data via free products, free amalgamated products and HNN extensions. The counterpoint to this, that is the question on whether a given group decomposes into the amalgamated product or HNN extension of other groups, will be of special interest and we will approach it by understanding their actions on trees. This second part concludes with the introduction of graphs of groups and the fundamental theorem of Bass-Serre theory.

In the last part, we will investigate the word problem and its solvability in specific classes of groups. The word problem asks if two words on the generators of some group G represent the same element in it. Even for finitely presentable groups, the word problem is not always solvable, that is decidable. We will get to know Hopfian and residually finite groups as examples of classes in which the words problem actually is solvable. If time permits, we will conclude the lecture with an introduction into hyperbolic groups.

The following are reading material.

- R C Lyndon and P E Schupp, Combinatorial group theory, 2001
- P de la Harpe, Topics in geometric group theory, 2000
- O Bogopolski, Introduction to group theory, 2008
- J Rotman, An introduction to the theory of groups, 1995
- W Magnus, A Karrass, and D Solitar, Combinatorial group theory, 2005
- D Robinson, A course in the theory of groups, 1993

2 Geometric group theory

2.1 Bass-Serre graphs

Definition 2.1.1. A **graph** X is a tuple consisting of a set of vertices X^0, a set of edges X^1, together with functions $\alpha, \omega : X^1 \to X^0$ and $\bar{\cdot} : X^1 \to X^1$, such that $\overline{\overline{e}} = e$ and $\alpha(\overline{e}) = \omega(e)$ for every $e \in X^1$. We call $\alpha(e)$ the **initial vertex**, $\omega(e)$ the **terminal vertex**, and $\overline{e}$ the **inverse vertex**.

A convention is that unless otherwise specified, we identify edges e and e' if $\alpha(e) = \alpha(e')$ and $\omega(e) = \omega(e')$. The following are translations of notions.

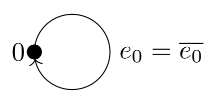
- A subgraph is an **induced** subgraph.
- A graph homomorphism ϕ from X to Y is a mapping from $X^i \to Y^i$ for $i = 0, 1$ such that $\phi(\alpha(e)) = \alpha(\phi(e))$ and $\phi(\overline{e}) = \overline{\phi(e)}$.
- Given $x \in X^0$, then we call the set $\{e \mid \alpha(e) = x\}$ the **star** of x, or $\operatorname{star} x$. The cardinality of $\operatorname{star} x$ is called the **valency** of x.
- A homomorphism $\phi : X \to Y$ is **locally injective** if and only if its restriction to $\operatorname{star} x$ is injective for all $x \in X^0$.
- An **orientation** of X is a choice of vertices $X^1_+ \subseteq X^1$ which picks exactly one of each pair $\{e, \overline{e}\}$.

Example 2.1.2.

- Fix $n \in \mathbb{N}_{\geq 1}$ for $n \neq 2$. Set

$$\mathcal{C}^0_n = \{0, \dots, n-1\}, \qquad \mathcal{C}^1_n = \{e_i, \overline{e_i} \mid i < n\}, \qquad \omega(e_i) = \alpha(e_{i+1}) = i + 1 \mod n_i, \qquad i < n.$$

  Then $\mathcal{C}_1$ is



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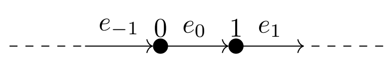
- $\mathcal{C}_\infty$ is given by

$$\mathcal{C}^0_\infty = \mathbb{Z}, \qquad \mathcal{C}^1_\infty = \{e_i, \overline{e_i} \mid i \in \mathbb{Z}\}, \qquad \omega(e_i) = \alpha(e_{i+1}).$$

  Then $\mathcal{C}_\infty$ is



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The graphs $\mathcal{C}_n$ and $\mathcal{C}_\infty$ for $n \neq 2$ are called **circuits**.

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A sequence $p = e_1 \dots e_n$ with $e_i \in X^1$ is called a **path** from $\alpha(e_1) = x_0$ to $x_n = \omega(e_n)$ if and only if $\omega(e_i) = \alpha(e_{i+1})$ for all $i < n$. We consider vertices to be paths of length zero. A path is called **reduced** if $\overline{e_i} \neq e_{i+1}$. If p is a path, then $p^{-1} = \overline{e_n} \dots \overline{e_1}$ is called its **inverse path**. A path is called a **closed path** if $\omega(e_n) = \alpha(e_1)$.

Note.

- If we have a path p given, we can naturally consider it to be a subgraph via

$$X^0_p = \{\alpha(e_i) \mid i < n\} \cup (\omega(e_n)), \qquad X^1_p = \{e_1, \dots, e_n, \overline{e_1}, \dots, \overline{e_n}\}.$$

- If $p = e_1 \dots e_n$ is closed, then a permutation of the form

$$e_{i+1} \dots e_n e_1 \dots e_i$$

  is called a **cyclic permutation**. p is called **cyclically reduced** if every cyclic permutation is reduced.

Exercise 2.1.3.

- Let $\phi : X \to Y$ be a morphism of graphs. Then ϕ is locally injective if and only if the image of any reduced path is reduced.
- If p is closed and reduced, then it contains a circuit as a substructure.

If $p = e_1 \dots e_n$ and $q = f_1 \dots f_m$ such that $\omega(e_n) = \alpha(f_1)$ then we denote by

$$pq = e_1 \dots e_n f_1 \dots f_m.$$

A graph X is **connected** if for any $x, y \in X^0$ there is a path from x to y. A connected graph without circuits is called a **tree**.

Exercise 2.1.4.

- X is a tree if and only if for all $x, y \in X^0$ there is a unique reduced path from x to y.
- If X is connected and T is a tree, then any $\phi : X \to T$ locally injective is already injective and X is a tree.

Lemma 2.1.5. *Let X be a connected graph and $T \subseteq X$ a maximal subtree of X, then $T^0 = X^0$.*

Proof. Otherwise, there is some $x \in X^0 \setminus T^0$. As X is connected, there is some path p starting in T, ending in x. As $x \notin T^0$, there exists an edge in p such that $\alpha(e) \in T^0$ and $\omega(e) \notin T^0$. But then

$$T' = \left(T^0 \cup \{\omega(e)\}, T^1 \cup \{e\}\right)$$

is again a tree, a contradiction. $\square$

Such a tree T is called a **spanning tree** for X.

2.2 Cayley graphs

Definition 2.2.1. Let G be a group and X a graph. We say that G **acts** on X if and only if it acts on X^0 and X^1 as sets, such that

- $g \cdot \alpha(e) = \alpha(g \cdot e)$, and
- $g \cdot \overline{e} = \overline{g \cdot e}$.

Note. This just means that

$$\begin{array}{cccc} \phi_g & : & X^0 & \longrightarrow & X^0 \\ & & x & \longmapsto & g \cdot x \end{array}$$

is a morphism of graphs for any $g \in G$.

Notation. gh is multiplication and $g \cdot h$ is action.

Remark. Given G and X arbitrary, then G acts on X by $g \cdot x = x$ and $g \cdot e = e$. Hence we will ask for nice properties of the action.

Definition 2.2.2. Assume G acts on a graph X. Then we say that G **acts without inversion of edges**, if $g \cdot e \neq \overline{e}$ for all $e \in X^1$. We say that G **acts freely** on X, if $g \cdot x = x$ if and only if $g = e_G$.

Definition 2.2.3. Let G be a group and $S \subseteq G \setminus \{e_G\}$.

- We say that S **generates** G, or G is **generated** by S, if there is no proper subgroup of G containing S. That is, the smallest subgroup H containing S equals G.
- If S has some property P, then we say that G is P**-ly generated**. For example, if S is finite, then G is finitely generated.
- If P is a property of subgroups, then S P**-ly generates** G, if the smallest subgroup of G containing S with property P, is already G. For example, if the smallest normal subgroup of G containing S is already G, then S normally generates G.

Example 2.2.4. $(\mathbb{Z}, +)$ is generated by $\{1\}$ or $\{-1\}$ or $\{-1, 1\}$ or $\{2, 3\}$ or $\mathbb{Z} \setminus \{0\}$.

Example 2.2.5. Let G be an infinite simple group. Then it is normally generated by any $g \in G \setminus \{e_G\}$. A question is can it be generated by g? No. G is cyclic and simple if and only if $G = \mathbb{Z}/p\mathbb{Z}$ for p prime. [1] A_∞ is an infinite simple group.

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Definition 2.2.6. Assume G is a group and $S \subseteq G \setminus \{e_G\}$. Then we define the graph $\Gamma(G, S)$ via

- the vertex set is $\Gamma(G, S)^0 = G$,
- the set of positive edges is $\Gamma(G, S)^1_+ = G \times S$,
- for e an edge, we have $\alpha((g, s)) = g$ and $\omega((g, s)) = gs$, and
- the inverse of (g, s) is $\overline{(g, s)} = \left(gs, s^{-1}\right)$, where
$$S^{-1} = \left\{s^{-1} \mid s \in S\right\}$$
is a set of new formal symbols. Thus $\left(g, s^{-1}\right) \notin G \times S$, even if as elements $s^{-1} = s' \in S$. If $s = s^{-1}$, this avoids troubles.

We consider $\Gamma(G, S)$ to be a labelled graph, where the label of (g, s) is s.

Exercise 2.2.7.

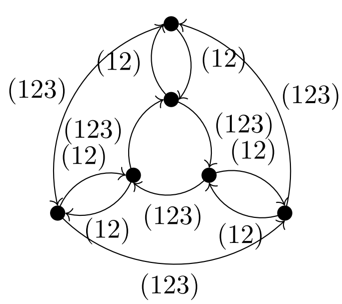
- $\Gamma(G, S)$ is connected if and only if S is a generating set for G.
- Otherwise set $H = \langle S \rangle \subsetneq G$. How does H relate to $\Gamma(G, S)$?

Definition 2.2.8. If G is a group and $S \subseteq G \setminus \{e_G\}$ generates G, then $\Gamma(G, S)$ is called the **Cayley graph** of G with respect to S.

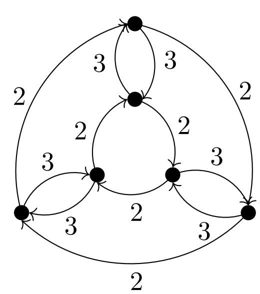
Exercise 2.2.9. Given S a connected graph. Is there a group G and $S \subseteq G \setminus \{e_G\}$ such that $X \cong \Gamma(G, S)$, where S is a generating set?

Example 2.2.10.

- Recall $\mathcal{C}_n$ and $\mathcal{C}_\infty$. Then
$$\mathcal{C}_n \cong \Gamma(\mathbb{Z}/n\mathbb{Z}, \{1\}), \qquad \mathcal{C}_\infty \cong \Gamma(\mathbb{Z}, \{1\}).$$
- Careful. Cayley graphs depend heavily on the choice of S. It is not always easy to determine whether it is cyclic. Consider
$$\Gamma_1 = \Gamma(\mathcal{S}_3, \{(123), (12)\}), \qquad \Gamma_2 = \Gamma(\mathbb{Z}/6\mathbb{Z}, \{2, 3\}).$$
Then



Γ_1



Γ_2

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Given Γ_i, is the group abelian? A group is abelian if and only if all its generators commute, that is $ab = ba$. For Γ_2, if $a = 2$ and $b = 3$, then $(2)(3) = (3)(2)$.

[1] Exercise

Lemma 2.2.11. *Every group G acts on its Cayley graph by left multiplication. The multiplication is free, label-preserving, and without inversion of edges. Furthermore, every ϕ_g is a label-preserving automorphism of $\Gamma(G,S)$.*

Proof. Define the action via $h \cdot g = hg$ for all $h \in G$ and $g \in \Gamma(G,S)^0$, and $h \cdot (g,s) = (hg,s)$. One checks easily that this defines an action. It is obviously label-preserving and hence without inversion of edges, as positive and negative edges have disjoint label sets. Now, if $h \cdot g = g$, then $hg = g$ and this $h = e_G$. Hence the action is free. Clearly, ϕ_h is injective, as $\phi_h(g_1) = \phi_h(g_2)$ if and only if $hg_1 = hg_2$ if and only if $g_1 = g_2$. For surjectivity, note that $g = hh^{-1}g$ and hence $g = h \cdot \left(h^{-1}g\right) = \phi_h\left(h^{-1}g\right)$. ☐

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Lemma 2.2.12. *Let G be some group and $S \subseteq G \setminus \{e_G\}$ a generating set. Denote by $\mathrm{Aut}_L\,\Gamma(G,S)$ the label-preserving automorphism group of its Cayley graph. Then*

$$G \cong \mathrm{Aut}_L\,\Gamma(G,S).$$

Proof. By 2.2.11 we know that

$$\begin{array}{rccl} \Phi & : & G & \longrightarrow & \mathrm{Aut}_L\,\Gamma(G,S) \\ & & h & \longmapsto & \phi_h \end{array}.$$

One easily checks that this is a group homomorphism. If $\phi_h = \phi_g$, then in particular they agree on the vertex e_G, that is $h = \phi_h(e_G) = \phi_g(e_G) = g$, so $g = h$ and Φ is injective. Now consider $\phi \in \mathrm{Aut}_L\,\Gamma(G,S)$ arbitrary. We claim that $\phi = \phi_h$ with $h = \phi(e_G)$. As ϕ is label-preserving and every vertex has exactly one outgoing and one incoming edge with label s, we know that $\phi((g,s)) = (\phi(g), s)$. Hence

$$\phi(\omega((g,s))) = \omega(\phi((g,s))) = \omega((\phi(g), s)) = \phi(g)\,s.$$

As $\Gamma(G,S)$ is connected, we get that two label-preserving automorphisms agree if and only if they agree on one vertex. Now, $\phi(e_G) = h = \phi_h(e_G)$, so $\phi = \phi_h$. ☐

Example 2.2.13. The group of all automorphisms of $\mathcal{C}_n$ is called the **dihedral group** and denoted by $\mathcal{D}_n$. Note that every such automorphism is uniquely determined by its image on e_0. Hence if we consider $\alpha(e_0) = e_1$ and $\beta(e_0) = \overline{e_{n-1}}$, then

$$\mathcal{D}_n = \left\{a^k, a^k b \;\middle|\; k < n\right\}, \qquad \mathcal{D}_\infty = \left\{a^k, a^k b \;\middle|\; k \in \mathbb{Z}\right\}.$$

Exercise 2.2.14.

- Draw the Cayley graphs of $\mathcal{D}_n$ with respect to $S = \{a,b\}$.
- Prove that $\mathcal{D}_3 \cong \mathcal{S}_3$.
- Determine the axis of the reflection and the representation a^k and $a^k b$ for given ϕ just by using $\omega(\phi(e_0))$ and $\alpha(\phi(e_0))$.

2.3 Words and paths

Note. If for some group element g, both $g = s_1$ and $g^{-1} = s_2$ are in S, then we distinguish the edges $e_1 = (e_G, s_1)$ and $e_2 = \left(e_G, s_2^{-1}\right)$ even though $\alpha(e_1) = e_G = \alpha(e_2)$ and $\omega(e_1) = s_1 = g = s_2^{-1} = \omega(e_2)$.

Definition 2.3.1. Let S be any set. We say that w is a **word** on S if and only if it is a finite sequence of the form

$$w = s_1^{\epsilon_1} \dots s_n^{\epsilon_n}, \qquad s_i \in S, \qquad \epsilon_i = -1, 1.$$

We call S an **alphabet** and elements of S are **letters**. If $S \subseteq G$, then every word in S considered as a product, defines some group element. We write

$$w \underset{G}{=} s_1^{\epsilon_1} \dots s_n^{\epsilon_n} \underset{G}{=} g,$$

and we say that w **represents** G.

Example 2.3.2. Consider $\mathbb{Z}$ with $S = \{s_0 = -1, s_1 = 1\}$. Then $w_1 = s_0 s_1 \neq s_1^{-1} s_1 = w_2$ but $w_1 \underset{G}{=} w_2$.

Remark 2.3.3. If S is a generating set for G, then for every $g \in G$, every word in S corresponds to a unique path $p_w(g)$ in the Cayley graph starting at g and ending at gh, where $h \underset{G}{=} w$.

Example 2.3.4. Let $\mathbb{Z} \times \mathbb{Z}$ and $S = \{a = (1,0), b = (0,1)\}$. Consider

$$w_1 = aabbab^{-1}, \qquad w_2 = baaa, \qquad w_3 = aba^{-1}a^{-1}.$$

Then $w_1 \underset{G}{=} w_2$ and $w_3 \underset{G}{=} ba^{-1} \underset{G}{=} a^{-1}b$.

Definition 2.3.5. A word $w = s_1^{\epsilon_1} \dots s_n^{\epsilon_n}$ on S is called **reduced** if and only if $s_i = s_{i+1}$ implies that $\epsilon_i = \epsilon_{i+1}$.

Consider $s \in G$ with $s^2 = 1$. Then $s \underset{G}{=} s^{-1}$ and $w = ss^{-1}$ is not reduced. But $w' = ss$ is reduced.

2.4 Free groups

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Fact 2.4.1 (Tits alternative)**.** If G is an infinite linear group, then either it is virtually solvable, that is there is a finite index subgroup which is solvable, or it contains a non-abelian free group as a subgroup.

Definition 2.4.2 (Free groups I)**.** Let G be a group and $S \subseteq G \setminus \{e_G\}$ be any subset of G. Then G is called **free on** S, or a **free group with basis** S, if and only if every element of G can be represented uniquely as a reduced word on S.

Remark. This implies that S generates G.

Example. Let G be finite. Then for all $s \in S$ there exists $n \in \mathbb{N}_{>0}$ such that $s^n \underset{G}{=} e_G$, so not unique.

Exercise 2.4.3 (Free groups II)**.** A group G is **free on** $S \subseteq G$ if and only if $\Gamma(G, S)$ of G with respect to S is a tree and $S \cap S^{-1} = \emptyset$, considered as an intersection in G.

Example 2.4.4. Consider the subgroup $F \subseteq \mathrm{SL}_2(\mathbb{Z})$ generated by

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Then F is free on $S = \{A, B\}$. First note that

$$A^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}, \qquad B^n = \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}.$$

Clearly, F acts on $\mathbb{R}^2$. Set

$$U = \{(x, y) \mid |x| < |y|\}, \qquad V = \{(x, y) \mid |x| > |y|\}.$$

Then

$$\begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2ny \\ y \end{pmatrix},$$

so $A^n(U) \subseteq V$ for all $n \geq 1$. Similarly, $B^n(V) \subseteq V$. Assume

$$w = A^{k_0} B^{l_0} \dots A^{k_{n-1}} B^{k_{n-1}} A^{k_n}, \qquad |k_i|, |l_i| > 0, \qquad k_0, l_0, k_n \geq 0$$

is an arbitrary word on S. Assume $w \underset{G}{=} e_G = I$. Now, if $|k_n|, |k_0| > 0$ then $w(U) \subseteq V$. But $U \cap V = \emptyset$, a contradiction. Otherwise consider w', the word which arises by conjugating by a high enough power of A, so $w' = A^N w A^{-N}$. Then w' is of the above form. But $w' \underset{G}{=} e_G$ if and only if $w \underset{G}{=} e_G$, a contradiction.

Remark. This proof generalises to the so-called ping pong lemma, telling when two subgroups $\langle A \rangle$ and $\langle B \rangle$ appear as a free product $\langle A \rangle * \langle B \rangle$.

Lecture 6 Wednesday 16/10/19

Lecture 6 is a problem class.

Lecture 7 Thursday 17/10/19

Proposition 2.4.5 (Free groups III)**.** *Let S be an arbitrary non-empty set. Then there is a* ***group*** $\mathrm{F}(S)$ ***which is free on*** *S.*

Proof.

- Set $S^{\pm} = \{s^{\epsilon} \mid \epsilon = -1, 1\}$. If $s_1 = s_2^{-1} \in S^{\pm}$, identify $s_1^{-1} = \left(s_2^{-1}\right)^{-1} = s_2$. Set $\mathrm{F}'(S)$ to be the set of all words on S. As usual, we denote the empty word by ϵ. Further, for two words u_1 and u_2 given by $u_i = s_{i,1}^{\epsilon_{i,1}} \dots s_{i,n_i}^{\epsilon_{i,n_i}}$ let $u_1 u_2 = s_{1,1}^{\epsilon_{1,1}} \dots s_{1,n_1}^{\epsilon_{1,n_1}} s_{2,1}^{\epsilon_{2,1}} \dots s_{2,n_2}^{\epsilon_{2,n_2}}$. Note that this is not a group, since no inverses.

- We will define an equivalence relation on $\mathrm{F}'(S)$. Say that $u \sim v$ if and only if there exists a finite sequence of words such that $u = u_0, \dots, u_n = v$ and each u_{i+1} arises from u_i by inserting or deleting a subword of the form ss^{-1} for $s \in S^{\pm}$. We say that u is reduced, if it does not contain a subword ss^{-1}.

- Claim that every equivalence class contains exactly one reduced word. Assume $u \sim v$. Then there exists $u = u_0, \dots, u_n = v$. Choose this sequence such that $\sum_{i=0}^{n} |u_i|$ is minimal, where $|u_i|$ denotes the word length of u_i. As u and v are reduced, we know that $|u_0| < |u_1|$ and $|u_n| < |u_{n-1}|$. Then there exists $0 < i < n$ such that $|u_{i-1}| < |u_i|$ and $|u_{i+1}| < |u_i|$. Say, u_i arises from u_{i-1} by adding ss^{-1} and u_{i+1} from u_i by deleting tt^{-1}. Now either ss^{-1} and tt^{-1} are disjoint in u_i, then replace the sequence $u_{i-1} u_i u_{i+1}$ by $u_{i-1} u_i' u_{i+1}$ where u_i' arises from u_{i-1} by deleting tt^{-1}, or not, then cancelling the subsequence $u_i u_{i+1}$ still gives a connecting sequence from u to v. In both cases we obtain a sequence of smaller length, a contradiction.

- Denote by $[u]$ the class $u / \sim$. We set $[u][v] = [uv]$. This is clearly independent of choice, that is if $u' \sim u$, then $u'v \sim uv$. Hence associativity is clear. Also, if $w = s_1^{\epsilon_1} \dots s_n^{\epsilon_n}$, then $[w] = [s_1^{\epsilon_1}] \dots [s_n^{\epsilon_n}]$. Hence $[S] = \{[s] \mid s \in S\}$ generates $\mathrm{F}(S)$. By the claim, every word has a unique reduced representation in $[S]$. Hence $\mathrm{F}(S)$ is free on $[S]$.

$\square$

Proposition 2.4.6 (Free groups IV)**.** *Assume F is a group and $S \subseteq F$ is any set. Then F is a* ***free group with basis*** *S if and only if F is the universal object in the class of groups with respect to S, that is whenever G is a group and $f : S \to G$ is any map, there exists a unique group homomorphism $\phi : F \to G$ extending f, so*

$$\begin{tikzcd} S \arrow[r, "f"] & G \\ F \arrow[u, "\mathrm{id}"] \arrow[ur, dashed, "\exists! f"'] & \end{tikzcd}.$$

Proof.

$\implies$ Assume F is free on S and $f : S \to G$ is any map. We first prove uniqueness of ϕ. By definition, for any $g \in F$ there exists a unique reduced word on S such that $w \underset{F}{=} g$, say $w = s_1^{\epsilon_1} \dots s_n^{\epsilon_n}$. Then, if $\phi : S \to G$ is a homomorphism of groups extending f, clearly,

$$\phi(g) = f(s_1)^{\epsilon_1} \dots f(s_n)^{\epsilon_n}.$$

Hence unique. Now for existence we prove that this actually is a homomorphism of groups, that is $\phi(gh) = \phi(g)\phi(h)$. Let $w_g \underset{F}{=} g$ and $w_h \underset{F}{=} h$ be reduced. Then $w_g w_h \underset{F}{=} gh$, but maybe not reduced. Say w_{gh} is the reduced word. If $w_g w_h$ is not reduced, then $w_g = w_g^1 w_g^2$ and $w_h = w_h^1 w_h^2$ such that $w_g^2 = \left(w_h^1\right)^{-1}$. Then clearly

$$\phi(gh) = f\left(w_g^1 w_h^2\right) = f\left(w_g^1\right) f\left(w_g^2\right) f\left(w_h^1\right)^{-1} f\left(w_h^2\right) = \phi(g)\phi(h).$$

Hence ϕ is indeed a group homomorphism.

$\Longleftarrow$ Let F be any group and $S \subseteq F$ such that for any group G and $f : S \to G$ there exists a unique homomorphism $\phi : F \to G$ extending f. First, consider $G_1 = \langle S \rangle \leq F$. Then the map $f : S \to F$ sending S to itself extends to some $\phi : F \to F$ with image G_1. But also, the homomorphism $\mathrm{id}_F : F \to F$ is a homomorphism extending f. By uniqueness, $\phi = \mathrm{id}_F$ whence $\langle S \rangle = G_1 = F$. Hence S generates F. Next, consider $G = \mathrm{F}(S)$ and $f : S \to \mathrm{F}(S)$ the natural embedding from S into $\mathrm{F}(S)$. We want to show that any element in F has a unique reduced representation on S. Indeed, assume $w = s_1^{\epsilon_1} \dots s_n^{\epsilon_n}$ is reduced for $n > 0$, but $w \underset{F}{=} e_F$. Then any homomorphism $\phi : F \to \mathrm{F}(S)$ extending f has to send w_G simultaneously to $e_{\mathrm{F}(S)}$ and to $w_{\mathrm{F}(S)}$, hence $e_{\mathrm{F}(S)} \underset{\mathrm{F}(S)}{=} w$, a contradiction.

$\square$

Lecture 8 Tuesday 22/10/19

Remark 2.4.7. Note that we also proved that F is free on $S \subseteq F$ if and only if $F = \mathrm{F}(S)$. Hence, for every set S there is a unique, up to isomorphism, group F which is free on S.

Lemma 2.4.8. *Every group G is the quotient of some free group.*

Proof. Consider $S \subseteq G$ some generating set of G. Consider the homomorphism $\phi : \mathrm{F}(S) \to G$. By 2.4.6 this exists. As S generates G, it is surjective. Now by the isomorphism theorem, we have $G \cong \mathrm{F}(S) / \operatorname{Ker} \phi$. $\square$

Remark 2.4.9. Let G be any group and $S \subseteq G$ a generating set. Then if S is infinite then $|S| = |G|$. Why? Note that $\left|\mathcal{P}^{\mathrm{fin}}(S)\right| = |S|$ for all infinite sets S, in ZF. As every element corresponds to a finite sequence in S, we get

$$|S| \leq |G| \leq \left|\mathcal{P}^{\mathrm{fin}}(S)\right| = |S| .$$

Definition 2.4.10. Let G be an arbitrary group. The **rank** of G is the smallest cardinal κ such that G arises as the quotient of a free group of rank κ. By 2.4.9 this is well-defined. We denote it by $\operatorname{rk} G$.

Lemma 2.4.11. *Let S and S' be any sets. Then $\mathrm{F}(S) \cong \mathrm{F}(S')$ if and only if $|S| = |S'|$.*

Proof.

$\Longrightarrow$ If $|S| = |S'|$, then there exists a bijection $f : S \to S'$. Let $\phi : \mathrm{F}(S) \to \mathrm{F}(S')$ be the unique homomorphism extending f, and $\psi : \mathrm{F}(S') \to \mathrm{F}(S)$ the unique homomorphism extending f^{-1}. Then $\psi \circ \phi : \mathrm{F}(S) \to \mathrm{F}(S)$ extends $\mathrm{id}_S : S \to \mathrm{F}(S)$. But clearly $\mathrm{id}_{\mathrm{F}(S)} : \mathrm{F}(S) \to \mathrm{F}(S)$ is another such homomorphism, whence by uniqueness $\psi \circ \phi = \mathrm{id}_{\mathrm{F}(S)}$, whence ϕ is an isomorphism.

$\Longleftarrow$ Assume $\mathrm{F}(S) \cong \mathrm{F}(S')$. If both S and S' are infinite, then

$$|S| = |\mathrm{F}(S)| = |\mathrm{F}(S')| = |S'| ,$$

by 2.4.9. Otherwise assume S is finite. Set $G = \mathbb{Z}/2\mathbb{Z}$. Recall that any homomorphism from $\mathrm{F}(S)$ to G is uniquely determined by its image on S. Hence, there is a one to one correspondence between the set of homomorphisms $\phi : \mathrm{F}(S) \to G$ and $\mathcal{P}(S)$, via

$$\phi \mapsto X_\phi = \{s \in S \mid \phi(s) = 1\} .$$

Note, as $\mathrm{F}(S) \cong \mathrm{F}(S')$, there are as many homomorphisms from $\mathrm{F}(S) \to G$ as from $\mathrm{F}(S') \to G$. Hence

$$2^{|S|} = |\mathrm{Hom}(\mathrm{F}(S), G)| = |\mathrm{Hom}(\mathrm{F}(S'), G)| = 2^{|S'|},$$

and as S was finite, $|S| = |S'|$.

$\square$

Definition 2.4.12. If G is free on S, we call $|S|$ the **rank** of G and say that S is a basis of G.

2.5 Presentations of groups

Definition 2.5.1. Let G be any group and $S \subseteq G$ a subset for G. Recall that $f : S \hookrightarrow G$. Let R be a set of words over S. We say that G **allows the presentation** $\langle S \mid R \rangle$ if and only if S generates G and R normally generates $\operatorname{Ker} \phi$. For brevity we write

$$G = \langle S \mid R \rangle .$$

We call the elements in G **generators** of G and the elements in R **relators**.

Lecture 9 Wednesday 23/10/19

Notation. $\langle R \rangle^{\mathrm{F}(S)}$ denotes the subgroup of $\mathrm{F}(S)$ normally generated by R, so

$$G \cong \mathrm{F}(S) / \langle R \rangle^{\mathrm{F}(S)} .$$

Generally if G is a group and $X \subseteq G$ is a subset, then $\langle X \rangle^G$ denotes the subgroup normally generated by X.

Note. Elements of R are called relators and identities in G are called **relations**. Every relator r codes a relation $r \underset{G}{=} e_G$.

Example. In $S = \{a, b\}$, ab^2a^{-1} is a relator and $ab^2 \underset{G}{=}$ is a relation, and $ab^2a^{-1} \underset{G}{=} e_G$ if and only if $ab^2 \underset{G}{=} a$.

Remark 2.5.2 (Free groups V)**.** A group F is **free on** S if and only if it allows the presentation $\langle S \mid \rangle$.

Exercise 2.5.3.

1. Show that for $r \in \langle R \rangle^{\mathrm{F}(S)}$ and $g, h \in \mathrm{F}(S)$ is arbitrary, then $grh \in \langle R \rangle^{\mathrm{F}(S)}$ if and only if $gh \in \langle R \rangle^{\mathrm{F}(S)}$.

2. Show that $g \in \mathrm{F}(S)$ is in $\langle R \rangle^{\mathrm{F}(S)}$ if and only if it is of the form

$$g = \prod_{i=1}^{n} u_i r_i u_i^{-1}, \qquad r_i \in R, \qquad u_i \in \mathrm{F}(S) .$$

Definition 2.5.4. Let $G = \langle S \mid R \rangle$. Then we call G **finitely presented** if both S and R are finite.

Example 2.5.5. Claim that the group $\mathcal{S}_3$ allows the presentation

$$\left\langle x, y \;\middle|\; x^2, y^2, (xy)^3 \right\rangle .$$

Let

$$\begin{array}{rccc} \phi & : & \mathrm{F}(\{x, y\}) & \longrightarrow & \mathcal{S}_3 \\ & & (x, y) & \longmapsto & ((12), (23)) \end{array} .$$

Clearly

$$\phi\left(x^2\right) = (12)^2 = e_{\mathcal{S}_3} = (23)^2 = \phi\left(y^2\right) .$$

Also, $xy = (132)$ and hence

$$\phi\left((xy)^3\right) = (132)^3 = e_{\mathcal{S}_3},$$

hence $(xy)^3 \in \operatorname{Ker} \phi$. Thus, by 2.5.3.2 and as ϕ is a homomorphism, we get

$$\left\langle \left\{ x^2, y^2, (xy)^3 \right\} \right\rangle^{\mathrm{F}(S)} \subseteq \operatorname{Ker} \phi.$$

Now assume $w = x^{k_1} y^{l_1} \dots x^{k_n} y^{l_n}$ on $\{x, y\}$ such that $w \underset{\mathcal{S}_3}{=} e_{\mathcal{S}_3}$, that is $w \in \operatorname{Ker} \phi$. By 2.5.3.1, we can cancel all occurences of $x^{\pm 2} y^{\pm 2}$. We obtain $w' = xyxy \dots$ such that $w' \underset{\mathcal{S}_3}{=} e_{\mathcal{S}_3}$ if and only if $w' \underset{\mathcal{S}_3}{=} e_{\mathcal{S}_3}$. Now cancel $(xy)^3$ and obtain a word of length at most six. Now, note the relator $(xy)^3$ yields the relation $xyx \underset{\mathcal{S}_3}{=} yxy$. Hence the only words left are of length at most three, that is

$$\epsilon, x, y, xy, yx, xyx = yxy.$$

Hence, w is in $\langle R \rangle^{\mathrm{F}(S)}$.

Exercise 2.5.6. For any set S and R a set of words on S, there exists some G with $G = \langle R \mid S \rangle$.

Remark 2.5.7. Let G be a group and $S \subseteq G$, and let R be a set of words on S. Then the following are equivalent.

1. $G = \langle S \mid R \rangle$.
2. A word w on S represents e_G in G if and only if $w \in \langle R \rangle^{\mathrm{F}(S)}$.

Proposition 2.5.8. *Let H be a group and $S \subseteq H$, and $R \subseteq \mathrm{F}(S)$. Then the following are equivalent.*

- $H = \langle S \mid R \rangle$.
- *For any group G and $f : S \to G$, there exists a homomorphism $\phi : H \to G$ extending f if and only if $f(r) = e_G$, with if $r = s_1^{\epsilon_1} \dots s_n^{\epsilon_n}$ then $f(r) = f(s_1)^{\epsilon_1} \dots f(s_n)^{\epsilon_n}$. In that case, ϕ is unique.*

Proof.

$\implies$ Assume $H = \langle S \mid R \rangle$ and $f : S \to G$. Clearly, as S generates H, any homomorphism $\phi : H \to G$ extending f has to send $h = s_1^{\epsilon_1} \dots s_n^{\epsilon_n}$ to $f(s_1)^{\epsilon_1} \dots f(s_n)^{\epsilon_n}$. Hence ϕ is uniquely determined, if it exists. Now, assume that ϕ is a well-defined homomorphism of groups. By 2.5.7, that $r \underset{H}{=} e_H$ for any $r \in R$, whence $\phi(r) = e_G = f(r)$. Now assume $f(r) = e_G$ for all $r \in R$. We want to show that ϕ is well-defined, that is if $w_1 \underset{H}{=} w_2$, then $\phi(w_1) \underset{G}{=} \phi(w_2)$. Clearly, this amounts to say that any word $w_1 w_2^{-1}$ which represents e_H is sent to e_G. By 2.5.7.2, w represents e_H if and only if $w \in \langle R \rangle^{\mathrm{F}(S)}$. By assumption $\phi(R) = \{e_G\}$ and 2.5.3.2, the claim follows.

$\impliedby$ Exercise. [2]

$\square$

Lecture 10 Thursday 24/10/19

Remark 2.5.9. We defined $G = \langle S \mid R \rangle$ through some $\phi : \mathrm{F}(S) \to G$. Note that ϕ and R do not determine each other uniquely.

Example.

- Consider
$$G = \left\langle x \;\middle|\; x^3 \right\rangle = \mathbb{Z}_3,$$
and
$$\begin{array}{cccc} \phi_1 & : & \mathrm{F}(\{x\}) & \longrightarrow & G \\ & & x & \longmapsto & 1 \end{array}, \qquad \begin{array}{cccc} \phi_1 & : & \mathrm{F}(\{x\}) & \longrightarrow & G \\ & & x & \longmapsto & 2 \end{array}.$$
- Consider
$$G = \left\langle x, y \;\middle|\; x^2, y^2, (xy)^3 \right\rangle = \mathcal{S}_3,$$
as we saw, given by
$$\begin{array}{cccc} \phi_1 & : & \mathrm{F}(\{x, y\}) & \longrightarrow & G \\ & & (x, y) & \longmapsto & ((12), (23)) \end{array}.$$
An easy calculation shows that
$$\mathcal{S}_3 = \left\langle x, y \;\middle|\; x^2, (xy)^3, x^{-1} y^{-1} (xy)^2 \right\rangle,$$
via the same homomorphism ϕ.

It is hard to read even basic algebraic properties from a presentation, such as whether or not a group is finite.

Example.

$\mathcal{S}_3 = \left\langle x, y \;\middle|\; x^2, y^2, (xy)^3 \right\rangle$, $\mathcal{A}_4 = \left\langle x, y \;\middle|\; x^3, y^3, (xy)^2 \right\rangle$ are finite, $\quad G = \left\langle x, y \;\middle|\; x^3, y^3, (xy)^3 \right\rangle$ is infinite.

[2]Exercise

Remark 2.5.10. More generally, a group in which every element has finite order is called **periodic**. Burnside asked, in 1902, whether every finitely generated periodic group is finite. This generalises to the following. Let S be any set and let $w(S)$ denote the set of all words on S. Define the groups

$$\mathrm{B}(m,n) = \langle x_1, \ldots, x_m \mid w(\{x_1, \ldots, x_m\})^n \rangle .$$

They are called **free Burnside groups**. Are the $\mathrm{B}(m,n)$ finite?

- Burnside proved for $n = 2, 3$.
- Sanov proved in 1940 for $n = 4$.
- Hall proved in 1958 for $n = 6$.
- Novikov and Adian disproved in 1968 for $m \geq 2$ and $n \geq 667$ odd.
- $\mathrm{B}(2,5) = \left\langle x, y \,\middle|\, w(x,y)^5 \right\rangle$ is open.

Exercise 2.5.11. Show that all $\mathrm{B}(1,n)$ and $\mathrm{B}(m,n)$ are finite.

Why bother? Most groups simply just come through their presentation and problems on groups, such as isomorphism problems and word problems, are best expressed through presentations. A very important class of groups are Coxeter groups. A group G is **Coxeter**, if

$$G = \left\langle s_1, \ldots, s_n \,\middle|\, s_i^2, (s_i s_j)^{m_{ij}}, i, j = 1, \ldots, n \right\rangle, \qquad 1 \leq m_{ij} \leq \infty.$$

2.6 Group actions on graphs

Recall the definition of a group acting on a graph. Many statements in Bass-Serre theory start with the assumption that G acts on X without inversion of edges.

Definition 2.6.1. Let X be a graph. We define its **barycentric subdivision** $\mathcal{B}(X)$ via

- $\mathcal{B}(X)^0 = \left\{ v_e \,\middle|\, e \in X^1 \right\}$,
- $\mathcal{B}(X)^1 = \left\{ e_1, e_2 \,\middle|\, e \in X^0 \right\}$ such that

$$\alpha(e_1) = \alpha(e), \qquad \omega(e_1) = \alpha(e_2) = v_e, \qquad \omega(e_2) = \omega(e),$$

- $\overline{e_1} = (\overline{e})_2$.

Now, if G acts on X, we can naturally extend the action via

$$g \cdot v_e = v_{ge}, \qquad g \cdot e_1 = (ge)_1, \qquad g \cdot e_2 = (ge)_2 .$$

Note. The action did not change on X^0.

Exercise 2.6.2. If G acts on X, then it acts as above on $\mathcal{B}(X)$ and this action is without inversion of edges.

2.6.1 Automorphisms of trees

Definition 2.6.3. Let X be a tree. A reduced path is called a **geodesic**. For $x, y \in X$, we denote the unique geodesic from x to y by $[x - y]$. Its length is called the **distance** between x and y and denoted by $\mathrm{d}(x,y)$.

Note. d defines a distance function and if $\sigma \in \operatorname{Aut} X$, then it acts on X as an isometry.

Note. Further, for $X_1, X_2 \subseteq X$ disjoint subtrees of X, there is a unique geodesic which starts in X_1, ends in X_2, and with all edges outside $X_1^1 \cup X_2^1$. We denote this by $X_1 - X_2$, and its length by $\mathrm{d}(X_1, X_2)$.

Definition 2.6.4. Let X be a tree and $\sigma \in \operatorname{Aut} X$. Then we set

$$|\sigma| = \min \left\{ \mathrm{d}(x, \sigma(x)) \mid x \in X^0 \right\},$$

and call it the **displacement length** of σ. Let

$$\widetilde{\sigma_{\min}} = \left\{ x \in X^0 \mid \mathrm{d}(x, \sigma(x)) = |\sigma| \right\}.$$

Then $\sigma_{\min}$ is the induced subgraph on $\widetilde{\sigma_{\min}}$. If $|\sigma| = 0$, we write $\sigma_{\min} = \mathring{\sigma}$, and if $|\sigma| > 0$, we write $\sigma_{\min} = \vec{\sigma}$.

Definition 2.6.5. If X is a tree and G acting without inversion of edges as a group of automorphisms, we say that X is a G**-tree**. Lecture 11 Tuesday 29/10/19

Proposition 2.6.6. *Let X be a G-tree, and $\sigma \in G$.*

1. *If $|\sigma| = 0$, then $\mathring{\sigma}$ is a tree. Furthermore, if $y \in X^0$, then let x be the final vertex of $[y - \mathring{\sigma}]$. Then*
$$\mathrm{d}(y, \sigma(y)) = 2\mathrm{d}(y, x), \qquad [y - \sigma(y)] = [y - x] \cup [x - \sigma(y)].$$

2. *Assume $|\sigma| > 0$. Then $\vec{\sigma} \cong \mathcal{C}_\infty$ and σ acts on $\vec{\sigma}$ by translation of length $|\sigma|$. If $y \in X^0$ is arbitrary and x is the terminal vertex of $[y - \vec{\sigma}]$, then*
$$\mathrm{d}(y, \sigma(y)) = 2\mathrm{d}(y, x) + |\sigma|, \qquad [y - \sigma(y)] = [y - x] \cup [x - \sigma(x)] \cup [\sigma(x) - \sigma(y)].$$

Proof.

1. Let $|\sigma| = 0$. Clearly, $\mathring{\sigma}$ is a tree as for x and y fixed by σ, clearly $[x - y]$ has to be fixed by σ, whence $[x - y] \in \mathring{\sigma}$. Consider $y \in X^0$ arbitrary and x as described. We prove the statement by induction on $\mathrm{d}(y, \mathring{\sigma})$. Clearly, if $\mathrm{d}(y, \mathring{\sigma}) = 0$, then $y \in \mathring{\sigma}$, and the statement holds, as all geodesics are the empty path. Now $\mathrm{d}(y, \mathring{\sigma}) = n + 1$. Let z be the terminal vertex of the first edge in $[y - x]$. Then $\mathrm{d}(z, \mathring{\sigma}) = \mathrm{d}(z, x) = n$, whence $[z - \sigma(z)] = [z - x] \cup [x - \sigma(z)]$. Now $\sigma(z)$ is a neighbour of $\sigma(y)$. Then either $\sigma(y) \notin [x - \sigma(z)]$, then we are done, as $\mathrm{d}(y, \sigma(y)) = \mathrm{d}(y, z) + \mathrm{d}(z, \sigma(z)) + \mathrm{d}(\sigma(z), \sigma(y)) = 2(n + 1) = 2\mathrm{d}(y, x)$. Otherwise, $\mathrm{d}(\sigma(y), x) = n - 1$. Clearly, $\left|\sigma^{-1}\right| = 0$ and $\mathring{\sigma^{-1}} = \mathring{\sigma}$. Then $\mathrm{d}(\sigma(y), x) = n - 1 < n + 1 = \mathrm{d}(x, y) = \mathrm{d}\left(\sigma^{-1}(\sigma(y)), x\right)$. Hence $\mathrm{d}\left(\sigma(y), \sigma^{-1}(\sigma(y))\right) = 2n > 2\mathrm{d}(\sigma(y), x)$, a contradiction.

2. Consider $x \in \vec{\sigma}$. We want to show that $\left[x - \sigma^2(x)\right] = [x - \sigma(x)] \cup \left[\sigma(x) - \sigma^2(x)\right]$. Assume for a contradiction that the terminal edge of $[x - \sigma(x)]$, call it e, is the first edge inverse of $\left[\sigma(x) - \sigma^2(x)\right]$. Let z be the terminal vertex of the first edge in $[x - \sigma(x)]$. Then clearly, $\sigma(z)$ is the terminal vertex of the first edge in $\left[\sigma(x) - \sigma^2(x)\right]$. By assumption, this is the initial vertex of e. But then $\mathrm{d}(z, \sigma(z)) = \mathrm{d}(x, \sigma(x)) - 2 < |\sigma|$, a contradiction. Note that if $\mathrm{d}(x, \sigma(x)) = 1$, then σ would invert the edge e, a contradiction. Hence $\left[x - \sigma^2(x)\right] = [x - \sigma(x)] \cup \left[\sigma(x) - \sigma^2(x)\right]$. Inductively, this produces a tree
$$T = \cdots - \sigma^{-1}(x) - x - \sigma(x) - \cdots \subseteq \vec{\sigma},$$
and σ acts on T by translation with length $|\sigma|$. Now consider $y \in X^0 \setminus T^0$. Note that $\sigma \in \operatorname{Aut} T$. Thus $[\sigma(x) - \sigma(y)]$ intersects T trivially. Hence, $\mathrm{d}(y, \sigma(y)) = \mathrm{d}(y, x) + \mathrm{d}(x, \sigma(x)) + \mathrm{d}(\sigma(x), \sigma(y)) = \mathrm{d}(y, x) + |\sigma| + \mathrm{d}(x, y) = 2\mathrm{d}(x, y) + |\sigma| > |\sigma|$, for all $y \notin T$. Hence $T = \vec{\sigma} \cong \mathcal{C}_\infty$.

$\square$

Definition 2.6.7. Let X be a G-tree and $\sigma \in G$. Then

- if $|\sigma| = 0$, then σ is called an **elliptic element**, and
- if $|\sigma| > 0$, then σ is called a **hyperbolic element**.

Remark 2.6.8. If G does invert an edge, it resembles more a rotation. Indeed, the extended action on $\mathcal{B}(X)$ yields that σ is elliptic.

Lecture 12 is a problem class. Lecture 12 Wednesday 30/10/19

Lecture 13 Thursday 31/10/19

Lemma 2.6.9. *Let X be a G-tree with trivial edge stabilisers, that is for all $e \in X^1$ and $g \in G$ such that $g \cdot e = e$, then $g = \mathrm{id}_G$. If $g, h \in G$ are hyperbolic such that $\left|\vec{g}^0 \cap \vec{h}^0\right| \geq |h| + |g| + 2$, then g and h commute.*

Proof. First note that h^{-1} also is hyperbolic with $|h| = \left|h^{-1}\right|$ and $\overrightarrow{h^{-1}} = \vec{h}$, only that h^{-1} moves the elements on $\vec{h}$ in the inverse direction. As g and h commute if and only if g and h^{-1} commute, we may assume that they translate the elements of $L = \vec{g} \cap \vec{h}$ in the same direction. Let $L^0 = \{x_i \mid k < i < j\}$ with $-\infty \leq k < j \leq \infty$. By assumption $(j-1) - k \geq |g| + |h| + 2$, thus there is some $i > k$ such that $i + |g| + |h| + 1 < j$. Thus consider $g^{-1}h^{-1}gh$, then

$$g^{-1}h^{-1}gh\left(x_i\right) = g^{-1}h^{-1}g\left(x_{i+|h|}\right) = g^{-1}h^{-1}\left(x_{i+|g|+|h|}\right) = g^{-1}\left(x_{i+|g|}\right) = x_i.$$

Similarly, $g^{-1}h^{-1}gh\left(x_{i+1}\right) = g^{-1}h^{-1}\left(x_{i+|g|+|h|+1}\right) = x_{i+1}$. Hence $g^{-1}h^{-1}gh$ fixes e with $\alpha(e) = x_i$ and $\omega(e) = x_{i+1}$, where $g^{-1}h^{-1}gh = e_G$, if and only if $gh = hg$. $\square$

Exercise 2.6.10. Let X be a G-tree.

- If $x \in X^0$ and $g \in G$ such that $\mathrm{d}\left(x, g^2 \cdot x\right) = 2\mathrm{d}\left(x, g \cdot x\right) > 0$, then g is hyperbolic and $x \in \vec{g}$.
- If $g \in G$ is elliptic, and $h \in G$ is arbitrary, then $g^h = hgh^{-1}$ is elliptic with $\mathring{g^h} = h\mathring{g}$.
- If $g \in G$ is hyperbolic, and $h \in G$ is arbitrary, then $g^h = hgh^{-1}$ is hyperbolic with $\vec{g^h} = h\vec{g}$ and $|g| = \left|g^h\right|$.

Proposition 2.6.11. *Let X be a G-tree.*

- *If g and h are elliptic with $\mathring{g} \cap \mathring{h} = \emptyset$, then gh is hyperbolic with $|gh| = 2\mathrm{d}\left(\mathring{g}, \mathring{h}\right)$. Furthermore, $\vec{gh}$ intersects $\mathring{g}$ and $\mathring{h}$ non-trivially.*
- *If g and h are hyperbolic with $\vec{g} \cap \vec{h} = \emptyset$, then gh is hyperbolic with $|gh| = |g| + |h| + 2\mathrm{d}\left(\vec{g}, \vec{h}\right)$.*

2.6.2 Nielson-Schreier theorem

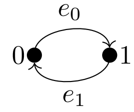
Recall that F is free of basis S if and only if $S \cap S^{-1} = \emptyset$ and $\Gamma(F, S)$ is a tree. Also, we know that F acts freely on $\Gamma(F, S)$. Hence, every free group acts freely on a tree. We want to prove the inverse, that every group acting freely on a tree is free. Then we deduce that subgroups of free groups are free.

Definition 2.6.12. Let X be a graph and G a group acting on X. For $x \in X^0 \cup X^1$ we denote by $\mathcal{O}(x)$ the orbit of x under G, that is $\mathcal{O}(x) = \{g \cdot x \mid g \in G\}$. Then we define the **factor graph** $G \backslash X$ via

- $(G \backslash X)^i = \left\{\mathcal{O}(x) \mid x \in X^i\right\}$ for $i = 0, 1$,
- $\alpha(\mathcal{O}(e)) = \mathcal{O}(\alpha(e))$ for $e \in X^1$, and
- $\overline{\mathcal{O}(e)} = \mathcal{O}(\overline{e})$ for $e \in X^1$.

Lecture 14 Tuesday 05/11/19

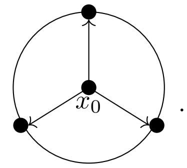
Remark 2.6.13. From now on we do not identify edges through their initial and terminal vertex any longer. In particular we can include $\mathcal{C}_2$ in the class of circuits where $\mathcal{C}_2^0 = \{0, 1\}$ and $\mathcal{C}_2^1 = \{e_0, \overline{e_0}, e_1, \overline{e_1}\}$, so $\mathcal{C}_2$ is



.

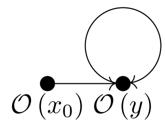
Hence a tree is a connected graph without any $\mathcal{C}_n$ for $n \in \mathbb{N}$. Now 2.4.3 changes to G is free on S if and only if $\Gamma(G, S)$ is a tree.

Example 2.6.14. Consider the graph

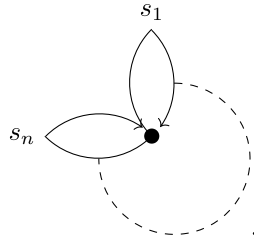


X

Let $G_1 = \{\mathrm{id}\}$, $G_2 = \mathbb{Z}_3$, and $G_3 = \mathcal{S}_4$ acting on X, where $\mathbb{Z}_3 \curvearrowright X$ fixing x_0. Then

$G_1 \backslash X \cong X$, $\qquad G_2 \backslash X \cong$  , $\qquad G_3 \backslash X \cong \mathcal{C}_1$.

Example 2.6.15. Let F be a free group on S. Then if $|S| = n$,

$F \backslash \Gamma(F, S) \cong$  ,

a rose with n petals.

Lemma 2.6.16. *The factor graph $G \backslash X$ is well-defined.*

Proof. Note $\mathcal{O}(x) = \mathcal{O}(x')$ if and only if $x \in \mathcal{O}(x)$. Consider $e' \in \mathcal{O}(e)$, that is there exists $g \in G$ such that $e' = g \cdot e$. Want to show that $\alpha(e') \in \mathcal{O}(\alpha(e))$ and $\overline{e'} \in \mathcal{O}(\overline{e})$. Recall that group actions give rise to graph homomorphisms, whence $\alpha(e') = \alpha(g \cdot e) = g \cdot \alpha(e) \in \mathcal{O}(\alpha(e))$. Same for $\overline{e'} = \overline{g \cdot e} = g \cdot \overline{e} \in \mathcal{O}(\overline{e})$. □

Definition 2.6.17. We call the map

$$\begin{array}{rccc} \mathrm{p} & : & X & \longrightarrow & G \backslash X \\ & & x & \longmapsto & \mathcal{O}(x) \end{array}$$

a **projection**, and if $y \in G \backslash X$, then any $x \in X$ with $\mathrm{p}(x) = y$ is called a **lift** of y.

Proposition 2.6.18. *If T' is a tree in $G \backslash X$, then there is some tree $T \subseteq X$ such that $\mathrm{p}|_T : T \to T'$ is an isomorphism. Then T is called a lift of T'.*

Proof. First note that for any $f \in (G \backslash X)^1$ and $x \in \mathcal{O}(\alpha(f))$, then there is some $e \in X^1$ such that $\mathrm{p}(e) = f$ and $\alpha(e) = x$. Denote by $\mathcal{T}$ the set of all subtrees of X which project injectively into T'. Note that for $T' \neq \emptyset$ the set $\mathcal{T} \neq \emptyset$ as any lift of any vertex in T' is in $\mathcal{T}$. We can partially order $\mathcal{T}$ by inclusion. Then every chain in $\mathcal{T}$ has a maximal element, being the union. So, by Zorn's lemma, we obtain a maximal element T. We claim that $T \cong T'$. Otherwise, there was some $f \in T'$ such that $\alpha(f) \in \mathrm{p}(T)$ and $\omega(f) \notin \mathrm{p}(T)$. Now let $x = \mathrm{p}^{-1}(\alpha(f))$. By the above, there exists $e \in X^1$ such that $\alpha(e) = x$ and $\mathrm{p}(e) = f$. But then $T \cup \{e, \omega(e)\}$ is still a tree mapped injectively to T', a contradiction. □

Theorem 2.6.19 (Free group VI)**.** *A group G is free if and only if it acts freely and without inversion of edges on a tree.*

Theorem 2.6.20 (Nielson-Schreier theorem)**.** *A subgroup of a free group is free.*

Proof. Let G be free on S and $H \leq G$. Then $\Gamma(G, S)$ is a tree and G acts on it freely without inversion of edges. Now clearly restricting the action to H yields that H acts freely on some tree. Hence H is free. □

Exercise 2.6.21. Assume G acts freely on a tree X, without inversion of edges. Set $Y = G\backslash X$. If Y is finite, then

$$\operatorname{rk} G = \left|Y_+^1\right| - \left|Y^0\right| + 1.$$

Lemma 2.6.22. *If G is a free group of finite rank, and $H \leq G$ of index n, then*

$$\operatorname{rk} H - 1 = n\left(\operatorname{rk} G - 1\right).$$

Proof. Let $\Gamma(G, S)$ be the Cayley graph. Then H acts on it via $h : g \mapsto gh$ and $h : (g, s) \mapsto (gh, s)$. Then

$$\left(H\backslash\Gamma(G, S)\right)^0 = \{gH \mid g \in G\} = G/H, \qquad \left(H\backslash\Gamma(G, S)\right)_+^1 = G/H \times S.$$

Hence, $\operatorname{rk} H = n \operatorname{rk} G - n + 1$. ☐

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Proof of 2.6.19. We already know that $\Gamma(G, S)$ is a tree for G free on S and G acts freely on it. Now assume G acts freely on a tree X. Set $X' = G\backslash X$. Let T' be a spanning tree for X'. By 2.6.18 there exists a lift T of T' such that $\mathrm{p}(T) \cong T'$. Pick an orientation of X'. We may assume G acts without inversion of edges, whence we can lift that orientation to X. Denote by $E' = X_+^1 \setminus T'^1$. As the action is free, for every $f \in E'$ there is a unique lift $e \in X_+^1$ with $\alpha(e) \in T$. Set

$$E = \{e \mid \mathrm{p}(e) \in E',\ \alpha(e) \in T\}.$$

Clearly, $\omega(e) \notin T$ for all $e \in E$. Further, all gT are disjoint for distinct g. Furthermore, the gT cover all vertices of X. Also, for any $e \in E$ there is a unique $g_e \in G$ such that $\omega(e) \in g_e T$. Now set $S = \{g_e \mid e \in E\}$. Define a graph T_G via

$$T_G^0 = \{(gT) \mid g \in G\}, \qquad T_G^1 = X_+^1 \setminus \left\{(gT)^1 \;\middle|\; g \in G\right\}.$$

For $f \in T_G^1$, $\alpha(f) = (gT)$ and $\omega(f) = (hT)$ if and only if $\alpha(f) \in (gT)^0$ and $\omega(f) \in (hT)^0$. We claim that $T_G \cong \Gamma(G, S)$. Note that T_G is a tree, so this concludes the proof. Note that

$$S = \left\{g \in X_+^1 \;\middle|\; \exists e \in X_+^1,\ \alpha(e) \in T,\ \omega(e) \in gT\right\}.$$

Hence, for $f \in T_G^1$, with $\alpha(f) = (gT)$ and $\omega(f) = (hT)$, whence for $e = g^{-1}(f)$ we have $\alpha\left(g^{-1}(f)\right) = g^{-1}(\alpha(f)) \in T$ and $\omega(e) = g^{-1}(\omega(e)) \in g^{-1}hT$. Thus $g^{-1}h \in S$. Define

$$\begin{array}{rrcl} \phi & : \quad T_G & \longmapsto & \Gamma(G, S) \\ & (gT) & \longmapsto & g \in \Gamma(G, S)^0 \\ & f & \longmapsto & (g, s) \end{array}, \qquad \alpha(f) \in gT, \qquad \omega(f) = hT, \qquad g^{-1}h = s.$$

One easily checks that ϕ is an isomorphism $T_G \xrightarrow{\sim} \Gamma(G, S)$. ☐

Recall for $H \leq G$ a subgroup of a free group, we studied $H \curvearrowright \Gamma(G, S)$. Now we want to understand what $H\backslash\Gamma(G, S)$ looks like and give another proof of Nielson-Schreier.

2.7 Fundamental groups of graphs

The following is the plan.

- Introduce fundamental groups.
- Show they always are free groups.
- View H as the fundamental group of $H\backslash\Gamma(G, S)$.
- Deduce that H is free.

Let X be a connected graph and $x \in X^0$. Let us denote the set of closed paths starting at x by $\mathcal{P}(X, x)$. We say that $p, q \in \mathcal{P}(X, x)$ are **homotopic**, if there exists $p = p_0, \dots, p_n = q$ such that p_{i+1} arises from p_i by deleting or inserting subpaths of the form $e\overline{e}$, for $e \in X^1$. This is an equivalence relation and every class has exactly one reduced representative. Denote the homotopy class of p by $[p]$.

Exercise 2.7.1. Define a multiplication $[p][q] = [pq]$. Show that this is well-defined and yields a group structure.

Definition 2.7.2. We denote the group of homotopy classes of $\mathcal{P}(X,x)$ by $\pi_1(X,x)$ and call it the **fundamental group** of X with respect to x.

Exercise 2.7.3. Let X be connected and $x, y \in X^0$. Then $\pi_1(X,x) \cong \pi_1(X,y)$.

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Example 2.7.4.

- If X is a tree, then $\pi_1(X,x) = \{e\}$.
- If $X \cong \mathcal{C}_n$ for $1 < n < \infty$, then $\pi_1(X,x) = (\mathbb{Z}, +)$.

Definition 2.7.5. Let X be a connected graph and $T \subseteq X$ a spanning tree of X. Then for every $y \in X^0$, and $x \in X^0$ fixed, there is a unique reduced path in T from x to y. Denote that path by p_y. Now, if $e \in X^1$ we denote by $p_e = p_{\alpha(e)} e p_{\omega(e)}^{-1} \in \mathcal{P}(X,x)$.

Note. $[p_{\overline{e}}] = [p_e]^{-1}$, and if $e \in T^1$, then $[p_e] = 1$.

Theorem 2.7.6. *Let X be connected and $T \subseteq X$ a spanning tree. Then $\pi_1(X,x)$ is free on*

$$S = \left\{[p_e] \mid e \in X_+^1 \setminus T^1\right\}.$$

Proof. First, if $p \in \mathcal{P}(X,y)$ and $p = e_1 \dots e_n$, then $[p] = [p_{e_1}] \dots [p_{e_n}]$. As $[p_e] = 1$ for all $e \in T$ we know that S generates $\pi_1(X,x)$. Now let

$$w = [p_{e_1}] \dots [p_{e_n}] = [p_{e_1} \dots p_{e_n}], \qquad e_i \notin T, \qquad e_{i+1} = \overline{e_i}$$

be a reduced word on S. Then $w = u_1 e_1 v_1 \dots u_n e_n v_n$, where $v_i u_{i+1} \in T$ and $e_i \notin T$. Hence the reduced presentation of w is some $t_1 e_1 \dots t_n e_n t_{n+1}$, where t_i might be empty. This is the empty word, if and only if $n = 0$ and $t_n = \epsilon$, whence $w = \epsilon$. ☐

Example 2.7.7. The fundamental group of the rose with n-many petals is the free group of rank n.

Remark 2.7.8 (Free group VII)**.** A group is free if and only if it is the fundamental group of a graph.

Let G be free on S and $H \le G$, then $H \curvearrowright \Gamma(G,S)$. Set $Y = H \backslash \Gamma(G,S)$.

Lemma 2.7.9. *The group H consists of all labels of paths in Y which start and end in H.*

Recall that

- $Y^0 = \{Hg \mid g \in G\}$ and $Y^1 = \{(Hg, s) \mid s \in S\}$, and
- $\mathrm{l}((g,s)) = s$, and if $p = e_1 \dots e_n$, then $\mathrm{l}(p) \underset{G}{=} \mathrm{l}(e_1) \dots \mathrm{l}(e_n)$.

Proof. First, let p be a path in Y with $\alpha(p) = \omega(p) = H$. Let $p = e_1 \dots e_n$. Note $\omega(e_{i+1}) = \alpha(e_i)\,\mathrm{l}(e_i)$. Hence,

$$H = \alpha(e_1) = \omega(e_n) = \alpha(e_n)\,\mathrm{l}(e_n) = \omega(e_{n-1})\,\mathrm{l}(e_n) = \dots = \alpha(e_1)\,\mathrm{l}(e_1) \dots \mathrm{l}(e_n).$$

Conversely, if $h \in H$, then note that every vertex in the factor has exactly one outgoing and one incoming edge of label s. Hence $h = s_1 \dots s_n$ for $s_i \in S^{\pm}$ gives rise to a unique reduced path p in Y such that $\alpha(p) = H$ and $\omega(p) = \alpha(p)\,\mathrm{l}(s_1) \dots \mathrm{l}(s_n) = H \cdot h = H$. ☐

Theorem 2.7.10. *Let G be free and $H \le G$ and $Y = H \backslash \Gamma(G,S)$. If T is a spanning tree of Y, then $H \cong \pi_1(Y,H)$ and free on $S = \left\{\mathrm{l}(p_e) \mid e \in Y_+^1 \setminus T^1\right\}$.*

Proof. We claim that

$$\begin{array}{rcrcl} \phi & : & \pi_1(Y,H) & \longrightarrow & G \\ & & [p] & \longmapsto & \mathrm{l}(p) \end{array}$$

is injective. Clearly ϕ is well-defined as homotopic paths have the same label, it is a homomorphism of groups, and by 2.7.9 its image is exactly H. Note that every homotopy class has a unique reduced representative, $[p] = [e_1] \dots [e_n]$ for $e_i \in Y_+^1 \setminus T^1$. This is mapped to the reduced word $\mathrm{l}(e_1) \dots \mathrm{l}(e_n) = s_1 \dots s_n$. Hence $\phi([p]) = e_G$ if and only if $[p] = 1$. ☐

2.8 Outlook

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The goal is a rough version of the fundamental structure theorem. Assume G acts without inversion of edges on a tree X. Then it is the fundamental group of some graph of groups (G, Y), such that $Y = G \backslash X$.

3 Bass-Serre theory

Bass-Serre theory is a fancy name of groups acting on graphs. Until now, free actions. Now, drop the freedom and study the algebraic structures appearing, especially if factor graphs of $G \circ G$ are simple. These structures correspond to freely amalgamated products and HNN extensions. We introduce them algebraically, and later view them as fundamental groups. Intuitively, graphs of groups are graphs for which each vertex and each edge is a group, and such that edge groups embed into vertex groups. From there we deduce the main theorem. Amalgamated products and HNN extensions are ways to build new groups out of two given ones. A pre-step will be free products.

3.1 Free products of groups

Given two groups G_1 and G_2, we already know one way of combining them,

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_i \in G_i\}.$$

Note. $G_i \hookrightarrow G_1 \times G_2$ but there are non-trivial relations between G_1 and G_2, such as $[g_1, g_2] = e_G$ for all $g_i \in G_i$.

Definition 3.1.1. Let G be a group and $A, B \leq G$ subgroups of G. Then we say that G is the **free product** of A and B, writing $G = A * B$, if and only if

- $G = \langle A, B \rangle$, and
- Whenever $g = g_0 \dots g_n$ such that $n \in \mathbb{N}$, and $g_i \in A \cup B \setminus \{e_G\}$ and $g_i \in A$ if and only if $g_{i+1} \in B$, then $g \underset{G}{\neq} e_G$.

Such a presentation of g is called a **normal form** of length $n+1$.

Remark 3.1.2.

- If $A = \{e_G\}$, then $G = A * B = B$. This is called a **trivial free product**. Everything else is called a **non-trivial free product**.
- $G = \mathbb{Z} * \mathbb{Z} \cong \mathbb{F}_2$.
- $G = \mathbb{F}_{n+1} \cong \mathbb{F}_n * \mathbb{Z}$.
- $G = A * B \cong B * A$.

Lemma 3.1.3. *Let A and B be groups. Then there is some G, and $\iota_A : A \hookrightarrow G$ and $\iota_B : B \hookrightarrow G$ embeddings such that $G = \iota_A(A) * \iota_B(B)$.*

Proof. By taking isomorphic copies if necessary, we may assume that $A \cap B = \{1\}$. Define the set of elements of G as all words $g = g_0 \dots g_n$ with $g_i \in A \setminus \{e_A\} \cup B \setminus \{e_B\}$, and $g_i \in A$ if and only if $g_{i+1} \in B$. For $h = h_0 \dots h_m \in G$ we set inductively

$$g \cdot h = \begin{cases} g_0 \dots g_n h_0 \dots h_m & g_n \in A \iff h_0 \in B \\ g_0 \dots g_{n-1} x h_1 \dots h_m & g_n \in A \iff h_0 \in A, \ x = g_n h_0 \neq e_A \\ g_0 \dots g_{n-1} h_1 \dots h_m & g_n \in A \iff h_0 \in A, \ g_n h_0 = e_A \end{cases}.$$

Identify the empty word with e_G. Clearly, A and B inject into G as words with length one or zero, and $G = A * B$. □

Exercise 3.1.4. Show that $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathcal{D}_\infty$.

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Proposition 3.1.5. *If $A = \langle S_A \mid R_A \rangle$ and $B = \langle S_B \mid R_B \rangle$, then if $S_A \cap S_B = \emptyset$,*

$$A * B = \langle S_A \cup S_B \mid R_A \cup R_B \rangle .$$

Proof. $A \cong \mathrm{F}(S_A) / \langle R_A \rangle^{\mathrm{F}(S_A)}$ and $\phi_A : \mathrm{F}(S_A) \to A$ such that $\operatorname{Ker} \phi_A = \langle R_A \rangle^{\mathrm{F}(S_A)}$. Similar for ϕ_B. We know that $\phi_A \cup \phi_B : S_A \cup S_B \to A * B$ has a unique extension to a homomorphism of groups, say $\psi : \mathrm{F}(S_A \cup S_B) \to A * B$. Clearly $S_A \cup S_B$ generate $A * B$. We claim that $\operatorname{Ker} \psi = \langle \operatorname{Ker} \phi_A \cup \operatorname{Ker} \phi_B \rangle$. Then $\operatorname{Ker} \psi \supseteq \operatorname{Ker} \phi_A \cup \operatorname{Ker} \phi_B$ is clear. Now let $w = g_1 \dots g_2$ be in normal form with respect to $\mathrm{F}(S_A) * \mathrm{F}(S_B)$. Then $\psi(w) = \psi(g_1) \dots \psi(g_n)$ is of alternating form. If $\psi(w) = e_{A*B}$, then there exists i such that $\psi(g_i) = e_{A*B}$. So $g_i \in \operatorname{Ker} \phi_A \cup \operatorname{Ker} \phi_B$. Inductively, as $g' = g_1 \dots g_{i-1} g_{i+1} \dots g_n$ has a smaller normal form, whence $\psi(g') = e_G$, we conclude inductively that $w \in \langle \operatorname{Ker} \phi_A \cup \operatorname{Ker} \phi_B \rangle$. □

Proposition 3.1.6. *Let G be a group and $A, B \le G$. Then $G = A * B$ if and only if for any group H, and $\phi_A : A \to H$ and $\phi_B : B \to H$ there is a unique homomorphism $\psi : G \to H$ extending ϕ_A and ϕ_B.*

Proof. Exercise. [3] □

Lemma 3.1.7 (Ping pong lemma)**.** *Let G be a group, $A, B \le G$ such that $|A| + |B| \ge 5$, and $H = \langle A, B \rangle$. Assume that G acts on some set X. If there are non-empty subsets $X_1, X_2 \subseteq X$ such that*

- *$X_i \not\subseteq X_j$ for $i \ne j$, and*
- *$a(X_2) \subseteq X_1$ for all $a \in A \setminus \{e_G\}$ and $b(X_1) \subseteq X_2$ for all $b \in B \setminus \{e_G\}$,*

then $H = A * B$.

Remark 3.1.8. If $|A| = |B| = 2$, then $A \cong B \cong \mathbb{Z}_2$ and $H \cong \mathcal{D}_n$ is the dihedral group for some $n \in \mathbb{N}$.

Proof. If A or B is trivial, then $H = A * B$. For example, if A is trivial $H = B = A * B$. Hence without loss of generality, $|A| \ge 3$ and $|B| \ge 2$. We want to show that no normal form with respect to A and B is trivial in G. Consider $w = a_1 b_1 \dots a_n b_n a_{n+1}$. Then

$$w(X_2) = a_1 b_1 \dots a_n b_n a_{n+1}(X_2) \subseteq \dots \subseteq a_1(X_2) \subseteq X_1.$$

So $w \ne e_G$. The same holds for $w = b w' b'$. If w starts with a letter in A and ends with a letter in B, or vice versa, take $a' \in A \setminus \{e_G, a\}$ and consider $w' = a'^{-1} w a'$, or $w' = a' w a'^{-1}$. Then $w' \ne e_G$ so $w \ne e_G$. Hence $H = A * B$. □

Example 3.1.9. The **projective special linear group** $\mathrm{PSL}_2(\mathbb{Z})$, the **modular group**. Clearly general linear groups act on vector spaces. To any vector space we can associate a projective space by factoring through a non-zero scalar multiplication. We denote them by $[x]$. If k^{n+1} is a vector space, we can identify the projective space with $k^n \cup \{\infty\}$ via

$$[x_1, \dots, x_{n+1}] = \left[\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}, 1\right] \mapsto \left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right), \qquad [x_1, \dots, x_n, 0] \mapsto \infty.$$

If $k = \mathbb{R}$, then this is denoted by $\mathbb{P}_n$. We set $\mathrm{PGL}_n(k) = \mathrm{GL}_n(k) / \lambda \mathrm{I}_n$. Recall that $\mathrm{SL}_2(\mathbb{Z})$ are those of determinant one. Then $\lambda \mathrm{I}_2 \cap \mathrm{SL}_2(\mathbb{Z}) = \{-\mathrm{I}_2, \mathrm{I}_2\}$. So, consequently $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) / \{-\mathrm{I}_2, \mathrm{I}_2\}$, and it acts on $\mathbb{P}_1$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\begin{pmatrix} x \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} ax + b \\ cx + d \end{pmatrix}\right] = \left[\begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix}\right].$$

Hence

$$\begin{array}{rcl} \mathrm{PSL}_2(\mathbb{Z}) \curvearrowright \mathbb{P}_1 & \longrightarrow & \mathbb{P}_1 \\ x & \longmapsto & \begin{cases} \dfrac{ax+b}{cx+d} & cx + d \ne 0 \\ \infty & cx + d = 0 \end{cases} , \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{array}$$

[3] Exercise

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Lemma 3.1.10. $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$.

Proof. A fact is that $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

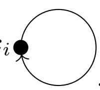
Hence $\mathrm{PSL}_2(\mathbb{Z})$ is generated by $[A] = A/\{\pm \mathrm{I}_2\}$ and $[B] = B/\{\pm \mathrm{I}_2\}$. Then

$$B^3 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \mathrm{I}_2 \equiv e_{\mathrm{PSL}_2(\mathbb{Z})}.$$

Similarly $A^2 = -\mathrm{I}_2$. Set $X_1 = \mathbb{R}_{<0}$ and $X_2 = \mathbb{R}_{>0}$. Then $[A] \cdot x = -1/x$, $[B] \cdot x = 1 - 1/x$, and $[B]^2 \cdot x = -1/(x-1)$. Now clearly $[A] \cdot X_i = X_j$, and $[B] \cdot X_1 \subsetneq X_2$ and $[B]^2 \cdot X_1 \subsetneq X_2$. Hence $\mathrm{PSL}_2(\mathbb{Z}) \cong \langle [A] \rangle * \langle [B] \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_3$. ☐

Remark. Free products appear naturally in topology. Let X_1 and X_2 be connected topological spaces, and $x_i \in X_i$. Then we can construct a new topological space X by gluing X_1 and X_2 on $x_1 = x_2$. Then $\pi_1(X) = \pi_1(X_1) * \pi_1(X_2)$.

Example. Let

$X_1 = X_2 = x_i$  .

Then $\pi_1(X_i, x_i) = \mathbb{Z}$ and $\pi_1(X, x_i) = \mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$.

If we glue X_1 and X_2 along subspaces of large dimension we naturally obtain amalgamated free products.

3.2 Amalgamated free products

In free products of G_1 and G_2 we have $G_1 \underset{G_1 * G_2}{\cap} G_2 = \{e_G\}$.

Definition 3.2.1. Let G_1 and G_2 be groups, and $A \leq G_1$ and $B \leq G_2$ such that $A \cong B$, say via $\phi : A \xrightarrow{\sim} B$. Then we call

$$G_1 * G_2 / \left\langle \left\{ a^{-1}\phi(a) \mid a \in A \right\} \right\rangle^{G_1 * G_2}$$

the **amalgamated free product** and denote it by

$$\langle G_1 * G_2 \mid \phi(a) = a, \ a \in A \rangle,$$

or $G_1 \underset{A=B}{*} G_2$ or $G_1 \underset{A}{*} G_2$, where in the last two cases the isomorphism should be mentioned, unless it is clear or irrelevant.

Remark 3.2.2.

- A corollary is that for any two G_1 and G_2 with subgroups $A \cong B$ there is some $G \cong G_1 \underset{A=B}{*} G_2$.
- If $A = \{e_G\}$, then $G_1 \underset{A}{*} G_2 = G_1 * G_2$. If $A = G_1$, then $G_1 \underset{A}{*} G_2 = G_2$.

Exercise 3.2.3.

- Assume $\phi_1, \phi_2 : A \to B$. Is
$$\langle G_1 * G_2 \mid \phi_1(a) = a \rangle \cong \langle G_1 * G_2 \mid \phi_2(a) = a \rangle?$$
- Assume $B' \cong B \leq G_2$. Is
$$G_1 \underset{A=B}{*} G_2 \cong G_1 \underset{A=B'}{*} G_2?$$

Exercise 3.2.4. If $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ with $S_1 \cap S_2 = \emptyset$, then

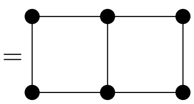
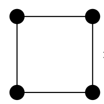
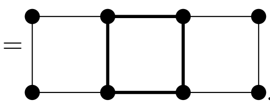
$$G_1 \underset{A=B}{*} G_2 = \left\langle S_1 \cup S_2 \;\middle|\; R_1 \cup R_2 \cup \left\{ a^{-1}\phi(a) \mid a \in A \right\} \right\rangle.$$

We will see that $G_i \overset{\iota_i}{\hookrightarrow} G_1 \underset{A}{*} G_2$, whence we identify, often, G_i with its image $\iota_i(G_i)$.

Example 3.2.5.

- Let $k \leq n, m$ and let $\mathbb{F}_k \leq \mathbb{F}_n, \mathbb{F}_m$, where $\mathbb{F}_k$ is the group generated by the first k-many generators. Then $\mathbb{F}_n \underset{\mathbb{F}_k}{*} \mathbb{F}_m \cong \mathbb{F}_{n+m-k}$. Indeed, if $\mathbb{F}_n = \langle a_1, \dots, a_n \mid \rangle$ and $\mathbb{F}_m = \langle b_1, \dots, b_m \mid \rangle$, and $\phi_i : a_i \mapsto b_i$ for $i \leq k$, by 3.2.4
$$\mathbb{F}_n \underset{\mathbb{F}_k}{*} \mathbb{F}_m = \left\langle a_1, \dots, a_n, b_1, \dots, b_m \mid a_i^{-1} b_i,\ i = 1, \dots, k \right\rangle.$$
By inverse Tietze 2 k-times, we get
$$\mathbb{F}_n \underset{\mathbb{F}_k}{*} \mathbb{F}_m = \langle a_{k+1}, \dots, a_n, b_1, \dots, b_m \mid \rangle \cong \mathbb{F}_{n+m-k}.$$

- Consider

$X =$  $\supset$  $= X_1$, $\quad X \underset{X_1}{\sqcup} X =$ .

Then $\pi_1(X) = \mathbb{F}_2$ and $\pi_1(X_1) = \mathbb{F}_1$, and
$$\pi_1 \left(X \underset{X_1}{\sqcup} X \right) = \mathbb{F}_3 \cong \mathbb{F}_2 \underset{\mathbb{F}_1}{*} \mathbb{F}_2.$$

Theorem 3.2.6 (Seifert-van Kampen)**.** *Let X be a path-connected topological space such that $X = X_1 \cup X_2$ for X_i path-connected. If $X_1 \cap X_2$ is path-connected, then*
$$\pi_1(X) = \pi_1(X_1) \underset{\pi_1(X_1 \cap X_2)}{*} \pi_1(X_2),$$
and furthermore
$$\begin{tikzcd} & \pi_1(X_1) \arrow[dr] \arrow[drr, bend left, "f"] & & \\ \pi_1(X_1 \cap X_2) \arrow[ur] \arrow[dr] & & \pi_1(X) \arrow[r, dashed, "\exists! h"] & \pi_1(Y) \\ & \pi_1(X_2) \arrow[ur] \arrow[urr, bend right, "g"'] & & \end{tikzcd}.$$

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Definition 3.2.7. Let G_1 and G_2 be groups, and $A \leq G_1$ and $B \leq G_2$, and such that $A \cong B$. Let T_A be the set of representatives of right cosets of A in G_1, and similarly T_B. Now, we call some tuple $(x_0, \dots, x_n)$ such that

- $x_0 \in A$, and
- $x_i \in \mathrm{T}_A \setminus \{e_{G_1}\} \cup \mathrm{T}_B \setminus \{e_{G_2}\}$, and $x_i \in \mathrm{T}_A$ if and only if $x_{i+1} \in \mathrm{T}_B$,

an A**-normal form**, with respect to T_A and T_B. We always pick e_{G_1} to represent A. Similarly, we can define a B**-normal form**.

Example 3.2.8. Let $G_1 = \left\langle a \mid a^{12} \right\rangle$ and $G_2 = \left\langle b \mid b^{15} \right\rangle$, let $A = \left\langle a^4 \right\rangle \cong \mathbb{Z}_3 \cong \left\langle b^5 \right\rangle = B$, and fix $\phi : a^4 \mapsto b^5$. Choose $\mathrm{T}_A = \left\{ e_{G_1}, a, a^2, a^3 \right\}$ and $\mathrm{T}_B = \left\{ e_{G_2}, b, b^2, b^3, b^4 \right\}$. Consider $g \in G = G_1 \underset{A=B}{*} G_2$, where
$$g = \left[a^3 b a^5 \right] = a^3 b a^5 / \left\langle a'^{-1} \phi(a') \mid a' \in A \right\rangle^{G_1 * G_2}.$$
Then
$$\begin{aligned} \left[a^3 b a^5 \right] &= \left[a^3 b a^4 a \right] && a^5 \in Aa \\ &= \left[a^3 b b^5 a \right] && a^4 = \phi\left(a^4\right) = b^5 \\ &= \left[a^3 b^5 b a \right] && b^6 \in Bb \\ &= \left[a^3 a^4 b a \right] && b^5 = \phi\left(a^4\right) = a^4 \\ &= \left[a^4 a^3 b a \right], \end{aligned}$$
and $\left(a^4, a^3, b, a\right)$ is an A-normal form.

Theorem 3.2.9. *The following are equivalent.*

- $G_1 \underset{A=B}{*} G_2$.
- *Every element $g \in G$ can be written uniquely as $g = x_0 \dots x_n$ where $(x_0, \dots, x_n)$ is in A-normal form.*

Proof.

$\Longrightarrow$ Assume $G \cong G_1 \underset{A=B}{*} G_2$. Recall that we identify the embedded images of G_1 and G_2 with the groups themselves. Fix $\mathrm{T}_A = \{x \mid x \in \mathrm{T}_A\}$ and $\mathrm{T}_B = \{y \mid y \in \mathrm{T}_B\}$. Then for every $g \in G_1$, there exist unique $a_g \in A$ and $x_g \in \mathrm{T}_A$ such that $g = a_g x_g \in A x_g$. Similarly for every $h \in G_2$, $h = b_h y_h$. Now, as $G = G_1 * G_2 / N$, we can write any $f \in G$ as $f = [g_1 h_1 \dots g_n h_n]$, where all but possibly g_1 and h_n are non-trivial. Now $h_n = b_{h_n} y_{h_n}$, and $[b_{h_n}] = [\phi(a_n)] = [a_n]$. Hence

$$\begin{aligned} [g_1 h_1 \dots g_n h_n] &= [g_1 h_1 \dots g_n a_n y_{h_n}] \\ &= [g_1 h_1 \dots a_{g_n a_n} x_{g_n a_n} y_{h_n}] && g_n a_n = a_{g_n a_n} x_{g_n a_n} \\ &= [g_1 h_1 \dots b_{n-1} x_{g_n a_n} y_{h_n}] && [b_{n-1}] = [\phi(a_{g_n a_n})] = [a_{g_n a_n}] \\ &= \dots \\ &= [a_{g_1 a_1} x_{g_1 a_1} y_{h_1 b_1} \dots x_{g_n a_n} y_{h_n}], \end{aligned}$$

and $(a_{g_1 a_1}, x_{g_1 a_1}, y_{h_1 b_1}, \dots, x_{g_n a_n}, y_{h_n})$ is an A-normal form. For uniqueness the idea is as follows. We define an action of G_1 and G_2 on W_A, the set of all A-normal forms. We extend this to G and show that if $g = x_0 \dots x_n$, then $g \cdot e_{G_1} = (x_0, \dots, x_n)$. Hence the normal form is unique. Note that there is a bijection between W_A and W_B via $f : (x_0, \dots, x_n) \mapsto (\phi(x_0), x_1, \dots, x_n)$. Now define $G_1 \curvearrowright \mathrm{W}_A$. Pick $z = (z_0, \dots, z_n) \in \mathrm{W}_A$. Then

$$g \cdot z = a_g x_g \cdot z = \begin{cases} (g z_0, z_1, \dots, z_n) & g \in A \\ (a_{g z_0}, x_{g z_0}, z_1, \dots, z_n) & g \notin A,\ z_1 \in G_2,\ z_1 \in \mathrm{T}_B \\ (g z_0 z_1, z_2, \dots, z_n) & g \notin A,\ z_1 \in G_2,\ g z_0 z_1 \in A \\ (a_{g z_0 z_1}, x_{g z_0 z_1}, z_2, \dots, z_n) & g \notin A,\ z_1 \in G_2,\ g z_0 z_1 \notin A \end{cases}.$$

This is an action. Similarly we define $G_2 \curvearrowright \mathrm{W}_B$. We shift that action to W_A via $h \cdot z = f^{-1}(h \cdot f(z))$ for $h \in G_2$ and $z \in \mathrm{W}_A$. Clearly, these actions extend to $G_1 * G_2 \curvearrowright \mathrm{W}_A$. Further for $z \in \mathrm{W}_A$ we have

$$\begin{aligned} \phi(a) a^{-1} \cdot z &= \phi(a) \cdot (a^{-1} z_0, z_1, \dots, z_n) = f^{-1}\left(\phi(a) \cdot f\left((a^{-1} z_0, z_1, \dots, z_n)\right)\right) \\ &= f^{-1}\left(\phi(a) \cdot \left(\phi(a)^{-1} \phi(z_0), z_1, \dots, z_n\right)\right) = f^{-1}(\phi(z_0), z_1, \dots, z_n) = (z_0, \dots, z_n) = z. \end{aligned}$$

Hence we can extend the action to $G = G_1 * G_2 / \left\langle \left\{ a^{-1} \phi(a) \mid a \in A \right\} \right\rangle^N$. If $g \in G$, we can write it as $g = z_0 \dots z_n$ where $(z_0, \dots, z_n)$ is an A-normal form. Then

$$g \cdot (e_{G_1}) = z_0 \dots z_n \cdot (e_{G_1}) = z_0 \dots z_{n-1} \cdot (e_{G_1}, z_n) = \dots = z_0 \cdot (e_{G_1}, z_1, \dots, z_n) = (z_0, \dots, z_n).$$

Thus every g has a unique A-normal form.

$\Longleftarrow$ Exercise. [4]

$\square$

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Corollary 3.2.10.

- *No A-normal form of length at least two can represent the trivial element in $G_1 \underset{A}{*} G_2$.*
- *We can identify $G_1 \underset{A}{*} G_2$ with W_A and multiplication given by the action just defined.*
- *Clearly, G_1 and G_2 embed onto $G_1 \underset{A}{*} G_2 = \mathrm{W}_A$ via $\iota_1(g) = (a_g, x_g)$ for $g \in G_1$ and $\iota_2(h) = \left(\phi^{-1}(b_h), y_h\right)$ for $h \in G_2$. Also $\iota_1(G_1) = \iota_2(G_2) = \iota_1(G_1) \cap \iota_2(G_2)$.*

[4] Exercise

Fact 3.2.11. Let G_1, G_2, A be groups such that A embeds via $i_k : A \hookrightarrow G_k$ for $k = 1, 2$. Then the following are equivalent.

- $G_1 \underset{i_1(A)=i_2(A)}{*} G_2 \cong G$.
- There are embeddings $j_k : G_k \hookrightarrow G$ with $j_1 \circ i_1 = j_2 \circ i_2$ such that for any H and homomorphisms $\phi_k : G_k \to H$ such that $\phi_1 \circ i_1 = \phi_2 \circ i_2$ there is a unique $\psi : G \to H$ such that $\psi \circ j_k = \phi_k$. Thus

$$
\begin{tikzcd}
 & G_1 \arrow[dr, hook, dashed, "j_1"] & \\
A \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & G \\
 & G_2 \arrow[ur, hook, dashed, "j_2"'] &
\end{tikzcd}, \qquad
\begin{tikzcd}
 & G_1 \arrow[dr, hook, "j_1"] \arrow[drr, bend left, "\phi_1"] & & \\
A \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & G \arrow[r, dashed, "\exists!\psi"] & H \\
 & G_2 \arrow[ur, hook, "j_2"'] \arrow[urr, bend right, "\phi_2"'] & &
\end{tikzcd}.
$$

3.3 Amalgamated products and trees

Definition 3.3.1. We call a graph X a **segment** if $X^0 = \{x, y\}$ and $X^1_+ = \{e\}$ such that $\alpha(e) = x$ and $\omega(e) = y$.

Theorem 3.3.2. *Let $G = G_1 \underset{A}{*} G_2$. Then there is a tree X such that G acts on X without inversion of edges and such that the factor graph Y is a segment. Furthermore, there is a lift $T = \{x, y, e\}$ of Y, such that* $\mathrm{Stab}\, x = G_1$*,* $\mathrm{Stab}\, y = G_2$*, and* $\mathrm{Stab}\, e = A$*.*

Proof. Let $X^0 = G/G_1 \cup G/G_2$ and $X^1_+ = G/A$, with $\alpha(gA) = gG_1$ and $\omega(gA) = gG_2$. First, let us show that X is connected. Let $g \in G$ and $g = z_0 \dots z_n$ in A-normal form. We show there is a path from G_1 to gG_i for $i = 1, 2$ by induction on n. If $n = 0$ then either $i = 1$ and $gG_i = z_0 G_1 = G_1$ and the empty path suffices, or $i = 2$ and $gG_i = G_2$, and $\alpha(A) = G_1$ and $\omega(A) = G_2$, so $e = A$ suffices. Now let $n > 0$ and set $g' = z_0 \dots z_{n-1}$. Then $z_n \in G_i$, whence $g'G_i = z_0 \dots z_{n-1} G_i = z_0 \dots z_{n-1} z_n G_i = gG_i$, and by assumption there is a path from G_1 to $g'G_1 = gG_1$. Now, assume there is a circuit $e_1 \dots e_n$. We may assume that $\alpha(e_1) = G_1$. As odd paths end in a coset of G_2, we know that n is even. Now there are $x_1 \in G_1$ and $x_i \in G_i \setminus A$ for $i > 1$, and $y_j \in G_2 \setminus A$ for $j \geq 1$ such that

$$
\begin{aligned}
\alpha(e_1) &= G_1 = x_1 G_1, & \omega(e_1) &= x_1 G_2 = x_1 y_1 G_2, \\
& & \vdots & \\
\alpha(e_n) &= x_1 y_1 \dots x_{\frac{n}{2}} y_{\frac{n}{2}} G_2, & \omega(e_n) &= x_1 y_1 \dots x_{\frac{n}{2}} y_{\frac{n}{2}} G_1 = G_1,
\end{aligned}
$$

a contradiction by uniqueness of A-normal forms. Now every vertex $g_1 G_1$ and $g_2 G_1$ are in the same orbit, as $g_2 g_1^{-1} \cdot g_1 G_1 = g_2 G_1$. The same for G_2. Hence $Y = [G_1] \xrightarrow{[A]} [G_2]$. Also all gA are in the same orbit. Then Y has a lift $G_1 \xrightarrow{A} G_A$ and $\mathrm{Stab}\, G_1 = G_1$, $\mathrm{Stab}\, G_2 = G_2$, and $\mathrm{Stab}\, A = A$. $\square$