M3P21 Geometry II: Algebraic Topology

Lectured by Dr Christian Urech
Typeset by David Kurniadi Angdinata

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Contents

0   Some underlying geometric notions

0.1   Introduction

Combines topological spaces with algebraic objects, groups.

- How to show that a torus is not homeomorphic to a sphere?

- How to show that $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n \neq m$?

Content is fundamental groups and homology. We will follow chapter one and two from

- A Hatcher, Algebraic topology, 2002

Prerequisites are the following.

- Point set topology. Topological spaces, continuous maps, product and quotient topologies, Hausdorff spaces, etc.

- Basic group theory. Normal subgroups and quotients, isomorphism theorems, free groups, presentation of groups, etc.

0.2   Homotopy

Let X, Y be topological spaces and $I = [0,1]$.

Definition. A **homotopy** is a continuous map $F : X \times I \to Y$. For every $t \in I$ we obtain a continuous map
$$
\begin{array}{rcl}
f_t : X & \to & Y \\
x & \mapsto & f_t(x) = F(x,t)
\end{array} .
$$

Definition. Two continuous maps $f_0, f_1 : X \to Y$ are **homotopic** if there exists a homotopy $F : X \times I \to Y$ such that
$$
f_0(x) = F(x,0), \qquad f_1(x) = F(x,1),
$$
for all $x \in X$. We write $f_0 \cong f_1$. (Exercise: this is an equivalence relation)

Definition. Let $A \subseteq X$ be a subspace. A **retraction** of X onto A is a continuous map $r : X \to A$ such that

- $r(X) = A$, and

- $r \mid_A = id_A$.

Example. If $X \neq \emptyset$, $p \in X$, then X retracts to p by the constant map $X \to \{p\}$.

Definition. A **deformation retraction** of X onto $A \subseteq X$ is a retraction that is homotopic to the identity. That is, there is a continuous map
$$
\begin{array}{rcl}
F : X \times I & \to & A \\
(x,t) & \mapsto & f_t(x)
\end{array} ,
$$
such that $f_0 = id_X$ and $f_1 : X \to A$ is the deformation retraction.

Example. The closed n-dimensional n-**disc**
$$
D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}
$$
deformation retracts to $(0, \ldots, 0) \in \mathbb{R}^n$. Let $f_t(x) = t \cdot x$. $t = 1$ gives $f_1 = id_{D^n}$ and $t = 0$ gives $f_0 : D^n \to (0, \ldots, 0)$.

Example. Let S^n be the n-**sphere**,
$$
\partial D^{n+1} = S^n = \{x \in \mathbb{R}^n \mid |x| = 1\}.
$$

The cylinder $S^n \times I$ deformation retracts to $S^n \times \{0\}$, by defining $f_t(x,r) = (x, t \cdot r)$.

An observation is if X is a topological space, and $f : X \to \{p\}$ for $p \in X$ is a deformation retraction of X to p, then X is path connected. Indeed, if $F : X \times I \to X$ is a homotopy from id_X to f and $x \in X$ is a point, then this gives a path

$$
\begin{array}{rcl}
I & \to & X \\
t & \mapsto & F(x, t)
\end{array}
$$

that connects x to p. This implies that not all retractions are deformation retractions.

Example. A retraction that is not a deformation retraction. Take a space that is not path connected and retract it to a point. Let $X = \{0, 1\}$ with discrete topology. $x \mapsto 0$ is a retraction, but not a deformation retraction because X is not path connected.

Definition. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \to X$ such that $fg \cong id_Y$ and $gf \cong id_X$. If there exists a homotopy equivalence between X and Y, X and Y are **homotopy equivalent** or they have the same **homotopy type**.

Lemma 0.1. *A deformation retraction $f : X \to A$ is a homotopy equivalence.*

Proof. Let $i : A \hookrightarrow X$ be the inclusion map. Then $fi = id_A$ and $if = f \cong id_X$ by definition. $\qquad\square$

Example. The disc with two holes is equivalent to ∞.

Example. $\mathbb{R}^n$ deformation retracts to a point, by $f_t(x) = t \cdot x$.

Definition.

- X is **contractible** if it is homotopy equivalent to a point.

- A continuous map is **nullhomotopic** if it is homotopy equivalent to a constant map.

0.3   Cell complexes

Example. The torus $S^1 \times S^1$ is the union of a point, two open intervals, and the open disc $Int(D^2)$.

These are called **cells**. Can think of discs D^n glued together.

Definition. A **CW-complex**, or **cell complex**, is a topological space X such that there exists a decomposition

$$
X = \bigcup_{n \in \mathbb{N}} X^n,
$$

where the X^n are constructed inductively in the following way.

- X^n is a discrete set.

- For each $n \geq 0$ there is an collection of closed n-discs $\{D_\alpha^n\}$ together with continuous maps $\phi_\alpha : \partial D_\alpha^n \to X^{n-1}$, such that

$$
X^n = \frac{X^{n-1} \sqcup \bigsqcup_\alpha D_\alpha^n}{\sim},
$$

where $x \sim \phi_\alpha(x)$ for all $x \in \partial D_\alpha^n$ for all α.

- A subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n.

Remark.

- As a set,

$$
X^n = X^{n-1} \sqcup \bigsqcup_\alpha e_\alpha^n,
$$

where each e_α^n is homeomorphic to an open n-disc. These e_α^n are called the n-**cells** of X.

- If $X = X^m$ for some m, then X is called **finite dimensional**. The minimal m such that $X = X^m$ is the **dimension** of X.

Lecture 2
Tuesday
15/01/19

Example.

- $[0, 1]$ is a CW-complex.

- $\mathbb{R}$ is a CW-complex.

- S^1 is a CW-complex.

- A graph is a CW-complex.

- $S^n = D^n / \partial D^n$ is a CW-complex. See worksheet 1.

Can also decompose CW-complexes.

- The sphere S^2 is one 0-cell, one 1-cell, and two 2-cells.

- The torus $S^1 \times S^1$ is one 0-cell, two 1-cells, and one 2-cell.

- The Möbius strip is two 0-cells, three 1-cells, and one 2-cell.

- The Klein bottle is one 0-cell, two 1-cells, and one 2-cell.

Definition. If X is a CW-complex with finitely many cells the **Euler characteristic** $\chi(X)$ of X is the number of even cells minus the number of odd cells.

Fact. $\chi(X)$ does not depend of the choice of cells decomposition.

Example.

- $\chi(S^n) = 0$ if n is odd and $\chi(S^n) = 2$ if n is even.

- $\chi(S^1 \times S^1) = 0$.

This is the generalisation of the following observation by Leonhard Euler. Let P be a convex polyhedron, where

- V is the number of vertices of P,

- E is the number of edges of P, and

- F is the number of faces of P.

Then $V - E + F = 2$.

Example. A topological space that is not a CW-complex. $X = \{0, 1\}$ with trivial topology does not contain any closed points.

Fact. CW-complexes are always Hausdorff.

1   The fundamental group

1.1   Paths and homotopy

Let X be a topological space. A **path** is a continuous map $f : I \to X$, where $I = [0, 1]$.

Definition. Two paths f_0, f_1 are **homotopic** if there exists a homotopy between f_0 and f_1 preserving the endpoints, that is a continuous map

$$F : \quad I \times I \quad \to \quad X$$
$$(s, t) \quad \mapsto \quad f_t(s) \quad ,$$

such that

$$f_t(0) = f_0(0), \qquad f_t(1) = f_0(1),$$

for all $t \in I$, and

$$F(s, 0) = f_0(s), \qquad F(s, 1) = f_1(s),$$

for all $s \in I$.

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set. Then all the paths in X are homotopic if they have the same endpoints.

Proof. Let $f_0, f_1 : I \to X$ be two paths such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$. Define

$$f_t(s) = (1 - t) f_0(s) + t f_1(s).$$

$\square$

Lemma 1.1. *Being homotopic is an equivalence relation on the set of paths with fixed endpoints. We will write $f_0 \cong f_1$ for two homotopic paths f_0 and f_1.*

Proof.

- f is homotopic to f.

- If f_0 is homotopic to f_1 by a homotopy f_t, then f_1 is homotopic to f_0 by the homotopy f_{1-t}.

- If f_0 is homotopic to f_1 by a homotopy f_t and $f_1 = g_0$ is homotopic to g_1 by a homotopy g_t, then f_0 is homotopic to g_1 by the homotopy

$$h_t = \begin{cases} f_{2t} & 0 \leq t \leq \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Then

$$H : \quad I \times I \quad \to \quad X$$
$$(s, t) \quad \mapsto \quad h_t(s)$$

is continuous because its restriction to the closed subsets $I \times [0, 1/2]$ and $I \times [1/2, 1]$ is continuous, since if the restriction to two closed subsets is continuous then the restriction to the union of these subsets is continuous.

$\square$

Let X be a topological space and $I = [0, 1]$. If $f : I \to X$ is a path, $[f]$ is the class of all paths on X homotopic to f.

Definition. Let $f, g : I \to X$ be two paths such that $f(1) = g(0)$. The **product path** $f \cdot g$ is the path

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \leq s \leq 1 \end{cases}.$$

A convention is that whenever we write $f \cdot g$ we implicitly assume $f(1) = g(0)$.

Lemma 1.2. *Let f_0, f_1, g_0, g_1 be paths on X such that $f_1 \cong f_0$ and $g_0 \cong g_1$. Then $f_0 \cdot g_0 \cong f_1 \cdot g_1$.*

Proof.

$$
\begin{array}{rcl}
I \times I & \to & X \\
(s,t) & \mapsto & (f_t \cdot g_t)(s)
\end{array}
$$

is a homotopy between $f_0 \cdot g_0$ and $f_1 \cdot g_1$.   □

Remark. Let $\phi : [0,1] \to [0,1]$ be continuous such that $\phi(0) = 0$ and $\phi(1) = 1$. If $f : I \to X$ is a path, then $f\phi \cong f$. This is a **reparametrisation**.

Proof. Define

$$
\phi_t(s) = (1-t)\phi(s) + ts,
$$

then $f\phi_t$ is a homotopy between $f\phi$ and f.   □

For $x \in X$, let the **constant path** at x be

$$
c_x : \begin{array}{rcl}
I & \to & X \\
s & \mapsto & x
\end{array} .
$$

For a path $f : I \to X$, define

$$
f^{-1} : \begin{array}{rcl}
I & \to & X \\
s & \mapsto & f(1-s)
\end{array} .
$$

Lemma 1.3. *Let $f, g, h : I \to X$ be paths. Then*

1. $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$,

2. $f \cdot c_{f(1)} \cong f$ *and* $c_{f(0)} \cdot f \cong f$, *and*

3. $f \cdot f^{-1} \cong c_{f(0)}$ *and* $f^{-1} \cdot f \cong c_{f(1)}$.

Proof.

1. $((f \cdot g) \cdot h)\phi = f \cdot (g \cdot h)$, where

$$
\phi(s) = \begin{cases}
\frac{s}{2} & s \in \left[0, \frac{1}{2}\right] \\
s - \frac{1}{4} & s \in \left[\frac{1}{2}, \frac{3}{4}\right] \\
2s - 1 & s \in \left[\frac{3}{4}, 1\right]
\end{cases} ,
$$

so $(f \cdot g) \cdot h \cong f \cdot (g \cdot h)$ by reparametrisation.

2. Again reparametrisation, by

$$
\psi(s) = \begin{cases}
2s & s \in \left[0, \frac{1}{2}\right] \\
1 & s \in \left[\frac{1}{2}, 1\right]
\end{cases} , \qquad
\chi(s) = \begin{cases}
0 & s \in \left[0, \frac{1}{2}\right] \\
2s - 1 & s \in \left[\frac{1}{2}, 1\right]
\end{cases} .
$$

3. Define

$$
H(s,t) = \begin{cases}
f(\max\{1 - 2s, t\}) & s \in \left[0, \frac{1}{2}\right] \\
f(\max\{2s - 1, t\}) & s \in \left[\frac{1}{2}, 1\right]
\end{cases} .
$$

H is continuous, and

$$
H(s, 0) = f^{-1} \cdot f, \qquad H(s, 1) = c_{f(1)}.
$$

The inverse is similar.

□

Definition. A **loop** with **basepoint** $x_0 \in X$ is a path $f : I \to X$ such that $f(0) = f(1) = x_0$.

Definition. Denote by $\pi_1(X, x_0)$ the set of homotopy classes $[f]$ of loops $f : I \to X$ with basepoint x_0.

Proposition 1.4. $\pi_1(X, x_0)$ *is a group with product* $[f][g] = [f \cdot g]$ *and neutral element* $c_{x_0} : I \to X$, *the constant path at* x_0.

Proof. Follows directly from Lemma 1.2 and Lemma 1.3. $\qquad\square$

Definition. $\pi_1(X, x_0)$ is the **fundamental group** of X at x_0.

Example. Let $X \subseteq \mathbb{R}^n$ be a convex set and $x_0 \in X$. Then $\pi_1(X, x_0) = 0$.

Proof. X is convex gives that all loops are homotopic to each other. $\qquad\square$

Example.

- The fundamental group of a space X with the trivial topology is trivial, since X is simply connected, because all maps $f : I \to X$ are continuous, so path connected and all paths are homotopic.

- The fundamental group of a space X with the discrete topology is trivial, since $f : I \to X$ continuous gives f constant.

Assume $x_0, x_1 \in X$ such that x_0 and x_1 are in the same path component of X. Let $h : I \to X$ be a path such that $h(0) = x_0$ and $h(1) = x_1$. Define

$$\begin{array}{rccc} \beta_h : & \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ & [f] & \mapsto & \left[h \cdot f \cdot h^{-1}\right] \end{array} \ .$$

This is well-defined by Lemma 1.2.

Proposition 1.5. $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ *is an isomorphism.*

Proof. It is a homomorphism.

$$\beta_h[f \cdot g] = \left[h \cdot f \cdot g \cdot h^{-1}\right] = \left[h \cdot f \cdot h^{-1}\right]\left[h \cdot g \cdot h^{-1}\right] = \beta_h[f] \cdot \beta_h[g],$$

and $\beta_h[c_{x_1}] = [c_{x_1}]$. It is bijective with $(\beta_h)^{-1} = \beta_{h^{-1}}$. $\qquad\square$

If X is path connected, we often write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Definition. X is **simply connected** if it is path connected and $\pi_1(X) = 0$.

Proposition 1.6. X *is simply connected if and only if there exists a unique homotopy class of paths between any two points of* X.

Proof.

$\implies$ There exists a path between any two points. Let f, g be two paths from x_0 to x_1 for $x_0, x_1 \in X$. $f \cdot g^{-1} \cong g \cdot g^{-1}$ gives $f \cong f \cdot g^{-1} \cdot g \cong g \cdot g^{-1} \cdot g \cong g$.

$\impliedby$ X is path connected. $x_1 = x_0$ gives that all loops at x_0 are homotopic to each other, so $\pi_1(X) = 0$.

$\qquad\square$

1.2   The fundamental group of the circle

Goal is to show that $\pi_1\left(S^1\right) \cong \mathbb{Z}$.

Definition. A **covering space** of a space X is a space $\widetilde{X}$ and a continuous map $p : \widetilde{X} \to X$ such that for each $x \in X$ there is an open $x \in U \subseteq X$ such that

- $p^{-1}\left(U\right) = \bigcup_{j \in J} \widetilde{U_j}$, where $\widetilde{U_j} \subseteq \widetilde{X}$ is open,

- $\widetilde{U_i} \cap \widetilde{U_j} = \emptyset$ if $i \neq j$, and

- $p\mid_{\widetilde{U_j}} : \widetilde{U_j} \to U_i$ is a homeomorphism for all $j \in J$.

Such a U is called **evenly covered**. The $\widetilde{U_j}$ are called **sheets**.

Example.

$$\begin{array}{rcl} p: & \mathbb{R} & \to \quad S^1 \\ & s & \mapsto \quad \left(\cos\left(2\pi s\right), \sin\left(2\pi s\right)\right) \end{array}$$

Definition. Let $p : \widetilde{X} \to X$ be a covering space. A **lift** of a continuous map $f : Y \to X$ is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$, so

$$\begin{array}{ccc} & & \widetilde{X} \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow p \\ Y & \underset{f}{\longrightarrow} & X \end{array}.$$

Proposition 1.7 (Unique lifting property). *Let $p : \widetilde{X} \to X$ be a covering space and $f : Y \to X$ be a continuous map. If there are two lifts $\widetilde{f_1}, \widetilde{f_2} : Y \to \widetilde{X}$ of f such that $\widetilde{f_1}\left(y\right) = \widetilde{f_2}\left(y\right)$ for some $y \in Y$ and if Y is connected, then $\widetilde{f_1} = \widetilde{f_2}$.*

Proof. Let $y \in Y$ and $U \subseteq X$ be an evenly covered neighbourhood of $f\left(y\right)$. Then

$$p^{-1}\left(U\right) = \bigcup_j \widetilde{U_j}.$$

Let $\widetilde{U_1}$ be the sheet such that $\widetilde{f_1}\left(y\right) \in \widetilde{U_1}$, and let $\widetilde{U_2}$ be the sheet such that $\widetilde{f_2}\left(y\right) \in \widetilde{U_2}$. Let $N \subseteq Y$ be open and $y \in N$ such that $\widetilde{f_1}\left(N\right) \subseteq \widetilde{U_1}$ and $\widetilde{f_2}\left(N\right) \subseteq \widetilde{U_2}$. We have $p\widetilde{f_1} = p\widetilde{f_2}$.

$$\widetilde{f_1}\left(y\right) = \widetilde{f_2}\left(y\right) \qquad \Longleftrightarrow \qquad \widetilde{U_1} = \widetilde{U_2} \qquad \Longleftrightarrow \qquad \widetilde{f_1}\mid_N = \widetilde{f_2}\mid_N .$$

Let

$$A = \left\{ y \in Y \mid \widetilde{f_1}\left(y\right) = \widetilde{f_2}\left(y\right)\right\},$$

so A is open and $Y \setminus A$ is open. Thus $A \neq \emptyset$ gives $A = Y$. $\square$

Proposition 1.8 (Homotopy lifting property). *Let $p : \widetilde{X} \to X$ be a covering space and $F : Y \times I \to X$ be a continuous map such that there exists a lift $\widetilde{f_0} : Y \times \{0\} \to \widetilde{X}$ of $F\mid_{Y \times \{0\}}$. Then there is a unique lift $\widetilde{F} : Y \times I \to \widetilde{X}$ of F such that $\widetilde{F}\mid_{Y \times \{0\}} = \widetilde{f_0}$.*

Proof. Let $y_0 \in Y$ and $t \in I$. There are open $y_0 \in N_t \subseteq Y$ and $t \in \left(a_t, b_t\right) \subseteq I$ such that $F\left(N_t \times \left(a_t, b_t\right)\right) \subseteq U \subseteq X$, where $U \subseteq X$ is open and evenly covered. Compactness of I gives that there exist

$$0 = t_0 < \cdots < t_m = 1,$$

and there exists $y_0 \in N \subseteq Y$ open such that $F\left(N \times \left[t_i, t_{i+1}\right]\right) \subseteq U_i \subseteq X$, where $U_i \subseteq X$ is open and evenly covered. We inductively construct a lift $\widetilde{F}\mid_{N \times I}$ of $F\mid_{N \times I}$.

- $\widetilde{F}\mid_{N\times[0,0]}=\widetilde{f_0}\mid_{N\times[0,0]}$ exists.

- Assume the lift has been constructed on $N\times[0,t_i]$. Let $\widetilde{U_i}\subseteq\widetilde{X}$ be such that $p\mid_{\widetilde{U_i}}\colon\widetilde{U_i}\to U_i$ such that $\widetilde{F}(y_0,t_i)\subseteq\widetilde{U_i}$. After shrinking N, may assume $\widetilde{F}(N\times\{t_i\})\subseteq\widetilde{U_i}$. Define $\widetilde{F}$ on $N\times[t_i,t_{i+1}]$ to be composition of F with the homeomorphism $p^{-1}\colon U_i\to\widetilde{U_i}$.

After finitely many steps we obtain a lift $\widetilde{F}\colon N\times I\to\widetilde{X}$, where $y_0\in N\subseteq Y$ is open, so for each $y\in Y$ there is a neighbourhood $N_y\subseteq Y$ such that $F\mid_{N_y\times I}\colon N_y\times I\to X$ lifts. For all $y\in Y$, $\{y\}\times I$ is connected and can be lifted, so Proposition 1.7 gives that the lift of $N\times I$ is unique. Thus there is a unique lift $\widetilde{F}\colon Y\times I\to\widetilde{X}$. $\hfill\square$

Example. Let X be a topological space and A be discrete. Then $p\colon X\times A\to X$ is a covering space. This is the **trivial covering**. (Exercise: show the unique lifting property and the homotopy lifting property for the trivial covering)

Corollary 1.9. *Let $f\colon I\to X$ be a path, $f(0)=x_0$, and $p\colon\widetilde{X}\to X$ be a covering space. For each $\widetilde{x_0}\in p^{-1}(x_0)$, there is a unique lift $\widetilde{f}\colon I\to\widetilde{X}$ such that $\widetilde{f}(0)=\widetilde{x_0}$.*

Proof. Proposition 1.8 for Y a point. $\hfill\square$

Theorem 1.10. *Let $x_0=(1,0)\in S^1$. $\pi_1\left(S^1,x_0\right)$ is the infinite cyclic group generated by the homotopy class of the loop*

$$\begin{array}{rcl}\omega\colon\quad I&\to&S^1\\ s&\mapsto&(\cos(2\pi s),\sin(2\pi s))\end{array}.$$

Remark.

- $[\omega]^n=[\omega_n]$, where

$$\omega_n(s)=(\cos(2\pi ns),\sin(2\pi ns)).$$

-
$$\begin{array}{rcl}p\colon\quad\mathbb{R}&\to&S^1\\ s&\mapsto&(\cos(2\pi s),\sin(2\pi s))\end{array}$$

  is a covering space.

- ω_n lifts to

$$\begin{array}{rcl}\widetilde{\omega_n}\colon\quad I&\to&\mathbb{R}\\ s&\mapsto&ns\end{array},$$

  such that $\widetilde{\omega_n}(0)=0$ and $\widetilde{\omega_n}(1)=n$.

Proof of Theorem 1.10.

- If $f\colon I\to S^1$ be a loop at x_0, then the homotopy lifting property gives that there exists a lift $\widetilde{f}\colon I\to\mathbb{R}$ such that $\widetilde{f}(0)=0$. Since $p\left(\widetilde{f}(1)\right)=f(1)=x_0$, then $\widetilde{f}(1)=n$ for some $n\in\mathbb{Z}$. $\widetilde{\omega_n}\colon I\to\mathbb{R}$ is another path such that $\widetilde{\omega_n}(0)=0$ and $\widetilde{\omega_n}(1)=n$, so $\widetilde{f}\cong\widetilde{\omega_n}$. Let $F\colon I\times I\to\mathbb{R}$ be a homotopy equivalence between $\widetilde{f}$ and $\widetilde{\omega_n}$. Then $pF\colon I\times I\to S^1$ gives a homotopy between $p\widetilde{f}=f$ and $p\widetilde{\omega_n}=\omega_n$.

- Let $m,n\in\mathbb{Z}$ and assume $\omega_m\cong\omega_n$. Let $F\colon I\times I\to S^1$ be a homotopy.

$$F(0,t)=\omega_m(t),\qquad F(1,t)=\omega_n(t),\qquad F(s,0)=F(s,1)=x_0,$$

  for all $s,t\in I$. The unique lifting property gives that $\widetilde{\omega_n},\widetilde{\omega_m}\colon I\to\mathbb{R}$ are unique lifts such that $\widetilde{\omega_n}(0)=0=\widetilde{\omega_m}(0)$. The homotopy lifting property gives that F lifts uniquely to a homotopy $\widetilde{F}\colon I\times I\to\mathbb{R}$ between $\widetilde{\omega_n}$ and $\widetilde{\omega_m}$, and $\widetilde{F}(s,1)\in\mathbb{Z}$ for all $s\in I$. Thus $\widetilde{F}(s,1)=n=m$, so $\omega_m\cong\omega_n$ if and only if $n=m$.

$\hfill\square$

Lecture 5 is a problem class.

Theorem 1.11. *Every non-constant polynomial $p \in \mathbb{C}[z]$ has a root in $\mathbb{C}$.*

Proof. May assume

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n.$$

Assume p has no roots in $\mathbb{C}$. For each $r \in \mathbb{R}_{\geq 0}$ we obtain a loop

$$
\begin{aligned}
f_r : \quad I \quad &\to \quad \mathbb{C} \\
s \quad &\mapsto \quad \frac{p\left(re^{2\pi is}\right)/p(r)}{\left|p\left(re^{2\pi is}\right)/p(r)\right|} \quad,
\end{aligned}
$$

so $|f_r(s)| = 1$. $f_r(0) = 1$ and $f_r(1) = 1$, so f_r is a loop based at 1. f_0 is the constant loop at 1. $f_r(s)$ depends continuously on r, so $f_r \cong f_0$ for all $r \in \mathbb{R}_{\geq 0}$ and $[f_r] = [f_0] = 0 \in \pi_1\left(S^1\right)$. Fix $r \in \mathbb{R}_{\geq 0}$ such that $r > 1$ and $r > |a_1| + \cdots + |a_n|$. For $|z| = r$ we have

$$|z^n| > (|a_1| + \cdots + |a_n|)\left|z^{n-1}\right| \geq \left|a_1 z^{n-1}\right| + \cdots + |a_n| \geq \left|a_1 z^{n-1} + \cdots + a_n\right|.$$

Hence, for $0 \leq t \leq 1$ the polynomial $p_t(z) = z^n + t\left(a_1 z^{n-1} + \cdots + a_n\right)$ has no root z with $|z| = r$. Define

$$F_r(t,s) = \frac{p_t\left(re^{2\pi is}\right)/p_t(r)}{\left|p_t\left(re^{2\pi is}\right)/p_t(r)\right|}.$$

$F_r(0,s) = \omega_n(s)$ and $F_r(1,s) = f_r(s)$, so $[\omega_n] = [f_r] = 0 \in \pi_1\left(S^1\right)$. Theorem 1.10 gives that $n = 0$, so p is constant. $\square$

See Hatcher Theorem 1.9 and Theorem 1.10 for more applications.

Proposition 1.12. *Let X, Y be topological spaces, $x_0 \in X$, and $y_0 \in Y$. Then*

$$\pi_1\left(X \times Y, (x_0, y_0)\right) \cong \pi_1\left(X, x_0\right) \times \pi_1\left(Y, y_0\right).$$

Proof. A map

$$
\begin{aligned}
f : \quad Z \quad &\to \quad X \times Y \\
z \quad &\mapsto \quad (g(z), h(z))
\end{aligned}
$$

is continuous if and only if $g : Z \to X$ and $h : Z \to Y$ are continuous. For $Z = I$,

$$\{ \text{ loops in } X \times Y \text{ based at } (x_0, y_0) \} \quad \longleftrightarrow \quad \{ \text{ loops in } X \text{ based at } x_0 \} \times \{ \text{ loops in } Y \text{ based at } y_0 \}.$$

Two loops

$$
\begin{aligned}
f_1 : \quad I \quad &\to \quad X \times Y & f_2 : \quad I \quad &\to \quad X \times Y \\
s \quad &\mapsto \quad (g_1(s), h_1(s)) \quad, & s \quad &\mapsto \quad (g_2(s), h_2(s))
\end{aligned}
$$

are homotopic if and only if $g_1 \cong g_2$ and $h_1 \cong h_2$, so there is a bijection

$$\pi_1\left(X \times Y, (x_0, y_0)\right) \cong \pi_1\left(X, x_0\right) \times \pi_1\left(Y, y_0\right).$$

$f_1 \cdot f_2 = (g_1 \cdot g_2, h_1 \cdot h_2)$ and the constant loop is mapped to the constant loop, so this is also a group isomorphism. $\square$

Example. The torus $S^1 \times S^1$ has

$$\pi_1\left(S^1 \times S^1\right) \cong \pi_1\left(S^1\right) \times \pi_1\left(S^1\right) \cong \mathbb{Z}^2.$$

1.3   Induced homomorphisms

Let X, Y be topological spaces, $x_0 \in X$, and $\phi : X \to Y$. An observation is that ϕ induces a homomorphism

$$\phi_* : \quad \pi_1(X, x_0) \quad \to \quad \pi_1(Y, \phi(x_0)) \\ [f] \quad \mapsto \quad [\phi f] \quad .$$

ϕ_* is well-defined, since if f_t is a homotopy between the loops f_0 and f_1 based at x_0, then ϕf_t is a homotopy of loops between ϕf_0 and ϕf_1. Moreover,

$$\phi(f \cdot g) = (\phi f) \cdot (\phi g),$$

and ϕ maps the constant path at x_0 to the constant path at $\phi(x_0)$, so ϕ is a homomorphism.

Proposition 1.13.

  1. *Let $\psi : X \to Y$ and $\phi : Y \to Z$ be continuous maps between topological spaces, $x_0 \in X$, and*

$$\psi_* : \pi_1(X, x_0) \to \pi_1(Y, \psi(x_0)), \qquad \phi_* : \pi_1(Y, \psi(x_0)) \to \pi_1(Z, \phi\psi(x_0)),$$

$$(\phi\psi)_* : \pi_1(X, x_0) \to \pi_1(Z, \phi\psi(x_0)).$$

  Then $(\phi\psi)_* = \phi_*\psi_*$.

  2. *Let $id_X : X \to X$ be the identity then*

$$(id_X)_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

  is the identity.

Proof.

  1. Let $f : I \to X$ be a loop at x_0, then

$$(\phi\psi)_*([f]) = [(\phi\psi)f] = [\phi(\psi f)] = \phi_*([\psi f]) = \phi_*\psi_*([f]).$$

  2. $(id_X)_*([f]) = [id_X f] = [f]$.

$$\square$$

These two observations yield in particular that if $\phi : X \to Y$ is a homeomorphism with inverse $\psi : Y \to X$, then

$$\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$$

is an isomorphism with inverse ψ_*.

Proposition 1.14. *Let $\phi : X \to Y$ be a homotopy equivalence. Then*

$$\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$$
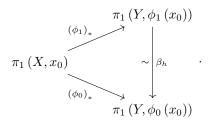
is an isomorphism for all $x_0 \in X$.

Recall that if $x_0, x_1 \in X$ and $h : I \to X$ is a path such that $h(0) = x_0$ and $h(1) = x_1$, then we obtain an isomorphism

$$\beta_h : \quad \pi_1(X, x_1) \quad \to \quad \pi_1(X, x_0) \\ [f] \quad \mapsto \quad [h \cdot f \cdot h^{-1}] \quad .$$

Lemma 1.15. *Let $\phi_t : X \to Y$ be a homotopy and $x_0 \in X$. Define the path*

$$h : \quad I \quad \to \quad Y$$
$$s \quad \mapsto \quad \phi_s(x_0) \quad ,$$

where $h(0) = \phi_0(x_0)$ and $h(1) = \phi_1(x_0)$. Then $(\phi_0)_* = \beta_h(\phi_1)_*$, that is the following diagram commutes.

Proof. For $t \in I$, define the path

$$h_t : \quad I \quad \to \quad X$$
$$s \quad \mapsto \quad h(ts) \quad ,$$

where $h_t(0) = \phi_0(x_0)$ and $h_t(1) = h(t) = \phi_t(x_0)$. Let f be a loop at x_0. Define

$$F_t = h_t \cdot (\phi_t f) \cdot h_t^{-1}.$$

Then F_t is a loop at $\phi_0(x_0)$, which is continuous in t. So F_t is a homotopy of loops between

$$F_0 = h_0 \cdot (\phi_0 f) \cdot h_0^{-1} \cong \phi_0 f, \qquad F_1 = h_1 \cdot (\phi_1 f) \cdot h_1^{-1} = h \cdot (\phi_1 f) \cdot h^{-1}.$$

Hence

$$(\phi_0)_*([f]) = [\phi_0 f] = \left[h \cdot (\phi_1 f) \cdot h^{-1} \right] = \beta_h([\phi_1 f]) = \beta_h(\phi_1)_*([f]).$$
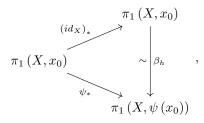
$\square$

Lemma 1.15 implies in particular the following.

Corollary 1.16. *If $\psi : X \to X$ is continuous and $\psi \cong id_X$, then*

$$\psi_* : \pi_1(X, x_0) \to \pi_1(X, \psi(x_0))$$

is an isomorphism for all $x_0 \in X$.

Proof. By Lemma 1.15 there is a path h from $\psi(x_0)$ to x_0 such that

so $\psi_* = \beta_h$ hence an isomorphism. $\square$

Proof of Proposition 1.14. Let $\phi : X \to Y$ be a homotopy equivalence. Let $\psi : Y \to X$ be a homotopy inverse of ϕ, that is $\phi\psi \cong id_Y$ and $\psi\phi \cong id_X$.

$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\phi(x_0)) \xrightarrow{\phi_*} \pi_1(Y, \psi\phi\psi(x_0)).$$

Have to show that ϕ_* is bijective. The observation above gives that $(\psi\phi)_* = \psi_*\phi_*$ is an isomorphism, so ϕ_* is injective and ψ_* is surjective. Similarly $(\phi\psi)_* = \phi_*\psi_*$ is an isomorphism, so ψ_* is injective and ϕ_* is surjective. $\square$

Lemma 1.17. *Let X be a topological space and $x_0 \in X$. Assume*

$$X = \bigcup_{\alpha \in \Lambda} A_\alpha,$$

such that

- *the A_α are all open and path connected,*

- *$x_0 \in A_\alpha$ for all $\alpha \in \Lambda$, and*

- *all the intersections $A_\alpha \cap A_\beta$ are path connected for all $\alpha, \beta \in \Lambda$.*

If f is a loop in X at x_0, then we can write $[f] = [h_1] \ldots [h_m]$, such that the h_i are loops at x_0, and each contained in a single A_{α_i}.

Proof. f is continuous, so for all $s \in I$ there is an open neighbourhood V_s such that $f(V_s)$ such that $f(V_s) \subseteq A_\alpha$ for some α. We can choose V_s to be an interval (a_s, b_s) such that $f([a_s, b_s]) \subseteq A_\alpha$. I is compact gives that a finite number of such intervals cover I, so there is a partition

$$0 = s_0 < \cdots < s_m = 1,$$

such that $f([s_{i-1}, s_i]) \subseteq A_{\alpha_i}$ for some α_i. Let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$, and rescaling. $f \cong f_1 \cdots \cdots f_m$ for $f_i \subseteq A_{\alpha_i}$ and $A_{\alpha_i} \cap A_{\alpha_j}$ is path connected. Let g_i be a path from x_0 to $f(s_i)$ in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$. Let g_0, g_m be the constant loops at x_0. $h_i = g_{i-1} \cdot f_i \cdot g_i^{-1}$ is a loop based at x_0 and $h_i \subseteq A_{\alpha_i}$. Thus

$$f \cong \left(g_0 \cdot f_1 \cdot g_1^{-1}\right) \cdot \cdots \cdot \left(g_{m-1} \cdot f_m \cdot g_m^{-1}\right),$$

so $[f] = [h_1] \ldots [h_m]$. $\qquad\square$

Example. Möbius strip M deformation retracts to S^1. Thus $\phi : M \to S^1$ is a homotopy equivalence, so $\pi_1(M) \cong \pi_1(S^1) \cong \mathbb{Z}$.

Example. There is no deformation retraction of S^1 to a point $p \in S^1$ because $\pi_1(S^1) \not\cong \pi_1(p)$.

Example. There is no retraction of the disc D^2 to its boundary $S^1 \subseteq D^2$.

Proof. Assume there is a retraction $r : D^2 \to S^1$, consider the embedding $i : S^1 \hookrightarrow D^2$. Then $ri = id_{S^1}$. Thus

$$\pi_1\left(S^1\right) \xrightarrow{\ i_* \ } \pi_1\left(D^2\right) \xrightarrow{\ r_* \ } \pi_1\left(S^1\right) ,$$
$$\underset{\mathbb{Z}}{\shortparallel} \qquad\qquad \underset{0}{\shortparallel} \qquad\qquad \underset{\mathbb{Z}}{\shortparallel}$$

so $r_* i_*\left(\pi_1\left(S^1\right)\right) = 0$ but $r_* i_* = (ri)_* = id_{\pi_1(S^1)}$, a contradiction. $\qquad\square$

Theorem 1.18 (Brouwer fixed point theorem). *Let $h : D^2 \to D^2$ be a continuous map. Then h has a fixed point, that is there exists $x \in D^2$ such that $h(x) = x$.*

Proof. Assume $h(x) \neq x$ for all $x \in D^2$. Define $r : D^2 \to S^1$ by defining $r(x)$ to be the intersection of the ray starting at $h(x)$ towards x with S^1. r is continuous, and if $x \in S^1$, then $r(x) = x$, so r is a retraction, a contradiction. $\qquad\square$

Lemma 1.17 gives that if $U_1, U_2 \subseteq X$ are open and path connected such that $U_1 \cup U_2 = X$ and $U_1 \cap U_2$ is path connected and $x_0 \in U_1 \cap U_2$, then every $[f] \in \pi_1(X, x_0)$ can be factorised as $[f] = [g_1][h_1] \ldots [g_n][h_n]$ such that the g_i are loops at x_0 contained in U_1 and the h_i are loops at x_0 contained in U_2. In other words, $i_1 : U_1 \hookrightarrow X$ and $i_2 : U_2 \hookrightarrow X$, so

$$(i_1)_* : \pi_1(U_1, x_0) \to \pi_1(X, x_0), \qquad (i_2)_* : \pi_1(U_2, x_0) \to \pi_1(X, x_0).$$

Lemma 1.17 gives that $(i_1)_*(\pi_1(U_1, x_0)) \cup (i_2)_*(\pi_1(U_2, x_0))$ generate $\pi_1(X, x_0)$.

Proposition 1.19. $\pi_1(S^n) = 0$ if $n \geq 2$.
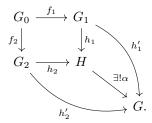
Proof. Let $U_1 = S^n \setminus \{(1, 0, \ldots, 0)\}$ and $U_2 = S^n \setminus \{(-1, 0, \ldots, 0)\}$. Then $U_1 \cong \mathbb{R}^n$ and $U_2 \cong \mathbb{R}^n$, by stereographic projection. $U_1 \cup U_2 = S^n$ and $U_1 \cap U_2$ is path connected. Let $x_0 \in U_1 \cap U_2$. $\pi_1(U_1, x_0) = 0$ and $\pi_1(U_2, x_0) = 0$, so Lemma 1.17 gives that $\pi_1(S^n, x_0)$. $\qquad\square$

1.4   Free products with amalgamation

Definition. If S is a set, then F_S is **the free group** on S. We can write any group G as a quotient of some free group F_S, $G = F/\langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ is the **normal closure** of $R \subseteq F_S$, the smallest normal subgroup of F_S containing R. We write $G = \langle S \mid R \rangle$. This is called a **presentation** of G.

Let G_0, G_1, G_2 be groups, and $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$ be homomorphisms.

Definition. A group H together with homomorphisms $h_1 : G_1 \to H$ and $h_2 : G_2 \to H$ such that $h_1 f_1 = h_2 f_2$ is an **amalgamated product** of G_1 and G_2 over G_0, if satisfies the following universal property. For every group G and all homomorphisms $h_1' : G_1 \to G$ and $h_2' : G_2 \to G$ such that $h_1' f_1 = h_2' f_2$, there exists a unique homomorphism $\alpha : H \to G$ such that $h_1' = \alpha h_1$ and $h_2' = \alpha h_2$.

$$
\begin{array}{ccc}
G_0 & \xrightarrow{\ f_1\ } & G_1 \\
{\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle h_1} \qquad {\scriptstyle h_1'} \\
G_2 & \xrightarrow{\ h_2\ } & H \\
& {\scriptstyle h_2'} & \qquad {\scriptstyle \exists! \alpha} \\
& & G.
\end{array}
$$

Theorem 1.20. *Given $f_1 : G_0 \to G_1$ and $f_2 : G_0 \to G_2$. Then there exists an amalgamated product, unique up to isomorphism. We denote it by $G_1 \underset{G_0}{*} G_2$.*

Proof. Worksheet 2.   □

1.5   The Seifert-van Kampen Theorem

Theorem 1.21 (Theorem of Seifert and van Kampen)**.** *Let X be a topological space and $U_1, U_2 \subseteq X$ be open and path connected such that $X = U_1 \cup U_2$ and $U_1 \cap U_2$ is path connected and let $x_0 \in U_1 \cap U_2$. Then*

$$
\pi_1(X, x_0) \cong \pi_1(U_1, x_0) \underset{\pi_1(U_1 \cap U_2, x_0)}{*} \pi_2(U_2, x_0).
$$

Thus

$$
\begin{array}{ccc}
U_1 \cap U_2 & \xhookrightarrow{\ i_1\ } & U_1 \\
{\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\
U_2 & \xhookrightarrow{\ j_2\ } & X
\end{array}
\qquad \Longrightarrow \qquad
\begin{array}{ccc}
\pi_1(U_1 \cap U_2, x_0) & \xrightarrow{\quad (i_1)_* \quad} & \pi_1(U_1, x_0) \\
{\scriptstyle (i_2)_*}\downarrow & & \downarrow{\scriptstyle (j_1)_*} \\
\pi_1(U_2, x_0) & \xrightarrow[\ (j_2)_*\ ]{} & \pi_1(U_1, x_0) \underset{\pi_1(U_1 \cap U_2, x_0)}{*} \pi_2(U_2, x_0)
\end{array}
\quad .
$$

A   Quotient topology

Recall that if X is a set with equivalence relation $\sim$, there is a quotient set $X/\sim$. The quotient map

$$\pi: \quad X \quad \to \quad \frac{X}{\sim}$$
$$x \quad \mapsto \quad [x]$$

is characterised by the following universal property. For every map $g : X \to Y$ such that

$$a \sim b \qquad \implies \qquad g\left(a\right) = g\left(b\right),$$

there exists a unique $f : X/\sim \to Y$ such that $g = f \cdot \pi$, so

$$
\begin{array}{c}
X \\
\pi \downarrow \quad \searrow g \\
\dfrac{X}{\sim} \xrightarrow[\exists! f]{} Y
\end{array}
\quad .
$$

Let X be a topological space and $\sim$ be an equivalence relation on X. We define a topology on $X/\sim$ by

$$U \subseteq \frac{X}{\sim} \text{ open} \qquad \Longleftrightarrow \qquad \pi^{-1}\left(U\right) \text{ open.}$$

Remark.

- This is the largest topology on $X/\sim$ such that π is continuous. Exercise 1 states that if Z is a topological space and $f : X/\sim \to Z$ is a map, then f is continuous if and only if $f\pi : X \to Z$ is continuous. This implies that the topological quotient $\pi : X \to X/\sim$ is characterised by the following universal property. For any topological space Z and a continuous $g : X \to Z$ such that

$$a \sim b \qquad \implies \qquad g\left(a\right) = g\left(b\right),$$

there exists a unique continuous map $f : X/\sim \to Z$ such that $gf \cdot \pi$, so

$$
\begin{array}{c}
X \\
\pi \downarrow \quad \searrow g \\
\dfrac{X}{\sim} \xrightarrow[\exists! f]{} Z
\end{array}
\quad .
$$

- The quotient map is in general not open. For example, if $\pi : [0,1] \to S^1$, then $[0,1] \subset [0,1)$ is open but $\pi\left([0,1)\right) \subseteq S^1$ is not open.

- If X is Hausdorff, in general $X/\sim$ is not Hausdorff.

- If $\sim$ is the trivial relation, then $\pi : X \to X/\sim$ is a homeomorphism. Exercise 3 states that if X, Y are topological spaces, X is compact, Y is Hausdorff, and $\pi : X \to Y$ is surjective and continuous, then π is a quotient, that is there exists $\sim$ on X and $\pi : X \to Y \cong X/\sim$ is a quotient map.

- In particular, if $\pi : X \to Y$ is bijective, then π is a homeomorphism. Exercise $4, 5, 6$ state that if f is continuous and surjective, $f\left(\partial D^n\right)$ is a point, and f is a bijection on $D^n \setminus \partial D^n$, then

$$
\begin{array}{c}
D^n \\
\pi \downarrow \quad \searrow f \\
\dfrac{D^n}{\partial D^n} \xrightarrow[\sim]{} S^1
\end{array}
\quad .
$$