M4P33 Algebraic Geometry

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Syllabus

Contents

0 Introduction

0.1 Bézout's theorem

Lecture 1 Monday 13/01/20

Here is an example of a theorem in algebraic geometry and an outline of a geometric method for proving it which illustrates some of the main themes in algebraic geometry.

Theorem 0.1 (Bézout)**.** *Let C be a plane algebraic curve $\{(x,y) \mid f(x,y) = 0\}$ where f is a polynomial of degree m. Let D be a plane algebraic curve $\{(x,y) \mid g(x,y) = 0\}$ where g is a polynomial of degree n. Suppose that C and D have no component in common, since if they had a component in common, then their intersection would obviously be infinite. Then $C \cap D$ consists of mn points, provided that*

- *we work over the complex numbers $\mathbb{C}$,*
- *we work in the projective plane, which consists of the ordinary plane together with some points at infinity, and*
- *we count intersections with the correct multiplicities, so if the curves are tangent at a point, it counts as more than one intersection.*

Consider the cases where C is a line of degree one and D has either degree one or two. The projective plane will be formally defined later in the course. We will not define intersection multiplicities in this course, but the idea is that multiple intersections resemble multiple roots of a polynomial in one variable.

Proof. We prove a special case, where C is the union of m lines, then use this to prove the general case of the theorem.

- First for the special case, suppose we have m lines in the plane, with equations

$$a_1x + b_1y + c_1 = 0, \qquad \dots, \qquad a_mx + b_my + c_m = 0.$$

  We can multiply these equations together to get

$$(a_1x + b_1y + c_1) \dots (a_mx + b_my + c_m) = 0.$$

  This is an equation of degree m and its solution set is the union of the lines. Each line intersects D in n points, counted with multiplicities, because we can rearrange the equation of the line into the form $x = \dots$ or $y = \dots$ then substitute into the equation for D. This usually gives a polynomial of degree n in one variable, and this has n roots if we count them correctly. There are also special cases to worry about where the line intersects D at infinity. Combining all the m lines, we deduce that their union intersects D in mn points.

- Now we deduce the general case from the special case. We let the curve C vary in a family of curves of degree m. What exactly we mean by varying in a family will be defined later in the course. As an example, consider the family of curves

$$\mathcal{F} : \left\{(x,y) \;\middle|\; x^2 - y^2 = t\right\},$$

  where t is a parameter, so for different values of t we get different curves. When the curve C varies in a family like this, the number of intersection points in $C \cap D$ does not change, counting with multiplicity. This is the core of the proof. It requires a lot of work to justify which we will not do here. For any degree m curve C, it is possible to find a family of curves which contains both C itself and a union of m lines X. For example, if C is the hyperbola defined by the equation $x^2 - y^2 = 1$, then it is found in the family $\mathcal{F}$, with $t = 1$. If we let $t = 0$ in this family, then the equation factors as $(x-y)(x+y)$ and this defines the union of two lines in the plane. We have already proved that $X \cap D$ has mn points, and we stated that $X \cap D$ has the same number of points as $C \cap D$ because C and X are in the same family. We conclude that $C \cap D$ has mn points.

$\square$

The idea that something stays the same everywhere, or almost everywhere, in a family of varying algebraic sets is a key theme in algebraic geometry. Note that this proof uses not just curves but also higher-dimensional algebraic sets. Instead of thinking about a family of curves such as $\mathcal{F}$, with coordinates (x,y) and a parameter t, we can regard x,y,t all as coordinates in three-dimensional space and consider the surface

$$\left\{(x,y,t) \mid x^2-y^2=t\right\}.$$

Then we use facts about this surface as part of the proof. We will not prove Bézout's theorem in this course. In particular, we will not define intersection multiplicities. But we will set up many of the tools needed to fill in the gaps in this outline proof.

0.2 Practical information about the course

The following are books.

- M Reid, Undergraduate algebraic geometry, 1988
- R Hartshorne, Algebraic geometry, 1977

During the course we will sometimes assume results from commutative algebra. Books which contain these results, and much much more, include the following.

- H Matsumura, Commutative ring theory, 1986
- M F Atiyah and I G Macdonald, Introduction to commutative algebra, 1969
- D Eisenbud, Commutative algebra: with a view toward algebraic geometry, 2011

The following is the course outline.

- Affine varieties.
  - Definition and examples.
  - Maps between varieties.
  - Translating between geometry and commutative algebra by the Nullstellensatz.
- Projective varieties.
  - Definition and examples.
  - Maps between varieties.
  - Rigidity and images of maps.
- Dimension.
  - Several different definitions, all equivalent, but useful for different purposes.
  - Calculating dimensions of examples.

What is not in the course?

- Schemes.
- Sheaves and cohomology.
- Curves, divisors, and the Riemann–Roch theorem.

1 Affine varieties

1.1 Affine algebraic sets

1.1.1 Affine space

Let k be an algebraically closed field. We are going to be thinking about solutions to polynomials, so everything is much simpler over algebraically closed fields. Number theorists might be interested in other fields, but you generally have to start by understanding the algebraically closed case first. In this course we will stop with the algebraically closed case too. Apart from being algebraically closed, it usually does not matter much which field we use to do algebraic geometry, except sometimes it matters whether the characteristic is zero or positive. In this course I will take care to mention results which depend on the characteristic, and sometimes we might consider only the characteristic zero case. You will not lose much if you just assume that $k = \mathbb{C}$ throughout the course, except when it will be explicitly something else. Indeed it is often useful to think about $k = \mathbb{C}$ because then you can use your usual geometric intuition. When I draw pictures on the whiteboard, I am usually only drawing the real solutions because it is hard to draw shapes in $\mathbb{C}^2$. This is cheating but it is often very useful. The real solutions are not the full picture but in many cases we can still see the important features there.

Definition 1.1. Algebraic geometers write $\mathbb{A}^n$ to mean k^n, and call it **affine n-space**.

You may think of this as just a funny choice of notation, but there are at least two reasons for it.

- When we write k^n, it makes us think of a vector space, equipped with operations of addition and scalar multiplication. But $\mathbb{A}^n$ means just a set of points, described by coordinates $(x_1, \dots, x_n)$ with $x_i \in k$, without the vector space structure.
- Because it usually does not matter much what our base field k is, as long as it is algebraically closed, it is convenient to have notation which does not prominently mention k.

On occasions when it is important to specify which field k we are using, we write $\mathbb{A}^n_k$ for affine n-space.

1.1.2 Definition and examples

Definition 1.2. An **affine algebraic set** is a subset $V \subseteq \mathbb{A}^n$ which consists of the common zeroes of some finite set of polynomials $f_1, \dots, f_m$ with coefficients in k. More formally, an affine algebraic set is a set of the form Lecture 2 Thursday 16/01/20

$$V = \{(x_1, \dots, x_n) \in \mathbb{A}^n \mid f_1(x_1, \dots, x_n) = \dots = f_m(x_1, \dots, x_n) = 0\}, \qquad f_1, \dots, f_m \in k[X_1, \dots, X_n].$$

Example. Examples.

- The empty set, defined by the polynomial $f_1 = 3$, for example.
- The whole space $\mathbb{A}^n$, defined by the polynomial $f_1 = 0$, or by the empty set of polynomials.
- Any finite subset $\{a_1, \dots, a_n\}$ in $\mathbb{A}^1$, defined by the polynomial $f_1 = (X - a_1) \dots (X - a_n)$.
- Any single-point set $\{(a_1, \dots, a_n)\}$ in $\mathbb{A}^n$, defined by the polynomials $f_i = X_i - a_i$. Note that this is different from the example of a finite set in $\mathbb{A}^1$, because that example had a single polynomial in one variable of degree n, while here we have n distinct polynomials in n variables of degree one.
- Any algebraic curve in $\mathbb{A}^n$, that is, a set of the form
$$\{(x_1, \dots, x_n) \in \mathbb{A}^n \mid f(x_1, \dots, x_n) = 0\}, \qquad f \in k[X_1, \dots, X_n].$$
- Embeddings of $\mathbb{A}^m$ in $\mathbb{A}^n$ where $m < n$,
$$\{(x_1, \dots, x_m, 0, \dots, 0) \in \mathbb{A}^n\} = \{(x_1, \dots, x_n) \in \mathbb{A}^n \mid x_{m+1} = \dots = x_n = 0\}.$$
More generally, the image of a linear map $\mathbb{A}^m \to \mathbb{A}^n$,
$$\{(x_1, \dots, x_n) \in \mathbb{A}^n \mid \text{some linear conditions}\}.$$

Example. Non-examples.

- Any infinite subset of $\mathbb{A}^1$, other than $\mathbb{A}^1$ itself, such as a line segment, a line with a double point, or an infinite discrete set. This is because a one-variable polynomial with infinitely many roots must be the zero polynomial. This also tells us that $\left\{x \in \mathbb{A}^1 \mid x \neq 0\right\}$ is not an affine algebraic set. However there is an affine algebraic set which is isomorphic to $\mathbb{A}^1 \setminus \{0\}$, namely $\left\{(x, y) \in \mathbb{A}^2 \mid xy - 1 = 0\right\}$. By looking at just the x coordinate, this set bijects to $\mathbb{A}^1 \setminus \{0\}$.
- A sine wave. If $\{(x, y) \mid y = \sin x\}$ were an affine algebraic set, then $\{(x, y) \mid y = \sin x,\ y = 0\}$ would also be an affine algebraic set because it is defined by imposing an extra polynomial condition, but the latter is an infinite discrete set.
- The example of the image of a linear map $\mathbb{A}^m \to \mathbb{A}^n$ does not generalise to images of maps where each coordinate is given by a polynomial. For example, consider the map

$$\begin{array}{cccc} \phi & : & \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2 \\ & & (x, y) & \longmapsto & (x, xy) \end{array}.$$

  The image of ϕ is $S = \mathbb{A}^2 \setminus \{(0, y)\} \cup \{(0, 0)\}$. To prove that S is not an affine algebraic set, consider a polynomial $g(X, Y) \in k[X, Y]$ which vanishes on S. For each fixed $y \in k$, the one-variable polynomial $g(X, y)$ vanishes at all $x \neq 0$. This implies that $g(X, y)$ is the zero polynomial. Thus $g(x, y) = 0$ for all $(x, y) \in k^2$, that is, g is the zero polynomial.

Remark 1.3. The words affine variety mean more or less the same thing as affine algebraic set but there is an ontological difference. Affine algebraic set means a subset which lives inside $\mathbb{A}^n$ and knows how it lives inside $\mathbb{A}^n$, while affine variety means an object in its own right which is considered outside of $\mathbb{A}^n$. I will try to use these words consistently, but the difference is quite subtle and books may not always use it consistently. For the first few weeks, we will talk about affine algebraic sets only. Note that some books, such as Reid and Hartshorne, have another difference between affine varieties and affine algebraic sets. They require varieties to be irreducible, which we will define next time. Other books, such as Shafarevich, do not require varieties to be irreducible. In this course we will not require varieties to be irreducible.

1.1.3 New algebraic sets from old

Now we prove that the union of two affine algebraic sets is an affine algebraic set. Consider two points $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ in $\mathbb{A}^n$. The two-point set $\{(a_1, \dots, a_n), (b_1, \dots, b_n)\}$ can be defined by taking the product for each possible pair of equations, one from each list, so $(X_i - a_i)(X_j - b_j) = 0$ for all $i, j \in \{1, \dots, n\}$.

Note. It is necessary to consider all the pairs between the lists, not just the ones with $i = j$, because otherwise we would be allowing points like $(a_1, \dots, a_{n-1}, b_n)$.

Lemma 1.4. *If $V, W \subseteq \mathbb{A}^n$ are affine algebraic sets, then their union $V \cup W \subseteq \mathbb{A}^n$ is also an affine algebraic set.*

Proof. We have to take the product for each possible pair of defining polynomials. If

$$V = \{\underline{x} \in \mathbb{A}^n \mid f_1(\underline{x}) = \dots = f_r(\underline{x}) = 0\}, \qquad W = \{\underline{x} \in \mathbb{A}^n \mid g_1(\underline{x}) = \dots = g_s(\underline{x}) = 0\},$$

then

$$V \cup W = \{\underline{x} \in \mathbb{A}^n \mid \forall 1 \leq i \leq r,\ \forall 1 \leq j \leq s,\ f_i(\underline{x}) g_j(\underline{x}) = 0\}.$$

Let us check that these equations really do define $V \cup W$. First, suppose that $\underline{x} \in V \cup W$. Then either $\underline{x} \in V$, so $f_i(\underline{x}) = 0$ for every i, so we can multiply by $g_j(\underline{x})$ to get $f_i(\underline{x}) g_j(\underline{x}) = 0$ for every i and j, or $\underline{x} \in W$, in which case the same argument works with g_j in place of f_i. The reverse direction is a little trickier. Suppose that we have $\underline{x} \in \mathbb{A}^n$ satisfying $f_i(\underline{x}) g_j(\underline{x}) = 0$ for all i and j. Looking just at f_1, we get

$$f_1(\underline{x}) g_1(\underline{x}) = 0 \implies f_1(\underline{x}) = 0 \text{ or } g_1(\underline{x}) = 0, \qquad \dots, \qquad f_1(\underline{x}) g_s(\underline{x}) = 0 \implies f_1(\underline{x}) = 0 \text{ or } g_s(\underline{x}) = 0.$$

Putting these all together, we get $f_1(\underline{x}) = 0$ or $g_j(\underline{x}) = 0$ for every j. We can do the same thing for f_2 to get $f_2(\underline{x}) = 0$ or $g_j(\underline{x}) = 0$ for every j, and so on for each f_i. Putting all these together, we get $f_i(\underline{x}) = 0$ for every i or $g_j(\underline{x}) = 0$ for every j. This says precisely that $\underline{x} \in V \cup W$. ☐

It is even easier to check that the intersection of finitely many affine algebraic sets is an affine algebraic set.

Lemma 1.5. *If $V, W \subseteq \mathbb{A}^n$ are affine algebraic sets, then their intersection $V \cap W \subseteq \mathbb{A}^n$ is also an affine algebraic set.*

Proof. Just combine the lists of defining equations. That is, say

$$V = \{\underline{x} \in \mathbb{A}^m \mid f_1(\underline{x}) = \dots = f_r(\underline{x}) = 0\}, \qquad W = \left\{\underline{y} \in \mathbb{A}^n \;\middle|\; g_1(\underline{y}) = \dots = g_s(\underline{y}) = 0\right\}.$$

Then $V \cap W$ is simply the set where all the polynomials in both lists vanish, that is

$$V \cap W = \{\underline{x} \in \mathbb{A}^n \mid f_1(\underline{x}) = \dots = f_r(\underline{x}) = g_1(\underline{x}) = \dots = g_s(\underline{x}) = 0\}.$$

□

Just a remark on one other way of constructing new affine algebraic sets from existing ones.

Lemma 1.6. *If $V \subseteq \mathbb{A}^m$ and $W \subseteq \mathbb{A}^n$ are affine algebraic sets, then their Cartesian product $V \times W \subseteq \mathbb{A}^{m+n}$ is an affine algebraic set.*

Proof. Write

$$V = \{\underline{x} \in \mathbb{A}^m \mid f_1(\underline{x}) = \dots = f_r(\underline{x}) = 0\}, \qquad W = \left\{\underline{y} \in \mathbb{A}^n \;\middle|\; g_1(\underline{y}) = \dots = g_s(\underline{y}) = 0\right\}.$$

Then

$$V \times W = \left\{\left(\underline{x}, \underline{y}\right) \in \mathbb{A}^{m+n} \;\middle|\; f_1(\underline{x}) = \dots = f_r(\underline{x}) = g_1(\underline{y}) = \dots = g_s(\underline{y}) = 0\right\}.$$

□

This looks a bit like the equations defining $V \cap W$, but here the f_i involve different variables from the g_j, while for $V \cap W$ both used the same variables.

1.1.4 Ideals and algebraic sets

Lecture 3 Friday 17/01/20

The union of infinitely many affine algebraic sets is not always an affine algebraic set. I do not mean that it is never an affine algebraic set, just that there exist counter-examples. Indeed, any subset of $\mathbb{A}^n$ can be written as a union of single-point sets. The intersection of infinitely many affine algebraic sets always an affine algebraic set. If we try to prove this by combining the lists of defining equations, we run into a problem. In our definition of affine algebraic sets we only allowed a finite list of polynomial equations. We introduce ideals to remove this restriction.

Definition 1.7. Recall from commutative algebra that, if R is a ring, an **ideal** is a subset $I \subseteq R$ with the properties that

- if $f, g \in I$, then $f + g \in I$, and
- if $f \in I$ and $q \in R$, then $qf \in I$.

Given any subset $S \subseteq R$, we define the **ideal generated by** S to be the smallest ideal which contains S, and denote it by $\langle S \rangle$. In particular, if S is the finite set $\{f_1, \dots, f_m\}$ then it generates the ideal

$$\langle f_1, \dots, f_m \rangle = \{q_1 f_1 + \dots + q_m f_m \mid q_1, \dots, q_m \in R\}.$$

Let us introduce some notation.

Definition 1.8. For any set $S \subseteq k[X_1, \dots, X_n]$, let

$$\mathbb{V}(S) = \{\underline{x} \in \mathbb{A}^n \mid \forall f \in S,\ f(\underline{x}) = 0\}.$$

Lemma 1.9. *If $S \subseteq k[X_1, \dots, X_n]$ generates the ideal I, then $\mathbb{V}(S) = \mathbb{V}(I)$.*

Proof. We have $S \subseteq I$ and so it is easy to see that $\mathbb{V}(I) \subseteq \mathbb{V}(S)$. Suppose that $\underline{x} \in \mathbb{V}(S)$, and $f \in \mathbb{V}(I)$. Then there are $f_1, \dots, f_m \in S$ and $q_1, \dots, q_m \in k[X_1, \dots, X_n]$ such that $f = q_1 f_1 + \dots + q_m f_m$. Since $f_1(\underline{x}) = \dots = f_m(\underline{x}) = 0$, it follows that $f(\underline{x}) = 0$. Since this holds for every $f \in I$, $\underline{x} \in \mathbb{V}(I)$. □

Theorem 1.10 (Hilbert basis theorem)**.** *From commutative algebra, if k is any field, then the polynomial ring $k[X_1, \dots, X_n]$ is Noetherian. That means that the following two equivalent conditions hold.*

- *Let I be an ideal in $k[X_1, \dots, X_n]$. Then there exists a finite set $\{f_1, \dots, f_m\} \subseteq k[X_1, \dots, X_n]$ which generates I.*
- *Let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of ideals in $k[X_1, \dots, X_n]$. Then there is some N such that $I_n = I_N$ for every $n > N$.*

Using the Hilbert basis theorem, we can deduce that the restriction to finite lists of polynomials in the definition of affine algebraic sets is unnecessary.

Corollary 1.11. $\mathbb{V}(S)$ *is an affine algebraic set for any set of polynomials $S \subseteq k[X_1, \dots, X_n]$.*

Proof. Let I be the ideal in $k[X_1, \dots, X_n]$ generated by S. By the Hilbert basis theorem, $k[X_1, \dots, X_n]$ is Noetherian and so we can choose a finite set $\{f_1, \dots, f_m\}$ which generates I. Then Lemma 1.9 tells us that $\mathbb{V}(S) = \mathbb{V}(I) = \mathbb{V}(f_1, \dots, f_m)$. ☐

Corollary 1.12. *The intersection of finitely many affine algebraic sets is an affine algebraic set.*

Proof. Combine the lists of defining polynomials for all the algebraic sets, and apply Corollary 1.11. ☐

We can also go in the other direction, from affine algebraic sets to ideals. Say $V_n = \mathbb{V}(I_n)$. Does $V_1 \supseteq V_2$ imply that $I_1 \subseteq I_2$? No. The problem is that there is more than one ideal defining the same algebraic set.

Example. Let $I_1 = \langle X \rangle$ and $I_2 = \langle X^2 \rangle$ in $k[X]$. We have $\mathbb{V}(I_1) = \{0\} = \mathbb{V}(I_2)$.

However, there is a natural choice we can make for one ideal canonically associated with an affine algebraic set, the set of all polynomials which vanish on that set.

Definition 1.13. Formally, if A is any subset of $\mathbb{A}^n$, usually A will be an affine algebraic set, we define

$$\mathbb{I}(A) = \{f \in k[X_1, \dots, X_n] \mid \forall \underline{x} \in A, \ f(\underline{x}) = 0\}.$$

Note. $\mathbb{I}(A)$ is an ideal in $k[X_1, \dots, X_n]$.

We have now defined two functions

$$\mathbb{V} : \{\text{ideals in } k[X_1, \dots, X_n]\} \to \{\text{affine algebraic sets in } \mathbb{A}^n\},$$

$$\mathbb{I} : \{\text{affine algebraic sets in } \mathbb{A}^n\} \to \{\text{ideals in } k[X_1, \dots, X_n]\}.$$

These functions are not inverses of each other. The example of $\langle X \rangle$ and $\langle X^2 \rangle$ shows that $\mathbb{I}(\mathbb{V}(\langle X^2 \rangle)) = \langle X \rangle \ne \langle X^2 \rangle$. But composing $\mathbb{V}$ and $\mathbb{I}$ in the other order gives the identity.

Lemma 1.14. *If V is an affine algebraic set, then $\mathbb{V}(\mathbb{I}(V)) = V$.*

Proof. It is clear that $V \subseteq \mathbb{V}(\mathbb{I}(V))$, and this works when V is any subset of $\mathbb{A}^n$, not necessarily algebraic. For the reverse inclusion, we have to use the hypothesis that V is an affine algebraic set. By the definition of affine algebraic sets, $V = \mathbb{V}(J)$ for some ideal $J \subseteq k[X_1, \dots, X_n]$. Suppose that $\underline{y} \notin V$. We shall show that $\underline{y} \notin \mathbb{V}(\mathbb{I}(V))$. Because $\underline{y} \notin V = \mathbb{V}(J)$, there exists $f \in J$ such that $f(\underline{y}) \ne 0$. By definition, $J \subseteq \mathbb{I}(V)$ and so $f \in \mathbb{I}(V)$. Hence $f(\underline{y}) \ne 0$ tells us that $\underline{y} \notin \mathbb{V}(\mathbb{I}(V))$. ☐

What is the geometric interpretation of the Hilbert basis theorem?

Note. It is clear that $\mathbb{V}$ and $\mathbb{I}$ reverse the direction of inclusions, so if $I_1 \subseteq I_2$, then $\mathbb{V}(I_2) \subseteq \mathbb{V}(I_1)$.

Hence the ascending chain condition for ideals translates into the descending chain condition for affine algebraic sets. The following statement is the translation into affine algebraic sets of the Hilbert basis theorem.

Lemma 1.15. *Let $V_1 \supseteq V_2 \supseteq \dots$ be a descending chain of affine algebraic sets in $\mathbb{A}^n$. Then there exists N such that $V_n = V_N$ for all $n > N$.*

Proof. The fact that $V_1 \supseteq V_2 \supseteq \dots$ implies that $\mathbb{I}(V_1) \subseteq \mathbb{I}(V_2) \subseteq \dots$. Because $k[X_1, \dots, X_n]$ is Noetherian, there exists N such that $\mathbb{I}(V_n) = \mathbb{I}(V_N)$ for all $n > N$. By Lemma 1.14, $V_n = \mathbb{V}(\mathbb{I}(V_n))$ for every n and so this proves the proposition. ☐

1.1.5 Statement of the Nullstellensatz

When does $\mathbb{I}(\mathbb{V}(I)) = I$? It turns out that the only reason that this can fail is where elements of the ideal I have n-th roots which are not in I, just as with the example of $I = \langle X^2 \rangle$ where $X^2 \in I$ has a square root X which is not in I. To state this precisely, we need to recall the definition of the radical of an ideal from commutative algebra.

Definition 1.16. Let I be an ideal in a ring R. The **radical** of I is

$$\operatorname{rad} I = \sqrt{I} = \{f \in R \mid \exists n > 0,\ f^n \in I\}.$$

We say that I is a **radical ideal** if $\operatorname{rad} I = I$.

Note. If I is any ideal, then $\operatorname{rad} I$ is always a radical ideal.

Theorem 1.17 (Hilbert's Nullstellensatz)**.** *Let I be any ideal in the polynomial ring $k[X_1, \dots, X_n]$ over an algebraically closed field k. Then*

$$\mathbb{I}(\mathbb{V}(I)) = \operatorname{rad} I.$$

This is a substantial theorem, fundamental to algebraic geometry. We will prove it in a few lectures' time, not because we need to develop more theory, just because I would like to introduce some more concepts first which will allow us to do more with examples.

Note. To calculate $\operatorname{rad} I$, we need to add in n-th roots of all elements of I, not just the generators.

Example. If $I = \langle X, Y^2 - X \rangle \subseteq k[X, Y]$, then we can rewrite this as $I = \langle X, Y^2 \rangle$ and so $\operatorname{rad} I = \langle X, Y \rangle \neq I$, even though neither of the original generators of I had any non-trivial n-th roots.

1.1.6 Basic facts about the Zariski topology

We have seen that affine algebraic sets in $\mathbb{A}^n$ satisfy the following conditions.

- $\mathbb{A}^n$ and $\emptyset$ are affine algebraic sets.
- A finite union of affine algebraic sets is an affine algebraic set.
- An arbitrary intersection of affine algebraic sets is an affine algebraic set.

The are precisely the conditions satisfied by the closed sets in a topological space. Therefore, we can define a topological space in which the underlying set is $\mathbb{A}^n$ and closed sets are the affine algebraic sets. This is called the **Zariski topology**. For any affine algebraic set $V \subseteq \mathbb{A}^n$, we define the **Zariski topology** on V to be the subspace topology on V induced by the Zariski topology on $\mathbb{A}^n$.