

M4P55 Commutative Algebra

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Syllabus

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0 Introduction

Lecture 1
Thursday
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The prerequisites are

- groups,
- rings,
- fields, and
- a solid linear algebra.

This course is good for

- algebraic geometry, and
- algebraic number theory.

The following are books.

- M Reid, Undergraduate commutative algebra, 1995
- M F Atiyah and I G Macdonald, Introduction to commutative algebra, 1969

The following is the structure of the course.

- Generalities on rings, such as ideals, and examples.
- Localisation of rings between a ring R and the fraction field K of R , such as \mathbb{Z} and \mathbb{Q} .
- Finiteness conditions of Noetherian rings and Artinian rings.
- Integral closure and normal rings, such as $\mathbb{Z}[i] \subset \mathbb{Q}(i)$ and $\mathbb{Z}[\sqrt{-3}] \subset \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right] \subset \mathbb{Q}(\sqrt{-3})$.
- Discrete valuation rings.
- Completion of rings with topology.

1 Rings and ideals

Definition 1.1. A **commutative ring** is a set $(A, +, \cdot, 0, 1)$ such that

1. $(A, +, 0)$ is an abelian group,
2. for all $x, y, z \in A$,
 - $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
 - $x \cdot y = y \cdot x$,
 - $x \cdot (y + z) = x \cdot y + x \cdot z$, and
3. for all $x \in A$, $x \cdot 1 = 1 \cdot x = x$.

Remark 1.2.

- One is uniquely determined by 3, since $1' = 1' \cdot 1 = 1$.
- If $1 = 0$, then $0 = x \cdot 0 = x \cdot 1 = x$, since

$$x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0,$$

so $x \cdot 0 = 0$. So every element is zero. Hence $R = \{0\}$.

Definition 1.3. A **homomorphism of rings** $f : A \rightarrow B$ is a map such that for all $x, y \in A$,

$$f(x + y) = f(x) + f(y), \quad f(xy) = f(x)f(y), \quad f(1) = 1.$$

Example. If $A \subset B$ is closed under $+$ and \cdot , and $1 \in A$, then

$$\begin{array}{ccc} A & \longrightarrow & B \\ x & \longmapsto & x \end{array}$$

is a homomorphism.

Remark 1.4.

- A composition of homomorphisms is a homomorphism.
- An **isomorphism** is a bijective homomorphism.

Definition 1.5. A subset I of a ring A is an **ideal** if I is a subgroup of the additive group $(A, +)$ which is closed under multiplication by elements of A , so $xI \subset I$ for any $x \in A$. Sometimes this is written as $I \triangleleft A$. In this case the **quotient group** A/I is naturally a ring, where $(x + I)(y + I)$ is defined as $xy + I$.

Proposition 1.6. Let I be an ideal of a commutative ring A . Then there is a natural bijection between the ideals $J \subset A$ such that $I \subset J$ and the ideals of A/I .

Proof. Let

$$\begin{array}{ccc} A & \longrightarrow & A/I \\ x & \longmapsto & x + I \end{array}$$

be the natural surjective map. Send J to its image under this map. □

Definition 1.7. If $f : A \rightarrow B$ is a homomorphism, then

$$\text{Ker } f = \{x \in A \mid f(x) = 0\}$$

is an ideal in A , and

$$\text{Im } f = f(A) \cong A / \text{Ker } f \subset B.$$

2 Polynomials and formal power series

Definition 2.1. Let R be a ring. The **polynomial ring** with coefficients in R is

$$R[x] = \{a_0 + \cdots + a_n x^n \mid a_i \in R, n \in \mathbb{Z}_{\geq 0}\}.$$

The addition is coefficient-wise, and the multiplication is given by the formula

$$\left(\sum_{i \geq 0} a_i x^i\right) \left(\sum_{j \geq 0} b_j x^j\right) = \sum_{i \geq 0} \left(\sum_{j+k=i, j \geq 0, k \geq 0} a_j b_k\right) x^i,$$

where all but finitely many coefficients are zero. Define

$$R[x_1, \dots, x_n] = R[x_1] \cdots [x_n] = \left\{ \sum_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n} \mid a_{i_1, \dots, i_n} \in R \right\},$$

where all but finitely many coefficients a_{i_1, \dots, i_n} are equal to zero.

Definition 2.2. The **ring of formal power series** with coefficients in R is

$$R[[t]] = \{a_0 + a_1 t + \dots \mid a_i \in R\}.$$

The addition is coefficient-wise, and the multiplication is given by the formula

$$\left(\sum_{i \geq 0} a_i t^i\right) \left(\sum_{j \geq 0} b_j t^j\right) = \sum_{i \geq 0} \left(\sum_{j+k=i, j \geq 0, k \geq 0} a_j b_k\right) t^i.$$

Define

$$R[[t_1, \dots, t_n]] = R[[t_1]] \cdots [[t_n]].$$

In $R[[t]]$ many products equal one unlike in $R[t]$, for example $(1-t)(1+t+\dots) = 1$.

3 Zero-divisors, nilpotents, units

Definition 3.1. Let A be a ring. An element $x \in A$ is a **zero-divisor** if $x \neq 0$ but $xy = 0$ for some $y \neq 0$ in A . A ring without zero-divisors is called an **integral domain**. An element $x \in A$ is **nilpotent** if $x^n = 0$ for some $n \in \mathbb{Z}_{>0}$. A **unit** $x \in A$ is an element such that $xy = 1$ for some $y \in A$. The units of A form a group under multiplication, denoted by A^* , or A^\times .

Definition 3.2. Let $x \in A$. Then the set

$$\langle x \rangle = \{xy \mid y \in A\}$$

is an ideal. Such ideals are called **principal ideals**.

Remark. $x \in A^*$ if and only if $\langle x \rangle = A$, and R is a field if and only if $R^* = R \setminus \{0\}$.

Proposition 3.3. Let A be a non-zero ring. Then the following are equivalent.

1. A is a field.
2. There are no ideals in A other than $\langle 0 \rangle$ and A .
3. Every non-zero homomorphism $f : A \rightarrow B$ is injective.

Proof.

1 \implies 2 Clear.

2 \implies 3 $\text{Ker } f \subset A$ is an ideal. Since $f \neq 0$, $\text{Ker } f \neq A$. Hence $\text{Ker } f = 0$.

3 \implies 1 Take any $x \neq 0$ in A . Look at $\langle x \rangle$. Define $B = A/\langle x \rangle$. Then take $f : A \rightarrow B$ to be the natural surjective map. If f is not identically zero, we get a contradiction with 3.

□

4 Prime ideals and maximal ideals

Definition 4.1. An ideal $I \subset A$ is called **prime** if $I \neq A$ and if whenever $xy \in I$, then $x \in I$ or $y \in I$. An ideal $J \subset A$ is called **maximal** if there is no ideal J' such that $J \subsetneq J' \subsetneq A$.

Notation. The set of prime ideals of A is called the **spectrum** of A and is denoted by $\text{Spec } A$.

Lemma 4.2. An ideal $I \subset A$ is prime if and only if A/I is an integral domain.

Proof. Obvious. □

Lemma 4.3. An ideal $J \subset A$ is maximal if and only if A/J is a field.

Proof. Obvious. □

Proposition 4.4. If $f : A \rightarrow B$ is a ring homomorphism and $I \subset B$ is a prime ideal, then $f^{-1}(I)$ is a prime ideal of A .

Proof. It is easy to see that $f^{-1}(I)$ is an ideal in A . Suppose $xy \in f^{-1}(I)$ for some $x, y \in A$. Then $f(x)f(y) = f(xy) \in I$. Since I is prime, $f(x) \in I$ or $f(y) \in I$, so $x \in f^{-1}(I)$ or $y \in f^{-1}(I)$. □

So we get a canonical map

$$\begin{aligned} f^* : \text{Spec } B &\longrightarrow \text{Spec } A \\ I \subset B &\longmapsto f^{-1}(I) \subset A \end{aligned}$$

Remark 4.5. If $f : A \rightarrow B$ is a ring homomorphism, then $f^{-1}(\mathfrak{p})$, where $\mathfrak{p} \subset B$ is a prime ideal, is a prime ideal. But this is false for maximal ideals. Let $A = \mathbb{Z}$, let $B = \mathbb{Q}$, and let $f(x) = x$. Then $\langle 0 \rangle \subset \mathbb{Q}$ is a maximal ideal and $f^{-1}(\langle 0 \rangle) = \langle 0 \rangle \subset \mathbb{Z}$ is not a maximal ideal. For example, $\langle 0 \rangle \subsetneq \langle 2 \rangle \subsetneq \mathbb{Z}$.

Theorem 4.6. Let A be a non-zero ring. Then A has at least one maximal ideal. In particular, $\text{Spec } A$ is not empty.

The proof is based on Zorn's lemma. Let S be a set. Then a **partial order** is a binary relation \leq such that

- $x \leq x$ for all $x \in S$,
- $x \leq y \leq z$ implies that $x \leq z$, and
- $x \leq y$ and $y \leq x$ imply that $x = y$,

where not all pairs are comparable. A **chain** $T \subset S$ is a subset in which every two elements are comparable.

Lemma 4.7 (Zorn). Suppose that S is a partially ordered set such that every chain $T \subset S$ has an upper bound, that is an element $t \in S$ such that $x \leq t$ for all $x \in T$. Then S has a maximal element, that is there exists $s \in S$ such that if $x \in S$ and $x \geq s$, then $x = s$.

Zorn's lemma is equivalent to the axiom of choice.

Proof of Theorem 4.6. Let Σ be the set of all ideals of A which are not equal to A . Then $\langle 0 \rangle \in \Sigma$, so $\Sigma \neq \emptyset$. Equip Σ with partial order given by inclusion. Enough to check the assumption of Zorn's lemma. Suppose T is a chain of ideals, so it is a collection of ideals J_i for $i \in T$. Consider instead

$$I = \bigcup_{i \in T} J_i.$$

Claim that T is a chain implies that I is an ideal. Then $x \in I$ implies that $x \in J_i$ for some i . Take any $x, y \in I$. Then $x \in J_i$ and $y \in J_k$ for some $i, k \in T$, so T is a chain, hence $i \leq k$ or $k \leq i$, so $J_i \subset J_k$ or $J_k \subset J_i$. Without loss of generality assume $J_i \subset J_k$. Then $x, y \in J_k$, so $x + y \in J_k \subset I$. Clearly, I is an upper bound. □

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Corollary 4.8. Any ideal of A is contained in a maximal ideal of A .

Proof. If $I \subset A$ is an ideal, apply Theorem 4.6 to A/I . □

Corollary 4.9. Any non-unit of A is contained in a maximal ideal.

Proof. Apply Corollary 4.8 to $\langle a \rangle$. □

Example. The maximal ideals of \mathbb{Z} are $\langle p \rangle$, where p is prime.

Definition 4.10. A ring A is **local** if A has exactly one maximal ideal.

Example. Any field is a local ring. If k is a field, then $k[[t]]$ is a local ring.

Lemma 4.11 (Prime avoidance). *Let A be a ring and let $\mathfrak{p} \subset A$ be a prime ideal. Suppose that I_1, \dots, I_n are ideals in A such that $\bigcap_{j=1}^n I_j \subset \mathfrak{p}$. Then $I_j \subset \mathfrak{p}$ for some j . If, moreover, $\bigcap_{j=1}^k I_j = \mathfrak{p}$, then $I_j = \mathfrak{p}$ for some j .*

Proof. Suppose that I_j is not a subset of \mathfrak{p} for any j . Then there exists $x_j \in I_j$ such that $x_j \notin \mathfrak{p}$. Hence

$$x_1, \dots, x_n \in I_1 \dots I_n \subset \bigcap_{j=1}^n I_j \subset \mathfrak{p},$$

so $x_1(x_2 \dots x_n) \in \mathfrak{p}$. Then $x_1 \notin \mathfrak{p}$ implies that $x_2 \dots x_n \in \mathfrak{p}$. Since \mathfrak{p} is prime we get a contradiction. For the second claim, we know that some $I_j \subset \mathfrak{p}$. But $\mathfrak{p} = \bigcap_{j=1}^k I_j \subset I_k$ for all k . Hence $\mathfrak{p} = I_j$. □

5 Nilradical and the Jacobson radical

Proposition 5.1. *The set $\mathcal{N}(A)$ consisting of all nilpotents of the ring A and zero is an ideal. Then $\mathcal{N}(A)$ is called the **nilradical** of A . The quotient $A/\mathcal{N}(A)$ has no nilpotents.*

Proof. Suppose $x \in A$ is nilpotent, so $x^n = 0$. For any $a \in A$, $(ax)^n = a^n x^n = 0$. Let x and y be nilpotents. Say $x^n = y^m = 0$. Then

$$(x+y)^{n+m} = \sum_{i,j \geq 0, i+j=n+m} a_{ij} x^i y^j, \quad a_{ij} \in A.$$

Clearly, either $i \geq n$ or $j \geq m$. Then $a_{ij} x^i y^j = 0$. Therefore, $(x+y)^{n+m} = 0$, hence $x+y \in \mathcal{N}(A)$. If $x + \mathcal{N}(A)$ is nilpotent in $A/\mathcal{N}(A)$, then $x^n + \mathcal{N}(A) = \mathcal{N}(A)$ is the trivial coset. Hence $x^n \in \mathcal{N}(A)$. Thus $(x^n)^m = 0$ for some m . □

Definition 5.2. A ring A such that $\mathcal{N}(A) = 0$ is called a **reduced ring**.

Proposition 5.3. $\mathcal{N}(A)$ is the intersection of all prime ideals of A .

Proof.

- ⊂ Let I be the intersection of all prime ideals of A . Let $f \in A$ be such that $f^n = 0$. Take any prime ideal $\mathfrak{p} \subset A$. We know that $f^n = 0 \in \mathfrak{p}$. Then $f(f \dots f) \in \mathfrak{p}$ and \mathfrak{p} prime implies that $f \in \mathfrak{p}$, so $f \in I$.
- ⊃ Let us prove the converse. Suppose f is not nilpotent, so $f^n \neq 0$ for all $n \geq 1$. We will show that there exists a prime ideal $\mathfrak{p} \subset A$ that does not contain f . Let us consider all ideals of A that do not contain f^m , where $m \in \mathbb{Z}_{>0}$. Let Σ be the set of ideals $J \subset A$ such that

$$J \cap \{f^m \mid m \geq 1\} = \emptyset.$$

The zero ideal $\langle 0 \rangle$ is in Σ . So $\Sigma \neq \emptyset$. Equip Σ with a partial order given by inclusion. Applying Zorn's lemma we obtain that Σ contains a maximal element. Call it \mathfrak{p} . By construction, $\mathfrak{p} \cap \{f^m \mid m \geq 1\} = \emptyset$, so $f \notin \mathfrak{p}$. It remains to prove that \mathfrak{p} is prime. Enough to prove that if $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$, then $xy \notin \mathfrak{p}$. Consider the ideal $\mathfrak{p} + \langle x \rangle \supsetneq \mathfrak{p}$. Since \mathfrak{p} is maximal in Σ , thus $\mathfrak{p} + \langle x \rangle$ is not in Σ . By definition of Σ there exists $n \geq 1$ such that $f^n \in \mathfrak{p} + \langle x \rangle$. Similarly, there exists $m \geq 1$ such that $f^m \in \mathfrak{p} + \langle y \rangle$. Then $(\mathfrak{p} + \langle x \rangle)(\mathfrak{p} + \langle y \rangle) \subset \mathfrak{p} + \langle xy \rangle$. In particular, $f^{n+m} = f^n \cdot f^m \in \mathfrak{p} + \langle xy \rangle$. If $xy \in \mathfrak{p}$, then $f^{n+m} \in \mathfrak{p}$, which is not possible. Therefore, $xy \notin \mathfrak{p}$. So \mathfrak{p} is a prime ideal that does not contain f . □

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Definition 5.4. The **Jacobson radical** $\mathcal{J}(A)$ is the intersection of all maximal ideals of A .

Proposition 5.5. $x \in \mathcal{J}(A)$ if and only if $1 - xy \in A^*$ for all $y \in A$.

Proof.

\Rightarrow Let $x \in \mathcal{J}(A)$. Suppose there exists $y \in A$ such that $1 - xy$ is not a unit. By Corollary 4.9 every non-unit is contained in a maximal ideal. Say $M \subset A$ is a maximal ideal and $1 - xy \in M$. But $x \in \mathcal{J}(A) \subset M$. Then $1 = (1 - xy) + xy \in M$, but then $M \neq A$. A contradiction.

\Leftarrow Given $x \in A$ such that $1 - xy \in A^*$ for all $y \in A$, we must have $x \in \mathcal{J}(A)$. If $x \notin \mathcal{J}(A)$, then there exists a maximal ideal $M \subset A$ such that $x \notin M$. Then $M + \langle x \rangle = A \ni 1$. Thus $1 = m + xy$, where $y \in A$. But by assumption $1 - xy \in A^*$, so $m \in A^*$. But then $M = A$. A contradiction.

□

Definition 5.6. Let I be an ideal of A . The **radical** of I is the set

$$\text{rad } I = \{x \in A \mid \exists n \geq 1, x^n \in I\}.$$

Proposition 5.7. The radical of I is the intersection of all prime ideals of A that contain I .

Proof. Apply Proposition 5.3 to A/I .

□

Definition 5.8. Let I be an indexing set. For each $i \in I$ we are given a ring R_i . Consider the product set $\prod_{i \in I} R_i$. This is $(x_i)_{i \in I}$ for $x_i \in R_i$. Define

$$0 = (0)_{i \in I} \in \prod_{i \in I} R_i, \quad 1 = (1)_{i \in I} \in \prod_{i \in I} R_i.$$

Define addition and multiplication coordinate-wise, so

$$(a_i)_{i \in I} + (b_i)_{i \in I} = (a_i + b_i)_{i \in I}, \quad (a_i)_{i \in I} \cdot (b_i)_{i \in I} = (a_i \cdot b_i)_{i \in I}, \quad (a_i)_{i \in I}, (b_i)_{i \in I} \in \prod_{i \in I} R_i.$$

Then $\prod_{i \in I} R_i$ is a ring, the **product of rings**.

A warning is if I has at least two elements, then $\prod_{i \in I} R_i$ has zero-divisors.

Example. $R_1 \times R_2$ has $(1, 0) \cdot (0, 1) = (0, 0) = 0$.

If $h_i : R \rightarrow R_i$ is a ring homomorphism for $i \in I$, then $(h_i)_{i \in I}$ is a ring homomorphism $R \rightarrow \prod_{i \in I} R_i$.

Remark 5.9. Let \mathfrak{p}_i for $i \in I$ be all prime ideals of R . Let $h_i : R \rightarrow R/\mathfrak{p}_i$. Then

$$h = (h_i)_{i \in I} : R \rightarrow \prod_{i \in I} R/\mathfrak{p}_i$$

is a homomorphism, and

$$\text{Ker } h = \bigcap_{i \in I} \text{Ker } h_i = \bigcap_{i \in I} \mathfrak{p}_i = \mathcal{N}(R).$$

So there is an injective map

$$R/\mathcal{N}(R) \hookrightarrow \prod_{i \in I} R/\mathfrak{p}_i,$$

a product of integral domains. Now take $f_j : R \rightarrow R/M_j$, so if we take the indexing set J to be the set of all maximal ideals of R , then we obtain an injective map

$$R/\mathcal{J}(R) \hookrightarrow \prod_{j \in J} R/M_j,$$

a product of fields.

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6 Localisation of rings

Example. Fix a prime p . Then

$$\mathbb{Z} \subset \left\{ \frac{m}{p^k} \mid m \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0} \right\} \subset \mathbb{Q}.$$

Definition 6.1. A subset S of a ring A is called a **multiplicative set** if $1 \in S$ and $0 \notin S$, and S is closed under multiplication.

Example 6.2.

- Let $a \in A$ be a non-nilpotent. Then $\{1, a, \dots\}$ is a multiplicative set.
- Let $\mathfrak{p} \subsetneq A$ be a prime ideal. Then $A \setminus \mathfrak{p}$ is a multiplicative set. Indeed, if $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$ then $xy \notin \mathfrak{p}$ by the definition of a prime ideal.
- If we have a family \mathfrak{p}_i for $i \in I$ of prime ideals, then $A \setminus \bigcup_{i \in I} \mathfrak{p}_i$ is a multiplicative set.
- A^* is a multiplicative set.
- All non-zero-divisors in A form a multiplicative set.
- Let $I \subsetneq A$ be an ideal. Then $1 + I = \{1 + x \mid x \in I\}$ is a multiplicative set.

Definition 6.3. Consider $A \times S$ and the equivalence relation on $A \times S$ defined as

$$(a, s) \sim (b, t) \iff \exists u \in S, u(at - bs) = 0.$$

Check that this is indeed an equivalence relation.¹ The following is some notation.

- The equivalence class of (a, s) is written as a/s . For example, if $t \in S$, then $a/s = at/st$.
- The set of equivalence classes is denoted by $S^{-1}A$.

Define

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

Need to check that these operations are well-defined.² Define $\frac{0}{1}$ as the zero of $S^{-1}A$, and $\frac{1}{1}$ as the one of $S^{-1}A$. Then $S^{-1}A$ is a ring, the **localisation of A with respect to a multiplicative set S** .

Lemma 6.4. *There is a ring homomorphism*

$$\begin{aligned} f : A &\longrightarrow S^{-1}A \\ x &\longmapsto \frac{x}{1} \end{aligned}.$$

This f is injective if and only if S has no zero-divisors.

Proof. If S contains a zero-divisor, say u , then there exists $a \in A$ for $a \neq 0$ such that $ua = 0$. Then

$$f(a) = \frac{a}{1} = \frac{au}{u} = \frac{0}{u} = 0.$$

So $\text{Ker } f$ contains a , hence f is not injective. If f has no zero-divisors, then $u \cdot a = u(a - 0) \neq 0$ if $a \neq 0$ and any $u \in S$. Hence $f(a) \neq 0$. \square

If A is an integral domain, then $\text{Ker } f = 0$. So $A \hookrightarrow S^{-1}A$.

¹Exercise

²Exercise

Example. Let $R = \mathbb{Z}$.

- If $S = \{1, a, \dots\}$, then

$$S^{-1}\mathbb{Z} = \left\{ \frac{n}{a^m} \mid n \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0} \right\}.$$

- If $S = \mathbb{Z} \setminus p\mathbb{Z}$, then

$$S^{-1}\mathbb{Z} = \left\{ \frac{n}{m} \mid p \nmid m \right\}.$$

- If $S = \mathbb{Z} \setminus \bigcup_{p_i \text{ prime}} p_i \mathbb{Z}$, then

$$S^{-1}\mathbb{Z} = \left\{ \frac{n}{m} \mid p_i \nmid m \right\}.$$

- If $S = \mathbb{Z}^* = \{\pm 1\}$, then $S^{-1}\mathbb{Z} = \mathbb{Z}$.
- If $S = \{\text{all non-zero elements}\}$, then $S^{-1}\mathbb{Z} = \mathbb{Q}$.
- If $S = \{1 + I \mid I \subset \mathbb{Z} \text{ ideal}\} = \{1 + nk \mid k \in \mathbb{Z}\}$, then

$$S^{-1}\mathbb{Z} = \left\{ \frac{m}{1 + nk} \mid m, k \in \mathbb{Z} \right\},$$

where n is fixed.

Example. Let $R = k[x]$, where k is a field.

- If $S = k[x]^* = k^*$, then $S^{-1}k[x] = k[x]$.
- If $S = \{\text{all non-zero elements}\}$, then

$$S^{-1}k[x] = k(x) = \left\{ \frac{f(x)}{g(x)} \mid g(x) \text{ arbitrary non-zero polynomial} \right\}.$$

Example 6.5. Let k be a field, and let $A = k[x, y] / \langle xy \rangle$. Note that A has zero-divisors, since $xy = 0$ in A , but $x \neq 0$ in A and $y \neq 0$ in A . Then $S = \{1, x, \dots\}$ is a multiplicative set, since $x^n \neq 0$ in A for $n = 1, 2, \dots$, because no power of the polynomial x is in $\langle xy \rangle$. What is $S^{-1}A$? Let $f : A \rightarrow S^{-1}A$. Then $a \in \text{Ker } f$ if and only if $a/1 = 0/1$, if and only if $u \cdot (a \cdot 1 - 0 \cdot 1) = 0$ for some $u \in S$, if and only if $ua = 0$. Let $a \neq 0$. Then $u = 1$ is not interesting. Take $u = x$ and $a = y$, then $xy = 0$, hence $y \in \text{Ker } f$. Then f is a homomorphism, hence $\text{Ker } f$ is an ideal. So $\langle y \rangle = yA \subset \text{Ker } f$. In general,

$$a = \sum_{i,j \geq 0} a_{ij} x^i y^j \equiv a_{00} + \sum_{i \geq 1} a_{i0} x^i + \sum_{j \geq 1} a_{0j} y^j \pmod{\langle xy \rangle}.$$

Then $\text{Ker } f = yA = \langle y \rangle$, since $\sum_{j \geq 1} a_{0j} y^j$ goes to zero, since it is annihilated by x , and $x^n \cdot \sum_{i \geq 0} a_i x^i$ is never zero in A . Thus $f(A) = k[x]$, and

$$S^{-1}A = \left\{ \frac{f(x)}{x^n} \mid f(x) \in k[x], n \geq 0 \right\} = k[x, x^{-1}] = \left\{ \sum_{i \in \mathbb{Z}, a_i = 0 \text{ for almost all } i} a_i x^i \mid a_i \in k \right\}.$$