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UROP Project

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Enumerative geometry on manifolds

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When we try to describe curves on a 2-sphere, things become difficult. If we work in $\mathbb{R}^3$, we increse the complexity. If we work in $\mathbb{R}^2$, indentifying some points on the sphere is inevitable, which over simplifies the question. To deal with a topological spaces like sphere are not easy, however we do have tools to describe such a space.

Def 2.1. An n-dimensional topological manifold is a topological space X together with a set of open sets on X U_α and maps $\phi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ such that

(1) $X \subseteq \bigcup_\alpha U_\alpha$
(2) Each ϕ_α is a homeomorphism from U_α to a open set $V_\alpha \subseteq \mathbb{R}^n$.

The pair (U_α, ϕ_α) is called a chart for X. The collection $\{(U_\alpha, \phi_\alpha)\}$ is called an atalas for X.

Each ϕ_α defines a system of coordinates on U_α, which is the usual coordinates $(x_1, x_2, x_3, ..., x_n)$ on $V_\alpha \subseteq \mathbb{R}^n$. Thiese are the local coordinates on U_α. Functions $f_{\alpha\beta} : V_\beta \longrightarrow V_\alpha$ defined by $f_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$are called the transition functions.

Now we can do calculations on manifolds use the local coordinates on U_α, and indentify calculations with different systems of coordinates using transion functions.

Eg 2.2. The 2-sphere S^2 is a manifold.pick north and south poles,$(0, 0, 1)$ and $(0, 0, -1)$ on $S^2 \subset \mathbb{R}^2$. We have one-to-one correspondences $\phi_1 : S^2 - \{(0, 0, 1)\} \longrightarrow \mathbb{R}^2$ and $\phi_2 : S^2 - \{(0, 0, -1)\} \longrightarrow \mathbb{R}^2$ defined by

$$\phi_1(x, y, z) = (\frac{x}{1-z}, \frac{y}{1-z}) \qquad \phi_2(x, y, z) = (\frac{x}{1+z}, \frac{y}{1+z})1$$

With inverses

$$\phi_1^{-1}(x, y) = (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1})$$
$$\phi_2^{-1}(x, y) = (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, -\frac{x^2+y^2-1}{x^2+y^2+1})$$

We can easly show these are homeomorphisms, so S^2 is a manifold.

To count the number of certain geometric varieties in a topological space, we need to have a way to describe all such varieties. We first consider linear ones.

Def 2.3. A Grassmannian $Gr(k, n)$ over $\mathbb{C}$ is the space of all k-dimensional linear subspace of $\mathbb{C}^n$.

Eg 2.4. $Gr(1,n)$ the projective spaces $\mathbb{P}^n$.

Eg 2.5. $Gr(2,4)$ the hyperplanes in $\mathbb{P}^4$.

Thm 2.6. *$Gr(k,n)$ is a manifold with dimension $k(n-k)$.*

Proof. □

Then we consider the case when our varieties has degree > 1.

Def 2.7. Let X be a complex manifold with a atalas $\{U_\alpha\}$. If we have holomorphic functions $g_{\alpha\beta}$ defined on each $U_\alpha \cap U_\beta$ such that $g_{\alpha\alpha} = 1$ and $g_{\alpha\gamma} = g_{\alpha\beta}g_{\beta\gamma}$ on each $U_\alpha \cup U_\beta \cap U_\gamma \neq \emptyset$, then we call $g_{\alpha\beta}$ transitions subordinate to $\{U_\alpha\}$.

Def 2.8. Let $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ be transition funtions subordinate to the open cover $U_{|alpha}$. $g \sim g'$ if there are nowhere vanishing holomorphic functions f_α on U_α such that $g_{\alpha\beta} = f_\alpha g'_{\alpha\beta} f_\beta^{-1}$. The set of equivalence classes is called the set of line bundles which can be trivialised over $\{U_\alpha\}$

Def 2.9. A line bundle on a topological manifold X with open cover $\{U_\alpha\}$ is a line bundle which can be trivialised over some open covering U_α. A line bundle L trivialised over U_α is equivalent to a line bundle L' trivialised over U'_β if they are equivalent after restricting all transition functions to the sets of open cover $\{U_\alpha \cap U'_\beta\}$

If we restricting to just complex manifold there is another way to view line bundles from a more geometric aspect.

Given transition functions $g_{\alpha\beta}$ subordinate to $\{U_\alpha\}$.We build a new complex manifold E of dimension $n+1$, where $n = \mathrm{dim}X$. Then E can be obtained from the cover $\{U_\alpha \times \mathbb{C}\}$ by indentifying $(p, v_\alpha) \in U_\alpha \times \mathbb{C}$ and $(q, v_\beta) \in U_\beta \times \mathbb{C}$ if $p = q$ and $v_\alpha = g_{\alpha\beta}(p) \cdot v_\beta$. From this relations we define charts on E:

$$\phi_\alpha \times 1 : U_\alpha \times \mathbb{C} \longrightarrow \mathbb{C}^n \times \mathbb{C} = \mathbb{C}^{n+1}$$

3 Transverse intersection and cohomology

4 Vector bundles and chern class

5 Applications