Imperial College London

M4R Project

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Enumerative geometry of lines

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Introduction

1 Grassmannian as projective variety

How can we parametrized all lines in a projective space $\mathbb{P}^n$? We know a line l in $\mathbb{P}^n$ corresponds to a 2-dimensional linear subspace of $\mathbb{A}^n$. So the the question transfers to parameterising linear subspaces of a vector space.

Definition 1.1. Let V be a vector space of dimension n, define $G(k, n)$ to be the set of all linear k-subspaces of V.

Example 1.2. A first example of such a space is projective spaces $\mathbb{P}^n$ or they can be denoted as $G(1, n+1)$. Lines in $\mathbb{P}^n$ form $G(2, n+1)$.

To avoid confusion, if Λ is a k-dimensional subspace of a n-dimension vector space V, then we use $[\Lambda] \in G(k, n)$ to denote the point corresponds to the subspace.

It can be shown that Grassmannian is a projective variety, so we can use all the tools in intersection theory to deal with it. The structure can be seen via a method called Plucker embedding.

Let $\Lambda \subset V$ be a k-dimensional subspace, then $\wedge^k \Lambda$ is a 1-dimensional subspace of $\wedge^n V$. More precisely, if $v_1, ..., v_k$ is a basis of Λ, then $\wedge^k \Lambda$ is the line spanned by $v_1 \wedge ... \wedge v_k$. This corresponds to a point of $\mathbb{P}\wedge^k V$.

(By $\mathbb{P}V$ we mean the quotient of V by the relation v w iff $v = \lambda w$. This is similar to the construction of projective space)

This gives a map of sets $G(k, n) \longrightarrow \mathbb{P}\wedge^k V \cong \mathbb{P}^{\binom{n}{k}-1}$. This map is called Plucker embedding.

Lemma 1.3. *This map $G(k, n) \longrightarrow \mathbb{P}\wedge^k V \cong \mathbb{P}^{\binom{n}{k}-1}$ is injective.*

Proof. let $v_1, ..., v_k$ be a basis of $\Lambda \subset V$, we can extend it to a basis of V by adding linear independent vectors $u_{k+1}, ...u_n$. Let $a = v_1 \wedge ... \wedge v_k$, then $\forall v \in V$, $v \wedge a = 0$ if and only if $(b_1 v_1 + ... b_n u_n) \wedge a = 0$ iff $b_{k+1} u_{k+1} \wedge a + ... + b_n u_n \wedge a = 0$ iff $v \in \Lambda$ so a determines Λ. $\square$

So $G(k, n)$ is isomorphic to its image in $\mathbb{P}\wedge^k V$. Lets call its image G. It left to show that its image is the common zero locus of some homogeneous polynomials. We can show this by express our k-subspace as a matrix.

Let $e_1, e_2, ..., e_n$ is a basis for V,we can identify V as k^n then any k-vector space is the span of k linear independent vectors in this basis. We can write them as a matrix:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \dots & a_{k,n} \end{pmatrix}$$

However, just like coordinates of points in a projective space, the matrix A is not unique. Since we can multiply on the left any invertible $k \times k$ matrix Ω without changing the row spaces, because the rows in ΩA are linear combinations of the rows in A. In this setting, $\bigwedge^k V$ is given by the set

$$\{e_{i1} \wedge ... \wedge e_{ik}\}_{1 \leq i1 < ... < ik \leq n}$$

After the Plucker embedding, this matrix get sent to the wedge product of row vectors:

$$v_1 \wedge ... \wedge v_k = \sum_{1 \leq i1 < ... < ik \leq n} D_{i1,...,ik} e_{i1} \wedge ... \wedge e_{ik}$$

Where $D_{i1,...,ik}$ is the determinant of k minors of the matrix A. However, just like coordinates of points in a projective space, the matrix A is not unique. Since we can multiply on the left any invertible $k \times k$ matrix Ω without changing the row spaces, because the rows in ΩA are linear combinations of the rows in A.If we choose another basis of Λ, then the result is multiply by the determinant of Ω. As Ω invertible, the determinant is never 0, so the product is always the same point in $\mathbb{P}\bigwedge^k V$.

Now we try to find these homogeneous polynomials that defines $Im(G(k,n))$. For any k-subspace, we can find k linearly independent vectors that span it. So $\omega \in \mathbb{P}\bigwedge^k V$ is in the image if and only if there exits $v_1, ..., v_k$(linearly independent) such that $\omega = v_1 \wedge ... \wedge v_k$.

Lemma 1.4. *$\omega \in \mathbb{P}\bigwedge^k V$ such that $\omega = v_1 \wedge ... \wedge v_k$ if and only if the map:*

$$V \xrightarrow{\wedge\omega} \mathbb{P}\bigwedge^{k+1} V$$

has kernel of dimension at least k.

Proof. ($\Rightarrow$) Trivial as v_i is basis of the kernel. ($\Leftarrow$) If the kernel has dimension at least k, then there exits $v_1, ..., v_k$(linearly independent), such that $v_i \wedge \omega = 0, 1 \leq i \leq k$.

Lets start from v_1, if $v_i \wedge \omega = 0$, then there exists $\omega' \in \mathbb{P}\bigwedge^{k-1} V$ such that $\omega = v_0 \wedge \omega'$. Repeat this process, we can see that $\omega = v_1 \wedge ... \wedge v_k$.

□

Then the image can be expressed as:

$$G = \{\omega \in \mathbb{P}\bigwedge^{k} V \mid \mathrm{rank}(V \xrightarrow{\wedge\omega} \mathbb{P}\bigwedge^{k+1} V) \leq n - k\}$$

This can be interpreted as the zero locus of degree $(n-k+1)$ homogeneous polynomials that are determinant of $(n-k+1)$ minors of the map $\wedge\omega : V \to \mathbb{P}\bigwedge^{k+1} V$ written as a matrix. So we can conclude that $G(k,n)$ is a variety.

2 From group of cycles to chow ring

A projective space can be covered by Zariski open subsets isomorphic to affine space. So we can define a local coordinates on projective space. Similarly, we can cover a Grassmannian $G = G(k, n)$ by Zariski open subsets as well. Too see this, fix an $(n-k)$-dimensional subspace $\Gamma \subset V$, and let U_Γ be the subset of k-subspaces that do not meet Γ:

$$U_\Gamma = \{L \ \in \ G \mid L \ \cap \ \Gamma = \emptyset\}$$

Lets show it is Zariski open: let $w_1, w_2, \dots, w_{n-k}$ be a basis of Γ and let $\gamma = w_1 \wedge \cdots \wedge w_{n-k}$then we have :

$$U_\Gamma = \{[\omega] \in G \subset \mathbb{P}(\wedge^k V) \mid \omega \wedge \gamma \neq 0\}$$

3 chern classes

References