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UROP PROJECT

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Enumerative geometry on manifolds

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Contents

1	Introduction	2
2	Manifolds and Grassmannian	3
3	Transverse intersection and cohomology	5
4	Enumerative problem with lines	8
5	Vector bundles and chern class	9
6	Applications	10

1 Introduction

2 Manifolds and Grassmannian

When we try to describe curves on a 2-sphere, things become difficult. If we work in \mathbb{R}^3 , we increase the complexity. If we work in \mathbb{R}^2 , indentifying some points on the sphere is inevitable, which over simplifies the question. To deal with a topological spaces like sphere are not easy, however we do have tools to describe such a space.

Def 2.1. An n -dimensional topological manifold is a topological space X together with a set of open sets on X U_α and maps $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ such that

- (1) $X \subseteq \bigcup_\alpha U_\alpha$
- (2) Each ϕ_α is a homeomorphism from U_α to a open set $V_\alpha \subseteq \mathbb{R}^n$.

The pair (U_α, ϕ_α) is called a chart for X . The collection $\{(U_\alpha, \phi_\alpha)\}$ is called an atlas for X .

Each ϕ_α defines a system of coordinates on U_α , which is the usual coordinates $(x_1, x_2, x_3, \dots, x_n)$ on $V_\alpha \subseteq \mathbb{R}^n$. These are the local coordinates on U_α . Functions $f_{\alpha\beta} : V_\beta \rightarrow V_\alpha$ defined by $f_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ are called the transition functions.

Now we can do calculations on manifolds use the local coordinates on U_α , and indentify calculations with different systems of coordinates using transition functions.

Eg 2.2. The 2-sphere S^2 is a manifold. pick north and south poles, $(0, 0, 1)$ and $(0, 0, -1)$ on $S^2 \subset \mathbb{R}^3$. We have one-to-one correspondences $\phi_1 : S^2 - \{(0, 0, 1)\} \rightarrow \mathbb{R}^2$ and $\phi_2 : S^2 - \{(0, 0, -1)\} \rightarrow \mathbb{R}^2$ defined by

$$\phi_1(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \quad \phi_2(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

With inverses

$$\begin{aligned} \phi_1^{-1}(x, y) &= \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right) \\ \phi_2^{-1}(x, y) &= \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, -\frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right) \end{aligned}$$

We can easily show these are homeomorphisms, so S^2 is a manifold.

To count the number of certain geometric varieties in a topological space, we need to have a way to describe all such varieties. We first consider linear ones.

Def 2.3. A Grassmannian $Gr(k, n)$ over \mathbb{C} is the space of all k -dimensional linear subspace of \mathbb{C}^n .

Eg 2.4. $Gr(1, n)$ the projective spaces \mathbb{P}^n .

Eg 2.5. $Gr(2, 4)$ the hyperplanes in \mathbb{P}^4 .

Thm 2.6. $Gr(k, n)$ is a manifold with dimension $k(n - k)$.

Proof.

□

Then we consider the case when our varieties has degree > 1 .

Def 2.7. Let X be a complex manifold with a atlas $\{U_\alpha\}$. If we have holomorphic functions $g_{\alpha\beta}$ defined on each $U_\alpha \cap U_\beta$ such that $g_{\alpha\alpha} = 1$ and $g_{\alpha\gamma} = g_{\alpha\beta}g_{\beta\gamma}$ on each $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, then we call $g_{\alpha\beta}$ transitions subordinate to $\{U_\alpha\}$.

Def 2.8. Let $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ be transition funtions subordinate to the open cover $U_{|alpha}$. $g \sim g'$ if there are nowhere vanishing holomorphic functions f_α on U_α such that $g_{\alpha\beta} = f_\alpha g'_{\alpha\beta} f_\beta^{-1}$. The set of equivalence classes is called the set of line bundles which can be trivialised over $\{U_\alpha\}$

Def 2.9. A line bundle on a topological manifold X with open cover $\{U_\alpha\}$ is a line bundle which can be trivialised over some open covering U_α . A line bundle L trivialised over U_α is equivalent to a line bundle L' trivialised over U'_β if they are equivalent after restricting all transition functions to the sets of open cover $\{U_\alpha \cap U'_\beta\}$

If we restricting to just complex manifold there is another way to view line bundles from a more geometric aspect.

Given transition functions $g_{\alpha\beta}$ subordinate to $\{U_\alpha\}$. We build a new complex manifold E of dimension $n + 1$, where $n = \dim X$. Then E can be obtained from the cover $\{U_\alpha \times \mathbb{C}\}$ by indentifying $(p, v_\alpha) \in U_\alpha \times \mathbb{C}$ and $(q, v_\beta) \in U_\beta \times \mathbb{C}$ if $p = q$ and $v_\alpha = g_{\alpha\beta}(p) \cdot v_\beta$. From this relations we define charts on E :

$$\phi_\alpha \times 1 : U_\alpha \times \mathbb{C} \longrightarrow \mathbb{C}^n \times \mathbb{C} = \mathbb{C}^{n+1}$$

3 Transverse intersection and cohomology

Using Grassmannian and line bundles, we can describe curves or other algebraic varieties in a certain space(Eg. \mathbb{C}^n). Some of the conditions we used to enumerate, such as intersection with a curve or containment in some hypersurfaces, are just some submanifolds of our parameter space.

Eg 3.1. Conics in \mathbb{P}^2 pass through the point $[1 : 0 : 0]$. Generally conics in \mathbb{P}^2 can be represented by degree 2 polynomials:

$$a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_0x_1 + a_4x_0x_2 + a_5x_1x_2 = 0$$

if we consider the parameter space $[a_0 : a_1 : a_2 : a_3 : a_4 : a_5]$, this is just P^5 . if the conics pass through $[1 : 0 : 0]$

$$a_0 = 0$$

And by definition $a_0 = 0$ is a submanifold of P^5 .

What if we have more than one conditions? Apparently, this means there are multiple submanifolds and what we need to consider is the intersection of these submanifolds. Then we changed the enumeration problem to a intersection problem. To get a integer result, the dimension of the intersection has to be 0, so we need an algebra to calculate the the "number"(which make sense with 0 dimension)of intersection while keeping track of the dimension.

Another problem arised,which is more primary, is that what kind of intersection we are considering. For example, a line can intersect with a surface in three ways, pass through the surface, tangent to the surface or contained in the surface. Which one is more general than the other? Obviously, the first case. We can see this by changing the position of the line in a small amount. The first case remain the same. If we move the line in the direction normal to the surface, the tangent line pass through the surface. The result is similar for the third case.

And we will solve this two problems in this chapter. Firstly, we revise the definition of homology of a topological space.

Def 3.2. Let X be a topological space. We define the group of singular k-chains $C_k(X)$ to be the free abelian group generated by the set of all continuous maps from the standard k-simplex to X :

$$C_k(X) = \left\{ \sum_i n_i f_i \mid n_i \in \mathbb{Z}, f_i \Delta_k \rightarrow X \right\}$$

As we known, each standard k-simplex has $k + 1$ faces, each of which is homeomorphic to the standard $(k - 1)$ -simplex. We use $j_{i,k} : \Delta_{k-1} \hookrightarrow \Delta_k^i$ to denote this inclusion map.

Def 3.3.

$$\begin{aligned} \partial_k : C_k(X) &\rightarrow C_{k-1}(X) \\ \sum_i n_i f_i &\mapsto \sum_{i,r} (-1)^r n_i (f_i \circ j_{r,k}) \end{aligned}$$

Lem 3.4. $\partial_{k-1} \circ \partial_k = 0$

And now we have $Im(\partial_{k+1}) \subset Ker(\partial_k)$. The group $Z_k(X) = Ker(\partial_k)$ is called the group of k -cycles and the group $B_k(X) = Im(\partial_{k+1})$ is the group of k -boundaries.

Def 3.5. The k^{th} homology group is the quotient

$$H_K(X) = \frac{Z_k(X)}{B_k(X)}$$

Now we want to associate k -dimensional submanifolds of X to homology classes in the k^{th} homology group. So we can keep track of the dimension of different submanifolds. To make things easier, we only consider the case when X is an oriented manifold and only the oriented submanifolds in X .

Def 3.6. An oriented triangulation of Z is a decomposition $Z = \cup Z_i$ of Z into a union of finitely many subsets Z_i such that the following hold:

There are orientation-preserving homeomorphisms

$$f_i : \Delta_k \rightarrow Z_i$$

Any intersection $Z_{i_1} \cap \dots \cap Z_{i_R}$ is identified via any of the f_{i_i} with a face of Δ_k

Now we want to show if Z is compact and oriented submanifold of a compact complex manifold, $[Z]$ is a homology class. We choose a triangulation as above $T = \sum_i f_i \in C_k(X)$.

Lem 3.7. $\partial_k T = 0$

Proof. $\partial_k T = \sum_i g_i, g_i \in C_{k-1}(X)$. As Z compact and oriented, each of the $(K-1)$ -chain is in the boundaries of exactly two k -chains on the opposite side. So each of the $(k-1)$ -chain appears in the sum exactly twice, once with a plus sign and once with a minus sign. So $\partial_k T = 0$. \square

And this is true even when we choose a different triangulation.

However, homology is not the best tool to describe the intersection of these submanifolds. Cohomology is the better tool and it is highly related to homology via Poincaré duality. We can define it by a lot of definition we used for homology.

Def 3.8. The group of k -cochains is

$$C^k(X) = Hom(C_k(X), \mathbb{Z})$$

The coboundary map is:

$$\begin{aligned} \delta_k : C^k(X) &\rightarrow C^{k+1}(X) \\ (\delta_k(f))(Z) &= f(\partial_{k+1}(Z)) \end{aligned}$$

where $f \in C^k(X)$ and $Z \in C_{k+1}(X)$

Lem 3.9. $\delta_k \circ \delta_{k-1} = 0$

Proof.

$$\begin{aligned} ((\delta_k \circ \delta_{k-1})(f))(Z) &= (\delta_k(\delta_{k-1}(f)))(Z) \\ &= (\delta_{k-1}(f)(\partial_k(Z))) \\ &= f(\partial_{k-1} \circ \partial_k(Z)) = 0 \end{aligned}$$

□

Similarly the group $\text{Ker}(\delta_k)$ is called the group of k-cocycles $Z^k(X)$ and the group $\text{Im}(\delta_k)$ is called the group of k-coboundaries $B^k(X)$. and we have $B^k(X) \subset Z^k(X)$.

Def 3.10. The k^{th} cohomology group is the quotient

$$H^k(X) = \frac{Z^k(X)}{B^k(X)}$$

And immediately there is a pairing between cohomology and homology:

$$\begin{aligned} \langle \cdot, \cdot \rangle : H^k(X) \times H_k(X) &\rightarrow \mathbb{Z} \\ \langle [f], [Z] \rangle &= f(Z) \end{aligned}$$

4 Enumerative problem with lines

5 Vector bundles and chern class

6 Applications