Imperial College London

UROP Project

Imperial College London

Department of Mathematics

Enumerative geometry on manifolds

Author:
Wei Jiang

Supervisor:
Dr. Arbesfeld Noah

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When we try to describe curves on a 2-sphere, things become difficult. If we work in $\mathbb{R}^3$, we increse the complexity. If we work in $\mathbb{R}^2$, indentifying some points on the sphere is inevitable, which over simplifies the question. To deal with a topological spaces like sphere are not easy, however we do have tools to describe such a space.

Def 2.1. An n-dimensional topological manifold is a topological space X together with a set of open sets on X U_α and maps $\phi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ such that

(1) $X \subseteq \bigcup_\alpha U_\alpha$
(2) Each ϕ_α is a homeomorphism from U_α to a open set $V_\alpha \subseteq \mathbb{R}^n$.

The pair (U_α, ϕ_α) is called a chart for X. The collection $\{(U_\alpha, \phi_\alpha)\}$ is called an atalas for X.

Each ϕ_α defines a system of coordinates on U_α, which is the usual coordinates $(x_1, x_2, x_3, ..., x_n)$ on $V_\alpha \subseteq \mathbb{R}^n$. Thiese are the local coordinates on U_α. Functions $f_{\alpha\beta} : V_\beta \longrightarrow V_\alpha$ defined by $f_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$are called the transition functions.

Now we can do calculations on manifolds use the local coordinates on U_α, and indentify calculations with different systems of coordinates using transion functions.

Eg 2.2. The 2-sphere S^2 is a manifold.pick north and south poles, $(0, 0, 1)$ and $(0, 0, -1)$ on $S^2 \subset \mathbb{R}^2$. We have one-to-one correspondences $\phi_1 : S^2 - \{(0, 0, 1)\} \longrightarrow \mathbb{R}^2$ and $\phi_2 : S^2 - \{(0, 0, -1)\} \longrightarrow \mathbb{R}^2$ defined by

$$\phi_1(x, y, z) = (\frac{x}{1-z}, \frac{y}{1-z}) \qquad \phi_2(x, y, z) = (\frac{x}{1+z}, \frac{y}{1+z})$$

With inverses

$$\phi_1^{-1}(x, y) = (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1})$$
$$\phi_1^{-1}(x, y) = (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, -\frac{x^2+y^2-1}{x^2+y^2+1})$$

We can easly show these are homeomorphisms, so S^2 is a manifold.

Def 2.3. A Grassmannian $Gr(k, n)$ over $\mathbb{C}$ is the space of all k-dimensional linear subspace of $\mathbb{C}^n$.

Eg 2.4. $Gr(1, n)$ the projective spaces $\mathbb{P}^n$.

Eg 2.5. $Gr(2, 4)$ the hyperplanes in $\mathbb{P}^4$.

Thm 2.6. $Gr(k, n)$ is a manifold with dimension $k(n-k)$.

Proof. $\square$

3 Transverse intersection and cohomology

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