# Wheatstone Bridge Sensitivity

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*Abstract*—**One of the important properties of a bridge is the sensitivity of the unbalance voltage to changes in some bridge parameter. In the self-balancing bridge, the uniformity of the null sensitivity, as well as the peak value of sensitivity, is relevant. The sensitivity of the four-arm Wheatstone bridge is reexamined from this point of view, with particular regard to the problem of the interchange of the generator and detector. It is shown that a single parameter set of universal curves suffices to describe the sensitivity variation of both configurations.**

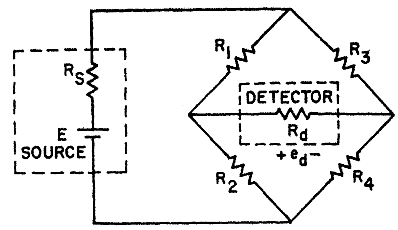


Fig. 1. Wheatstone bridge.

## I. Introduction

IT IS WELL KNOWN that the Wheatstone bridge balance is unaffected when the source and detector are interchanged, but that the null sensitivity is affected by this interchange. For a fixed set of bridge arms one of the two arrangements is superior. The criterion for obtaining maximum sensitivity has been expressed in a number of ways; for example, "considering the battery and the galvanometer, the one having the higher resistance should join the junction of the two highest resistance bridge arms to the junction of the two lowest" [1]. The classical analysis of this problem is complicated by two peripheral factors.

1) Maximum sensitivity is obtained at maximum source voltage, which in turn is determined by the allowable power dissipation. Hence, the two circuit arrangements should be compared not on the basis of equal voltages, but on the basis of equal peak-power dissipation.
2) The measurement accuracy depends on the galvanometer damping, which in turn is determined by the bridge resistance seen at the galvanometer terminals. Hence a comparison of sensitivity values of various bridge arrangements is of limited value unless the galvanometer-damping ratios are identical.

The intertwining of these three factors, null balance sensitivity, power dissipation in the bridge arms, and galvanometer damping, makes it difficult to establish a single formula for the optimum bridge arrangement [2], [3].

The present paper concerns itself with the self-balancing or servo-driven bridge employed to measure an unknown resistance varying over a *wide* range of values. The electronic amplifier used in the null detection circuit effectively isolates the bridge from the detector, so that the bridge resistance level has only a minor influence on the overall damping. Such bridges are commonly operated from a fixed voltage source. The sensitivity problem in such self-balancing bridges can therefore be treated in a much simpler fashion that the classical problem of the battery-operated laboratory bridge.

The accuracy of a servo-driven bridge depends among other things on the servo loop gain, and the null sensitivity of the bridge itself is a factor in this overall gain function. Uniform measurement accuracy for all values of the unknown resistor requires a uniform value of loop gain, hence, uniform bridge sensitivity. It would appear that the optimum arrangement for a self-balancing bridge is one that yields a *uniform* value of sensitivity over the range of resistance values of the unknown. *Maximizing* the value of the sensitivity would appear to be a secondary objective.

The material that follows is an analysis of the Wheatstone bridge circuit, Fig. 1, from this point of view. It is convenient to define a *dimensionless* sensitivity function $S_{R_i}$ by expressing the unbalance voltage $e_d$ as a fraction of the source voltage $E$. Thus,

$$S_{R_i} = \left|\frac{\partial e_d/E}{\partial R_i/R_i}\right| \quad \text{at null} \tag{1}$$

designates the null sensitivity to a fractional change in the resistance $R_i$.

## II. Analysis of Ideal Bridge

The simple case of zero source impedance and detector conductance $R_s = 0$ and $R_d = \infty$ is analyzed first in order to get an overview of the problem. From Fig. 1

$$\frac{e_d}{E} = \frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4} = \frac{R_2R_3 - R_1R_4}{(R_1 + R_2)(R_3 + R_4)}. \tag{2}$$

Without loss of generality $R_4$ is chosen as the unknown resistor to be measured. The analysis is restricted to a bridge with two fixed and one adjustable bridge arm. There are now two configurations.

1) The two fixed resistors are connected across the source; i.e., $R_1$ and $R_2$ are fixed, $R_3$ is adjustable.

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2) The two fixed arms are connected across the detector; i.e., $R_1$ and $R_3$ are fixed, $R_2$ is adjustable.

These two configurations are equivalent to interchanging source and detector. [A third case, fixed $R_2$ and $R_3$ with adjustable $R_1$, can be shown to yield results similar to 2).]

*Case 1: $R_3$ Adjustable*

The sensitivity of the bridge null to a misadjustment in the balancing resistor $R_3$ is

$$\frac{\partial e_d}{\partial R_3} = \frac{R_4 E}{(R_3 + R_4)^2}. \tag{3}$$

Hence the dimensionless sensitivity function is

$$S_{R_3} = \left|\frac{R_3 R_4}{(R_3 + R_4)^2}\right|_{R_1R_4=R_2R_3} = \frac{(R_1/R_2)}{[1 + (R_1/R_2)]^2}. \tag{4}$$

Since $R_1$ and $R_2$ are fixed bridge arms, the sensitivity is constant, independent of the value of the unknown resistor $R_4$, which is being balanced. The magnitude of this uniform sensitivity depends only on the bridge ratio $R_1/R_2$. Maximum sensitivity is obtained for equal bridge arms $R_1 = R_2$ and is

$$S_{\max} = \tfrac{1}{4}. \tag{5}$$

*Case 2: $R_2$ Adjustable*

This case requires the computation of

$$\frac{\partial e_d}{\partial R_2} = \frac{R_1 E}{(R_1 + R_2)^2} \tag{6}$$

$$S_{R_2} = \left|\frac{R_1 R_2}{(R_1 + R_2)^2}\right|_{R_1R_4=R_2R_3} = \frac{R_3/R_4}{[1 + (R_3/R_4)]^2}. \tag{7}$$

The bridge sensitivity is *not* constant, but depends on the value of the unknown $R_4$; the peak value of sensitivity is again $\frac{1}{4}$ and occurs for the equal arm bridge $R_4 = R_3$.

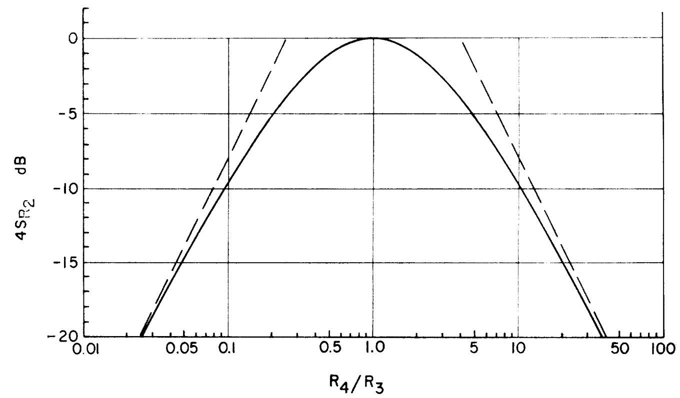


Fig. 2. Sensitivity variation of ideal bridge, fixed arms across detector.

## III. General Wheatstone Bridge

When the source and detector resistances are of the same order of magnitude as the bridge arm, the analysis of the ideal bridge no longer suffices, and the general bridge must be considered. The determination of the unbalanced voltage and of its derivatives is very tedious; a simpler method of computing the null sensitivity is by means of the compensation theorem [4]. The details of either analysis method are available in textbooks and will not be repeated here [1], [4].

*Case 1: Fixed $R_1$ and $R_2$, and Variable $R_3$*

The relevant sensitivity function is

$$S_{R_3} = \frac{R_1R_2R_dR_4}{[(R_1+R_2+R_s)R_4+R_2R_s][R_1R_4+(R_1R_2+R_1R_d+R_2R_d)]} \tag{8}$$

and it vanishes for both zero and infinite $R_4$. Its maximum is found by the usual differentiation procedure:

$$S_{\max} = \frac{\dfrac{(R_1/R_2)}{[1 + (R_1/R_2)]^2}}{\left\{\sqrt{\dfrac{R_s}{R_d}\dfrac{R_1/R_2}{[1 + (R_1/R_2)]^2}} + \sqrt{\left[\dfrac{1 + R_s}{R_1 + R_2}\right]\left[1 + \dfrac{1}{(1/R_1 + 1/R_2)R_d}\right]}\right\}^2} \tag{9}$$

The normalized sensitivity $4S_{R_2}$ is shown in Fig. 2 on logarithmic coordinates. The plot shows a superficial resemblance to a frequency response plot of $j\omega/(1 + j\omega)^2$, that is, at small and large values of $R_4/R_3$ the slopes of the curve approaches asymptotes with a 20 dB/decade roll-off. However, at the "break" the correction factor is $-12$ dB rather than $-6$ dB. Hence the gain characteristic of Fig. 2 is much flatter than that of a frequency response characteristic. The decrease in gain 1 decade on either side of the peak is only 10 dB; it would be 17 dB for a frequency response curve of the same asymptotic roll-off.

From the point of view of uniform sensitivity, Case 1 is clearly superior. We can expect a similar result when source resistance and detector conductance are small compared to the bridge-arm parameters.

and it occurs at a value of $R_4$ equal to

$$R_{\text{opt}} = \sqrt{\frac{R_sR_d}{R_1/R_2}}\sqrt{\frac{1 + \dfrac{1}{(1/R_1 + 1/R_2)R_d}}{1 + R_s/(R_1 + R_2)}}. \tag{10}$$

It is now possible to rearrange (8) in the form

$$\frac{S_{R_3}}{S_{\max}} = \frac{2 + \alpha + 1/\alpha}{R_4/R_{\text{opt}} + R_{\text{opt}}/R_4 + \alpha + 1/\alpha}, \tag{11}$$

where

$$\alpha = \sqrt{\frac{(R_1/R_2)(R_s/R_d)}{[1 + (R_1/R_2)]^2}} \cdot \frac{1}{\sqrt{[1 + (R_s/R_1 + R_2)][1 + 1/(1/R_1 + 1/R_2)R_d]}} \tag{12}$$

is a fixed parameter. Note that the complicated sensitivity formula of (8) has been condensed to a relationship between normalized variables in terms of a *single* parameter. This relationship (11) forms a set of "universal curves" for the Wheatstone bridge sensitivity, and is plotted in Fig. 3. For the ideal bridge $\alpha$ is zero.

The value of $S_{\max}$ (9) is always less than the peak sensitivity of the ideal bridge as given by (4). The value of $R_4$ at which this maximum sensitivity occurs $R_{\text{opt}}$ is of the *order* of $\sqrt{R_s R_d}$, the geometric mean of source and detector resistances.

For the sensitivity function, one can define the customary half-power condition as the range of resistance for which the sensitivity is within $-3$ dB of its peak value. Fig. 4 is a plot of this *half*-range (i.e., the ratio of the resistance at the upper $-3$-dB point to the center resistance) with $\alpha$ as the parameter. For very small $\alpha$, the half-power point occurs at very large values of $R_4/R_{\text{opt}}$. For this case (11) can be approximated by

$$\frac{S_{R_3}}{S_{\max}} = \frac{1}{1 + \alpha(R_4/R_{\text{opt}})} \tag{13}$$

and the $-3$-dB resistance range becomes

$$\frac{R_4}{R_{\text{opt}}} = \frac{0.414}{\alpha}, \qquad \alpha \ll 1 \tag{14}$$

which is shown asymptotically in Fig. 4.

The meaning of *small* source resistance and conductor admittance is

$$R_s \ll R_1 + R_2 \tag{15a}$$

$$\frac{1}{R_d} \ll \frac{1}{R_1} + \frac{1}{R_2}, \tag{15b}$$

which can be memorized by noting that the source resistance shares a common loop with the *fixed* bridge arms, while the detector shares a common node (see Fig. 5). For this case (9), (10), and (12) are approximated by

$$S_{\max} \approx \frac{1}{\left\{\sqrt{\frac{(1 + R_1/R_2)^2}{R_1/R_2}} + \sqrt{R_s/R_d}\right\}^2} \tag{16}$$

$$R_{\text{opt}} \approx \sqrt{\frac{R_s R_d}{R_1/R_2}} \tag{17}$$

$$\alpha \approx \sqrt{\frac{(R_1/R_2)(R_s/R_d)}{(1 + (R_1/R_2))^2}}. \tag{18}$$

The maximum sensitivity is again provided, approximately, by the equal arm bridge $R_1 = R_2$; it is slightly lower than $\frac{1}{4}$. The corresponding parameters are

$$R_{\text{opt}} \approx \sqrt{R_s R_d} \tag{19a}$$

$$\alpha \approx \tfrac{1}{2}\sqrt{R_s/R_d}. \tag{19b}$$

For $\alpha = 0$, (9), (10)–(15), and (17)–(19) do not apply to the singular conditions where $R_s = 0$ *or* $R_d = \infty$, except that the ideal bridge $R_s = 0$ *and* $R_d = \infty$ corresponds to the horizontal line. The singular cases are treated in the Appendix.

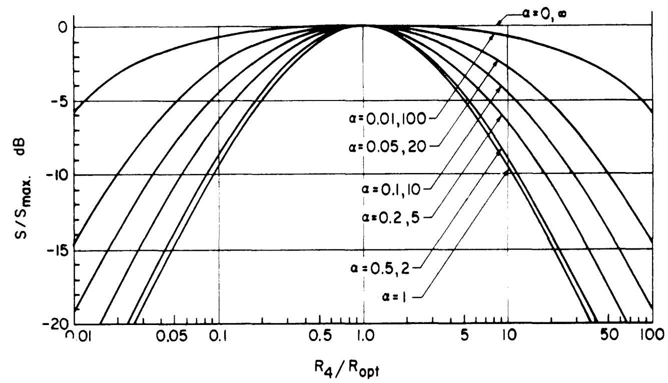


Fig. 3. Universal curves $S/S_{\max}$ versus $R/R_{\text{opt}}$.

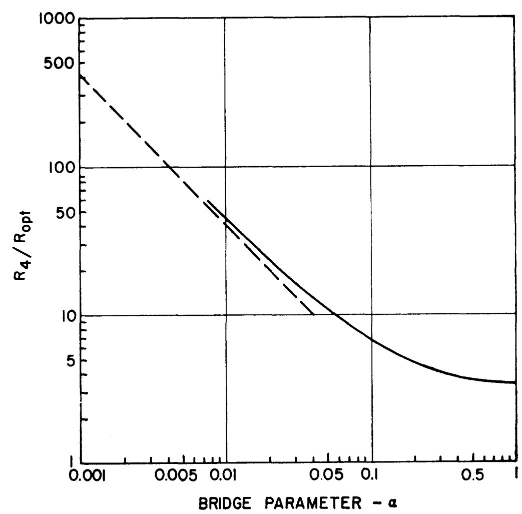


Fig. 4. Resistance half-range for —3-dB sensitivity.

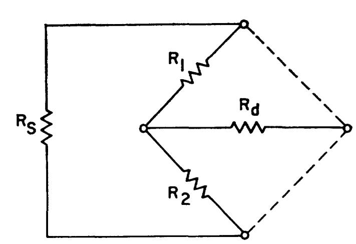


Fig. 5.

*Case 2: Fixed $R_1$ and $R_3$, and Variable $R_2$*

The sensitivity function is

$$S_{R_2} = \frac{R_1 R_3 R_4 R_d}{[(R_1 + R_3 + R_d)R_4 + R_3 R_d][R_1 R_4 + (R_1 R_3 + R_1 R_s + R_3 R_s)]} \tag{20}$$

and it again vanishes for zero and infinite $R_4$. The extremum is

$$S_{\max} = \frac{1}{\{1 + \sqrt{[1 + (R_1 + R_3)/R_d][1 + (1/R_1 + 1/R_3)R_s]}\}^2} \tag{21}$$

$$R_{\text{opt}} = R_3\sqrt{\frac{1 + (1/R_1 + 1/R_3)R_s}{1 + (R_1 + R_3)/R_d}}. \tag{22}$$

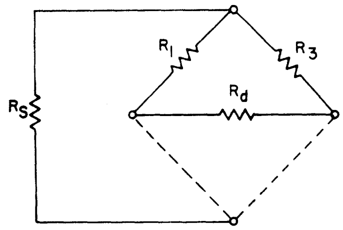


Fig. 6.

Equation (8) can again be rewritten

$$S_{R_2} = \frac{2 + \beta + (1/\beta)}{(R_4/R_{\text{opt}}) + (R_{\text{opt}}/R_4) + \beta + (1/\beta)}, \tag{23}$$

where

$$\beta = \frac{1}{\sqrt{[1 + (R_1 + R_3)/R_d][1 + (1/R_1 + 1/R_3)R_s]}} \tag{24}$$

is a fixed bridge parameter, which is unity for the ideal bridge. Since (11) and (23) have the same form, the universal curves of Fig. 3 apply as well to (23). Note that the maximum sensitivity is always less than $\frac{1}{4}$.

For this configuration, small source resistance and detector conductance means

$$\frac{1}{R_s} \gg \frac{1}{R_1} + \frac{1}{R_3} \tag{25a}$$

$$R_d \gg R_1 + R_3, \tag{25b}$$

which can be memorized by noting that the source shares a common node with the fixed bridge arms, while the detector shares a common loop (Fig. 6). For this case, $\beta \approx 1$ and (21)–(23) reduce to the ideal bridge case.

## IV. Discussion of Results

Since both cases are represented by the universal curves of Fig. 3, their relative appraisal on the basis of sensitivity would appear to be straightforward. All that is required is to compute the bridge parameters $\alpha$ and $\beta$. The case with the smaller value of the parameter yields the more uniform sensitivity function. (In case the computation produces a value of $\alpha$ greater than unity, it must be replaced by $1/\alpha$; $\beta$ is always smaller than unity.) As an example, consider the equal arm bridge with detector resistance equal to the bridge arm and source resistance equal to half the bridge arm. The results for this example are the following.

*Case 1:* $R_1 = R_2 = R_d = R$; $R_s = \frac{1}{2} R$

Equations (12), (9), and (10) yield

$$\alpha = 0.258; \qquad S_{\max} = 0.0845; \qquad R_{\text{opt}} = 0.775R.$$

*Case 2*[1]*:* $R_1 = R_3 = R_d = R$; $R_s = \frac{1}{2}R$

Equations (24), (21), and (22) yield

$$\beta = 0.408; \qquad S_{\max} = 0.0840; \qquad R_{\text{opt}} = 0.815R.$$

[1] Also computed by Frank [4].

Case 1 is slightly superior both in terms of uniformity of sensitivity and in its peak value. But this may be deceptive. In order to realize this superiority the following are necessary.

1) The unknown resistance range to be measured can be viewed as a *ratio* of maximum to minimum value, not as a *difference*. If the difference in the extreme resistance values is the criterion, then the magnitude of $R_{\text{opt}}$ becomes relevant. In the above example, Case 2 has a larger $R_{\text{opt}}$ and therefore, a smaller drop-off in sensitivity for the same resistance difference.

2) It must be possible to select the bridge parameters so that $R_{\text{opt}}$ is the geometric mean of the maximum and minimum values of the unknown.

In addition, many other factors not considered in this analysis contribute to the overall accuracy of the self-balancing bridge; for example, possible accuracy and resolution problems with regards to the value of the servo-driven balancing resistor.

One must then conclude that the universal curves of Fig. 3 are at best a starting point and guide to the selection of bridge parameters. With this in view, we now work out a specific and mathematically tractable example.

## V. Example: Equal Arm Bridge

Assume the following constraints on the bridge design are accepted: the two fixed bridge arms are identical, and their resistance equals the geometric means of the source and detector resistances. Let

$$\frac{R_d}{R_s} = m^2 \tag{26a}$$

$$R_k = mR_s = \frac{R_d}{m}, \tag{26b}$$

where $R_k$ represents the two fixed bridge arms. It is now possible to determine the bridge performance in terms of the single parameter $m$.

*Case 1:* $R_1 = R_2 = R_k$

The correction terms are

$$\frac{R_s}{R_1 + R_2} = \frac{1}{2m}; \qquad \frac{1}{(1/R_1 + 1/R_2)R_d} = \frac{1}{2m}. \tag{27}$$

Equations (9), (10), and (12) yield

$$S_{\max} = \frac{1/4}{[(1/2m) + (1 + 1/2m)]^2} = \frac{1}{4(1 + 1/m)^2} \tag{28a}$$

$$R_{\text{opt}} = \sqrt{R_s R_d} = R_k \tag{28b}$$

$$\alpha = \frac{1/2m}{1 + 1/2m} = \frac{1}{2m + 1}. \tag{28c}$$

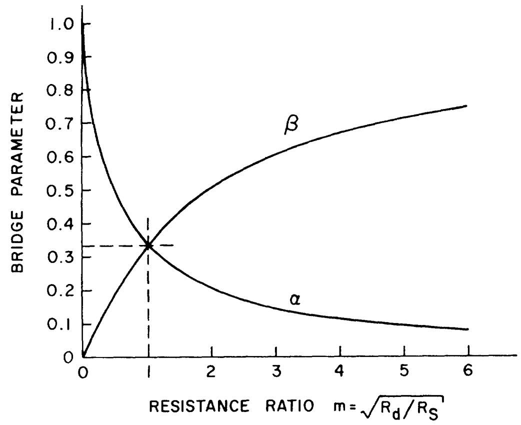


Fig. 7. Bridge parameter variation for equal arm bridge.

*Case 2:* $R_1 = R_3 = R_k$

The correction terms are

$$\frac{R_1 + R_3}{R_d} = \frac{2}{m};$$
$$\left(\frac{1}{R_1} + \frac{1}{R_3}\right)R_s = \frac{2}{m}. \tag{29}$$

Equations (21), (22), and (24) yield

$$S_{\max} = \frac{1}{[1 + (1 + 2/m)]^2} = \frac{1}{4(1 + 1/m)^2} \tag{30a}$$

$$R_{\text{opt}} = R_k \tag{30b}$$

$$\beta = \frac{1}{1 + 2/m}. \tag{30c}$$

The two cases have the same peak sensitivity, and these maxima occur at the same value of resistance! However, the shape of the sensitivity function differs, since that depends on $\alpha$ and $\beta$. The two cases are identical for $\alpha = \beta$. As is to be expected, this works out to $m = 1$, to identical source and detector resistance. For the usual $m > 1$, where the source resistance is smaller than the detector resistance, Case 1 is always superior. Only for $m < 1$ would Case 2 show more uniform sensitivity. Equations (28c) and (30c) are plotted in Fig. 7.

## Appendix

The analysis of Case 1 breaks down when either $R_s = 0$ or $R_d = \infty$. The two cases are treated below.

$R_s = 0$, *$R_d$ Finite*

Equation (8) becomes

$$S_{R_3} = \frac{\dfrac{R_1/R_2}{[1 + (R_1/R_2)]^2}}{1 + \dfrac{(1 + R_4/R_2)}{(1/R_1 + 1/R_2)R_d}}. \tag{31}$$

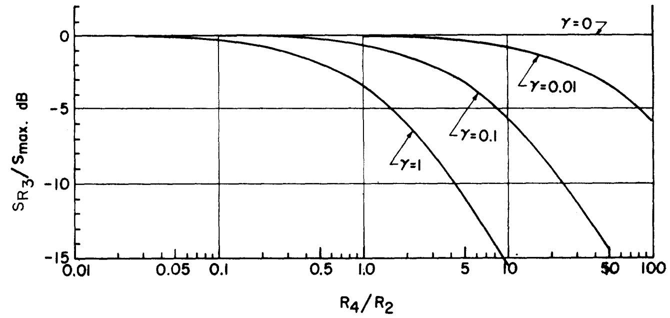


Fig. 8. Sensitivity variation, singular cases, fixed arms across source.

This function decreases monotonically from a maximum at $R_4 = 0$. The maximum can be obtained from (9) as well as from (31). One can rewrite (31)

$$\frac{S_{R_3}}{S_{\max}} = \frac{1}{1 + (\gamma/1 + \gamma)(R_4/R_2)} \tag{32}$$

where

$$\gamma = \frac{1}{(1/R_1 + 1/R_2)R_d}. \tag{33}$$

Equation (30) is plotted in Fig. 8.

$R_d = \infty, R_s \neq 0$

Equation (8) becomes

$$S_{R_3} = \frac{\dfrac{R_1/R_2}{(1 + R_4/R_2)^2}}{1 + \dfrac{(1 + R_2/R_4)R_s}{R_1 + R_2}} \tag{34}$$

which rises monotonically to a peak value at $R_4 = \infty$. This peak value is again obtainable from (9) as well as from (34). The latter can be rewritten

$$\frac{S_{R_3}}{S_{\max}} = \frac{1}{1 + \delta/(1 + \delta)(R_2/R_4)}, \tag{35}$$

where

$$\delta = \frac{R_s}{R_1 + R_2}.$$

Equations (32) and (35) have the same form except that the independent variables are reciprocal. Hence Fig. 8 can be used to display both (32) and (35).

## References

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